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Analytical Approximation of Periodic Ateb-Functions via Elementary Functions

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Abstract
We consider the problem of analytic approximation of periodic Ateb- functions, widely used in nonlinear dynamics. Ateb-functions are the result of the following procedure. Initial ODE contains only the inertial and non-linear terms. It can be integrated, which leads to an implicit solution. To obtain explicit solutions we are led to necessity to inverse incomplete Beta functions. As a result of this inversion we obtain the special Ateb-functions. Their properties are well known, but the use of Ateb- functions is difficult in practice. In this regard, the problem arises of the Ateb functions approximation by smooth elementary functions. For this purpose in the present article the asymptotic method is used with a quantity $1/(\alpha+1)$ *as a small parameter, were* $\alpha \geq 1$ *– exponent of nonlinearity. We also investigated the analytical approximation of Ate-b functions' period Comparison of simulation results, obtained by the approximate expression, with the results of numerical solution of the corresponding Cauchy problem shows their sufficient accuracy for practical purposes, even for* $\alpha = 1$.

Keywords
Ateb function, elementary function,· asymptotic approximation, Beta function

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Introduction

The equation of the form

$$\frac{d^2x}{dt^2} + x|x|^{\alpha-1} = 0, \quad \alpha > 0 \tag{1}$$

is often found in the problems of nonlinear dynamics [1], so its solution in a standard form is of great interest. It can be integrated using the function Cs and Sn, introduced by Liapunov [2,3]. These functions are an inverse of incomplete Beta function [4]. Much later, the similar functions (up to normalization) have been proposed by Rosenberg, who called them Ateb- functions (cam and sam) [5,6]. Under this title they entered the modern practice [7-9]. However, it is inconvenient to work with these objects, and therefore there is a problem of the approximate analytical approximation of Ateb-functions. This approximation is based on the use of nonsmooth (sawtooth) functions, proposed in [10] (see also [11]). The Ateb- functions approximations by elementary smooth functions have been proposed in [12, 13] (see also [14]). This paper deals with the generalization of the results of [13], with the approximation of period for periodic Ateb- functions and with the comparison of various approximations.

The paper is organized as follows. Asymptotic procedure is described in Section 1. Approximations of period are analyzed in Section 2. Error estimations are considered in Section 3. Finally, Section 4 presents the concluding remarks.

1. Asymptotic procedure

We assume further $1 \le \alpha$ (case $0 < \alpha < 1$ was investigated in [15]). Let us first consider the following initial conditions for the equation (1)

$$x(0) = 1, \quad \frac{dx}{dt}|_{t=0} = 0 \tag{2}$$

The first integral of Cauchy problem (1), (2) can be written as follows:

$$\left(\frac{dx}{dt}\right)^2 = \frac{2}{\alpha + 1}\left(1 - x|x^a|\right) \tag{3}$$

Integration allows us to reduce the expression (3) to the form

$$\lambda^{1/2} t = \pm \int_1^{0 \le x \le 1} \frac{d\xi}{\sqrt{1 - \xi^{2/\lambda}}} = \pm\left[\int_0^{0 \le x \le 1} \frac{d\xi}{\sqrt{1 - \xi^{2/\lambda}}} - \int_0^1 \frac{d\xi}{\sqrt{1 - \xi^{2/\lambda}}}\right] \tag{4}$$

where $\lambda = 2/(\alpha + 1)$.

The definite integral in the right-hand side of the expression (4) is calculated elementarily,

$$\int_0^1 \frac{d\xi}{\sqrt{1 - \xi^{2/\lambda}}} = \frac{\lambda}{2} B\left(\frac{\lambda}{2}, \frac{1}{2}\right) \tag{5}$$

where $B(...,...)$ is the complete Beta- function [16].

