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Isogeometric Approximation Methods Using the Interlineation Operators

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Abstract
Interlineation of functions of two or more variables is approximation of the functions by their traces or traces of some differential operators on the fixed system of lines. The given paper presents the analysis of the building methods of interlineation operators, that preserve the differentiability class and have the same traces as the approximated function

Keywords
Approximation, preservation class, interlineation

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Introduction

The paper provides a review of the main results obtained for the construction of operator's interlineation for functions of two or three variables [1-9]. These operators preserve the properties of the approximated function. Usually the properties of the approximated function are known due to experimental data. Used operators for the approximation of differentiable functions have the following properties:

1. If the approximated function f is continuous together with its partial derivatives up to order r, then building interlineation operators Of have the same differential properties with the approximated function in interlineation lines, that is: $f \in C^r(D) \Rightarrow Of \in C^r(D)$

2. If the function $f(x,y)$ is given by its traces and traces of its partial derivatives with respect to y on system of non-intersecting curves, then its interlineation operator Of has, together with partial derivatives the same traces on these lines.

3. Iinterlineation operators for functions of three variables have similar properties with preservation of the geometric properties.

1. Generalized D'Alembert formula that preserves class of the differentiability with the traces of the approximated function and its derivatives on one line

Let us assume $f(x,y) \in C^r\left(R^2\right), r \ge 1;\ f^{(0,s)}(x,y) = \dfrac{\partial^s f(x,y)}{\partial y^s},\ \gamma(x) \in C^r(R)$.

Taylor operator for one variable function $f(x,y) \in C^r\left(R^2\right)$

$$T_N f(x,y) = \sum_{s=0}^{N} f^{(0,s)}\left(x,\gamma(x)\right)\frac{\left(y-\gamma(x)\right)^s}{s!},\ f^{(0,s)}\left(x,\gamma(x)\right) = \left.\frac{\partial^s f(x,y)}{\partial y^s}\right|_{y=\gamma(x)},$$

has the following properties:

$$\left.\frac{\partial^q T_N f(x,y)}{\partial y^q}\right|_{y=\gamma(x)} = \left.\frac{\partial^q f(x,y)}{\partial y^q}\right|_{y=\gamma(x)}, 0 \le q \le N,$$

$$f \in C^r(R^2) \bigcap f^{(0,s)} \in C^{r-s}(R^2), 0 \le s \le N \le r \Rightarrow$$
$$T_N f \in C^{r-N}(R^2) \bigcap T_N f \notin C^r(R^2).$$

Hence, this operator does not preserve the differentiablity class $C^r(R^2)$ of the function $f(x,y)$.

Let us introduce the operator which is the generalization of D'Alembert formula for the case $N \ge 2$.

$$O_N f(x,y) = \sum_{\ell=1}^{N} \lambda_{0,\ell} f\left(x + \beta_{0,\ell}\left(y - \gamma(x)\right), \gamma(x)\right) +$$
$$+\sum_{s=1}^{N}\sum_{\ell=1}^{N} \lambda_{s,\ell} \int_0^{x+\beta_{s,\ell}(y-\gamma(x))} f^{(0,s)}\left(t, \gamma(t)\right) \frac{(x+\beta_{s,\ell}(y-\gamma(x)-t)^{s-1}}{(s-1)!} dt,$$

In where $\lambda_{s,\ell}, s = \overline{0,N}, \ell = \overline{0,N}$ for each value $s \in [0,N]$ are found by menas of solving the system of linear algebraic equations

$$\sum_{\ell=1}^{N} \lambda_{s,\ell} \left(\beta_{s,\ell}\right)^p = \delta_{p,s}, 0 \le p \le N.$$

These systems have single solution for each s and $\beta_{s,\ell} \ne \beta_{s,\ell\ell}, \ell \ne \ell\ell$ since their determinants

$$\det\left[\beta_{s,\ell}^p\right]_{\ell=\overline{0,N}}^{p=\overline{0,N}} \ne 0, s = \overline{0,N}$$

are Vandermonde determinants.

