PERIODIC POINTS IN SHIFTS OF FINITE TYPE OVER GROUPS WITH CONNECTIONS TO GROWTH

A Dissertation

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ABSTRACT

We develop several tools and techniques for constructing or proving the non-existence of weakly and strongly aperiodic shifts of finite type on groups. Additionally, inspired by the classification of all groups of polynomial growth and its implications for SFT-periodicity, we consider the task of surveying groups according to their possible exponential growth rates. We also propose a theory of algebraic shift spaces and pose several questions for future investigations.

Our main results are as follows: we prove that weak and strong SFT-periodicity is a commensurability invariant for all finitely generated groups, and we show that the extension of a group with a strongly aperiodic SFT by another such group has a strongly aperiodic SFT as well, provided the kernel is finitely generated. On the topic of exponential growth rates, we provide data for the growth spectrum of the free group on two generators, showing in particular that the growth spectrum is unbounded and has infinitely many limit points.

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1. INTRODUCTION

The study of tilings of the plane and its higher dimensional analogues is ancient, being a natural consideration in classical geometry and art, and saw systematic treatments as early as the 17th century, when Johannes Kepler published his *Harmonices Mundi*. In the 1900s, the work of Harold Coxeter and M. C. Escher established the theory of tilings as a topic of interest in contemporary mathematics as well as modern culture. In particular, the problem of finding aperiodic tile sets—that is, finite sets of shapes that can be used to tile the plane, but only in configurations that admit no translation symmetries—was first posed, appearing as "an amusing combinatorial problem" in a paper by Hao Wang in 1961 [31].

Motivated by connections to questions of decidability, Wang described the problem in the following way: assume we are given a finite set of square tiles with colored edges, such as those shown in Figure 1.1. Tiles may not be rotated or reflected. Such a tile set is *solvable* if one can cover the plane with copies of the given tiles subject to the restriction that adjoining edges must have the same color.

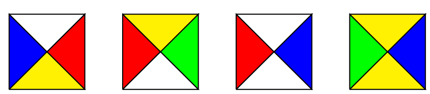


Figure 1.1: A set of Wang tiles

For example, the set of Wang tiles in Figure 1.1 is solvable, since the tiles can be used to form the 3×3 block of tiles shown in Figure 1.2, and this block can be repeated infinitely in both the vertical and horizontal direction. Wang called such a block a *cyclic rectangle* and conjectured that every solvable tile set admits a cyclic rectangle.

In 1966, Robert Berger proved Wang's conjecture false by constructing a solvable finite

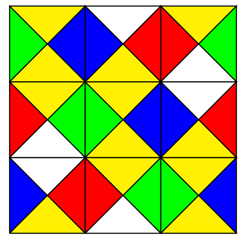


Figure 1.2: A cyclic rectangle of Wang tiles

tile set that admitted no cyclic rectangle [2]. In other words, Berger showed that there exist aperiodic tile sets for 2-dimensional Euclidean space.

We remark that the dependence of the definition on the use of colors is superficial, as one can add bumps and notches to the edges of a square to achieve the same constraints, as demonstrated in Figure 1.3. The key distinction is that allowed configurations are prescribed by finitely many local rules; such sets of configurations can be recognized as *shifts of finite type* and are the primary topic of this study. The precise definitions are below.

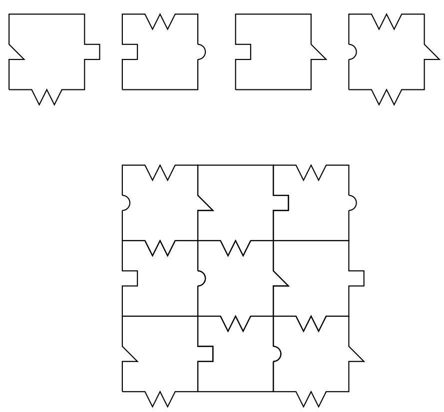


Figure 1.3: The tiles and cyclic rectangle from Figures 1.1 and 1.2 represented by shape alone

One may consider Wang's problem in one dimension rather than two: in this context, a tile set is considered solvable if one can simply lay down tiles in an infinite line, rather than a two-dimensional grid, subject to the same constraints. Unlike in two dimensions, the one-dimensional version of Wang's conjecture is true: any solvable tile set admits a periodic solution. This follows from a standard exercise in symbolic dynamics (see Theorem 3.3.3 for a proof).

The fact that the lattices $\mathbb{Z}$ and $\mathbb{Z}^2$ exhibit different behavior when it comes to the existence of periodic tilings is interesting. We say that $\mathbb{Z}^2$ has a *strongly aperiodic shift of finite type* but $\mathbb{Z}$ does not. One may similarly inspect $\mathbb{Z}^3$ and higher dimensions; it was shown by Culik and Kari that $\mathbb{Z}^n$ admits a strongly aperiodic shift of finite type whenever $n \geq 2$ [10], extending Berger's result. In general, the theory of aperiodic tilings in dimensions two and higher is not just an abstract consideration: in the 1980s, the theory of aperiodic tilings of Euclidean space was applied to chemistry and materials science, becoming known as the study of quasicrystals.

One need not restrict consideration to tilings of flat space. More generally, the following question arises: given a discrete group G (bearing in mind the case when G is a cocompact lattice in a locally compact group, as with $\mathbb{Z}^n \leq \mathbb{R}^n$ above), can we define a finite set of tiles for the vertices of G that can tile all of G, but only aperiodically? With some groups, such as $\mathbb{Z}$ and the infinite dihedral group D_∞, the answer is "no" (see Theorem 4.1.4); for others, such as $\mathbb{Z}^2$ and the discrete Heisenberg group $H_3(\mathbb{Z})$, the answer is "yes" (see [9]); and in still other cases, such as when $G = F_2$, the free group on two generators, the answer depends on the exact definition of "aperiodic" (see Theorem 3.3.5).

We briefly introduce the precise terminology needed to describe our results; a more complete account may be found in Section 3. Let A be a finite set (called an *alphabet*), which is to be thought of as a set of labels or colors, and let G be a group. The set A^G of all labellings of G—that is, all functions from G to A—is called the *full shift* over G with alphabet A . The group G acts on A^G by *translation*: there is a natural right G-action

$x \mapsto x^g$, where $x^g(h)$ is defined to be $x(gh)$ for $h \in G$. In our original example of Wang tiles from Figure 1.1, we have $G = \mathbb{Z}^2$, and A consists of the four tiles shown. Translation of x by an element $(m,n) \in \mathbb{Z}^2$ then corresponds to moving the tile at (m,n) to the origin and all others accordingly. Equivalently, one may view this shift as standing at position (m,n) and declaring it to be the new origin.

A subset $X \subset A^G$ is a *shift of finite type* (SFT) if and only if it can be defined in the following way: there exists a finite set $\Omega \subset G$ and a collection of *forbidden patterns* $P \subset A^\Omega$ such that, given any $x \in A^G$, $x \in X$ if and only if none of the forbidden patterns appear among the blocks $g\Omega$ in the labelling x (that is, among the patterns $x^g\big|_\Omega$). In the Wang tile example, we might take Ω to be the plus shape $\{(\pm1,0),(0,\pm1),(0,0)\}$ and our forbidden pattern set P to consist of any tilings of Ω violating the coloring constraints. We note that one may define a shift of finite type in terms of "allowed patterns" as well.

We are concerned with the existence of periodic configurations in shifts of finite type. A configuration $x \in A^G$ is called *weakly periodic* if there exists some $g \in G$, $g \neq 1$, such that $x^g = x$. We call x *strongly periodic* if the orbit $\{x^g : g \in G\}$ of x under translation is finite. In the Wang tile example, configurations formed from some cyclic rectangle are strongly periodic.

We call a group G *weakly* (respectively, *strongly*) SFT-periodic if every nonempty SFT over G contains a weakly (resp. strongly) periodic configuration, no matter which finite alphabet is chosen. For example, $\mathbb{Z}$ is strongly SFT-periodic, F_2 is weakly but not strongly SFT-periodic, and $\mathbb{Z}^2$ is neither.

We remark that G fails to be weakly SFT-periodic if and only if there exists a nonempty shift of finite type X over G such that $x^g \neq x$ for all $x \in X$, $g \neq 1 \in G$. Such a SFT is called a *strongly aperiodic shift of finite type*. (One may similarly define a *weakly aperiodic shift of finite type* over any group which fails to be strongly SFT-periodic.) Determining which groups have aperiodic shifts of finite type is an eclectic task that uses techniques from and has implications for group theory, dynamical systems, geometry, and computability theory. In this report we develop several tools and techniques for the task, leading to new classification results for the existence of periodic configurations in shifts of finite type. A summary of our main results is below.

Theorem 4.1.4 (appearing in [5])**.** Let G_1 and G_2 be finitely generated commensurable groups.

(1) If G_1 is weakly SFT-periodic, then G_2 is weakly SFT-periodic.

(2) If G_1 is strongly SFT-periodic, then G_2 is strongly SFT-periodic.

Theorem 4.1.6. Let G be a group and let N be a finitely generated normal subgroup of G. If N and G/N admit strongly aperiodic shifts of finite type, then G admits a strongly aperiodic shift of finite type.

As an example application, Theorem 4.1.6 gives another proof that $\mathbb{Z}^{2n}$ admits a strongly aperiodic shift of finite type. Additionally, en route to Theorem 4.1.6 we discover an interesting connection to the theory of the growth of groups:

Theorem 4.2.8. Suppose G has polynomial growth of degree d. Then G is strongly SFT-periodic if and only if $d = 1$.

It also follows from Theorem 4.1.4 and [9] that when $d = 2, 3, 4$ in the above statement, G is not even weakly SFT-periodic, i.e. G has a strongly aperiodic SFT. Recently, Jeandel has extended this result, discovering that all groups of nonlinear polynomial growth have strongly aperiodic SFTs [21]. This completes the classification of groups of polynomial growth according to SFT-periodicity (see Section 4.2).

The aforementioned classification relies heavily on Gromov's theorem for groups of polynomial growth. Some SFT-periodicity results are known for groups of intermediate growth as well, such as a construction of Marcinkowski and Nowak that shows the Grigorchuk group fails to be strongly SFT-periodic (along with certain other groups of intermediate growth) [25].

Naturally, one next turns to groups of exponential growth, seeking to extract invariants which might enable us to derive similar SFT-periodicity results. One example of an invariant which proves fruitful is the number of ends of a group: Cohen has shown that all finitely generated groups with at least two ends are weakly SFT-periodic [7]. In this work, the invariant we propose to study is the *(exponential) growth spectrum* $\mathrm{Gspec}(G)$ of a group G, which is a generalization of the uniform exponential growth rate. For example, we obtain the following result for F_2, the prototypical example of a group that is weakly but not strongly SFT-periodic:

Theorem 5.3.4. $\mathrm{Gspec}(F_2)$ is unbounded and has infinitely many limit points in the interval $[3,7]$.

We conclude this study by briefly outlining some new topics in shift spaces over groups, namely those concerned with algebraic shift spaces. In addition to being connected to our work with shifts of finite type, these theories engender some new and interesting questions.

2. PRELIMINARIES FROM COMBINATORIAL GROUP THEORY

In this section we establish some terminology and notation we will use when working with group presentations. We also briefly present some classical results from combinatorial and geometric group theory we will need; see the surveys [22] and [11] for a more in-depth treatment.

2.1 Marked groups and the word metric

Let F_m denote the free group on m generators, say with basis $\{a_1, \ldots, a_m\}$. A *word* in F_m is a product of the form

$$w = a_{\nu(1)}^{\varepsilon_1} a_{\nu(2)}^{\varepsilon_2} \cdots a_{\nu(n)}^{\varepsilon_n}, \quad \text{where } \nu(i) \in \{1, \ldots, m\},\ n \geq 0,\ \varepsilon \in \{-1, +1\}. \tag{2.1}$$

(By convention, an empty product in a group G, such as when $n = 0$ in (2.1), evaluates to the identity element $1_G \in G$.) Each element w of F_m can be uniquely represented by a product of the form (2.1) such that $a_{\nu(i)}^{\varepsilon_i} a_{\nu(i+1)}^{\varepsilon_{i+1}} \neq 1$ for $i = 1, \ldots, n-1$. Products of this form are called *reduced words*. The *length* of such a w is defined to be $|w| := n$. The unique word of length 0, denoted by ε, is called the *empty word* and, as mentioned above, represents the identity element $1_{F_m} \in F_m$.

Notice that if g is an element of F_m such that $g = a_{\nu(1)}^{\varepsilon_1} \cdots a_{\nu(n)}^{\varepsilon_n}$ for some $\nu(i) \in \{1, \ldots, m\}$, $n \geq 0$, $\varepsilon \in \{-1, +1\}$, where the product is not necessarily reduced, then $|g| \leq n$. In particular, if u and v are words in F_m, then $|uv| \leq |u| + |v|$. If in fact $|uv| = |u| + |v|$, then there is no cancellation between u and v upon taking their product $w = uv$, and we say $w = uv$ *as a reduced product*.

If $g_1, \ldots, g_m$ are elements of a group G, not necessarily distinct, there exists a unique homomorphism $\pi : F_m \to G$ such that $\pi(a_j) = g_j$, $j = 1, \ldots, m$. That is, $\pi(a_{\nu(1)}^{\varepsilon_1} \cdots a_{\nu(n)}^{\varepsilon_n}) =$

$g_{\nu(1)}^{\varepsilon_1} \cdots g_{\nu(n)}^{\varepsilon_n}$ for words $w = a_{\nu(1)}^{\varepsilon_1} \cdots a_{\nu(n)}^{\varepsilon_n} \in F_m$ and $\pi(\varepsilon) = 1_G$. If π is surjective (i.e. $S := \{g_1, \ldots, g_m\}$ generates G), we call the pair (G, π) a *marked group*. If π is understood, we will sometimes write $w_1 \overset{G}{=} w_2$ whenever $w_1, w_2 \in F_m$ are words such that $\pi(w_1) = \pi(w_2)$, or $g \overset{G}{=} w$ whenever w is a word such that $\pi(w) = g$.

By abuse of terminology, if S generates G, we will typically refer to this marked group as "the marked group (G, S)." In other words, a marked group can be viewed either as a quotient of a free group or simply as a group with a distinguished generating set. In this document we will use both perspectives interchangeably.

In general, whenever we refer to "the free group F_m," we implicitly assume that a basis of m elements $A = \{a_1, \ldots, a_m\}$ for F_m has been fixed and that words, reduced words, and all related concepts are defined with respect to this basis.

A marked group (G, π) can be visualized via the following fundamental tool:

Definition. Let (G, π) be a marked group, where $\pi : F_m \to G$ and F_m is a free group with basis A. The *(right) Cayley graph* of G with respect to π is the directed graph $\mathscr{G}$ with vertex set $\mathscr{V} = G$ and edge set $\mathscr{E} = \{(g, g\pi(a)) : g \in G,\ a \in A\}$.

Example 2.1.1. Let G be the symmetric group on the letters $\{1, 2, 3\}$ with generators $g_1 = (123)$, $g_2 = (12)$. The Cayley graph of G with respect to $\{g_1, g_2\}$ is shown in Figure 2.1 on the left. In drawing a Cayley graph, if some g_j has order 2, we will often replace the two edges (g, gg_j) and (gg_j, g) with a single undirected edge, as shown on the right.

Informally, we often identify a finitely generated group G with its Cayley graph $\mathscr{G}$ (with respect to an implicit set of generators), allowing us to consider G as a geometric object in its own right. For example, the undirected combinatorial distance on $\mathscr{G}$ equips G with a metric, which also arises in the following algebraic way.

Definition. Let (G, π) be a marked group, where $\pi : F_m \to G$. The *word length* of $g \in G$

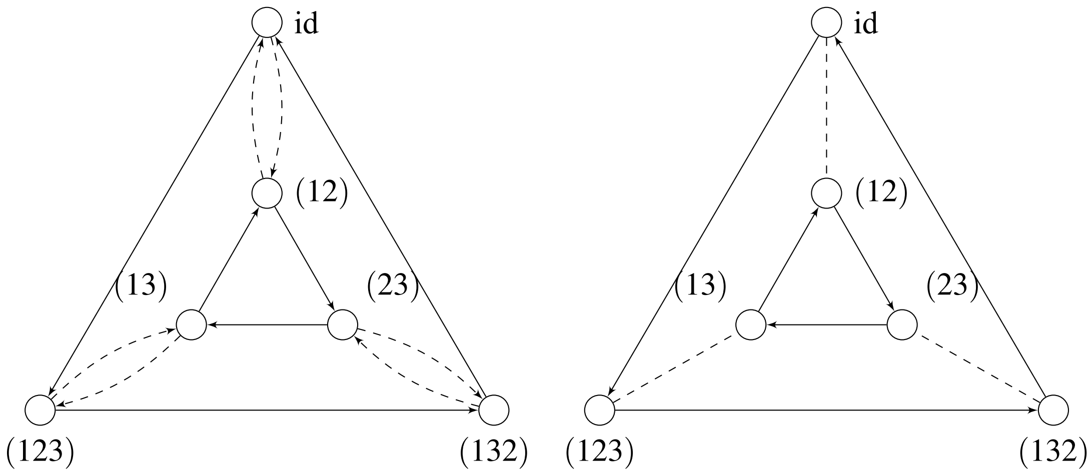


Figure 2.1: The Cayley graph of the symmetric group on three letters

with respect to π is

$$|g|_\pi := \min\{|w| : w \in \pi^{-1}(g)\}. \tag{2.2}$$

In other words, the length of g is the length of the shortest word in F_m representing g.

Remark 2.1.2. If G is generated by the finite set S, the word length equation (2.2) for the marked group (G,S) can be rewritten as

$$|g|_S := \min\{n : \text{ there exist } s_1, s_2, \ldots, s_n \in S \cup S^{-1} \text{ such that } g = s_1 s_2 \cdots s_n\}.$$

Proposition 2.1.3. Let (G,π) be a marked group. The word length operator $|\cdot|_\pi$ with respect to π has the following properties:

(1) $|g|_\pi = 0$ if and only if $g = 1_G$.

(2) $|g|_\pi = |g^{-1}|_\pi \quad (g \in G)$.

(3) $|gh|_\pi \leq |g|_\pi + |h|_\pi \quad (g,h \in G)$.

Proof. Let $\pi : F_m \to G$.

(1) $\pi(\varepsilon) = 1_G$, and ε is the only word of length 0 in F_m.

(2) It suffices to note that for a reduced word $w = a_1^{\varepsilon_1} a_2^{\varepsilon_2} \cdots a_n^{\varepsilon_n}$ in F_m, $w^{-1} = a_n^{-\varepsilon_n} a_{n-1}^{-\varepsilon_{n-1}} \cdots a_1^{-\varepsilon_1}$ is also reduced, and $|w| = n = |w^{-1}|$.

(3) Suppose $w = a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n}$ and $v = b_1^{\delta_1} \cdots b_r^{\delta_r}$ are reduced words in F_m such that $\pi(w) = g$ and $\pi(v) = h$. Then

$$|gh|_\pi \leq |wv| = |a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n} b_1^{\delta_1} \cdots b_r^{\delta_r}| \leq n + r.$$

As this holds for all $w \in \pi^{-1}(g)$ and $v \in \pi^{-1}(h)$, $|gh|_\pi \leq |g|_\pi + |h|_\pi$. □

Corollary 2.1.4. Let (G, π) be a marked group. Then the function $d_\pi : G \times G \to \mathbb{R}_{\geq 0}$ defined by

$$d_\pi(g,h) := |g^{-1}h|_\pi \quad (g,h \in G)$$

is a metric, called the *word metric* with respect to π. Moreover, d_π is *left-invariant*, i.e. $d_\pi(g,h) = d_\pi(kg,kh)$ for all $g,h,k \in G$.

