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Method of Determination of Natural Frequencies and Forms of Nonlinear Vibrations for Layered Cylindrical Panels

M.V. Marchuk[1, 2*], T.V. Goriachko[1], V.S. Pakosh[1], and O.F. Lesyk[3]

Abstract
The technique of finding a finite number of first natural frequencies for geometrically nonlinear vibrations of layered elongated cylindrical panels at discrete consideration of components is proposed and verified The influence of the radius of curvature on the natural frequencies of three- and five-layered panels is investigated The dependence between the volume of filler three-layer panels and the lowest natural frequency has been established

Keywords
Nonlinear vibrations, elongated layered panel, perturbations method, natural frequencies

[1]Pidstryhach Institute for Applied Problems of Mechanics and Mathematics, NASU, L'viv, Ukraine
[2]National University "L'viv Polytechnic", L'viv, Ukraine
[3]Ternopil' National Economic University, Ternopil', Ukraine
* **Corresponding author:** mv_marchuk@ukr.net

Introduction

The layered panels with an arbitrary form of guides are bearing elements of constructions and technical means of various purposes. Since during the operation their components have different functional purpose, their thickness and mechanical characteristics are usually significantly differ among themselves. This leads to the need for a discrete approach for determining the amplitude and frequency characteristics of these objects.

In the proposed report the method for determining a finite number of natural frequencies and forms of layered cylindrical panels for geometrically nonlinear deformation is substantiated and verified. To describe the stress-strain state of each layer the dynamic geometrically nonlinear theory of elasticity is used. The quadratic approximation of displacement in each layer along the normal coordinates is proposed. In the tangential direction linear finite elements are used. The discrete variational problem is constructed. It is solved via authors' modification of perturbation method. The solutions of series of problems concerning the amplitude-frequency characteristics of plates-strips and elongated cylindrical panels with different number of layers are obtained. These solutions are compared with the results of other authors and for different theories. Sufficiently good coincidence gives us the perspective for the method developed.

1. The problem statement for a particular component of a layered panel

We consider a curved anisotropic elastic layer with thickness h in a natural mixed system of coordinates $\alpha_1, \alpha_2, \alpha_3$ on the median surface. This surface is formed by the motion of the line $\alpha_1 = 0; \alpha_3 = 0$ on the segment of arbitrary guiding. It is supposed that the layer is significantly larger along the axis α_2 in compare to the length of the section arc $\alpha_2 = 0$ of the median surface $\alpha_3 = 0$. So we have an elongated panel. If the conditions of fixing the ends of the panel $\alpha_1 = \pm\alpha_1^0$ and the initial conditions are independent of the coordinate α_2, then due to small influence of conditions of fixing the edges $\alpha_2 = \pm\alpha_2^0$, the functions, that determine the characteristics of geometrically nonlinear vibration processes in the plane of the median section, are dependent from α_1, α_3. To find these functions we have [9]:

– motion equations

$$div\hat{S} = \rho\frac{\partial^2 U}{\partial t^2}; \tag{1}$$

– elasticity relations

$$\hat{\Sigma} = \tilde{A}\otimes\hat{\varepsilon}; \tag{2}$$

– deformation relation between the strain tensor components $\hat{\varepsilon}$ and the components of the elastic displacement vector $\vec{U} = u_i\vec{e}_i\vec{e}_j$

$$\varepsilon_{ij} = \frac{1}{2}(\nabla_i u_j + \nabla_j u_i + \nabla_i u^k \nabla_j u_k); \tag{3}$$

– relation between the components S^{ij} of the nonsymmetrical Kirchhoff stress tensor $\hat{S}$ and the components σ^{ik} of the symmetric Piola stress tensor $\hat{\Sigma}$

$$S^{ij} = \sum_k \sigma^{ik}(\delta_k^j + \nabla_k u^j). \tag{4}$$

In equations (1) and (2) $\tilde{A}$ is the tensor of elastic properties of anisotropic layer, and ρ is its density.

