ON THE SOLUTIONS IN THE GLOBAL ATTRACTOR OF THE

INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

A Dissertation

by

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ABSTRACT

We study the global attractor for the solutions of the incompressible Navier-Stokes equations (NSE) equipped with appropriate boundary conditions.

A challenge in the cases when zero is not in the global attractor is to find sharp lower bound on the energy. A related challenging problem is to show that zero is in the attractor if and only if the external force is zero. We show that if zero were in the global attractor, then all its elements, as well as the external force, must be smooth functions. By exploring a particular family of function classes, we show that the set of nonzero external forces for which zero could be in the global attractor is meagre (of the first Baire category in a Fréchet topology).

The weak global attractor of three dimensional Navier-Stokes equations is a complex geometric object. An interesting challenging question is to measure its complexity. Invoking the fact that topology on the weak global attractor can be metrizable, we use a physically reasonable metric function to obtain explicit estimate for the Kolmogorov ε-entropy of the weak global attractor in terms of the physical parameter associated with the fluid flow.

We also study the existence of the nonstationary solutions in the global attractor of the space periodic two dimensional NSE which have constant energy and enstropy per unit mass for all time. A subclass of such solutions whose geometric structures have a supplementary stability property is defined and explored. We prove that the wave vectors of the active mode of this subclass must satisfy a finite Galerkin system. The nonexistence of solutions in this subclass is proved for the particular case when the external force has a special property.

DEDICATION

To my beloved wife, Ximei Liang, my parents Jican Zhang and Shuizhi Yin, and my daughters, Sarah L Zhang and Kaitlyn H Zhang

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1. INTRODUCTION

In this thesis, we study some properties of the global attractor of the incompressible Navier-Stokes equations (NSE). These equations describe the dynamics of incompressible, viscous fluid flows. The global attractor, which is a uniquely determined compact invariant set in an appropriate phase space attracting all the solutions, captures all the information about the long time behavior of the solutions, and is therefore an appropriate object to be studied.

In Chapter 3, we study the consequence of assuming zero to be in the global attractor of 2D NSE. It is a well-known that all solutions in the global attractor of the incompressible 2D NSE can be extended to analytic functions, in time, in a uniform strip $\mathcal{S}$ containing the real axis in the complexified time plane (see, e.g. $[2, 11, 13, 17, 20]$). Moreover, these solutions are known to be uniformly bounded in the norm $|A \cdot|$, where A is the Stokes operator, $|\cdot| = |\cdot|_{L^2(\Omega)}$, and $\Omega = [0, L]^2$ is the spatial domain, with periodic boundary conditions. On the other hand, as remarked in $[4]$, if $0 \in \mathcal{A}$, then by inserting 0 into the NSE, one sees that g must be in the domain $\mathcal{D}(A)$, consequently, the solution is in fact in $\mathcal{D}(A^2)$. This in turn implies that g is in $\mathcal{D}(A^2)$, and so on by induction. Thus, g must be in $\mathcal{D}(A^m)$ for any $m \in \mathbb{N}$. Asking that 0 be in the attractor is a highly restrictive assumption. We carry out rather intensive estimates during the inductive process described above to establish uniform bounds in $|A^m \cdot|$ on a strip $\mathcal{S}$ of a specific width δ for all $m \in \mathbb{N}$. A special function class, denoted by $\mathcal{C}(\sigma)$, is identified and studied. In particular, we show that if $0 \in \mathcal{A}$, then all elements in $\mathcal{A}$, as well as g, are in $C^\infty(\Omega)$.

In Chapter 4, we concentrate on a conjecture proposed by P. Constantin, namely, 0 is in the global attractor $\mathcal{A}$ if and only if the body force g is zero. By connecting

the functional class $\mathcal{C}(\sigma)$ and the usual Gevrey class of spatial analytic functions, we show that the set of nonzero forces for which $0 \in \mathcal{A}$ is meagre (of the first Baire category in a Fréchet topology). Moreover, we provide an explicit criterion for zero to be in the global attractor.

In Chapter 5, we use the Kolmogorov ε-entropy to measure the complexity of the weak global attractor $\mathcal{A}_{\mathrm{w}}$ of the 3D NSE. To do that, we first introduce a particular metric (which has no apparent physical meaning) that generates the weak topology on $\mathcal{A}_{\mathrm{w}}$ and use it to obtain an upper bound for the Kolmogorov ε-entropy of $\mathcal{A}_{\mathrm{w}}$. This bound is expressed explicitly in terms of the physical parameters of the fluid flow. We then apply the squeezing property to establish an explicit upper estimate on the Kolmogorov ε-entropy of $\mathcal{A}_{\mathrm{w}}$, using a more physically relevant metric.

In Chapter 6, we study the geometric properties of the nonstationary solutions in the global attractor of the space periodic 2D NSE with their energy and enstropy per unit mass being constant for every $t \in (-\infty, \infty)$. Such solutions, due to the hypothetical existence of such solutions,were called "ghost solutions". We introduce and study geometric structures shared by all ghost solutions. This study led us to consider a subclass of ghost solutions for which those geometric structures have a supplementary stability property.

2. PRELIMINARIES

2.1 General mathematical settings

The equations to be considered are the following d-dimensional incompressible Navier-Stokes equations (NSE) with periodic boundary conditions in $\Omega = [0, L]^d$ (where $d = 2, 3$)

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f,$$
$$\nabla \cdot u = 0,$$
$$u(x, 0) = u_0(x),$$
$$\int_\Omega u \, dx = 0, \qquad \int_\Omega f \, dx = 0. \tag{2.1}$$

where $u : \mathbb{R}^d \to \mathbb{R}^d$, $p : \mathbb{R}^d \to \mathbb{R}$ are the unknown Ω-periodic functions, the velocity field and the presure, respectively, and $\nu > 0$ is the viscosity of the fluid, $L > 0$ is the period, are given constants, and f is the "body" force (see, e.g., [2, 20, 21] for more details).

Denote by $H^m(\Omega)$ the usual Sobolev spaces which consists of all functions of $L^2(\Omega)$ with distributional derivatives up to order $m \in \mathbb{N}$ that belong to $L^2(\Omega)$. The phase space H (respectively, V) is defined as the subspace of $[L^2(\Omega)]^d$ (respectively, $[H^1(\Omega)]^d$) consisting of all elments in the closure of the set of $\mathbb{R}^d$-valued trigometric polynomials v satisfying

$$\nabla \cdot v = 0, \tag{2.2}$$

and

$$\int_\Omega v(x) dx = 0. \tag{2.3}$$

The scalar product in H and V are taken to be

$$(u, v) = \int_\Omega u(x) \cdot v(x) dx, \text{ and } ((u, v)) = \int_\Omega \sum_{i=1}^d \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_i} dx,$$

respectively, with associated norm $|u| = (u, u)^{1/2}$ and $\|u\| = ((u, u))^{1/2}$.

Let $\mathcal{P}_\sigma : [L^2(\Omega)]^d \to H$ be the orthogonal projection (called the Helmholtz-Leray projection). Define the Stokes operator $A = -\mathcal{P}_\sigma \Delta$ ($=-\Delta$, under periodic boundary conditions) with domain $\mathcal{D}(A) = V \cap [H^2(\Omega)]^d$. As an operator from H to H, the Stokes operator A has a positive definite compact inverse. As a consequence, the real Hilbert space H has an orthonormal basis $\{\omega_j\}_{j=1}^\infty$ consisting of eigenfunctions of A, namely, $A\omega_j = \lambda_j \omega_j$ with $0 < \lambda_1 = \left(\frac{2\pi}{L}\right)^d \leq \lambda_2 \leq \lambda_3 < \cdots$. The powers A^s are defined by

$$A^s v = \sum_{j=1}^\infty \lambda_j^s (v, \omega_j) \omega_j, \quad s \in \mathbb{R},$$

where $(\cdot, \cdot)$ is the L^2- scalar product. The domain of A^s is denoted by $\mathcal{D}(A^s)$. System (2.1) can be written as the differential evolutionary equation

$$\frac{du}{dt} + \nu Au + B(u, u) = g, \quad u \in H, \tag{2.4}$$

where the bilinear operator B and the driving force g are defined as $B(u, v) = \mathcal{P}_\sigma ((u \cdot \nabla)v)$ and $g = \mathcal{P}_\sigma f$, respectively. Here, we assume that f is time independent and hence (2.4) is an autonomous equation.

Since the boundary conditions are periodic, we may express an element $u \in H$ as a Fourier series expansion

$$u(x) = \sum_{k \in \mathbb{Z}^d \smallsetminus \{0\}} \hat{u}(k) e^{i\kappa_0 k \cdot x},$$

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where $\kappa_0 = 2\pi/L$,

$$\hat{u}(0) = 0, \tag{2.5}$$

$$(\hat{u}(k))^* = \hat{u}(-k), \tag{2.6}$$

and, due to incompressibility,

$$k \cdot \hat{u}(k) = 0. \tag{2.7}$$

Parseval's identity reads as

$$|u|^2 = L^d \sum_{k \in \mathbb{Z}^d \smallsetminus \{0\}} \hat{u}(k) \cdot \hat{u}(-k) = L^d \sum_{k \in \mathbb{Z}^d \smallsetminus \{0\}} |\hat{u}(k)|^2,$$

(we assume it will be clear from the context when $|\cdot|$ refers to the length of a vector in $\mathbb{C}^d$) as well as

$$(u, v) = L^d \sum_{k \in \mathbb{Z}^d \smallsetminus \{0\}} \hat{u}(k) \cdot \hat{v}(-k).$$

For $v \in H$, we define $P_n v$ to be the orthogonal projector of H onto the

$$\text{span}\{\omega_j | A\omega_j = \lambda_j \omega_j, j \le n\}. \tag{2.8}$$

2.2 The global attractor $\mathcal{A}$ of 2D NSE

Recall that the global attractor $\mathcal{A}$ of the 2D NSE (2.4) is the collection of all elements u_0 in H for which there exists a solution $u(t)$ of the NSE (2.4), for all $t \in \mathbb{R}$, such that $u(0) = u_0$ and $\sup_{t \in \mathbb{R}} |u(t)| < \infty$ (see, e.g., [2, 20]).

Equivalent definitions of $\mathcal{A}$ can also be given. As it is well-known, for any $u_0 \in H$, $f \in H$, there exists a unique continuous function u from $[0, \infty)$ into H such that $u(0) = u_0$, $u(t) \in \mathcal{D}(A)$, $t \in (0, \infty)$, and u satisfies the NSE (2.4), for all $t \in (0, \infty)$. Therefore, one can define the map $S(t) : H \to H$ by

$$S(t)u_0 = u(t), \tag{2.9}$$

where $u(t)$ is the solution. Since $S(t_1)S(t_2) = S(t_1 + t_2)$, the family $\{S(t)\}_{t \geq 0}$ is called the "solution" semigroup. Furthermore, a compact set $\mathcal{B}$ of H is called absorbing if for any bounded set $\tilde{\mathcal{B}} \subset H$ there is a time $\tilde{t} = \tilde{t}(\tilde{\mathcal{B}}) \geq 0$ such that $S(t)\tilde{\mathcal{B}} \subset \mathcal{B}$ for all $t \geq \tilde{t}$. The global attractor can be now defined alternatively by the formula

$$\mathcal{A} = \bigcap_{t \geq 0} S(t)\mathcal{B}, \tag{2.10}$$

where $\mathcal{B}$ is any absorbing compact subset of H.

2.3 Weak global attractor $\mathcal{A}_\mathrm{w}$ of the 3D NSE

To define the global attractor for 3D NSE, we first have to introduce the concept of Leray-Hopf weak solutions.

Definition 2.3.1 (see, e.g.,[10])**.** A (Leray-Hopf) weak solution on a time interval $J \subset \mathbb{R}$ is defined as a function $u = u(t)$ on J with values in H that satisfies the following properties:

(i) $u \in L^\infty_\mathrm{loc}(J; H) \cap L^2_\mathrm{loc}(J; V)$;

(ii) $\partial_t u \in L^{4/3}_\mathrm{loc}(J; V')$;

(iii) $u \in C(J; H_\mathrm{w})$, i.e., u is weakly continuous in H, which means that for every $v \in H$, the function $t \mapsto (u(t), v)$ is a continuous function from J into $\mathbb{R}$;

(iv) u satisfies (2.4) in the distribution sense on J with values in V';

(v) For almost all t' in J, u satisfies the following energy inequality:

$$\frac{1}{2}|u(t)|^2 + \nu \int_{t'}^{t} |A^{\frac{1}{2}}u(s)|^2 ds \leq \frac{1}{2}|u(t')|^2 + \int_{t'}^{t} (g, u(s)) ds \qquad (2.11)$$

for all t in J with $t > t'$. The allowed times t' are characterized as the points of strong continuity from the right, in H, for u, and their set denoted by $J'(u)$, whose complement has zero Lebesgue measure.

(vi) If J is closed and bounded on the left, with its left end point denoted by t_0, then the solution is continuous in H at t_0 from the right, i.e., $u(t) \to u(t_0)$ in H as $t \to t_0^+$.

In the following discussion, weak solution always means Leray-Hopf weak solution. The weak global attractor for 3D NSE is defined as follows.

Definition 2.3.2. Let

$$\mathcal{W} = \{u \in C(\mathbb{R}; H_{\mathrm{w}}) : u \text{ weak solution on } \mathbb{R} \text{ with } \sup_{t \in \mathbb{R}} |u(t)| < \infty\} \qquad (2.12)$$

The global weak attractor is defined to be

$$\mathcal{A}_{\mathrm{w}} = \{u_0 \in H : \exists u \in \mathcal{W}, u(0) = u_0\} \qquad (2.13)$$

The weak global attractor $\mathcal{A}_w$ can be considered as the smallest compact set in the weak topology of the phase space H which attracts all the weak solutions in the weak topology of H.

Remark 2.3.3. Notice that, since well-posedness (in the Hadamard sense) has only been established for the 2D NSE, but not yet for 3D case ([2, 21]), no well-defined semigroup associated with the solutions of the dynamical system determined by 3D

NSE is known to exist. Therefore, we do not have the equivalent definition of $\mathcal{A}_w$ using the concept of semigroup as we have done in 2D case in (2.10).

2.4 Complexification of the Navier-Stokes equations

Now consider the NSE (2.4) with complexified time and the corresponding solutions in $H_{\mathbb{C}}$ as in [2, 11]. Define

$$H_{\mathbb{C}} = \{u + iv : u, v \in H\},$$

and that $H_{\mathbb{C}}$ is a Hilbert space with respect to the following inner product

$$(u + iv, u' + iv')_{H_{\mathbb{C}}} = (u, u')_H + (v, v')_H + i[(v, u')_H - (u, v')_H],$$

where $u, u', v, v' \in H$. The extension $A_{\mathbb{C}}$ of A is given by

$$A_{\mathbb{C}}(u + iv) = Au + iAv,$$

for $u, v \in \mathcal{D}(A)$; thus $\mathcal{D}(A_{\mathbb{C}}) = \mathcal{D}(A)_{\mathbb{C}}$. Similarly, $B(\cdot, \cdot)$ can be extended to a bounded bilinear operator from $\mathcal{D}(A_{\mathbb{C}}^{1/2}) \times \mathcal{D}(A_{\mathbb{C}})$ to $H_{\mathbb{C}}$ by the formula

$$B_{\mathbb{C}}(u + iv, u' + iv') = B(u, u') - B(v, v') + i[B(u, v') + B(v, u')],$$

for $u, v \in \mathcal{D}(A^{1/2}), u', v' \in \mathcal{D}(A)$.

The Navier-Stokes equation with complex time is defined as

$$\frac{du(\zeta)}{d\zeta} + \nu A_{\mathbb{C}} u(\zeta) + B_{\mathbb{C}}(u(\zeta), u(\zeta)) = g, \tag{2.14}$$

Remark 2.4.1. We emphasize that some of the estimates and calculations produced

in the following chapters are formal but can be justified rigorously by obtaining them first for the Gelerkin approximation system and then for the full NSE by passing to the limit, using the appropriate compactness and convergence theorem, such as Aubin compactness theorem (see, e.g., [22]).

3. CONSEQUENCES OF ZERO BEING IN THE GLOBAL ATTRACTOR OF INCOMPRESSIBLE NAVIER-STOKES EQUATIONS[*]

In this chapter, we consider the two dimensional Navier-Stokes equations (2.1) with $d = 2$. Our main objective is to study the consequences of assuming zero to be in the global attractor.

3.1 Specific preliminaries and known facts

The following inequalities will be needed for the discussion in this chapter.

$$\kappa_0 |u| \le |A^{\frac{1}{2}} u|, \quad \text{for } u \in \mathcal{D}(A^{\frac{1}{2}}), \tag{3.1}$$

$$|u|_{L^4(\Omega)} \le c_L |u|^{\frac{1}{2}} |A^{\frac{1}{2}} u|^{\frac{1}{2}}, \quad \text{for } u \in \mathcal{D}(A^{\frac{1}{2}}), \tag{3.2}$$

$$|u|_\infty \le c_A |u|^{\frac{1}{2}} |Au|^{\frac{1}{2}}, \quad \text{for } u \in \mathcal{D}(A). \tag{3.3}$$

known respectively as the Poincaré, Ladyzhenskaya and Agmon inequalities. Both c_L and c_A are absolute constants (see, e.g., [2, 21]).

Another inequality that will be used frequently is the following Young's inequality,

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}, \tag{3.4}$$

for any $a, b \in \mathbb{R}$ with $a \ge 0, b \ge 0$, and any $p, q \in \mathbb{N}^+$, $1 < p, q < \infty$ satisfying $1/p + 1/q = 1$.

Our estimates will depend on the Grashof number

$$G = \frac{|g|}{\nu^2 \kappa_0^2}. \tag{3.5}$$

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[*]Part of this section is reproduced with permission from "Time analyticity with higher norm estimates for the 2D Navier-Stokes equations" by C. Foias, M. S. Jolly, R. Lan, R. Rupam, Y.Yang and B. Zhang, IMA Journal of Applied Mathematics (2014), hxu014 [6].

We also recall that (see Proposition 2.1 in [4]) if $G < c_L^{-2}$ then $\mathcal{A}$ consists of unique steady state. Throughout this chapter we will assume that G satisfies

$$G \geq \frac{1}{c_L^2}. \tag{3.6}$$

We recall the following algebraic properties of the bilinear operator $B(u,v)$. One has from [2,4], for every $u, v, w \in \mathcal{D}(A)$,

$$(B(u,v), w) = -(B(u,w), v), \tag{3.7}$$

$$(B(u,u), Au) = 0, \tag{3.8}$$

$$(B(Av, v), u) = (B(u, v), Av), \tag{3.9}$$

$$(B(u,v), Av) + (B(v,u), Av) + (B(v,v), Au) = 0. \tag{3.10}$$

From (3.9) and (3.10), it easily follows that if $u \in \mathcal{D}(A^{3/2})$ then $B(u,u) \in \mathcal{D}(A)$ and that

$$AB(u,u) = B(u, Au) - B(Au, u). \tag{3.11}$$

Multiply equation (2.4) by u and Au, respectively, integrate over Ω, and apply the relations (3.7), (3.8) and the Poincare inequality (3.1) and Young's inequality (3.4), we have

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \nu\kappa_0^2|u|^2 \leq \frac{1}{2}\frac{d}{dt}|u|^2 + \nu|A^{\frac{1}{2}}u|^2 = (g,u) \leq \frac{|g|^2}{2\nu\kappa_0^2} + \frac{\nu\kappa_0^2}{2}|u|^2, \tag{3.12}$$

$$\frac{1}{2}\frac{d}{dt}|A^{\frac{1}{2}}u|^2 + \nu|Au|^2 = (g, Au) \leq \frac{|g|^2}{2\nu} + \frac{\nu|Au|^2}{2}. \tag{3.13}$$

Equations (3.12) and (3.13) are called the balance equations for the energy and enstrophy, respectively. Applying Gronwall's lemma to (3.12) and (3.13) we obtain,

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for all $t \geq t_0$, that

$$|u(t)|^2 \leq e^{-\nu \kappa_0^2 (t-t_0)} |u(t_0)|^2 + (1 - e^{-\nu \kappa_0^2 (t-t_0)}) \nu^2 G^2, \qquad (3.14)$$

$$|A^{\frac{1}{2}} u(t)|^2 \leq e^{-\nu \kappa_0^2 (t-t_0)} |A^{\frac{1}{2}} u(t_0)|^2 + (1 - e^{-\nu \kappa_0^2 (t-t_0)}) \nu^2 \kappa_0^2 G^2. \qquad (3.15)$$

From (3.15) we see that the closed ball B of radius $2\nu \kappa_0 G$ in $\mathcal{D}(A^{1/2})$ is, by Rellich Lemma, a compact subset of H and hence it is an absorbing set in H. Therefore, we can define the global attractor $\mathcal{A}$ of the 2D NSE as in (2.10).

In the next lemma we list several necessary estimates involving $B(\cdot, \cdot)$.

Lemma 3.1.1. *The following hold in the appropriate space,*

$$|(B(u,u), A^2 u)| \leq 2c_L^2 |Au| |A^{\frac{3}{2}} u| |A^{\frac{1}{2}} u|, \ u \in \mathcal{D}(A^2), \qquad (3.16)$$

$$|(B(u,u), A^3 u)| \leq \sqrt{2}(\sqrt{2} c_L^2 + c_A) |u|^{\frac{1}{2}} |Au|^{\frac{1}{2}} |A^{\frac{3}{2}} u| |A^2 u|, \ u \in \mathcal{D}(A^3). \qquad (3.17)$$

Relations (3.16) and (3.17) can be established using (3.2), (3.3), (3.7)-(3.10) and (3.11).

3.2 Supplementary estimates

In the following, we focus on the complexified NSE (2.14).

We now obtain estimates for the nonlinear terms with complexified time, observing that neither relations (3.7)-(3.10) nor Lemma 3.1.1 hold in this case. We will use the Ladyzhenskaya and Agmon inequalities as described before, noting the additional factor 2.

$$|u|_{L^4} \leq 2c_L |u|^{\frac{1}{2}} |A^{\frac{1}{2}} u|^{\frac{1}{2}}, \qquad \text{where} \ \ u \in \mathcal{D}(A^{\frac{1}{2}})_{\mathbb{C}}, \qquad (3.18)$$

and

$$|u|_\infty \le 2c_A|u|^{\frac{1}{2}}|Au|^{\frac{1}{2}}, \qquad \text{where} \ \ u \in \mathcal{D}(A)_{\mathbb{C}}. \tag{3.19}$$

In the present complex case, the analogue of Lemma 3.1.1 is

Lemma 3.2.1. *For* $u \in \mathcal{D}(A^m)_{\mathbb{C}}$*, where* $m = 1, 2$ *or* 3*,*

$$|(B(u,u), Au)| \le 4c_L^2|u|^{\frac{1}{2}}|A^{\frac{1}{2}}u||Au|^{\frac{3}{2}}, \tag{3.20}$$

$$|(B(u,u), A^2u)| \le 2(2c_L^2 + c_A)|u|^{\frac{1}{2}}|Au|^{\frac{3}{2}}|A^{\frac{3}{2}}u|, \tag{3.21}$$

$$|(B(u,u), A^3u)| \le 2(2c_L^2 + c_A)|u|^{\frac{1}{2}}|Au|^{\frac{3}{2}}|A^{\frac{5}{2}}u|. \tag{3.22}$$

Proof. To obtain the first inequality, use L^4 norms and (3.18) for the first two terms and the L^2 norm for the third term. For the second inequality, we use integration by parts to get

$$|B(u,u), A^2u)| \le \sum_{j=1,2} [|(B(D_ju,u), D_jAu)| + |(B(u, D_ju), D_jAu)|]. \tag{3.23}$$

Using L^4, L^4, L^2 and (3.18), we have

$$\sum_j |(B(D_ju,u), D_jAu)| \le 4c_L^2|A^{\frac{1}{2}}u||Au||A^{\frac{3}{2}}u| \tag{3.24}$$

and similaryly, use L^∞, L^2, L^2 and (3.19) to obtain

$$\sum_j |(B(u, D_ju), D_jAu)| \le 2c_A|u|^{\frac{1}{2}}|Au|^{\frac{3}{2}}|A^{\frac{3}{2}}u|, \tag{3.25}$$

for $u \in \mathcal{D}(A^{\frac{3}{2}})$.

Now apply the interpolating inequality $|A^{\frac{1}{2}}u| \le |Au|^{\frac{1}{2}}|u|^{\frac{1}{2}}$ in (3.24) to arrive at

(3.21).

For the last inequality, we use the same method as above. Integrating by parts, using L^4, L^4, L^2 and (3.18) for the first term on the right-hand side, and L^∞, L^2, L^2 and (3.19) for the second term on the right-hand side, and then interpolating, we have

$$|B(u,u), A^3 u)| \le \sum_{j=1,2} [|(B(D_j u, u), D_j A^2 u)| + |(B(u, D_j u), D_j A^2 u)|]$$
$$\le 4c_L^2 |A^{\frac{1}{2}} u| \|Au\| |A^{\frac{5}{2}} u| + 2c_A |u|^{\frac{1}{2}} |Au|^{\frac{3}{2}} |A^{\frac{5}{2}} u|$$
$$\le 2(2c_L^2 + c_A)|u|^{\frac{1}{2}} |Au|^{\frac{3}{2}} |A^{\frac{5}{2}} u|,$$

thus obtaining (3.22).