It is not hard to see that d_π is exactly the (undirected) combinatorial distance in the Cayley graph of G with respect to π. The left-invariance of d_π indicates that left-translation in G by a group element $k \in G$ is an isometry, which in turn shows that there is nothing special about 1_G as a basepoint—the local behavior in a neighborhood of any $k \in G$ is the same as the local behavior around the origin 1_G. We will avail ourselves of this fact in Section 3, which is concerned with sets of labellings of Cayley graphs defined by local rules.

Having equipped a marked group (G, π) with a metric d_π, we will need a precise definition of a "shortest path" in the space (G, d_π).

Definition. Let (G, π) be a marked group, where $\pi : F_m \to G$ and F_m is free with basis A.

A *geodesic path* is a function $\gamma\colon\{0,1,\ldots,n\}\to G$ such that

$$d_\pi(\gamma(i),\gamma(j)) = |i-j| \tag{2.3}$$

for all $i,j\in\{0,1,\ldots,n\}$.

If γ instead has domain $\mathbb{Z}_{\geq 0}$ but still satisfies (2.3) for all $i,j\in\mathbb{Z}_{\geq 0}$, γ is a *geodesic ray*.

A *geodesic word* is a word $w = a_1^{\varepsilon_1}\cdots a_n^{\varepsilon_n} \quad (a_i\in A,\ n\geq 0,\ \varepsilon_i\in\{-1,+1\})$ in F_m such that the map

$$\gamma\colon\{0,1,\ldots,n\}\to G \text{ defined by } \gamma(i) \overset{G}{=} a_1^{\varepsilon_1}\cdots a_i^{\varepsilon_i} \tag{2.4}$$

is a geodesic path.

Equivalently, geodesic words are the minimal length word representatives for elements of G:

Proposition 2.1.5. Let (G,π) be a marked group, where $\pi : F_m\to G$. A word $w\in F_m$ is a geodesic word if and only if $|w| = |\pi(w)|_\pi$.

Proof. Write $w = a_1^{\varepsilon_1}\cdots a_n^{\varepsilon_n}$, where the a_i are elements of the implicit basis of F_m. Let γ be as in (2.4). If w is geodesic, then

$$|w| = n = d_\pi(\gamma(0),\gamma(n)) = |\gamma(n)|_\pi = |\pi(w)|_\pi.$$

On the other hand, suppose $|w| = |\pi(w)|_\pi$. Let $i,j\in\{0,1,\ldots,n\}$ be given with $i\leq j$; we wish to show that $d_\pi(\gamma(i),\gamma(j)) = |\pi(a_{i+1}^{\varepsilon_{i+1}}\cdots a_j^{\varepsilon_j})|_\pi$ equals $j-i$. Evidently $|\pi(a_{i+1}^{\varepsilon_{i+1}}\cdots a_j^{\varepsilon_j})|_\pi \leq j-i$, so suppose for the sake of contradiction that $|\pi(a_{i+1}^{\varepsilon_{i+1}}\cdots a_j^{\varepsilon_j})|_\pi < j-i$. Then there is some $v\in F_m$ with $|v|<j-i$ and $\pi(v) = \pi(a_{i+1}^{\varepsilon_{i+1}}\cdots a_j^{\varepsilon_j})$, yielding the word

$$w' = a_1^{\varepsilon_1}\cdots a_i^{\varepsilon_i} v a_{j+1}^{\varepsilon_{j+1}}\cdots a_n^{\varepsilon_n}$$

with $|w'| < n$ and $\pi(w') = \pi(w)$, a contradiction. $\square$

Remark 2.1.6. If $\gamma : \{0, 1, \ldots, n\} \to G$ is a geodesic path and $g_0 \in G$ is any fixed group element, then the path $\gamma' : \{0, 1, \ldots, n\} \to G$ defined by $\gamma'(i) = g_0\gamma(i)$ is also geodesic, by Corollary 2.1.4. That is, translation preserves geodesicity.

3. SHIFTS OF FINITE TYPE AND PERIODIC POINTS

In this section we construct the theoretical framework needed to discuss precisely the tiling problems described in Section 1. We first define shift spaces and shifts of finite type over a group G and show how elements of these spaces can be visualized as labellings of G which are constrained by a set of local rules. Next, we develop a set of algebraic tools for working with shift spaces. We conclude with definitions for periodic points in shift spaces, a topic which will be expounded upon in Section 4. Some of this material has appeared in [5].[1]

3.1 Shift spaces

We begin with the basic definitions of shift spaces. Much of the material in this section is standard in the literature and can be found, for example, in [6].

Let S and A be sets with A finite. We denote by A^S the set of all functions $f : S \to A$. If $T \subset S$ and $f \in A^S$, we denote by $f\big|_T$ the function $T \to A$ given by restriction: $f\big|_T(t) := f(t) \quad (t \in T)$.

If G is a group, we refer to A^G as the set of all *configurations over G (with alphabet A)*. We equip A^G with the prodiscrete topology, i.e. the product topology with each factor of A having the discrete topology. Notice that A^G is compact Hausdorff by Tychonoff's theorem. The following proposition gives some properties of this topology.

Proposition 3.1.1. Let G be a group and let A be a finite set.

(1) If $x \in A^G$ is a configuration, a neighborhood basis for x is given by the collection of *basic neighborhoods*

$$N_x(\Omega) = \left\{ y \in A^G : y\big|_\Omega = x\big|_\Omega \right\} \quad (\Omega \subset G,\ \Omega \text{ finite}).$$

[1]Available for electronic retrieval at `http://nyjm.albany.edu/j/2015/21-36v.pdf`.

(2) A subbasis for the topology on A^G is given by the collection of the *cylinder sets*

$$C(g,a) := \{x \in A^G : x(g) = a\} \quad (g \in G,\ a \in A).$$

(3) Suppose $\{B_n\}_{n=1}^{\infty}$ is a collection of finite subsets of G such that $B_n \subset B_{n+1}$ for all $n \geq 1$ and $\bigcup_{n=1}^{\infty} B_n = G$ (in particular, G is countable). Let $\lambda : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$ be a decreasing function such that $\lambda(\infty) := \lim_{n\to\infty} \lambda(n) = 0$. Then the topology of A^G can be defined by the following metric:

$$d(x,y) = \lambda\left(\sup\left\{n : x\big|_{B_n} = y\big|_{B_n}\right\}\right),$$

where by convention $\sup \varnothing = 0$.

Proof. For $g \in G$, we denote by $\pi_g : A^G \to A$ the projection map $x \mapsto x(g)$.

(1) $N_x(\Omega) = \bigcap_{\omega\in\Omega} \pi_\omega^{-1}(x(\omega))$ is open in the prodiscrete topology. Suppose now that $x \in U \subset A^G$, where U is open. Then there exists a basic open set

$$B = \pi_{g_1}^{-1}(A_1) \cap \pi_{g_2}^{-1}(A_2) \cap \cdots \cap \pi_{g_m}^{-1}(A_m)$$

in A^G with $x \in B \subset U$ (here $A_i \subset A$ for each i), and it follows that $x \in N_x(\{g_1, g_2, \ldots, g_m\}) \subset B$.

(2) Each $C(g,a)$ is open, since $C(g,a) = \pi_g^{-1}(\{a\})$. On the other hand, given $x \in A^G$ and $\Omega = \{\omega_1, \ldots, \omega_m\} \subset G$,

$$N_x(\Omega) = C(\omega_1, x(\omega_1)) \cap \cdots \cap C(\omega_m, x(\omega_m))$$

and by (1) we have a subbasis.

(3) First we show each $C(g,a)$ is open under the specified metric. If $x \in C(g,a)$ and n

is large enough so that $g \in B_n$, we have $x \in B_x(\lambda(n)) \subset C(g,a)$ (where $B_x(r)$ denotes the open ball of radius r around x).

On the other hand, $B_x(r)$ is open in the prodiscrete topology for $x \in A^G$ and $r > 0$: we can find n large enough so that $\lambda(n) < r$, giving $N_x(B_n) \subset B_x(r)$. We conclude that d metrizes the prodiscrete topology. □

In addition to its topological structure, A^G comes equipped with a right G-action $x \mapsto x^g$, called the *shift action*, defined by the following equation:

$$(x^g)(h) := x(gh) \quad (x \in A^G,\ g,h \in G).$$

In the case of a finitely generated group G, this shift can be visualized as follows. View A as a set of colors and $x \in A^G$ as a coloring of the right Cayley graph of G with basepoint the identity element $1 \in G$. Then applying a shift $x \mapsto x^g$ corresponds to changing the basepoint of the coloring to g. (One can also define a *left* G-action by $(g.x)(h) := x(g^{-1}h)$, which corresponds to translating the color at 1 to g and all other colors relative to this translation. In the theory of shift spaces over groups, the two are functionally equivalent; in this document we will use the right action unless otherwise stated.)

Example 3.1.2. Let $A = \{\bullet, \circ, \bullet\}$ and $G = \langle r, \sigma \mid r^3, \sigma^2, r\sigma = \sigma r^2 \rangle$ be the symmetric group on 3 letters. An example of a configuration in A^G and its shift by σ are illustrated in Figure 3.1.

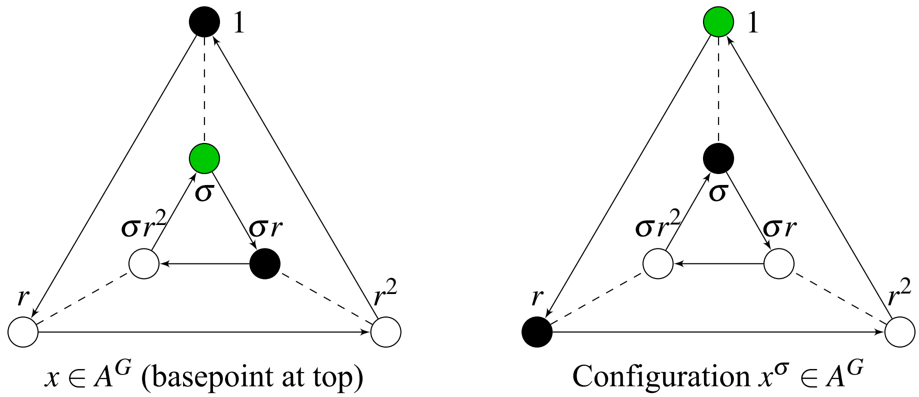


Figure 3.1: Configurations in the symmetric group on three letters

In general, we will primarily be concerned with infinite discrete groups G, although in some sections, such as Section 3.2.3, finite groups will play a role.

Proposition 3.1.3. Let G be a group and A a finite set. Fix $g \in G$. Then the map $f : A^G \to A^G$ defined by $f(x) = x^g$ is a homeomorphism.

Proof. To see that f is continuous, it suffices to note that

$$f^{-1}(C(h,a)) = C(gh,a).$$

The inverse function to f is given by $x \mapsto x^{g^{-1}}$ and is continuous by the same reasoning. □

The shift action and the prodiscrete topology work together in a natural way. This is further illustrated by the concept of a *shift space*, which we now define.

Definition. Let G be a group and A a finite set. A *shift space* (or *subshift*, or simply *shift*) is a topologically closed subset of configurations $X \subset A^G$ such that $x^g \in X$ for all $x \in X$ and $g \in G$ (this latter condition is called *shift invariance*).

We say A is the *alphabet of* X and G is the *underlying group of* X. We will also say X is a shift space *over* G.

If $X = A^G$, we call X the *full shift*.

Example 3.1.4 (Infinite dihedral group)**.** Let $G = D_\infty = \langle a, \sigma \mid \sigma a = a^{-1}\sigma, \sigma^2 = 1 \rangle$ be the infinite dihedral group. Under this presentation, we obtain the "ladder" Cayley graph of D_∞ shown in Figure 3.2:

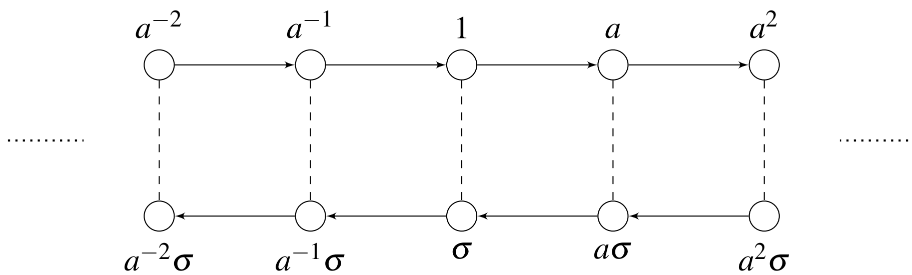
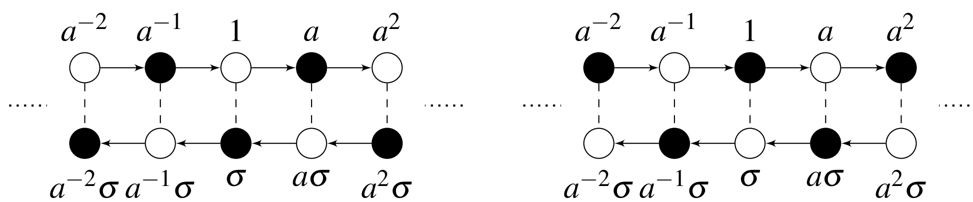


Figure 3.2: The Cayley graph of D_∞

Suppose $A = \{\circ, \bullet\}$ and $X \subset A^G$ consists of the two configurations shown in Figure 3.3:



Configuration x_0 Configuration x_1

Figure 3.3: Two configurations over D_∞

We see that $x_0^a = x_1 = x_0^\sigma$ and $x_1^a = x_0 = x_1^\sigma$. Thus X is shift-invariant, and of course X is topologically closed by virtue of being a finite set in a Hausdorff space. We will refer to the shift space X defined above as the *checkerboard shift* on D_∞ and will return to it in later examples.

One can also define shift spaces in terms of forbidden or allowed patterns:

Definition. Let G be a group and A a finite set. A *pattern* is a function $p : \Omega \to A$, where Ω is a finite subset of G called the *shape* of p. If $x \in A^G$, we say *p appears in x (at g)* if there is some $g \in G$ such that $(x^g)\big|_\Omega = p$, i.e. such that $x(g\omega) = p(\omega)$ for all $\omega \in \Omega$.

Proposition 3.1.5. Let G be a group and A a finite set, and suppose $X \subset A^G$ is a set of configurations. Let $\mathbf{P} = \bigcup_{\substack{\Omega \subset G \\ \Omega \text{ finite}}} A^\Omega$ be the set of all possible patterns.

The following are equivalent:

(i) X is a shift space.

(ii) There exists a set of patterns $P \subset \mathbf{P}$ such that

$$X = \{x \in A^G : \text{ no } p \in P \text{ appears in } x\}. \tag{3.1}$$

(iii) There exists a set of patterns $P' \subset \mathbf{P}$ such that if we let

$$S = \{\Omega \subset G : \Omega \text{ is the shape of some } p \in P'\},$$

then

$$X = \{x \in A^G : \text{ for each } g \in G \text{ and } \Omega \in S, (x^g)\big|_\Omega \in P'\}. \tag{3.2}$$

Remark 3.1.6. The patterns in (3.1) are called a *defining set of forbidden patterns for X* and the patterns in (3.2) are called a *defining set of allowed patterns for X*. When the underlying group and alphabet are understood, we will indicate (3.1) and (3.2) by the shorthand

$$X = \mathscr{X}_P \qquad \text{and} \qquad X = \mathscr{O}(P'), \tag{3.3}$$

respectively. The correspondence between these two patterns sets is as follows: Suppose $P = \bigcup_{\Omega \in S} A^{\Omega} \setminus P'$. We then have the following equality:

$$
\begin{aligned}
& A^G \setminus \mathscr{X}_P \\
&= \{x \in A^G : \text{ some } p \in P \text{ appears in } x\} \\
&= \{x \in A^G : \text{ there exist } g \in G,\ p \in P,\ \Omega \subset G \text{ such that } (x^g)\big|_{\Omega} = p\} \\
&= \{x \in A^G : \text{ for some } g \in G,\ \Omega \in S, (x^g)\big|_{\Omega} \in A^{\Omega} \setminus P'\} \\
&= A^G \setminus \mathscr{O}(P').
\end{aligned}
$$

Thusd $\mathscr{X}_P = \mathscr{O}(P')$.

Proof of Proposition 3.1.5. (i) $\Rightarrow$ (iii): Let $S = \{\Omega \subset G : \Omega \text{ finite}\}$ be the set of all shapes and let

$$P' = \left\{ (x^g)\big|_{\Omega} : x \in X,\ g \in G,\ \Omega \in S \right\}.$$

Define $X' = \mathscr{O}(P')$; we wish to show that $X = X'$. It is clear that $x \in X \Rightarrow x \in X'$ by definition. Conversely, suppose $x \in X'$. Then for any shape $\Omega \in S$, $x\big|_{\Omega} = (y^g)\big|_{\Omega}$ for some $y \in X$, $g \in G$, and since $y^g \in X$ as well, we can conclude that $N_x(\Omega)$ intersects the closed set X for all Ω. Hence $x \in X$.

(iii) $\Rightarrow$ (ii): Suppose there exist patterns P' with shapes S such that $X = \mathscr{O}(P')$. Let $P = \bigcup_{\Omega \in S} A^{\Omega} \setminus P'$. Remark 3.1.6 then shows that $\mathscr{X}_P = \mathscr{O}(P')$.

(ii) $\Rightarrow$ (i): Suppose $X = \mathscr{X}_P$ for some P. To see that X is shift-invariant, select $x \in X$, and notice that no $p \in P$ appears in x if and only if no $p \in P$ appears in x^g for all $g \in G$. To see that X is topologically closed, note that $y \in A^G \setminus X$ implies $(y^g)\big|_{\Omega} = p$ for some $g \in G$, $p \in P$, giving $N_y(g\Omega) \subset A^G \setminus X$. Thus $A^G \setminus X$ is open. $\square$

We see that shift spaces are characterized by being defined in terms of local rules for configurations. The most important shift spaces for our purposes are those which can be defined by *finitely many* local rules:

Definition. Let X be a shift space. If there exists a finite defining set of forbidden patterns for X, then X is called a *shift of finite type* (SFT).

Example 3.1.7. The checkerboard shift from Example 3.1.4 is a shift of finite type and can be defined by the set of four forbidden patterns shown in Figure 3.4:

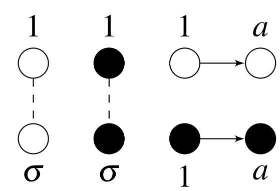


Figure 3.4: Nearest-neighbor forbidden patterns for the checkerboard shift on D_∞

Notice that different forbidden pattern sets may define the same SFT. The checkerboard shift can also be defined by the set of six forbidden patterns shown in Figure 3.5:

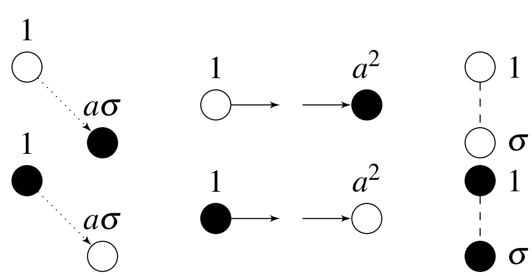


Figure 3.5: Locked shift forbidden patterns for the checkerboard shift on D_∞

The following proposition shows that in a shift of finite type, one may assume that the defining patterns all have the same shape.

