INVESTIGATION OF A MARKOV CHAIN ON FERRERS BOARDS

A Thesis

by

WILLIAM BARHAM LINZ

Submitted to the Office of Graduate and Professional Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

Chair of Committee,	Catherine Yan
Committee Members,	Joel Zinn
	Michael Longnecker
Head of Department,	Emil Straube

May 2016

Major Subject: Mathematics

ABSTRACT

This thesis is an investigation of some of the basic combinatorial, algebraic and probabilistic properties of a Markov chain on Ferrers Boards (*i.e.*, a Markov chain whose states are permutations on a given Ferrers Board). This is an extension of extensive work done over the last fifty years to understand the properties of a Markov chain known as the Tsetlin library. We will review the extensive literature surrounding the Tsetlin library, which also allows for the problem to be contextualized as a particularly nice model of a procedure for searching a database of files. Some of the specific questions we will explore include the transitivity of the Tsetlin library (in fact, we will prove that the extended library is transitive and at most n steps are needed to reach any state from an arbitrarily chosen state); the Tsetlin library's relation to permutation inversions and some other combinatorial statistics; and finally the computation of the Tsetlin library's stationary distribution and eigenvalues in some easy cases.

Although our analysis of the combinatorial aspects of the extended Tsetlin library is complete, we have been unable to fully describe the probabilistic aspects of the Tsetlin library. We are able to describe the stationary distribution for specific easy cases, but further analysis for more complicated cases has proven difficult. Computations have been done using the mathematical software Maple to determine if any patterns may be discerned from specific examples of the more complicated cases. However, the data indicates that the actual stationary distribution differs from our conjectured formula for the stationary distribution, which gives a need for further analysis in future work. We have also not been able to describe the eigenvalues or convergence to stationary for even the simplest Ferrers boards, but we do have var-

ious computations which we hope will be the basis for future exploration of these topics.

DEDICATION

I dedicate this thesis to my family, for their constant support and love over the years.

ACKNOWLEDGEMENTS

I would like to extend my heartfelt gratitude to my thesis advisor, Dr. Catherine Yan, for the mentoring she has provided me over the last three years starting from my days as an undergraduate.

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1. INTRODUCTION AND BACKGROUND MATERIAL

This thesis is designed to be as self-contained and comprehensive as possible. We therefore begin with a review of basic notions and definitions.

1.1 Review of Permutations and Ferrers Boards

1.1.1 Permutations

Throughout this section, we let $[n] = \{1, 2, 3, \ldots, n\}$, the set consisting of the first n positive integers. A *permutation* on $[n]$ is some ordering (or arrangement) of the elements of $[n]$. More formally, a permutation may be defined as a bijective function $\sigma : [n] \to [n]$ (where we identify the permutation σ corresponding to an arrangement $i_1 i_2 \cdots i_n$ with the bijective function σ such that $\sigma(j) = i_j$ for each $j \in [n]$). For example, considering the set [4], 1234 and 3142 are permutations of size 4 (size, of course, is defined as the number of elements of the permutation). The study of permutations and their properties is very ancient and a fundamental part of combinatorial theory.

We now present a more pictorial way to view permutations; namely, by viewing a permutation of size n as a placement of n tokens on an $n \times n$ grid so that no two tokens are in the same row or column. More illuminatingly, this placement may be described as a placement of n nonattacking rooks on an $n \times n$ chessboard (since a rook in chess can only attack along rows or columns). We present an example to demonstrate this placement.

In Figure 1.1, we have placed the permutation 3142 on a 4×4 grid with bullet points denoting the elements of the permutation. The placement is accomplished by identifying the horizontal columns with the placement of elements of the permutation (with the ith element in the permutation corresponding to the ith column in the

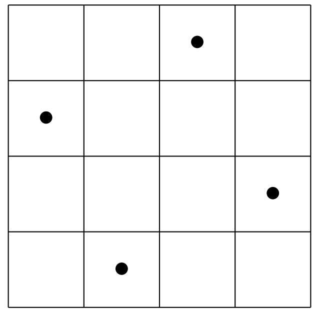


Figure 1.1: The permutation 3142 on a 4×4 grid

board) and by identifying the vertical rows of the board with the values placed in the permutation (so that if a token is placed in the jth row from the bottom row, it corresponds to the element j in the permutation). For example, in the permutation 3142, as 3 is in the first position, in the first column of the board a token will be placed on the third row up from the bottom of the board; since 1 is in the second position of 3142, in the second column a token will be placed in the first row up from the bottom; and so forth. It is clear from this description that the set of permutations of $[n]$ is in one-to-one correspondence with the placement of n nonattacking rooks on an $n \times n$ grid.

