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Semi-Inverse Method in the Nonlinear Dynamics

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Abstract
The semi-inverse method based on using an internal small parameter of the nonlinear problems is discussed on the examples of the chain of coupled pendula and of the forced pendulum. The efficiency of such approach is highly appeared when the non-stationary dynamical problems are considered. In the framework of this method we demonstrate that both the spectrum of nonlinear normal modes and the interaction of them can be analysed successfully.

Keywords
nonlinear dynamics, semi-inverse method, nonlinear normal modes

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Introduction

The majority of physical, mechanical and engineering problems dealing with nonlinear processes can not be solved by the direct methods. The main goal of the researchers is the construction of a simplified model, which can be solved by an approximate method without loosing the physical content. The asymptotic methods, in particular, the averaging ones, based on using a small parameter, are the well developed tools, which are effectively applied in the nonlinear dynamics [1–3]. Moreover, there is a popular jest, that a nonlinear problem is unsolvable, if a small parameter can not be found. In this context we would like to separate the problems with an external small parameter /ESP/ connected with a small perturbation of the solvable system, and the problems, the small parameter of which is an internal one /Internal Small Parameter – ISP/. Besides, in the latter an ISP can be a priori unclear /i.e. ISP is hidden/. The most evident example of the system with ESP can be found in the celestial mechanics. It is a small perturbation of the stationary orbit of a planet by a remote heavenly body.

The example of the system with the ISP can be found in the theory of nonlinear oscillations. Let's consider two weakly coupled pendula. The small parameter is determined by the coupling rigidity, and it is the internal parameter because the stationary dynamics of the system has to be considered in the terms of normal modes, but not of the pendula. The small coupling has to be accounted in the main asymptotic approximation The resonant interaction of the nonlinear normal modes /NNMs/ of the well known Fermi-Pasta-Ulam nonlinear lattice gives an example of the hidden ISP [4]. Really, considering the slow evolution of the interacting NNMs, we have to remove the fast motion and to analyse the envelopes of NNMs. Such an approach leads to the multiscale expansion, the small parameter of which turns out to be the NNMs' frequency splitting. This parameter depends on the length of the lattice and on the wave number.

In this communication we would like to discuss the effective method of the solution of nonlinear problems which do not content any small parameter in their initial formulation. This method is based on the averaging procedure, the validity of which is proven in the process of solution finding. The source of the ISP as well as its value are dictated by the solution obtained and are verified by the comparison with the numerical simulation data. Omitting the general discussion of the method, we will demonstrate its efficiency in the application to both stationary and non-stationary dynamical processes.

1. The chain of coupled pendula - the discrete Frenkel-Kontorova model

Let us consider the periodic system of N coupled pendula. This system has a wide application in physics modelling an array of the particles in the field of the local (on site) potential [5]. The best known examples of such systems is the Frenkel– Kontorova model describing a dislocation or interstitial atom in the crystal lattice. Replacing the pendula by the nonlinear oscillators with two-well on site potential, we obtain the Klein–Gordon model, which is used for the description of the displacement disorder in the crystals. In the linear approximation these models lead to equations of the same type (linear discrete Klein–Gordon equations), the properties of which in the continual limit are well studied. However, in relatively small systems, the discreteness plays an important role, and the traditional approach consists in the application of the Nonlinear Normal Modes (NNMs) concept.

Let us write the hamiltonian of the system with length N in the dimensionless form:

$$H=\sum_{j=1}^{N}[\frac{1}{2}(\frac{dq_j}{dt})^2+\frac{\beta}{2}(q_{j+1}-q_j)^2+(1-\cos q_j)] \tag{1}$$

Here the dimensionless time t is normalized by the gap frequency and the coupling parameter β is generally not small.

The respective equations of motion

$$\frac{d^2q_j}{dt^2}-\beta\Delta_2 q_j+\sin q_j=0, \tag{2}$$

where $\Delta_2 q_j=q_{j+1}-2q_j+q_{j-1}$, may be represented in the terms of the complex variables

$$\Psi_j=\frac{1}{\sqrt{2}}(\frac{1}{\sqrt{\omega}}\frac{dq_j}{dt}+i\sqrt{\omega}q_j), \tag{3}$$

where ω is a frequency of the oscillations, which will be defined latter.