Proposition 3.1.8. Let X be a SFT with alphabet A and underlying group G. Then there exists a single finite subset $\Omega \subset G$ and a set of patterns $P \subset A^\Omega$ such that

$$X = \mathscr{X}_P = \mathscr{O}(A^\Omega \setminus P). \tag{3.4}$$

Proof. Suppose $X = \mathscr{X}_{\widehat{P}}$, where $\widehat{P} = \{p_1 : \Omega_1 \to A,\ p_2 : \Omega_2 \to A,\ \dots,\ p_m : \Omega_m \to A\}$. Let $\Omega = \bigcup_{i=1}^m \Omega_i$ and $P = \{p \in A^\Omega : p\big|_{\Omega_i} = p_i \text{ for some } i\}$. Then Ω is finite and (3.4) holds (note that we invoke Remark 3.1.6 for the second equality). □

In particular, we see that a shift space X is a SFT if and only if there exists a finite defining set of allowed patterns for X. We remark here that the technique used in the proof of Proposition 3.1.8 of enlarging the forbidden pattern shapes will be used often and without comment in the remainder of this document. We also record here two more easy consequences of the definitions that will be used implicitly in the sequel.

Proposition 3.1.9. Let G be a group.

1. If G is finite, every shift space over G is a SFT.
2. If X_1, X_2 are SFTs over G and the same alphabet A, then $X_1 \cap X_2$ is also a SFT.

Proof. (1) Suppose G is finite and $X \subset A^G$ is a shift space. By Proposition 3.1.5, X has a defining set of forbidden patterns $P \subset \mathbf{P} := \bigcup_{\Omega \subset G} A^\Omega$. But $|P| \leq |\mathbf{P}| < \infty$, so X has a finite defining set of forbidden patterns.

(2) Suppose $X_1 = \mathscr{X}_P$ and $X_2 = \mathscr{X}_{P'}$. Then $X_1 \cap X_2 = \mathscr{X}_{P \cup P'}$, because $x \in X_1 \cap X_2$ if and only if no $p \in P$ appears in x and no $p' \in P'$ appears in x. □

Example 3.1.10 (Odd shift)**.** Not every shift space is a shift of finite type. Suppose $G = \langle a_1, \dots, a_m \mid \{r_i\}_{i \in I} \rangle$ is a group with finite generating set $S = \{a_1, \dots, a_m\}$. such that all defining relators r_i have even word length, and set $A = \{\circ, \bullet\}$. We define a forbidden pattern set

$$P = \{p : \{g,h\} \to A \ : \ p(g) = p(h) = \bullet,\ g,h \in G, \text{ and } d_S(g,h) \text{ is odd}\}.$$

$X := \mathscr{X}_P$ is the *odd shift* over G with respect to S. The configurations in X are exactly those in which every path of the form $\bullet\!-\!\circ\!-\!\circ\!-\cdots-\!\circ\!-\!\bullet$ has an odd number of white nodes appearing between the two black nodes.

If G is infinite, X is not a SFT. Indeed, suppose $P \subset A^{\Omega}$ is a finite set of allowed patterns for X and let $N = \max\{d_S(\omega_1,\omega_2) : \omega_1,\omega_2 \in \Omega\}$. Let K be an odd integer larger than N and choose $g_0 \in G$ such that $|g_0|_S = K$ (such a g_0 is guaranteed to exist as long as G is infinite). Then the configuration $x \in A^G$ defined by

$$x(g) = \begin{cases} \bullet & \text{if } g = 1, \\ \bullet & \text{if } g = g_0, \\ \circ & \text{otherwise} \end{cases}$$

is not in X, since $d_S(1,g_0)$ is odd. However, for any $g \in G$, $x^g\big|_{\Omega}$ must be an allowed pattern since it has at most one black node. This implies $x \in \mathscr{O}(P)$, a contradiction.

3.2 Algebraic tools for working with shift spaces

In this section we develop some useful algebraic results for constructing new shift spaces out of old. The constructions described here will be employed in Section 4 to prove results about SFTs when the underlying group is changed.

3.2.1 Products of shifts

Suppose A and B are two finite alphabets and S is an arbitrary set. Viewing $A \times B$ as a new alphabet, the set of functions $(A \times B)^S$ can be identified with $A^S \times B^S$ via the map

$\Phi : A^S \times B^S \to (A \times B)^S$ defined by

$$(\Phi(x,y))\,(s) = (x(s), y(s)). \tag{3.5}$$

We refer to this correspondence as *stacking*. Note that Φ is a bijection, so in particular, if the underlying set is a group G, the product of two full shifts A^G and B^G may be identified with the full shift $(A \times B)^G$. Indeed, the stacking map $\Phi : A^G \times B^G \to (A \times B)^G$ is a homeomorphism, since we have for each $g \in G$ the following commutative diagram of continuous maps:

$$\begin{tikzcd}
 & A^G \times B^G \arrow[dl] \arrow[d, "\Phi"'] \arrow[dr] & \\
A^G \arrow[d, "\pi_g"'] & (A \times B)^G \arrow[d, "\pi_g"'] & B^G \arrow[d, "\pi_g"'] \\
A & A \times B \arrow[l] \arrow[r] & B
\end{tikzcd}$$

Moreover, this correspondence respects the G-shift action in the sense that $(\Phi(x,y))^g = \Phi(x^g, y^g)$. Hence if two shift spaces (possibly not full shifts) have the same underlying group, we may, without complications, regard their product as a shift space. We also have the following:

Proposition 3.2.1. Suppose X_1, X_2 are SFTs with alphabets A, B respectively and underlying group G. Then $X_1 \times X_2$ is a SFT with alphabet $A \times B$ and underlying group G.

Proof. Let $X_1 = \mathscr{O}(P_1)$ and $X_2 = \mathscr{O}(P_2)$ for allowed pattern sets $P_1 \subset A^\Omega$, $P_2 \subset B^\Omega$. We show that $X_1 \times X_2 = \mathscr{O}(P_1 \times P_2)$:

$$\begin{aligned}
&(x_1, x_2) \in X_1 \times X_2 \\
\Leftrightarrow\ &(x_1^g)\big|_\Omega \in P_1 \text{ and } (x_2^g)\big|_\Omega \in P_2 \text{ for all } g \in G \\
\Leftrightarrow\ &(x_1^g, x_2^g)\big|_\Omega \in P_1 \times P_2 \text{ for all } g \in G \\
\Leftrightarrow\ &(x_1, x_2) \in \mathscr{O}(P_1 \times P_2).
\end{aligned}$$

□

When the two underlying groups are different, the situation is slightly more complicated. We will consider this case in the course of investigating shift spaces over quotient groups in Section 3.2.3.

3.2.2 Shift space rigidity

In this section we investigate the topic of forcing rigidity on SFTs. Let G be a group and A an alphabet. Given $g \in G$, we will use the notation

$$\mathrm{Fix}_{A^G}(g) := \{x \in A^G : x^g = x\}$$

for the set of *g-periodic configurations*, and similarly, if $S \subset G$, we define

$$\mathrm{Fix}_{A^G}(S) := \bigcap_{g \in S} \mathrm{Fix}_{A^G}(g) = \{x \in A^G : x^g = x \text{ for all } g \in S\}.$$

to be the set of *S-periodic* configurations. Below are some basic properties of the fixed point sets defined above.

Proposition 3.2.2. Let G be a group and A an alphabet, and suppose $S \subset G$.

(1) $\mathrm{Fix}_{A^G}(S)$ is topologically closed.

(2) $\mathrm{Fix}_{A^G}(S) = \mathrm{Fix}_{A^G}(\langle S\rangle)$, where $\langle S\rangle$ is the subgroup of G generated by S.

(3) If N is a normal subgroup of G, then $\mathrm{Fix}_{A^G}(N)$ is a shift space. Moreover, if N is finitely generated, then $\mathrm{Fix}_{A^G}(N)$ is a SFT.

Proof. (1) It suffices to show that $\mathrm{Fix}_{A^G}(g)$ is closed for an arbitrary $g \in G$. But A^G is Hausdorff, which means that the diagonal $\Delta := \{(x,x) : x \in A^G\}$ is closed in $A^G \times A^G$. Thus if $F : A^G \to A^G \times A^G$ is the continuous map defined by $F(x) = (x, x^g)$, then $\mathrm{Fix}_{A^G}(g) = F^{-1}(\Delta)$ is closed.

(2) Suppose $x \in \mathrm{Fix}_{A^G}(S)$ and $s = s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n} \in \langle S\rangle$ ($s_i \in S$, $\varepsilon_i \in \{-1,+1\}$). Noticing that $x^g = x$ implies $x = x^{g^{-1}}$, we have

$$x^{s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}} = x^{s_2^{\varepsilon_2} \cdots s_n^{\varepsilon_n}} = \cdots = x.$$

So $x \in \mathrm{Fix}_{A^G}(\langle S\rangle)$. The other inclusion is clear.

(3) To show $\mathrm{Fix}_{A^G}(N)$ is a shift space, it suffices to show $\mathrm{Fix}_{A^G}(N)$ is shift-invariant by (1). Suppose $x \in \mathrm{Fix}_{A^G}(N)$ and $g \in G$. Then

$$(x^g)^n = x^{gn} = x^{(gng^{-1})g} = (x^{gng^{-1}})^g = x^g \text{ for any } n \in N,$$

so $x^g \in \mathrm{Fix}_{A^G}(N)$.

Now suppose N is a finitely generated normal subgroup of G. Let S be a finite generating set for N, and set

$$P' = \{p : \{1,s\} \to A : s \in S, p(1) = p(s)\}.$$

Unraveling definitions, we have

$$\mathscr{O}(P') = \{x \in A^G : \text{ for each } g \in G \text{ and } s \in S,\ x(g) = x(gs)\}.$$

But

$$\mathrm{Fix}_{A^G}(N) = \{x \in A^G : x^n = x \text{ for all } n \in N\},$$

and

$$\begin{aligned}
& x^n = x && \text{for all } n \in N \\
\Leftrightarrow\quad & x(ng) = x(g) && \text{for all } n \in N, g \in G \\
\Leftrightarrow\quad & x(g(g^{-1}ng)) = x(g) && \text{for all } n \in N, g \in G \\
\Leftrightarrow\quad & x(gn') = x(g) && \text{for all } n' \in N, g \in G \\
\Leftrightarrow\quad & x(gs) = x(g) \text{ and } x(g) = x(gs^{-1}) && \text{for all } s \in S, g \in G \\
\Leftrightarrow\quad & x(gs) = x(g) && \text{for all } s \in S, g \in G,
\end{aligned}$$

where we have used the fact that every $n' \in N$ is a product of elements of $S \cup S^{-1}$ in the second-to-last equivalence. So $\mathrm{Fix}_{A^G}(N) = \mathscr{O}(P')$ is a shift of finite type. $\square$

Example 3.2.3. We remark that if H is not a normal subgroup of G, $\mathrm{Fix}_{A^G}(H)$ need not be shift-invariant and hence may not be a shift space, even if H is of finite index in G. For example, let G be the infinite dihedral group D_∞, let $A = \{\circ, \bullet\}$, and let $H = \langle a^3, \sigma\rangle = \{a^{3n}\sigma^i : n \in \mathbb{Z},\ i \in \{0,1\}\}$. H is evidently of finite index in G but not normal. The configuration x shown in Figure 3.6 is in $\mathrm{Fix}_{A^G}(H)$:

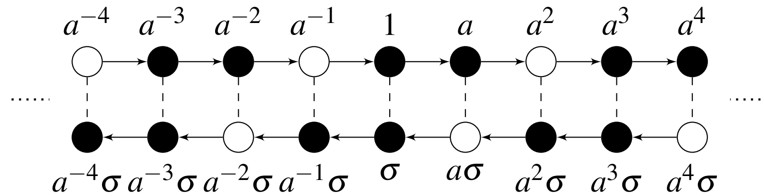


Figure 3.6: Configuration $x = x^\sigma = x^{a^3}$

However, x^a shown in Figure 3.7 is not in $\mathrm{Fix}_{A^G}(H)$:

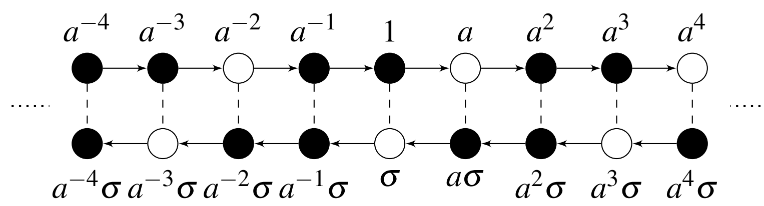


Figure 3.7: Configuration $x^a = (x^a)^\sigma$

Remark 3.2.4. The proof of Proposition 3.2.2 shows, more generally, that if $x \in \mathrm{Fix}_{A^G}(H)$ for some subgroup $H \leq G$, then $x^g \in \mathrm{Fix}_{A^G}(H)$ for all $g \in N_G(H)$, the normalizer of H in G.

We have shown that given a normal subgroup N with certain finiteness conditions, we can define a SFT over G whose configurations are at least N-periodic. More subtle is the following question: can we define a SFT whose configurations are at least N-periodic and also *at most* N-periodic? We show that the answer is "yes."

More precisely, let G be a group and A an alphabet. If $S \subset G$, we define

$$\mathrm{Wide}_{A^G}(S) := \{x \in A^G : x^g = x \text{ implies } g \in \langle S \rangle\}$$

to be the set of S-*wide configurations*. Clearly

$$\mathrm{Wide}_{A^G}(S) = \mathrm{Wide}_{A^G}(\langle S \rangle) = \{x \in A^G : \mathrm{Stab}_G(x) \leq \langle S \rangle\},$$

where $\mathrm{Stab}_G(x) := \{g \in G : x^g = x\}$ is the *stabilizer of x in g*. In other words, configurations in $\mathrm{Wide}_{A^G}(S)$ are those that are "at most S-periodic."

Ideally, one would like to prove structural results about $\mathrm{Wide}_{A^G}(S)$ similar to those for $\mathrm{Fix}_{A^G}(S)$. Unfortunately, the set $\mathrm{Wide}_{A^G}(S)$ by itself does not exhibit much structure in general. For example, just as with H-periodic configurations, the set of H-wide configurations need not be shift-invariant if H is not normal. For a counterexample, the configuration in Figure 3.7 is in $\mathrm{Wide}_{A^G}(\langle a^3, a\sigma\rangle)$, but the configuration in Figure 3.6 is not because $\sigma \notin \langle a^3, a\sigma\rangle = \{a^{i+3n}\sigma^i : n \in \mathbb{Z},\ i \in \{0,1\}\}$.

Moreover, even N normal of finite index does not guarantee $\mathrm{Wide}_{A^G}(N)$ is topologically closed. Take $G = \mathbb{Z}$, $A = \{\circ, \bullet\}$, and $N = 2\mathbb{Z} \trianglelefteq G$. Then the configurations

$$x_n \in A^G \ (n \in \mathbb{Z}_{>0}) \quad \text{defined by} \quad x_n(i) = \begin{cases} \bullet & |i| \leq n, \\ \circ & |i| > n \end{cases}$$

are in $\mathrm{Wide}_{A^G}(N)$, but the constant configuration x defined by $x(i) = \bullet$ for all $i \in \mathbb{Z}$, which is in $\overline{\mathrm{Wide}_{A^G}(N)}$, is not in $\mathrm{Wide}_{A^G}(N)$.

However, if we combine the periodicity restrictions of $\mathrm{Fix}_{A^G}(N)$ with restrictions that force configurations to be N-wide, we obtain the following special SFT, an instance of which was seen in Example 3.1.4:

Theorem 3.2.5. Let G be a group, and suppose $N \trianglelefteq G$ is a finitely generated normal subgroup of G of finite index. Then there exists a finite alphabet T and a nonempty shift of finite type X over G with alphabet T such that

$$X \subset \mathrm{Fix}_{T^G}(N) \cap \mathrm{Wide}_{T^G}(N), \quad \text{i.e. } x \in X \implies x^g = x \text{ if and only if } g \in N.$$

Proof. Choose as alphabet a set $T = \{t_1, t_2, \ldots, t_n\}$ of distinct right coset representatives for N in G with $t_1 = 1$. Let

$$P = \{p : \{1,t\} \to T : t \in T \setminus \{1\},\ p(1) = p(t)\}.$$

Noticing that $X := \mathrm{Fix}_{T^G}(N) \cap \mathscr{X}_P$ is a shift of finite type by Proposition 3.2.2(3), it suffices to show that X is nonempty and that $X \subset \mathrm{Wide}_{T^G}(N)$. Let $x \in T^G$ be defined by

$$x(nt) = t \quad (n \in N,\ t \in T).$$

That is, x sends each group element to its coset representative. Now, $x \in \mathrm{Fix}_{T^G}(N)$, since

$$x^n(n't) = t = x(n't) \quad (n' \in N,\ t \in T)$$

for all $n \in N$. Moreover, $x \in \mathscr{X}_P$: selecting $g = nt \in G$ $(n \in N, t \in T)$ and $t' \in T \setminus \{1\}$, we have

$$x^g(t') = x(ntt') \neq x(nt) = x^g(1)$$

(tt' and t cannot be in the same right coset of N, since N is normal), showing that $x \in \mathscr{X}_P$. Hence X is nonempty.

It remains to show that $X \subset \mathrm{Wide}_{T^G}(N)$. If $x \in \mathrm{Fix}_{T^G}(N) \cap \mathscr{X}_P$ and $x^g = x$, then

$$x(1) = x^g(1) = x(nt) = x(t) \quad (g = nt,\ n \in N,\ t \in T),$$

which gives $t = 1$ since $x \in \mathscr{X}_P$. That is, $x \in \mathrm{Wide}_{T^G}(N)$. □

Example 3.2.6. If $G = D_\infty$ and $N = \langle a^2, a\sigma \rangle$, we can choose the right coset representatives $T = \{1, \sigma\}$ for N in G. If we identify 1 with ○ and σ with ●, then the SFT constructed in Theorem 3.2.5 is exactly the checkerboard shift from Example 3.1.4, and the defining set of forbidden patterns is the second pattern set illustrated in Example 3.1.7.

The SFT constructed in Theorem 3.2.5 will be used in Section 4.1.

The last characterization of rigidity we will introduce in this section is concerned with deterministically reconstructing configurations from partial data. This is related to our considerations of N-periodicity above and is best illustrated by some simple examples.

Example 3.2.7. Let $G = \mathbb{Z}$ and $A = \{0,1\}$. For brevity, we will depict configurations $x \in A^{\mathbb{Z}}$ by $\cdots x(-2)x(-1).x(0)x(1)x(2)\cdots$, using the decimal point to locate the 0th entry. Let

$$X = \{\cdots 110.110\,110 \cdots,\ \cdots 101.101\,101 \cdots,\ \cdots 011.011\,011 \cdots\}.$$

Then X is a shift of finite type with allowed pattern set $\{110, 101, 011\}$.

