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Geometrically Nonlinear Vibrations of Functionally Graded Shallow Shells

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Abstract
An original method for investigation of geometrically nonlinear vibrations of functionally graded shallow shells and plates with complex planform is presented Shells under consideration are made from a composite of ceramics and metal. Power law of volume fraction distribution of materials through thickness is chosen. Mathematical statement is implemented in the framework of the refined geometrically nonlinear theory of the shallow shells of the first order (Timoshenko type). The proposed approach combines the application of the R-functions theory (RFM), variational Ritz method, procedure by Bubnov-Galerkin and Runge-Kutta method Due to use of this combined algorithm it is possible to reduce the initial nonlinear system of motion equations with partial derivatives to a nonlinear system of ordinary differential equations. Investigation task of functionally graded shallow shells with arbitrary planform and different types of boundary conditions is carried out by the proposed method Test problems and numerical results have been presented for one-mode approximation in time. In future, the developed method may be extended to investigation of geometrically nonlinear forced vibrations of functionally graded shallow shells with complex planform.

Keywords
Functionally graded shallow shells, nonlinear vibrations, theory of the R-functions, method by Ritz.

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Introduction

Design and construction of modern aerospace elements make use of a wide range of composite materials, in particular, functionally graded materials (FGMs). Considering that the basic advantages of FGMs are associated with their heat and corrosion resistance in the high temperature environment, it might be noted that most scientific works at first dealt with investigation of thermal stresses and strains. In later works [1-6] the problems of strength, vibration and stability of FGM elements under mechanical load were analysed. An extensive survey on literature of nonlinear static and dynamic behaviour of FGM plates and shallow shells is represented in [1, 5]. Finite Element Method is the most frequently used in the context. Using variational methods one is usually limited to rectangular shape of the plan, and as a rule, to simple type of boundary conditions along the whole boundary (clamped or simply supported edge). Taking into account the possibilities of the R-function theory the method for investigation of nonlinear vibration of FGM shallow shells with arbitrary planform and different boundary conditions is proposed.

1. Mathematical statement of problem

According to the refined geometrically nonlinear theory of shallow shells, the displacements at any point of shell u_1, u_2, u_3 can be represented as [2, 4]:

$$u_1 = u + z\psi_x, \quad u_2 = v + z\psi_y, \quad u_3 = w, \tag{1}$$

where u, v are middle surface displacements along the axes *Ox* and *Oy* respectively, w is the transverse deflection of the shell along the axis *Oz*, ψ_x, ψ_y are angles of rotations of the normal to the middle surface about axes *Ox* and *Oy*.

Relations for deformations $\{\varepsilon\}=\{\varepsilon_{11};\ \varepsilon_{22};\ \varepsilon_{12}\}^T$, $\{\chi\}=\{\chi_{11};\ \chi_{22};\ \chi_{12}\}^T$ are expressed as:

$$\varepsilon_{ij} = \varepsilon_{ij}^{L} + \varepsilon_{ij}^{NL}, \quad (i, j = 1,2),$$

where

$$\varepsilon_{11}^{L} = u,_x + w/R_x \quad \varepsilon_{22}^{L} = v,_y + w/R_y \quad \varepsilon_{12}^{L} = u,_y + v,_x,$$

$$\varepsilon_{11}^{NL} = \frac{1}{2} w,_x^2, \quad \varepsilon_{22}^{NL} = \frac{1}{2} w,_y^2; \quad \varepsilon_{12}^{NL} = w,_x\, w,_y,$$

$$\varepsilon_{13} = w,_x + \psi_x, \quad \varepsilon_{23} = w,_y + \psi_y, \chi_{11} = \psi_{x},_x, \quad \chi_{22} = \psi_{y},_y \quad \chi_{12} = \psi_{x},_y + \psi_{y},_x. \tag{2}$$

Not the presence of nonlinearity in the strain-displacement relations via the quadratic term.