Taking into account definition (3) one can rewrite equations (2)as follows:

$$\begin{gathered} i\frac{d\Psi_j}{dt}+\frac{\omega}{2}\left(\Psi_j+\Psi_j^*\right)+\frac{\beta}{2\omega}\Delta_2\left(\Psi_j-\Psi_j^*\right) \\ +\frac{1}{\sqrt{2\omega}}\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\left(\frac{1}{\sqrt{2\omega}}\right)^{2k+1}\left(\Psi_j-\Psi_j^*\right)^{2k+1}=0 \end{gathered} \tag{4}$$

Let's represent the solution of equation (4) in the form:

$$\Psi_j(t)=\psi_j e^{i\omega t}, \tag{5}$$

where ψ_j is the amplitude.

One can show that function (5) is the solution of the resonant equation, which is obtained by averaging of equation (4) over the period $2\pi/\omega$, if the relation

$$\frac{\beta}{2\omega}\Delta_2\psi_j-\frac{\omega}{2}\psi_j+\frac{1}{\sqrt{2\omega}}J_1\left(\sqrt{\frac{2}{\omega}}|\psi_j|\right)\frac{\psi_j}{|\psi_j|}=0, \tag{6}$$

is satisfied. /Here J_1 is the Bessel function of the first kind/.

Equation (6) has the sense of dispersion relation, if the function ψ_j is represented as follows:

$$\psi_j=\sqrt{X}e^{-i\kappa j}, \tag{7}$$

where $X=const$ and $\kappa=2\pi k/N$ with $k=0,1,2\ldots,N-1$.

Now one can understand the origin of the frequency ω. Really, one can see that functions (5), (7) describe the nonlinear normal mode with the wave number κ. Then the parameter ω is its frequency.

According to definition (3), the amplitude X is expressed via the amplitude of the displacement Q:

$$X=\frac{\omega Q^2}{2} \tag{8}$$

Taking into account this relation and equation (7), one can rewrite dispersion relation (6) as follows:

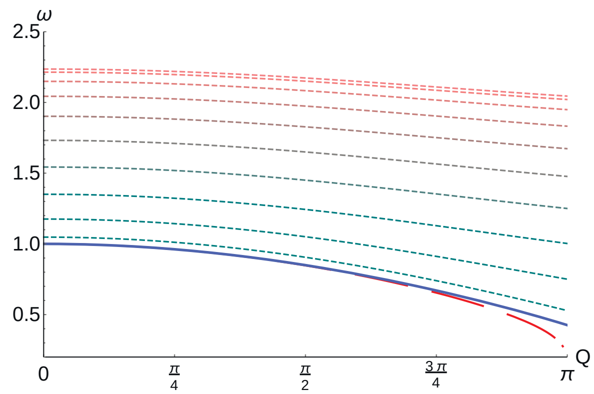

Figure 1. NNMs frequencies calculated according to dispersion ratio (9) for the chain with 20 particles and the wave numbers $k=1,\dots 10$. Solid blue and long-dashed red curves correspond to the uniform (zone-bounding) mode frequencies, calculated according to equations (11) and (10), respectively. The coupling parameter is $\beta = 1.0$.

$$\omega^2 = 2\left(\frac{1}{Q}J_1\left(Q\right) + 2\beta sin^2\frac{\kappa}{2}\right). \tag{9}$$

If the wave number is zero – $\kappa = 0$, equation (9) describes the amplitude dependence of the uniform /gap/ mode, when all pendula oscillate in phase:

$$\omega = \sqrt{\frac{2}{Q}J_1(Q)}. \tag{10}$$

In such a case, this frequency has to correspond to oscillation frequency of single pendulum, the exact value of which is well known:

$$\omega = \frac{\pi \sin\left(Q/2\right)}{2F\left(Q/2, 1/\sin^2\left(Q/2\right)\right)}. \tag{11}$$

The comparison of these frequencies is shown in figure 1.

One can see, that the frequencies (10) and (11) are well accorded, excluding a vicinity of the limiting oscillation amplitude $Q = \pi$.