$\square$

Concerning the existence of $\mathcal{D}(A^{\frac{1}{2}})_\mathbb{C}$-valued analytic extensions of the solutions of the NSE, one can consult Section 7 in Part I of [20] and Chapter 12 in [2]. However, for our presentation, we need the following observations.

Remark 3.2.2. Many of the differential relations to follow are of the form

$$\frac{d}{d\rho}\Phi(u(t_0 + \rho e^{i\theta})) = \Psi(u(t_0 + \rho e^{i\theta})),$$

where $\Phi(u)$ and $\Psi(u)$ are explicit functions of u in a specified subspace of $H_\mathbb{C}$. Often the definition of $\Psi(u)$ involves many terms. Therefore in the sequel we will make the following abuse of notation

$$\frac{d}{d\rho}\Phi(u(t_0 + \rho e^{i\theta})) = \Psi(u).$$

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Remark 3.2.3. Let $m \in \mathbb{N}$. To solve equation (2.14) in a strip $\mathcal{S}(\delta_m)$ and to insure that $u(\zeta)$ is a $\mathcal{D}(A^{m/2})_{\mathbb{C}}$-valued analytic function (equivalently, $A^{m/2}u(\zeta)$ is $H_{\mathbb{C}}$-valued analytic), the proof for the case $m = 1$ presented in [2] and [20] shows that it suffices to establish the following fact:

For any $t_0 \in \mathbb{R}$, $\theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]$ and solution $u(\zeta)$ of equation (2.14) in $\mathcal{S}(\delta_m)$, the solution of the equation

$$\frac{d}{d\rho}u(t_0 + \rho e^{i\theta}) + \nu(\cos\theta)Au + B(u, u) = g, \quad g \in \mathcal{D}(A^{\frac{m-1}{2}}) \qquad (3.26)$$

satisfies, for

$$0 \le \rho \le \frac{\delta_m}{\sin\pi/4} = \sqrt{2}\delta_m,$$

the following conditions

$$u(t_0 + \rho e^{i\theta}) \in \mathcal{D}(A^{\frac{m+1}{2}})_{\mathbb{C}},$$

and $\sup|A^{\frac{m+1}{2}}u(t_0 + \rho e^{i\theta})|$ is finite and independent of t_0, ρ and θ.

This can be rigorously established with "the Galerkin approximation, for which analyticity in time is a classical result because it is a finite-dimensional system with a polynomial nonlinearity. The crucial part, then, is to obtain suitable a priori estimates for the solution in a complex time region that is independent of the size of the Galerkin approximation." (see [9] Chapter II, Section 8, Page 63) and then passing to the limit invoking Vitali's Theorem.

3.3 One of the main results and its proof

Let $H_{\mathbb{C}}$ be the complex Hilbert space $H \otimes \mathbb{C} = H + iH$. Similarly, for any linear subspace D of H we denote $D \otimes \mathbb{C}$ by $D_{\mathbb{C}}$. For $\delta > 0$ we define, as mentioned in the introduction, the strip

$$\mathcal{S}(\delta) := \{\zeta \in \mathbb{C} : |\mathfrak{I}(\zeta)| < \delta\}. \qquad (3.27)$$

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Theorem 3.3.1. *If $0 \in \mathcal{A}$, then there exists $\delta > 0$ such that for any $m \in \mathbb{N}$ there exists $\tilde{R}_m \in [0, \infty)$ such that for any solution $u(\cdot)$ in $\mathcal{A}$, the function $A^{\frac{m}{2}} u(\zeta)$ is $H_{\mathbb{C}}$-valued analytic in the strip $\mathcal{S}(\delta)$, and $|A^{\frac{m}{2}} u(\zeta)| \leq \tilde{R}_m \nu \kappa_0^m$, where $u(\zeta)$ satisfies the NSE, (2.14), with complexified time.*

The specific form of the estimates for δ and $\tilde{R}_m$ are provided in Proposition 3.3.15 below.

Corollary 3.3.2. *If $0 \in \mathcal{A}$, then $\mathcal{A} \cup \{g\} \subset C^{\infty}_{\mathrm{per}}([0, L]^2)$.*

Proof of Theorem 3.3.1

Using the procedure described in Remark 3.2.3, we will prove Theorem 3.3.1 by induction on m. In order to start a uniform recurrent process we need $m \geq 3$. In the following Lemmas 3.3.3, 3.3.6 and 3.3.7 we obtain the necessary estimates for $m = 1, 2, 3$. We stress that the case $m = 1$ was treated in Theorem 12.1 in [2], while the cases $m = 1, 2$ were already established in [4], Theorem 11.1, although with different estimates.

The cases for $m = 1$

Lemma 3.3.3. *If $u(\cdot)$ is a solution of the NSE in the attractor $\mathcal{A}$, then*

(i) $u(\cdot)$ can be extended to a $\mathcal{D}(A^{\frac{1}{2}})_{\mathbb{C}}$ -valued analytic function in the strip $\mathcal{S}(\delta_1)$, where

$$\delta_1 := \frac{1}{16 \cdot 24^3 c_L^8 \nu \kappa_0^2 G^4} \tag{3.28}$$

and

$$|A^{\frac{1}{2}} u(\zeta)| \leq \tilde{R}_1 \nu \kappa_0, \quad \forall \ \zeta \in \mathcal{S}(\delta_1), \tag{3.29}$$

where

$$\tilde{R}_1 = \sqrt{2} G. \tag{3.30}$$

(ii) Moreover, defining

$$R_2 = 2137G^3c_L^4, \qquad (3.31)$$

we have

$$|Au(t)| \leq R_2\nu\kappa_0^2, \quad \forall t \in \mathbb{R}. \qquad (3.32)$$

Proof. First, according to Remark 3.2.3 to prove the statement (i) it is sufficient to establish estimates (3.29) and (3.30) for δ_1 chosen as in (3.28).

Taking the inner product of both sides in (3.26) with $Au(t_0 + \rho e^{i\theta})$, we obtain

$$\frac{1}{2}\frac{d}{d\rho}|A^{\frac{1}{2}}u(t_0 + \rho e^{i\theta})|^2 = \Re(e^{i\theta}(g, Au)) - \nu\cos\theta|Au|^2 - \Re(e^{i\theta}(B(u,u), Au))$$

$$\leq |g||Au| - \frac{\nu}{\sqrt{2}}|Au|^2 + |(B(u,u), Au))|.$$

Using the Cauchy-Schwarz inequality and then the Young's inequality (3.4) with $p = q = 2$, for the term $|g||Au|$, we obtain

$$\frac{d}{d\rho}|A^{\frac{1}{2}}u(t_0 + \rho e^{i\theta})|^2 + \frac{\nu}{\sqrt{2}}|Au|^2 \leq \sqrt{2}\frac{|g|^2}{\nu} + |2(B(u,u), Au)|.$$

We use Lemma 3.2.1 for the bilinear term in the above relation to get the following inequality

$$\frac{d}{d\rho}|A^{\frac{1}{2}}u(t_0 + \rho e^{i\theta})|^2 + \frac{\nu}{\sqrt{2}}|Au|^2 \leq \sqrt{2}\frac{|g|^2}{\nu} + 8c_L^2|u|^{\frac{1}{2}}|A^{\frac{1}{2}}u||Au|^{\frac{3}{2}}.$$

Using again Young's inequality (3.4) with $p = 4$ and $q = 4/3$ for the last term, we

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have

$$8c_L^2|u|^{\frac{1}{2}}|A^{\frac{1}{2}}u||Au|^{\frac{3}{2}} \leq \frac{3}{4}\left(\nu^{\frac{3}{4}}\left(\frac{\sqrt{2}}{3}\right)^{\frac{3}{4}}|Au|^{\frac{3}{2}}\right)^{\frac{4}{3}} + \frac{1}{4}\left(\frac{1}{\nu^{\frac{3}{4}}}\left(\frac{3}{\sqrt{2}}\right)^{\frac{3}{4}}8c_L^2|u|^{\frac{1}{2}}|A^{\frac{1}{2}}u|\right)^4,$$

and hence,

$$\frac{d}{d\rho}|A^{\frac{1}{2}}u(t_0 + \rho e^{i\theta})|^2 + \frac{\nu}{2\sqrt{2}}|Au|^2 \leq \sqrt{2}\frac{|g|^2}{\nu} + \frac{8^3 27 c_L^8}{\nu^3\sqrt{2}}|u|^2|A^{\frac{1}{2}}u|^4. \qquad (3.33)$$

From (3.1) and (3.33), we obtain

$$\frac{d}{d\rho}|A^{\frac{1}{2}}u(t_0 + \rho e^{i\theta})|^2 \leq \sqrt{2}\frac{|g|^2}{\nu} + \frac{8^3 27 c_L^8}{\nu^3 \kappa_0^2\sqrt{2}}|A^{\frac{1}{2}}u|^6.$$

The above inequality has the form

$$\frac{d\phi}{d\rho} \leq \gamma + \beta\phi^3, \qquad (3.34)$$

where

$$\phi(\rho) := |A^{\frac{1}{2}}u(t_0 + \rho e^{i\theta})|^2, \quad \gamma = \sqrt{2}\frac{|g|^2}{\nu}, \quad \beta = \frac{24^3 c_L^8}{\nu^3 \kappa_0^2\sqrt{2}}.$$

Integrating (3.34), we obtain

$$\int_{\phi(0)}^{\phi(\rho)} \frac{d\phi}{(\gamma^{\frac{1}{3}} + \beta^{\frac{1}{3}}\phi)^3} \leq \int_{\phi(0)}^{\phi(\rho)} \frac{d\phi}{(\gamma + \beta\phi^3)} \leq \rho,$$

and hence

$$\frac{1}{2\beta^{\frac{1}{3}}(\gamma^{\frac{1}{3}} + \beta^{\frac{1}{3}}\phi(0))^2} - \frac{1}{2\beta^{\frac{1}{3}}(\gamma^{\frac{1}{3}} + \beta^{\frac{1}{3}}\phi(\rho))^2} \leq \rho. \qquad (3.35)$$

Thus, if

$$\rho \leq \frac{1}{4\beta^{\frac{1}{3}}(\gamma^{\frac{1}{3}} + \beta^{\frac{1}{3}}\phi(0))^2} \tag{3.36}$$

then $\phi(\rho)$ satisfies

$$\gamma^{\frac{1}{3}} + \beta^{\frac{1}{3}}\phi(\rho) \leq \sqrt{2}(\gamma^{\frac{1}{3}} + \beta^{\frac{1}{3}}G^2(\nu\kappa_0)^2),$$

that is

$$|A^{\frac{1}{2}}u(t_0 + \rho e^{i\theta})|^2 \leq (\sqrt{2} - 1)\left(\frac{\gamma}{\beta}\right)^{\frac{1}{3}} + \sqrt{2}G^2(\nu\kappa_0)^2 \tag{3.37}$$

$$\leq \frac{2^{\frac{1}{3}}(|g|\nu\kappa_0)^{2/3}}{24c_L^{8/3}} + \sqrt{2}G^2(\nu\kappa_0)^2$$

$$\leq \left(\sqrt{2} + \frac{2^{\frac{1}{3}}}{24}\right)G^2(\nu\kappa_0)^2 \leq 2G^2(\nu\kappa_0)^2.$$

Note that in the third inequality above we used (3.6).

If δ_1 is defined as in (3.28) and if $\rho \leq \sqrt{2}\delta_1$, then (3.36) holds. Consequently, (3.37) also holds. That is

$$|A^{\frac{1}{2}}u(t_0 + \rho e^{i\theta})| \leq \tilde{R}_1\nu\kappa_0,$$

where $\tilde{R}_1$ is defined in (3.30).

Since $\theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ and $t_0 \in \mathbb{R}$ are arbitrary, we infer

$$|A^{\frac{1}{2}}u(\zeta)| \leq \tilde{R}_1\nu\kappa_0, \quad \text{for } \zeta \in \mathcal{S}(\delta_1).$$

With this estimate, the proof of statement (i) is concluded.

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It remains to prove statement (ii). Integrating (3.33) and applying (3.6), we obtain

$$\frac{\nu}{2\sqrt{2}} \int_0^{\sqrt{2}\delta_1} |Au(t_0 + \rho e^{i\pi/4})|^2 d\rho \le |A^{\frac{1}{2}}u(t_0)|^2 + \int_0^{\sqrt{2}\delta_1} (\gamma + \beta|A^{\frac{1}{2}}u(t_0 + \rho e^{i\pi/4})|^6) d\rho,$$

$$\le 2G^2\nu^2\kappa_0^2 + \sqrt{2}\gamma\delta_1 + 8\sqrt{2}G^6\nu^6\kappa_0^6\beta\delta_1$$

$$= \left[2 + \frac{1}{8 \cdot 24^3 c_L^8 G^4} + \frac{1}{2}\right] G^2\nu^2\kappa_0^2$$

$$\le \left[2 + \frac{1}{8 \cdot 24^3} + \frac{1}{2}\right] G^2\nu^2\kappa_0^2 \le 2\sqrt{2}G^2\nu^2\kappa_0^2,$$

i.e.,

$$\int_0^{\sqrt{2}\delta_1} |Au(t_0 + \rho e^{i\pi/4})|^2 d\rho \le 8G^2\nu\kappa_0^2. \tag{3.38}$$

Since $Au(\zeta)$ is an analytic function in $D(t_0, \delta_1) := \{s_1 + is_2 : |s_1 - t_0|^2 + s_2^2 \le \delta_1^2\}$, it satisfies the mean value property

$$Au(t_0) = \frac{1}{\pi\delta_1^2} \iint_{D(t_0,\delta_1)} Au(s_1 + is_2) ds_1 ds_2,$$

from which we deduce

$$|Au(t_0)| \le \frac{1}{\pi\delta_1^2} \iint_{D(t_0,\delta_1)} |Au(s_1 + is_2)| ds_1 ds_2.$$

In order to exploit estimate (3.38), we replace the disk $D(t_0, \delta_1)$ by the polygon $abcdef$ as shown in Figure 3.1.
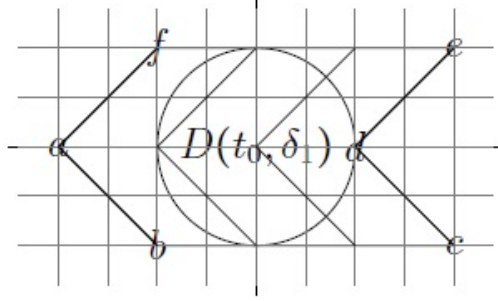
Figure 3.1: Polygon abcdef

Now, by using Schwarz reflection principle (see, e.g., [12]), we obtain

$$|Au(t_0)| \le \frac{1}{\pi\delta_1^2} \int_{abcdef} |Au(s_1 + is_2)| ds_1 ds_2$$

$$= \frac{2}{\sqrt{2}\pi\delta_1^2} \int_{t_0-2\delta_1}^{t_0+\delta_1} \int_0^{\sqrt{2}\delta_1} |Au(t + \rho e^{i\pi/4})| d\rho dt$$

$$\le \frac{2}{\sqrt{2}\pi\delta_1^2} \int_{t_0-2\delta_1}^{t_0+\delta_1} \left(\int_0^{\sqrt{2}\delta_1} |Au(t + \rho e^{i\pi/4})|^2 d\rho \right)^{\frac{1}{2}} (\sqrt{2}\delta_1)^{\frac{1}{2}} dt$$

$$\le \frac{12 \cdot 2^{\frac{1}{4}}}{\pi} \left(\frac{G^2\nu\kappa_0^2}{\delta_1} \right)^{\frac{1}{2}},$$

that is

$$|Au(t_0)| \le \frac{6 \cdot 2^8 \cdot 3\sqrt{3}}{2^{\frac{1}{4}}\pi} G^3 c_L^4 \nu\kappa_0^2 \le R_2 \nu\kappa_0^2.$$

This completes the proof of the statement (ii) and Lemma 3.3.3. □

Corollary 3.3.4. *For all $u^0 \in \mathcal{A}$, we have*

$$|A^{\frac{1}{2}}u^0| \le R_1\nu\kappa_0, \quad R_1 := G, \tag{3.39}$$

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and

$$|Au^0| \le R_2 \nu \kappa_0^2, \tag{3.40}$$

where R_2 is given in (3.31).

Proof. Let $u^0 \in \mathcal{A}$ and denote by $u(t)$, $t \in \mathbb{R}$, the solution of the NSE satisfying $u(0) = u^0$. Then, according to Lemma 3.3.3, (3.32) holds; in particular, for $t = 0$. This yields (3.40). The estimate (3.39) follows from (3.15) with $t = 0$ and $t_0 \to -\infty$. □

Corollary 3.3.5. *If $0 \in \mathcal{A}$ then $g \in \mathcal{D}(A^{\frac{1}{2}})$, and*

$$|A^{\frac{1}{2}}g| \le \frac{\tilde{R}_1 \nu \kappa_0}{\delta_1}, \tag{3.41}$$

where δ_1 and $\tilde{R}_1$ are defined as in (3.28) *and* (3.30)*, respectively.*

Proof. Let $u(t)$, $t \in \mathbb{R}$, be a solution of NSE such that $u(0) = 0$. According to Theorem 11.1 in [4], if $0 \in \mathcal{A}$, then $g \in \mathcal{D}(A)$. We evaluate the NSE at $t_0 = 0$ and $\theta = 0$ to obtain

$$\left. \frac{du(\zeta)}{d\zeta} \right|_{t_0=0} = g.$$

Since $u(\zeta)$ is a $\mathcal{D}(A^{\frac{1}{2}})_\mathbb{C}$-valued analytic function, its derivative $\frac{du(\zeta)}{d\zeta} \in \mathcal{D}(A^{\frac{1}{2}})_\mathbb{C}$ for all $\zeta \in \mathcal{S}(\delta_1)$. Thus, $g \in \mathcal{D}(A^{\frac{1}{2}})$, and then from

$$A^{\frac{1}{2}}g = A^{\frac{1}{2}}\frac{du(\zeta)}{d\zeta}|_{\zeta=0} = \frac{dA^{\frac{1}{2}}u(\zeta)}{d\zeta}|_{\zeta=0} = \frac{1}{2\pi i} \int_{\partial D(0,\delta)} \frac{A^{\frac{1}{2}}u(z)}{z^2} dz,$$

where $\delta \in (0, \delta_1)$, we obtain

$$|A^{\frac{1}{2}}g| \le \frac{\tilde{R}_1 \nu \kappa_0}{\delta}. \tag{3.42}$$

Letting $\delta \to \delta_1$ in (3.42), we deduce (3.41) □

The case for $m = 2$

Lemma 3.3.6. *If $0 \in \mathcal{A}$ and if $u(t)$, $t \in \mathbb{R}$, is any solution of the NSE in $\mathcal{A}$, then $u(t)$ can be extended to a $\mathcal{D}(A)_{\mathbb{C}}$-valued analytic function $u(\zeta)$, for $\zeta \in \mathcal{S}(\delta_2)$, where*

$$\delta_2 := \min\left\{\delta_1, 16^{-1}\left[(2c_L^2 + c_A)^{\frac{8}{3}}\tilde{R}_1^{\frac{8}{3}}\left(\frac{\nu\kappa_0^2}{8\delta_1^2}\right)^{\frac{2}{3}} + (2c_L^2 + c_A)^4 \tilde{R}_1^2 R_2^2 (\nu\kappa_0^2)^2\right]^{-\frac{1}{2}}\right\}, \quad (3.43)$$

and δ_1, $\tilde{R}_1$ and R_2 are defined in (3.28), (3.30) and (3.31), respectively. Furthermore,

$$|Au(\zeta)| \le \tilde{R}_2 \nu\kappa_0^2, \quad \text{for } \zeta \in \mathcal{S}(\delta_2), \quad (3.44)$$

where

$$\tilde{R}_2 := \left(\frac{3(\sqrt{2} \cdot 16^2 \cdot 24^6 c_L^{16})^{2/3}}{4(2c_L^2 + c_A)^{4/3}}G^6 + 4R_2^2\right)^{\frac{1}{2}}. \quad (3.45)$$

Proof. Applying again the short procedure in Remark 3.2.3, we take the inner product of the NSE with the function $A^2 u(t_0 + \rho e^{i\theta})$ and obtain

$$\frac{1}{2}\frac{d}{d\rho}|Au(t_0 + \rho e^{i\theta})|^2 = \Re(e^{i\theta}(g, A^2 u)) - \nu(\cos\theta)|A^{\frac{3}{2}}u|^2 - \Re(e^{i\theta}(B(u,u), A^2 u)).$$

Using Corollary 3.3.5 and proceeding as in the proof of Lemma 3.3.3, we obtain

$$\frac{1}{2}\frac{d}{d\rho}|Au(t_0 + \rho e^{i\theta})|^2 \le |(A^{\frac{1}{2}}g, A^{\frac{3}{2}}u)| - \frac{\nu}{\sqrt{2}}|A^{\frac{3}{2}}u|^2 + |(B(u,u), A^2 u)|.$$

Using Young's inequality (3.4) with $p = q = 2$ for the term $|(A^{\frac{1}{2}}g, A^{\frac{3}{2}}u)|$, we obtain

$$\frac{d}{d\rho}|Au(t_0 + \rho e^{i\theta})|^2 \le \sqrt{2}\frac{|A^{\frac{1}{2}}g|^2}{\nu} - \frac{\nu}{\sqrt{2}}|A^{\frac{3}{2}}u|^2 + 2|(B(u,u), A^2 u)|.$$

We use Lemma 3.2.1 to obtain

$$\frac{d}{d\rho}|Au(t_0 + \rho e^{i\theta})|^2 \le \sqrt{2}\frac{|A^{\frac{1}{2}}g|^2}{\nu} - \frac{\nu}{\sqrt{2}}|A^{\frac{3}{2}}u|^2 + 4(2c_L^2 + c_A)|u|^{\frac{1}{2}}|Au|^{\frac{3}{2}}|A^{\frac{3}{2}}u|.$$

Using Young's inequality again, we obtain

$$\frac{d}{d\rho}|Au(t_0 + \rho e^{i\theta})|^2 + \frac{1}{2\sqrt{2}}\nu|A^{\frac{3}{2}}u|^2 \le \sqrt{2}\frac{|A^{\frac{1}{2}}g|^2}{\nu} + \frac{8\sqrt{2}(2c_L^2 + c_A)^2}{\nu}|u||Au|^3.$$

Using Poincaré's inequality, the bound on $|A^{\frac{1}{2}}u|$ obtained in Lemma 3.3.3 and again Corollary 3.3.5 we obtain

$$\frac{d}{d\rho}|Au(t_0 + \rho e^{i\theta})|^2 + \frac{1}{2\sqrt{2}}\nu|A^{\frac{3}{2}}u|^2 \le \sqrt{2}\frac{\tilde{R}_1^2 \nu \kappa_0^2}{\delta_1^2} + 8\sqrt{2}(2c_L^2 + c_A)^2\tilde{R}_1|Au|^3. \qquad (3.46)$$

As before, we ignore the term containing $|A^{\frac{3}{2}}u|^2$ to get the inequality

$$\frac{d\phi_2(\rho)}{d\rho} \le \gamma_2 + \beta_2(\phi_2(\rho))^{\frac{3}{2}}, \qquad (3.47)$$

where

$$\phi_2(\rho) = |Au(t_0 + \rho e^{i\theta})|^2, \quad \gamma_2 = \sqrt{2}\frac{\tilde{R}_1^2 \nu \kappa_0^2}{\delta_1^2}, \quad \beta_2 = 8\sqrt{2}(2c_L^2 + c_A)^2\tilde{R}_1.$$

From (3.47), we obtain the analogue of the relation (3.35), namely

$$\frac{2}{\beta_2((\gamma_2/\beta_2)^{2/3} + \phi_2(0))^{\frac{1}{2}}} - \frac{2}{\beta_2((\gamma_2/\beta_2)^{2/3} + \phi_2(\rho))^{\frac{1}{2}}} \le \rho.$$

We observe that if

$$\rho < \frac{1}{\beta_2((\gamma_2/\beta_2)^{2/3} + \phi_2(0))^{\frac{1}{2}}},$$

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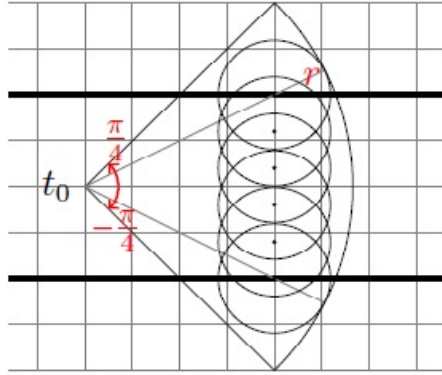
Figure 3.2: Sector in the band

then

$$((\gamma_2/\beta_2)^{2/3} + \phi_2(\rho))^{\frac{1}{2}} \le 2((\gamma_2/\beta_2)^{2/3} + \phi_2(0))^{\frac{1}{2}}$$

and hence,

$$|Au(t_0 + \rho e^{i\theta})|^2 \le 3(\gamma_2/\beta_2)^{2/3} + 4|Au(t_0)|^2.$$

By (3.32), we obtain

$$|Au(t_0 + \rho e^{i\theta})|^2 \le \left(\frac{3(\sqrt{2} \cdot 16^2 \cdot 24^6 c_L^{16})^{2/3}}{4(2c_L^2 + c_A)^{4/3}} G^6 + 4R_2^2 \right) \nu^2 \kappa_0^4,$$

Thus, if we define δ_2 by (3.43) and $\tilde{R}_2$ by (3.45), then we obtain (3.44).   □

The case for $m = 3$

We now consider the case $m = 3$ after which we can proceed by induction for all $m > 3$. Let $\delta_3 = \delta_2/2$, where δ_2 is defined as in Lemma 3.3.6 and let $r = (2\sqrt{2} - \sqrt{5})\delta_3$. Then, given any ζ in $\mathcal{S}(\delta_3)$, there is a real t_0 such that $D(\zeta, r)$ is in the sector of $D(t_0, 2\sqrt{2}\delta_3)$ where θ varies from $-\pi/4$ to $\pi/4$, as shown in Figure 3.2.