Notice that in defining a configuration $x \in X$, as soon as one pattern is chosen to appear at a fixed position n_0, then all of x is uniquely determined. For example, if .110 appears at the origin, the pattern at position 1 must be 101, the pattern at position 2 must be 011, and so on. More succinctly, x can be quickly reconstructed using the rule

$$x(n+2) = x(n+1) + x(n) \pmod 2 \quad (n \in \mathbb{Z}).$$

Example 3.2.8. Let $G = \mathbb{Z}^2$ and $A = \{\square, \blacksquare\}$. Let $X \subset A^G$ consist of the four configurations obtained by translating the tiling shown in Figure 3.8 around the plane:

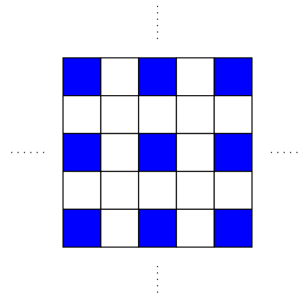
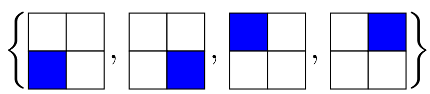


Figure 3.8: A configuration with orbit size 4

An immediate attempt to give a defining set of allowed patterns for X by parroting the technique of Example 3.2.7 might yield



but this set is not sufficient since it allows for configurations such as the one shown in Figure 3.9.

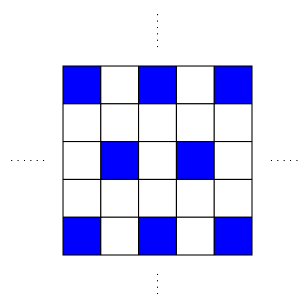
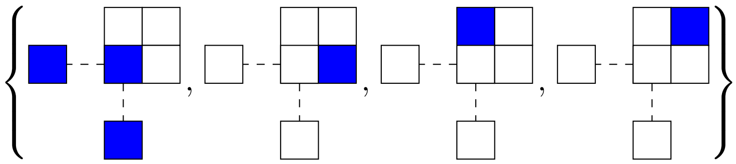


Figure 3.9: An invalid configuration

One needs to slightly extend the allowed pattern shape to enforce the appropriate rigidity. For example, the following allowed pattern set suffices:



With these examples in mind, we introduce the following terminology:

Definition. Let X be a shift space with underlying group G and alphabet A. A finite subset $\Omega \subset G$ is called *fundamental shape* for X if the restriction map $r : X \to A^{\Omega}$ defined by $r(x) = x\big|_{\Omega}$ is injective. In this case, $r(X)$ is called a set of *fundamental patterns* for X.

We show below that a shift space X has a fundamental pattern set if and only if it is finite, so we have not introduced a new class of shift spaces. Nevertheless, as we have seen, not every fundamental pattern set can serve as an allowed pattern set, and vice versa. Hence it is of interest to find fundamental shapes and fundamental pattern sets of minimal size as well as fast algorithms to reconstruct a configuration from a given fundamental pattern. We will mention this topic again in Section 6.

The examples above also indicate that shift spaces with fundamental pattern sets must exhibit periodicity. The following proposition makes this explicit.

Proposition 3.2.9. Let G be a group and A and alphabet, and suppose $X \subset A^G$ is a shift space. The following are equivalent:

(i) X has a set of fundamental patterns.

(ii) X is finite.

(iii) There exists a normal subgroup $N \trianglelefteq G$ of finite index in G such that $X \subset \mathrm{Fix}_{A^G}(N)$.

Proof. (i) $\Rightarrow$ (ii): By definition, if X has a set of fundamental patterns, then there is an injection from X into a finite set.

(ii) $\Rightarrow$ (iii): Suppose $X = \{x_1, \ldots, x_n\}$. As X is shift-invariant, each orbit $x_i^G := \{x_i^g : g \in G\}$ is finite. Thus all of the stabilizer subgroups $\mathrm{Stab}_G(x_i)$ are of finite index in G, making their intersection $\mathrm{Stab}_G(X) := \bigcap_{i=1}^n \mathrm{Stab}_G(x_i)$ finite index in G. By a standard group-theoretic result, there exists a normal subgroup $N \trianglelefteq G$ of finite index in G such that $N \subset \mathrm{Stab}_G(X)$. It follows that $X \subset \mathrm{Fix}_{A^G}(N)$.

(iii) $\Rightarrow$ (i): Suppose $X \subset \mathrm{Fix}_{A^G}(N)$, where N is of finite index. Then any T such that $NT = G$ is a fundamental shape for X, since whenever $x, y \in X$ and $x\big|_T = y\big|_T$, we have $x(nt) = x(t) = y(t) = y(nt)$ for all $g = nt \in G$. $\square$

From our investigations thus far, it is intuitively clear that a finite shift space over a finitely generated group is a SFT. We delay recording a formal proof of this fact until after we have investigated shift spaces over quotient groups, which, as we will see, are the same as the fixed point sets $\mathrm{Fix}_{A^G}(N)$ we have been considering.

3.2.3 Shift spaces over quotient groups

The following proposition lists the fundamental facts about shift spaces over quotient groups.

Proposition 3.2.10. Let G be a group and A an alphabet, and suppose N is a normal subgroup of G. The quotient map $\pi : G \to G/N$ induces a function $\Pi : A^{G/N} \to A^G$, defined by

$$(\Pi(x))(g) = x(\pi(g)) \quad \text{(i.e. } \Pi(x) = x \circ \pi)$$

satisfying the following properties:

(1) Π is injective.

(2) Π is G-equivariant in the sense that $(\Pi(x))^g = \Pi(x^{\pi(g)}) \quad (x \in A^{G/N},\ g \in G)$.

(3) $\Pi\left(A^{G/N}\right) = \mathrm{Fix}_{A^G}(N)$.

(4) Π is continuous and thus gives a homeomorphism between $A^{G/N}$ and $\mathrm{Fix}_{A^G}(N)$.

Proof. (1) Suppose $\Pi(x) = \Pi(y)$. As π is surjective, $x \circ \pi = y \circ \pi$ implies $x = y$.

(2) Given $x \in A^{G/N}$, $g \in G$, we have

$$(\Pi(x))^g(h) = x(\pi(gh)) = x^{\pi(g)}(\pi(h)) = (\Pi(x^{\pi(g)}))(h) \quad (h \in G).$$

(3) On the one hand, given $x \in A^{G/N}$,

$$(\Pi(x))^n = \Pi(x^{\pi(n)}) = \Pi(x) \quad (n \in N)$$

by (2), so $\Pi(x) \in \mathrm{Fix}_{A^G}(N)$. On the other hand, if $y \in \mathrm{Fix}_{A^G}(N)$ is given, the configuration $x \in A^{G/N}$ defined by $x(\pi(g)) = y(g)$ for $g \in G$ is well-defined, since y is constant on cosets of N. So $y = \Pi(x) \in \Pi\left(A^{G/N}\right)$.

(4) Under Π, the preimage of a cylinder set is a cylinder set:

$$\Pi^{-1}(C(g,a)) = C(\pi(g),a) \quad (g \in G,\ a \in A).$$

So Π is continuous; by (1) and $A^{G/N}$ compact, Π is an embedding. □

Notice that Proposition 3.2.10 yields another proof of the fact from Proposition 3.2.2(3) that $\mathrm{Fix}_{A^G}(N)$ is a shift space when N is normal. In fact, Π gives a one-to-one correspondence between shift spaces with alphabet A and underlying group G/N and shift spaces contained in $\mathrm{Fix}_{A^G}(N)$. Moreover, if N is finitely generated, this correspondence preserves the property of being a SFT:

Proposition 3.2.11. Let G be a group, A an alphabet, and N a finitely generated normal subgroup of G. Suppose $X \subset A^{G/N}$ is a shift space. Then X is a SFT if and only if $\Pi(X)$ is a SFT, where Π is as in Proposition 3.2.10.

Proof. First suppose $X = \mathscr{O}(P)$, where $P \subset A^{\Omega}$ is a finite set of allowed patterns with shape $\Omega \subset G/N$. Let $\Omega' \subset G$ be a finite set such that $\pi|_{\Omega'} : \Omega' \to \Omega$ is a bijection, and set $P' = \{p \circ \pi|_{\Omega'} : p \in P\} \subset A^{\Omega'}$. Then $\Pi(X) = \mathscr{O}(P') \cap \Pi\left(A^{G/N}\right)$, because for $x \in A^{G/N}$,

$$\begin{aligned}
& \Pi(x) \in \mathscr{O}(P') \\
\Leftrightarrow \quad & \text{for all } g \in G,\ \Pi(x)^g|_{\Omega'} \in P' \\
\Leftrightarrow \quad & \text{for all } g \in G,\ x^{\pi(g)} \circ \pi|_{\Omega'} \in P' \\
\Leftrightarrow \quad & \text{for all } g \in G,\ x^{\pi(g)} \circ \pi|_{\Omega'} = p \circ \pi|_{\Omega'} \text{ for some } p \in P \\
\Leftrightarrow \quad & \text{for all } g \in G,\ x^{\pi(g)}|_{\Omega} = p \text{ for some } p \in P \\
\Leftrightarrow \quad & \text{for all } g \in G,\ x^{\pi(g)} \in P \\
\Leftrightarrow \quad & x \in \mathscr{O}(P).
\end{aligned}$$

By Proposition 3.2.2, $\Pi(X) = \mathscr{O}(P') \cap \mathrm{Fix}_{A^G}(N)$ is a SFT.

Conversely, suppose $\Pi(X)$ is a SFT. Then $\Pi(X) = \mathscr{O}(P')$ for a finite set of allowed patterns $P' \subset A^{\Omega'}$. Let $\Omega = \pi(\Omega') \subset G/N$ and

$$P = \{p \in A^{\Omega} : p \circ \pi|_{\Omega'} \in P'\}.$$

Then, by the same computation as above, $X = \mathscr{O}(P)$, where we have used the fact that

$$x^{\pi(g)} \circ \pi|_{\Omega'} = p' \text{ for some } p' \in P' \implies x^{\pi(g)}|_{\Omega} = p \text{ for some } p \in P$$

because such a p' must be constant on cosets of N and we can define p by

$$p(\pi(\omega')) = p'(\omega') \quad (\omega' \in \Omega').$$

$\square$

Corollary 3.2.12. Let G be a finitely generated group and A an alphabet. If $X \subset A^G$ is a finite shift space, then X is a SFT.

Proof. By Proposition 3.2.9, there exists a normal subgroup $N \trianglelefteq G$ of finite index in G such that $X \subset \mathrm{Fix}_{A^G}(N)$. Let $\Pi : A^{G/N} \to \mathrm{Fix}_{A^G}(N)$ be the canonical homeomorphism and $Y = \Pi^{-1}(X)$. Since Y is a shift space over a finite group, Y is a SFT.

Now, as a finite index subgroup of a finitely generated group, N is finitely generated, which implies by Proposition 3.2.11 that $X = \Pi(Y)$ is a SFT. $\square$

We can now return to the question raised in 3.2.1 of taking products of shift spaces with different underlying groups. Suppose $X \subset A^G$ and $Y \subset B^H$ are shift spaces, where G, H are groups and A, B are finite alphabets. Under the identifications $\Pi_1 : A^G \to \mathrm{Fix}_{A^{G\times H}}(H) \subset A^{G\times H}$ and $\Pi_2 : B^H \to \mathrm{Fix}_{B^{G\times H}}(G) \subset B^{G\times H}$, we can identify $X \times Y$ with the shift space $\Pi_1(X) \times \Pi_2(Y)$ contained in $(A \times B)^{G\times H}$. Moreover, if X and Y are SFTs and G and H are finitely generated, then by Proposition 3.2.11 and Proposition 3.2.1, $X \times Y$ is a SFT.

3.2.4 Higher block shift

Having seen how shift spaces behave under taking products and quotients of the underlying groups, we now consider the case of passing to a finite index subgroup. In this section we will show that by a change of alphabet, a shift space over a group G can be faithfully encoded as a shift space over $H \leq G$ whenever H is of finite index in G.

Definition. Let G be a group and A an alphabet. Suppose H is a finite index subgroup of G and $T \subset G$ is a finite set such that $HT = G$. The *higher block map* (of A^G, relative to (H, T)) is the function $\psi_{H,T} : A^G \to B^H$ defined by

$$(\psi_{H,T}(x))\,(h) := (x^h)\big|_T.$$

That is, $\psi_{H,T}(x)$ is the configuration $z \in B^H$ such that

$$z(h)(t) = x(ht) \quad (h \in H,\ t \in T). \tag{3.6}$$

Intuitively, $\psi_{H,T}$ replaces the label $a \in A$ at each element $h \in H$ with the entire T-pattern $b \in A^T$ centered at h. The next result shows that $\psi_{H,T}$ embeds A^G into B^H as a SFT.

Proposition 3.2.13. Let G be a group, A an alphabet, H a finite index subgroup of G, and $T \subset G$ a finite set such that $HT = G$. Let $B = A^T$.

(1) $\psi_{H,T}$ is injective.

(2) $\psi_{H,T}$ is H-equivariant, i.e. $(\psi_{H,T}(x))^h = \psi_{H,T}\left(x^h\right) \quad \left(x \in A^G,\ h \in H\right)$.

(3) $\psi_{H,T}$ is continuous.

(4) $\psi_{H,T}\left(A^G\right)$ is a shift of finite type in B^H.

Proof. (1) If $\psi_{H,T}(x_1) = \psi_{H,T}(x_2)$, (3.6) shows that $x_1(ht) = x_2(ht)$ for all $h \in H$, $t \in T$. Since $HT = G$, $x_1 = x_2$.

(2) Let $x \in A^G$, $h \in H$. Set $z = \psi_{H,T}(x)$, $z' = \psi_{H,T}\left(x^h\right)$. Then

$$(z^h(k))(t) = z(hk)(t) = x(hkt) = x^h(kt) = (z'(k))(t) \quad (k \in H,\ t \in T).$$

(3) Given $h \in H$ and $b : T \to A$,

$$\psi_{H,T}^{-1}(C(h,b)) = \bigcap_{t \in T} C(ht, b(t)).$$

Since the preimage of an arbitrary cylinder set is open, $\psi_{H,T}$ is continuous. Thus A^G can be embedded as a shift space in B^H.

(4) Let $E = H \cap TT^{-1}$ and

$$P = \{p \in B^E : \text{for all } h \in E, t \in T \text{ such that } h^{-1}t \in T,\ p(1)(t) = p(h)(h^{-1}t)\}. \tag{3.7}$$

We claim $\psi_{H,T}\left(A^G\right) = \mathscr{O}(P)$. For $\subset$, suppose $\psi_{H,T}(x) = z \in B^H$ and $k \in H$ are given. For all $h \in E$, $t \in T$ such that $h^{-1}t \in T$,

$$(z^k)\big|_E(1)(t) = x(kt) = x(khh^{-1}t) = (z^k)\big|_E(h)(h^{-1}t),$$

showing $z \in \mathscr{O}(P)$. For $\supset$, suppose $z \in \mathscr{O}(P)$ is given. Define $x \in A^G$ by $x(ht) = z(h)(t) \quad (h \in H,\ t \in T)$. To see that x is well-defined, notice that $ht = h_2t_2$ implies $tt_2^{-1} = h^{-1}h_2 \in E$ and

$$z(h)(t) = (z^h)\big|_E(1)(t) = (z^h)\big|_E(h^{-1}h_2)(h_2^{-1}ht) = z(h_2)(t_2).$$

It is clear that $z = \psi_{H,T}(x)$. □

The shift of finite type $\psi_{H,T}\left(A^G\right)$ defined in Proposition 3.2.13(4) will be called the *higher block shift* (of A^G) relative to (H,T).

Remark 3.2.14. The defining set of allowed patterns in (3.7) is designed to account for the possibility that the T-shape centered at one element of H may overlap with the T-shape centered at another element of H. If this happens, the two corresponding T-patterns are required to agree on the overlap.

However, in some cases $H \cap TT^{-1}$ is reduced to the single element 1, which means there is no overlap—for example, this occurs if T is a set of distinct right coset representatives. If $H \cap TT^{-1} = \{1\}$, then the condition in (3.7) for $p \in B^E$ to be in P is always satisfied. In other words, if $H \cap TT^{-1} = \{1\}$, then $\mathscr{O}(P)$ is all of B^H and $\psi_{H,T}$ is surjective.

3.3 Periodic configurations

Having developed several tools for working with shift spaces and algebraic perturbations of shift spaces, we now define the primary periodicity properties with which we will concern ourselves in this study.

Definition. Let G be a group and A an alphabet, and let $x \in A^G$.

(1) x is *weakly periodic* if there exists $g \neq 1$ in G such that $x^g = x$.

(2) x is *strongly periodic* if the orbit x^G of x is finite.

In other words, a weakly periodic configuration x is one with nontrivial stabilizer $\mathrm{Stab}_G(x)$, and a strongly periodic configuration x is one whose stabilizer subgroup $\mathrm{Stab}_G(x)$ is of finite index in G. If G is infinite, strongly periodic configurations are also weakly periodic.

Recall that our primary goal in this project is to investigate how group-theoretic properties of a group G influence the dynamical properties of shift spaces over G. In particular, we would like to know how shifts of finite type behave, and a natural invariant to consider is the existence and type of periodic points in SFTs over G.

Definition. Let G be a group.

1. If every nonempty shift of finite type over G contains a weakly periodic configuration, G is *weakly SFT-periodic*.
2. If every nonempty shift of finite type over G contains a strongly periodic configuration, G is *strongly SFT-periodic*.

We can now make our main question precise:

Question 3.3.1. Which finitely generated groups are strongly SFT-periodic, and which are weakly SFT-periodic?

As with configurations, an infinite strongly SFT-periodic group is also weakly SFT-periodic. Notice that SFT-periodicity does not depend on a particular choice of finite alphabet. This is a natural simplification to make, because the topological dynamics of A^G are equivalent to those of B^G even if A and B are different finite alphabets.

A group G is not weakly SFT-periodic if and only if there exists a nonempty shift of finite type X over G such that $x^g = x$ implies $g = 1$ for all $x \in X,\ g \in G$. Such a SFT is called a *strongly aperiodic SFT* in the literature, and the task of finding strongly aperiodic SFTs over various groups is an ongoing investigation. We will see in Section 4.1 that our general classification techniques can be used to explicitly define strongly aperiodic SFTs.

In order to introduce the task of classifying all finitely generated groups according to the SFT-periodicity hierarchy, we conclude this section by presenting some well-known classification results. In Section 4.3 we will give a more thorough review of the current classification progress.

Proposition 3.3.2. Let G be a finite group. Then G is strongly SFT-periodic but not weakly SFT-periodic.

Proof. If X is a SFT over G, then every $x \in X$ is strongly periodic since X is finite. In particular, G is strongly SFT-periodic.

To see G is not weakly SFT-periodic, take $A = G$ as the finite alphabet, let $x_0 \in A^G$ be the identity function, and define $X = \{x_0^g : g \in G\}$ to be the orbit of x_0. X is a shift space (A^G has the discrete topology) so by Corollary 3.2.12, X is a SFT, and it is clear that $x^g = x$ implies $g = 1$ for all $x \in X$, $g \in G$. (This result also follows from Theorem 3.2.5.) $\square$

Thus finite groups are trivially pathological in this context, and they will be of no further concern to us. The simplest infinite group is $\mathbb{Z}$, which can easily be classified:

Proposition 3.3.3. $\mathbb{Z}$ is strongly SFT-periodic.

Proof. Let X be a nonempty SFT over $\mathbb{Z}$. We may suppose $X = \mathscr{O}(P)$ for some finite set of allowed patterns $P \subset A^{\Omega}$, where $\Omega = \{-n, -n+1, \ldots, n-1, n\}$ for an appropriately large n. Pick $x \in X$. By the pigeonhole principle, some $p \in P$ must appear infinitely often in x, say $x^r\big|_{\Omega} = x^m\big|_{\Omega} = p$ for some $r < m \in \mathbb{Z}$. We may assume $m - r > 2n$ (this condition ensures that the matching blocks do not overlap, a technical detail which avoids unnecessary complications). By shifting x if necessary, we may also assume $r = 0$.