To understand the discrepancy one can assume that equation (6) is the stationary version of more general equation

$$i\frac{\partial \psi_j}{\partial \tau} + \frac{\beta}{2\omega}\Delta_2\psi_j - \frac{\omega}{2}\psi_j + \frac{1}{\sqrt{2\omega}}J_1\left(\sqrt{\frac{2}{\omega}}|\psi_j|\right)\frac{\psi_j}{|\psi_j|} = 0. \tag{12}$$

Equation (12) can be obtained under assumption that a specific time scale for the variation of ψ_j is much more than the period of modes - $2\pi/\omega$. In spite of that it is not clearly appeared, the small parameter is the ratio of mode period to the envelope variation time. The frequency of pendulum oscillations converges to zero in the vicinity of the limiting oscillation amplitude $Q = \pi$ and the separation of the times turns out to be invalid.

Equation (12) describes a slow variation of the envelope function ψ_j that results from the nonlinear interaction with surrounding modes. Therefore, one can use it for analysing the resonant modes interaction.

Equations (12) correspond to the Hamilton function in the form:

$$H_a = \sum_{j=1}^{N}\frac{\beta}{2\omega}|\psi_{j+1} - \psi_j|^2 + \frac{1}{2}\sum_{j=1}^{N}\left(J_0(\sqrt{\frac{2}{\omega}}|\psi_j|) - \omega|\psi_j|^2\right) \tag{13}$$

The additional integral of motion is the "occupation" number:

$$X = \frac{1}{N}\sum_{j=1}^{N} |\psi_j|^2. \tag{14}$$

In such a case the key small parameter is a difference between the mode frequencies. Let us consider the non-stationary dynamics of the chain following from the interaction of the low frequency modes with the wave numbers $\kappa_0 = 0$ and $\kappa_1 = 2\pi/N$. It was early shown [6] that the combination of these modes leads to dividing the system on two domains, inside of which the particles move almost coherently. These domains we will designate as the "coherent domains". It is convenient to introduce the domain coordinates:

$$\begin{aligned}\chi_1 &= \frac{1}{\sqrt{2N}}\sum_{j=1}^{N} \psi_j \left[1 + (\cos\kappa_1 j + \sin\kappa_1 j)\right]\\ \chi_2 &= \frac{1}{\sqrt{2N}}\sum_{j=1}^{N} \psi_j \left[1 - (\cos\kappa_1 j + \sin\kappa_1 j)\right].\end{aligned} \tag{15}$$

/It is useful to note, that the domain coordinates (15) are transformed into coordinates of particles, if the chain length $N = 2$./ The inverse transformation to the complex amplitude of the particles is written as follows:

$$\psi_j = \frac{1}{\sqrt{2N}}\left[(\chi_1 + \chi_2) + (\chi_1 - \chi_2)(\cos\kappa_1 j + \sin\kappa_1 j)\right]. \tag{16}$$

One can see that the amplitude vectors $(\chi_1, \chi_2) = (1, 0)$ and $(\chi_1, \chi_2) = (0, 1)$ correspond to the maximum of pendulum displacements in the one and second domains, respectively. (It is easy to check out that the domain coordinates χ_j coincide with the coordinates of the pendula in the case $N = 2$.)

After substituting the expression (16) into equation (13), the equations of motion for the domain coordinates in the terms of complex amplitudes may be obtained immediately by variation of the hamiltonian with respect to the domain coordinates χ_1, χ_2. However, the resulting equations are lengthy enough and don't allow to analyse the chain dynamics clearly. Therefore, one should simplify the procedure. First of all, one can see that transformation (16) preserves the integral of occupation number (14):

$$X = |\chi_1|^2 + |\chi_2|^2. \tag{17}$$

With using the integral (17), the dimension of the system's phase space may be reduced by introducing the relative amplitudes of the domain coordinates χ_j [4]:

$$\chi_1 = \sqrt{X}\cos\theta e^{i\delta_1}, \quad \chi_2 = \sqrt{X}\sin\theta e^{i\delta_2}. \tag{18}$$