Using (3.46) and the notation from (3.47) we obtain, for $\theta \in [-\pi/4, \pi/4]$,

$$\frac{\nu}{2\sqrt{2}} \int_0^{2\sqrt{2}\delta_3} |A^{\frac{3}{2}}(t_0 + \rho e^{i\theta})|^2 d\rho \le |Au(t_0)|^2 + \int_0^{2\sqrt{2}\delta_3} (\gamma_2 + \beta_2 |Au|^3) d\rho \qquad (3.48)$$

$$\le N_2 (\nu \kappa_0^2)^2,$$

where

$$N_2 := R_2^2 + \frac{2\delta_2 \tilde{R}_1^2}{\delta_1^2 \nu \kappa_0^2} + 16(2c_L^2 + c_A)^2 \tilde{R}_1 \tilde{R}_2^3 \delta_2 \nu \kappa_0^2. \qquad (3.49)$$

Using the mean value theorem for the analytic function $A^{\frac{3}{2}} u(\zeta)$ (as we did before for $Au(\zeta)$) in $D(\zeta, r)$ we obtain

$$\begin{aligned}
|A^{\frac{3}{2}} u(\zeta)| &\le \frac{1}{\pi r^2} \iint_{\{s_1 + is_2 \in D(\zeta, r)\}} |A^{\frac{3}{2}} u(s_1 + is_2)| ds_1 ds_2 \\
&\le \frac{1}{\pi r^2} \int_{-\pi/4}^{\pi/4} \int_0^{2\sqrt{2}\delta_3} |A^{\frac{3}{2}} u(t_0 + \rho e^{i\theta})| \rho d\rho d\theta \\
&\le \frac{1}{\pi r^2} \int_{-\pi/4}^{\pi/4} d\theta \left(\int_0^{2\sqrt{2}\delta_3} |A^{\frac{3}{2}} u(t_0 + \rho e^{i\theta})|^2 d\rho \right)^{\frac{1}{2}} \left(\int_0^{2\sqrt{2}\delta_3} \rho^2 d\rho \right)^{\frac{1}{2}} \\
&\le \frac{1}{\pi r^2} \frac{\pi}{2} (2\sqrt{2} N_2 \nu \kappa_0^4)^{\frac{1}{2}} \frac{(2\sqrt{2}\delta_3)^{\frac{3}{2}}}{\sqrt{3}} \\
&= \frac{4}{\sqrt{3}(2\sqrt{2} - \sqrt{5})^2} N_2^{\frac{1}{2}} \frac{\nu^{\frac{1}{2}} \kappa_0^2}{\delta_3^{\frac{1}{2}}} < 4 N_2^{\frac{1}{2}} \frac{\nu^{\frac{1}{2}} \kappa_0^2}{\delta_3^{\frac{1}{2}}}.
\end{aligned}$$

Now to obtain for $A^{\frac{3}{2}} u(t)$, $t \in \mathbb{R}$, an estimate analogous to (3.32), we use an argument similar to that involving the polygon $abcdef$ in the proof of Lemma 3.3.3; we note that now the roles of δ_1 and (3.38) are played by δ_3 and (3.48), respectively.

In this manner, we obtain

$$|A^{\frac{3}{2}} u(t)| \le R_3 \nu \kappa_0^3, \qquad (3.50)$$

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where

$$R_3 := \frac{12\sqrt{2}}{\pi} \left(\frac{N_3}{\delta_3 \nu \kappa_0^2} \right)^{\frac{1}{2}}, \tag{3.51}$$

and

$$N_3 := R_2^2 + \frac{2\delta_3 \tilde{R}_1^2}{\delta_1^2 \nu \kappa_0^2} + 16(2c_L^2 + c_A)^2 \tilde{R}_1 \tilde{R}_2^3 \delta_3 \nu \kappa_0^2.$$

We sum up the results obtained above in the following

Lemma 3.3.7. *If $0 \in \mathcal{A}$ and if $u(t), t \in \mathbb{R}$, is any solution of the NSE in $\mathcal{A}$, then $u(t)$ can be extended to a $\mathcal{D}(A^{\frac{3}{2}})_{\mathbb{C}}$-valued analytic function $u(\zeta)$, for $\zeta \in \mathcal{S}(\delta_3)$, where*

$$\delta_3 := \frac{\delta_2}{2}, \tag{3.52}$$

and δ_2 is defined as in (3.43), for which the following estimates holds

$$|A^{\frac{3}{2}} u(\zeta)| \le \tilde{R}_3 \nu \kappa_0^3, \quad \text{for } \zeta \in \mathcal{S}(\delta_3),$$

where

$$\tilde{R}_3 := 4 \frac{N_2^{\frac{1}{2}}}{\delta_3^{\frac{1}{2}} \nu^{\frac{1}{2}} \kappa_0}, \tag{3.53}$$

and N_2 is defined in (3.49).

Moreover, $u(t)$ satisfies the relation (3.50).

Remark 3.3.8. Lemmas 3.3.3, 3.3.6 and 3.3.7 establish the validity of Theorem 3.3.1 for the case $m \in \{1, 2, 3\}$.

Two lemmas

In this part we present an extension of the estimates given in Lemma 3.1.1 and Lemma 3.2.1 to the powers $A^m (m \in \mathbb{N}, m > 3)$ of A. For this purpose, we will adapt

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a method of P. Constantin (first used by W. Chen [1] to estimate $|(B(u,v), e^{A^\beta} w)|$, $\beta \in (0,1))$ to the case where A^β is replaced by $m \ln(\frac{A^{1/2}}{\kappa_0})$.

Lemma 3.3.9. *Let* $u \in \mathcal{D}(A^{\frac{m}{2}}), v \in \mathcal{D}(A^{\frac{m+1}{2}})$, $w \in \mathcal{D}(A^m)$, *and* $m \in \mathbb{N}$, $m > 3$, *then*

$$|(B(u,v), A^m w)| \le 2^m c_A \left(|u|^{\frac{1}{2}} |Au|^{\frac{1}{2}} |A^{\frac{1+m}{2}} v| + |A^{\frac{m}{2}} u||A^{\frac{1}{2}} v|^{\frac{1}{2}} |A^{\frac{3}{2}} v|^{\frac{1}{2}} \right) |A^{\frac{m}{2}} w|.$$

Proof. Fix $m \in \mathbb{N}$, $m > 3$. To simplify the exposition, we denote $\tilde{u} := A^{\frac{m}{2}} u$ and $u \in \mathcal{D}(A^{\frac{m}{2}})$.

Then for any $u \in \mathcal{D}(A^{\frac{m}{2}}), v \in \mathcal{D}(A^{\frac{m+1}{2}})$, $w \in \mathcal{D}(A^m)$, we have

$$\begin{aligned}
|(B(u,v), A^m w)| &\le L^2 \kappa_0^{1+2\alpha} \sum_{\substack{h,j,k \in \mathbb{Z}^2 \smallsetminus \{0\} \\ h+j+k=0}} |\hat{u}(h)||j||\hat{v}(j)||\hat{w}(k)||k|^{2\alpha} \\
&= L^2 \kappa_0^{1+m} \sum_{\substack{h,j,k \in \mathbb{Z}^2 \smallsetminus \{0\} \\ h+j+k=0}} |\hat{u}(h)||j||\hat{v}(j)||\hat{\tilde{w}}(k)||k|^m \\
&\le L^2 \kappa_0^{1-m} \sum_{\substack{h,j,k \in \mathbb{Z}^2 \smallsetminus \{0\} \\ h+j+k=0}} |\hat{\tilde{u}}(h)||j||\hat{\tilde{v}}(j)||\hat{\tilde{w}}(k)|(|h|+|j|)^m |h|^{-m}|j|^{-m} \\
&= L^2 \kappa_0^{1-m} \sum_{\substack{h,j,k \in \mathbb{Z}^2 \smallsetminus \{0\} \\ h+j+k=0}} |\hat{\tilde{u}}(h)||j||\hat{\tilde{v}}(j)||\hat{\tilde{w}}(k)|e^{m[\ln(|h|+|j|)-\ln|h|-\ln|j|]} \\
&= L^2 \kappa_0^{1-m} \sum_{\substack{h,j,k \in \mathbb{Z}^2 \smallsetminus \{0\} \\ h+j+k=0 \\ |h| \le |j|}} \cdots + L^2 \kappa_0^{1-m} \sum_{\substack{h,j,k \in \mathbb{Z}^2 \smallsetminus \{0\} \\ h+j+k=0 \\ |h| > |j|}} \cdots \\
&=: I_1 + I_2,
\end{aligned}$$

where the notation is self-explanatory.

For I_1, since $\ln(|h| + |j|) - \ln|h| - \ln|j|$ is decreasing with respect to $|j|$, we have

$$I_1 \le L^2 \kappa_0^{1-m} \sum_{\substack{h,j,k \in \mathbb{Z}^2 \setminus \{0\} \\ h+j+k=0 \\ |h| \le |j|}} |\hat{\tilde{u}}(h)||j||\hat{\tilde{v}}(j)||\hat{\tilde{w}}(k)|e^{m[\ln 2 - \ln|h|]}$$

$$= L^2 \kappa_0 \sum_{\substack{h,j,k \in \mathbb{Z}^2 \setminus \{0\} \\ h+j+k=0 \\ |h| \le |j|}} |\hat{u}(h)||j||\hat{\tilde{v}}(j)||\hat{\tilde{w}}(k)|e^{m \ln 2}$$

$$\le 2^m L^2 \kappa_0 \sum_{\substack{h,j,k \in \mathbb{Z}^2 \setminus \{0\} \\ h+j+k=0}} |\hat{u}(h)||j||\hat{\tilde{v}}(j)||\hat{\tilde{w}}(k)|.$$

By estimating I_2 in a same way, we obtain

$$|(B(u,v), A^m w)| \le 2^m L^2 \kappa_0 \sum_{\substack{h,j,k \in \mathbb{Z}^2 \setminus \{0\} \\ h+j+k=0}} [|\hat{u}(h)||j||\hat{\tilde{v}}(j)| + |\hat{\tilde{u}}(h)||j||\hat{v}(j)|]|\hat{\tilde{w}}(k)|.$$

We define the auxiliary functions U and $\tilde{U}$ by $U := \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} |\hat{u}(k)|e^{i\kappa_0 k \cdot x}$ and $\tilde{U} := \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} |\hat{\tilde{u}}(k)|e^{i\kappa_0 k \cdot x}$ and in a similar way, the functions $V, \tilde{V}, W, \tilde{W}$. Then we have that

$$|(B(u,v), A^m w)| \le 2^m \int_{[0,L]^2} [(U \cdot (-\Delta)^{\frac{1}{2}} \tilde{V}) + (\tilde{U} \cdot (-\Delta)^{\frac{1}{2}} V)]\tilde{W} d^2 x$$

$$\le 2^m [|U|_{L^\infty}|(-\Delta)^{\frac{1}{2}} \tilde{V}|_{L^2} + |\tilde{U}|_{L^2}|(-\Delta)^{\frac{1}{2}} V|_{L^\infty}]|\tilde{W}|_{L^2}$$

Using Agmon's inequality, we obtain

$$|(B(u,v), A^m w)| \le 2^m c_A [|U|_{L^2}^{\frac{1}{2}}|(-\Delta)U|_{L^2}^{\frac{1}{2}}|(-\Delta)^{\frac{1}{2}} \tilde{V}|_{L^2}$$

$$+ |\tilde{U}|_{L^2}|(-\Delta)^{\frac{1}{2}} V|_{L^2}^{\frac{1}{2}}|(-\Delta)^{\frac{3}{2}} V|_{L^2}^{\frac{1}{2}}]|\tilde{W}|_{L^2}$$

$$= 2^m c_A \left(|u|^{\frac{1}{2}}|Au|^{\frac{1}{2}}|A^{\frac{1+m}{2}} v| + |A^{\frac{m}{2}} u||A^{\frac{1}{2}} v|^{\frac{1}{2}}|A^{\frac{3}{2}} v|^{\frac{1}{2}} \right)|A^{\frac{m}{2}} w|.$$

$\square$

Lemma 3.3.10. *Let $u \in \mathcal{D}(A^{\frac{m}{2}})_{\mathbb{C}}, v \in \mathcal{D}(A^{\frac{m+1}{2}})_{\mathbb{C}}, w \in \mathcal{D}(A^m)_{\mathbb{C}}$ and $m > 3$, then*

$$|(B(u,v), A^m w)| \leq 2^{m+\frac{3}{2}} c_A \left(|u|^{\frac{1}{2}} |Au|^{\frac{1}{2}} |A^{\frac{1+m}{2}} v| + |A^{\frac{m}{2}} u| |A^{\frac{1}{2}} v|^{\frac{1}{2}} |A^{\frac{3}{2}} v|^{\frac{1}{2}} \right) |A^{\frac{m}{2}} w|.$$

The proof of Lemma 3.3.10 is omitted since it is similar to the one above, and the only difference is in the constant.

Induction for $m \geq 3$

The standing assumption in this part is that $0 \in \mathcal{A}$.

Under this assumption we will obtain, for all $m > 3$ and for any solution $u(t)$, $t \in \mathbb{R}$, in $\mathcal{A}$, estimates of the form

$$|A^{\frac{m}{2}} u(t)| \leq R_m \nu \kappa_0^m, \quad \text{for } t \in \mathbb{R}, \tag{3.54}$$

and for its analytic extension

$$|A^{\frac{m}{2}} u(\zeta)| \leq \tilde{R}_m \nu \kappa_0^m, \quad \text{for } \zeta \in \mathcal{S}(\delta), \tag{3.55}$$

where $\delta = \delta_3$ (see (3.52)).

Note that (3.54), (3.55) were already established for $m = 1, 2, 3$ in Section 5. For the values of R_m, $\tilde{R}_m$ see (3.30), (3.31), (3.45), (3.51), (3.53). Therefore we will assume that (3.54) and (3.55) are valid for some $m \in \mathbb{N}, m \geq 3$ and prove by induction (starting with $m = 3$) that they are also valid for $m + 1$.

For this we need the following

Lemma 3.3.11. *Assume* $0 \in \mathcal{A}$, *then*

$$g \in \mathcal{D}(A^{\frac{m}{2}}),$$

and

$$G_{m+1} := \frac{|A^{\frac{m}{2}} g|}{\nu^2 \kappa_0^{m+2}} \leq \frac{\tilde{R}_m}{\nu \kappa_0^2 \delta}. \tag{3.56}$$

Proof. Let $u(t)$ be the solution of the NSE satisfying $u(0) = 0$ and let $u(\zeta)$ be its $\mathcal{D}(A^{\frac{m}{2}})_{\mathbb{C}}$-analytic extension in $\mathcal{S}(\delta)$. Then we have $g = \frac{du}{dt}|_{t=0} \in \mathcal{D}(A^{\frac{m}{2}})$, and since $u(\zeta)$ satisfies (3.55) for $t \in \mathbb{R}$, also that

$$\left| A^{\frac{m}{2}} \frac{du(\zeta)}{d\zeta}|_{\zeta=t} \right| = \left| \frac{1}{2\pi i} \int_{|\xi - t| = \delta} \frac{A^{\frac{m}{2}} u(\xi)}{(\xi - t)^2} d\xi \right| \leq \frac{\tilde{R}_m \nu \kappa_0^m}{\delta},$$

for $t \in \mathbb{R}$, in particular $t = 0$. Therefore, $G_{m+1} \leq \frac{\tilde{R}_m}{\nu \kappa_0^2 \delta}$. $\qquad \square$

Lemma 3.3.12. *Let* $u(t), t \in \mathbb{R}$, *be any solution of the NSE in* $\mathcal{A}$. *If* $u(t)$ *satisfies* (3.54), *and its analytic extension satisfies* (3.55) *for some* $m \geq 3$, *then* (3.54) *also holds for* $m + 1$, *i.e.*

$$\left| A^{\frac{m+1}{2}} u(t) \right| \leq R_{m+1} \nu \kappa_0^{m+1}, \quad \forall\ t \in \mathbb{R}, \tag{3.57}$$

provided R_{m+1} *is defined by*

$$R_{m+1}^2 := \frac{36}{\pi^2} \left(\frac{1}{\delta \nu \kappa_0^2} + \frac{4}{\nu^2 \kappa_0^4 \delta^2} + 2\sqrt{2} \Gamma_m \right) \tilde{R}_m^2, \tag{3.58}$$

where Γ_m *is given in* (3.61) *and* (3.63), *below.*

Remark 3.3.13. From the definition of δ (see (3.52), (3.43), (3.28)) and assumption (3.6), one easily obtains that $1 \geq \delta \nu \kappa_0^2$, which implies that

$$R_{m+1}^2 > \tilde{R}_m^2. \tag{3.59}$$

Proof of Lemma 3.3.12. Let $t_0 \in \mathbb{R}$ be arbitrary and $\rho \in [0, \sqrt{2}\delta)$.

Once again, we follow the procedure outlined in Remark 3.2.3. So, taking inner product in both sides of (3.26) with $A^m u$, we get

$$\frac{1}{2}\frac{d}{d\rho}|A^{\frac{m}{2}}u(t_0 + \rho e^{i\theta})|^2 \le |(g, A^m u)| - \nu\cos\theta|A^{\frac{m+1}{2}}u|^2 + |(B(u,u), A^m u)|.$$

For $m = 3$, since

$$(B(u,u), A^3 u) = \sum_{j=1,2}\left[(B(D_j u, u), D_j A^2 u) + (B(u, D_j u), D_j A^2 u)\right]$$

$$= \sum_{j=1,2}\left[-(B(D_j D_j u, u), A^2 u) - 2(B(D_j u, D_j u), A^2 u) - (B(u, D_j D_j u), A^2 u)\right],$$

we have

$$|(B(u,u), A^3 u)| \le 4c_L^2|A^{\frac{1}{2}}u|^{\frac{1}{2}}|Au||A^{\frac{3}{2}}u|^{\frac{1}{2}}|A^2 u| + 8c_L^2|A^{\frac{1}{2}}u|^{\frac{1}{2}}|A^{\frac{3}{2}}u|^{\frac{1}{2}}|Au||A^2 u|$$

$$+ 4c_L^2|u|^{\frac{1}{2}}|A^{\frac{1}{2}}u|^{\frac{1}{2}}|A^{\frac{3}{2}}u|^{\frac{1}{2}}|A^2 u|^{\frac{3}{2}}$$

$$\le 16c_L^2|u|^{\frac{1}{2}}|A^{\frac{1}{2}}u|^{\frac{1}{2}}|A^{\frac{3}{2}}u|^{\frac{1}{2}}|A^2 u|^{\frac{3}{2}}$$

$$\le \frac{16c_L^2}{\kappa_0^{1/2}}|A^{\frac{1}{2}}u||A^{\frac{3}{2}}u|^{\frac{1}{2}}|A^2 u|^{\frac{3}{2}},$$

and consequently

$$\frac{1}{2}\frac{d}{d\rho}|A^{\frac{3}{2}}u(t_0 + \rho e^{i\theta})|^2 + \frac{3\nu\cos\theta}{4}|A^2 u|^2 \le \frac{1}{\nu\cos\theta}|Ag|^2 + |(B(u,u), A^3 u)| \qquad (3.60)$$

$$\le \frac{|Ag|^2}{\nu\cos\theta} + \frac{1}{4}\left(\left(\frac{3}{\nu\cos\theta}\right)^{3/4}\frac{16c_L^2}{\kappa_0^{\frac{1}{2}}}|A^{\frac{1}{2}}u||A^{\frac{3}{2}}u|^{\frac{1}{2}}|\right)^4 + \frac{3}{4}\left(\left(\frac{\nu\cos\theta}{3}\right)^{3/4}|A^2 u|^{\frac{3}{2}}\right)^{4/3}$$

$$\le \frac{|Ag|^2}{\nu\cos\theta} + \frac{3^3 \cdot 2^{\frac{31}{2}}c_L^8}{\nu^3\kappa_0^2}|A^{\frac{1}{2}}u|^4|A^{\frac{3}{2}}u|^2 + \frac{\nu\cos\theta}{4}|A^2 u|^2.$$

It follows that

$$\frac{1}{2}\frac{d}{d\rho}|A^{\frac{3}{2}}u(t_0+\rho e^{i\theta})|^2 + \frac{1}{2}\nu\cos\theta|A^2u|^2 \le \frac{2}{\nu\sqrt{2}}|Ag|^2 + \Gamma_3\nu\kappa_0^2|A^{\frac{3}{2}}u|^2,$$

where

$$\Gamma_3 := 3^3 \cdot 2^{\frac{31}{2}} c_L^8 \tilde{R}_1^2. \tag{3.61}$$

For $m > 3$, by Young's inequality and Lemma 3.3.10, we obtain

$$\frac{1}{2}\frac{d}{d\rho}|A^{\frac{m}{2}}u(t_0+\rho e^{i\theta})|^2 + \frac{3}{4}\nu\cos\theta|A^{\frac{m+1}{2}}u|^2 \tag{3.62}$$

$$\le \frac{1}{\nu\cos\theta}|A^{\frac{m-1}{2}}g|^2 + 2^{m+\frac{3}{2}}c_A|u|^{\frac{1}{2}}|Au|^{\frac{1}{2}}|A^{\frac{1+m}{2}}u||A^{\frac{m}{2}}u|$$

$$+ 2^{m+\frac{3}{2}}c_A|A^{\frac{1}{2}}u|^{\frac{1}{2}}|A^{\frac{3}{2}}u|^{\frac{1}{2}}|A^{\frac{m}{2}}u|^2,$$

and

$$\frac{1}{2}\frac{d}{d\rho}|A^{\frac{m}{2}}u(t_0+\rho e^{i\theta})|^2 + \frac{1}{2}\nu\cos\theta|A^{\frac{m+1}{2}}u|^2$$

$$\le \frac{1}{\nu\cos\theta}|A^{\frac{m-1}{2}}g|^2 + \frac{1}{\nu\cos\theta}\left(2^{m+\frac{3}{2}}c_A|u|^{\frac{1}{2}}|Au|^{\frac{1}{2}}|A^{\frac{m}{2}}u|\right)^2$$

$$+ 2^{m+\frac{3}{2}}c_A|A^{\frac{1}{2}}u|^{\frac{1}{2}}|A^{\frac{3}{2}}u|^{\frac{1}{2}}|A^{\frac{m}{2}}u|^2$$

$$\le \frac{1}{\nu\cos\theta}|A^{\frac{m-1}{2}}g|^2 + \left(\frac{\{2^{m+\frac{3}{2}}c_A\}^2}{\nu\cos\theta}\tilde{R}_1\nu\tilde{R}_2\nu\kappa_0^2 + 2^{m+\frac{3}{2}}c_A\sqrt{\tilde{R}_1\nu\kappa_0\tilde{R}_3\nu\kappa_0^3}\right)|A^{\frac{m}{2}}u|^2$$

$$\le \frac{2}{\nu\sqrt{2}}|A^{\frac{m-1}{2}}g|^2 + \Gamma_m\nu\kappa_0^2|A^{\frac{m}{2}}u|^2,$$

where

$$\Gamma_m := 2^{m+\frac{3}{2}}c_A\left[2^{m+2}c_A\tilde{R}_1\tilde{R}_2 + \sqrt{\tilde{R}_1\tilde{R}_3}\right]. \tag{3.63}$$

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Since $\cos\theta \geq \frac{\sqrt{2}}{2}$, for $\theta \in [-\pi/4, \pi/4]$, we obtain

$$\frac{1}{2}\frac{d}{d\rho}|A^{\frac{m}{2}}u(t_0 + \rho e^{i\theta})|^2 + \nu\frac{\sqrt{2}}{4}|A^{\frac{m+1}{2}}u(\zeta)|^2 \leq \frac{\sqrt{2}}{\nu}|A^{\frac{m-1}{2}}g|^2 + \nu\kappa_0^2\Gamma_m|A^{\frac{m}{2}}u|^2. \qquad (3.64)$$