Suppose that volume fraction of the ceramic phase $V = V(z,h,k)$ is defined according to the power-law:

$$V = \left(\frac{2z+h}{2h}\right)^k, \tag{3}$$

where h is a thickness of the shell, k is the parameter that governs the material variation in the thickness direction. It varies from 0 to infinity $(0 \le k \le \infty)$. In the case $k = 0$, it is the whole structure is ceramic, in case $k = \infty$, it is the whole structure is a metallic one. Then general formula for the determination of the elastic modulus E, the Poisson ratio ν and the density ρ of the composite as a function of z and temperature T is represented by

$$P(z,T) = (P_c(T) - P_m(T))\left(\frac{z}{h} + \frac{1}{2}\right)^k + P_m(T), \tag{4}$$

where $P_c(T), P_m(T)$ are the corresponding characteristics of ceramics and metal.

Functionally graded materials are widely used in the high-temperature environments and their mechanical characteristics might be different in depending on temperature changing. So the dependence should be included in the calculation of exact solutions. We make use of these dependencies shown in [5, 10]:

$$P_j = P_0\left(P_{-1}T^{-1} + 1 + P_1 T + P_2 T^2 + P_3 T^3\right),$$

where $P_0, P_{-1}, P_1, P_2, P_3$ are the coefficients defined for each certain material.

The relations between stress and strain resultants in matrix form are given by the following formulas

$$\{N\} = [A]\{\varepsilon^0\} + [B]\{\chi\}, \quad \{M\} = [B]\{\varepsilon^0\} + [D]\{\chi\}, \tag{5}$$

where $\{N\} = \{N_{11}, N_{22}, N_{12}\}^T$ are forces per unit edge length in the middle surface of a shell, $\{M\} = \{M_{11}, M_{22}, M_{12}\}^T$ are bending and twisting moments per unit edge length, components of the vectors $\{\varepsilon^0\} = \{\varepsilon_{11}^0, \varepsilon_{22}^0, \varepsilon_{12}^0\}^T$ and $\{\chi\} = \{\chi_{11}, \chi_{22}, \chi_{12}\}^T$ are defined by expressions (2). Elements of the matrixes $[A]$, $[B]$, $[D]$ have the following form:

$$([A],[B],[D]) \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)[C](1, z, z^2)dz, \text{ where } [C] = \frac{1}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}. \tag{6}$$

If the Poisson ratios of the constituent phases are such that $\nu_m = \nu_c$, then elements A_{ij}, B_{ij}, D_{ij} of the matrixes in formula (6) may be calculated easily and relation (5) will have the following type:

$$\{N\}=[C](E_1\{\varepsilon^0\}+E_2\{\chi\}),\quad \{M\}=[C](E_2\{\varepsilon^0\}+E_3\{\chi\}), \tag{7}$$

where

$$E_1=\left(E_m+\frac{E_c-E_m}{k+1}\right)h,\quad E_2=\frac{(E_c-E_m)kh^2}{2(k+1)(k+2)},$$

$$E_3=\left(\frac{E_m}{12}+(E_c-E_m)\left(\frac{1}{k+3}-\frac{1}{k+2}+\frac{1}{4(k+1)}\right)\right)h^3.$$

The density of the composite ρ is also determined as a result of the integration through the shell thickness. The result is

$$\rho=\left(\rho_m+\frac{\rho_c-\rho_m}{k+1}\right)h.$$

Potential and kinetic energy are given by the fomulas:

$$U=\frac{1}{2}\iint_{\Omega}(N_{11}\varepsilon_{11}+N_{22}\varepsilon_{22}+N_{12}\varepsilon_{12}+M_{11}\chi_{11}+M_{22}\chi_{22}+M_{12}\chi_{12})d\Omega+$$

$$+\frac{1}{2}\iint_{\Omega}\left(Q_x(w_{,x}+\psi_x)+Q_y(w_{,y}+\psi_y)\right)d\Omega$$

$$T=\frac{1}{2}\iint_{\Omega}I_0\left(u_{,t}^2+v_{,t}^2+w_{,t}^2\right)+2I_1\left(u_{,t}\psi_{x,t}+v_{,t}\psi_{y,t}\right)+I_2\left(\psi_{x,t}^2+\psi_{y,t}^2\right)dxdy,$$

where

$$I_o=\left(\rho_m+\frac{\rho_c-\rho_m}{k+1}\right)h,\qquad I_1=\int_{-h/2}^{h/2}\rho(z)\,zdz=\frac{(\rho_c-\rho_m)k}{2(k+1)(k+2)}h^2,$$

$$I_2=\int_{-h/2}^{h/2}\rho(z)z^2dz=\left(\frac{\rho_m}{12}+(\rho_c-\rho_m)\left(\frac{1}{k+3}-\frac{1}{k+2}+\frac{1}{4(k+1)}\right)\right)h^3.$$

Then expressions account for rotary inertia and shear deformation. Transverse forces Q_x, Q_y are defined as:

$$Q_x=K_s^2A_{33}\varepsilon_{13},\quad Q_y=K_s^2A_{33}\varepsilon_{23},$$

where K_s^2 is a shear coefficient assumed equal to $5/6$.