Define $y \in A^{\mathbb{Z}}$ to be the periodic extension of $x\big|_{\{0,1,\ldots,m-1\}}$ to $\mathbb{Z}$. That is, if $\phi : \mathbb{Z} \to \{0, 1, \ldots, m-1\}$ is the map which sends each number to its remainder mod m, then $y(k) = x(\phi(k)) \quad (k \in \mathbb{Z})$. We then notice that $y^k\big|_{\Omega} = x^{\phi(k)}\big|_{\Omega}$ for all $k \in \mathbb{Z}$, which shows $y \in X$. Thus X contains a strongly periodic configuration. $\square$

The fact that there are no nonempty aperiodic SFTs over $\mathbb{Z}$ was well-known early in the development of one-dimensional symbolic dynamics, but the situation for higher dimensions remained unclear for several decades. In 1966, Roger Berger constructed an aperiodic set of Wang tiles (see 1), proving that $\mathbb{Z}^2$ is not weakly SFT-periodic [2]. The generalization to dimensions higher than 2 was completed by Culik and Kari in 1995 [10], who modified an aperiodic set of Wang tiles to form an aperiodic set of Wang "cubes," thus completing the classification for free abelian groups:

Theorem 3.3.4 (Culik, Kari 1995)**.** If $n \geq 2$, $\mathbb{Z}^n$ is not weakly SFT-periodic.

Notice that weak and strong SFT-periodicity coincide in the case of free abelian groups. It will follow from the results of Section 4.1 that they coincide for all (infinite) finitely generated abelian groups. Hence it is natural to wonder if weak and strong periodicity always coincide for infinite groups, which would render the distinction ultimately unnecessary. However, this is not the case: it follows from the work of Piantadosi [29], [30] that F_m is weakly SFT-periodic but not strongly SFT-periodic whenever $m \geq 2$. We provide our own proof of this fact in order to illustrate a parallel with the proof of Proposition 3.3.3.

Theorem 3.3.5. F_2 is weakly SFT-periodic but not strongly SFT-periodic.

Proof. Let $\{a,b\}$ be a free basis for F_2.

First, suppose $X \subset A^{F_2}$ is a nonempty SFT. We may assume $X = \mathscr{O}(P)$ for some finite set of allowed patterns $P \subset A^{\Omega}$, where $\Omega = \{w \in F_2 : |w| \leq n\}$ for an appropriately large n. Pick $x \in X$. By the pigeonhole principle, there must be some $p \in P$ that appears infinitely often among the patterns $x^{a^i}\big|_{\Omega} \quad (i \in \mathbb{Z})$. Translating x by some a^i if necessary, we may assume that

$$x\big|_{\Omega} = x^{a^m}\big|_{\Omega} = p \text{ for some } m > 2n, \text{ so } x(w) = x(a^m w) \text{ for all } w \in F_2 \text{ with } |w| \leq n. \quad (3.8)$$

Let $\phi : \mathbb{Z} \to \{0,1,\ldots,m-1\}$ take each number to its remainder mod m. We define a configuration $y \in A^{F_2}$ by

$$y(a^k w) = x(a^{\phi(k)} w) \quad \text{for reduced products } a^k w,\ k \in \mathbb{Z}.$$

Intuitively, we have restricted x to the block

$$\{a^k w \in F_2 : k = 0,1,\ldots,m-1,\ w \text{ does not start with } a^{\pm 1}\}$$

and taken y to be the periodic extension of this block along the geodesic axis $\{a^l : l \in \mathbb{Z}\}$. Indeed, $y^{a^m} = y$, showing y is weakly periodic.

We must also verify that $y \in X$, which we will do by showing that $y^{a^k w}\big|_{\Omega} = x^{a^{\phi(k)} w}\big|_{\Omega}$ for all reduced products $a^k w \in F_2$, i.e. $y(a^k w w') = x(a^{\phi(k)} w w')$ whenever $|w'| \leq n$. The only case in which this equality is nontrivial is when $w' = w^{-1} a^j w''$ as a reduced product for some $j \neq 0$. But since

$$x(a^{\phi(k)+j} w'') = x(a^{\phi(k+j)} w'') \text{ provided } |a^j w''| \leq n$$

by (3.8), the result follows.

We now show F_2 is not strongly SFT-periodic by constructing a nonempty SFT Z over F_2 whose configurations all have infinite orbit. Let $A = \{\circ, \bullet, \textcolor{blue}{\bullet}\}$, $\Omega = \{1, a, b, a^{-1}, b^{-1}\}$, and let $P \subset A^{\Omega}$ be the set of allowed patterns in Figure 3.10.

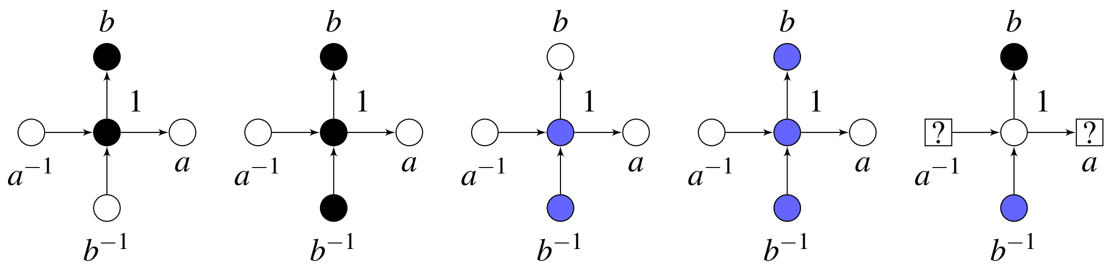


Figure 3.10: Allowed patterns for the SFT $Z \subset A^{F_2}$. A $\boxed{?}$ indicates all possible colors should be allowed in the corresponding position.

Set $Z = \mathscr{O}(P)$. It is not hard to see that Z is nonempty—an example configuration $z \in Z$ is shown in Figure 3.11. Now, any configuration $x \in Z$ must contain a $\circ$ node somewhere, say at $g \in F_2$. But then by our choice of P, $x(gb^l)$ is $\bullet$ for all $l > 0$ and $\textcolor{blue}{\bullet}$ for all $l < 0$. It follows that the configurations x^{gb^l} $(l \in \mathbb{Z})$ are all distinct (each has a $\circ$ node at b^{-l} but nowhere else along the b^l axis). Thus F_2 is not strongly SFT-periodic. $\square$

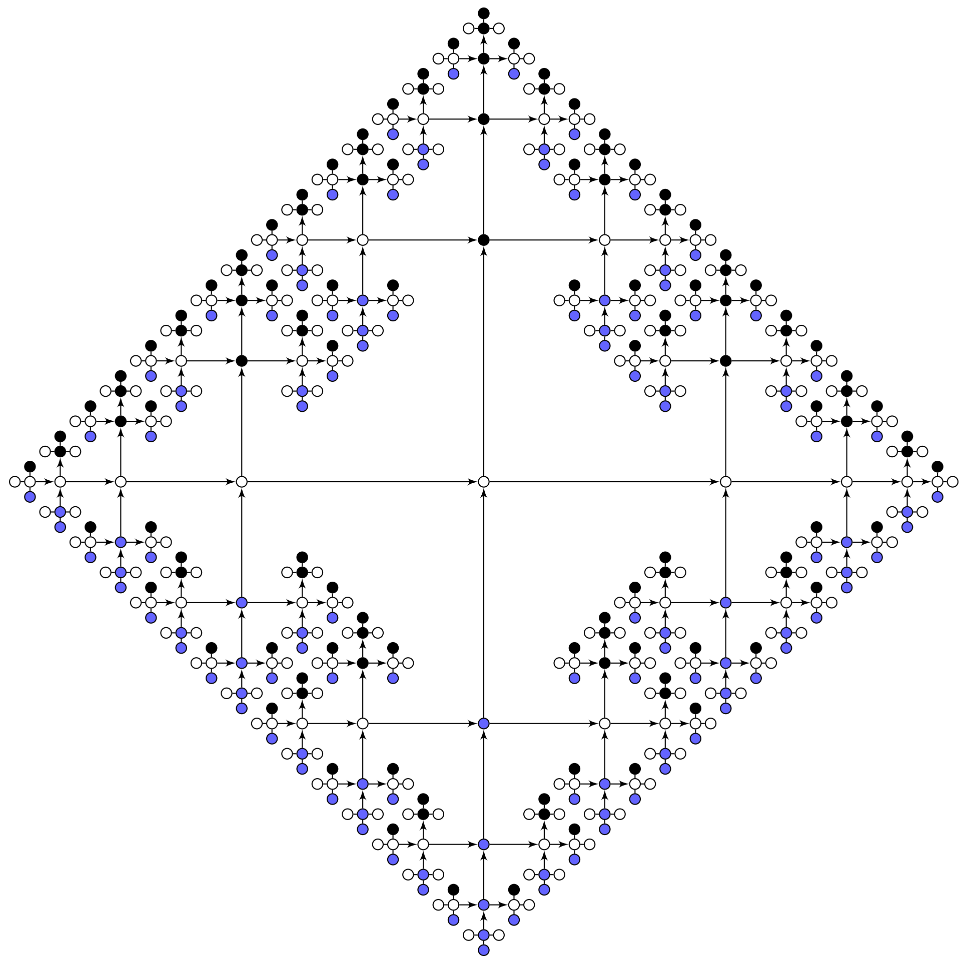


Figure 3.11: A configuration $z \in Z$. Notice that z is periodic along the a^l axis but aperiodic along the b^l axis.

Modifying the proof above for the case of F_m, $m > 2$, would not be difficult, but simpler is to recall that F_m is a finite index subgroup of F_2 for any $m \geq 2$ and invoke Theorem 4.1.4 from the next section. Thus we obtain the general case:

Corollary 3.3.6. F_m is weakly SFT-periodic but not strongly SFT-periodic for $m \geq 2$.

4. PERIODICITY AND GROWTH OF GROUPS

Our primary program of research in this study is the search for ways to classify each finitely generated group according to whether it is strongly SFT-periodic, weakly SFT-periodic, or neither. In Section 3.3 we focused on the specific groups $\mathbb{Z}^n$ and F_m, but in this section we will develop broad criteria which can be used to classify entire families of groups. We begin by showing that the properties of weak and strong SFT-periodicity are preserved under taking finite index subgroups and extensions. We then demonstrate that there is a connection between the growth of a group and its SFT-periodicity; in particular, we will see that groups of polynomial growth can be completely classified. We conclude by summarizing the current status of the classification program in the literature. Some of this material has appeared in [5].[1]

4.1 Periodicity and commensurability

Our work up to this point has hinted that the essential properties of a SFT over a group are left unchanged by operations that are finite in some sense. One explicit equivalence relation that can be thought of as a finite transformation is as follows.

Definition. Let G_1 and G_2 be groups. If there exist finite index subgroups $H_1 \leq G_1$ and $H_2 \leq G_2$ such that H_1 is isomorphic to H_2, then G_1 and G_2 are called *commensurable*.

Remark 4.1.1. Commensurability is an equivalence relation on the class of groups. To see that the relation is transitive, notice that if H is a finite index subgroup of both G_1 and G_2 and K is a finite index subgroup of both G_2 and G_3, then $H \cap K$ is a finite index subgroup of both G_1 and G_3.

The first main result of this section is that the commensurability relation over finitely

[1] Available for electronic retrieval at `http://nyjm.albany.edu/j/2015/21-36v.pdf`.

generated groups preserves weak and strong SFT-periodicity, a fact proved by the author and Penland in [5]. We will use the tools of Section 3 to prove this theorem in two stages.

Lemma 4.1.2. Let G be a group and let H be a finite index subgroup of G. If H is weakly (strongly) SFT-periodic, then G is weakly (strongly) SFT-periodic.

Proof. We first suppose H is weakly SFT-periodic. Let X be a nonempty SFT over G with alphabet A. We may assume $X = \mathscr{X}_P$, where $P \subset A^\Omega$ for a finite $\Omega \subset G$ with $1_G \in \Omega$. Let T' be a set of distinct right coset representatives for H in G with $1 \in T'$, and set $T = T'\Omega$. Since $HT = G$, we may define a higher block map $\psi_{H,T} : A^G \to B^H$ as in Section 3.2.4. We will show $\psi_{H,T}(X)$ is a SFT in B^H.

Let P' be the pattern set defined as follows:

$$P' = \{p' : \{1\} \to B : \text{there exist } p \in P, t \in T' \text{ such that } p'(1)(t\omega) = p(\omega) \text{ for all } \omega \in \Omega\}.$$

Suppose $z = \psi_{H,T}(x)$ for some $x \in A^G$. On the one hand, if $p' \in P'$ appears in z at h, then there exist $p \in P$, $t \in T'$ such that $p'(1)(t\omega) = p(\omega)$ for all $\omega \in \Omega$. That is, $z(h)(t\omega) = p(\omega)$ for all $\omega \in \Omega$, which implies $x^{ht}\big|_\Omega = p$. This shows $\psi_{H,T}(X) \subset \mathscr{X}_{P'}$.

On the other hand, if $p \in P$ appears in x at $g = ht \in G \quad (h \in H,\ t \in T')$, the same computation shows that $z^h\big|_{\{1\}} \in P'$. Thus $\mathscr{X}_{P'} \cap \psi_{H,T}\left(A^G\right) \subset \psi_{H,T}(X)$. We conclude that $\psi_{H,T}(X) = \psi_{H,T}\left(A^G\right) \cap \mathscr{X}_{P'}$, which is a SFT by Proposition 3.2.13(4).

Now, by hypothesis there exists $z = \psi_{H,T}(x) \in B^H$ such that $\mathrm{Stab}_H(z)$ is nontrivial. But $\psi_{H,T}$ is H-equivariant and injective by Proposition 3.2.13, so $\mathrm{Stab}_H(z) \subset \mathrm{Stab}_G(x)$ and $\mathrm{Stab}_G(x)$ is nontrivial as well. In conclusion, G is weakly SFT-periodic.

If in fact H is strongly SFT-periodic, then z can be chosen so that $\mathrm{Stab}_H(z)$ is of finite index in H (and hence in G), implying $\mathrm{Stab}_G(x)$ is of finite index in G as well. So G is strongly SFT-periodic. $\square$

Lemma 4.1.2 showed that one can take a SFT over G and contract it down to a SFT over H without introducing additional periodicity. The next lemma reverses the process, showing one can take a SFT over H and expand to a SFT over G, again without introducing additional periodicity.

Lemma 4.1.3. Let G be a finitely generated group and let H be a finite index subgroup of G. If G is weakly (strongly) SFT-periodic, then H is weakly (strongly) SFT-periodic.

Proof. Since H contains a finite index subgroup that is normal in G, by Lemma 4.1.2 we may assume without loss of generality that H is normal in G. Let T be a set of left coset representatives for H in G with $1 \in T$.

Suppose X is a nonempty SFT over H with alphabet A. We can find a finite subset $\Omega \subset H$ and a defining set of allowed patterns $P \subset A^{\Omega}$ for X. Now, considering Ω as a subset of G, we obtain a SFT $X' = \mathscr{O}(P)$ over G with alphabet A. X' is also nonempty: choose $x \in X$ and define $x' \in X'$ by the formula $x'(th) = x(h) \quad (t \in T, h \in H)$. Then every Ω-pattern appearing in x' also appears in x, because

$$(x')^g(\omega) = x'(g\omega) = x'(th\omega) = x(h\omega) = x^h(\omega) \quad (g = th \in G,\ \omega \in \Omega).$$

Moreover, $y'\big|_H \in X$ whenever $y' \in X'$.

Now, by Theorem 3.2.5, there exists a nonempty SFT Y over G whose configurations y satisfy $y^g = y$ if and only if $g \in H$. It follows from Proposition 3.2.1 that $X' \times Y$ is a nonempty SFT over G; moreover, given a configuration $(x', y) \in X' \times Y$, $\mathrm{Stab}_G(x', y) \subset \mathrm{Stab}_H(x'\big|_H)$. We may conclude that H is weakly (strongly) SFT-periodic whenever G is. $\square$

Together, these two lemmas comprise the proof of the following result.

Theorem 4.1.4. Let G_1 and G_2 be finitely generated commensurable groups.

(1) If G_1 is weakly SFT-periodic, then G_2 is weakly SFT-periodic.

(2) If G_1 is strongly SFT-periodic, then G_2 is strongly SFT-periodic.

There is a slightly more general notion of commensurability, called *commensurability up to finite kernels*, where, in addition to allowing the operation of taking finite index subgroups, we also allow taking a quotient by a finite group, and consider all groups which can be reached using these steps or their inverses. More precisely, we have the following definition:

Definition. Let G_1 and G_2 be groups. G_1 and G_2 are called *commensurable up to finite kernels* if there exists a sequence of homomorphisms

$$G_1 = H_1 \to H_2 \leftarrow H_3 \to \cdots H_{n-1} \leftarrow H_n = G_2$$

where each homomorphism has finite kernel and finite index image.

If G_1 and G_2 are commensurable, then they are commensurable up to finite kernels. It can be shown that the converse holds when G_1 and G_2 are residually finite, although not in general—see [11, IV.27-30] for more information.

It is a natural question to ask if SFT-periodicity is in fact preserved under commensurability up to finite kernels. Using our machinery, we may obtain a partial result in this direction:

Lemma 4.1.5. Let G be a group and let N be a finitely generated normal subgroup of G. If G is strongly SFT-periodic, then G/N is strongly SFT-periodic. Moreover, if N is not weakly SFT-periodic (i.e. has a strongly aperiodic SFT) and G is weakly SFT-periodic, then G/N is weakly SFT-periodic.

Proof. Suppose X is a nonempty SFT over G/N with alphabet A, and let $\Pi : A^{G/N} \to A^G$

be as in Section 3.2.3. By Proposition 3.2.11, $\Pi(X)$ is a nonempty SFT over G. Notice that for all $x \in A^{G/N}$, $\pi^{-1}\left(\mathrm{Stab}_{G/N}(x)\right) = \mathrm{Stab}_G(\Pi(x))$ by Proposition 3.2.10.

If G is strongly SFT-periodic, then there exists $\Pi(x) \in \Pi(X)$ such that $\mathrm{Stab}_G(\Pi(x))$ is of finite index in G. It follows that $\mathrm{Stab}_{G/N}(x)$ is of finite index in G/N, showing that G/N is strongly SFT-periodic.

Now suppose G is weakly SFT-periodic but N is not. Then there exists a nonempty SFT Y over N that is $\{1_N\}$-wide, i.e. for all $y \in Y$, $y^n = y$ for some $n \in N$ only if $n = 1$. As in the proof of Lemma 4.1.3, Y extends to a nonempty SFT Y' over G which has the following property: if $y \in Y'$ and $y^n = y$ for some $n \in N$, then $n = 1$. We then obtain the SFT $\Pi(X) \times Y'$ over G, which by hypothesis contains some weakly periodic configuration $(\Pi(x), y)$. Thus there exists $g \notin N$ satisfying $(\Pi(x), y)^g = (\Pi(x), y)$, from which it follows that $x^{\pi(g)} = x$ with $\pi(g) \neq 1_{G/N}$. This shows G/N is weakly SFT-periodic. □

In particular, since finite groups are finitely generated and not weakly periodic, Lemma 4.1.5 together with Lemma 4.1.2 shows that whenever $\pi : G \to H$ is a group homomorphism with finite kernel and finite index image, G being weakly (strongly) SFT-periodic implies H is weakly (strongly) SFT-periodic. In Section 4.2, we will see another application of Lemma 4.1.5. We also have the following restatment, which is important in its own right and seems not to have been known.

