Proceedings of the 5th International Conference on Nonlinear Dynamics
ND-KhPI2016
September 27-30, 2016, Kharkov, Ukraine

Resonance Behavior of the Forced Dissipative Spring-Pendulum System

Kateryna Yu. Plaksiy[1], Yuri V. Mikhlin[2*]

Abstract
Dynamics of the dissipative spring-pendulum system under periodic external excitation in the vicinity of external resonance and simultaneous external and internal resonances is studied. Analysis of the system resonance behaviour is made on the base of the concept of nonlinear normal vibration modes (NNMs), which is generalized for systems with small dissipation. The multiple scales method and subsequent transformation to the reduced system with respect to the system energy, an arctangent of the amplitudes ratio and a difference of phases of required solutions are applied. Equilibrium positions of the reduced system correspond to nonlinear normal modes. So-called Transient nonlinear normal modes (TNNMs), which exist only for some certain levels of the system energy are selected. In the vicinity of values of time, corresponding to these energy levels, these TNNMs temporarily attract other system motions. Interaction of nonlinear vibration modes under resonance conditions is also analysed. Reliability of obtained analytical results is confirmed by numerical and numerical-analytical simulation.

Keywords
External and internal resonances, Nonlinear normal modes

[1,2] National Technical University "Kharkov Polytechnic Institute", Kharkov, Ukraine
*** Corresponding author:** muv@kpi.kharkov.ua

Introduction

Presence of external and internal resonances leads to the increase of vibration amplitudes, which is an undesirable phenomenon, in particular, in engineering practice. Resonances cause complex behavior of the nonlinear system. Namely, few vibration modes can exist simultaneously; some vibration modes can lose their stability, and the new vibration modes can appear as a result of bifurcation; transfer of energy from one subsystem to another one can be observed; the energy localization can be realized etc. Resonance effects are analyzed in numerous publications, in particular, in books [1-3]. The transfer of energy caused by internal resonance in nonlinear systems is discussed in various publications, in particular, in [4-7].

Nonlinear normal modes (NNMs) are important elements of the nonlinear systems behavior. The Kauderer-Rosenberg concept of NNMs [8,9], first proposed for conservative systems, is based on construction of trajectories in the system configuration space. Theory of NNMs for conservative and non-conservative systems, and different applications of this theory are presented in different publications, in particular, in [2, 10-12]. In nonlinear dissipative systems due to exponential decrease of vibration amplitudes the classical NNMs by Kauderer-Rosenberg cannot exist, but some similar vibration regimes can. These are some generalization of the NNMs.

The so-called *reduced system* can be constructed by means of introducing new variables, one of them is concerned with the system energy, other ones are arctangent of the ratio of amplitudes and the difference of phases. This reduced system was used earlier in some preceding publications for non-dissipative systems [13,14] and for dissipative ones [15,16]. It permits to show some important elements of the nonlinear system resonance behavior. Important characteristics of dynamical process can be analyzed by such system. Besides, one can observe appearance of the so-called *transient nonlinear normal modes* (TNNMs) existing only for some specific values of the system energy. They temporarily attract other motions of the system when the system energy is close to these energy values.

Here the spring-pendulum system under external periodic excitation is investigated. The small mass pendulum can be considered as the vibration absorber. Both the case of external resonance on the first fundamental frequency, and the case of simultaneous external and internal resonances are analyzed. The vibration modes evolution and the energy transfer from one mode to another one are considered. The TNNMs are determined and their influence to transient of the system is analyzed. Obtained analytical results are verified by numerical and numerical-analytical simulation.

1. Forced resonance vibrations of dissipative spring-pendulum system

One considers the spring-pendulum system with small dissipation under external periodic excitation, which is shown in Fig. 1.