The equations of motion in the framework of the refined geometrically nonlinear theory of the shallow shells of the first order have been obtained in Ref [4, 5].

2. Method of solution

The proposed method of investigation of geometrically nonlinear vibration of FGM shallow shells assumes the solution of linear problem at the first step [7]. The variational structural method (RFM) is applied to seek for such a solution. This approach is based on an application of the R-function theory and method by Ritz. Linear solution problem for FGM shallow shells was provided in [8, 9].

It is assumed that inertia forces in the middle surface of a shell are ignored at solving the nonlinear problem. Introduce unknown functions in the expansion of eigenfunctions $w_1^{(e)}(x,y)$, $u_1^{(e)}(x,y)$, $v_1^{(e)}(x,y)$, $\psi_{x1}^{(e)}(x,y)$, $\psi_{y1}^{(e)}(x,y)$. They correspond to the main vibration form:

$$\begin{cases} u(x,y,t) = y_1(t)u_1^{(e)}(x,y) + y_1^2(t)u_2(x,y), \\ v(x,y,t) = y_1(t)v_1^{(e)}(x,y) + y_1^2(t)v_2(x,y), \\ w(x,y,t) = y_1(t)w_1^{(e)}(x,y), \\ \psi_x(x,y,t) = y_1(t)\psi_{x1}^{(e)}(x,y) + y_1^2(t)\psi_{x2}(x,y), \\ \psi_y(x,y,t) = y_1(t)\psi_{y1}^{(e)}(x,y) + y_1^2(t)\psi_{y2}(x,y). \end{cases} \tag{8}$$

Coefficient of this expansion is function $y(t)$ depending on time. Functions $u_2, v_2, \psi_{x2}, \psi_{y2}$ might be solutions of the following system of differential equations:

$$\begin{cases} L_{11}u_2(x,y) + L_{12}v_2(x,y) + L_{14}\psi_{x2}(x,y) + L_{15}\psi_{y2}(x,y) = NL_1 w_1^{(e)}(x,y) \\ L_{21}u_2(x,y) + L_{22}v_2(x,y) + L_{24}\psi_{x2}(x,y) + L_{25}\psi_{y2}(x,y) = NL_2 w_1^{(e)}(x,y) \\ L_{41}u_2(x,y) + L_{42}v_2(x,y) + L_{44}\psi_{x2}(x,y) + L_{45}\psi_{y2}(x,y) = NL_4 w_1^{(e)}(x,y) \\ L_{51}u_2(x,y) + L_{52}v_2(x,y) + L_{54}\psi_{x2}(x,y) + L_{55}\psi_{y2}(x,y) = NL_5 w_1^{(e)}(x,y) \end{cases} \tag{9}$$

where

$$NL_1(w) = -L_{11}(w)\frac{\partial w}{\partial x} - L_{12}(w)\frac{\partial w}{\partial y}, \quad NL_2(w) = -L_{12}(w)\frac{\partial w}{\partial x} - L_{22}(w)\frac{\partial w}{\partial y},$$

$$NL_4(w) = -L_{41}(w)\frac{\partial w}{\partial x} - L_{42}(w)\frac{\partial w}{\partial y}, \quad NL_5(w) = -L_{51}(w)\frac{\partial w}{\partial x} - L_{52}(w)\frac{\partial w}{\partial y}.$$