Theorem 1. Operator $O_N f(x,y)$ has the following properties

$$f \in C^r(R^2) \cap f^{(0,s)}(x,\gamma(x)) \in C^{r-s}(R), s = \overline{0,N} \Rightarrow O_N f(x,y) \in C^r(R^2),$$

$$\left.\frac{\partial^q O_N f(x,y)}{\partial y^q}\right|_{y=\gamma(x)} = \left.\frac{\partial^q f(x,y)}{\partial y^q}\right|_{y=\gamma(x)}, 0 \le q \le N, N \le r.$$

Let us introduce operator $D_N f(x,y)$, being the integral generalization for the operator $O_N f(x,y)$:

$$D_N f(x,y) = \int_{-1}^{1} G_0(\beta) f\left(x + \beta\left(y - \gamma(x)\right), \gamma(x)\right) d\beta +$$
$$+\sum_{s=1}^{N}\int_{-1}^{1} G_s(\beta) \int_0^{x+\beta(y-\gamma(x))} f^{(0,s)}\left(t, \gamma(t)\right) \frac{\left(x+\beta\left(y-\gamma(x)\right)-t\right)^{s-1}}{(s-1)!} dt d\beta,$$

where $G_s(\beta) \in C^r[-1,1], s = \overline{0,N}$ are given functions.

Theorem 2. Operator $D_N f(x,y)$ has the following properties:

$$f \in C^r(R^2) \bigcap \gamma \in C^r(R) \Rightarrow D_N f(x,y) \in C^r(R^2),$$

$$\left.\frac{\partial^q D_N f(x,y)}{\partial y^q}\right|_{y=\gamma(x)} = \left.\frac{\partial^q f(x,y)}{\partial y^q}\right|_{y=\gamma(x)}, 0 \le q \le N, N \le r,$$

if auxiliary functions satisfy the following condition:

$$\int_{-1}^{1} G_s(\beta) \beta^p d\beta = \delta_{s,p}, 0 \le s, p \le N.$$

So operators $O_N f(x,y)$ and $D_N f(x,y)$, unlike Taylor operator, $T_N f(x,y)$ have differentiablity class and traces of function, $f(x,y)$ and its derivatives up to N -th order.

2. Hermite interlineation of two-variable function on the given system of skew lines preserving the $C^r(R^2)$ class.

An important generalization of the Taylor formula for the several points case is called Hermit interpolation polynomial and belongs, as any other polynomial, to the class $C^\infty(R^n)$. In order to construct Hermite interlineation operators the approximated function traces and its partial derivatives up to given order $N \geq 1$ on the given system of parrallel straight lines are used. Operators

$$E_{MN} f(x,y) = \sum_{k=1}^{M}\sum_{s=0}^{N} f^{(0,s)}(x,y_k) h_{k,s}(y) \frac{(y-y_k)^s}{s!},$$

$$f^{(0,s)}(x,y_k) = \left. \frac{\partial^s f(x,y)}{\partial y^s} \right|_{y=y_k}; h_{k,s}^{(p)}(y_\ell) = \delta_{k,\ell}\delta_{p,0}, p = \overline{0,N-s};$$

$$\left. \frac{\partial^p E_N f(x,y)}{\partial y^p} \right|_{y=y_\ell} = f^{(0,p)}(x,y_\ell), k,\ell = \overline{1,M}; p,s = \overline{0,N},$$

are the interpolation operators for functions of one variable. They have the differentiation order, being completely determined by differential properties of auxiliary functions $h_{k,s}(y)\frac{(y-y_k)^s}{s!}, k = \overline{1,M}; s = \overline{0,N}$ (algebraic, trigonometric, generalized polynomials, spline functions, etc.) and differential properties of the indicated traces Therefore, if

$$f^{(0,s)}(x,y_k) \in C^{r-s}(R), s = \overline{0,N}, r \geq N \geq 1, k = \overline{1,M},$$