Boundary conditions on the front surface of the panel $\alpha_3 = \pm h/2$ in the case it belongs to the layered structure are shown below, and initial conditions have the form

$$u_i(\alpha_1, \alpha_3, t)\Big|_{t=t_0} = v_i^0(\alpha_1, \alpha_3), \quad \frac{\partial u_i(\alpha_1, \alpha_3, t)}{\partial t}\Bigg|_{t=t_0} = v_i^1(\alpha_1, \alpha_3), \quad i = 1, 3, \tag{5}$$

$$\left|v_3^0(\alpha_1, \alpha_3)\right| >> \left|v_1^0(\alpha_1, \alpha_3)\right|, \quad (\alpha_1, \alpha_3) \in \Omega = [-\alpha_1^0, \alpha_1^0]\times[-h/2, h/2]. \tag{6}$$

2. The layered panels

Assume that a panel consists of N layers (see Fig. 1). Each k-th layer is considered as a separate thin panel with its own mechanical and material characteristics.

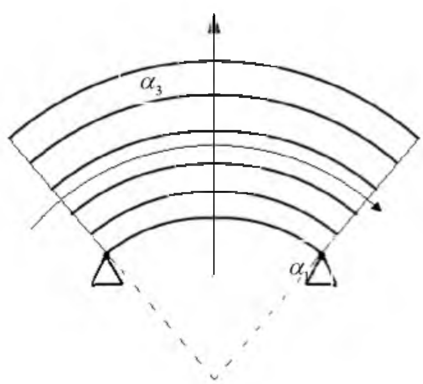

Figure 1. Layered cylindrical panel with hinges fixed on the elongated edges

Hooke's law is different for each layer:

$$\sigma^{(k)} = [Q^k]\varepsilon^k, \quad k = 1,...,N, \tag{7}$$

where $[Q^k]$ is tensor of elastic properties of anisotropic k-th layer.

Assuming that the value of α_3 coordinate at the top of k-th layer is h_k, and $h_0 = -h/2$, the equations (1) for a layered structure are written as

$$\sum_{i=1}^{3} \nabla_i S^{(k)ij} = \rho \frac{\partial^2 u_j^{(k)}}{\partial t^2}, \tag{8}$$

$$(\alpha_1, \alpha_3) \in \Omega = [-\alpha_1^0, \alpha_1^0] \times [h_{k-1}, h_k], \quad k = 1,...,N.$$

The contact conditions between the layers are

$$u_i^{(k-1)}(\alpha_1, h_{k-1}, t) = u_i^{(k)}(\alpha_1, h_k, t), \quad i = 1, 2, 3, \tag{9}$$

$$S^{(k-1)3i}(\alpha_1, h_{k-1}, t) = S^{(k)3i}(\alpha_1, h_k, t), \quad |\alpha_1| \le \alpha_1^0, \quad k = 2,...,N, \tag{10}$$

and on the lower and upper facial surfaces of the layered structure we have

$$S^{(m)31}(\alpha_1, h_m, t) = S^{(m)33}(\alpha_1, h_m, t) = 0, \quad |\alpha_1| \le \alpha_1^0, \quad m = 0, N. \tag{11}$$

At the elongated ends of the panel $\alpha_1 = \pm\alpha_1^0$ under the conditions of fixing the hinge on the lower surface of the front $\alpha_2 = -h/2$ the boundary conditions have the form

$$S^{(k)1i}(a, \alpha_3, t) = 0, \quad k = 1, N, \tag{12}$$

$$u_i^{(N)}(a, \pm h/2, t) = 0, \quad |\alpha_3| \le h/2, \quad i = 1, 3, \quad a = \pm\alpha_1. \tag{13}$$

3. Approximations

Assuming that each k-th layer is thin, quadratic approximations along α_3 coordinate are used for components of elastic displacement vector u_1 and u_3 [10]:

$$u_i^{(k)}(\alpha_1, \alpha_3) = \sum_{j=0}^{2} u_{ij}^{(k)}(\alpha_1) p_j(\alpha_3), \quad i = 1, 3, \tag{14}$$

where

$$p_0(\alpha_3) = \frac{1}{2} - \frac{2\alpha_3 - h_{k-1} - h_k}{2(h_k - h_{k-1})},$$

$$p_1(\alpha_3) = \frac{1}{2} + \frac{2\alpha_3 - h_{k-1} - h_k}{2(h_k - h_{k-1})}, \quad p_2(\alpha_3) = 1 - \left(\frac{2\alpha_3 - h_{k-1} - h_k}{h_k - h_{k-1}}\right)^2, \quad \alpha_3 \in [h_{k-1}, h_k].$$