Linear operators L_{ij} $(i, j = \overline{1,5})$ in equations (9) are given by

$$L_{11} = A_{11}\frac{\partial^2}{\partial x^2} + A_{33}\frac{\partial^2}{\partial y^2}, \quad L_{12} = L_{21} = (A_{12} + A_{33})\frac{\partial^2}{\partial x \partial y}, \quad L_{14} = L_{41} = B_{11}\frac{\partial^2}{\partial x^2} + B_{33}\frac{\partial^2}{\partial y^2},$$

$$L_{15} = L_{51} = L_{24} = L_{42} = (B_{12} + B_{33})\frac{\partial^2}{\partial x \partial y}, \quad L_{22} = A_{22}\frac{\partial^2}{\partial y^2} + A_{33}\frac{\partial^2}{\partial x^2}, \quad L_{25} = L_{52} = B_{22}\frac{\partial^2}{\partial y^2} + B_{33}\frac{\partial^2}{\partial x^2},$$

$$L_{44} = D_{11}\frac{\partial^2}{\partial x^2} + D_{33}\frac{\partial^2}{\partial y^2} - K_s^2 A_{33}\psi_x, \quad L_{45} = L_{54} = (D_{12} + D_{33})\frac{\partial^2}{\partial x \partial y},$$

$$L_{55} = D_{22}\frac{\partial^2}{\partial y^2} + D_{33}\frac{\partial^2}{\partial x^2} - K_s^2 A_{33}\psi_y.$$

System (9) is supplemented by the corresponding boundary conditions. Solution of this problem is carried out by means of variational method by Ritz and RFM. Taking into account such a choice of functions $u_2(x,y)$, $v_2(x,y)$, $\psi_{x2}(x,y)$, $\psi_{y2}(x,y)$ and substituting expressions (8) in the equation of motion and applying the Bubnov-Galerkin procedure, the following nonlinear differential equation of the second order is obtained:

$$\ddot{y}(t) + \omega_L^2 y_1(t) + y_1^2(t)\beta + y_1^3(t)\gamma = 0. \tag{10}$$

Values for coefficients of equation (10) have been obtained in analytical form. They are expressed through the double integrals of unknown functions:

$$\beta = \frac{-1}{m_1\left\|w_1^{(e)}\right\|^2}\iint_\Omega \left(L_{31}u_2 + L_{32}v_2 + L_{34}\psi_{x2} + L_{35}\psi_{y2} + N_{11}^{L_1}\frac{\partial^2 w_1^{(e)}}{\partial x^2} + N_{22}^{L_1}\frac{\partial^2 w_1^{(e)}}{\partial y^2} + 2N_{12}^{L_1}\frac{\partial^2 w_1^{(e)}}{\partial x \partial y} \right) w_1^{(e)} dxdy,$$

$$\gamma = \frac{-1}{m_1\left\|w_1^{(e)}\right\|^2}\iint_\Omega \left(NL_{33}w_1^{(e)} + N_{11}^{(L_2)}\frac{\partial^2 w_1^{(e)}}{\partial x^2} + N_{22}^{(L_2)}\frac{\partial^2 w_1^{(e)}}{\partial y^2} + 2N_{12}^{(L_2)}\frac{\partial^2 w_1^{(e)}}{\partial x \partial y} \right) w_1^{(e)} dxdy.$$

Here

$$L_{13} = -L_{31} = -(k_1 A_{11} + k_2 A_{12})\frac{\partial}{\partial x},\ L_{23} = -L_{32} = -(k_1 A_{21} + k_2 A_{22})\frac{\partial}{\partial y},$$

$$L_{34} = -L_{43} = (K_s^2 A_{33} + k_1 B_{11} + k_2 B_{12})\frac{\partial}{\partial x},\ L_{35} = -L_{53} = (K_s^2 A_{33} + k_1 B_{12} + k_2 B_{22})\frac{\partial}{\partial y},$$

$$NL_{33}(w) = -\frac{1}{2}\left(A_{11}\left(\frac{\partial w}{\partial x}\right)^2 + A_{12}\left(\frac{\partial w}{\partial y}\right)^2\right)\frac{\partial^2 w}{\partial x^2} - \frac{1}{2}\left(A_{21}\left(\frac{\partial w}{\partial x}\right)^2 + A_{22}\left(\frac{\partial w}{\partial y}\right)^2\right)\frac{\partial^2 w}{\partial y^2} - 2A_{33}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y},$$