then

$$E_{MN} f(x,y) \in C^{r-N}(R^2) \Rightarrow E_{MN} f(x,y) \notin C^r(R^2).$$

Let us introduce operator

$$O_{M,N} f(x,y) = \sum_{k=1}^{M} h_{M,k,0}(x,y) \sum_{\ell=0}^{N} \lambda_{0,\ell} f\left(x+\beta_{0,\ell}(y-\gamma_k(x)), \gamma_k(x)\right) +$$

$$+\sum_{k=1}^{M}\sum_{s=1}^{N} h_{M,k,s}(x,y) \sum_{\ell=0}^{N} \lambda_{s,\ell} \int_0^{x+\beta_{s,\ell}(y-\gamma_k(x))} f^{(0,s)}(t,\gamma_k(t)) \frac{\left(x+\beta_{s,\ell}(y-\gamma_k(x))-t\right)^{s-1}}{(s-1)!} dt,$$

where $\beta_{s,\ell} \in [-b,b], s = \overline{0,N}, \ell = \overline{0,N}$ are given different numbers (real or complex), unknown parameters $\lambda_{s,\ell}, s = \overline{0,N}, \ell = \overline{0,N}$ for each value of $s = \overline{0,N}$ are found by means of solving the system of linear algebraic equations $\sum_{\ell=1}^{N} \lambda_{s,\ell} (\beta_{s,\ell})^p = \delta_{p,s}, 0 \leq p \leq N$ provided condition $-b \leq \beta_{s,0} < \beta_{s,1} < \cdots < \beta_{s,N} \leq b,\; s = \overline{0,N}; 1 \leq b \leq \infty$.

Theorem 3. Operators $O_{M,N} f$ have the following properties:

$$f \in C^r(R^2) \cap f^{(0,s)} \in C^{r-s}(R), 0 \leq s \leq N \leq r \Rightarrow O_{MN} f \in C^r(R^2),$$

$$\left. \frac{\partial^q O_{M,N} f(x,y)}{\partial y^q} \right|_{y=\gamma_l(x)} = \left. \frac{\partial^q f(x,y)}{\partial y^q} \right|_{y=\gamma_l(x)} = f^{(0,q)}(x,\gamma_l(x)),$$

$q = \overline{0,N}, N \leq r, l = \overline{1,M}$.

Let us introduce operators

$$D_{M,N} f(x,y) = \sum_{k=1}^{M} h_{M,k,0}(x,y) \int_{-b}^{b} G_{k,0}(\beta) f\left(x+\beta(y-\gamma_k(x)), \gamma_k(x)\right) d\beta +$$

$$+\sum_{k=1}^{M}\sum_{s=1}^{N} h_{M,k,s}(x,y)\int_{-b}^{b} G_{k,s}(\beta)\int_{0}^{x+\beta(y-\gamma_k(x))} f^{(0,s)}(t,\gamma_k(t))\frac{(x+\beta(y-\gamma_k(x)-t)^{s-1}}{(s-1)!}dtd\beta .$$

Theorem 4. Operators $D_{M,N}f$ have the following properties:

$$f\in C^r(R^2)\bigcap f^{(0,q)}(x,\gamma_l(x))\in C^{r-q}(R)\Rightarrow D_{M,N}f\in C^r(R^2),$$

$$\left.\frac{\partial^q D_{M,N}f(x,y)}{\partial y^q}\right|_{y=\gamma_l(x)}=\left.\frac{\partial^q f(x,y)}{\partial y^q}\right|_{y=\gamma_l(x)}=$$

$$=f^{(0,q)}(x,\gamma_l(x)),0\le q\le N,N\le r;l=\overline{1,M} ,$$

if the following conditions

$$\int_{-b}^{b} G_{k,s}(\beta)\beta^p d\beta=\delta_{s,p},0\le s,p\le N;k=\overline{1,M}$$

hold true

Hence, operators $O_{MN}f(x,y)$ and $D_{MN}f(x,y)$, unlike Hermit operators $E_{MN}f(x,y)$, preserve the diferentiablity class of the function $f(x,y)$ and its traces and traces of its derivatives up to $N>0$ order on the system of skew lines, despite the traces $f^{(0,s)}(x,\gamma_k(x))\notin C^r(R)$.