Later we use a minus sign in the right-hand side of equation (4). Using the change of variable $\xi = sin^{\lambda}\theta$, we obtain the expression (4) in the form

$$-\lambda^{1/2} t + \frac{\lambda}{2} B\left(\frac{\lambda}{2}, \frac{1}{2}\right) = \lambda \int_0^{0 \le \theta \le \pi/2} sin^{-1+\lambda}\theta \ d\theta . \tag{6}$$

We consider separately the integrand

$$sin^{-1+\lambda}\theta = \theta^{-1+\lambda}\left(\frac{\theta}{sin\theta}\right)^{1-\lambda} = \theta^{-1+\lambda}\left[\frac{\theta}{sin\theta} - \lambda \ln\frac{\theta}{sin\theta} + \ldots\right]$$

Using Maclaurin series for the function $\theta/\sin\theta$ one obtains

$$sin^{-1+\lambda}\theta = \theta^{-1+\lambda} + \frac{\theta^{1+\lambda}}{3!} + \ldots + O(\lambda) .$$

Next, we consider $\lambda \ll 1$ (as will be shown below, this assumption is not restrictive, and found further approximate expression can be used even for $\lambda \ge 0.5$). Then the main contribution in the integration makes the first term of this expression, so to a first approximation one obtains

$$-\lambda^{1/2} t + \frac{\lambda}{2} B\left(\frac{\lambda}{2}, \frac{1}{2}\right) \approx \theta^{\lambda},$$

i.e. $\theta \approx \left[-\lambda^{1/2} t + \frac{\lambda}{2} B\left(\frac{\lambda}{2}, \frac{1}{2}\right)\right]^{1/\lambda}$.

In the original variables we obtain

$$x(t) \approx \sin^{\lambda}\left(\left[-\lambda^{1/2}t + \frac{\lambda}{2}B\left(\frac{\lambda}{2},\frac{1}{2}\right)\right]^{1/\lambda}\right) \tag{7}$$

We make the change of variable $\tau = \lambda^{1/2}t$, then

$$x(\tau) \approx \sin^{\lambda}\left(\left[\frac{\lambda}{2}B\left(\frac{\lambda}{2},\frac{1}{2}\right) - \tau\right]^{1/\lambda}\right) \tag{8}$$

Solution (8) should be used for a quarter of the period T of the original solution, further it must be extended periodically.

Let us now consider the solution of equation (1) with initial conditions

$$x(0) = 0 \tag{9}$$

$$\frac{dx}{dt} = 1 \quad \text{for} \quad x = 1 \tag{10}$$

An approximate solution of Cauchy problem (1), (9), (10) has the form

$$x(t) \approx \sin^{\lambda}\left(\left[\lambda^{1/2}t\right]^{1/\lambda}\right) \tag{11}$$

or

$$x(\tau) \approx \sin^{\lambda}\left(\tau^{1/\lambda}\right). \tag{12}$$

Note that the expression (12) gives one approximate expression for inverse of incomplete Beta function [16] from the $\lambda = 1$ (sinus) to $\lambda = \infty$ (linear function).

2. Approximation of period

From the expression (12) we obtain an approximate formula for the period T of Ateb- function

$$T = 4\left(\frac{\pi}{2}\right)^{\lambda} \tag{13}$$

Let us consider the problem of approximation of period detail. The exact expression of this period is [16]

$$\frac{T}{4} = \frac{\lambda}{2}B\left(\frac{\lambda}{2},\frac{1}{2}\right) = \frac{\lambda}{2}\frac{\Gamma\left(\frac{\lambda}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\lambda}{2}\right)} \tag{14}$$

where $\Gamma(...)$ is the Gamma function [16].

Using formulas [16]

$$\Gamma\left(\frac{1}{2}\right) = \pi^{1/2}$$

$$\Gamma\left(\frac{1}{2}+\frac{\lambda}{2}\right) = \frac{2\pi^{1/2}\Gamma\left(\frac{\lambda}{2}\right)}{2^{\frac{2}{n+1}}\Gamma(\lambda)}$$

and asymptotic approximations for $\lambda \ll 1$ [16]

$$\Gamma(\lambda) \sim \frac{1}{\lambda}$$

one obtains

$$\frac{T}{4} = 2^{\lambda-2}\lambda\frac{\Gamma^2(2\lambda)}{\Gamma(\lambda)} \sim 1 + \lambda \ln 2 \quad (15)$$

Formula (15) can be corrected in such a way

$$\frac{T}{4} \sim 1 + \lambda\left(\frac{\pi}{2} - 1\right) \quad (16)$$

Let us estimate the accuracy of the various approximations period. The numerical results are shown in Fig. 1. It can be seen that the proposed approximate formulas provide a reasonably accurate approximation of the period.