It follows that

$$\nu\frac{\sqrt{2}}{2}\int_0^{\sqrt{2}\delta}|A^{\frac{m+1}{2}}u(t_0 + \rho e^{i\theta})|^2 d\rho \leq |A^{\frac{m}{2}}u(t_0)|^2 + \frac{4\delta}{\nu}|A^{\frac{m-1}{2}}g|^2$$

$$+ 2\nu\kappa_0^2\Gamma_m\int_0^{\sqrt{2}\delta}|A^{\frac{m}{2}}u(\zeta)|^2|_{\zeta=t_0+\rho e^{i\theta}}d\rho$$

$$\leq R_m^2\nu^2\kappa_0^{2\alpha} + 4\delta G_m^2\nu^3\kappa_0^{2(m+1)} + 2\nu\kappa_0^2\Gamma_m\sqrt{2}\delta\tilde{R}_m^2\nu^2\kappa_0^{2\alpha}$$

and

$$\int_0^{\sqrt{2}\delta}|A^{\frac{m+1}{2}}u(t_0 + \rho e^{i\theta})|^2 d\rho \leq \sqrt{2}R_m^2\nu\kappa_0^{2\alpha} + 4\sqrt{2}\delta G_m^2\nu^2\kappa_0^{2(m+1)} + 4\Gamma_m\delta\tilde{R}_m^2\nu^2\kappa_0^{2(m+1)}$$

$$=: N_m.$$

Since $u(\zeta)$ is $\mathcal{D}(A^{\frac{m}{2}})_{\mathbb{C}}$-valued analytic in $\mathcal{S}(\delta)$, we obtain (as we have done in the

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proof of Lemma 3.3.3) the following successive relations

$$
\begin{aligned}
|A^{\frac{m+1}{2}}u(t_0)| &= |\frac{1}{\pi\delta^2}\iint_{D(t_0,\delta)} A^{\frac{m+1}{2}}u(\zeta)d\Im(\zeta)d\Re(\zeta)| \\
&\le \frac{1}{\pi\delta^2}\iint_{D(t_0,\delta)} |A^{\frac{m+1}{2}}u(\zeta)|d\Im(\zeta)d\Re(\zeta) \\
&\le \frac{1}{\pi\delta^2}\iint_{abcdef} |A^{\frac{m+1}{2}}u(\zeta)|d\Im(\zeta)d\Re(\zeta) \\
&= \frac{2}{\sqrt{2}\pi\delta^2}\int_{t_0-2\delta}^{t_0+\delta} dt \int_0^{\sqrt{2}\delta} |A^{\frac{m+1}{2}}u(\zeta)|\|_{\zeta=\tau e^{i\pi/4}+t}d\tau \\
&\le \frac{2}{\sqrt{2}\pi\delta^2}\int_{t_0-2\delta}^{t_0+\delta} dt \left(\int_0^{\sqrt{2}\delta} |A^{\frac{m+1}{2}}u(\zeta)|^2|_{\zeta=\tau e^{i\pi/4}+t}d\tau\right)^{\frac{1}{2}} (\sqrt{2}\delta)^{\frac{1}{2}} \\
&\le \frac{2}{\sqrt{2}\pi\delta^2}3\delta(N_m\sqrt{2}\delta)^{\frac{1}{2}} \\
&= \frac{6\cdot 2^{\frac{1}{4}}}{\sqrt{2\pi}}(\frac{N_m}{\delta})^{\frac{1}{2}},
\end{aligned}
$$

that is, using (3.56),

$$
\begin{aligned}
|A^{\frac{m+1}{2}}u(t_0)|^2 &\le \frac{18\sqrt{2}}{\pi^2}\frac{N_m}{\delta} \\
&= \frac{18\sqrt{2}}{\pi^2}\left(\frac{\sqrt{2}R_m^2\nu\kappa_0^{2\alpha}}{\delta} + 4\sqrt{2}G_m^2\nu^2\kappa_0^{2(m+1)} + 4\Gamma_m\tilde{R}_m^2\nu^2\kappa_0^{2(m+1)}\right) \\
&\le \frac{36}{\pi^2}\nu^2\kappa_0^{2(m+1)}\left(\frac{R_m^2}{\delta\nu\kappa_0^2} + 4\frac{\tilde{R}_{m-1}^2}{\nu^2\kappa_0^4\delta^2} + 2\sqrt{2}\Gamma_m\tilde{R}_m^2\right) \\
&\le \frac{36}{\pi^2}\nu^2\kappa_0^{2(m+1)}\left(\frac{1}{\delta\nu\kappa_0^2} + \frac{4}{\nu^2\kappa_0^4\delta^2} + 2\sqrt{2}\Gamma_m\right)\tilde{R}_m^2 \\
&= R_{m+1}^2\nu^2\kappa_0^{2(m+1)},
\end{aligned}
$$

where R_{m+1} is defined in (3.58).

We conclude the proof by observing that $t_0 \in \mathbb{R}$ is arbitrary. $\qquad\square$

We next extend the result in Lemma 3.3.12 to the strip $\mathcal{S}(\delta)$.

Lemma 3.3.14. *If $0 \in \mathcal{A}$ and if the solution $u(t), t \in \mathbb{R}$, of the NSE in $\mathcal{A}$ satisfies*

(3.54), (3.55) *for* m, *then* $u(\zeta)$ *is a* $\mathcal{D}(A^{\frac{m+1}{2}})_{\mathbb{C}}$-*valued analytic function, and* (3.55) *also holds for* $m + 1$. *In particular, we have*

$$|A^{\frac{m+1}{2}} u_\kappa(\zeta)| \le \tilde{R}_{m+1} \nu \kappa_0^{m+1}, \quad \forall\ \zeta \in \mathcal{S}(\delta),$$

where

$$\tilde{R}_{m+1}^2 := \beta^{\Gamma_{m+1}} \frac{36\sqrt{2}}{\pi^2} \Gamma_m (1 + \varepsilon_m) \tilde{R}_m^2, \tag{3.65}$$

$$\beta := e^{2\sqrt{2}\delta\nu\kappa_0^2}, \tag{3.66}$$

$$\varepsilon_m = \frac{1}{2\sqrt{2}\Gamma_m \delta\nu\kappa_0^2} + \frac{\sqrt{2}}{\Gamma_m \nu^2 \kappa_0^4 \delta^2} + \frac{\pi^2}{72\nu^2 \kappa_0^4 \delta^4 \Gamma_m \Gamma_{m+1}}, \tag{3.67}$$

and Γ_m *is defined in* (3.63).

Moreover, the following inequality holds

$$\tilde{R}_{m+1}^2 > R_{m+1}^2.$$

Proof. Let $t_0 \in \mathbb{R}$ be arbitrary and $\rho \in [0, \sqrt{2}\delta)$. By virtue of Remarks 3.2.3, we can assume that $u(\zeta)$ is $\mathcal{D}(A^{\frac{m+1}{2}})_{\mathbb{C}}$-valued analytic. Taking the inner product of (2.14) with $A^{m+1}u$, as in the proof of Lemma 3.3.12, we get

$$\frac{1}{2}\frac{d}{d\rho}|A^{\frac{m+1}{2}} u(t_0 + \rho e^{i\theta})|^2 + \nu\frac{\sqrt{2}}{4}|A^{\frac{m+2}{2}} u(\zeta)|^2 \tag{3.68}$$

$$\le \frac{\sqrt{2}}{\nu}|A^{\frac{m}{2}} g|^2 + \nu\kappa_0^2 \Gamma_{m+1}|A^{\frac{m+1}{2}} u|^2,$$

where the Lemma 3.3.11 is used.

It follows that

$$\frac{d}{d\rho}|A^{\frac{m+1}{2}}u(t_0+\rho e^{i\theta})|^2 \le \frac{4}{\nu\sqrt{2}}|A^{\frac{m}{2}}g|^2 + 2\nu\kappa_0^2\Gamma_{m+1}|A^{\frac{m+1}{2}}u|^2.$$

Since $\rho \in [0,\sqrt{2}\delta)$, we have, by (3.56),

$$
\begin{aligned}
|A^{\frac{m+1}{2}}u(\zeta)|^2|_{\zeta=t_0+\rho e^{i\theta}} &\le e^{2\nu\kappa_0^2\Gamma_{m+1}\rho}|A^{\frac{m+1}{2}}u(t_0)|^2 + \frac{\frac{4}{\nu\sqrt{2}}|A^{\frac{m}{2}}g|^2}{2\nu\kappa_0^2\Gamma_{m+1}}(e^{2\nu\kappa_0^2\Gamma_{m+1}\rho}-1)\\
&\le e^{2\nu\kappa_0^2\Gamma_{m+1}\rho}\left[|A^{\frac{m+1}{2}}u(t_0)|^2 + \frac{\sqrt{2}|A^{\frac{m}{2}}g|^2}{\nu^2\kappa_0^2\Gamma_{m+1}}\right]\\
&\le e^{2\sqrt{2}\delta\nu\kappa_0^2\Gamma_{m+1}}\left[R_{m+1}^2\nu^2\kappa_0^{2(m+1)} + \frac{\sqrt{2}}{\Gamma_{m+1}}G_{m+1}^2\nu^2\kappa_0^{2\alpha+2}\right]\\
&\le \beta^{\Gamma_{m+1}}\left\{\frac{36}{\pi^2}\left[\frac{1}{\delta\nu\kappa_0^2} + \frac{4}{\nu^2\kappa_0^4\delta^2} + 2\sqrt{2}\Gamma_m\right] + \frac{\sqrt{2}}{\nu^2\kappa_0^4\delta^2\Gamma_{m+1}}\right\}\tilde{R}_m^2\nu^2\kappa_0^{2\alpha+2}\\
&= \beta^{\Gamma_{m+1}}\frac{72\sqrt{2}}{\pi^2}\Gamma_m(1+\varepsilon_m)\tilde{R}_m^2\nu^2\kappa_0^{2\alpha+2}\\
&=: \tilde{R}_{m+1}^2\nu^2\kappa_0^{2\alpha+2},
\end{aligned}
$$

where

$$\tilde{R}_{m+1}^2 := \beta^{\Gamma_{m+1}}\frac{72\sqrt{2}}{\pi^2}\Gamma_m(1+\varepsilon_m)\tilde{R}_m^2. \tag{3.69}$$

$\square$

While (3.69) shows that $\tilde{R}_m$ increases with m, the next result provides an explicit upper bound.

Proposition 3.3.15. *For $m>3$,*

$$\tilde{R}_{m+1}^2 \le C(g)\beta_1^{4^{m+1}}\beta_2^{(m+1)^2+\frac{9}{2}(m+1)},$$

where

$$C_1 := \prod_{m=3}^{\infty}(1 + \varepsilon_m), \quad C_3 := 4[2^{\frac{5}{2}}c_A^2\tilde{R}_1\tilde{R}_2 + 2^{\frac{1}{2}}c_A\sqrt{\tilde{R}_1\tilde{R}_3}],$$

$$C_2 := 3^3 2^{-7} c_L^8 \tilde{R}_1^2 \prod_{\gamma=3}^{\infty}(1 + \eta_\gamma), \quad \eta_\gamma = \frac{\sqrt{\tilde{R}_1\tilde{R}_3}}{2^{\gamma+2}c_A\tilde{R}_1\tilde{R}_2},$$

$$\beta_1 := \beta^{C_3}, \quad \beta_2 := \max\{\frac{72\sqrt{2}}{\pi^2}, c_A^2\tilde{R}_1\tilde{R}_2\},$$

$$C(g) := C_1 C_2 \tilde{R}_3^2 \beta_2^{-19/2},$$

where β and ε_m are defined in (3.66) and (3.67), respectively.

Proof. Since $\sum_{m=3}^{\infty}\varepsilon_m$ is convergent, we have $C_1 := \prod_{m=3}^{\infty}(1 + \varepsilon_m) < \infty$. Due to the definition of Γ_m in (3.63), we have

$$\prod_{\gamma=3}^{m}\Gamma_\gamma = 3^3 2^{\frac{31}{2}} c_L^8 \tilde{R}_1^2 \prod_{\gamma=4}^{m} 2^{2\gamma+7/2}(c_A^2\tilde{R}_1\tilde{R}_2)\left(1 + \frac{\sqrt{\tilde{R}_1\tilde{R}_3}}{2^{\gamma+2}c_A\tilde{R}_1\tilde{R}_2}\right)$$

$$=: 3^3 2^{\frac{31}{2}} c_L^8 \tilde{R}_1^2 \prod_{\gamma=4}^{m} 2^{2\gamma+7/2}(c_A^2\tilde{R}_1\tilde{R}_2)[1 + \eta_\gamma]$$

$$= 3^3 2^{\frac{31}{2}} c_L^8 \tilde{R}_1^2 2^{m^2+\frac{9}{2}m-\frac{45}{2}}(c_A^2\tilde{R}_1\tilde{R}_2)^{m-3} \prod_{\gamma=4}^{m}[1 + \eta_\gamma]$$

$$< 3^3 2^{-7} c_L^8 \tilde{R}_1^2 2^{m^2+\frac{9}{2}m}(c_A^2\tilde{R}_1\tilde{R}_2)^{m-3} \prod_{\gamma=4}^{\infty}[1 + \eta_\gamma]$$

$$=: 2^{m^2+\frac{9}{2}m}(c_A^2\tilde{R}_1\tilde{R}_2)^{m-3}C_2,$$

and

$$\sum_{\gamma=4}^{m+1} \Gamma_\gamma = \sum_{\gamma=4}^{m+1} [2^{2\gamma+7/2} c_A^2 \tilde{R}_1 \tilde{R}_2 + 2^{\gamma+\frac{3}{2}} c_A \sqrt{\tilde{R}_1 \tilde{R}_3}]$$

$$= 2^{7/2} c_A^2 \tilde{R}_1 \tilde{R}_2 \sum_{\gamma=4}^{m+1} 2^{2\gamma} + 2^{\frac{3}{2}} c_A \sqrt{\tilde{R}_1 \tilde{R}_3} \sum_{\gamma=4}^{m+1} 2^\gamma$$

$$= 2^{7/2} c_A^2 \tilde{R}_1 \tilde{R}_2 \frac{4^{m+2} - 4^4}{3} + 2^{\frac{3}{2}} c_A (2^{m+2} - 2^4) \sqrt{\tilde{R}_1 \tilde{R}_3}$$

$$\le \left(2^{\frac{5}{2}} c_A^2 \tilde{R}_1 \tilde{R}_2 + 2^{\frac{1}{2}} c_A \sqrt{\tilde{R}_1 \tilde{R}_3} \right) 4^{m+2}$$

$$=: C_3 4^{m+1}.$$

It follows from the recursion relation (3.69) that

$$\tilde{R}_{m+1}^2 := \beta^{\Gamma_{m+1}} \frac{72\sqrt{2}}{\pi^2} \Gamma_m (1 + \varepsilon_m) \tilde{R}_m^2$$

$$= \beta^{\sum_{\gamma=4}^{m+1} \Gamma_\gamma} \left(\frac{72\sqrt{2}}{\pi^2}\right)^{m-2} \prod_{\gamma=3}^m \Gamma_\gamma \prod_{\gamma=3}^m (1 + \varepsilon_\gamma) \tilde{R}_3^2$$

$$\le \beta^{C_3 4^{m+1}} \left(\frac{72\sqrt{2}}{\pi^2}\right)^{m-2} 2^{m^2 + \frac{9}{2} m} (c_A^2 \tilde{R}_1 \tilde{R}_2)^{m-3} C_2 C_1 \tilde{R}_3^2$$

$$\le \beta^{C_3 4^{m+1}} C_2 C_1 \tilde{R}_3^2 \max\{\frac{72\sqrt{2}}{\pi^2}, 2, c_A^2 \tilde{R}_1 \tilde{R}_2\}^{m^2 + \frac{13}{2} m - 5}$$

$$=: C(g) \beta_1^{4^{m+1}} \beta_2^{(m+1)^2 + \frac{9}{2}(m+1)}.$$

$\square$

Remark 3.3.16. Theorem 3.3.1 is now a direct consequence of Lemmas 3.3.3, 3.3.6, 3.3.7, 3.3.14 and Proposition 3.3.15.

3.4   The class $\mathcal{C}(\sigma)$

The estimates for $\{|A^{\frac{m}{2}} u(t)|, t \in \mathbb{R}\}$ can be slightly improved by shrinking the width δ_m of the strip $\mathcal{S}(\delta_m)$ in the induction argument for $m > 3$.

Theorem 3.4.1. *Let $0 \in \mathcal{A}$ and let*

$$\delta_{m+1} := \frac{3}{4}\delta_m$$

for $m \in \mathbb{N}, m \geq 3$. Then for any solution in $u(t) \in \mathcal{A}, t \in \mathbb{R}$, one has

$$|A^{\frac{m+1}{2}}u(\zeta)| \leq \tilde{R}_{m+1}\nu\kappa_0^{m+1}, \quad \forall \, \zeta \in \mathcal{S}(\delta_{m+1}), \tag{3.70}$$

for all $m \geq 3$, where the constants $\tilde{R}_{m+1}, m \geq 3$, are redefined in the following way

$$\tilde{R}_{m+1}^2 := \frac{1024\sqrt{2}}{\pi^2}\Gamma_m(1 + \xi_m)\tilde{R}_m^2, \tag{3.71}$$

with

$$\xi_m = \frac{1}{4\sqrt{2}\nu\kappa_0^2\delta_{m+1}\Gamma_m} + \frac{1}{\sqrt{2}\nu^2\kappa_0^4\delta_m\delta_{m+1}\Gamma_m}.$$

Furthermore, we have the following estimate

$$\tilde{R}_{m+1}^2 \leq \tilde{C}(g)\beta_3^{\frac{3}{2}(m+1)^2}, \tag{3.72}$$

where

$$\beta_3 := \max(\frac{1024\sqrt{2}}{\pi^2}, c_A^2\tilde{R}_1\tilde{R}_2), \quad \tilde{C}(g) := C_2C_4\tilde{R}_3^2\beta_3^{-3/8},$$

and

$$C_4 := \prod_{\gamma=3}^{\infty}(1 + \xi_\gamma).$$

Proof. As done in the proof of Lemma 3.3.7, we can easily prove that under the new definition (3.71), the relation (3.70) is true.

Then, we obtain (as in the proof of Proposition 3.3.15)

$$
\begin{aligned}
\tilde{R}_{m+1}^2 &:= \frac{1024\sqrt{2}}{\pi^2}\Gamma_m(1+\xi_m)\tilde{R}_m^2 \\
&< \left(\frac{1024\sqrt{2}}{\pi^2}\right)^{m-2}\tilde{R}_3^2\prod_{\gamma=3}^m\Gamma_\gamma\prod_{\gamma=3}^\infty(1+\xi_\gamma) \\
&\le \left(\frac{1024\sqrt{2}}{\pi^2}\right)^{m-3}2^{m^2+\frac{9}{2}m}(c_A^2\tilde{R}_1\tilde{R}_2)^{m-2}C_2C_4\tilde{R}_3^2 \\
&\le \tilde{C}(g)\beta_3^{\frac{3}{2}(m+1)^2}.
\end{aligned}
$$

$\square$

The estimates in Theorem 3.4.1 identify the role of the subset of $C_{\mathrm{per}}^\infty([0,L]^2)\cap H$ defined below (see [6] for more details).

Definition 3.4.2.

$$
\mathcal{C}(\sigma) := \left\{u\in C_{\mathrm{per}}^\infty([0,L]^2)\cap H : \exists\, c_0 = c_0(u)\in\mathbb{R}\text{ such that }\frac{|A^{\frac{m}{2}}u|^2}{\nu^2\kappa_0^{2m}}\le c_0 e^{\sigma m^2},\, m\in\mathbb{N}\right\}.
$$

Remark 3.4.3. The main conclusion of Theorem 3.4.1 can be given in the following succinct formulation

$$
0\in\mathcal{A}\Rightarrow\mathcal{A}\subset\mathcal{C}\left(\frac{3}{2}\ln\beta_3\right),
$$

where $\frac{3}{2}\ln\beta_3 = O(\ln G)$.

Remark 3.4.4. An equivalent definition of the class $\mathcal{C}(\sigma)$ is

$$
\mathcal{C}(\sigma) = \left\{u\in C^\infty([0,L]^2)\cap H : |u|_{\mathcal{C}_\sigma} := \sup\{|A^{\frac{m}{2}}u|e^{-\frac{\sigma}{2}m^2},\, m\in\mathbb{N}\} < \infty\right\}. \qquad (3.73)
$$

It is easy to check that $u\mapsto|u|_{\mathcal{C}_\sigma}$ is a norm on $\mathcal{C}(\sigma)$. Obviously, $\mathcal{C}(\sigma)$ equipped with this norm is a Banach space.

Moreover, Theorem 3.4.1 has the following corollary

Corollary 3.4.5. *If* $0 \in \mathcal{A}$, *then* $g \in \mathcal{C}(\frac{5}{2} \ln \beta_3)$, *where* $\frac{5}{2} \ln \beta_3 = O(\ln G)$.

Proof. Since $\delta_m = \frac{1}{2^{m-3}} \delta_3 > \frac{1}{2^m} \delta_3$, by (3.56) and (3.72) we get

$$|A^{\frac{m}{2}} g| \leq \tilde{R}_m \nu \kappa_0^m \frac{2^m}{\delta_3} \leq \tilde{R}_m \nu \kappa_0^m \frac{\beta_3^m}{\delta_3} \leq \frac{\tilde{R}_m \nu \kappa_0^m}{\delta_3 \beta_3^{-\frac{1}{2}}} \beta_3^{\frac{1}{2}m^2}.$$

and then

$$\frac{|A^{\frac{m}{2}} g|^2}{\nu^4 \kappa_0^{2\alpha+4}} \leq \frac{\tilde{C}(g)}{\nu^2 \kappa_0^4 \delta_3^2 \beta_3^{-1}} \beta_3^{\frac{5}{2}m^2},$$

where both sides are dimensionless.

Consequently, it follows that

$$g \in \mathcal{C}(\frac{5}{2} \ln \beta_3).$$

$\square$

4. ON AN EXPLICIT CRITERION FOR ZERO BEING IN THE GLOBAL ATTRACTOR*

One conjecture proposed by P. Constantin is that, zero is in the global attractor $\mathcal{A}$ of the 2D NSE if and only if the force g is zero. In this chapter, we will focus on discussing if it is possible that the immobile fluid is in the global attractor of the 2D NSE if the body force is not potential.

.

4.1 Constantin-Chen Gevrey classes

We first define the general Constantin-Chen Gevrey ($\mathcal{CCG}$) classes [1].

Given a function $\phi(\chi)$ with the following properties:

$$\phi'(\chi) > 0,$$

$$\phi''(\chi) < 0,$$

for all $\chi \in [1, \infty)$, we define the general Constantin-Chen Gevrey ($\mathcal{CCG}$) class $E(\phi)$ as the collection of all $u \in C^\infty([0, L]^2) \cap H$ for which $|e^{\phi(\kappa_0^{-1} A^{\frac{1}{2}})} u|$ is finite, that is,

Definition 4.1.1. $E(\phi) = \{u \in H : |e^{\phi(\kappa_0^{-1} A^{\frac{1}{2}})} u| < \infty\}$,

where

$$\left(e^{\phi(\kappa_0^{-1} A^{\frac{1}{2}})} u\right)\hat{}(k) := e^{\phi(|k|)} \hat{u}(k), \quad \forall\, k \in \mathbb{Z}^2 \smallsetminus \{0\}.$$

A typical example of a $\mathcal{CCG}$ class is $\tilde{\phi}(\chi) = \beta \ln \chi$, for $\beta > 0$. Actually, $E(\tilde{\phi}) =$

$H \cap \mathcal{D}(A^{\beta/2})$. The proofs of the two estimates for the bilinear term $|(B(u,v), A^\gamma w)|$ given in previous chapter is based on this typical $\mathcal{CCG}$ class.

In this section we will investigate the relation between the class $\mathcal{C}(\sigma)$ and $E(\phi_b)$, where

$$\phi_b(\chi) = b[\ln(\chi + e)]^2, \quad b > 0. \tag{4.1}$$

For convenience, we take the following notation

Definition 4.1.2. For $b > 0$, we define $E_b := E(\phi_b)$, $E^b u := e^{\phi_b(\kappa_0^{-1} A^{1/2})} u$, $|u|_b := |E^b u|$.