In such a case the parameter X specifies the total excitation of the system, while the "angle" θ shows the relative excitation of the coherent domains. In fact, the energy of the system doesn't depend on the absolute values of the phases δ_1 and δ_2, but it is the function of their difference $\Delta = \delta_1 - \delta_2$ only:

$$\begin{aligned}H(\theta,\Delta) &= -\frac{\omega}{2}X + \frac{\beta X \sin^2(\kappa_1/2)}{\omega}(1 - \cos\Delta\sin 2\theta) - \sum_{j=1}^{N} J_0(\xi_j)\\ \xi_j &= \sqrt{\frac{2X}{\omega}\frac{1 + \cos 2\theta\,(\cos(\kappa_1 j) + \sin(\kappa_1 j)) + \sin(\kappa_1 j)\cos(\kappa_1 j)\,(1 - \cos\Delta\sin 2\theta)}{N}}\end{aligned} \tag{19}$$

The first term in hamiltonian (19) corresponds to the kinetic parts of the oscillation energy. The second term describes the linear splitting of the NNMs frequencies. And the third term is associated with the nonlinear interaction of the NNMs.

Hamiltonian (19) allows to analyse the phase portrait of the system and to define the bifurcations of the phase trajectories under different excitation level that is specified by the excitation number X.

This procedure was well discussed for the coupled nonlinear oscillators [7, 8] as well as for the nonlinear chains [4, 6] in the small-amplitude approximation. However, the dispersion relation (9) and the asymptotic equation (12) are not restricted by any assumptions with respect to a smallness of the oscillation amplitudes. In such a case one can find the bifurcations at any given amplitude as a function of the chain parameters.

One can show that the stationary points ($\theta = \pi/4, \Delta = 0$) and ($\theta = \pi/4, \Delta = \pi$) of system (19) correspond to the NNMs, while the values of θ, which are equal to 0 and $\pi/2$ relate to the coherent domains χ_1 and χ_2, respectively. The "domain states" are the parts of the trajectory, which divides the attraction areas of the NNMs (it is named the Limiting Phase Trajectory - LPT). It is known [6], that the uniform (zone-bounding) mode can lose its stability at the certain conditions, if the nonlinearity is soft. In such a case, two new stationary points, which relate to the weakly localized states, appeared. If the oscillation amplitudes increase or the coupling parameter can be decreased, the separatrix passing through the unstable NNM expands and it can reach the domain states $\theta = 0$ and $\theta = \pi$. At this moment any paths from one domain state to another one turn out to be forbidden. Thus, the energy, initially concentrated in a one of domains, becomes to be captured in it.

Taking into account all mentioned above, it is easy to formulate the bifurcation conditions. The lost of stability occurs when hamiltonian (19) plateaus near the lowest NNM:

$$\frac{\partial^2 H(\theta,\Delta)}{\partial \theta^2}\ |_{(\theta=\pi/4,\Delta=0)} = 0. \tag{20}$$

Solving equation (20) with respect to coupling parameter β, one can obtain the instability threshold:

$$\beta_{ins} = \frac{J_2(Q)}{2\sin^2(\kappa_1/2)}. \tag{21}$$

The global bifurcation occurs when the energy of unstable stationary point ($\theta = \pi/4, \Delta = 0$) becomes equal to the energy of "domain states" ($\theta = \pi/2, \Delta = \pm\pi/2$) and ($\theta = 0, \Delta = \pm\pi/2$). Under this condition the solution of respective equation leads to the localization threshold as follows:

$$\begin{gathered}\beta_{loc} = 2\frac{\frac{1}{N}\sum_{j=1}^{N} J_0\left(\frac{Q}{\sqrt{2}} f_j\right) - J_0(Q)}{Q^2 \sin^2(\kappa_1/2)} \\ f_j = 1 + \cos(\kappa_1 j) + \sin(\kappa_1 j)\end{gathered} \tag{22}$$

First fact of worth is that instability threshold (21) as well as localization one (22) are in inverse proportion to the squared $\sin(\kappa_1/2)$. Taking into account that $\kappa_1 \sim 1/N$, one can conclude that the crucial values of coupling grow while the chain length increases. However, it is clear that the real parameter, which determines the resonant conditions, is a value of the gap between zone-bounding and the first non-uniform modes. This value is defined by the "effective coupling constant" that is the production $\beta \sin^2(\pi/N)$ [6]. Such conclusion coincides with that the producion $\beta_{ins} \sin^2 \kappa_1/2$ does not depend on the length of the chain and is equal to ε_{ins} for the pair of coupled pendula [8]. One should notice, that threshold (22) coincides exactly with the values that were obtained in the work [8], if $N = 2$. Figure 2(a) shows the "effective" threshold values for the instability and localization bifurcations.