Theorem 4.1.6. Let G be a group and let N be a finitely generated normal subgroup of G. If N and G/N admit strongly aperiodic shifts of finite type, then G admits a strongly aperiodic shift of finite type.

It remains to demonstrate that SFT-periodicity can be passed from H back to G whenever $\pi : G \to H$ is a group homomorphism with finite kernel and finite index image. It seems likely that we must require G to be finitely generated to obtain the result.

Conjecture 4.1.7. Let G be a finitely generated group and let N be a finite normal subgroup of G. If G/N is weakly (strongly) SFT-periodic, then G is weakly (strongly) SFT-periodic.

4.2 Growth of groups

The study of growth of groups is centered around the following question: given a marked group (G,S), how quickly do balls in the Cayley graph of G with respect to S expand as one increases the radius? The concept of the growth of a group can be traced back to the 1950's and 60's, where it appeared in the study of the fundamental groups of three-dimensional manifolds [26] and Riemannian geometry [27].

As we have indicated, a primary thrust of investigation in this study is to obtain results for classifying each finitely generated group as strongly SFT-periodic, weakly SFT-periodic, or neither. To that end, several group invariants and equivalence relations have shown themselves to be useful in this classification, such as the commensurability relation introduced in Section 4.1. In this section, we recount the basic facts about group growth and demonstrate that there is a connection between the growth of a group and its SFT-periodicity. More details on growth, including proofs of the routine results listed here can be found in the surveys [23] and [11], as well as Chapter 6 of [6].

Definition. Let (G,S) be a marked group. The *growth function of G with respect to S* is the function $\gamma_{G,S} : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ defined by

$$\gamma_{G,S}(n) = \mathrm{card}\{g \in G : |g|_S \leq n\},$$

and the *spherical growth function of G with respect to S* is defined on $\mathbb{Z}_{\geq 0}$ by

$$\sigma_{G,S}(n) = \mathrm{card}\{g \in G : |g|_S = n\}.$$

In other words, $\gamma_{G,S}(n)$ measures the number of elements at distance at most n from a fixed basepoint in the Cayley graph of the marked group (G,S). We note that

$$\gamma_{G,S}(n) = \sum_{i=0}^{n} \sigma_{G,S}(i) \quad (n \geq 0) \text{ and}$$

$$\sigma_{G,S}(n) = \gamma_{G,S}(n) - \gamma_{G,S}(n-1) \quad (n \geq 1).$$

One can eliminate the reliance on a particular generating set S by introducing the following equivalence relation for growth functions:

Definition. Let $\gamma_1, \gamma_2 : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$ be positive functions on the natural numbers. If there exists $C \in \mathbb{Z}_{>0}$ such that $\gamma_1(n) \leq C\gamma_2(Cn)$ for all $n \in \mathbb{Z}_{>0}$, we write $\gamma_1 \preceq \gamma_2$. If $\gamma_1 \preceq \gamma_2$ and $\gamma_2 \preceq \gamma_1$, we say γ_1 and γ_2 are *equivalent* and write $\gamma_1 \sim \gamma_2$.

If $\gamma_1 \preceq \gamma_2$ but γ_1 and γ_2 are not equivalent, we write $\gamma_1 \precneqq \gamma_2$.

Proposition 4.2.1. Let S, S' be two finite generating sets for the group G. Then $\gamma_{G,S} \sim \gamma_{G,S'}$.

Notice that $\preceq$ is a preorder on functions $\mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$, making $\sim$ an equivalence relation, and that $\preceq$ induces a partial order on $\sim$-equivalence classes (henceforth called *growth types*).

Example 4.2.2 (Power growth)**.** If $\gamma : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$ can be defined by a polynomial function $p(n)$ of degree d, then $\gamma \sim n^d$. More generally, given $r, s > 0$, $n^r \preceq n^s$ if and only if $r \leq s$, implying that $n^r \nsim n^s$ whenever $r \neq s$.

Example 4.2.3 (Exponential growth)**.** Let $\alpha, \beta > 1$. Then $\alpha^n \sim \beta^n$. Thus, unlike polynomial functions, exponential functions all have the same growth type.

Example 4.2.4 (Intermediate growth)**.** It is clear that $n^d \preceq e^n$ for any $d > 0$; that is, power growth is always dominated by exponential growth. It is less obvious that there exist functions γ such that

$$n^d \precneqq \gamma(n) \precneqq e^n \text{ for all } d > 0, \tag{4.1}$$

but it can be shown that any function $\gamma_\alpha : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$ defined by $\gamma_\alpha(n) = e^{n^\alpha}$ for $0 < \alpha < 1$ satisfies (4.1). Functions or growth types satisfying (4.1) are said to have *intermediate growth*. In fact, $\alpha \neq \alpha'$ implies $\gamma_\alpha \nsim \gamma_{\alpha'}$.

We are, of course, concerned not with the growth types of arbitrary functions but with the growth types of functions of the form $\gamma_{G,S}(n)$, where (G,S) is a marked group. These functions have some nice properties, such as being strictly increasing when G is infinite and having at most exponential growth. The precise statements are listed below.

Proposition 4.2.5. Let (G,S) be a marked group.

(1) $\gamma_{G,S} \sim 1$ if and only if G is finite.

(2) If G is infinite, then $\sigma_{G,S}(n) > 0$ for all n. Thus $n \preceq \gamma_{G,S}$.

(3) $\gamma_{G,S} \preceq e^n$.

It turns out that the three growth types introduced in Examples 4.2.2 - 4.2.4 are exactly the three possible types of growth functions for groups. More precisely, we have the following striking results:

Theorem 4.2.6 (Gromov 1981 [19])**.** Let G be a finitely generated group. G has polynomial growth if and only if G is virtually nilpotent (that is, G has a nilpotent subgroup of finite index). Moreover, if $\gamma_G(n) \sim n^d$, then d is an integer.

Theorem 4.2.7 (Grigorchuk 1984 [17])**.** There exists a group of intermediate growth.

In [5], Penland and the author first observed that Gromov's theorem gives a close connection between groups of polynomial growth and SFT-periodicity. From our work above we have the following result:

Theorem 4.2.8. Suppose G has polynomial growth of degree d. Then G is strongly SFT-periodic if and only if $d = 1$.

Theorem 4.2.8 follows from the fact that a group is commensurable with $\mathbb{Z}$ if and only if it has linear growth, along with following lemma, which is a generalization of [5, Corollary 14]. (Note that nilpotent groups are automatically polycylic, although the converse may not hold.)

Lemma 4.2.9. Suppose G is virtually nilpotent but not virtually cyclic. Then G is not strongly SFT-periodic.

Proof. We may assume G is nilpotent. Since G is not virtually cyclic, there exists a surjective homomorphism $f : G \to \mathbb{Z}^2$, and as every subgroup of a finitely generated nilpotent group is finitely generated, $N := \ker f$ is finitely generated. By Lemma 4.1.5, G is not strongly periodic. $\square$

Recently, Emmanuel Jeandel was able to extend Theorem 4.2.8 by showing that virtually polycyclic groups (excluding virtually cyclic ones) are not even weakly periodic [21]. Thus we have a complete classification of groups of polynomial growth:

Theorem 4.2.10. Let G be a finitely generated group with polynomial growth of degree d.

(1) If $d = 0$, then G is strongly SFT-periodic but not weakly SFT-periodic (Proposition 3.3.5).

(2) If $d = 1$, then G is strongly (and weakly) SFT-periodic.

(3) If $d \geq 2$, then G is neither weakly nor strongly SFT-periodic, i.e. G has a strongly aperiodic shift of finite type (Jeandel).

We remark that this classification relies heavily on Gromov's theorem. Naturally, one next turns to groups of exponential growth, although here the situation is more complicated. For example, Cohen has shown that all finitely generated groups with at least two ends (this class includes free groups) are weakly SFT-periodic [7]. On the other hand, Mozes has produced strongly aperiodic SFTs on cocompact lattices in higher rank semisimple Lie groups [28]. Thus, although all groups of exponential growth have equivalent growth rates under the relation $\sim$ described above, their SFT-periodicity behavior might vary. Hence we seek to extract finer invariants from groups of exponential growth which might enable us to produce classification results. In Section 5, we will introduce the *(exponential) growth spectrum* of a group of exponential growth, which is a generalization of the uniform exponential growth rate.

4.3 Current classification status

We conclude this section by tabulating the current results in the literature for SFT-periodicity. In some cases, we have listed redundant results for clarity.

4.3.1 Rigidity results

1. If G_1, G_2 are finitely generated commensurable groups and G_1 is weakly (strongly) SFT-periodic, then G_2 is weakly (strongly) SFT-periodic [5].

2. If N is a finitely generated normal subgroup of G and G is strongly SFT-periodic, then G/N is strongly SFT-periodic [5].

3. If N is a finitely generated normal subgroup of G and both N and G/N have strongly aperiodic SFTs, then G has a strongly aperiodic SFT (Theorem 4.1.6).

4. If G_1, G_2 are finitely presented torsion-free groups that are quasi-isometric to each other and G_1 has a strongly aperiodic SFT, then G_2 has a strongly aperiodic SFT [7].

4.3.2 Completely classified groups

Tables 4.1, 4.2, and 4.3 list groups whose SFT-periodicity have been completely classified.

Table 4.1: Strongly SFT-periodic groups

$\mathbb{Z}$ (folklore)
All two-ended groups
All groups of linear growth

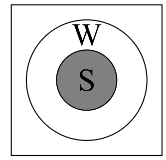


Table 4.2: Weakly but not strongly SFT-periodic groups

F_n, $n \geq 2$ [29]
$\mathrm{SL}_2(\mathbb{Z})$

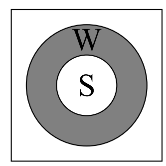
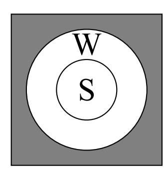


Table 4.3: Groups that are not weakly SFT-periodic

$\mathbb{Z}^n$, $n \geq 2$ [2], [10]
All virtually polycyclic groups [21]
All groups with polynomial growth of degree ≥ 2
Surface groups [8]
$\mathbb{Z} \times T$, where T is Thompson's group [20]
Cocompact lattices in higher rank semisimple Lie groups [28]



4.3.3 Partially classified groups

Tables 4.4 and 4.5 list groups whose SFT-periodicity have been partially classified.

Table 4.4: Weakly SFT-periodic groups

Groups with at least two ends [7]
Recursively presented groups with undecidable word problem [20]

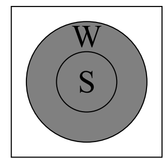
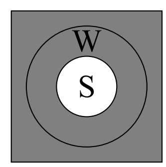


Table 4.5: Groups that are not strongly SFT-periodic

Amenable groups having some *p* not dividing the index of any subgroup, such as the Grigorchuk group [25]
Nonamenable groups [3]
Baumslag-Solitar groups [1]



4.3.4 Conjectures

Conjecture 4.3.1 ([5])**.** A group is strongly SFT-periodic if and only if it is virtually cyclic.

5. EXPONENTIAL GROWTH RATES OF GROUPS

Group growth was introduced in Section 4.2, where it was shown that groups of polynomial growth have been completely classified according to SFT-periodicity. In this section we consider groups of exponential growth.

In geometric applications, exponential growth rates are particularly important; for example, Manning showed that if Γ is the fundamental group of a Riemannian manifold of unit diameter and ω its exponential growth rate with respect to an appropriate set of generators, then $\log \omega$ is a lower bound for the topological entropy of the geodesic flow of the manifold [24]. The exponential growth rate of a group is also naturally associated with properties of random walks on its Cayley graph; see [16] for a survey of this topic and others related to exponential growth.

We have seen in Section 4.2 that there exists a group of polynomial growth of degree n if and only if $n \in \{0, 1, 2, 3, \ldots\}$. By analogy, a natural question to ask is the following: which real numbers ω can be realized as the exponential growth rate of some group with respect to some generating set? In contrast to the case of polynomial growth, this question has seen surprisingly little scrutiny. In this section we develop some tools to provide partial answers to this question.

5.1 Computing exponential growth rates

Definition. Let (G, S) be a marked group. The *exponential growth rate* of G with respect to S is defined to be

$$\omega(G,S) = \limsup_{n\to\infty} \sqrt[n]{\gamma_{G,S}(n)}.$$

Needing to use a limit superior in the definition above is slightly inconvenient. The following aid to computing exponential growth rates is well-known (see [11, VI.51, 53(v), 56]).

Proposition 5.1.1. Let G be an infinite group with finite generating set S. Then

$$\omega(G,S) = \lim_{n\to\infty} \sqrt[n]{\gamma_{G,S}(n)} = \lim_{n\to\infty} \sqrt[n]{\sigma_{G,S}(n)}.$$

In particular, both limits exist.

Example 5.1.2. The simplest case in which the exponential growth rate $\omega(G,S)$ can be computed is when G is free with basis S. In this case, given a geodesic word $g = s_1 s_2 \cdots s_n \quad (s_i \in S \cup S^{-1})$, a word g' has length $n+1$ with prefix g if and only if $g' = gs$ for some $s \in S \cup S^{-1}$, $s \neq s_n^{-1}$. Thus

$$\sigma_{G,S}(n+1) = (2|S|-1)\sigma_{G,S}(n) \tag{5.1}$$

for all $n \geq 1$ and we have

$$\omega(G,S) = \lim_{n\to\infty} \sqrt[n]{\sigma_{G,S}(n)} = \lim_{n\to\infty} \sqrt[n]{(2|S|-1)^{n-1}(2|S|)} = 2|S|-1.$$

We remark that (5.1) is a recurrence relation. Growth series, which will be introduced in Section 5.1.1 below, can be used to systematically exploit recurrence relations to compute exponential growth rates—see Example 5.1.4 for a simple example.

Exponential growth rates provide a definition of exponential growth alternate to the one given in Section 4.2:

Proposition 5.1.3. Let G be a group with generating set S. Then $\gamma_{G,S}$ has exponential growth if and only if $\omega(G,S) > 1$.

We remark that since the property of having exponential growth is invariant under change of generators, this implies G has exponential growth if and only if $\omega(G,S) > 1$ for *all* generating sets S of G.

Proof of Proposition 5.1.3. Suppose $\gamma_{G,S}(n) \sim \alpha^n$, where $\alpha > 1$. Then there exists a constant $C > 0$ such that

$$\frac{1}{C}\alpha^n \leq \gamma_{G,S}(Cn) \quad (n \geq 1).$$

Raising to the power of $1/Cn$ and taking the limit as $n \to \infty$, we obtain

$$1 < \alpha^{1/C} \leq \omega(G,S).$$

On the other hand, if $\omega(G,S) > 1$, there exists $\alpha > 1$ such that $\sqrt[n]{\gamma_{G,S}(n)} \geq \alpha$ for all n sufficiently large, i.e. $\gamma_{G,S}(n) \geq \alpha^n$. Since the growth of $\gamma_{G,S}$ cannot be greater than exponential, we conclude $\gamma_{G,S}(n) \sim \alpha^n$. $\square$

Henceforth, G will denote a finitely generated infinite group and S an arbitrary finite generating set for G.

We seek methods to compute or at least approximate $\omega(G,S)$. In some cases this can be done directly, as in Example 5.1.2, where we explicitly computed $\sigma_{G,S}(n)$. In many cases, however, exact knowledge of $\sigma_{G,S}(n)$ may be difficult to obtain and is not necessary to compute $\omega(G,S)$. For example, if F_2 is free on $\{x,y\}$, we will be able to quickly show that $\omega(F_2,\{x,y,x^2\}) = 2\sqrt{2}+1$, though the exact growth function of F_2 with respect to $\{x,y,x^2\}$ is tedious to compute (see Example 5.1.6).

5.1.1 Growth series

The *growth series of G with respect to S* is the power series

$$\Gamma_{G,S}(z) = \sum_{n=0}^{\infty} \gamma_{G,S}(n) z^n$$

and the *spherical growth series of G with respect to S* is

$$\Sigma_{G,S}(z) = \sum_{n=0}^{\infty} \sigma_{G,S}(n) z^n.$$

Note that $\Sigma(z) = (1-z)\Gamma(z)$, so these series have the same radius of convergence. Moreover, it is immediate from the definition of exponential growth rate that this radius of convergence is $\omega(G,S)^{-1}$. Hence $\Gamma(z)$ and $\Sigma(z)$ are virtually interchangeable.

If $\Sigma(z)$ is a (reduced) rational function $P(z)/Q(z)$, then $\omega(G,S)$ can be computed as the inverse of the root of $Q(z)$ that is nearest to zero. (By a fact from analysis sometimes called Pringsheim's theorem (see [14, Theorem IV.6]), we may assume the dominant singularity of $\Sigma(z)$ is positive real since the coefficients of Σ are nonnegative.)

Example 5.1.4. Suppose (G,S) is a marked group satisfying

$$\sigma_{G,S}(n+1) = K\sigma_{G,S}(n) \quad \text{for all } n \geq N, \tag{5.2}$$

where K and N are constants. Proposition 5.1.1 can be used to show $\omega(G,S) = K$, but alternatively, we may multiply (5.2) by z^{n+1} and sum from N to ∞ to obtain

$$\sum_{n=N+1}^{\infty} \sigma_{G,S}(n) z^n = Kz \sum_{n=N}^{\infty} \sigma_{G,S}(n) z^n.$$

Rearranging, we obtain

$$\sum_{n=N+1}^{\infty} \sigma_{G,S}(n) z^n = \frac{K\sigma_{G,S}(N) z^{N+1}}{1-Kz},$$

showing that $\omega(G,S) = (1/K)^{-1} = K$.

For more information about the technique used here of determining the asymptotic behavior of a combinatorial sequence from its generating function, see [14].

It is often simpler to find a closed form for $\Sigma_{G,S}(z)$ than to compute $\sigma_{G,S}$. Moreover, it is sometimes possible to compute growth series by decomposing a group as a direct product or free product:

Proposition 5.1.5. Let (G_1, S_1), (G_2, S_2) be marked groups with spherical growth series $\Sigma_1(z) = \Sigma_{G_1,S_1}(z)$ and $\Sigma_2(z) = \Sigma_{G_2,S_2}(z)$.

(1) If $S \subset G_1 \times G_2$ is the generating set $(S_1 \times \{1_{G_2}\}) \cup (\{1_{G_1}\} \times S_1)$, then

$$\Sigma_{G_1 \times G_2, S}(z) = \Sigma_1(z)\Sigma_2(z).$$

(2) If $S \subset G_1 \times G_2$ is the generating set $(S_1 \cup \{1_{G_1}\}) \times (S_2 \cup \{1_{G_2}\})$, then

$$\Sigma_{G_1 \times G_2, S}(z) = \Sigma_1(z) \circ \Sigma_2(z),$$

where $\circ$ denotes term-by-term multiplication, i.e. $\left(\sum a_n z^n\right)\left(\sum b_n z^n\right) = \sum a_n b_n z^n$.

(3) If $S \subset G_1 * G_2$ is the generating set $S_1 \cup S_2$, then

$$\Sigma_{G_1 * G_2, S}(z) = \frac{\Sigma_1(z)\Sigma_2(z)}{1 - (\Sigma_1(z) - 1)(\Sigma_2(z) - 1)}.$$

(For proofs of (1) and (3), see [11, VI.4]. (2) is clear.)