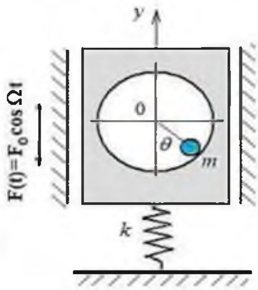

Figure 1. Spring-pendulum system

Equations of motion of the system are the following:

$$\begin{cases} \ddot{u} + \omega_u^2 u + \varepsilon\eta_u \dot{u} - \varepsilon\mu(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) = \varepsilon^2 f\cos\tau \\ \ddot{\theta} + \varepsilon\eta_\theta\dot{\theta} + p^2\sin\theta - \ddot{u}\sin\theta = 0 \end{cases} \tag{1}$$

where $u = \dfrac{y}{R}$, $\tau = \Omega t$, $\omega = (k/(M+m))^{0.5}$, $p^2 = \dfrac{g}{R\Omega}$, $\mu = m/(m+M)$, $\omega_u^2 = 1/\Omega^2$, $f = \dfrac{F_0}{(M+m)R\omega^2\Omega^2}$, $\eta_u = \dfrac{\beta_u}{(M+m)\Omega}$, $\eta_\theta = \dfrac{\beta_\theta}{m\Omega}$, β_x and β_0 are coefficients of dissipation.

There are two NNMS by Kauderer-Rosenberg in the system (1) without dissipation: the localized x-mode of vertical vibrations ($x = x(t)$, $\theta = 0$), and the non-localized θ-mode (or pendulum mode), when amplitudes for vertical and angle coordinates are of the same order. When the dissipation is present, such modes are not the NNMs by Kauderer-Rosenberg due to exponential decrease of vibration amplitudes. We will investigate such motions of the system (1) which are close to NNMs of the corresponding conservative spring-pendulum system.

One uses the multiple scale method [17]. Introducing the transformations $u \to \varepsilon u$, $\theta \to \varepsilon\theta$, we will determine a solution in the form of the following asymptotic series:

$$\begin{cases} \varepsilon u = \varepsilon u_0 + \varepsilon^2 u_1 + \ldots \\ \varepsilon\theta = \varepsilon\theta_0 + \varepsilon^2\theta_1 + \ldots \end{cases} \tag{2}$$

Here x_i and θ_i are functions of the independent variables as $T_n = \varepsilon^n t$ (n=0,1,2,...). To analyze the system (1) dynamics in vicinity of the external resonance on the first fundamental frequency, the detuning parameter Δ is introduced by the following resonance condition:

$$\omega_u^2 = 1 + \varepsilon\Delta \tag{3}$$

The following standard transformations are used:

$$\frac{d}{dt}=\frac{\partial}{\partial T_0}+\varepsilon\frac{\partial}{\partial T_1}+\varepsilon^2\frac{\partial}{\partial T_2}+\dots$$
$$\frac{d^2}{dt^2}=\frac{\partial^2}{\partial T_0^2}+2\varepsilon\frac{\partial}{\partial T_0}\frac{\partial}{\partial T_1}+\varepsilon^2\left(\frac{\partial^2}{\partial T_1^2}+2\frac{\partial}{\partial T_0}\frac{\partial}{\partial T_2}\right)+\dots \tag{4}$$

Using relations (2-4) and expansions in power series for sin0 and cos0, one has the following partial differential equations in the first and second approximations by the small parameter ε:

$$\varepsilon^1:\quad \frac{\partial^2 u_0}{\partial T_0^2}+u_0=0,\qquad \frac{\partial^2\theta_0}{\partial T_0^2}+p^2\theta_0=0 \tag{5}$$

$$\varepsilon^2:\quad \frac{\partial^2 u_1}{\partial T_0^2}+u_1=-\Delta u_0-2\frac{\partial^2 u_0}{\partial T_1\partial T_0}+\mu\left(\frac{\partial^2\theta_0}{\partial T_0^2}\right)\theta_0+\mu\left(\frac{\partial\theta_0}{\partial T_0}\right)^2-\eta_u\frac{\partial u_0}{\partial T_0}+\frac{f}{2}(e^{iT_0}+e^{-iT_0})$$
$$\frac{\partial^2\theta_1}{\partial T_0^2}+p^2\theta_1=-2\frac{\partial^2\theta_0}{\partial T_1\partial T_0}+\left(\frac{\partial^2 u_0}{\partial T_0^2}\right)\theta_0-\eta_\theta\frac{\partial\theta_0}{\partial T_0} \tag{6}$$

Solution of the system (5) is written as

$$\begin{cases} u_0=C_u(T_1)e^{iT_0}+\overline{C}_u(T_1)e^{-iT_0}\\ \theta_0=C_\theta(T_1)e^{piT_0}+\overline{C}_\theta(T_1)e^{-piT_0}\end{cases} \tag{7}$$