$N_{ij}^{(L_1)}$, $N_{ij}^{(L_2)}$ $(i, j = 1,2)$ are defined by the following expressions:

$$\{N^{(L_1)}\} = \{N_{11}^{(L_1)};\ N_{22}^{(L_1)};\ N_{12}^{(L_1)}\}^T = [A]\{\varepsilon^{L_1}\}^T + [B]\{\chi\},\ \{\varepsilon^{L_1}\} = \left\{\frac{\partial u_1}{\partial x}; \frac{\partial v_1}{\partial y}; \frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right\};$$

$$\{N^{(L_2)}\} = \{N_{11}^{(L_2)};\ N_{22}^{(L_2)};\ N_{12}^{(L_2)}\}^T = [A]\{\varepsilon^{L_2}\}^T,\ \{\varepsilon^{L_2}\} = \left\{\frac{\partial u_2}{\partial x}; \frac{\partial v_2}{\partial y}; \frac{\partial u_2}{\partial y} + \frac{\partial v_2}{\partial x}\right\}.$$

Method by Runge-Kutta was used for solution of equation (10).

3. Numerical results

In order to validate of the proposed method and the computer code, in the framework of system POLE-RL [10] test problem has been solved. Free vibrations of clamped and simply supported FGM cylindrical and spherical square shell panels were studied. Numerical values of the natural frequencies for clamped and simply supported functionally graded cylindrical and spherical square shell panels were compared with the published results in works [12, 13].

The following material properties are used:

aluminum: $E_m = 70GPa$, $\nu_m = 0.3$, $\rho_m = 2707\ Kg/m^3$; (11)

alumina: $E_c = 380GPa$, $\nu_c = 0.3$, $\rho_c = 3000\ Kg/m^3$. (12)

The non-dimensional frequency is given as: $\bar{\omega}_L = \omega_L a^2\sqrt{\frac{\rho_m h}{D}}$, where $D = \frac{E_m h^3}{12(1-\nu_m^2)}$.

In *Table 1* the fundamental frequencies of square clamped FG cylindrical shell panels with side-to-thickness ratios $h/a = 0.1$ are presented considering various side-to-radius ratio R/a and power law exponents *k*.

Table 1. Fundamental frequencies of CCCC square cylindrical shell panels, *h/a=0.1* for various *R/a* and *k* $\left(\bar{\omega}_L = \omega_L a^2\sqrt{\frac{\rho_m h}{D}}\right)$

k	Source	R/a=1	R/a=5	R/a=10	R/a=50	Plate
0	RFM	96.6235	73.6575	72.8029	72.5271	72.5156
	Ref.[12]	96.0131	73.6436	72.8141	72.5465	72.5353
	Ref.[13]	94.4973	71.8861	71.0394	70.766	70.7546
0.5	RFM	81.3031	60.8468	60.0817	59.8397	59.8315
	Ref.[12]	80.3049	60.6568	59.9353	59.7178	59.7142
	Ref.[13]	79.5689	63.1896	62.4687	62.238	62.2291
1	RFM	72.8309	54.0093	53.3031	53.0821	53.0755
	Ref.[12]	71.9167	53.9340	53.2759	53.0841	53.0835
	Ref.[13]	71.2453	56.5546	55.8911	55.6799	55.6722
10	RFM	53.1347	41.5894	41.1738	41.0456	41.0424
	Ref.[12]	52.278	41.0985	40.7046	40.5923	40.5229
	Ref.[13]	51.3803	33.6611	33.1474	32.9812	32.9743
∞	RFM	43.6591	33.2824	32.8964	32.7717	32.7665
	Ref.[12]	43.3815	33.2743	32.8995	32.7786	32.7735
	Ref.[13]	44.2962	32.4802	32.0976	31.9741	31.9689

Table 2 presents the fundamental frequency of a square clamped FG spherical shell panel with constituents aluminum (11) and alumina (12), and side-to-thickness ratio $h/a = 0.1$, considering various side-to-radius ratios R/a, and several power law exponents *k*.

Table 2. Fundamental frequencies of CCCC square spherical shell panels, *h/a*=0.1 for various *R/a* and *k*