For finding the unknown coefficients $u_{ij}^{(k)}(\alpha_1)$ in (14), approximation by the tangential coordinate α_1 was used on one-dimensional isoperimetric linear finite elements [10]:

$$u_{ij}^{(k)(e)} = \sum_{j,m}^{2} u_{ijm}^{(k)(e)}(\alpha_1)\phi_m^{(e)}(\xi), \quad \xi = \frac{2\alpha_1}{\alpha_{12}^{(e)} - \alpha_{11}^{(e)}} - 1, \tag{15}$$

where e is the number of finite elements of k-th layer; $u_{ijm}^{(k)(e)} = u_{ij}^{(k)}(\alpha_{1m}^{(e)})$, $m = 1, 2$ are the values on nodes $\alpha_{1m}^{(e)}(\alpha_1)$ of finite element; $\phi_1^{(e)}(\xi) = \frac{1}{2}(1-\xi)$; $\phi_2^{(e)}(\xi) = \frac{1}{2}(1+\xi)$.

4. The discretized problem

Considered above differential formulation of the problem of geometrically nonlinear free vibrations for single layer is equivalent to the problem of minimizing the functional L [10]:

$$L = -\int_{\Omega} \sum_i \sum_j u_i \frac{\partial S^{ij}}{\partial x_j} d\Omega - \int_{\Omega} \rho \frac{\partial^2 U^T}{\partial t^2} \cdot U d\Omega =$$

$$= -\int_{\Omega} \sum_i \sum_j S^{ij} \frac{\partial u_i}{\partial x_j} d\Omega - \int_{\Omega} \rho \frac{\partial^2 U^T}{\partial t^2} \cdot U d\Omega \to \min . \tag{16}$$

Boundary conditions (11), (12) and the contact conditions (9), (10) are natural ones for the variation formulation of the problem (16) [10], but the conditions (13) must be taken into account during solving.

In a case of layered panel we obtain:

$$L = \sum_{k=1}^{K} \left(-\int_{\Omega_k} \sum_i \sum_j S_k^{ij} \frac{\partial u_i}{\partial x_j} d\Omega - \int_{\Omega_k} \rho_k \frac{\partial^2 U^T}{\partial t^2} \cdot U d\Omega \right) \to \min. \tag{17}$$

After substituting (14), (15) in (17) (with using (8)) and composing the results together we obtain:

$$L^{\Delta} = \{u\}^T K_L \{u\} + \{u\}^T K_{NL}(u)\{u\} + \{u\}^T M\{\ddot{u}\} \to \min , \tag{18}$$

where $\{u\} = \{u\}(t)$ – vector of values of the coefficients $u_{ijm}^{(k)(e)}$ at the nodes on the finite-element of k-th layer; K_L – linear, and K_{NL} – nonlinear components of stiffness matrix; M – mass matrix [5].

For solving discretized problem (18) perturbation method described in [5, 6] is used

5. Numerical results

5.1. Verification of the proposed technique

Consider a cylindrical five-layer panel, the edges of which are fixed by hinges at the bottom of the front plane (see Fig. 1.). The panel has the following geometrical and mechanical characteristics:

$$l = 1 \text{ m}; \; h = 0,01 \text{ m}$$

$$E_1 = 40E_2, \quad G_{12} = G_{13} = 0,6E_2, \quad G_{23} = 0,5E_2, \quad \nu_1 = 0,25.$$

In order to analyze the reliability of the results we applied the proposed technique to the problem, the solutions of which are known [4]. Consider a cylindrical panel with radius of curvature $K = 0$. Partitioning into 50 finite elements by coordinate α_1 was used for calculation of the values of natural frequencies.

In Table 1 the dimensionless values ω_{NL} / ω_L obtained at the dimensionless amplitudes $\frac{w_{\max}}{h}$ for free vibrations of five-layered panel are compared with the results from the work [4].