The solution (7) is substituted to the system (6), and secular terms are eliminated. One has

$$\begin{cases} 2i\dfrac{\partial C_u}{\partial T_1}+C_u\Delta+i\eta_u C_u-\dfrac{f}{2}=0\\ 2\dfrac{\partial C_\theta}{\partial T_1}+\eta_\theta C_\theta=0\end{cases} \tag{8}$$

Then the change of variables, $C_u=a_u e^{i\beta_u}$, $C_\theta=a_\theta e^{i\beta_\theta}$ leads to the system of equations written with respect to amplitudes a_u, a_θ and phases β_u, β_θ of the solution (7):

$$\begin{cases} a_u'=-\dfrac{\eta_x}{2}a_x-\dfrac{f}{4}\sin\beta_u\\ a_\theta'=-\dfrac{\eta_\theta}{2}a_\theta\\ \beta_u'=\dfrac{\Delta}{2}-\dfrac{f}{4a_u}\cos\beta_u\\ \beta_\theta'=0\end{cases} \tag{9}$$

The change of variables $a_u=\dfrac{\sqrt{\mu}}{2}K\cos\psi$, $a_\theta=K\sin\psi$ gives us the *reduced system* written with respect to the energy *K*, the arctangent of the ratio of amplitudes ψ and the phases β_u, β_θ:

$$K'=-K\left(\frac{\eta_u}{2}\cos^2\psi+\frac{\eta_\theta}{2}\sin^2\psi\right)-\frac{f}{4}\sin\beta_u\cos\psi$$
$$\psi'=\sin\psi\left(\frac{\eta_u-\eta_\theta}{2}\cos\psi+\frac{f}{4K}\sin\beta_u\right) \tag{10}$$

$$\beta_u' = \frac{\Delta}{2} - \frac{f}{4K\cos\psi}\cos\beta_u \qquad \beta_\theta' = 0$$

The additional equation with respect to the phase difference $\varphi = \beta_u - 2\beta_\theta$ can be written as

$$\varphi' = \Delta - \frac{f}{2K\cos\psi}\cos\beta_u \tag{11}$$

Equilibrium positions of the equations (10) and (11), which describe "vibrations in unison", correspond to nonlinear normal modes of the original system (1).

Relation $\sin\psi = 0$ corresponds to localized vibrations of the elastic spring. This localized mode exists for any levels of the energy K; it is characterized by the straight line $\psi = 0$ in the plane (ψ,φ). Then it is necessary to analyze trajectories in the plane (ψ,φ).

Localization of energy on the pendulum is impossible due to design of the system under consideration. Such localization could be described by the condition $\cos\psi = 0$. It corresponds to the straight line $\psi = \pi/2$ in the plane (ψ,φ). One has from the second equation of the system (10) that this equilibrium position exists if $\pm\frac{f}{4K}\sin\beta_u = 0$. It is possible if the phase $\beta_u = 0 + \pi n$. This vibration mode could be realized for high levels of energy which are not considered here. In this case the energy equation is the following: $K' = -\frac{\eta_\theta}{2}K$. So, the energy decreases, so, the localized on pendulum vibrations are unstable, and this hypothetical vibration mode cannot be realized.

In a case when together $\cos\psi \neq 0$ and $\sin\psi \neq 0$, the mode of coupled vibrations of the system (1) appears. Condition of the mode existence is obtained from the second equation of the reduced system (10) of the form:

$$\cos\psi = \pm\frac{f}{2K\sqrt{(\eta_u - \eta_\theta)^2 + \Delta^2}}$$

In order to analyze trajectories in the space (ψ,φ) the reduced system (10) is integrated by the Runge-Kutta method of the 4-th order when $0 \le \psi(0) \le \pi/2$ for the following initial value and values of the system parameters: $K(0) = 0.5$, $\eta_u = 0.4$, $\eta_\theta = 0.7$, $\Delta = 0.2$ and $f = 0.2$. Trajectories in the plane (ψ,φ), which are presented in Fig. 2, approach the line $\psi = 0$ corresponding to the vibration mode with localization on the spring. So, this vibration mode is stable in vicinity of the resonance, and the stable modes of coupled vibrations are not appeared.