The evolution of the phase trajectories of the system can be conveniently analysed in the Poincare map for the domain coordinates (fig. 2b-d). Figure 2(b) shows the Poincare map for the chain with 16 coupled pendula and the coupling parameter $\beta > \beta_{ins}$. It is well seen that two stationary points corresponding to the modes with wave numbers $\kappa = 0$ and $\kappa = 2\pi/N$. Figures 2(c, d) show the Poincare map at the localization threshold and below of it. In figure 2(c), one can see that the separatrix, crossing the unstable stationary point, surrounds two new stable stationary points, which were created as a result of instability of the lower mode with the wave number $\kappa = 0$. In contrast, the separatrix in figure 2(d) circumscribes the stationary point corresponding to mode with wave number $\kappa = 2\pi/N$, and no path from one localization state to another one exists.

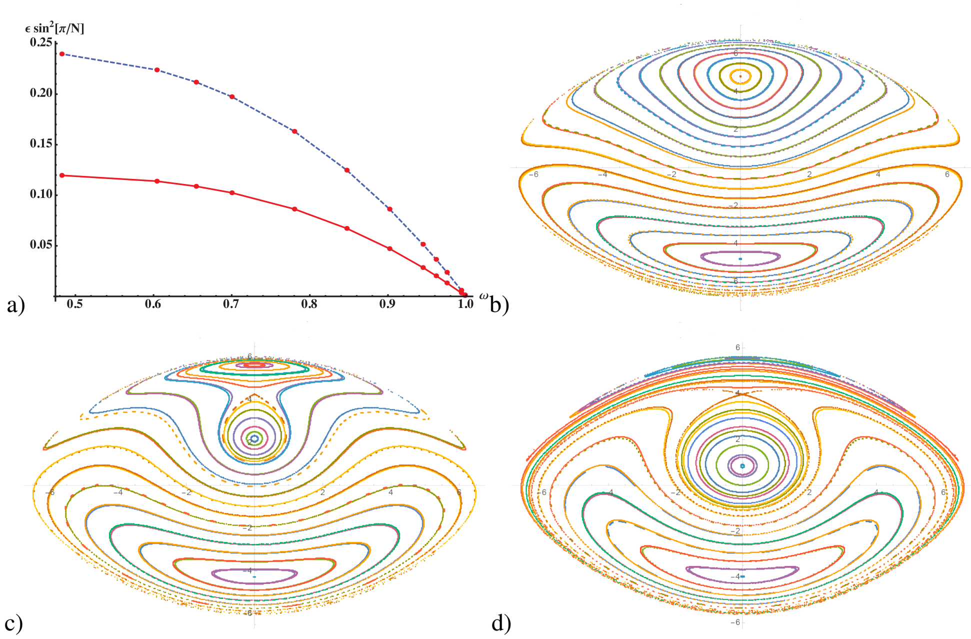

Figure 2. (a) Effective thresholds (21), (22) (blue dashed and red solid curves, respectively) vs oscillation frequency ω for the chain with 16 pendula. (b – d) Poincare maps for the chain with 16 pendula at oscillation amplitude $Q = 3\pi/4$ and different coupling parameters: (b) $\beta > \beta_{ins}$, (c) $\beta = \beta_{loc}$, (d) $\beta < \beta_{loc}$.

2. Forced pendulum

We discuss the undamped dynamics of a pendulum excited by an harmonic excitation and undergoing unidirectional motion. Corresponding equation of motion is

$$\frac{d^2q}{dt^2} + \sin q = f \sin \Omega t, \tag{23}$$

where q is the angular coordinate of the pendulum, f and Ω are the harmonic forcing amplitude and frequency.