3. Error estimations

To further we choose the approximation of the period in form (13). Then the formula (8) can be written as

$$x(\tau) \approx \sin^{\lambda}\left(\left[\left(\frac{\pi}{2}\right)^{\lambda} - \tau\right]^{\frac{1}{\lambda}}\right) \quad (17)$$

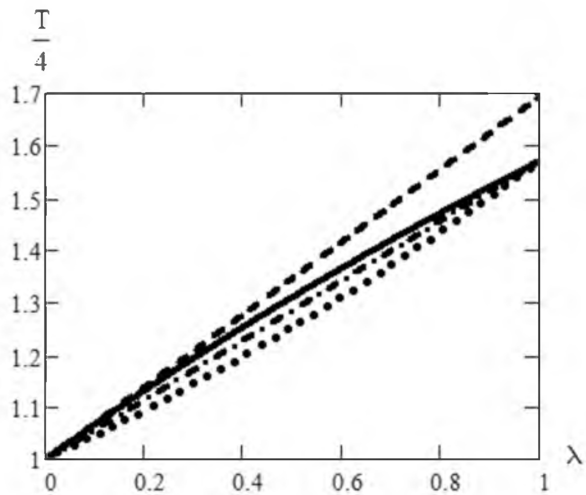

Figure 1. Approximations of period: solid line corresponds to formula (14) (exact values); dotted line – to formula (13); dashed line – to formula (15); dashed dotted line – to formula (16)

Let us cite the approximation which was obtained yet in [12] under the assumption $\lambda \ll 1$

$$x(\tau) = 1 - \lambda \ln\left(\cosh\frac{\tau}{\lambda}\right) \quad (18)$$

Let us compare the results of calculations using formulas (17), (18) with the solution of the Cauchy problem

$$\frac{d^2x}{d\tau^2}+\frac{1}{\lambda}x|x|^{\frac{2}{\lambda}-2}=0 \tag{19}$$

$$x(0)=1 \quad \frac{dx}{d\tau}|_{\tau=0}=0 \tag{20}$$

The graphs shown in Fig. 2-6, confirm the relatively high accuracy of approximations (17) and (18).

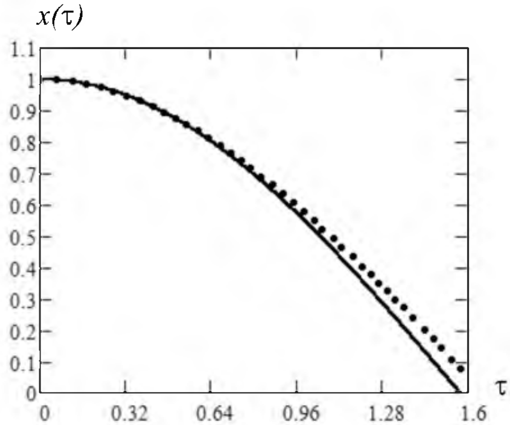

Figure 2 Comparison of solution of Cauchy problem (19), (20) (solid line) with approximation (17) (dashed line) and (18) (dotted line) for $\lambda = 1$

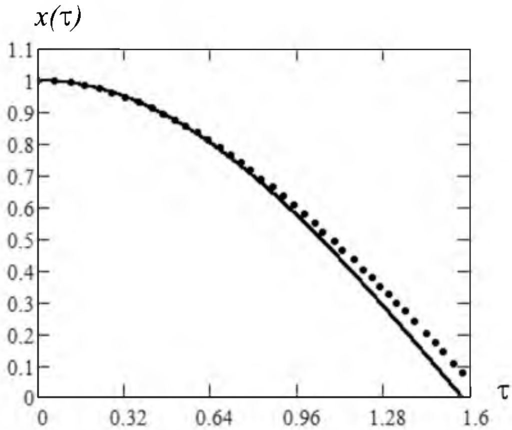

Figure 3 Comparison of solution of Cauchy problem (19), (20) (solid line) with approximation (17) (dashed line) and (18) (dotted line) for $\lambda = 0.5$