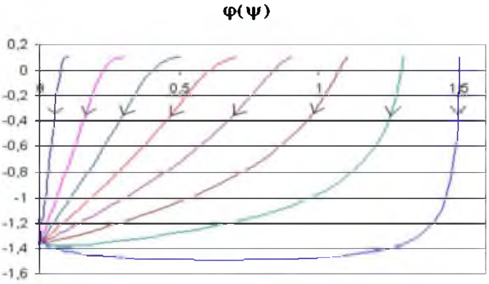

Figure 2. Trajectories in the plane (ψ,φ)

In order to illustrate behavior of the spring-pendulum system (1) near resonance one integrates the system (1) by the method of Runge-Kutta of the 4-th order on the interval $t \in [0,1000]$ for the following initial conditions: $a_u(0) = 0.2$, $a_\theta(0) = 0.1$, $\beta_u(0) = 0.1$, $\beta_\theta(0) = 0.02$, and for $\eta_u = 0.3$, $\eta_\theta = 0.2$, $\Delta = 0.1$, $\mu = 0.3$, $f = 0.5$, $p = 1.7$. The solution of the first approximation by the small parameter can be written as $u_o = 2a_u\cos(t + \beta_u)$, $\theta_o = 2a_\theta\cos(pt + \beta_\theta)$. In Figs. 3-5 coordinates are presented in time, and in the system configuration space.

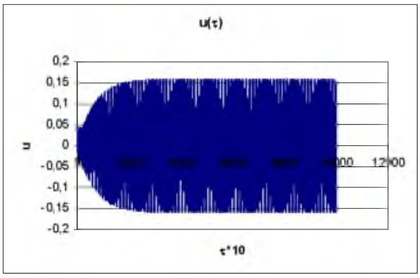

Figure 3. Dependence $u(\tau)$

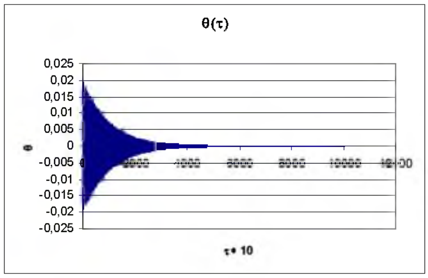

Figure 4. Dependence $\theta(\tau)$

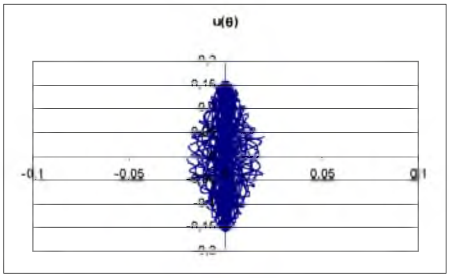

Figure 5. Dependence $u(\theta)$

Analysis of the Figures 2-5 shows that the localized mode of spring vibrations is stable in the resonance case; this confirms the preceding analysis of the reduced system. In particular, trajectories in Fig. 5 approach with time the straight line $\theta = 0$ corresponding to this localized mode.

2. Case of simultaneous external and internal resonances

In order to consider motions in vicinity of both external resonance, and internal one, we introduce two detuning parameters Δ_1 and Δ_2 as

$$\omega_u^2 = 1 + \varepsilon\Delta_1 \tag{12}$$

$$p^2 = \frac{1}{4} + \varepsilon\Delta_2 \tag{13}$$

The relation (12) corresponds to the vicinity of external resonance, and the relation (13) corresponds to the vicinity of the main parametric resonance of the system (1).

Using relations (10), (11) and (3), and expansions in power series for sinθ and cosθ, one has the following equations of the first and second approximations by the small parameter:

$$\varepsilon^1: \qquad \frac{\partial^2 u_0}{\partial T_0^2} + u_0 = 0\,, \quad \frac{\partial^2 \theta_0}{\partial T_0^2} + \frac{1}{4}\theta_0 = 0 \tag{14}$$

$$\varepsilon^2: \qquad \frac{\partial^2 u_1}{\partial T_0^2} + u_1 = -\Delta_1 u_0 - 2\frac{\partial^2 u_0}{\partial T_1 \partial T_0} + \mu\left(\frac{\partial^2 \theta_0}{\partial T_0^2}\right)\theta_0 + \mu\left(\frac{\partial \theta_0}{\partial T_0}\right)^2 - \eta_u \frac{\partial x_0}{\partial T_0} + \frac{f}{2}\left(e^{iT_0} + e^{-iT_0}\right)$$

$$\frac{\partial^2 \theta_1}{\partial T_0^2} + \theta_1 = -\Delta_2 \theta_0 - 2\frac{\partial^2 \theta_0}{\partial T_1 \partial T_0} + \left(\frac{\partial^2 u_0}{\partial T_0^2}\right)\theta_0 - \eta_\theta \frac{\partial \theta_0}{\partial T_0} \tag{15}$$