$$\left(\overline{\omega}_L = \omega_L a^2 \sqrt{\frac{\rho_m h}{D}}\right)$$

k	Source	R/a=1	R/a=5	R/a=10	R/a=50	Plate
0	RFM	123.3867	75.3315	73.2306	72.5443	72.5156
	Ref.[12]	122.3533	75.2810	73.2322	72.5633	72.5353
	Ref.[13]	120.9210	73.5550	71.4659	70.7832	70.7546
0.5	RFM	104.6467	62.3415	60.4591	59.8532	59.8315
	Ref.[12]	103.1490	62.0789	60.2831	59.7265	59.7142
	Ref.[13]	102.5983	64.6114	62.8299	62.2519	62.2291
1	RFM	94.0795	55.3867	53.6488	53.0936	53.0755
	Ref.[12]	92.6962	55.2302	53.5864	53.0895	53.0835
	Ref.[13]	92.2147	57.8619	56.2222	55.6923	55.6722
10	RFM	66.7355	42.4042	41.3755	41.0518	41.0424
	Ref.[12]	65.7018	41.8796	40.8883	40.5946	40.5929
	Ref.[13]	64.8773	34.6658	33.4057	32.9916	32.9743
∞	RFM	55.7515	34.0388	33.0896	32.7795	32.7665
	Ref.[12]	55.2827	34.0141	33.0884	32.7862	32.7735
	Ref.[13]	56.2999	33.2343	32.2904	31.9819	31.9689

The test examples presented here confirm a good agreement of results obtained and published ones for cases considered.

To illustrate the strength of the proposed method, several shells with complex plan-form are analyzed. The effect of boundary conditions, the shape of the plan, curvatures on the fundamental frequencies has been examined. Now the free vibrations of simply supported FGM shallow shell with rectangular cuts are analyzed. The shape of this shell is shown in Figure 1.

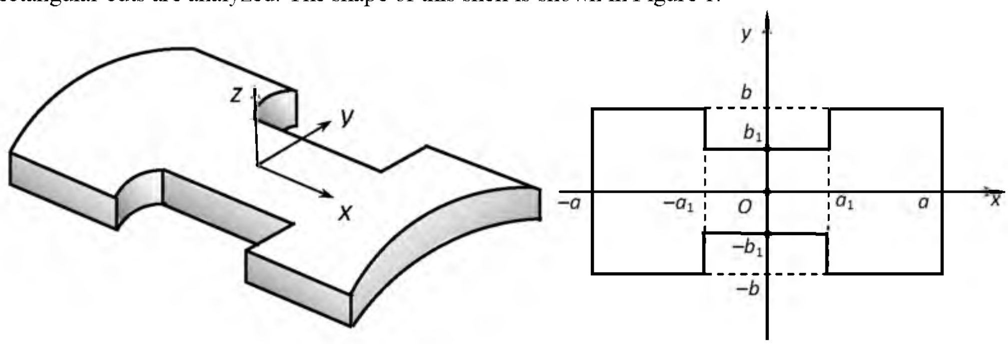

Figure 1. FGM shell panel with rectangular cuts

The geometrical parameters are: $\frac{b}{a} = 1$; $\frac{a_1}{2a} = 0.25$; $\frac{b_1}{2a} = 0.35; 0.4; 0.45$. The shell consists of two composites: aluminum (11) and alumina (12).

The boundary conditions for FDST are the following (contour is simply supported in tangent direction and clamped in transvers direction):

$$u = \psi_x = 0,\ \forall(x,y) \in \partial\Omega_1,\quad v = \psi_y = 0,\ \forall(x,y) \in \partial\Omega_2,\quad w = 0,\ \forall(x,y) \in \partial\Omega.$$

To satisfy the main boundary conditions it is necessary to construct the following solution structure:

$$U = \omega_1 P_1,\ V = \omega_2 P_2,\ W = \omega\ P_3,\ \Psi_x = \omega_1 P_4,\ \Psi_y = \omega_2 P_5,$$

where P_1, P_2, P_3, P_4, P_5 - are indefinite components;

$\omega_1 = 0$ - is the equation of parts of boundary domain parallel to the axis OX;

$\omega_2 = 0$ - is the equation of parts of boundary domain parallel to the axis OY;

$\omega\ = 0$ - is the equation of the whole domain.