Example 5.1.6. Consider $\mathbb{Z}$ with the standard generating set $S = \{1\}$. Evidently $\sigma_{\mathbb{Z},S}(0) = 1$ and $\sigma_{\mathbb{Z},S}(n) = 2$ for $n \geq 1$. Thus

$$\Sigma_{\mathbb{Z},S}(z) = 1 + 2z + 2z^2 + \cdots = 1 + 2z\left(\frac{1}{1-z}\right) = \frac{1+z}{1-z}.$$

Similarly, if we equip $\mathbb{Z}$ with the generating set $S_2 = \{1, 2\}$, by induction $|g|_{S_2} = n$ for $n \geq 1$ if and only if $g \in \{2n, 2(n-1)+1, -2n, -2(n-1)-1\}$. Thus we have $\sigma_{\mathbb{Z},S_2}(0) = 1$ and

$\sigma_{\mathbb{Z},S_2}(n) = 4$ for $n \geq 1$, yielding

$$\Sigma_{\mathbb{Z},S_2}(z) = \frac{1+3z}{1-z}.$$

Thus if F_2 is free on $\{x,y\}$, we can use Proposition 5.1.5(3) to compute that

$$\sigma_{F_2,\{x,y,x^2\}}(z) = \frac{\Sigma_{\mathbb{Z},S}(z)\Sigma_{\mathbb{Z},S_2}(z)}{1-(\Sigma_{\mathbb{Z},S}(z)-1)(\Sigma_{\mathbb{Z},S_2}(z)-1)} = \frac{(1+z)(1+3z)}{1-2z-7z^2}.$$

The root of $Q(z) = 1-2z-7z^2$ nearest to 0 is $(2\sqrt{2}-1)/7$. Hence $\omega(F_2,\{x,y,x^2\}) = 7/(2\sqrt{2}-1) = 2\sqrt{2}+1 \approx 3.828$.

5.1.2 Regular languages

Let A be a finite alphabet. In this section the elements of A will typically be referred to as *letters*. The *free monoid over A* is the set A^* of all finite sequences of letters:

$$A^* := \{a_1 a_2 \cdots a_m : m \geq 0,\ a_j \in A \text{ for } j = 1,\ldots,m\}.$$

Elements of A^* are called *words (over A)*. The *length* of a word $w = a_1 \cdots a_m$ is $|w| := m$. The unique word of length 0 is called the *empty word* and will be denoted by ε. With ε as identity, A^* is a monoid under the operation of concatenation, i.e. $(a_1 \cdots a_m)\cdot(b_1 \cdots b_n) := a_1 \cdots a_m b_1 \cdots b_n$. We remark that A^* can also be regarded as a submonoid of the free group on the generators A.

A *language* $\mathscr{L}$ (with alphabet A) is a subset of A^*. As with groups, we can define the spherical growth series of a language to be $\Sigma_{\mathscr{L}}(z) = \sum_n \sigma_{\mathscr{L}}(n) z^n$, where $\sigma_{\mathscr{L}}(n)$ is the number of words in $\mathscr{L}$ of length n; we also have an exponential growth rate $\omega(\mathscr{L}) = \lim_{n\to\infty} \sqrt[n]{\sigma_{\mathscr{L}}(n)}$.

A *finite state automaton* consists of an alphabet A together with a finite set $S = \{s_1, s_2, \ldots, s_n\}$

of states, a subset $S_a \subset S$ of which are designated as *accept states*, a distinguished *initial state* $s_1 \in S$, and a *transition function* $\mu : S \times A \to S$. A finite state automata can be depicted graphically, as in Figure 5.1 below.

Given a finite state automaton $\mathfrak{A} = (A, S, S_a, s_1, \mu)$ as above, the *language defined by* $\mathfrak{A}$ is the set of all words $a_1 a_2 \cdots a_r \in A^*$ such that there exist states $t_0 = s_1, t_1, \ldots, t_r$ with $\mu(t_{i-1}, a_i) = t_i$ for $i = 1, \ldots, r$ and $t_r \in S_a$. If a language can be defined by a finite state automaton, it is called a *regular language*.

Example 5.1.7. Consider the finite state automaton $\mathfrak{A}$ defined by the data below:

$$A = \{x, y, \bar{x}, \bar{y}\} \qquad S = \{1,2,3,4,5,6\}$$
$$S_a = \{1,2,3,4,5\} \quad s_1 = 1$$

μ	x	$\bar{x}$	y	$\bar{y}$
1	2	3	4	5
2	2	6	4	5
3	6	3	4	5
4	6	6	4	6
5	6	6	6	5
6	6	6	6	6

A graphical representation of $\mathfrak{A}$ is shown in Figure 5.1. $\mathfrak{A}$ can be thought of as a directed graph whose (labeled) arrows are the transitions $(s, a) \xrightarrow{a} \mu(s, a)$. The language of the automaton then corresponds to the set of paths in the directed graph starting from s_1 and ending in an accepted state. We will always assume there exists a path from the start state s_1 to any other state $s \neq s_1$ (else s could simply be removed from the automaton). We have omitted the state 6 from the picture since it is an absorbing node and is not an accept state; one may regard it as a *fail state*, as no path which travels to state 6 can result in an accepted word.

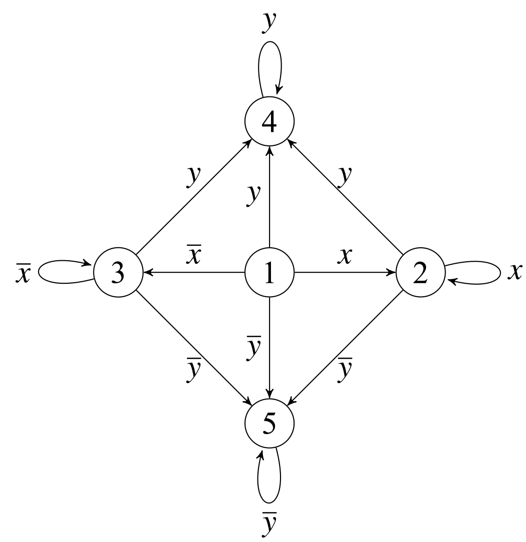


Figure 5.1: The finite state automaton $\mathfrak{A}$ from Example 5.1.7

The language defined by the automaton in this example accepts exactly those words that are of the form $s^m t^n$, where $s \in \{x, \overline{x}\}$ and $t \in \{y, \overline{y}\}$ and $m, n \geq 0$.

The language defined by the automaton in Example 5.1.7 is clearly somehow representative of the free abelian group of rank 2. Let us make this precise.

Definition. Let (G,S) be a marked group, and regard S as a formal alphabet. If there exists some language $\mathscr{L}$ with alphabet S such that the function $\pi : \mathscr{L} \to G$ sending each word $w \in \mathscr{L}$ to its representative in G is a bijection, then $\mathscr{L}$ is called a *normal form* for (G,S). Moreover, if $|w| = |\pi(w)|_S$ for all $w \in \mathscr{L}$ (i.e. every word w is geodesic), then $\mathscr{L}$ is called a *geodesic normal form* for (G,S).

Remark 5.1.8. In Example 5.1.7, $\mathscr{L}$ is a geodesic normal form for $(\mathbb{Z}^2, \{(\pm 1, 0), (0, \pm 1)\})$ under the correspondence

$$x \mapsto (1,0), \quad \overline{x} \mapsto (-1,0),$$
$$y \mapsto (0,1), \quad \overline{y} \mapsto (0,-1).$$

In general, if there exists a geodesic normal form $\mathscr{L}$ for a marked group (G,S), the language has the same growth series (and hence the same exponential growth rate) as the marked group:

$$\Sigma_{\mathscr{L}}(z) = \Sigma_{G,S}(z).$$

(See Example 5.1.10 for the growth series of the language above.)

Returning to the general case, let $\mathscr{L}$ be a language defined by a finite state automaton $\mathfrak{A} = (A,S,S_a,s_1,\mu)$. To count words in $\mathscr{L}$, let $M = (m_{ij})$ be the $n \times n$ transition matrix of $\mathfrak{A}$, i.e. $m_{ij} = \text{card}\{(s_i,a) \in S \times A \mid \mu(s_i,a) = s_j\}$. Then the $(1,j)$-th entry of M^n is the number of length-n paths from s_1 to s_j, and we conclude that

$$\sigma_{\mathscr{L}}(n) = \sum_{s_j \in S_a} (M^n)_{1j} = aM^n b, \tag{5.3}$$

where $a = (1,0,\dots 0)$ and b is the column vector with 1's in the positions corresponding to accept states and 0's elsewhere.

Thus we are led to the following result (this proof comes from [12]):

Proposition 5.1.9. Let $\mathscr{L}$ be a regular language. The spherical growth series of $\mathscr{L}$ is a rational function with integer coefficients.

Proof. Let M be defined as above. Using (5.3), the spherical growth series of $\mathscr{L}$ is

$$\sum_{n=0}^{\infty}(aM^n b)z^n = a\left(\sum_{n=0}^{\infty}(zM)^n\right)b = a(I-zM)^{-1}b = \frac{P(z)}{\det(I-zM)}, \tag{5.4}$$

where $P(z)$ is some polynomial with integer coefficients. □

Example 5.1.10. Equation (5.4) can also be used to compute growth series. For example, the transition matrix M for the automaton $\mathfrak{A}$ described in Example 5.1.7 is

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

With this M, one can compute

$$(I - zM)^{-1} = \begin{pmatrix} 1 & -\frac{z}{z-1} & -\frac{z}{z-1} & \frac{z(z+1)}{(z-1)^2} & \frac{z(z+1)}{(z-1)^2} & -\frac{4z^2(z+2)}{(z-1)^2(4z-1)} \\ 0 & -\frac{1}{z-1} & 0 & \frac{z}{(z-1)^2} & \frac{z}{(z-1)^2} & -\frac{z(5z+1)}{(z-1)^2(4z-1)} \\ 0 & 0 & -\frac{1}{z-1} & \frac{z}{(z-1)^2} & \frac{z}{(z-1)^2} & -\frac{z(5z+1)}{(z-1)^2(4z-1)} \\ 0 & 0 & 0 & -\frac{1}{z-1} & 0 & \frac{3z}{(z-1)(4z-1)} \\ 0 & 0 & 0 & 0 & -\frac{1}{z-1} & \frac{3z}{(z-1)(4z-1)} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{4z-1} \end{pmatrix}$$

so that by (5.4),

$$\Sigma_{\mathscr{L}}(n) = 1 - \frac{z}{z-1} - \frac{z}{z-1} + \frac{z(z+1)}{(z-1)^2} + \frac{z(z+1)}{(z-1)^2} = \left(\frac{z+1}{z-1}\right)^2.$$

This is exactly what we expect, given Proposition 5.1.5, Example 5.1.6, and Remark 5.1.8. We remark that the computation can be simplified slightly by omitting state 6 from the matrix, as it represents a strongly connected component (of size one) which cannot reach any accept states.

Factoring z out of the determinant in (5.4) suggests the following:

Proposition 5.1.11. Suppose $\mathscr{L}$ is a language defined by an automaton $\mathfrak{A}$ with transition matrix M. Let λ be the spectral radius of M (i.e. the largest magnitude among the eigenvalues of M). Then $\omega(\mathscr{L}) \leq \lambda$.

Proof. Let $\|\cdot\|$ denote the 2-norm on matrices. Then, defining a and b as in (5.3),

$$\omega(\mathscr{L}) = \lim_{n\to\infty} \sqrt[n]{aM^n b} = \lim \sqrt[n]{\|aM^n b\|} \leq \lim \sqrt[n]{\|a\|\|M^n\|\|b\|} = \lim \sqrt[n]{\|M^n\|} = \lambda, \quad (5.5)$$

where we have used Gelfand's formula in the last equality. □

We would like the spectral radius to in fact equal the exponential growth rate. However, Example 5.1.10 shows that this may not occur: $\mathbb{Z}^2$ has exponential growth rate 1, but the spectral radius of M is 4. The anomaly occurs because of the "fail state," which cannot reach any accept states. Thus we require a partition of the states into those which can reach some accept state and those which cannot.

Proposition 5.1.12. Suppose $\mathscr{L}$ is a language defined by an automaton $\mathfrak{A}$ with initial state s_1 and other states $s_2, \ldots, s_m$. Also suppose that the states of $\mathfrak{A}$ have been partitioned into $s_1, \ldots, s_k$, each of which can reach some accept state, and $s_{k+1}, \ldots, s_m$, which cannot any accept state. Write the transition matrix with respect to $\{s_1, \ldots, s_m\}$ as

$$M = \begin{pmatrix} M_a & A \\ 0 & B \end{pmatrix},$$

where M_a is $k \times k$. Let λ be the spectral radius of M_a. Then $\omega(\mathscr{L}) = \lambda$.

Proof. First, notice that if we let

$$M_2 = \begin{pmatrix} M_a & 0 \\ 0 & I \end{pmatrix},$$

then $aM^n b = aM_2^n b$ for all $n \geq 0$, where a, b are as in (5.3). Thus the computation in (5.5) shows that $\omega(\mathscr{L}) \leq \lambda$, since λ is the spectral radius of M_2 as well.

By the theory of nonnegative matrices (e.g. see [15, Vol. 2, p.66]), we may take λ to be real and we are guaranteed an eigenvector y of length k with $y \geq 0$, $y \neq 0$ such that $M_a y = \lambda y$. By scaling, we may furthermore assume that each entry $y_i \leq 1$ and that $y_{i_0} = 1$ for one specific index i_0. Extend y to be an m-vector y' by appending $m-k$ zeros to the end.

Now, for each $s_i \in \{s_1, \ldots, s_k\}$, there exists some path from s_i to an accept state, say of length r_i. In other words, the ith entry of $M^{r_i}b$ is at least 1. By hypothesis, we then have

$$(M^{r_1} + M^{r_2} + \cdots + M^{r_k})\, b \geq y'.$$

Denote by e_{i_0} the row k-vector with a 1 in position i_0 and zeros elsewhere, and let e'_{i_0} be the row m-vector of the same form. Since we assume the starting state s_1 can reach any other state, there is q large enough that

$$aM^q \geq e'_{i_0}.$$

Thus we are led to the following conclusion:

$$\sum_{i=1}^{k} aM^{q+n+r_i}b \geq e'_{i_0}M^n y' = e_{i_0}M_a^n y = e_{i_0}\lambda^n y = \lambda^n.$$

Hence it suffices to show that

$$\omega(\mathscr{L}) = \lim_{n\to\infty} \sqrt[n]{\sum_{i=1}^{k} aM^{q+n+r_i}b}.$$

For brevity, let $\omega = \omega(\mathscr{L})$ and $x_n^{(i)} = aM^{q+n+r_i}b$. Suppose $\varepsilon > 0$ is given. Since $\lim_{n\to\infty} \sqrt[n]{x_n^{(i)}} =$

ω for each i, whenever n is sufficiently large we have

$$\begin{aligned}
&(1-\varepsilon)\omega \leq \sqrt[n]{x_n^{(i)}} \leq (1+\varepsilon)\omega, && \text{i.e.} \\
&(1-\varepsilon)^n\omega^n \leq x_n^{(i)} \leq (1+\varepsilon)^n\omega^n && \text{for all } i, \text{ implying} \\
&k(1-\varepsilon)^n\omega^n \leq \sum_{i=1}^{k} x_n^{(i)} \leq k(1+\varepsilon)^n\omega^n. &&
\end{aligned}$$

Taking nth roots to the limit yields

$$(1-\varepsilon)\omega \leq \lim_{n\to\infty} \sqrt[n]{\sum_{i=1}^{k} aM^{q+n+r_i}b} \leq (1+\varepsilon)\omega.$$

Since ε was arbitrary, we have the result. □

(Proposition 5.1.12 is stated in [13], although without the complete hypothesis and details; we have provided our own proof.)

Proposition 5.1.12 gives a fast way to compute the exponential growth rates of groups which have geodesic normal forms.

5.2 Growth spectra

Our focus is on the following open question:

Question 5.2.1. Which real numbers $\alpha > 1$ can be realized as exponential growth rates $\omega(G,S)$?

To approach this question, we first introduce some notation and describe some known results. If G is a finitely generated group and d is a positive integer, we define

$$\mathrm{Gspec}_d(G) = \{\omega(G,S) : |S| \leq d, \ S \text{ generates } G\}$$

and

$$\operatorname{Gspec}(G) = \bigcup_{d\geq 1} \operatorname{Gspec}_d(G).$$

$\operatorname{Gspec}(G)$ is called the *growth spectrum* of G. We also define $\Omega_d = \bigcup_G \operatorname{Gspec}_d(G)$, where the union is over all groups that can be generated by at most d elements, and $\Omega = \bigcup_{d\geq 1} \Omega_d$.

Question 5.2.1 is then simply "what is Ω?" and has been asked by de la Harpe [11, VI.61], among others. The following partial results are known:

Theorem 5.2.2. (1) $\Omega_d \subset [1, 2d-1]$, and $1, 2, \ldots, 2d-1$ are accumulation points in Ω_d.

(2) Ω is dense in $[1, \infty)$.

(3) Ω_2 is uncountable.

(1) and (2) are due to Grigorchuk and de la Harpe [18]. (3) is due to Erschler in [13].

In this section our investigations will focus on the growth spectra of specific groups G.

Question 5.2.3. Fix a finitely generated group G.

(1) Under what algebraic operations is $\operatorname{Gspec}(G)$ closed?

(2) Does $\operatorname{Gspec}(G)$ have limit points?

(3) Is $\operatorname{Gspec}(G)$ anywhere dense?

For example, we have the following result for (1):

Proposition 5.2.4. If $\alpha \in \operatorname{Gspec}(G)$, then $\alpha^r \in \operatorname{Gspec}(G)$ for all $r \geq 1$. In particular, if G has exponential growth, then $\operatorname{Gspec}(G)$ is unbounded.

Proof. Let $\alpha = \omega(G,S)$. We may assume $1 \in S$ and $S^{-1} = S$, so we have $\gamma_{G,S}(n) = |S^n|$. Thus

$$\omega(G,S^r) = \lim_{n\to\infty} \sqrt[n]{|(S^r)^n|} = \lim_{n\to\infty} \left(\sqrt[rn]{|S^{rn}|} \right)^r = (\omega(G,S))^r.$$

$\square$

On the algebraic structure of $\mathrm{Gspec}(G)$, we have the following result:

Theorem 5.2.5 ([4])**.** Let G be a hyperbolic group. Then $\mathrm{Gspec}(G)$ consists entirely of integer algebraic numbers.

Notice, however, that Theorem 5.2.2(3) implies that there exist exponential growth rates that are not algebraic numbers.

5.3 On the growth spectrum of F_2

Our goal in this section is to establish some groundwork for computing the structure of $\mathrm{Gspec}(F_2)$, the free group on two generators. The techniques we use can be adapted to other marked groups of exponential growth in some cases.

Let us begin by computing the exponential growth rate of F_2 with respect to $\{x, x^{2k+1}, y, y^{2j+1}\}$, where $k, j \geq 0$. (We consider only odd exponents to reduce the number of technical details needed to be taken into account, but our considerations can be reworked for the case of even exponents as well.)

First, we design an automaton $\mathfrak{A}$ to give a geodesic normal form for $\langle x \rangle$ with respect to the generating set $\{x, x^{2k+1}\}$. The automaton is shown in Figure 5.2.