Solution of the system (14),

$$\begin{cases} u_0 = C_u(T_1)e^{iT_0} + \overline{C}_u(T_1)e^{-iT_0} \\ \theta_0 = C_\theta(T_1)e^{\frac{1}{2}iT_0} + \overline{C}_\theta(T_1)e^{-\frac{1}{2}iT_0} \end{cases} \tag{16}$$

is substituted to equations (15). Then secular terms are eliminated. Introducing, as in the preceding section, sequential changes of variables, $C_u = a_u e^{i\beta_u}$, $C_\theta = a_\theta e^{i\beta_\theta}$ and $a_u = \frac{\sqrt{\mu}}{2}K\cos\psi$, $a_\theta = K\sin\psi$, one has the *reduced system* written with respect to the energy K, the arctangent of the ratio of amplitudes ψ and the phases β_u, β_θ:

$$\begin{aligned} K' &= -K\left(\frac{\eta_u}{2}\cos^2\psi + \frac{\eta_\theta}{2}\sin^2\psi\right) - \frac{f}{2\sqrt{\mu}}\sin\beta_u\cos\psi \\ \psi' &= \sin\psi\left(\frac{\sqrt{\mu}}{2}K\sin(2\beta_\theta - \beta_u) + \frac{\eta_u - \eta_\theta}{2}\cos\psi + \frac{f}{2\sqrt{\mu K}}\sin\beta_u\right) \\ \beta_u' &= \frac{\Delta_1}{2} + \frac{\sqrt{\mu}}{2}\frac{K\sin^2\psi}{\cos\psi}\cos(2\beta_\theta - \beta_u) - \frac{f}{2\sqrt{\mu K}\cos\psi}\cos\beta_u \\ \beta_\theta' &= \Delta_2 + \frac{\sqrt{\mu}}{2}K\cos\psi\cos(2\beta_\theta - \beta_u) \end{aligned} \tag{17}$$

The additional equation with respect to the phase difference $\varphi = \beta_u - 2\beta_\theta$ can be written as

$$\varphi' = \frac{\Delta_1}{2} - 2\Delta_2 + \sqrt{\mu}K\left(\frac{1}{2}\frac{\sin^2\psi}{\cos\psi} - \cos\psi\right)\cos(2\beta_\theta - \beta_u) - \frac{f}{2\sqrt{\mu K}\cos\psi}\cos\beta_u \tag{18}$$

One considers equilibrium positions for the second equation of the system (17) and equation (18). Condition $\sin\psi \equiv 0$ corresponds to the localized mode of the spring vibrations. This mode exists for all values of the energy K; it is described by the straight line $\psi = 0$ in the plane (ψ, φ).

For a case when both $\cos\psi \neq 0$, and $\sin\psi \neq 0$, it is possible to observe a mode of coupled vibrations of the system (1). Condition of the mode existence can be obtained from the second equation of the reduced system (17): $\cos\psi = \frac{\sqrt{\mu}}{\eta_\theta - \eta_u}K\sin(2\beta_\theta - \beta_u) + \frac{f}{2\sqrt{\mu K}(\eta_\theta - \eta_u)}\sin\beta_u$.

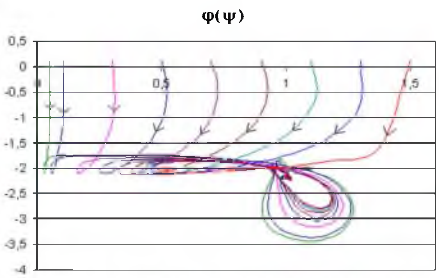

Figure 6. Trajectories in plane (φ,ψ)