By introducing the complex amplitude of the pendulum oscillations as

$$\begin{aligned} \psi &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{\omega}}\frac{dq}{dt} + i\sqrt{\omega}q\right) \\ q &= \frac{-i}{\sqrt{2\omega}}(\Psi - \Psi^*), \quad \frac{dq}{dt} = \sqrt{\frac{\omega}{2}}(\psi + \psi^*) \end{aligned} \tag{24}$$

one can rewrite equation (23) as follows

$$i\frac{d\psi}{dt} + \frac{\omega}{2}(\psi + \psi^*) + \frac{1}{\sqrt{2\omega}}\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\left(\frac{1}{2\omega}\right)^k (\psi - \psi^*)^{2k+1} = \frac{f}{2\sqrt{2\omega}}\left(e^{i\Omega t} - e^{-i\Omega t}\right) \tag{25}$$

To find the stationary solution of equation (25) one should assume that $\omega = \Omega$ and

$$\Psi = \sqrt{X}e^{i\Omega t}, \tag{26}$$

where $X = const$ is the amplitude.

Substituting solution (26) into equation (25) and multiplying the result on $\exp(-i\Omega t)$, one can obtain after integration over period $2\pi/\Omega$

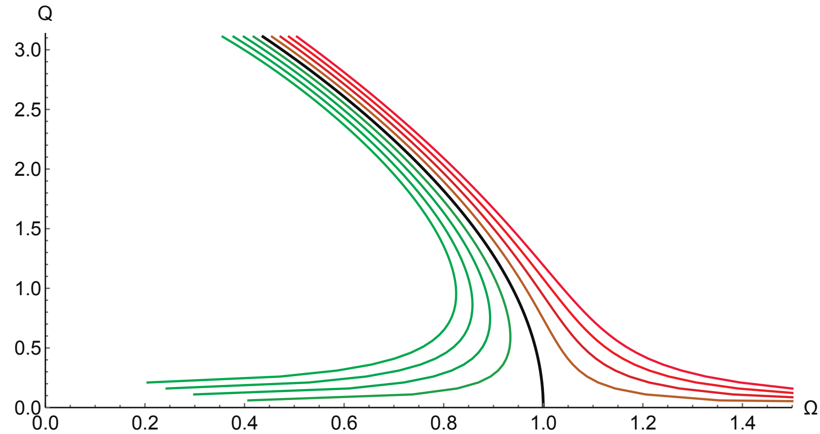

Figure 3. The driving frequency – amplitude relationship for different values of the force: $f = -0.2, \ldots, 0.2$ (colors – from red to green). Black curve shows the backbone frequency of free oscillations ($f = 0$).

$$-\frac{\Omega}{2}\sqrt{X} + \frac{1}{\sqrt{2\Omega}} J_1\left(\sqrt{\frac{2X}{\Omega}}\right) = \frac{f}{2\sqrt{2\Omega}} \tag{27}$$

With taking into account relation (8) between the complex and real amplitudes, the forcing frequency can be expressed via the amplitude of stationary oscillations as follows:

$$\Omega = \sqrt{\frac{1}{Q}\left(2J_1(Q) - f\right)}. \tag{28}$$

The amplitude-frequency relationship (28) is shown in figure 3 for the force $f = -0.2, \ldots, 0.2$.

Figure 3 shows that all brunches under free oscillation (black) curve have two stationary states - stable and unstable ones. The condition

$$\frac{d\Omega}{dQ} = 0 \tag{29}$$

determines the high boundary of the frequency range, where three stationary states occur.

However, if we interest ourself in the non-stationary processes in the motion of the driven pendulum, one should consider a solution

$$\Psi = \varphi e^{i\omega t}, \tag{30}$$

where φ is a slowly changing function and $\omega = \Omega - s$ ($s << \Omega$).

Substituting solution (30) into equation (25) and assuming that the detuning parameter s is small enough, one can consider φ as a function of the slow time $\tau = st$. The latter is supposed to be a new variable that is independent on the ”fast’ time t.

Then, multiplying equation (25) by the $e^{-i\omega t}$ and integrating it with respect to the ”fast” time t, the condition, which provides excluding the resonance (secular) terms, is obtained as:

$$is\frac{\partial\varphi}{\partial\tau} - \frac{\omega}{2}\varphi + \frac{1}{\sqrt{2\omega}} J_1\left(\sqrt{\frac{2}{\omega}}|\varphi|\right)\frac{\varphi}{|\varphi|} = \frac{f}{\sqrt{2\omega}} e^{i\tau}. \tag{31}$$

It is easy to check that the function $\varphi = \sqrt{X} e^{i\tau}$ is the solution of equation (31) if the frequency ω satisfies the relation

$$s - \frac{\omega}{2} + \frac{1}{\sqrt{2X\omega}} J_1\left(\sqrt{\frac{2X}{\omega}}\right) - \frac{f}{\sqrt{2X\omega}} = 0 \tag{32}$$

This equation permits to derive a very simple expression for the frequency of pendulum oscillations as a function of their amplitude:

$$\omega = -s + \sqrt{\frac{1}{Q}\left(2J_1\left(Q\right) - f\right) + s^2} \tag{33}$$

One can see that the limit $s \to 0$ leads to expression (28).