Table 1. Comparison of the obtained results with given in [4]

$\frac{w_{\max}}{h}$	ω_{NL}/ω_L	
	[4]	**Proposed technique**
0,2	1,0313	1,0401
0,4	1,1198	1,1214
0,6	1,2536	1,2695
0,8	1,4199	1,4418
1,0	1,6086	1,6588
1,2	1,8127	1,8627

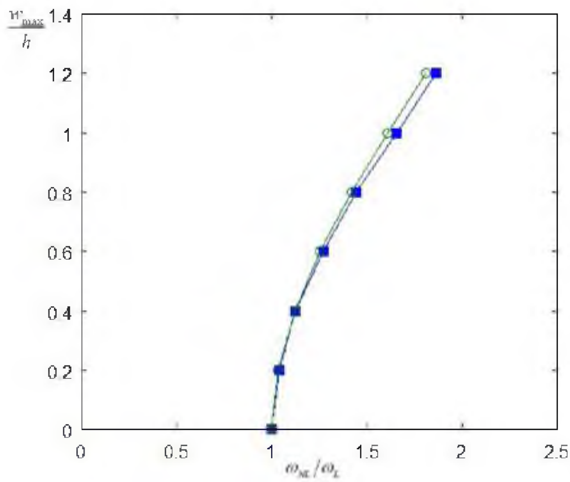

Figure 2. Comparison of backbone curves obtained via the method of perturbation and results of work [4].

Fig. 2 shows the backbone curves [11], constructed using the proposed technique (■) and the results given in the work [4] (o).

Also, the influence of the radius of curvature K on the free vibrations of the panel is investigated. Fig. 3 shows the dependence of the lowest natural frequency on the radius of curvature of five-layered panels made of carbon fiber.

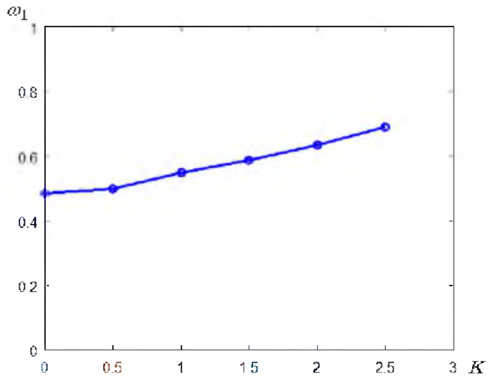

Figure 3. Dependence of the lowest natural frequency ($\omega_1 = \omega_0 \cdot 10^{-6}, Hz$) by the radius of curvature of the cylindrical panels

The maximum relative error in the Table 1 does not exceed 3%, which shows the effectiveness of the proposed technique. Comparative analysis of the graphs on Fig. 2 shows the reliability of the results obtained using the proposed technique. Also it was established, that the main amplitude of natural vibrations increases with increasing radius curvature of the panel.

5.2. Three-layered panel

We considered a layered plate-strip with elongated edges that are fixed with stationary hinges on the bottom plane (see Fig. 4). Geometrical characteristics of plane are $l = 1\,m$, $h = 0{,}1\,m$. It consists of three layers with following characteristics:

1) Rubber – $E = 0.1 \cdot 10^9\ N/m^2$, $\nu = 0{,}49$;
2) Steel – $E = 210 \cdot 10^9\ N/m^2$, $\nu = 0{,}3$.

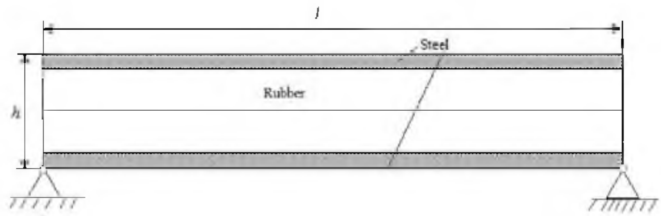

Figure 4. Panel with three layers

In Table 2 first five natural frequencies are shown for a panel consisting of three layers where steel layers have thickness 0.01m and rubber layer has thickness 0.08m.

Table 2. The first five natural frequencies of three-layered panel

n	ω_n, Hz
1	283000
2	1019000
3	1457300
4	1839600
5	2615200

In Table 3 dependency between first natural frequencies and thickness of middle layer (rubber layer) thickness is shown.