In order to construct trajectories on the plane (ψ, φ) one integrates the reduced system (17) by using the Runge-Kutta method of 4[th] order, when the arctangent of the ratio of amplitudes changes on the interval $0 \le \psi(0) \le \pi/2$; the initial value is the following: $(0) = 0.5$; besides, $\eta_u = 0.3$, $\eta_\theta = 0.2$, $\Delta_1 = 0.2$, $\Delta_2 = 0.1$, $\mu = 0.4$, $f = 0.35$. Trajectories in the plane (φ,ψ) are shown in Fig.6 for the case of simultaneous external and internal resonances. Each trajectory has a loop near some quasi-equilibrium state of the reduced system. This state moves in the plane (φ,ψ) and corresponds to the *transient nonlinear normal mode* (TNNM) which exists only for specific value of the system energy, that is, in some moment of time corresponding to this energy level. The TNNM is attractive, and other motions of the system approach some TNNM near the mentioned moment of time. Then the TNNMs disappears, and trajectories in the plane (φ,ψ) approach to the equilibrium position corresponding to the stable NNM of coupled vibrations. So, the mode of the localized vibrations of spring is not stable.

To illustrate a behavior of the spring-mass system (1) in vicinity of the resonance one calculates the system with respect to amplitude and phase of the solution (16) by the Runge-Kutta method on the interval $t \in [0,5000]$ for the following initial values: $a_u(0) = 0.05,\ a_\theta(0) = 0.01,\ \beta_u(0) = 0.1,$ $\beta_\theta(0) = 0.2$ and for $\eta_u = 0.3,\ \eta_\theta = 0.2, \varDelta_1 = 0.2,\ \varDelta_2 = 0.1,\ \mu = 0.4,\ f = 0.35.$

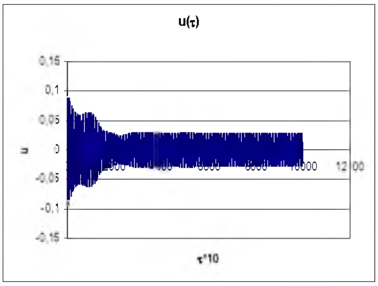

Figure 7. Dependence $u(\tau)$

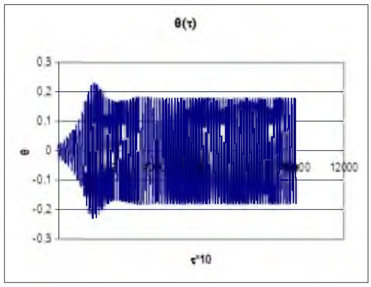

Figure 8. Dependence $\theta(\tau)$

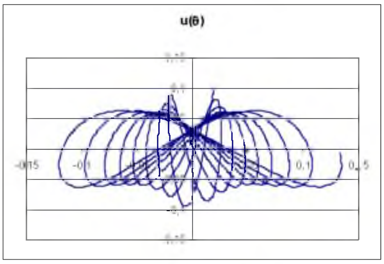

Figure 9a

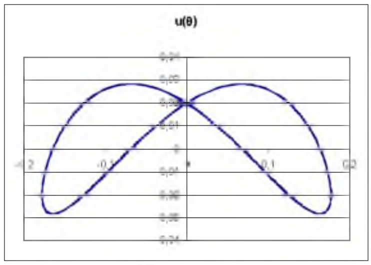

Figure 9b

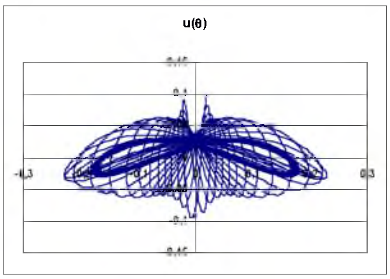

Figure 9c

Figure 9. Trajectories $u(\theta)$ in configuration plane for $t \in [0,100]$ (Fig.9a); $t \in [4800,5000]$ (Fig.9b); $t \in [0,5000]$ (Fig.9c)

Analysis of obtained results shows that in vicinity of the simultaneous external and internal resonances the *transient nonlinear normal mode* of coupled vibrations arises. At the beginning of the transient process the motions of the system are close to this TNNM. Then, due to vanishing of this mode, motions of the system tend to the stable mode of coupled vibrations. Trajectory of this stable mode can be observed in Fig. 12b where vibrations for large values of time are shown. This stable mode is close to the localized mode of the pendulum vibrations, and this fact can be used in the problem of vibration absorption. The numerical simulation fully confirms results obtained by analysis of the reduced system.