The non-stationary dynamics of the forced pendulum can be tackled by introducing the phase $\delta(\tau)$ and the amplitude $a = \sqrt{\omega/2}q$, such that the slowly varying function φ can now be expressed as $\varphi = ae^{i\delta(\tau)}$. Having introduced the phase shift $\Delta = \tau - \delta$, the equation of motion (31) can be written as

$$sa\dot{\Delta} + \frac{1}{\sqrt{2\omega}}J_1\left(\sqrt{\frac{2}{\omega}}a\right) - \left(s + \frac{\omega}{2}\right)a = \frac{f}{2\sqrt{2\omega}}\cos\Delta \tag{34}$$

$$s\dot{a} = \frac{f}{2\sqrt{2\omega}}\sin\Delta \tag{35}$$

The corresponding integral of motion reads

$$H = \frac{s}{2}\left[a^2\left(s + \frac{\omega}{2}\right) + \frac{af\cos\Delta}{\sqrt{2\omega}} + J_0\left(\sqrt{\frac{2}{\omega}}a\right) - 1\right] \tag{36}$$

In order to study the non-stationary processes one should consider the phase portrait of the system (36) /figs (4) (a-d)/. There are three control parameters in the considered system: detuning s, forcing f, and frequency ω. They are coupled with the stationary oscillation amplitude Q by relation (28). Let's fix detuning s and the force amplitude f and will vary the pendulum frequency ω.

Figure (4)(a) shows the phase portrait in terms of amplitude a and phase shift Δ, which is typical for the "low" frequency region. Three stationary points associate with stable and unstable brunches in figure (3). It is important that the attraction areas of the stable states are demarcated by two specific trajectories. The first one is the separatrix that passes through the unstable state and surrounds the large-amplitude stationary point.

The initial conditions with zero amplitudes (without dependence on the initial phase shift) leads to the motion along the closed trajectory, bounding the attraction area of the stable stationary points at $\Delta = 0$ /red curve in fig. (4)(a)/. The principal difference of this trajectory from the separatrix is the finite time of its passing. Due to this trajectory is the most distant one from the small-amplitude stationary point, we refer it as the Limiting Phase Trajectory (LPT). The LPT corresponds to the most intensive energy taking off by the pendulum from the energy source /at discussed initial conditions/. All other trajectories require non-zero initial conditions. Increasing the frequency ω is accompanied by enlarging of areas inside both the LPT and the separatrix. This process is finished when the LPT coincides with the separatrix /fig. (4)(b)/. At this moment the separatrix is becoming homoclinic and it makes up two LPTs, one of them surrounds the stable state with phase shift $\Delta = 0$ and another one envelopes the stationary point with $\Delta = \pi$. The latter turns out to be essentially larger than the first one. Therefore the amplitudes of non-stationary oscillations increase stepwise and the energy flow from the energy source to the pendulum respectively grows. The further rise of the frequency ω leads to weak decreasing of LPT as well as the separatrix up to the annihilation of latter at the frequency, the value of which is determined by equation (29) /see figs 4(c, d)/.

In order to estimate the threshold of the stepwise changing of oscillation amplitude one should note that the value of Hamiltonian (36) turns out to be equal to zero at the bifurcation point. Thus, solving the equation

$$H(Q,\omega)|_{\Delta=0} = 0$$

jointly with equation (33) with respect to ω and Q at fixed values f and s, one can calculate the threshold frequency as a function of detuning s and forcing f. Figure (5) shows the threshold values of ω at detuning $s = 0.1$ as the function of forcing f (red curves). The bifurcation value of ω that leads to the annihilation of unstable stationary point, is represented in figure (5) by blue curves.