Theorem 4.1.3. *If $v \in H$ satisfies*

$$\left| e^{b(\ln(\kappa_0^{-1} A^{\frac{1}{2}} + a))^2} v \right| < \infty, \quad a \geq e, \quad b > 0 \tag{4.2}$$

then

$$v \in \mathcal{C}\left(\frac{1}{2b}\right). \tag{4.3}$$

Proof. Noting the following relation

$$\begin{aligned}
\left| A^{\frac{\alpha}{2}} v \right| &= \left| A^{\frac{\alpha}{2}} e^{-b(\ln(\kappa_0^{-1} A^{\frac{1}{2}} + a))^2} e^{b(\ln(\kappa_0^{-1} A^{\frac{1}{2}} + a))^2} v \right| \\
&\leq \left| A^{\frac{\alpha}{2}} e^{-b(\ln(\kappa_0^{-1} A^{\frac{1}{2}} + a))^2} \right|_{\mathrm{op}} \left| e^{b(\ln(\kappa_0^{-1} A^{\frac{1}{2}} + a))^2} v \right|,
\end{aligned}$$

since

$$\begin{aligned}
\left| A^{\frac{\alpha}{2}} e^{-b(\ln(\kappa_0^{-1} A^{\frac{1}{2}} + a))^2} u \right|^2 &= \sum_{k \in \mathbb{Z}^2 \smallsetminus \{0\}} |k|^{2\alpha} e^{-2b(\ln(|k|+a))^2} |\hat{u}(k)|^2 \\
&\leq \sup_{k \in \mathbb{Z}^2 \smallsetminus \{0\}} |k|^{2\alpha} e^{-2b(\ln(|k|+a))^2} \sum_{k \in \mathbb{Z}^2 \smallsetminus \{0\}} |\hat{u}(k)|^2 \\
&= \sup_{k \in \mathbb{Z}^2 \smallsetminus \{0\}} |k|^{2\alpha} e^{-2b(\ln(|k|+a))^2} |u|^2,
\end{aligned}$$

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we obtain that

$$|A^{\frac{\alpha}{2}}e^{-b(\ln(\kappa_0^{-1}A^{\frac{1}{2}}+a))^2}|_{\mathrm{op}}^2 \leq \sup_{k \in \mathbb{Z}^2 \smallsetminus \{0\}} |k|^{2\alpha} e^{-2b(\ln(|k|+a))^2}$$

$$\leq \sup_{k \in \mathbb{Z}^2 \smallsetminus \{0\}} (|k|+a)^{2\alpha} e^{-2b(\ln(|k|+a))^2}$$

$$\leq \sup_{1+a \leq x} e^{2\alpha \ln x} e^{-2b(\ln x)^2} = e^{\frac{\alpha^2}{2b}}.$$

and (4.3) follows from Definition 3.4.2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 4.1.4. Using Theorem 4 in [15], it is easy to verify that if $G^s([0,L]^2)$ denotes the Gevrey class s ($s > 0$, defined in [15]; see also [18]), then the following relation holds

$$\bigcup_{s>0} G^s([0,L]^2) \subset \bigcap_{\sigma>0} \mathcal{C}(\sigma).$$

The "reverse" inclusion relation between the classes E_b and $\mathcal{C}(\sigma)$ is given in Theorem 4.1.6.

Proposition 4.1.5. *If $u \in \mathcal{C}(\sigma)$ for some $\sigma > 0$, i.e.,*

$$\exists\, c_0 > 0, \quad s.t. \quad |A^{\frac{\alpha}{2}}u|^2 \leq c_0 e^{\sigma\alpha^2}(\nu\kappa_0^\alpha)^2, \quad \forall \alpha \in \mathbb{N}, \tag{4.4}$$

then for fixed $\varepsilon \in [0,1]$, there exists $b := \frac{1}{2^{4+2\varepsilon}\sigma}$ such that

$$|e^{b[\ln(\kappa_0^{-1}A^{1/2}+e)]^{1+\varepsilon}}u| < \infty. \tag{4.5}$$

In particular, we have

$$|e^{b[\ln(\kappa_0^{-1}A^{1/2}+e)]^{1+\varepsilon}}u|^2 \leq \frac{4}{3}c(\varepsilon)|u|^2 + c_0^{\frac{1}{3}}(c_1|A^{\frac{1}{2}}u|)^{\frac{2}{3}}\nu^{\frac{4}{3}}\kappa_0^{-\frac{2}{3}}, \tag{4.6}$$

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where

$$c(\varepsilon) := e^{2b[\ln(e^2+e)]^{1+\varepsilon}}, \quad c_1 = \sum_{m \geq 2} \frac{1}{e^m} = \frac{1}{e^2 - e}.$$

Proof. First, by the definition of $e^{\phi(A^{1/2})u}$, we have

$$\left|e^{b[\ln(\kappa_0 A^{1/2}+e)]^{1+\varepsilon}} u\right|^2 = \sum_{m=0}^{\infty} \sum_{e^m \leq |k| < e^{m+1}} e^{2b[\ln(|k|+e)]^{1+\varepsilon}} |\hat{u}(k)|^2$$

$$= \sum_{m=0,1} + \sum_{m \geq 2} =: I_1 + I_2.$$

For I_1, it is easy to see that

$$I_1 = \sum_{m=0,1} \sum_{e^m \leq |k| < e^{m+1}} e^{2b[\ln(|k|+e)]^{1+\varepsilon}} |\hat{u}(k)|^2$$

$$\leq e^{2b[\ln(e^2+e)]^{1+\varepsilon}} \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} |\hat{u}(k)|^2$$

$$= e^{2b[\ln(e^2+e)]^{1+\varepsilon}} |u|^2$$

$$= c(\varepsilon) |u|^2 < \infty,$$

while for I_2, using the definition of the class $\mathcal{C}(\sigma)$ and Young's inequality we can

infer

$$I_2 = \sum_{m \geq 2} \sum_{e^m \leq |k| < e^{m+1}} e^{2b[\ln(|k|+e)]^{1+\varepsilon}} |\hat{u}(k)|^2$$

$$= \sum_{m \geq 2} \sum_{e^m \leq |k| < e^{m+1}} (|k| + e)^{2b[\ln(|k|+e)]^{\varepsilon}} |\hat{u}(k)|^2$$

$$\leq \sum_{m \geq 2} \sum_{e^m \leq |k| < e^{m+1}} (|k| + e)^{2b(m+2)^{\varepsilon}} |\hat{u}(k)|^2$$

$$\leq \sum_{m \geq 2} \sum_{e^m \leq |k| < e^{m+1}} |k|^{4b(m+2)^{\varepsilon}} |\hat{u}(k)|^2$$

$$= \sum_{m \geq 2} |A^{b(m+2)^{\varepsilon}}(P_{m+1} - P_m)u|^2 \kappa_0^{-4b(m+2)^{\varepsilon}}$$

$$\leq \sum_{m \geq 2} |A^{2b(m+2)^{\varepsilon}}u| |(P_{m+1} - P_m)u| \kappa_0^{-4b(m+2)^{\varepsilon}}$$

$$\leq \nu \sum_{m \geq 2} c_0^{1/2} e^{\frac{\sigma}{2} 16 b^2 (m+2)^{2\varepsilon}} |(P_{m+1} - P_m)u|$$

$$\leq \nu \sum_{m \geq 2} c_0^{1/2} e^{2^{3+2\varepsilon} \sigma b^2 m^{2\varepsilon}} |(P_{m+1} - P_m)u|^{\frac{1}{2}} |(P_{m+1} - P_m)u|^{\frac{1}{2}}$$

$$\leq \nu \left(\sum_{m \geq 2} c_0^2 e^{2^{5+2\varepsilon} \sigma b^2 m^{2\varepsilon}} |(P_{m+1} - P_m)u|^2 \right)^{1/4} \left(\sum_{m \geq 2} |(P_{m+1} - P_m)u|^{2/3} \right)^{3/4}$$

$$=: \nu I_{21}^{1/4} I_{22}^{3/4}$$

We now derive estimates for I_{21} and I_{22}. For I_{21}, we obtain

$$I_{21} = \sum_{m \geq 2} c_0^2 e^{2^{5+2\varepsilon}\sigma b^2 m^{2\varepsilon}} |(P_{m+1} - P_m)u|^2$$

$$\leq \sum_{m \geq 0} c_0^2 e^{2^{5+2\varepsilon}\sigma b^2 m^{2\varepsilon}} |(P_{m+1} - P_m)u|^2$$

$$= \sum_{m \geq 0} c_0^2 e^{2^{5+2\varepsilon}\sigma b^2 m^{2\varepsilon}} \sum_{e^m \leq |k| < e^{m+1}} |\hat{u}(k)|^2$$

$$\leq c_0^2 \sum_{m \geq 0} \sum_{e^m \leq |k| < e^{m+1}} e^{2^{5+2\varepsilon}\sigma b^2 [\ln(|k|+e)]^{2\varepsilon}} |\hat{u}(k)|^2$$

$$\leq c_0^2 \sum_{m \geq 0} \sum_{e^m \leq |k| < e^{m+1}} e^{2^{5+2\varepsilon}\sigma b^2 [\ln(|k|+e)]^{1+\varepsilon}} |\hat{u}(k)|^2,$$

since

$$2\varepsilon \leq 1 + \varepsilon, \quad \text{i.e.,} \quad \varepsilon \leq 1.$$

Defining b as

$$2^{4+2\varepsilon}\sigma b = 1, \quad \text{i.e.,} \quad b = \frac{1}{2^{4+2\varepsilon}\sigma},$$

we immediately get

$$I_{21} \leq c_0^2 \sum_{m \geq 0} \sum_{e^m \leq |k| < e^{m+1}} e^{2b[\ln(|k|+e)]^{1+\varepsilon}} |\hat{u}(k)|^2$$

$$= c_0^2 |e^{b[\ln(\kappa_0 A^{1/2}+e)]^{1+\varepsilon}} u|^2.$$

For I_{22}, we set $v = A^{1/2}u$ and apply Hölder's inequality as follows:

$$I_{22} = \sum_{m \geq 2} |(P_{m+1} - P_m)u|^{2/3}$$

$$= \sum_{m \geq 2} |A^{-1/2}(P_{m+1} - P_m)A^{1/2}u|^{2/3}$$

$$= \sum_{m \geq 2} \left(\sum_{e^m \leq |k| < e^{m+1}} \frac{1}{\kappa_0^2 |k|^2} |\hat{v}(k)|^2 \right)^{1/3}$$

$$\leq \frac{1}{\kappa_0^{2/3}} \sum_{m \geq 2} \frac{1}{e^{2m/3}} \left(\sum_{e^m \leq |k| < e^{m+1}} |\hat{v}(k)|^2 \right)^{1/3}$$

$$\leq \frac{1}{\kappa_0^{2/3}} \left(\sum_{m \geq 2} \frac{1}{e^m} \right)^{2/3} \left(\sum_{m \geq 2} \sum_{e^m \leq |k| < e^{m+1}} |\hat{v}(k)|^2 \right)^{1/3}$$

$$\leq \left(c_1 \frac{|A^{1/2}u|}{\kappa_0} \right)^{2/3} < \infty.$$

Therefore, we have

$$|e^{b[\ln(\kappa_0^{-1}A^{1/2}+e)]^{1+\varepsilon}}u|^2 \leq I_1 + I_2$$

$$\leq I_1 + I_{21}^{1/4} I_{22}^{3/4} \nu$$

$$= I_1 + \left(|e^{b[\ln(A^{1/2}+e)]^{1+\varepsilon}}u|^2 \right)^{1/4} c_0^{1/2} \nu I_{22}^{3/4}$$

$$\leq I_1 + \frac{1}{4} |e^{b[\ln(A^{1/2}+e)]^{1+\varepsilon}}u|^2 + \frac{3}{4} c_0^{2/3} \nu^{4/3} I_{22},$$

and then

$$|e^{b[\ln(\kappa_0^{-1}A^{1/2}+e)]^{1+\varepsilon}}u|^2 \leq \frac{4}{3} I_1 + c_0^{2/3} I_{22}$$

$$\leq \frac{4}{3} c(\varepsilon)|u|^2 + (c_0 c_1)^{2/3} \nu^{4/3} \kappa_0^{-2/3} |A^{1/2}u|^{2/3}.$$

$\square$

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By the proceeding proposition, taking $\varepsilon = 1$ we obtain the following.

Theorem 4.1.6. *If $u \in \mathcal{C}(\sigma)$, then $u \in E_b$, where $b := \frac{1}{64\sigma}$.*

Combining Corollary 3.4.5 and Theorem 4.1.6, we obtain the following result.

Corollary 4.1.7. *If $0 \in \mathcal{A}$, then $g \in E_b$, where $b = \frac{1}{160 \ln \beta_3}$.*

4.2 The topological properties of the class $\mathcal{C}(\sigma)$

In this section we use the space $\mathcal{F} := \mathcal{C}^\infty \cap H$ with the Fréchet topology defined by the following metric

$$d(u,v) := \sum_{\alpha=1}^{\infty} \frac{1}{2^\alpha} \frac{|A^{\frac{\alpha}{2}}(u-v)|}{1 + |A^{\frac{\alpha}{2}}(u-v)|}. \tag{4.7}$$

Let

$$E_{b,n} := \{ u \in E_b, |u|_b \le n \}. \tag{4.8}$$

Lemma 4.2.1. $E_{b,n}$ *is nowhere dense in* $(\mathcal{F}, d)$.

Proof. First, we prove $i_b : E_{b,n} \to (\mathcal{F}, d)$ is compact. Clearly, for all $\alpha \in \mathbb{N}$, there exist a constant c_α such that

$$|A^{\frac{\alpha}{2}} u| \le c_\alpha |u|_b. \tag{4.9}$$

For any sequence $\{u_n\} \subset E_{b,n}$, we have that $\{|A^{\frac{\alpha}{2}} u_n|\}$ is bounded by (4.9). Therefore, there exists a subsequence $\{u_{n_m}\}$ which is convergent in $\mathcal{D}(A^{\frac{\alpha-1}{2}})$. Since this is true for any fixed $\alpha \in \mathbb{N}$, by the diagonal process, we obtain a subsequence, denoted by the same notation $\{u_{n_m}\}$ for convenience, which is convergent for any $\alpha \in \mathbb{N}$. Hence it is convergent in $\mathcal{C}^\infty$ with the metric $d(\cdot, \cdot)$. Therefore, i_b is compact. Then, it follows that $\overline{i_b(E_{b,n})}$ is compact in $(\mathcal{F}, d)$.

Secondly, suppose $i_b(E_{b,n})$ is not nowhere dense. Then there exists a ball $\overline{B(x_0, \varepsilon)} \subset \overline{i_b(E_{b,n})}$. Clearly, $\overline{B(x_0, \varepsilon)}$ is compact.

If $x_0 = 0$, this contradicts the extension of the classical Riesz's Lemma for normed spaces to locally convex topological vector space, since $\mathcal{C}^\infty$ is infinite-dimensional space.

If $x_0 \neq 0$, consider a convex open neighborhood $N_{x_0} \subset \overline{B(x_0, \varepsilon)}$. We have that $\overline{N_{x_0}}$ and $-\overline{N_{x_0}}$ both are compact and convex. Let $f : \overline{N_{x_0}} \times (-\overline{N_{x_0}}) \ni (x_1, x_2) \mapsto \frac{x_1 + x_2}{2} \in \mathcal{C}^\infty \cap H$. Clearly, f is continuous. Therefore the range $R(f)$ is compact. Since $\frac{1}{2}(\overline{N_{x_0}} + (-\overline{N_{x_0}}))$ is an open neighborhood of 0 in $R(f)$. For the same reason as above, we get a contradiction. $\qquad\qquad\square$

Due to the above lemma, it follows that

Theorem 4.2.2. $\cup_{m=1}^\infty \cup_{n=1}^\infty E_{\frac{b}{m}, n}$ *is of first (Baire) category in* $(\mathcal{F}, d)$.

From the above theorem and Corollary 4.1.7, we have that the conjecture that $g \neq 0$ implies $0 \notin \mathcal{A}$ is "almost" true in the following sense.

Theorem 4.2.3. *If* $0 \in \mathcal{A}$, *then* $g \in E_b$ *(b is defined in Corollary 4.1.7) where* E_b *has the property that* $E_b = \cup_{n=1}^\infty E_{b,n}$ *is of first (Baire) category in* $(\mathcal{F}, d)$.

4.3 An explicit criterion

In previous section, we found that generically 0 is not in the attractor $\mathcal{A}$ since if $0 \in \mathcal{A}$, then g must be in the set E_b which is of first category (See Theorem 4.2.3). One immediately asks the following question: if $g \in E_b$, will $0 \in \mathcal{A}$? We partially answer this question by presenting a concrete criterion that is both sufficient and necessary for $0 \in \mathcal{A}$.

To present our result, we need some preparation. First, we can choose $\delta > 0$ and $M > 0$, such that for every $u_0 \in \mathcal{A}$, $S(t)u_0$ is extendable to a holomorphic function on $\mathcal{S}(\delta) = \{z \in \mathbb{C} : |\Im z| < \delta\}$ with values in E_b, and $|S(t)u_0|_b \leq M$ for all $t \in \mathcal{S}(\delta)$.

Let $u_0 = 0 \in \mathcal{A}$; let $u(t) = S(t)u_0$ be the solution of the NSE; we use the conformal mapping (see [5])

$$\phi : \mathcal{S}(\delta) \to \Delta = \{T \in \mathbb{C} : |T| < 1\}$$

defined by the following formula

$$T = \phi(t) = \frac{\exp(\pi t/2\delta) - 1}{\exp(\pi t/2\delta) + 1}, \quad t \in \mathcal{S}(\delta)$$

with inverse given by

$$t = \phi^{-1}(T) = \frac{2\delta}{\pi}[\log(1 + T) - \log(1 - T)].$$

The function $U(T) = u(t)$ satisfies the ODE

$$\frac{dU}{dT} = \delta_0 \psi(T)\{g - \nu AU - B(U,U)\}, \quad T \in \Delta \qquad (4.10)$$

with initial value

$$U(0) = u_0$$

where

$$\psi(T) = \frac{1}{2}\left(\frac{1}{1 + T} + \frac{1}{1 - T}\right) = \frac{1}{1 - T^2}$$

and $\delta_0 = 4\delta/\pi$.

By the analyticity of the function $U(T)$, we may express it in a Taylor series

$$U(T) = U_0 + U_1 T + U_2 T^2 + \cdots . \tag{4.11}$$

Note that $U_0 = u_0$. The convergence radius of the series (4.11) is at least 1 if $u_0 \in \mathcal{A}$, and it may be less than 1 if $u_0 \notin \mathcal{A}$.

Combining the series expansion form (4.11) for $U(T)$ and the ODE (4.10), we get

$$\frac{d}{dT}\left(\sum_{n=0}^{\infty} U_n T^n\right) = \frac{\delta_0}{1 - T^2}\left[g - \nu A \sum_{n=0}^{\infty} U_n T^n - \sum_{n=0}^{\infty}\sum_{h+k=n} B(U_h, U_k)\right]$$

from which we get the following criterion for $0 \in \mathcal{A}$ (see [8]).

Theorem 4.3.1. $0 \in \mathcal{A}$ *if and only if the Taylor series*

$$\sum_{n=0}^{\infty} U_n T^n, \quad T \in \Delta \tag{4.12}$$

converges in $|\cdot|_b$ for all $T \in \Delta$ and the sum $U(T) = \sum_{n=0}^{\infty} U_n T^n$, for $|T| < 1$, satisfies an estimate $|U(T)|_b \leq M$, for some $M > 0$, where U_n are computed recursively according to

$$U_0 = 0, \quad U_1 = \delta_0 g, \quad U_2 = -\frac{\nu \delta_0^2}{2} A g$$

and for $n \geq 2$

$$U_{n+1} = \frac{n-1}{n+1} U_{n-1} - \frac{\nu \delta_0}{n+1} A U_n - \frac{\delta_0}{n+1} \sum_{\substack{h+k=n \\ h,k \geq 1}} B(U_k, U_h). \tag{4.13}$$

Remark 4.3.2. Several remarks are in order.

1. Notice that all the U_n's defined in the Theorem 4.3.1 depend only on g.

2. The application of the criterion given in Theorem 4.3.1 does not seem to be an easy task in general. We illustrate its use in the next section in the special case of forcing a single eigenvector of A.

4.4 The case of Kolmogorov forcing

An an application for the criterion given in Theorem 4.3.1, we show that if the force $g \neq 0$ is an eigenvector of the Stokes operator A, with corresponding eigenvalue $\lambda > 0$, then 0 cannot be in $\mathcal{A}$.

If $0 \in \mathcal{A}$, where $Ag = \lambda g$, then noting that

$$B(g, g) = 0 , \tag{4.14}$$

the following lemma immediately follows from the the recursive relation (4.13) given in Theorem 4.3.1.

Lemma 4.4.1. *For the coefficients U_n, we have*

$$U_n = p_n(\lambda)g, \quad n = 1, 2, 3, \cdots,$$

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where $p_n(\cdot)$ are polynomials satisfying the following relations:

$$p_1(\lambda) = \delta_0, \tag{4.15}$$

$$p_2(\lambda) = -\frac{\nu}{2}\lambda\delta_0^2, \tag{4.16}$$

$$p_{N+1}(\lambda) = \frac{N-1}{N+1}p_{N-1}(\lambda) - \frac{\nu\delta_0\lambda}{N+1}p_N(\lambda), N = 2, 3, \cdots. \tag{4.17}$$

Proof. By Theorem 4.3.1, we can obtain (4.15) and (4.16) easily. Assume by induction that $U_n = p_n(\lambda)g$ is valid for all $n \le N$, where $N \ge 2$. Then by (4.13),

$$(N+1)U_{N+1} = (N-1)U_{N-1} - \nu\delta_0 AU_N - \delta_0 \sum_{h+k=N} B(U_k, U_h)$$

$$= (N-1)p_{N-1}(\lambda)g - \nu\delta_0\lambda p_N(\lambda)g - \delta_0 \sum_{h+k=N} p_h(\lambda)p_k(\lambda)B(g, g)$$

$$= (N-1)p_{N-1}(\lambda)g - \nu\delta_0\lambda p_N(\lambda)g.$$

Therefore,

$$U_{N+1} = p_{N+1}(\lambda)g,$$

where,

$$p_{N+1}(\lambda) = \frac{N-1}{N+1}p_{N-1}(\lambda) - \frac{\nu\delta_0\lambda}{N+1}p_N(\lambda).$$

The proof is completed by the induction hypothesis. □

From the above lemma and Theorem 4.3.1, we conclude that if $0 \in \mathcal{A}$, then the solution $u(t)$ is of a special form, namely, $u(t) = \phi(t)g$, where $\phi(t)$ is a bounded real-valued function on $\mathbb{R}$. Clearly the function $\phi(t)$ must satisfy the following ODE:

$$\frac{d\phi}{dt} + \nu\lambda\phi = 1,$$

from which it follows that

$$\phi(t) = \frac{1}{\nu\lambda} + (\phi(0) - \frac{1}{\nu\lambda})e^{-\nu\lambda t}.$$

Boundedness of the solution $u(t)$ for all negative time implies that $\phi(0) = \frac{1}{\nu\lambda}$, and hence $u(t) \equiv \frac{g}{\nu\lambda}$. This contradicts $u(0) = \phi(0)g = 0$. Therefore, in this case, using the criterion and dynamics analysis, we obtain that 0 is not in $\mathcal{A}$.

5.   ON ESTIMATING THE KOLMOGOROV ε-ENTROPY OF THE WEAK GLOBAL ATTRACTOR OF INCOMPRESSIBLE THREE DIMENSIONAL NAIVER-STOKES EQUATIONS

In this chapter, we restrict our considerations to the three dimensional incompressible Navier-Stokes equation (2.1) ($d = 3$).

5.1   Specific preliminaries

We first list some inequalities needed in this chapter (see, e.g., [2, 21]),

$$k_0|w| \le |A^{1/2}w|, \ \text{for} \ w \in V, \tag{5.1}$$

$$\|w\|_\infty \le c_A|A^{1/2}w|^{1/2}|Aw|^{1/2}, \text{for} \ w \in \mathcal{D}(A), \tag{5.2}$$

known, respectively, as Poincaré and Agmon inequalities, with c_A being nondimensional constant, and the inequality satisfied by the bilinear term,

$$\|B(u,u)\|_{V'} \le c_S|u|^{1/2}\|u\|^{3/2}, \quad \forall u \in V, \tag{5.3}$$

where c_S is a nondimensional constant which depends only on constants coming from the Sobolev embedding theorem and additional abstract numbers.

Similar to (3.5), in the 3D case, an important nondimensional parameter associated with the strength of the driving force g is the Grashof number [10]

$$G = \frac{|g|}{\nu^2 \kappa_0^{3/2}} = \frac{|g|}{\nu^2 \lambda_1^{3/4}}. \tag{5.4}$$

A related nondimensional parameter that will be used in our paper is

$$G_* = \frac{|A^{-1/2}g|}{\nu^2 \kappa_0^{1/2}}. \tag{5.5}$$

By Poincaré inequality (5.1), one has

$$G_* = \frac{|A^{-1/2}g|}{\nu^2 \kappa_0^{1/2}} \le \frac{\kappa_0^{-1}|g|}{\nu^2 \kappa_0^{1/2}} = G,$$

where the equality occurs if and only if g is the first eigenvector of A.