Conclusions

Dynamics of the dissipative spring-pendulum system (1) under the periodic excitation in vicinity of the external resonance on the first fundamental frequency and the simultaneous external and internal resonance is analyzed. Analysis of the resonance dynamics in these systems is made using the concept of nonlinear normal modes which was generalized to dissipative systems. Transfer to the *reduced system* written with respect to the system energy, the arctangent of amplitudes ratio and the difference of phases is used in this analysis. We can see that in region of the external resonance on the first fundamental frequency the vibration mode of localization on the spring is stable, and the mode of the coupled vibrations is absent. But in a case of simultaneous external and internal resonances coupled vibrations are observed, and the localized modes are absent. Besides, in the vicinity of the resonance so-called *transient nonlinear normal modes* (TNNMs), which exist only for certain levels of the system energy, appear. Although each TNNM exists only for some moment of time, it attracts other motions of the system before this moment. When this mode disappears, motions of the system attract to the stable nonlinear normal mode. Reliability of obtained analytical results is verified by numerical simulation. Besides, the obtained results can be useful in problem of the elastic vibrations extinguishing with the help of nonlinear absorbers.

References

[1] Nayfeh A.H., Mook D.T. *Nonlinear Oscillations*. New York: Wiley Inerscience; 1979.
[2] Avramov K.V., Mikhlin Yu.V. *Nonlinear Dynamics of Elastic Systems.V 1. Models, Methods, Phenomena (Second Edition)*. Moscow-Izhevsk: IKI; 2015 (in Russian).
[3] Manevich A.I., L.I. Manevitch L.I. *The Mechanics of Nonlinear Systems with Internal Resonances*. London: Imperial College Press; 2005.
[4] Mercer C.A., Rees P.L., Fahy V.J. Energy flow between two weakly coupled oscillators subject to transient excitation. *Journal of Sound and Vibration*, Vol. 15(3), pp. 373-379, 1971.
[5] Starzhinskii V.M. *Applied Methods of Nonlinear Vibrations.* Moscow: Nauka; 1977 (in Russian).
[6] Nayfeh S.A., Nayfeh A.H.. Energy transfer from high to low-frequency modes in a flexible structure via modulation. *ASME Journal Vibration and Acoustics*, Vol. 116, pp. 203-207, 1994.
[7] Vakakis A., Gendelman O., Bergman L., McFarland D., Kerschen G., Lee Y. *Nonlinear Targeted Energy Transfer in Mechanical and Structural Systems*., Berlin: Springer Science; 2008.
[8] Kauderer H. *Nichtlineare Mechanik*, Berlin: Springer-Verlag; 1958.
[9] Rosenberg R. Nonlinear vibrations of systems with many degrees of freedom. In: *Advances of Applied Mechanics*, 9, pp.156–243, 1966.
[10] Vakakis A., Manevitch L., Mikhlin Yu., Pilipchuk V., Zevin A. *Normal Modes and Localization in Nonlinear Systems*, New York: Wiley Interscience; 1996.
[11] Mikhlin Yu.V., Avramov K.V. Nonlinear normal modes for vibrating mechanical systems. Review of theoretical developments. *Applied Mechanics Review*, Vol. 63, pp. 4-20, 2010.
[12] Mikhlin Yu.V., Avramov K.V. Review of applications of nonlinear normal modes for vibrating mechanical systems. *Applied Mechanics Review*, Vol. 65 (2) (20 pages). 2013.
[13] Wang F., Bajaj A., Kamiya K.. *Nonlinear Normal Modes and Their Bifurcations for an Inertially-Coupled Nonlinear Conservative System*. Purdue University; 2005.
[14] Pilipchuk V.N. *Nonlinear Dynamics: Between Linear and Impact Limits.* Berlin: Springer-Verlag; 2010.
[15] Plaksiy K.Y., Mikhlin Yu.V. Dynamics of nonlinear dissipative systems in the vicinity of resonance. *Journal of Sound and Vibration*, Vol. 334, pp. 319-337, 2015.
[16] Plaksiy K.Y., Mikhlin Yu.V. Resonance behavior of the limited power-supply system coupled with the nonlinear absorber. *Mathematics in Engineering, Science and Aerospace*, Vol.6 (3), pp. 475-495, 2015.
[17] Nayfeh A.H. *Perturbation Methods.* New York: Wiley; 1973.