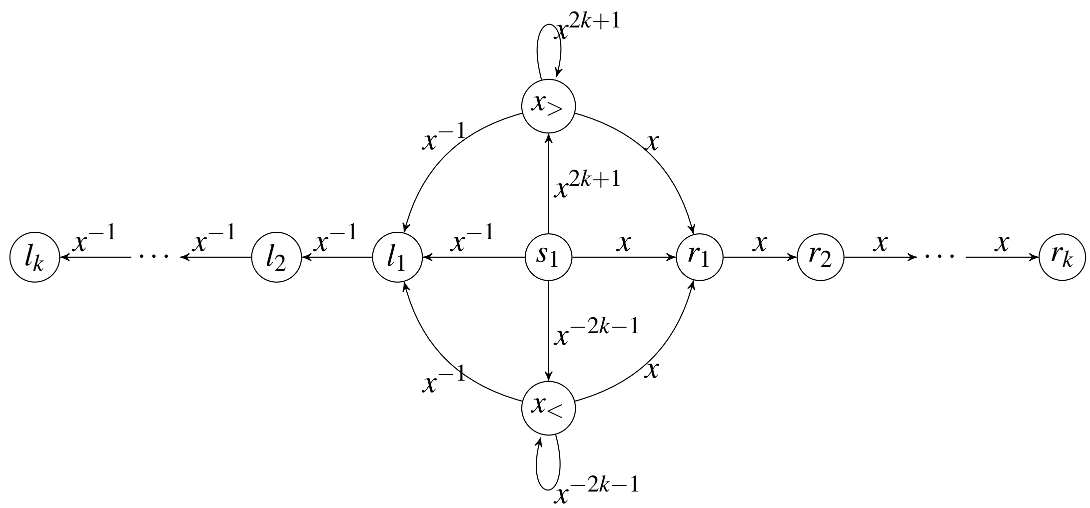


Figure 5.2: Automaton $\mathfrak{A}$ for $(\langle x\rangle, \{x, x^{2k+1}\})$

In general, in reading such a diagram we assume every state shown is an accept state unless noted otherwise. We also omit the fail state and any arrows leading to it. Proposition 5.1.12 shows that we may ignore the fail state in constructing the transition matrix.

Writing the states of $\mathfrak{A}$ in the order $s_1, x_<, x_>, l_1, \ldots, l_k, r_1, \ldots, r_k$, the transition matrix (in blocks) is

$$M = \begin{pmatrix} A & B & B \\ 0 & J & 0 \\ 0 & 0 & J \end{pmatrix},$$

where A is 3×3, B is $3 \times k$, J is $k \times k$, and

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix},$$

that is, J has 1's above the diagonal and 0's elsewhere.

Now, we may compute the growth series using (5.4). Making the substitution $w = z^{-1}$ to clarify the computation, $(I - zM)^{-1} = -z^{-1}(M - z^{-1}I)^{-1} = -w(M - wI)^{-1}$ equals

$$-w \begin{pmatrix} -w & 1 & 1 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1-w & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1-w & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -w & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & -w & \cdots & 0 & 0 & 0 & \cdots & 0 \\ & \vdots & & & & \ddots & & & & \vdots & \\ 0 & 0 & 0 & 0 & 0 & \cdots & -w & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -w & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -w & \cdots & 0 \\ & \vdots & & & & & & & & \ddots & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -w \end{pmatrix}^{-1}.$$

However, since we intend to premultiply by a, only the first row of $(I - zM)^{-1}$ matters, and this can be computed as

$$\Big(1, \frac{1}{w-1}, \frac{1}{w-1}, \frac{w+1}{(w-1)w}, \frac{w+1}{(w-1)w^2}, \ldots, \frac{w+1}{(w-1)w^k}, \\ \frac{w+1}{(w-1)w}, \frac{w+1}{(w-1)w^2}, \ldots, \frac{w+1}{(w-1)w^k}\Big).$$

In conclusion,

$$\Sigma_{(\langle x\rangle,\{x,x^{2k+1}\})}(z) = a(I-zM)^{-1}b = 1+\frac{2}{w-1}+2\left(\frac{w+1}{w-1}\right)\left(\frac{1}{w}+\frac{1}{w^2}+\cdots+\frac{1}{w^k}\right)$$
$$= \frac{(z+1)(1+z-2z^{k+1})}{(z-1)^2}.$$

Now, we may compute the spherical growth series for $(F_2,\{x,x^{2k+1},y,y^{2j+1}\})$ using the obvious decomposition as a free product and applying Proposition 5.1.5(3):

$$\Sigma_{F_2,\{x,x^{2k+1},y,y^{2j+1}\}} = \frac{(z+1)\left(2z^{j+1}-z-1\right)\left(2z^{k+1}-z-1\right)}{4z^{j+k+3}+4z^{j+k+2}-8z^{j+2}-8z^{k+2}-z^3+5z^2+5z-1} \tag{5.6}$$

The exponential growth rate of F_2 with respect to $\{x,x^{2k+1},y,y^{2j+1}\}$ can be computed from this formula. A table of numerical approximations for small j,k is below:

	$j=0$	1	2	3	4	5
$k=0$	3	4	4.18439	4.22417	4.23328	4.23541
1		5.37228	5.56632	5.59866	5.60433	5.60533
2			5.75538	5.78597	5.79117	5.79206
3				5.81615	5.82125	5.82212
4					5.82633	5.8272
5						5.82807

The experimental data suggests that the exponential growth rates approach algebraic numbers as $j,k\to\infty$. In fact, we can compute these limiting values using the following lemma:

Lemma 5.3.1. Suppose $p(x)$, $q(x)$ are (real) polynomials and $0<C<D<1$ are constants. If for each $j\geq 0$ the polynomial $r_j(x)=x^jq(x)+p(x)$ has a root $\lambda_j\in[C,D]$, then there exists a subsequence of λ_j converging to a root of $p(x)$ as $j \to \infty$.

Moreover, if $p(x)$ has only one real root λ in $[C,D]$, then in fact $\lambda_j \to \lambda$.

Proof. We have $p(\lambda_j) = -\lambda_j^j q(\lambda_j)$ so that $|p(\lambda_j)| \leq |\lambda_j|^j M$, where $M = \max\{|q(x)| : x \in [C,D]\}$. Since $\lambda_j \leq D < 1$ for all j, we conclude $p(\lambda_j) \to 0$ as $j \to \infty$. Now, there exists a convergent subsequence λ_{j_k} of λ_j, by compactness of $[C,D]$; say $\lambda_{j_k} \to \lambda \in [C,D]$. As $p(\lambda_{j_k}) \to 0$, λ is a root of $p(x)$.

Moreover, if λ is the only root of $p(x)$ in the interval $[C,D]$, then *every* convergent subsequence of λ_j must converge to λ, and this implies $\lambda_j \to \lambda$. □

Thus we can find the limiting values along the rows of the table:

Proposition 5.3.2. For each $k \geq 0$, $\lim_{j\to\infty} \omega(F_2, \{x, x^{2k+1}, y, y^{2j+1}\})$ is the inverse of the unique real root of $-8z^{k+2} - z^3 + 5z^2 + 5z - 1$ between $\frac{1}{7}$ and $\frac{1}{3}$.

Proof. Fix k. Note that $\omega_j := \omega(F_2, \{x, x^{2k+1}, y, y^{2j+1}\})$ is the inverse of the dominant singularity of (5.6). We may assume $3 \leq \omega_j \leq 7$, because

$$3 = \omega(F_2, \{x,y\}) \leq \omega_j \leq \omega(F_4, \{a_1, a_2, a_3, a_4\}) = 7.$$

Let λ_j be the singularity of (5.6) closest to 0, so that $\omega_j = 1/\lambda_j$. We know that λ_j is a root of the denominator, which we can write as

$$r_j(z) = z^j q(z) + p(z),$$

where

$$p(z) = -8z^{k+2} - z^3 + 5z^2 + 5z - 1$$

and $q(z)$ is some polynomial (depending only on k). Since $1/7 \leq \lambda_j \leq 1/3$, we may apply Lemma 5.3.1 to conclude that λ_j converges to the unique real root λ of $p(z)$ in the interval

$[\frac{1}{7}, \frac{1}{3}]$. $\square$

For example, we have $\lim_{j\to\infty} \omega(F_2, \{x, y, y^{2j+1}\}) = 2 + \sqrt{5} \approx 4.23607$. In fact, we may apply Lemma 5.3.1 a second time to obtain the following:

Proposition 5.3.3. $\lim_{k\to\infty} \lim_{j\to\infty} \omega(F_2, \{x, x^{2k+1}, y, y^{2j+1}\}) = 3 + 2\sqrt{2} \approx 5.82843$.

Proof. The root of $-z^3 + 5z^2 + 5z - 1$ nearest to zero is $3 - 2\sqrt{2}$. Hence if we apply Lemma 5.3.1 to Proposition 5.3.2, we find that

$$\lim_{k\to\infty} \lim_{j\to\infty} \omega(F_2, \{x, x^{2k+1}, y, y^{2j+1}\}) = \frac{1}{3 - 2\sqrt{2}} = 3 + 2\sqrt{2}.$$

$\square$

In summary, we have answered Question 5.2.3(2) for F_2.

Theorem 5.3.4. $\mathrm{Gspec}(F_2)$ is unbounded and has infinitely many limit points in the interval $[3, 7]$.

6. OTHER TOPICS IN SHIFT SPACES

6.1 Algebraic shift spaces

Let G be a group. In this section we impose additional structure on the shift spaces we consider. Namely, we now fix as our alphabet a finite commutative ring k with 1 and restrict our attention to the shift space k^G. With pointwise addition and multiplication, k^G becomes a k-algebra. These shift spaces are similar to those studied in Chapter 8 of [6], although the considerations there diverge from our own.

Proposition 6.1.1. Addition and multiplication on k^G are continuous and commute with the shift action, i.e.

$$(x+y)^g = x^g + y^g \quad \text{and} \quad (xy)^g = x^g y^g \quad \text{for all } x,y \in k^G, g \in G.$$

Proof. Fix $x,y \in k^G$. To show addition is continuous at (x,y) it suffices to pick a coordinate $g \in G$ and show that there are open sets $A,B \subset k^G$ such that $(x,y) \in A \times B$ and $(a+b)(g) = (x+y)(g)$ for all $a \in A$, $b \in B$. But this is immediate if one takes $A = C(g,x(g))$ and $B = C(g,y(g))$. Similarly, multiplication is a continuous map $k^G \times k^G \to k^G$.

The second assertion of the proposition is immediate. $\square$

Consider now the group ring $k[G] = \left\{\sum_{g\in G} a_g g : a_g \in k, \text{ only finitely many } a_g \text{ nonzero}\right\}$. The action of G on the configuration space induces an action of $k[G]$ on k^G:

$$\text{if } f = \sum_{g\in G} a_g g \in k[G], \text{ then } x^f = \sum a_g(x^g) \text{ for } x \in k^G. \tag{6.1}$$

We list some basic properties of how the $k[G]$-action interacts with the algebraic structure of $k[G]$ and k^G. In particular, the proposition below shows that k^G is a (right) $k[G]$-module.

Proposition 6.1.2. Let $x, y \in k^G$, $f, t \in k[G]$, and $c \in k$.

(1) $(x+y)^f = x^f + y^f$ and $(cx)^f = c(x^f)$.

(2) $x^{f+t} = x^f + x^t$ and $x^{ft} = (x^f)^t$.

(3) The map $z \mapsto z^f$ is a continuous map $k^G \to k^G$.

Proof. (1) and (3) are immediate corollaries of Proposition 6.1.1 (hence $k[G]$ acts by continuous linear endomorphisms on k^G). For (2), write $f = \sum a_g g$ and $t = \sum b_g g$ for $a_g, b_g \in k$. Then

$$x^{f+t} = \sum_{g \in G} (a_g + b_g) x^g = x^f + x^t$$

and

$$(x^f)^t = \sum_{h \in H} b_h \left(\sum_{g \in G} a_g x^g \right)^h = \sum_{\substack{g \in G \\ h \in H}} a_g b_h x^{gh} = x^{ft},$$

where we have used Proposition 6.1.1 in the second-to-last equality. $\square$

6.2 Zero sets

Definition. Let f be an element of the group ring $k[G]$. The *zero set of f* is

$$V(f) = \{x \in k^G : x^f \equiv 0\}.$$

f will sometimes be referred to as the *relation* of $V(f)$. More generally, if S is an arbitrary subset of $k[G]$, we define the *zero set of S* to be $V(S) = \bigcap_{f \in S} V(f)$. If $V(S)$ is an addition a shift space, we call $V(S)$ a *zero set shift*.

The following example shows that zero sets of algebraic shifts are not merely an abstract consideration.

Example 6.2.1 (Linear recurrence relations)**.** Consider a homogeneous linear recurrence relation over k, say

$$q_m = a_{n-1}q_{m-1} + a_{n-2}q_{m-2} + \cdots + a_0 q_{m-n}, \quad a_i \in k,\ a_0 \text{ invertible in } k. \tag{6.2}$$

This linear recurrence relation is *invertible*, meaning that any sequence $q_1, q_2, q_3, \ldots$ in k satisfying (6.2) for all $m \geq n+1$ can be extended in a unique way to a bi-infinite sequence $\ldots, q_{-2}, q_{-1}, q_0, q_1, \ldots$ satisfying (6.2) for all $n \in \mathbb{Z}$. Now, if we let X be the generator of $\mathbb{Z}$, X represents the "left shift" on the configuration space $k^{\mathbb{Z}}$, and we see that the set of all bi-infinite sequences satisfying (6.2) is exactly the zero set of

$$f = X^n - a_{n-1}X^{n-1} - \cdots - a_1 X - a_0 \in k[\mathbb{Z}].$$

If we regard X as an indeterminate, we have recovered the definition of the *characteristic polynomial* of the recurrence relation (6.2) from the classical theory of recurrence relations.

Remark 6.2.2. It is clear that zero sets are topologically closed k-submodules of the configuration space. Not every zero set is a zero set shift, however. In the spirit of the previous sections of this document, we present a simple example that illustrates this fact.

Example 6.2.3. Let k be the Bari group $\mathbb{Z}/2\mathbb{Z}$ and let G be the symmetric group S_3 equipped with generators $\sigma = (12)$ and $r = (123)$; the Cayley graph of $(S_3, \{\sigma, r\})$ is in Figure 6.1 below. (Note that we are using the *left* Cayley graph in this example, in contrast to the right Cayley graphs of previous examples, as it better illustrates the group ring action.) Define $f = 1 - \sigma \in k[G]$; then $V(f)$ consists exactly of those labellings such that the inner triangle of the graph matches the outer triangle. If we define x as below, we see that $x \in V(f)$ but $x^r \notin V(f)$.

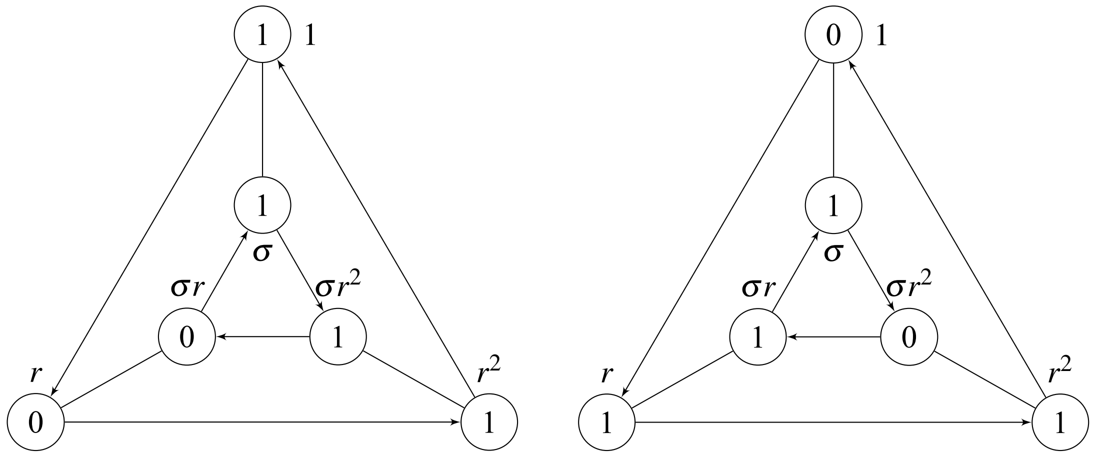


Figure 6.1: Left: Configuration $x \in V(f)$. Right: Configuration $x^r \notin V(f)$.

This shows $V(f)$ is not shift-invariant. Similar examples with infinite groups, such as the infinite dihedral group, are easily obtained.

Shift-invariance of zero sets is obstructed by the fact that elements of the group ring $k[G]$ may not commute with each other. Hence two ways of overcoming this difficulty immediately present themselves: consider zero sets only of elements of $Z(k[G])$, or content ourselves with shift-invariance under only a subgroup of G, which amounts to changing the alphabet and the underlying group via a recoding.

6.3 Zero sets of central relations

If we specialize to zero sets $V(f)$ whose relations f are in $Z(k[G])$, we obtain the following natural results.

Proposition 6.3.1. Suppose $f \in Z(k[G])$. Then $V(f)$ is a shift of finite type.

Proof. Write $f = a_1 g_1 + \cdots + a_n g_n$ for $a_i \in k$, $g_i \in G$. Set $\Omega = \{g_1, \ldots, g_n\}$ and define an allowed pattern set $P \subset k^\Omega$ by

$$P = \{p : \Omega \to k : a_1 p(g_1) + \cdots + a_n p(g_n) = 0\}.$$

Then it is quickly seen that $V(f) = \mathscr{O}(P)$. $\square$

Example 6.2.1 shows that the classical theory of linear recurrence relations can be encoded in the framework of zero set shifts. A well-known result from recurrence relations states that if R_1 and R_2 are homogenous linear recurrence relations with characteristic polynomials p_1 and p_2, respectively, then the solution set of R_1 is contained within the solution set of R_2 if and only if p_1 divides p_2. We are able to recover here the "only if" direction from our definitions with no extra effort.

Proposition 6.3.2. Suppose $f, g, h \in k[G]$ and $f = gh$. Then $V(g) \subset V(f)$.

Further restrictions may be applied under the umbrella of $Z(k[G])$, such as requiring f to be in $k[Z(G)]$ or even requiring G to be abelian.

6.4 Questions

In the context of this study, there are many questions one can ask about algebraic shift spaces, such as the following:

Question 6.4.1. Is every configuration in $V(f)$ periodic/strongly periodic?

Question 6.4.2. Which SFTs over G (in general) can be represented as zero sets?

Let $f \in \mathscr{G}$. $V(f)$ is always a nonempty shift of finite type, since $0 \in V(f)$. If we define $V'(f)$ to be the set of all $x \in V(f)$ such that no entry of x is zero, then $V'(f)$ is also a shift of finite type, but it may be empty. If so, f is called *0-inevitable*, but if $V'(f)$ is nonempty, then f is called *0-avoiding*.

Question 6.4.3. Under what conditions can we guarantee that f is 0-avoiding?

In Section 3.2.2 we remarked that it is of interest from a computational point of view to construct shift spaces with fundamental shapes from which configurations can quickly be reconstructed. The prototype of this paradigm is the *linear feedback shift register*, which generates a sequence based on a recurrence relation and an initial seed.

Question 6.4.4. When does $V(f)$ have a fundamental shape?

7. CONCLUSION

At its core, the theory of shift spaces over groups—and in particular the sub-topic of periodic points—is very much in the spirit of geometric group theory: one takes a space characterized by its local behavior (in this case, a shift space), equips it with an action by a group G, and asks how the global properties of G are reflected in the transformations it effects.

We have seen, however, that the applications of this theory are not merely group-theoretic: various other areas, such as symbolic dynamics and computability theory, take a hand as well. In this study we have investigated various aspects of shift spaces over groups, encountering surprising behaviors from time to time as well as expected ones. It is clear that there remains much territory to explore, but it is our hope that the groundwork laid here will prove useful for all excursions to come.

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