5.2   Metrics on the weak global attractor

It is shown in [10] that

$$\mathcal{A}_{\mathrm{w}} \subset \{u \in H : |u(t)| \le G_* \nu \kappa_0^{-1/2}, \quad \forall t \in \mathbb{R}\}, \tag{5.6}$$

and $\mathcal{A}_{\mathrm{w}}$ is a totally bounded set in H_{w}. Recall the definition of the set $\mathcal{W}$ given in (2.12), clearly,

$$|u(t)| \le R_0, \quad \forall t \in \mathbb{R}, \ \forall u \in \mathcal{W}, \tag{5.7}$$

where

$$R_0 = \frac{|g|_{L^2}}{\nu \lambda_1} = \frac{\nu G}{\lambda_1^{1/4}}. \tag{5.8}$$

Therefore the weak topology of H for bounded sets is metrizable on $\mathcal{A}_{\mathrm{w}}$. Among the metrics that generate the weak topology on $\mathcal{A}_{\mathrm{w}}$, we first choose the following one,

$$\mathrm{d}_{\mathrm{w}}(u,v) := \nu^{-1} \kappa_0^{1/2} |e^{-e^{\kappa_0^{-1} A^{1/2}}} (u - v)|, \tag{5.9}$$

for $u, v \in \mathcal{A}_{\mathrm{w}}$.

Two other metric functions that will be used are

$$\mathrm{d_n}(u,v) = \nu^{-1}\kappa_0^{3/2}|A^{-1/2}(u-v)|, \tag{5.10}$$

and

$$\mathrm{d_s}(u,v) = \nu^{-1}\kappa_0^{1/2}|u-v|. \tag{5.11}$$

Note that the metrics $\mathrm{d_s}$, $\mathrm{d_n}$ and $\mathrm{d_w}$ do not have physical dimensions.

5.3 Kolmogorov ε-entropy

For a metric space (X,d), let $B_d(x_0,\rho)$ denote the ball with radius ρ centered at x_0,

$$B_d(x_0,\rho) := \{x \in X : d(x,x_0) < \rho\}.$$

Recall that a set F in the metric space (X,d) is totally bounded if, for any $\eta > 0$, there exists finitely many open balls of radius η whose union covers F. It is well known that all compact sets are totally bounded and that a metric space is compact if and only if it is complete and totally bounded.

The following definitions were introduced by Kolmogorov in [14].

Definition 5.3.1. Suppose F is a totally bounded, non-empty set in a metric space (X,d), and let $\varepsilon > 0$ be any real number.

(i) The system γ of sets $U \subset X$ is said to be an ε-covering of the set F, if the diameter of each $U \in \gamma$, $\sup_{u_0,u_1 \in U} d(u_0,u_1)$, is no greater than 2ε and

$$F \subset \bigcup_{U \in \gamma} U.$$

(ii) The Kolmogorov ε–entropy of the set F, denoted by $\mathcal{H}_\varepsilon(F)$, is defined as

$$\mathcal{H}_\varepsilon(F) := \ln \mathcal{N}_\varepsilon(F),$$

and $\mathcal{N}_\varepsilon(F) = \min\{\operatorname{card}(\gamma) : \gamma \text{ is an } \varepsilon\text{-covering of } F\}$, where $\operatorname{card}(\gamma)$ is the cardinal of the system γ.

(iii) The functional dimension of the set F is defined as

$$\operatorname{df}(F) := \lim_{\varepsilon \to 0^+} \frac{\ln \mathcal{H}_\varepsilon(A)}{\ln \ln \epsilon^{-1}} \tag{5.12}$$

5.4 An upper estimate of the Kolmogorov ε-entropy of $\mathcal{A}_\mathrm{w}$ using the metric d_w

We start with the following lemma. Recall the definition of $P_K H$ defind in (2.8).

Lemma 5.4.1. *Given* $r > 0$. *If* K *is chosen to be the integer satisfying* (5.13) *below, then* $\mathcal{N}_{2r}(B_{\mathrm{d_s}}(0, G_*)) \le \mathcal{N}_r(B_{\mathrm{d_s}}(0, G_*) \cap P_K H)$.

Proof. For any $u_1, u_2 \in B_{\mathrm{d_s}}(0, G_*)$, denote $u = u_1 - u_2$. By the definition of d_w in (5.9),

$$\begin{aligned}
\mathrm{d}_\mathrm{w}(u_1, u_2)^2 &= \nu^{-2} \kappa_0 |e^{-e^{\kappa_0^{-1} A^{1/2}}} u|^2 \\
&= \nu^{-2} \kappa_0 |e^{-e^{\kappa_0^{-1} A^{1/2}}} P_K u|^2 + \nu^{-2} \kappa_0 |e^{-e^{\kappa_0^{-1} A^{1/2}}} (I - P_K) u|^2 \\
&\le \nu^{-2} \kappa_0 |e^{-e^{\kappa_0^{-1} A^{1/2}}} P_K u|^2 + 2 e^{-2 e^K} G_*^2,
\end{aligned}$$

where $K \ge 1$ is an integer, and I denotes the identity operator.

Consequently, if K is chosen to be large enough such that

$$\ln\left(\ln \frac{\sqrt{2} G_*}{r}\right) \le K \le 1 + \ln\left(\ln \frac{\sqrt{2} G_*}{r}\right). \tag{5.13}$$

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then

$$\mathrm{d_w}(u_1, u_2)^2 \le \nu^{-2}\kappa_0 \big| e^{-e^{\kappa_0^{-1}A^{1/2}}} P_K u \big|^2 + r^2. \tag{5.14}$$

By (5.14), it follows that for any r−covering of $B_{\mathrm{d_s}}(0, G_*) \cap P_K H$, we can find a $2r$−covering of $B_{\mathrm{d_s}}(0, G_*)$ having the same number of sets. This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

A special finite covering of the set $B_{\mathrm{d_s}}(0, G_*) \cap P_K H$ with respect to the metric $\mathrm{d_s}$, defined in (5.11), and an upper bound of the cardinal number of this covering are given in the following lemma.

Lemma 5.4.2. *For any $\eta > 0$ and integer $K \ge 1$, we have,*

$$B_{\mathrm{d_s}}(0, G_*) \bigcap P_K H \subset \bigcup_{u_0 \in S} \overline{B_{\mathrm{d_s}}(u_0, \eta)}$$

where $S \subset B_{\mathrm{d_s}}(0, G_*)$ and the cardinal of S satisfies the estimate

$$card\ (S) \le \left(\frac{2G_*}{\eta} + 1 \right)^{\dim P_K H}.$$

Proof. Clearly,

$$B_{\mathrm{d_s}}(0, G_*) \cap P_K H = \{ u \in P_K H : |u| \le G_* \nu \kappa_0^{-1/2} \}.$$

Notice that $P_K H$ is a Banach space of finite dimension. For fixed $R > 0$, let $u_1, \cdots, u_{N_\eta}$ (N_η is called metric entropy, which is an upper bound for covering number) be a maximum set of points in $B_{\mathrm{d_s}}(0, R)$, the ball of radius $R > 0$ in $P_K H$ with $|u_i - u_j| > \eta$, for $i \ne j$, then the closed balls of radius $\eta/2$ centered at the $u_j's$ are disjoint, and their union lies within the ball of radius $R + \eta/2$ centered at the origin.

Consequently,

$$N_\eta \cdot (\eta/2)^{\dim P_K H} \le (R + \eta/2)^{\dim P_K H},$$

and thus,

$$\mathcal{N}_\eta(B_{\mathrm{d_s}}(0,R)) \le N_\eta \le \left(\frac{R + \eta/2}{\eta/2}\right)^{\dim P_K H} = \left(1 + \frac{2R}{\eta}\right)^{\dim P_K H}. \qquad (5.15)$$

The result follows by applying (5.15) with $R = G_*$. $\qquad\square$

Remark 5.4.3. An estimate for $\dim(P_K H) := \mathrm{card}\ \{k \in \mathbb{Z}^3 \smallsetminus \{0\} : |k| \le K\}$, the dimension of the space $P_K H$, is (see page 43-44 in [2]),

$$2\left(\frac{4\pi}{3}(K - \frac{\sqrt{3}}{2})^3 - 1\right) \le \dim(P_K H) \le 2\left(\frac{4\pi}{3}(K + \frac{\sqrt{3}}{2})^3 - 1\right),$$

Using (5.13), it follows that

$$\dim(P_K H) \le 2\left(\frac{4\pi}{3}(\frac{\sqrt{3}}{2} + 1 + \ln\ln\frac{\sqrt{2}G_*}{r})^3 - 1\right).$$

Lemma 5.4.4. *For any $v \in H$, and real number $\rho > 0$, the following holds,*

$$B_{\mathrm{d_s}}(v, \rho) \subset B_{\mathrm{d_w}}(v, \rho).$$

Proof. The result follows from the following inequalities

$$\left|e^{-e^{\kappa_0^{-1} A^{1/2}}} v\right| \le |v| \sup_{|k| \ge 1} e^{-e^{|k|}} = e^{-e}|v| \le |v|.$$

$\qquad\square$

Due to Lemma 5.4.2 and Lemma 5.4.4, the following is true,

$$B_{d_s}(0, G_*) \cap P_K H \subset \bigcup_{u_0 \in S} B_{d_w}(u_0, \eta).$$

Now, for any fixed $\varepsilon > 0$, based on the above lemmas, we are ready to the get a estimate on the Kolmogorov ε–entropy of the weak attractor $\mathcal{A}_w$ endowed with the metric d_w.

Collecting the above discussions, we obtain

Theorem 5.4.5. *An upper estimate of the Kolmogorov ε–entropy for the weak global attractor $\mathcal{A}_w$, endowed with the metric d_w, is given by the following explicit formula,*

$$\mathcal{H}_\varepsilon(\mathcal{A}_w) \leq 2\left(\frac{4\pi}{3}(\frac{\sqrt{3}}{2} + 1 + \ln\ln\frac{4G_*}{\varepsilon})^3 - 1\right)\ln(\frac{4\sqrt{2}G_*}{\varepsilon} + 1),$$

An immediate consequence of Theorem 5.4.5 is the following estimate regarding the functional dimension of $\mathcal{A}_w$.

Corollary 5.4.6. *The functional dimension of $\mathcal{A}_w$, endowed with the metric d_w, is bounded above by 1, i.e., $df(\mathcal{A}_w) \leq 1$.*

Remark 5.4.7. The upper bound given in Corollary 5.4.6 is consistent with a general result obtained in [19].

5.5   An upper estimate of the Kolmogorov ε-entropy of $\mathcal{A}_w$ using the metric d_n

With the metric d_w, we successfully estimated $\mathcal{H}_\varepsilon(\mathcal{A}_w)$ and $df(\mathcal{A}_w)$. However, notice that there is no simple physical meaning of the metric d_w, therefore, we will also use a more physical metric to measure the Kolmogorov ε-entropy of $\mathcal{A}_w$.

5.5.1 H^1-*bounded subsets of* $\mathcal{A}_w$

We define, for any given non-negative real number α, the set

$$\mathcal{A}_{\mathrm{w},\alpha} = \{u \in \mathcal{A}_{\mathrm{w}}; \, \|u\| \leq (1+\alpha)\nu\lambda_1^{1/4}G\}.$$

In the following, we will always denote by $t_0 \in \mathbb{R}$ an initial time for trajectories and consider $\mathcal{A}_{\mathrm{w}}$ defined as in (2.13).

The following proposition provides an estimate of the distance between any point in the attractor $\mathcal{A}_{\mathrm{w}}$ and the set $\mathcal{A}_{\mathrm{w},\alpha}$, with respect to the metric $\mathrm{d_w}$ defined in (5.9).

Proposition 5.5.1. *Let $\alpha \geq 0$ and $u_0 \in \mathcal{A}_{\mathrm{w}}$. Then, there exists $u_1 \in \mathcal{A}_{\mathrm{w},\alpha}$ such that*

$$\mathrm{d_n}(u_0, u_1) \leq r_\alpha,$$

where

$$r_\alpha = \frac{G}{\alpha^2}(2 + \alpha + c_S(1+\alpha)^{3/2}G) \tag{5.16}$$

and c_S is the same nondimensional constant from (5.3).

Proof. Consider $u \in \mathcal{W}$ such that $u(t_0) = u_0$. Define

$$t = t_0 + \frac{1}{\nu\lambda_1\alpha^2}.$$

Given $\varepsilon > 0$, let $t' \in \mathbb{R}$ be a point of strong continuity of u such that $t' \in (t_0 - \varepsilon, t_0]$. We first show that there exists $t_1 \in [t', t]$ such that $u(t_1) \in \mathcal{A}_{\mathrm{w},\alpha}$.

After applying the Cauchy-Schwarz, Poincaré and Young inequalities to the second term on the right-hand side of the energy inequality (2.11), we obtain that

$$\nu \int_{t'}^{t} \|u(s)\|^2 \mathrm{d}s \leq |u(t')|^2 + \frac{|g|^2}{\nu\lambda_1}(t - t').$$

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Then, from the uniform bound (5.7) and the definition of t, it follows that

$$\nu \int_{t'}^{t} \|u(s)\|^2 \mathrm{d}s \;\leq\; \frac{|g|^2}{\nu^2 \lambda_1^2} + \frac{|g|^2}{\nu \lambda_1}(t-t')$$

$$\leq\; \frac{|g|^2}{\nu \lambda_1}\alpha^2(t-t') + \frac{|g|^2}{\nu \lambda_1}(t-t').$$

Therefore,

$$\frac{1}{t-t'} \int_{t'}^{t} \|u(s)\|^2 \mathrm{d}s \leq (1+\alpha^2)\frac{|g|^2}{\nu^2 \lambda_1} = (1+\alpha^2)\nu^2 \lambda_1^{1/2} G^2, \tag{5.17}$$

which implies that there exists $t_1 \in [t',t]$ such that

$$\|u(t_1)\| \leq (1+\alpha)\nu \lambda_1^{1/4} G,$$

i.e., $u(t_1) \in \mathcal{A}_{\mathrm{w},\alpha}$.

Since $t_0, t_1 \in [t',t]$, we have

$$\|u(t_0) - u(t_1)\|_{V'} \leq \int_{t'}^{t} \left\| \frac{\mathrm{d}u}{\mathrm{d}s}(s) \right\|_{V'} \mathrm{d}s. \tag{5.18}$$

Moreover, from the functional equation (2.4), it follows that

$$\int_{t'}^{t} \left\| \frac{\mathrm{d}u}{\mathrm{d}s}(s) \right\|_{V'} \mathrm{d}s \leq \int_{t'}^{t} (\|g\|_{V'} + \nu\|u(s)\| + \|B(u(s),u(s))\|_{V'}) \mathrm{d}s$$

Now, the estimate (5.3) for the nonlinear term together with (5.7) and (5.8) yields

$$\int_{t'}^{t} \left\| \frac{\mathrm{d}u}{\mathrm{d}s}(s) \right\|_{V'} \mathrm{d}s \le \int_{t'}^{t} \left(\|g\|_{V'} + \nu \|u(s)\|_{H^1} + c_S \frac{\nu^{1/2} G^{1/2}}{\lambda_1^{1/8}} \|u(s)\|^{3/2} \right) \mathrm{d}s$$

$$\le \|g\|_{V'}(t - t') + \nu \left(\int_{t'}^{t} \|u(s)\|^2 \mathrm{d}s \right)^{1/2} (t - t')^{1/2} +$$

$$+ c_S \frac{\nu^{1/2} G^{1/2}}{\lambda_1^{1/8}} \left(\int_{t'}^{t} \|u(s)\|^2 \mathrm{d}s \right)^{3/4} (t - t')^{1/4}$$

$$= (t - t') \left[\|g\|_{V'} + \nu \left(\frac{1}{t - t'} \int_{t'}^{t} \|u(s)\|^2 \mathrm{d}s \right)^{1/2} + \right.$$

$$\left. + c_S \frac{\nu^{1/2} G^{1/2}}{\lambda_1^{1/8}} \left(\frac{1}{t - t'} \int_{t'}^{t} \|u(s)\|^2 \mathrm{d}s \right)^{3/4} \right], \quad (5.19)$$

where in the second inequality we applied Hölder's inequality to each term in the previous integral.

From the Poincaré inequality, it follows that

$$\|g\|_{V'} \le \frac{|g|_{L^2}}{\lambda_1^{1/2}} = \nu^2 \lambda_1^{1/4} G. \quad (5.20)$$

Denote $u_1 = u(t_1)$. Thus, using (5.17), (5.20) and the definitions of t and t', we obtain from (5.18) and (5.19) that

$$\mathrm{d}_\mathrm{n}(u_0, u_1) \le$$

$$\le \frac{\lambda_1^{3/4}}{\nu} (t - t')(\nu^2 \lambda_1^{1/4} G + (1 + \alpha)\nu^2 \lambda_1^{1/4} G + c_S(1 + \alpha)^{3/2} \nu^2 \lambda_1^{1/4} G^2)$$

$$= \nu \lambda_1 G(t - t')(2 + \alpha + c_S(1 + \alpha)^{3/2} G)$$

$$\le \nu \lambda_1 G \left(\frac{1}{\nu \lambda_1 \alpha^2} + \varepsilon \right) (2 + \alpha + c_S(1 + \alpha)^{3/2} G). \quad (5.21)$$

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But since $\varepsilon > 0$ is arbitrary, we may take $\varepsilon \to 0^+$ above and conclude that

$$d_n(u_0, u_1) \le \frac{G}{\alpha^2}(2 + \alpha + c_S(1 + \alpha)^{3/2}G).$$

$\square$

Remark 5.5.2. Consider $u_0 \in V$ and let u be a local strong solution of (2.4) defined on some interval $[t_0, t_1)$ and satisfying $u(t_0) = u_0$. As noticed in [10, Remark 2], if we denote

$$y(t) = \nu^{2/3}|g|^{2/3} + \|u(t)\|^2, \tag{5.22}$$

and $y_0 = y(t_0)$, then we may estimate the time t_1 of existence of u as

$$t_1 \ge t_0 + T(y_0),$$

where

$$T(y_0) = \frac{3\nu^3}{16c_0^2 y_0^2} \tag{5.23}$$

and c_0 is a certain universal constant. Moreover, we have

$$y(t) \le 2y_0, \quad \forall t \in [t_0, t_0 + T(y_0)]. \tag{5.24}$$

In particular, if $u_0 \in \mathcal{A}_{w,\alpha}$, then

$$y_0 \le \nu^{2/3}|g|^{2/3} + (1 + \alpha)^2\nu^2\lambda_1^{1/2}G^2 = \nu^2\lambda_1^{1/2}(G^{2/3} + (1 + \alpha)^2G^2).$$

which implies, from (5.22), (5.23) and (5.24), that

$$\|u(t)\| \le \sqrt{2}\nu\lambda_1^{1/4}(G^{1/3} + (1 + \alpha)G), \tag{5.25}$$

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for all $t \in \left[t_0, t_0 + \frac{3}{16c_0^2\nu\lambda_1(G^{2/3}+(1+\alpha)^2G^2)^2}\right]$.

For each non-negative real number α, we denote

$$R_\alpha = (1+\alpha)\nu\lambda_1^{1/4}G, \qquad (5.26)$$

$$R_\alpha' = \sqrt{2}\nu\lambda_1^{1/4}(G^{1/3} + (1+\alpha)G) \qquad (5.27)$$

and

$$\tau_\alpha = \frac{3}{16c_0^2\nu\lambda_1(G^{2/3} + (1+\alpha)^2G^2)^2}, \qquad (5.28)$$

where c_0 is the same universal constant from (5.23).

Note that $\mathcal{A}_{w,\alpha} \subset B_V(0, R_\alpha)$. Moreover, if $u_0 \in \mathcal{A}_{w,\alpha}$ and u is a local strong solution satisfying $u(t_0) = u_0$, then from Remark 5.5.2 we have that

$$\|u(t)\| \leq R_\alpha', \quad \forall t \in [t_0, t_0 + \tau_\alpha]. \qquad (5.29)$$

Consider the mapping

$$\begin{aligned} S(t) : V &\to V \\ u_0 &\mapsto u(t), \end{aligned}$$

where $u(t)$ is the value at time t of a solution of (2.4). From (5.29) and the well-known result of uniqueness of strong solutions among the class of weak solutions, it follows that $S(t)$ is well-defined on $\mathcal{A}_{w,\alpha}$, for every $t \in [t_0, t_0 + \tau_\alpha]$.

For our purpose, we need the following squeezing property of trajectories of the 3D Navier-Stokes equations. The proof can be found in [3, Theorem 2.1].

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Theorem 5.5.3 (Squeezing Property). *Let u and v be two strong solutions of* (2.4) *defined on some interval $[t_0, t_0 + T)$ and satisfying*

$$\|u(t)\| \le R, \quad \|v(t)\| \le R, \quad \forall t \in [t_0, t_0 + T).$$

Then, there exist nondimensional constants c_1, c_2 depending only on R, T, g, ν and Ω, such that, for any given $m \in \mathbb{N}$, if for every $t \in [t_0, t_0 + T)$,

$$|(I - P_m)(u(t) - v(t))| \ge |P_m(u(t) - v(t))|$$

then,

$$|u(t) - v(t)| \le c_1 |u(t_0) - v(t_0)| e^{-c_2 \nu \lambda_{m+1}(t - t_0)}.$$

In the following lemma, we give an estimate of the number $\mathcal{N}_\delta(S(t)\mathcal{A}_{w,\alpha})$, for $t \in (t_0, t_0 + \tau_\alpha]$. The proof follows the same steps as in [3, Proposition 2.1], and hence is omitted.

Lemma 5.5.4. *Let $t \in (t_0, t_0 + \tau_\alpha]$ and consider the set $S(t)\mathcal{A}_{w,\alpha}$. Then, given $\delta > 0$, we have*

$$\mathcal{N}_\delta(S(t)\mathcal{A}_{w,\alpha}) \le \left(\frac{16 R'_\alpha}{\nu \lambda_1^{1/4} \delta} \right)^{m_\delta}, \tag{5.30}$$

where

$$m_\delta \le c'_2 \left[\ln \left(\frac{c'_1 R_\alpha}{\nu \lambda_1^{1/4} \delta} \right) \right]^{3/2}, \tag{5.31}$$

and c'_1, c'_2 are universal constants.

The following result shows that, by restricting t to a possibly smaller interval than $(t_0, t_0 + \tau_\alpha]$, we can guarantee that, for every $u_0 \in \mathcal{A}_{w,\alpha}$, the distance between

$S(t)u_0$ and u_0 remains smaller than the number r_α defined in (5.16).

Lemma 5.5.5. *Let α be a non-negative real number and consider*

$$T_\alpha = \min\left\{\tau_\alpha, \frac{r_\alpha}{\nu\lambda_1 G + \lambda_1^{3/4}R'_\alpha + c_S\nu^{-1/2}\lambda_1^{5/8}G^{1/2}(R'_\alpha)^{3/2}}\right\}, \tag{5.32}$$

where τ_α is defined in (5.28), r_α is defined in (5.16) and c_0 is the same universal constant from (5.23). If $t \in (t_0, t_0 + T_\alpha]$, then

$$d_n(S(t)u_0, u_0) \leq r_\alpha, \quad \forall u_0 \in \mathcal{A}_{w,\alpha}.$$

Proof. Let $u_0 \in \mathcal{A}_{w,\alpha}$ and consider $u \in \mathcal{W}$ such that $u(t_0) = u_0$. Then, we have

$$d_n(S(t)u_0, u_0) = \frac{\lambda_1^{3/4}}{\nu}\|u(t) - u(t_0)\|_{V'} \leq$$
$$\leq \frac{\lambda_1^{3/4}}{\nu}\int_{t_0}^t \left\|\frac{du}{ds}\right\|_{V'} ds \leq \frac{\lambda_1^{3/4}}{\nu}\int_{t_0}^t (\|g\|_{V'} + \nu\|u\| + \|B(u,u)\|_{V'})ds \tag{5.33}$$

Now, since $t \leq t_0 + T_\alpha \leq t_0 + \tau_\alpha$, we have from (5.29) that

$$\|u(s)\| \leq R'_\alpha, \quad \forall s \in [t_0, t]. \tag{5.34}$$

Thus, using (5.3) and (5.34) in (5.33), we obtain that

$$d_n(S(t)u_0, u_0) \leq T_\alpha[\nu\lambda_1 G + \lambda_1^{3/4}R'_\alpha + c_S\nu^{-1/2}\lambda_1^{5/8}G^{1/2}(R'_\alpha)^{3/2}] \leq r_\alpha,$$

where the last inequality follows from the definition of T_α. $\qquad\square$

5.5.3 Covering of $\mathcal{A}_w$

Given any covering of $S(t)\mathcal{A}_{w,\alpha}$, $t \in (t_0, t_0 + T_\alpha]$, with balls of radius $\delta > 0$, a covering for $\mathcal{A}_w$ can be obtained by increasing the radius of these same balls to $2r_\alpha + \delta$. Therefore, $\mathcal{N}_\delta(S(t)\mathcal{A}_{w,\alpha})$ is an upper bound of $\mathcal{N}_{2r_\alpha+\delta}(\mathcal{A}_w)$.

Lemma 5.5.6. *Let $\alpha \geq 0$ and $t \in (t_0, t_0 + T_\alpha]$, with T_α as given in (5.32). Then, for every $\delta > 0$, the following inequality holds*

$$\mathcal{N}_{2r_\alpha+\delta}(\mathcal{A}_w) \leq \mathcal{N}_\delta(S(t)\mathcal{A}_{w,\alpha}), \tag{5.35}$$

where r_α is defined in (5.16).