Using the R-function operations these equations are constructed:

$\omega_1 = (f_3 \vee_0 f_4) \wedge_0 f_1 \vee_0 (f_9 \vee_0 f_{10})$,

$\omega_2 = (f_3 \vee_0 f_4) \wedge_0 f_2 \vee_0 (f_5 \vee_0 f_6) \vee_0 (f_7 \vee_0 f_8)$,

$\omega\ = (f_1 \vee_0 f_2) \wedge_0 (f_3 \vee_0 f_4)$,

where $f_1 = \dfrac{(a^2 - x^2)}{2a} \geq 0$, $f_2 = \dfrac{(b^2 - y^2)}{2b} \geq 0$, $f_3 = \dfrac{(x^2 - a_1^2)}{2a_1} \geq 0$, $f_4 = \dfrac{(b_1^2 - y^2)}{2b_1} \geq 0$,

$f_5 = \dfrac{((x + a_1)^2 + (y - y_0)^2 - r_1^2)}{2r_1} \geq 0$, $f_6 = \dfrac{((x - a_1)^2 + (y - y_0)^2 - r_1^2)}{2r_1} \geq 0$,

$f_7 = \dfrac{((x + a_1)^2 + (y + y_0)^2 - r_1^2)}{2r_1} \geq 0$, $f_8 = \dfrac{((x - a_1)^2 + (y + y_0)^2 - r_1^2)}{2r_1} \geq 0$,

$f_9 = \dfrac{(x^2 + (y - b_1)^2 - r_2^2)}{2r_2} \geq 0$, $f_{10} = \dfrac{(x^2 + (y + b_1)^2 - r_2^2)}{2r_2} \geq 0$.

In order to get correct results for the case considered, a solution about gradual increase of cut size is obtained. So, the first domain was constructed with values $\dfrac{a_1}{2a} = 0.05$ and $\dfrac{b_1}{2a} = 0.48$. This shell with rectangular cuts tends to square shell panel quite close and hence it is clear that results for square shell panel and shell with small rectangular cuts are very close as well (see values from two first columns of *Table 3*).

So, these rectangular cuts have been expanded from size of cuts $\dfrac{a_1}{2a} = 0.1$ and $\dfrac{b_1}{2a} = 0.45$ by gradually increasing the size of the cuts given by $\dfrac{a_1}{2a} = 0.25$ and $\dfrac{b_1}{2a} = 0.3$. *Table 3* presents the fundamental frequency of simply supported FG cylindrical shell with side-to-thickness ratio $h/2a = 0.1$, side-to-radius ratios $R/2a = 10$, different shapes of domain and several power law exponents *k*.

Table 3. Fundamental frequencies of SSSS cylindrical shell panels with rectangular cuts, $h/2a = 0.1$, $R/2a = 10$ for various *k* and cut sizes $\left(\overline{\omega}_L = \omega_L a^2 \sqrt{\dfrac{\rho_m h}{D}}\right)$

k	**Cut** $a_1/2a = 0$ $b_1/2a = 0$	**Cut** $a_1/2a = 0.25$ $b_1/2a = 0.49$	**Cut** $a_1/2a = 0.25$ $b_1/2a = 0.45$	**Cut** $a_1/2a = 0.25$ $b_1/2a = 0.4$	**Cut** $a_1/2a = 0.25$ $b_1/2a = 0.3$	**Cut** $a_1/2a = 0.25$ $b_1/2a = 0.25$
0	42.43	42.88	48.13	60.85	89.08	115.25
0.2	38.78	39.28	44.12	55.76	81.71	105.82
0.5	34.81	35.37	39.74	50.20	73.65	95.48
1	30.80	31.37	35.27	44.53	65.38	84.83
2	27.40	27.90	31.36	39.59	58.10	75.34
10	24.18	24.28	27.19	34.42	50.19	64.74
∞	19.17	19.38	21.75	27.49	40.25	52.08

From the experiments conducted here for simply supported FGM shallow shells with different sizes of rectangular cuts, the following conclusion can be drawn: the fundamental frequencies increase with increasing the value of the cut parameter, which is fully explained by geometrical statement of problem.

Conclusions

An original method for investigation of geometrically nonlinear vibrations of functionally graded shallow shells and plates with complex planform is presented. The proposed approach combines the application of the R-functions theory (RFM), the Ritz variational method, the Bubnov-Galerkin procedure and the method by Runge-Kutta. The proposed approach for the refined geometrically nonlinear theory of shallow shells of the first order has been implemented in the system POLE-RL [10]. Test problems proved a good agreement with compare results from literature. New numerical results for the natural frequencies of FGM cylindrical and spherical shallow shells with complex planform and different types of boundary conditions have been obtained. In future, the method developed here could be extended to investigate geometrically nonlinear vibrations of FGM shallow shells with complex planform under transverse load.

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