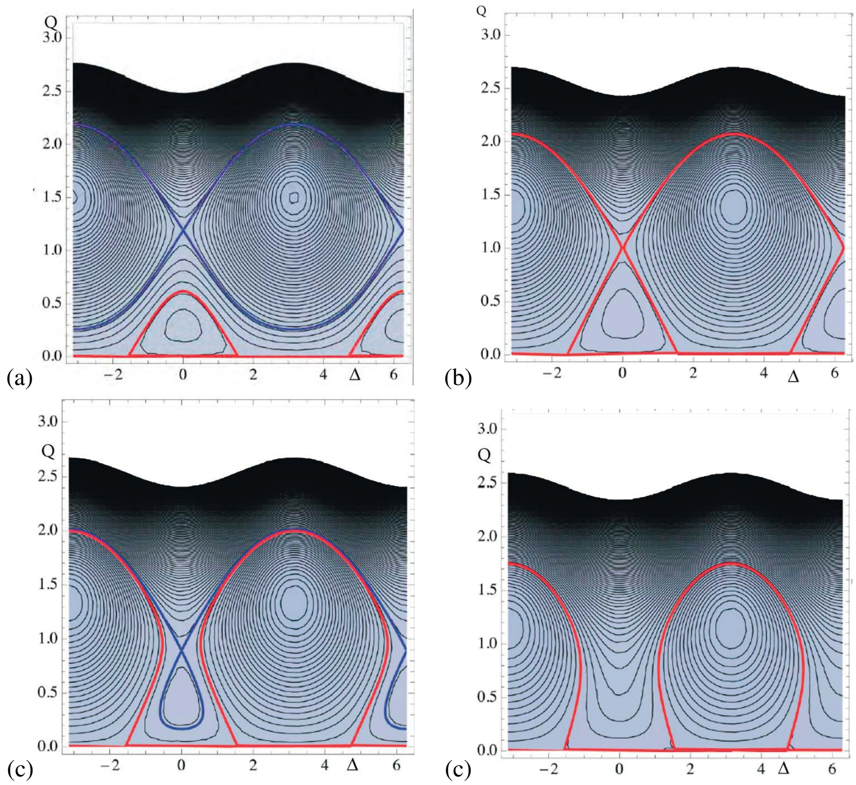

Figure 4. Evolution of the Δ-Q phase portrait for $f = 0.06$ for increasing frequency ω. (a) Before the first transition, $\omega = 0.79$; (b) after the first transition, $\omega = 0.8106$; (c) at the second transition, $\omega = 0.82$; (d) after the second transition, , $\omega = 0.85$.

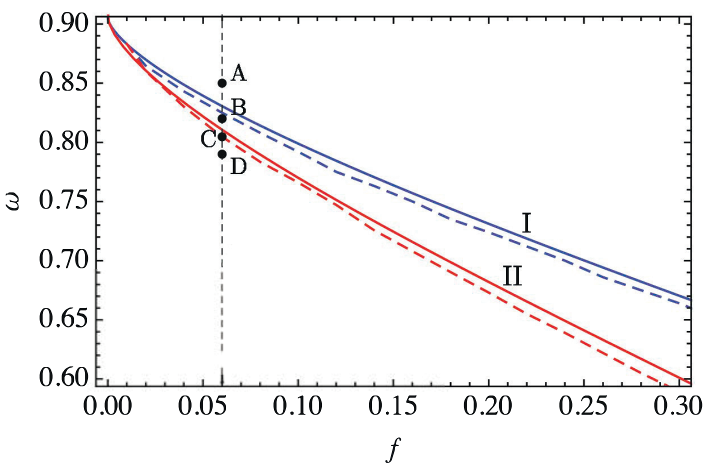

Figure 5. Analytic dynamical transitions thresholds on the (f-ω) plane for $s = 0.1$. First (blue) and second (red) thresholds, analytical (solid), numerical (dashed), and points corresponding to the phase portraits shown in fig. (4).

3. Conclusion

In conclusion, one should mention that the efficiency of the semi-inverse method discussed above depends strongly on the possibility of revealing the small parameter, which determines the time scale separation. It is important to notice that the value of small parameter and its physical content can be changed depended on the problem solved even in the framework of the same dynamical system. However, the verification of the small parameter existence has to be made *post factum* by the analytical or numerical methods.

Acknowledgments

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