Proof. Let $u_0 \in \mathcal{A}_w$. By Proposition 5.5.1, there exists $u_1 \in \mathcal{A}_{w,\alpha}$ such that

$$d_n(u_0, u_1) \leq r_\alpha. \tag{5.36}$$

Moreover, since $t \in (t_0, t_0 + T_\alpha]$, by Lemma 5.5.5, we have

$$d_n(S(t)u_1, u_1) \leq r_\alpha. \tag{5.37}$$

Now, given $\delta > 0$, consider a covering of $S(t)\mathcal{A}_{w,\alpha}$ with balls $B_w(y_j, \delta)$, $j = 1, \ldots, N$, centered at points $y_j \in \mathcal{A}_{w,\alpha}$. Then, there exists y_j such that

$$d_n(S(t)u_1, y_j) \leq \delta. \tag{5.38}$$

Therefore, from (5.36), (5.37) and (5.38), we obtain that

$$d_n(u_0, y_j) \leq d_n(u_0, u_1) + d_n(u_1, S(t)u_1) + d_n(S(t)u_1, y_j) \leq 2r_\alpha + \delta.$$

Thus, $u_0 \in B_{d_n}(y_j, 2r_\alpha + \delta)$. This proves (5.35). $\qquad\square$

Following from Lemmas 5.5.4 and 5.5.6, we can easily find an upper bound the Kolmogorov ε-entropy of $\mathcal{A}_{\mathrm{w}}$, for any $\varepsilon > 0$.

Theorem 5.5.7. *For every $\varepsilon > 0$, the following inequality holds*

$$\mathcal{H}_\varepsilon(\mathcal{A}_{\mathrm{w}}) \leq c \left[\ln \left(\frac{c}{\varepsilon} + \frac{c}{\varepsilon^3} \right) \right]^{3/2} \ln \left(\frac{c}{\varepsilon} + \frac{c}{\varepsilon^3} \right), \tag{5.39}$$

where c is a nondimensional constant depending on G.

Proof. Let $\varepsilon > 0$. Note that, by the definition of r_α in (5.16), we have

$$r_\alpha \sim \frac{c_S G^2}{\alpha^{1/2}} \quad \text{as } \alpha \to \infty.$$

Therefore, there exists in particular a constant M such that

$$r_\alpha \leq M \frac{c_S G^2}{\alpha^{1/2}}, \quad \forall \alpha \geq 0. \tag{5.40}$$

Choose α given by

$$\alpha = \left(\frac{4M c_S G^2}{\varepsilon} \right)^2. \tag{5.41}$$

With this choice of α, it follows from (5.40) that

$$r_\alpha \leq \frac{\varepsilon}{4}.$$

Now, choose $\delta = \varepsilon/2$. Thus, we have

$$2r_\alpha + \delta \leq \varepsilon,$$

and, using Lemmas 5.5.4 and 5.5.6, we obtain that

$$\mathcal{N}_\varepsilon(\mathcal{A}_\mathrm{w}) \leq \mathcal{N}_{2r_\alpha+\delta}(\mathcal{A}_\mathrm{w}) \leq \left(\frac{16R'_\alpha}{\nu\lambda_1^{1/4}\delta}\right)^{m_\delta},$$

with m_δ satisfying (5.31). Therefore,

$$\mathcal{H}_\varepsilon(\mathcal{A}_\mathrm{w}) = \ln(\mathcal{N}_\varepsilon(\mathcal{A}_\mathrm{w})) \leq c'_2\left[\ln\left(\frac{2c'_1 R_\alpha}{\nu\lambda_1^{1/4}\varepsilon}\right)\right]^{3/2}\ln\left(\frac{32R'_\alpha}{\nu\lambda_1^{1/4}\varepsilon}\right),$$

where c'_1 and c'_2 are the same nondimensional constants from (5.31).

Now, plugging the choice of α given in (5.41) into the definitions of R_α and R'_α given in (5.26) and (5.27), respectively, one immediately obtains (5.39). $\qquad\square$

Using the estimate of $\mathcal{H}_\varepsilon(\mathcal{A}_\mathrm{w})$ obtained in Theorem 5.5.7 into the definition of the functional dimension $\mathrm{df}(\mathcal{A}_\mathrm{w})$ given in (5.12), one easily obtains the estimate (5.42) given below.

Theorem 5.5.8. $\mathcal{A}_\mathrm{w}$ *has finite functional dimension with respect to the metric* d_n. *Moreover,*

$$\mathrm{df}(\mathcal{A}_\mathrm{w}) \leq \frac{5}{2}. \tag{5.42}$$

6. ON SOLUTIONS OF TWO DIMENSIONAL NAVIER-STOKES EQUATIONS WITH CONSTANT ENERGY AND ENSTROPY*

In this chapter, we study the solutions in the global attractor of two dimensional Navier-Stokes equations which projects onto one single point onto the energy-enstrophy plane..

6.1   Specific preliminaries

After the rescaling by

$$\tilde{u} = \frac{u}{\nu\kappa_0}, \tilde{t} = \nu\kappa_0^2 t, \tilde{x} = \kappa_0 x, \tilde{\Omega} = [0, 2\pi]^2, \tilde{g} = \frac{g}{\nu^2 \kappa_0^3},$$

the corresponding dimensionless functional form of (2.4) is

$$\frac{d\tilde{u}}{d\tilde{t}} + \tilde{A}\tilde{u} + \tilde{B}(\tilde{u}, \tilde{u}) = \tilde{g},$$

where $\tilde{A}$ and $\tilde{B}$ are defined through the rescaled Laplacian and gradient operators. Henceforth, for simplicity, we assume in this chapter that $\nu = 1$, $L = 2\pi$ and $\kappa_0 = 1$. The functional form then becomes

$$\frac{du}{dt} + Au + B(u, u) = g \qquad (6.1)$$

after dropping the tildes.

Also, recall the following form of Navier-Stokes equations in terms of the Fouier

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series coefficients,

$$\frac{d}{dt}\hat{u}(k,t) = \hat{g}(k) - |k|^2 \hat{u}(k,t) - \hat{B}(u,u)(k), \text{ for } k \in \mathbb{Z}^2 \smallsetminus \{0\}, \tag{6.2}$$

where $\hat{B}(u,v)(k)$ is given by

$$\hat{B}(u,v)(k) = i \sum_{j \in \mathbb{Z}^2 \smallsetminus \{0\}} [(\hat{u}(k-j) \cdot j)\hat{v}(j) - \frac{(\hat{u}(k-j) \cdot j)(\hat{v}(j) \cdot k)}{k \cdot k} k], \tag{6.3}$$

for $k \in \mathbb{Z}^2 \smallsetminus \{0\}$.

Throughout this chapter , we assume that

$$Ag = \lambda g, \text{ where } \lambda \in sp(A) \text{ and } \lambda > 1. \tag{6.4}$$

Since for eigenvector g of Stokes operator A, we have $B(g,g) = 0$. This yields that there exists $u_* := g/\lambda$ that satisfies $Au + B(u,u) = g$, the stationary NSE. For $\lambda = 1$, see [16]. Then, the energy and enstropy balances are

$$\frac{1}{2}\frac{d}{dt}|u|^2 = -\|u\|^2 + (g,u), \tag{6.5}$$

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 = -|Au|^2 + (g,Au) = -|Au|^2 + \lambda(g,u). \tag{6.6}$$

Well known algebraic properties of B(see, e.g., [4]) include the orthogonality relation

$$(B(u,v),w) = -(B(u,w),v), \forall u \in H, v,w \in V, \tag{6.7}$$

in particular,

$$(B(u,v),v) = 0, \forall u \in H, v \in V, \tag{6.8}$$

as well as,

$$(B(u,u),Au) = 0, \forall u \in D(A). \tag{6.9}$$

Also the strong form of enstrophy invariance holds (see, e.g., [4]),

$$(B(Av,v),u) = (B(u,v),Av), \forall u \in H, v \in D(A). \tag{6.10}$$

We first recall the definition of ghost solutions first introduced in [7].

Definition 6.1.1. A ghost solution is a nonstationary solution $u(\cdot) \in \mathcal{A}$, such that

$$\dot{e}(t) \equiv \dot{E}(t) \equiv 0, \forall t \in \mathbb{R},$$

where $e := |u(t)|^2$ and $E := \|u(t)\|^2$ are referred to as the energy and enstrophy, respectively.

Using (6.5) and (6.6), one immediately finds that a ghost solution satisfies

$$E = |A^{1/2}u|^2 = (g,u) = \frac{1}{\lambda}|Au|^2, \tag{6.11}$$

hence,

$$P := |Au|^2 = (Au,g) = \lambda E. \tag{6.12}$$

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We also recall the following elementary relations regarding the ghost solutions, whose proof can be found in [7].

Proposition 6.1.2. *If $u(t) \in \mathcal{A}$ satisfies* (6.11)*, then the following hold,*

$$(\dot{u}, g) = (\dot{u}, Au) = (\dot{u}, u) = 0,$$

$$|B(u, u)|^2 + |Au|^2 = |\dot{u}|^2 + |g|^2,$$

$$(B(u, u), g) = |B(u, u)|^2 - |\dot{u}|^2 = |g|^2 - |Au|^2,$$

$$|B(u, u) - g/2|^2 = |\dot{u} \pm g/2|^2,$$

$$\frac{d}{dt}(|B(u, u)|^2) = \frac{d}{dt}(|\dot{u}|^2),$$

$$(B(u, u), \dot{u}) + |\dot{u}|^2 = 0.$$

6.2   Necessary inequalities for solutions to be ghosts

Lemma 6.2.1. *For any ghost solution $u(t) \in \mathcal{A}$, one has*

$$P = |Au(t)|^2 < |g|^2 =: G^2,$$

Proof. It follows from Proposition 6.1.2 that

$$|B(u,u)|^2 - |\dot{u}|^2 = G^2 - P, \qquad (6.13)$$

and that

$$|\dot{u}|^2 + (B(u,u), \dot{u}) = 0,$$

which implies

$$|\dot{u}|^2 \le |B(u,u)||\dot{u}| \le \frac{|B(u,u)|^2}{2} + \frac{|\dot{u}|^2}{2},$$

hence,

$$|\dot{u}|^2 \le |B(u,u)|^2,$$

therefore, (6.13) gives $P \le G^2$.

Moreover, the equality occurs only when the equality occurs for the Cauchy-Schwarz inequality; so, if $P = G^2$, then

$$B(u,u) = c\dot{u}, \ \ \text{with} \ |c| = 1,$$

and by Proposition 6.1.2

$$0 = |\dot{u}|^2 + (B(u,u), \dot{u})$$
$$= (1+c)|\dot{u}|^2.$$

If $c = -1$, then

$$\dot{u} + B(u, u) = 0,$$

so by Proposition 6.1.2, $Au = g$, and, consequently, $u = A^{-1}g = g/\lambda = u_*$.

If $\dot{u} = 0$, then $B(u, u) = 0$, so, $Au = g$, which also gives $u = g/\lambda$.

$\square$

Lemma 6.2.2. *The following inequality holds for the energy* $e = |u|^2$ *and enstrophy* $E = |A^{1/2}u|^2$,

$$E \leq \lambda e \leq \lambda E.$$

Proof. Indeed, by (6.11)

$$0 \leq |Au - \lambda u|^2$$
$$= |Au|^2 - 2\lambda|A^{1/2}u|^2 + \lambda^2|u|^2$$
$$= \lambda(\lambda e - E),$$

hence, $\lambda e - E \geq 0$. For the upper bound $\lambda e \leq \lambda E$, recall the Poincaré inequality, namely, $|u| \leq |A^{1/2}u|$.
$\square$

In the discussion to follow, we will frequently use the sign of the differences of such two quantities, as P and G^2, E and λe. For convenience, we put them together in the next theorem.

Theorem 6.2.3. *The following are equivalent,*

$P < G^2$,

$E < \lambda e$,

$E^2 < eG^2$,

$E^2 < eP$.

Moreover, if any one of the above inequlities is replaced by equality, then the other three will also become equalities, and the equality occurs only if $u = u_*$.

Proof. From Lemma 6.2.1, we know that $P \le G^2$, with equality if and only if $u = u_*$.

From Lemma 6.2.2, clearly, $E \le \lambda e$.

If $E = \lambda e$, then, from the proof of Lemma 6.2.2, we see that

$$Au = \lambda u,$$

so that u is an eigenvector of the operator A, hence $B(u, u) = 0$. By taking inner product of the NSE with u, using Proposition 6.1.2, we get

$$(Au, g) = (g, g), \tag{6.14}$$

that is,

$$P = G^2,$$

thus, $u = u_*$, as desired.

Conversely, if $u = u_*$, then, since $g = \lambda u_* = \lambda u$, it holds that $E = (u, g) = \lambda(u, u) = \lambda e$.

It follows from the Cauchy-Schwarz inequality that $E = |A^{1/2}u|^2 = (u, Au) \le |u||Au| \le e^{1/2}P^{1/2}$, hence also $E^2 \le eG^2$, since $P \le G^2$.

If $E^2 = eP$, then by the condition for equality in the Cauchy-Schwarz inequality, $Au = \mu u$, for some μ. Then, combining $P = (Au, g) = \mu(u, g) = \mu E$ with (6.11), we

have $\mu = \lambda$. For the same reason as given in case (ii), we have $u = u_*$.

If $E^2 = eG^2$, then, since $E^2 \leq eP$, we have $G^2 \leq P$, hence it must happen that $P = G^2$.

Conversely, if $u = u_*$, then both $E^2 = eG^2$ and $E^2 = eP$ will hold.

$\square$

Henceforth, the following relations are always true for the ghost solutions defined in our paper, namely,

$$P < G^2, E < \lambda e, E^2 < eP, E^2 < eG^2. \tag{6.15}$$

Notice that these inequalities are strict because the ghost solutions are nonstationary. The parabola bound, namely, $E^2 \leq eG^2$ in our notation, has been obtained in [4] for all solutions in the global attractor of 2D NSE, regardless of the force.

Remark 6.2.4. In [23], we showed the uniqueness of the ghost solutions in the sense that there exists a unitary transformation between any two possible ghost solutions.

6.3 Chained ghost solutions

let $P_{012} = P_{012}(t)$ denote the orthogonal projection of H onto the space $H_{012} = \text{span}\{g, u, Au\} = \mathbb{R}g + \mathbb{R}u + \mathbb{R}Au..$ For $\omega \in H$, let $P_{012}\omega = \omega_1 g + \omega_2 u + \omega_3 Au$, then

$$\begin{pmatrix} (\omega, g) \\ (\omega, u) \\ (\omega, Au) \end{pmatrix} = M \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}, \tag{6.16}$$

where, using (6.11), (6.12), we have

$$M = M(e, E) = \begin{pmatrix} G^2 & E & P \\ E & e & E \\ P & E & P \end{pmatrix}, \tag{6.17}$$

with determinant

$$det(M) = (\lambda e - E)E(G^2 - P) > 0.$$

If we take $\omega = B(u, u)$, the system (6.16) becomes

$$\begin{pmatrix} G^2 - P \\ 0 \\ 0 \end{pmatrix} = M \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix},$$

so

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = M^{-1} \begin{pmatrix} G^2 - P \\ 0 \\ 0 \end{pmatrix},$$

that is,

$$P_{012} B(u(t), u(t)) = (g, u(t), Au(t)) M^{-1} \begin{pmatrix} G^2 - P \\ 0 \\ 0 \end{pmatrix}.$$

By (6.17), ω_1, ω_2, ω_3 are time independent. Actually,

$$P_{012}B(u,u) = g - Au,$$

and

$$|P_{012}B(u,u)|^2 = G^2 - P.$$

Also, notice that, since

$$\dot{u}(t) + B(u(t), u(t)) = g - Au(t) \in H_{012},$$

we have

$$(1 - P_{012})B(u(t), u(t)) = -\dot{u}(t), \forall t \in \mathbb{R}.$$

If we take $\omega = A^2 u$, then (6.16) becomes

$$M \begin{pmatrix} \gamma \\ \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \lambda P \\ P \\ |A^{3/2}u|^2 \end{pmatrix},$$

which, using (6.17), gives,

$$\begin{cases} G^2\gamma + E\beta + P\alpha = \lambda P \\ E\gamma + e\beta + E\alpha = P \\ P\gamma + E\beta + P\alpha = |A^{3/2}u|^2 \end{cases} \tag{6.18}$$

Solving (6.18), one gets

$$\gamma = \frac{\lambda P - |A^{3/2}u|^2}{G^2 - P},$$

$$\beta = \frac{\lambda P - |A^{3/2}u|^2}{\lambda e - E},$$

$$\alpha = \frac{P}{E} - \frac{\lambda P - |A^{3/2}u|^2}{G^2 - P} - \frac{e}{E}\frac{\lambda P - |A^{3/2}u|^2}{\lambda e - E}.$$

That is,

$$P_{012} A^2 u = (g, u, Au) M^{-1} \begin{pmatrix} \lambda P \\ P \\ |A^{3/2}u|^2 \end{pmatrix}$$

$$= \gamma g + \beta u + \alpha Au.$$

Notice that, by (6.11)

$$|A^{3/2}u - \lambda A^{1/2}u|^2 = |A^{3/2}u|^2 - \lambda P \geq 0,$$

with equality if and only if

$$A^{3/2}u = \lambda A^{1/2}u,$$

which, by (6.11) and Theorem 6.2.3(ii), holds if and only if

$$u = u_* = g/\lambda.$$

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Thus

$$\gamma < 0, \beta < 0, \text{and } \alpha > \lambda > 0;$$

and one of the above inequalities becomes an equality if and only if $u = u_*$.

Summarizing the above discussion, we obtain the following proposition.

Proposition 6.3.1. *For a ghost solution $u(\cdot)$,*

(i) If $A^2 u(t) \in H_{012}$, for some $t \in \mathbb{R}$, then $A^2 u(t) = \gamma g + \beta u(t) + \alpha A u(t)$ with

$$\gamma = \gamma(t) = \frac{\lambda P - |A^{3/2} u(t)|^2}{G^2 - P}, \tag{6.19}$$

$$\beta = \beta(t) = \frac{\lambda P - |A^{3/2} u(t)|^2}{\lambda e - E}, \tag{6.20}$$

$$\begin{aligned}
\alpha = \alpha(t) &= \frac{P}{E} - \frac{\lambda P - |A^{3/2} u(t)|^2}{G^2 - P} - \frac{e}{E} \frac{\lambda P - |A^{3/2} u(t)|^2}{\lambda e - E} \tag{6.21} \\
&= \lambda - \gamma(t) - \frac{e}{E} \beta(t).
\end{aligned}$$

(ii) $|A^{3/2} u(t)|^2 = \lambda P$ at some $t \in \mathbb{R}$, if and only if $u = u_*$.

Remark 6.3.2. It is easy to check that $\alpha^2 + 4\beta > 0$ is always true. Indeed, using (6.19), (6.20), (6.21),

$$\begin{aligned}
\alpha^2 + 4\beta &= (\lambda - \gamma - \frac{e}{E} \beta)^2 + 4\beta \tag{6.22} \\
&= \left(\frac{1}{G^2 - P} + \frac{e/E}{2e - E} \right)^2 \rho^2 \\
&\quad + 4\rho \left(\frac{1 - e/E}{2e - E} - \frac{1}{G^2 - P} \right) + 4,
\end{aligned}$$

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where

$$\rho := \lambda P - |A^{3/2}v|^2 < 0.$$

A simple calculation shows that the discriminant of the quadratic form on the right hand side of (6.22) is always negative, therefore, $\alpha^2 + 4\beta$ is always positive.

In the following, we will discuss the projection of the function A^2u onto the space H_{012}. We define E_μ as the eigensapce projector corresponding to the eigenvalue $\mu \in sp(A)$.

Theorem 6.3.3. *If the set $C := \{t \in \mathbb{R} : A^2u(t) \in H_{012}\} \subset \mathbb{R}$ has an accumulation point $t^* \in \mathbb{R}$, then $\alpha(t), \beta(t), \gamma(t)$ are constants in time.*

Proof. For any $t \in C$, from Proposition 6.3.1, we have $A^2u(t) = \gamma(t)g + \beta(t)u(t) + \alpha(t)Au(t)$. Then for any $\mu \neq \lambda$, $\mu \in sp(A)$,

$$(\mu^2 - \alpha(t)\mu - \beta(t))E_\mu u(t) = 0.$$

By Remark 6.3.2 it follows that that there are two real roots

$$\mu_\pm(t) = \frac{\alpha(t) \pm \sqrt{\alpha(t)^2 + 4\beta(t)}}{2}.$$

Now, analyticity of solutions in global attractor ([20]) implies that $\mu_\pm(t)$ are both analytic in t. Since the set C is assumed to have an accumulation point, we conclude that $\mu_\pm(t)$ are time independent, so $\alpha(t) = \mu_+(t) + \mu_-(t)$, $\beta(t) = -\mu_+(t)\mu_-(t)$ (and hence also $\gamma(t)$), are all time independent. $\qquad\square$

Theorem 6.3.4. *If the set C has an accumulation point $t^* \in \mathbb{R}$, then $A^2u(t) \in H_{012}$ for all $t \in \mathbb{R}$.*

Proof. Denote $R(t) = A^2 u(t) - \gamma(t)g - \beta(t)u(t) - \alpha(t)Au(t)$, analyticity of solutions in the global attractor ([20]) implies that $R(t)$ is analytic in time. Since C is assumed to have an accumulation point, we conclude that $R(t)$ are time independent, so $R(t) \equiv 0$, for all $t \in \mathbb{R}$. $\qquad\square$

Definition 6.3.5. A chained ghost solution is a ghost solution satisfying the following chained relation

$$A^2 u(t) = \gamma g + \beta u(t) + \alpha Au(t), \forall t \in \mathbb{R}, \qquad (6.23)$$

for some time independent coefficients γ, β, and α.

Remark 6.3.6. For any given chained ghost solution $u(t)$, if it exists, it follows easily that we also have the chained relation for $A^n u(t)$, $\forall n \in \mathbb{N}_+$, that is,

$$A^n u(t) = \gamma_n g + \beta_n u(t) + \alpha_n Au(t), \forall t \in \mathbb{R},$$

where γ_n, β_n, and α_n are all time independent.

From now on, we will focus on chained ghost solutions.

Theorem 6.3.7. *The chained ghost solution $u(t)$ has the following decomposition,*

$$u(t) = E_{\mu_+}u(t) + E_{\mu_-}u(t) + \eta g \qquad (6.24)$$

$$= u_+(t) + u_-(t) + \eta g, \forall t \in \mathbb{R}$$

where $\eta = E/G^2$.

Moreover,

$$|u_+|^2 = \frac{\lambda - \mu_-}{\mu_+(\mu_+ - \mu_-)} E(1 - P/G^2), \qquad (6.25)$$

$$|u_-|^2 = \frac{\lambda - \mu_+}{\mu_-(\mu_- - \mu_+)}E(1 - P/G^2). \tag{6.26}$$

Proof. If $(1 - P_{012})A^2u(t)$ vanishes, then

$$A^2u = \alpha Au + \beta u + \gamma g,$$

where α, β, and γ are as given in Definition 6.3.5. Equivalently,

$$(A^2 - \alpha A - \beta)u = \gamma g. \tag{6.27}$$

Clearly, (6.27) implies

$$(\lambda^2 - \alpha\lambda - \beta)E_\lambda u = \gamma g. \tag{6.28}$$

Here we discuss different possibilities.

(i) When $\lambda^2 - \alpha\lambda - \beta = 0$, whence $\gamma = 0$. Then (6.18) becomes

$$\begin{cases} E\beta + P\alpha = \lambda P \\ e\beta + E\alpha = P \\ E\beta + P\alpha = |A^{3/2}u|^2. \end{cases} \tag{6.29}$$

By (6.12), the first and second equations in (6.29) give

$$(\lambda e - E)\beta = 0,$$

so that by Theorem 6.2.3(ii), we have $\beta = 0$; Then the second and the third equations

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in (6.29), together with (6.12), imply

$$\lambda P = |A^{3/2}u|^2,$$

which contradicts Proposition 6.3.1 (ii) . Therefore, $\lambda^2 - \alpha\lambda - \beta$ will never vanish.

So, we need to consider the following case only that

(ii) $\lambda^2 - \alpha\lambda - \beta$ is never zero.

From (6.28),

$$E_\lambda u = \frac{\gamma}{\lambda^2 - \alpha\lambda - \beta} g;$$

and for all $\mu \in sp(A)$, with $\mu \neq \lambda$,

$$(\mu^2 - \alpha\mu - \beta)E_\mu u = 0. \tag{6.30}$$

By Remark 6.3.2, $\Delta := \alpha^2 + 4\beta > 0$, then, (6.30) gives

$$(\mu - \mu_+)(\mu - \mu_-)E_\mu u = 0,$$

where $\mu_\pm := \frac{1}{2}(\alpha \pm p)$, with $p = \Delta^{1/2}$. Necessarily, $\lambda \notin \{\mu_+, \mu_-\}$. Also, notice that if $\mu \notin \{\mu_+, \mu_-\}$, then $E_\mu u = 0$, thus the decomposition of u in terms of the eigenvectors of the operator A only contains three possible nonzero components. Without loss of generality, we may assume that $\mu_+ > \mu_-$. Therefore, in this case, we have,

$$u = u_+ + u_- + \eta g, \tag{6.31}$$

where $u_+ = E_{\mu_+}u$ and $u_- = E_{\mu_-}u$.