3. Hermit interpolation at the points of the skew lines system in cylindrical coordinates.

3.1. Construction of the interlineation operators

Let the lines system M be given in parametric form

$$\Gamma_k:\{(r,\phi,z):r=r_k(\phi)=f(\phi,z_k(\phi)),z=z_k(\phi),0\le\phi\le 2\pi\},k=\overline{1:M} ,$$

$$r_k(\phi),z_k(\phi)\in C^v[0,2\pi] .$$

The traces of function (generally, unknown) are

$$r=f(\phi,z)\in C^v(D),D=\{(\phi,z):0\le\phi\le 2\pi,0\le z\le H\}$$

and its derivatives up to N order for the variable z on these lines:

$$\frac{\partial^p f}{\partial z^p}(\phi,z_k(\phi))=f_{k,p}(\phi)\ \ ,k=\overline{1,M}\ \ ,p=\overline{0:N} .$$

Let us introduse the designations $g_k(\phi,z,\beta)=\phi+\beta(z-z_k(\phi))$ and system of fucntions $h_{k,s}(\phi,z),G_s(\beta),k=\overline{1,M};s=\overline{0,N}$. Suppose that these functions have the following properties

$$\left.\frac{\partial^q}{\partial z^q}h_{k,s}(\phi,z)\right|_{\Gamma_l}=\delta_{k,l}\delta_{q,N-s};k,l=\overline{1:M};q,s=\overline{0:N}$$

$$\int_{0}^{2\pi} G_s(\beta)\beta^m d\beta=\delta_{0,m};s,m=\overline{0,N} .$$

Theorem 5. If $f_{k,s}(\phi)\in C^{v-s}[0,2\pi],s=\overline{0,N},N\le v$, then $\forall\beta\in[-1,1]$

$$U_{k,0}(\phi,z,\beta)=f_{k,0}(g_k(\phi,z,\beta))\in C^v(D^*),D^*=\{(\phi,z,\beta):[0,2\pi]\times[0,H]\times[-1,1] ,$$

$$U_{k,s}(\phi,z,\beta)=\int_{0}^{g_k(\phi,z,\beta)} f_{k,s}(u)\frac{(g_k(\phi,z,\beta)-u)^{s-1}}{(s-1)!}du\in C^v(D^*).$$

Theorem 6. If the assumption (9)-(10) in [7] is true, then functions

$$V_{k,0}(\phi,z,\beta)=\int_{-1}^{1} G_0(\beta)f_{k,0}(g_k(\phi,z,\beta))d\beta\in C^v(D^*)$$

$$V_{k,s}(\phi,z)=\int_{-1}^{1} G_s(\beta)\int_{0}^{g_k(\phi,z,\beta)} f_{k,s}(u)\frac{(g_k(\phi,z,\beta)-u)^{s-1}}{(s-1)!}dud\beta$$

have the following properties

$$V_{k,s}(\phi,z)\in C^{v}\left(D^{*}\right), k=\overline{1:M}, s=\overline{0:N}$$

$$\left.\frac{\partial^{q}}{\partial z^{q}}V_{k,0}(\phi,z)\right|_{\Gamma_k}=\begin{cases}f_{k,0}(\phi), q=0\\ 0, 1\le q\le N\end{cases}$$

$$\left.\frac{\partial^{q}}{\partial z^{q}}V_{k,s}(\phi,z)\right|_{\Gamma_k}=\begin{cases}0, 0\le q\le s-1\\ f_{k,s}(\phi), q=s\\ \left.\int_{-1}^{1}G_s(\beta)\frac{\partial^{q-s}}{\partial z^{q-s}}f_{k,s}\left(g_k(\phi,z,\beta)\right)d\beta\right|_{\Gamma_k}=0\\ s<q\le N; q,s\le N\end{cases}$$