Table 3. Influence the relationships for thicknesses of rubber layer and panel to the first natural frequencies

$\frac{h_{rubber}}{h}$	ω_1, Hz
0.9	225650
0.8	283000
0.7	372770
0.6	490850
0.5	635100

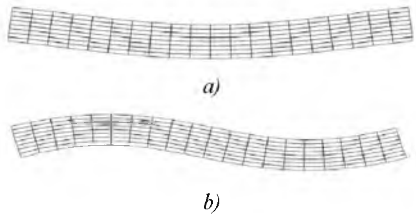

Figure 5. Eigenmodes of the panel: *a)* – the first mode; *b)* – second.

The Fig. 5 shows eigenmodes of the panel for first (*a*) and second (*b*) modes of the panel consisting of three layers where the steel layers have the thickness 0.01m and the rubber layer has the thickness 0.08m.

Table 4. The influence of curvature of the panel on its natural frequencies

K	ω_1, Hz
0	283000
0.5	254200
1	232000
2	218700

In Table 4 dependency between the radius of curvature and first natural frequency of the panel that consists of three layers where the steel layers have thickness 0.01m and the rubber has thickness 0.08m is shown.

For the panel considered above we can make the following conclusions:

1. the more matrix (rubber) component is included in the panel, the less is the first natural frequency;
2. the bigger the curvature radius is, the smaller the first natural frequency becomes.

Conclusions

The maximum relative error in the Table 1 does not exceed 3%, indicating the effectiveness of the proposed method. Comparative analysis of the graphs in Fig. 2 shows the reliability of the results using the developed method. Is also established that main amplitude of natural vibrations increases with increasing curvature of the panel.

In the future, should be carried out similar researches for various physical and mechanical characteristics of the components and conditions of fixing the ends of elongated layered panels.

References

[1] Amabili M. *Nonlinear Vibrations and Stability of Shells and Plates*, Cambridge: Cambridge Univ. Press, 2008.

[2] Amabili M. Nonlinear vibrations of rectangular plates with different boundary conditions: theory and experiments. *Comput. Struct.*, 2004, Vol. 82, p. 2587–2605.

[3] Kubenko V.D., Kovalchuk P.S., Podchasov N.P. *Nonlinear Vibrations of Cylindrical Shells*, K.: Vyshcha Shkola, 1989.

[4] Kurpa L.V., Budnikov N.A. Investigation of forced nonlinear vibrations of laminated shallow shells using a multimode approximation. *Bulletin of Donetsk Univ. Series A. Natural Sciences*, 2013, Vol. 1, p. 55–60.

[5] Lewandowski R. Analysis of strongly nonlinear free vibration of beams using perturbation method, *Civil and Env. Eng. Reports*, 2005, Vol. 1, p. 153–168.

[6] Marchuk M., Goriachko T. and Pakosh V. Geometrically Nonlinear Free Transversal Vibrations of Thin-Walled Elongated Panels with Arbitrary Generatrix. *Vibrations in Physical Systems*, 2014, Vol. 26, p. 153–160.

[7] Marchuk M., Mukha I. and Goriachko T. A comparative analysis of the characteristics of geometrically nonlinear stress-strain state of composite plates and cylindrical panels on the basis refied model and elasticity theory. *Modeling and Stability: Abstracts of Conference Reports*, K: Taras Shevchenko National Univ. of Kyiv, 2011.

[8] Marchuk M., Pakosh V., Lesyk O. and Hurayewska I. Influence of Pliability to Transversal Deformations of Shear and Compression on Correlation Frequency from Amplitude for Nonlinear Oscillations of Composite Plates. *Vibrations in Physical Systems*, 2006, Vol. 22, p. 251–256.

[9] Morozov N.F. *Selected Two-dimensional Problem of Elasticity Theory*, Leningrad: Publishing House of Leningrad. University Press, 1978.

[10] Reddy J. N. *An Introduction to Nonlinear Finite Element Analysis*, Oxford Univ. Press, 2004.

[11] Vol'mir A.S. *Nonlinear Dynamics of Plates and Shells*, M.: Nauka, 1972.