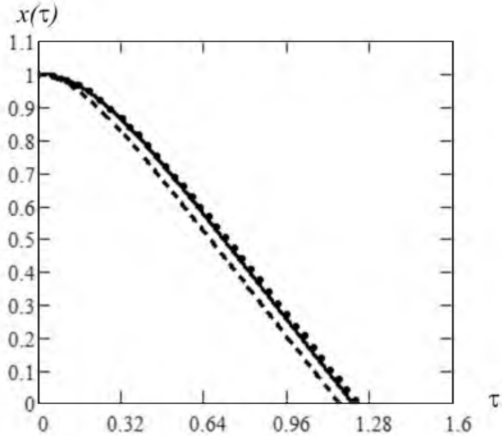

Figure 4 Comparison of solution of Cauchy problem (19), (20) (solid line) with approximation (17) (dashed line) and (18) (dotted line) for $\lambda = 1/3$

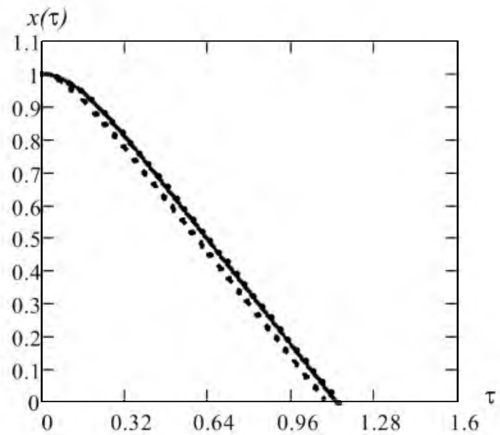

Figure 5 Comparison of solution of Cauchy problem (19), (20) (solid line) with approximation (17) (dashed line) and (18) (dotted line) for $\lambda = 0.2$

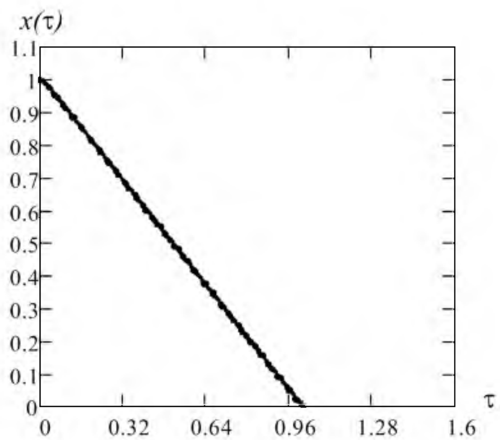

Figure 6 Comparison of solution of Cauchy problem (19), (20) (solid line) with approximation (17) (dashed line) and (18) (dotted line) for $\lambda = 0.02$

Interestingly, that the asymptotic expressions which are obtained on the assumption $\lambda \ll 1$, give sufficiently accurate results over the entire range $1 \le \lambda < \infty$. This once again confirms the words of Crighton [17]: „All experience suggests that asymptotic solutions are useful numerically far beyond their nominal range of validity, and can often be used directly."

It is also interesting that there may be quite accurate approximation of Ateb- function by different elementary functions.

Comparison of the results of calculations by the formula (12) with the solution of Cauchy problem

$$\frac{d^2x}{d\tau^2} + \frac{1}{\lambda} x|x|^{\frac{2}{\lambda}-2} = 0 \tag{21}$$

$$x(0) = 0 \qquad \left.\frac{dx}{d\tau}\right|_{x=1} = 0 \tag{22}$$

confirms the sufficient accuracy of approximation (12).

Comparison of solutions matching Cauchy problems were also conducted with the calculation according to the formulas (12), (18), (19) for $\alpha \le 1$. It turns out that formulas (12), (18) and (19) can be used with sufficient accuracy for $0.5 \le \alpha \le 1$ also.

Conclusions

Approximate analytical expressions of the sine and cosine Ateb functions can be written as follows

$$sam(\tau) \approx \sin^{\lambda}\left(\left[\left(\frac{\pi}{2}\right)^{\lambda} - \tau\right]^{\frac{1}{\lambda}}\right)$$

$$cam(\tau) \approx \sin^{\lambda}(\tau)^{\frac{1}{\lambda}} \tag{23}$$

To approximate the cosine Ateb- function the expression (18) can also be used.

To calculate the derivative of the sine and cosine Ateb- functions one can use the first known expression [1,6], then the approximation (21).

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