The Hermit interlineation operator is follows

$$O_{MN}f(\phi,z)=\sum_{k=1}^{M}h_{k,0}(\phi,z)\int_{-1}^{1}G_0(t)f_{k,0}\left(\phi+t\left(z-z_k(\phi)\right)\right)dt+$$

$$+\sum_{k=1}^{M}\sum_{s=1}^{N}h_{k,s}(\phi,z)\int_{-1}^{1}G_s(\beta)\int_{0}^{\phi+\beta(z-z_k(\phi))}f_{k,s}(u)\frac{\left(\phi+\beta\left(z-z_k(\phi)\right)-u\right)^{s-1}}{(s-1)!}du\,d\beta$$

Theorem 7. Operator $O_{MN}f(\phi,z)$ has the following properties

$$f_{k,s}(\phi)\in C^{v-s}[0,2\pi], k=\overline{1,M}; s=\overline{0,N}\Rightarrow O_{MN}f(\phi,z)\in C^{v}(D)$$

and to obtain

$$\frac{\partial^{p}O_{MN}f}{\partial z^{p}}\left(\phi,z_k(\phi)\right)=f_{k,p}(\phi), k=\overline{1,M}, p=\overline{0,N}$$

it is enough to fullfill the conditions (10) [7] and

$$h_{k,s}(\phi,z)\in C^{v}(D), k=\overline{1,M}; s=\overline{0,N},$$

$$\frac{\partial^{q}h_{k,p}}{\partial z^{q}}\left(\phi,z_l(\phi)\right)=\delta_{k,l}\delta_{p,q}, k,l=\overline{1,M}; p,q=\overline{0,N}$$

3.2. Construction of Hermit type operator interpolation on nonregular net of nodes, located on arbitrary system of non-intersecting lines of the surface preserving $C^{v}(D)$ class.

These operators have the following form

$$E_{MN}f(\phi,z)=\sum_{k=1}^{M}h_{k,0}(\phi,z)\int_{-1}^{1}G_0(\beta)sp_{k,0}\left(\phi+\beta\left(z-sp_k(\phi)\right)\right)d\beta+$$

$$+\sum_{k=1}^{M}\sum_{s=1}^{m}h_{k,s}(\phi,z)\int_{-1}^{1}G_s(\beta)\int_{0}^{\phi+\beta(z-sp_k(\varphi))}sp_{k,s}(u)\frac{(\phi+\beta(z-sp_k(\phi)-u)^{s-1}}{(s-1)!}du\,d\beta$$

where

$sp_{k,s}(\phi)\in C^{v-s}[0,2\pi]$,

$sp^{(p)}{}_{k,s}\left(\phi_j\right)=r_{k,j}^{p}\delta_{0,p}; sp_k\left(\phi_j\right)=r_{k,j}^{0}, k=\overline{1,M}; p,s=\overline{1,N}; j=\overline{0,Q_k}$.

Theorem 8. If theorem 7 statements are true, then operator $E_{MN}f(\phi,z)$ is defined with help of the followng relations:

$$sp_{k,s}(\phi)\in C^{v-s}[0,2\pi], k=\overline{1,M}; s=\overline{0,N}\Rightarrow E_{MN}f(\phi,z)\in C^{v}(D)$$

$$sp_{k,s}\left(\phi_j\right)=r_{k,j}^{(s)}, k=\overline{1:M}; s=\overline{0:N}; j=\overline{1:Q_k}\Rightarrow\frac{\partial^{p}E_{Mm}f}{\partial z^{p}}\left(\phi_j,z_{k,j}\right)=r_{k,j}^{(p)},$$

$k=\overline{1,M}; s=\overline{0,N}; j=\overline{1,Q_k}$.

Conclusions

Methods of construction interlineation operators described in the given paper, can be used in a variational structure methods for solving boundary value problems in case if derivatives on the domain border are not equal to zero and do not belong to the required differentiability class (in particular, this can be in case of Neumann inhomogeneous boundary conditions for Poisson's equation or for the Dirichlet boundary value problem for biharmonic equation, etc.).

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