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To determine the value of η, we notice that

$$B(u,u) = (1 - \lambda\eta)g - \mu_+ u_+ - \mu_- u_-. \tag{6.32}$$

Taking the inner product of (6.32) with g and using (6.11), (6.12), Proposition 6.1.2 , we have $\eta = E/G^2$, that is,

$$u = u_+ + u_- + \frac{E}{G^2}g. \tag{6.33}$$

Now, using (6.33) and

$$(B(u,u),u) = 0$$

gives

$$|A^{1/2}u|^2 = \lambda\frac{|A^{1/2}u|^4}{G^2} + \mu_+|u_+|^2 + \mu_-|u_-|^2; \tag{6.34}$$

and,

$$(B(u,u), Au) = 0,$$

gives

$$\lambda|A^{1/2}u|^2 = \lambda^2\frac{|A^{1/2}u|^4}{G^2} + \mu_+^2|u_+|^2 + \mu_-^2|u_-|^2; \tag{6.35}$$

From (6.34) and (6.35), we get,

$$\mu_+|u_+|^2(\lambda - \mu_+) + \mu_-|u_-|^2(\lambda - \mu_-) = 0, \tag{6.36}$$

which implies

$$\mu_- < \lambda < \mu_+. \tag{6.37}$$

Using (6.34), (6.35) to solve for $|u_-|^2$ and $|u_+|^2$ yields (6.25), (6.26). □

6.4   A finite Galerkin system

The decomposition (6.31) in Theorem 6.3.7 could be used to exploit the relations between different wavevectors when NSE is written in the form of Fourier modes, as shown in the next theorem.

Theorem 6.4.1. *Any chained ghost solution $u(t)$ satisfies the following Galerkin system,*

$$\frac{d}{dt}\hat{u}(k,t) = \hat{g}(k) - |k|^2\hat{u}(k,t) - [Q_k(\hat{u},\hat{u}) - \frac{Q_k(\hat{u},\hat{u}) \cdot k}{|k|^2}k], \tag{6.38}$$

where

$$Q_k(\hat{u},\hat{u}) = \sum_{\substack{h \in \mathbb{Z}^2 \setminus \{0\} \\ |h|^2, |k-h|^2 \in \{\lambda, \mu_+, \mu_-\}}} i(\hat{u}(h,t) \cdot k)\hat{u}(k-h,t), \ \text{for } |k|^2 \in \{\lambda, \mu_+, \mu_-\}$$

and

$$\hat{u}(k,t) = 0, \text{ if } |k|^2 \notin \{\lambda, \mu_+, \mu_-\},$$

$$\hat{g}(k) = 0, \text{ if } |k|^2 \neq \lambda.$$

Proof. It follows immediately from the NSE (6.1),

$$
\begin{cases}
\lambda \eta g + E_\lambda B(u, u) = g \\
\frac{d}{dt} u_+ + \mu_+ u_+ + E_{\mu_+} B(u, u) = 0 \\
\frac{d}{dt} u_- + \mu_- u_- + E_{\mu_-} B(u, u) = 0
\end{cases}
\tag{6.39}
$$

Notice that (6.39) is a finite system of differential algebraic equation involving the unknown components for u_+ and u_-, where $\mu_- < \lambda < \mu_+$. Clearly, $E_\mu B(u, u) = 0$, for all $\mu \notin \{\lambda, \mu_+, \mu_-\}$. When (6.39) is written in terms of the Fourier coefficients, invoking (6.2) and (6.3), we get (6.38).

$\square$

Remark 6.4.2. Denote $\hat{E} = E_{\mu_+} + E_{\mu_-} + E_\lambda$, then we could see from the proof of Theorem 6.4.1 that

$$
(1 - \hat{E}) B(u, u) \equiv 0,
$$

for u being a chained ghost solution.

Theorem 6.4.1 implies the following geometric characterization for chained ghosts.

Theorem 6.4.3. *If $u(t)$ is a solution of the Galerkin system in the interval $(0, \bar{t})$ for some $\bar{t} > 0$, satisfying $(1 - \hat{E}) B(u(t), u(t)) \equiv 0$, for any $t \in (0, \bar{t})$, then $u(t)$ can be extended to be a chained ghost solution.*

Proof. Taking the inner product of (6.38) with $\hat{u}(t)$ in the Fourier space gives that $|\hat{u}(t)|$ is constant in $t \in (0, \bar{t})$; taking the inner product of (6.38) with $A\hat{u}(t)$ in the Fourier space gives $|A^{1/2}\hat{u}(t)|$ is also constant in $t \in (0, \bar{t})$.

By the analyticity of $u(t)$ and $(1 - \hat{E}) B(u(t), u(t)) \equiv 0$, for any $t \in (0, \bar{t})$, we conclude that $u(t)$ can be extended to be a chained ghost solution. $\square$

6.5 The case when $\lambda = 2$

In the particular case when $\lambda = 2$, by (6.37), we must have $\mu_- = 1$, and $\mu_+ = -\beta$. Hence, solving $A^2 u = \gamma g + \beta u + \alpha A u$, using (6.33), we have,

$$\gamma g + \beta u + \alpha A u = \gamma g + \beta \left(\frac{E}{G^2} g + u_+ + u_- \right)$$
$$+ \alpha \left(\frac{2E}{G^2} g - \beta u_+ + u_- \right)$$
$$= A^2 u = \frac{4E}{G^2} g + \beta^2 u_+ + u_-.$$

from which one gets,

$$\alpha + \beta = 1, \tag{6.40}$$

and

$$\frac{4E}{G^2} = \gamma + \beta \frac{E}{G^2} + 2\alpha \frac{E}{G^2}. \tag{6.41}$$

It follows from the definition of α, β, and γ, expressions (6.40) and (6.41), that,

$$2P - |A^{3/2} u|^2 = \frac{-1}{(1 - e/E)(2e - E)^{-1} - (G^2 - P)^{-1}}. \tag{6.42}$$

After replacing (6.42), we get,

$$\beta = \frac{-1}{1 - \frac{e}{E} - \frac{2e - E}{G^2 - P}}. \tag{6.43}$$

Expression (6.43) tells us where we should look in order to identify a possible chained ghost when $\lambda = 2$.

Recall that for ghost solutions, we have the decomposition (6.24), namely,

$$u(t) = u_+(t) + u_-(t) + \eta g, \ \forall t \in \mathbb{R}$$

where $u_+ \in E_{\mu_+}$, $u_- \in E_{\mu_-}$ and $g \in E_\lambda$, with $\mu_- < \lambda < \mu_+$. Using the equation (6.39), we could easily identify the value of μ_+, when $\lambda = 2$.

Proposition 6.5.1. *Assume $\lambda = 2$, then $\mu_+ = 5$, for any chained ghost $u(t)$, with the decomposition* (6.24)*.*

Proof. Taking the dot product of the equation in (6.39) for the component u_+ with u_+, one gets,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}|u_+(t)|^2 + \mu_+|u_+|^2 &= -(E_{\mu_+}B(u(t), u(t)), u_+(t)) \\
&= -(B(u, u), u_+) \\
&= -(B(u_+ + u_- + \eta g, u_+ + u_- + \eta g), u_+) \\
&= -\eta(B(u_-, g), u_+) - \eta(B(g, u_-), u_+),
\end{aligned}
$$

where, in the last line above, we use that $B(v, v) = 0$, for any eigenvector v of the operator A and the relation (6.7).

Consider the term $(B(u,g), u_+)$ and express it in terms of its Fourier coefficients,

$$(B(u_-, g), u_+) = \sum_{\substack{h,j,k \in \mathbb{Z}^2 \smallsetminus \{0\} \\ h+j+k=0}} (\hat{u}_-(h) \cdot j)(\hat{g}(j) \cdot \hat{u}_+(k)). \tag{6.44}$$

On the right-hand side of expression (6.44), for $\hat{u}_-(h)$ not to be zero, we need

$$h \in S_1 := \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}, \tag{6.45}$$

and for $\hat{g}(j)$ not to be zero, we need

$$j \in S_2 := \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}, \tag{6.46}$$

then, if $|k|^2 \neq 5$, there is no combination of h from (6.45) and j from (6.46) satisfying $h + j + k = 0$, therefore, each term on the right hand of (6.44) will be zero if $|k|^2 \neq 5$. Similar arguments work for the Fourier expansion of $(B(g, u_-), u_+)$. Hence, if $\mu_+ \neq 5$, then $(B(g, u_-), u_+) \equiv 0$ and $(B(u_-, g), u_+) \equiv 0$, it follows then

$$\frac{1}{2} \frac{d}{dt} |u_+(t)|^2 + \mu_+ |u_+(t)|^2 = 0.$$

Since $u(t)$ is bounded for all $t \in \mathbb{R}$ we have $|u_+(t)|^2 \equiv 0$, which, combined with (6.36), also implies $|u_-(t)|^2 \equiv 0$, so, $u(t) = \eta g$, a steady state. Therefore, the only possible value for μ_+ is 5. □

6.6   Nonexistence of chained ghost solutions when $\lambda = 2$

In this section, we will prove that actually, chained ghost solutions do not exist when $\lambda = 2$. The proof relies heavily on the relations (6.38). To begin with, by the

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divergence free convergence condition (2.7) we may set,

$$\hat{u}(k,t) := i\alpha(k,t)\frac{k^\perp}{|k|}$$

$$\hat{g}(k) := i\gamma(k)\frac{k^\perp}{|k|}$$

$$\hat{B}(u,u)(k) := i\beta(u,u)(k)\frac{k^\perp}{|k|} = [Q_k(\hat{u},\hat{u}) - \frac{Q_k(\hat{u},\hat{u}) \cdot k}{|k|^2}k],$$

where $\alpha(k,t), \gamma(k)$ and $\beta(u,u)(k,t)$ are scalar functions; and, if $k = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$, then

$$k^\perp = \begin{bmatrix} -k_2 \\ k_1 \end{bmatrix}.$$

Notice that the reality condition (2.6) implies

$$\alpha(-k,t) = \overline{\alpha(k,t)}. \tag{6.47}$$

We take the dot product on both sides of (6.38) with $-i\frac{k^\perp}{|k|}$, using the above notation, to rewrite (6.38) as

$$\frac{d}{dt}\alpha(k,t) = \gamma(k) - |k|^2\alpha(k,t) + \sum_{\substack{h,j\in\mathbb{Z}^2\setminus\{0\} \\ h+j=k \\ |h|^2,|j|^2\in\{\lambda,\mu_+,\mu_-\}}} \frac{\alpha(h,t)\alpha(j,t)(h^\perp \cdot j)(k \cdot j)}{|h||k||j|}, \text{ for } |k|^2 \in \{\lambda,\mu_+,\mu_-\}. \tag{6.48}$$

Denote the third term on the right hand side of (6.48) as R_k. Since $\mu_+ = 5, \lambda = 2, \mu_- =$

1, then,

$$\alpha(k,t) = 0, \text{if } |k|^2 \notin \{1,2,5\} \tag{6.49}$$

$$\gamma(k) = 0, \text{if } |k|^2 \neq 2 \tag{6.50}$$

Also, denote

$$S_3 := \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}.$$

The following theorem was proved in [23].

Theorem 6.6.1. *When* $\lambda = 2$, *there do not exist chained ghost solutions.*

Proof. Note that by (6.48), (6.49), (6.50) we have $R_k = 0$, if $|k|^2 \notin \{1,2,5\}$. Let k be such that $|k|^2 \notin \{1,2,5\}$. In the following, we can get some algebric conditions from this fact.

We will examine all those combinations of the indices h, j, k for which the single term $R_k(h,j) := \alpha(h,t)\alpha(j,t)(h^\perp \cdot j)(k \cdot j)$ is not zero. Then, h, j and k must be in $S_1 \cup S_2 \cup S_3$. Moreover, the possible k's satisfying $|k|^2 \notin \{1,2,5\}$ and $h + j = k$ are those k's for which $|k|^2 \in \{4,8,9,10,13,18,20\}$.

It is easy to see that if h and j are from the same index set, say S_l, $l = 1,2,3$, then $R_k = 0$, so we do not have to consider the possibilities that $|k|^2 = 18$ and $|k|^2 = 20$, since then both h and j must be from the set S_3. Thus, we are left to consider the cases when $|k|^2 \in \{4,8,9,10,13\}$.

Case 1: $|k|^2 = 4$; then

$$k \in \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \end{bmatrix} \right\}.$$

If $k = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, then the possible combinations are:

(i). $h = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $j = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, for which $R_k(h, j) = -\frac{4}{\sqrt{5}}\alpha\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, t\right)\alpha\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, t\right)$;

(ii). $h = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $j = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, for which $R_k(h, j) = \frac{4}{\sqrt{5}}\alpha\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}, t\right)\alpha\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, t\right)$;

(iii). $h = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $j = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, for which $R_k(h, j) = 0$, since $k \cdot j = 0$;

(iv). $h = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, for which $R_k(h, j) = 0$;

(v). $h = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $j = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$;

(vi). $h = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $j = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$;

Observe that the contributions of the last two possibilities sum to be zero, since the product $\alpha(h, t)\alpha(j, t)$ does not change when the role of h and j are changed, and $h^{\perp} \cdot j = -j^{\perp} \cdot h$. Therefore,

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$$R_k = \frac{4}{\sqrt{5}}[(\alpha(\begin{bmatrix} 0 \\ -1 \end{bmatrix}, t))(\alpha(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, t)) - (\alpha(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, t))(\alpha(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, t))] = 0,$$

or,

$$(\alpha(\begin{bmatrix} 0 \\ -1 \end{bmatrix}, t))(\alpha(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, t)) - (\alpha(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, t))(\alpha(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, t)) = 0. \qquad (6.51)$$

If $k = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, we use similar arguments to obtain that

$$R_k = \frac{4}{\sqrt{5}}[(\alpha(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, t))(\alpha(\begin{bmatrix} -1 \\ 2 \end{bmatrix}, t)) - (\alpha(\begin{bmatrix} -1 \\ 0 \end{bmatrix}, t))(\alpha(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, t))] = 0,$$

or,

$$(\alpha(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, t))(\alpha(\begin{bmatrix} -1 \\ 2 \end{bmatrix}, t)) - (\alpha(\begin{bmatrix} -1 \\ 0 \end{bmatrix}, t))(\alpha(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, t)) = 0. \qquad (6.52)$$

If $k = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$, we have

$$(\alpha(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, t))(\alpha(\begin{bmatrix} -2 \\ -1 \end{bmatrix}, t)) - (\alpha(\begin{bmatrix} 0 \\ -1 \end{bmatrix}, t))(\alpha(\begin{bmatrix} -2 \\ 1 \end{bmatrix}, t)) = 0. \qquad (6.53)$$

If $k = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$, we have

$$(\alpha(\begin{bmatrix} -1 \\ 0 \end{bmatrix},t))(\alpha(\begin{bmatrix} 1 \\ -2 \end{bmatrix},t)) - (\alpha(\begin{bmatrix} 1 \\ 0 \end{bmatrix},t))(\alpha(\begin{bmatrix} -1 \\ -2 \end{bmatrix},t)) = 0. \tag{6.54}$$

Case 2: $|k|^2 = 8$; then

$$(\alpha(\begin{bmatrix} 1 \\ 0 \end{bmatrix},t))(\alpha(\begin{bmatrix} 1 \\ 2 \end{bmatrix},t)) - (\alpha(\begin{bmatrix} 0 \\ 1 \end{bmatrix},t))(\alpha(\begin{bmatrix} 2 \\ 1 \end{bmatrix},t)) = 0. \tag{6.55}$$

$$(\alpha(\begin{bmatrix} 0 \\ 1 \end{bmatrix},t))(\alpha(\begin{bmatrix} -2 \\ 1 \end{bmatrix},t)) - (\alpha(\begin{bmatrix} -1 \\ 0 \end{bmatrix},t))(\alpha(\begin{bmatrix} -1 \\ 2 \end{bmatrix},t)) = 0. \tag{6.56}$$

$$(\alpha(\begin{bmatrix} 0 \\ -1 \end{bmatrix},t))(\alpha(\begin{bmatrix} 2 \\ -1 \end{bmatrix},t)) - (\alpha(\begin{bmatrix} 1 \\ 0 \end{bmatrix},t))(\alpha(\begin{bmatrix} 1 \\ -2 \end{bmatrix},t)) = 0. \tag{6.57}$$

$$(\alpha(\begin{bmatrix} -1 \\ 0 \end{bmatrix},t))(\alpha(\begin{bmatrix} -1 \\ -2 \end{bmatrix},t)) - (\alpha(\begin{bmatrix} 0 \\ -1 \end{bmatrix},t))(\alpha(\begin{bmatrix} -2 \\ -1 \end{bmatrix},t)) = 0. \tag{6.58}$$

Case 3: $|k|^2 = 9$; then

$$(\alpha(\begin{bmatrix} 2 \\ 1 \end{bmatrix},t))(\alpha(\begin{bmatrix} 1 \\ -1 \end{bmatrix},t)) - (\alpha(\begin{bmatrix} 2 \\ -1 \end{bmatrix},t))(\alpha(\begin{bmatrix} 1 \\ 1 \end{bmatrix},t)) = 0. \tag{6.59}$$

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$$\left(\alpha\left(\begin{bmatrix}-1\\2\end{bmatrix},t\right)\right)\left(\alpha\left(\begin{bmatrix}1\\1\end{bmatrix},t\right)\right) - \left(\alpha\left(\begin{bmatrix}1\\2\end{bmatrix},t\right)\right)\left(\alpha\left(\begin{bmatrix}-1\\1\end{bmatrix},t\right)\right) = 0. \tag{6.60}$$

$$\left(\alpha\left(\begin{bmatrix}-2\\-1\end{bmatrix},t\right)\right)\left(\alpha\left(\begin{bmatrix}-1\\1\end{bmatrix},t\right)\right) - \left(\alpha\left(\begin{bmatrix}-2\\1\end{bmatrix},t\right)\right)\left(\alpha\left(\begin{bmatrix}-1\\-1\end{bmatrix},t\right)\right) = 0. \tag{6.61}$$

$$\left(\alpha\left(\begin{bmatrix}1\\-2\end{bmatrix},t\right)\right)\left(\alpha\left(\begin{bmatrix}-1\\-1\end{bmatrix},t\right)\right) - \left(\alpha\left(\begin{bmatrix}-1\\-2\end{bmatrix},t\right)\right)\left(\alpha\left(\begin{bmatrix}1\\-1\end{bmatrix},t\right)\right) = 0. \tag{6.62}$$

Case 4: $|k|^2 = 10$; then similarly,

$$\left(\alpha\left(\begin{bmatrix}0\\1\end{bmatrix},t\right)\right)\left(\alpha\left(\begin{bmatrix}1\\2\end{bmatrix},t\right)\right) = 0. \tag{6.63}$$

$$\left(\alpha\left(\begin{bmatrix}1\\0\end{bmatrix},t\right)\right)\left(\alpha\left(\begin{bmatrix}2\\1\end{bmatrix},t\right)\right) = 0. \tag{6.64}$$

$$\left(\alpha\left(\begin{bmatrix}0\\1\end{bmatrix},t\right)\right)\left(\alpha\left(\begin{bmatrix}-1\\2\end{bmatrix},t\right)\right) = 0. \tag{6.65}$$

$$\left(\alpha\left(\begin{bmatrix}-1\\0\end{bmatrix},t\right)\right)\left(\alpha\left(\begin{bmatrix}-2\\1\end{bmatrix},t\right)\right) = 0. \tag{6.66}$$

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$$(\alpha(\begin{bmatrix} 0 \\ -1 \end{bmatrix}, t))(\alpha(\begin{bmatrix} 1 \\ -2 \end{bmatrix}, t)) = 0. \tag{6.67}$$

$$(\alpha(\begin{bmatrix} 0 \\ -1 \end{bmatrix}, t))(\alpha(\begin{bmatrix} -1 \\ -2 \end{bmatrix}, t)) = 0. \tag{6.68}$$

$$(\alpha(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, t))(\alpha(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, t)) = 0. \tag{6.69}$$

$$(\alpha(\begin{bmatrix} -1 \\ 0 \end{bmatrix}, t))(\alpha(\begin{bmatrix} -2 \\ -1 \end{bmatrix}, t)) = 0. \tag{6.70}$$

Case 5: $|k|^2 = 13$; then similarly, we have

$$(\alpha(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, t))(\alpha(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, t)) = 0. \tag{6.71}$$

$$(\alpha(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, t))(\alpha(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, t)) = 0. \tag{6.72}$$

$$(\alpha(\begin{bmatrix} -1 \\ 2 \end{bmatrix}, t))(\alpha(\begin{bmatrix} -1 \\ 1 \end{bmatrix}, t)) = 0. \tag{6.73}$$

$$(\alpha(\begin{bmatrix} -2 \\ 1 \end{bmatrix}, t))(\alpha(\begin{bmatrix} -1 \\ 1 \end{bmatrix}, t)) = 0. \tag{6.74}$$

$$(\alpha(\begin{bmatrix} -1 \\ -2 \end{bmatrix}, t))(\alpha(\begin{bmatrix} -1 \\ -1 \end{bmatrix}, t)) = 0. \tag{6.75}$$

$$(\alpha(\begin{bmatrix} 1 \\ -2 \end{bmatrix}, t))(\alpha(\begin{bmatrix} 1 \\ -1 \end{bmatrix}, t)) = 0. \tag{6.76}$$

$$(\alpha(\begin{bmatrix} 1 \\ -1 \end{bmatrix}, t))(\alpha(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, t)) = 0. \tag{6.77}$$

$$(\alpha(\begin{bmatrix} -2 \\ -1 \end{bmatrix}, t))(\alpha(\begin{bmatrix} -1 \\ -1 \end{bmatrix}, t)) = 0. \tag{6.78}$$

Now, we can analyze the system of equations (6.51)-(6.78). Notice that the system holds for all $t \in \mathbb{R}$. In the following, for notational simplicity, we drop the explicit dependence of time t, and write $\alpha(k,t)$ as $\alpha(k)$, for any fixed $t \in \mathbb{R}$. Based on the system of equtions (6.51)-(6.78), we can make the following lemma.

Lemma 6.6.2. *If there is a $k_0 \in S_3$ such that $\alpha(k_0) \neq 0$, then $\alpha(k) = 0$, for all $k \in S_1 \cup S_2$.*

Proof of lemma 6.6.2. Indeed, one could check all possible choices of k_0 to see that

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the claim is true. Here, we assume, say, $\alpha\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \neq 0$, then also $\alpha\left(\begin{bmatrix} -1 \\ -2 \end{bmatrix}\right) \neq 0$, by the

reality condition (6.47). (6.63) leads to $\alpha\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 0 = \alpha\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$. From (6.75), we have

$\alpha\left(\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) = 0 = \alpha\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$. From (6.58), $\alpha\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = 0 = \alpha\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$. From (6.62), we see

$\alpha\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = 0 = \alpha\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$. Thus $\alpha(k) = 0$, for all $k \in S_1 \cup S_2$. $\qquad \square$

Consequently, either $\alpha(S_1 \cup S_2) \equiv 0$, or $\alpha(S_3) \equiv 0$, thus, either $u_+ \equiv 0$, or $u_- \equiv 0 = g$.

Since $g \neq 0$, then $u_+ \equiv 0$, so $|u_+|^2 \equiv 0$. From (6.25), it leads to $P = G^2$, so $u = u^* = g/\lambda$,

which is a contradiction. Therefore, there does not exist chained ghost solutions when

$\lambda = 2$. $\qquad \square$

7. CONCLUSION

In this dissertation, we first studied the consequences of assuming zero to be in the global attractor of the two dimensional incompressible Navier-Stokes equations. We showed that in this case, both the body force and the solutions must be of enough smoothness. Moreover, they belong to a special function class which is closely related to the usual Gevrey function classes. Based on such connection, we proved that for a generic nonzero forcing term, zero is not in the global attractor. An explicit criterion for zero to be in the global attrator is provided and is used to show that zero is indeed not in the global attractor if the forcing term is Kolmogorov forcing.

We also studied the solutions in the global attractor whose projection onto the energy-enstrophy plane is a single point. Geometric properties of such solutions (they are called "ghost solutions") are also explored. A subfamily of the ghost solutions is defined to be those ghost solutions that satisfy additional chained relation. We derived a finite Galerkin ODE system for which the solutions in such subfamily must satisfy. In a particular case when the body force is the Kolmogorov forcing corresponding to the second eigenvalue of the Stokes operator, we showed, based on the derived finite Galerkin system, that no chained ghost solutions can exist.

For incompressible three dimensional Navier-Stokes equations, the complexity of the weak global attractor is measured by using Kolmogorov-ε entropy. Two different metrics generating the weak topology on the weak global attractor are provided and used to give two different estimates on the upper bound for the Kolmogorov-ε entropy. These two upper bounds are a little bit different, however both of these two estimates give us the finiteness of the functional dimension of the weak global attractor.

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