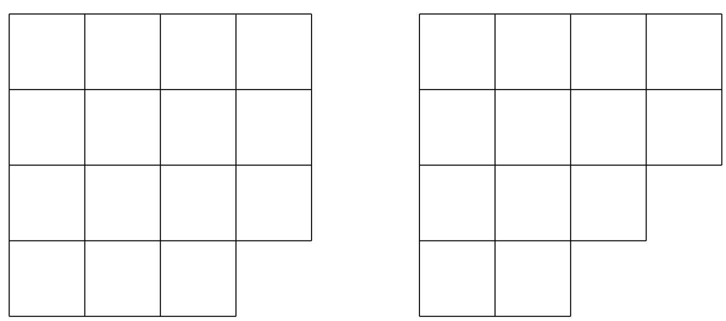
1.1.2 Ferrers Boards

We are now able to define the concept of a Ferrers Board, which in some sense distorts the permutation setup. We let the rows of an $n \times n$ board be labeled ascendingly r_1, r_2, …, r_n, (so that r_1 corresponds to the bottom row), and also let $|r_i|$ denote the number of squares in row r_i.

Definition 1. *A Ferrers Board is an $n \times n$ board (or shape) such that $|r_1| \leq |r_2| \leq \cdots \leq |r_n|$ and so that $|r_i| \geq i$ for each $i \in [n]$. Equivalently (although somewhat informally), a Ferrers board B is an $n \times n$ shape with a missing section λ in the lower right corner, so that λ has no "holes" and so that the main southwest-northeast* *diagonal is in B. We may write $B = (n \times n) - \lambda$.*

Since the classical case of a full board is achieved when $\lambda = \emptyset$, it is clear that the Ferrers board description is a generalization of the previous setting. Permutations on Ferrers boards are defined in the analogous way as in the full board, as a collection of n nonattacking rooks. In general, the set of permutations on a Ferrers board $B = (n \times n) - \lambda$ does not form a subgroup of the symmetric group S_n, so we must study the combinatorial properties of permutations on Ferrers boards more closely.

We will also make a distinction between rectangular and nonrectangular Ferrers boards. A *rectangular* Ferrers board is a board in which λ is a rectangle, *i.e.* $\lambda = a \times b$ for some $a, b \geq 0$. A *nonrectangular* Ferrers board is a board in which λ is not rectangular. Examples of both of these kinds of boards are given in Figure 1.2.



(a) A rectangular Ferrers Diagram

(b) A non-rectangular Ferrers Diagram

Figure 1.2: Examples of rectangular and nonrectangular Ferrers boards.

Although many of our combinatorial results will apply to arbitrary Ferrers boards, previous work and new computations indicate that rectangular Ferrers boards should be easier in principle to study, as many some properties which are held in the classical full board case are only preserved by rectangular Ferrers boards.

We will also sometimes use the following combinatorial structure as the setting for the problem. This structure is equivalent to a Ferrers board (and we will give an example demonstrating this equivalence), but proceeds from a different viewpoint.

Suppose that the set $[2n] = \{1, 2, \ldots, 2n\}$ is divided into two sets of n elements. One set L is a of "left arcs," the members of which are denoted by L_i, $1 \leq i \leq n$, while the second set R is a set of "right arcs," the members of which are denoted R_i, for $1 \leq i \leq n$. We use the symbol $<$ to mean "to the left of." Then, a *pattern* P is an arrangement of the left arcs and right arcs so that the following holds:

1. If $i < j$ (the usual ordering on the natural numbers), then $L_i < L_j$ and $R_i < R_j$.

2. For each $i \in [n]$, $L_i < R_i$

For example, if $n = 4$, $L_1L_2R_1L_3R_2L_4R_3R_4$ is one possible pattern. If the rows of a Ferrers boards are labeled $\{r_i\}_{i=1}$ ascendingly, and the columns are labeled $\{c_i\}_{i=1}$ left-to-right, then under the maps $L_i \leftrightarrow c_i$ and $R_j \leftrightarrow r_j$, we can transform any pattern P on $[2n]$ into a Ferrers board B of size n, and this transformation is a one-to-one correspondence. We give an example to illuminate this correspondence:

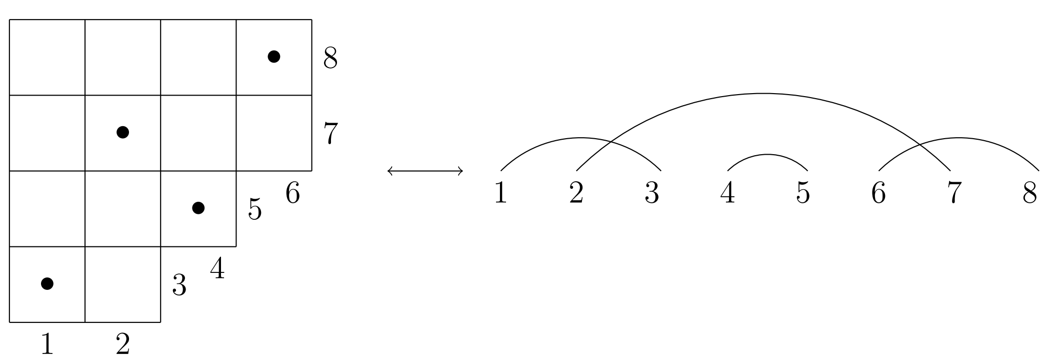


Figure 1.3: Equivalence between Ferrers Board of size 4 and Pattern on [8]

In Figure 1.3, the left and right arcs are drawn how they are usually pictorially represented (like arcs of a circle). Figure 1.3 also shows the equivalence between permutations on a Ferrers board and what are called *matchings* on a pattern P. We now subsequently define matchings on patterns P.

A *matching* is a set of n pairs $\{(L_i, R_j)\}$ such that in each pair $L_i < R_j$ and such that each L_i and R_j occur in exactly one pair (although not necessarily in the same pairs). There is an easy one-to-one correspondence between matchings M and permutations σ: $(L_i, R_j) \in M \leftrightarrow \sigma(i) = j$.

Although we will mostly work with Ferrers boards or the traditional permutation formulation, many of the combinatorial results will be easier to state (and prove) in terms of matchings.

1.2 Review of Markov Chains

We will now give a brief review of the elementary definitions of and relating to Markov chains. This will then enable us to define the Tsetlin library.

A *Markov chain* is a sequence of trials such that the probability the chain is in a given state E after k trials is only dependent on the state the chain is in after $k-1$ trials [3]. More precisely, if X_1, X_2, …, X_n denote the states of a finite system at time $t = 1$, $t = 2$, …, $t = n$, then the defining characteristic of Markov chains (considering the X_i's as random variables) [8]:

$$P(X_n = x_n |\, X_1 = x_1, X_2 = x_2, \ldots X_{n-1} = x_{n-1}) = P(X_n = x_n |\, X_{n-1} = x_{n-1}).$$

We will always assume that the Markov chain has stationary distribution variables - that is, $P(X_n = x_n |\, X_{n-1} = x_{n-1})$ is independent of the choice of n (so when considering a transition from state i to state j, we are free to assume that the chain is initially in state i). If i and j are two states of a Markov chain, then we can let $p_{ij} = P(X_n = j | X_{n-1} = i)$ for each choice of i and j. These are the transition probabilities of the Markov chain, and we can form $T = (p_{ij})$, the *transition matrix* of the Markov chain. The eigenvalues of T are referred to as the eigenvalues of the Markov chain.

The *stationary distribution* of a Markov chain is a probability vector π (*i.e.* a vector whose entries sum to 1) such that $\pi T = \pi$. The stationary distribution need not in general exist or be unique. Under some suitable technical hypotheses, the vector π may be understood as the limiting distribution to which the Markov chain converges. For the Markov chain, we will consider, the Tsetlin library, the existence and uniqueness of a stationary distribution is guaranteed, since the chain is finite, aperiodic and irreducible [3].

1.3 Three Equivalent Definitions for the Tsetlin Library

We will first define the Tsetlin library in the classical case of a full board (with $\lambda = \emptyset$) and then generalize to the case of arbitrary Ferrers boards. The original definition was in the case of permutations as list of elements, and since the full board is symmetric, there is little confusion about how the equivalent definitions relate to each other. In the case of an arbitrary Ferrers board, however, certain care must be taken, as an arbitrary Ferrers board will not in general possess the same symmetry as the full board. In addition to the description of the Tsetlin library on Ferrers boards, we will give equivalent definitions in terms of lists of elements (the typical word formulation) and also a description in terms of matchings on certain combinatorial patterns.

1.3.1 The Classical Tsetlin Library

Suppose that B_1, B_2, B_3, $\ldots$, B_n are a set of n books placed on a shelf in some order. At any time t, a librarian may select a book B_j with some probability p_j

($\sum_{i=1}^{n} p_i = 1$), and place it at the left end of the shuffle, moving other books to the right as needed to fill in the gap now created. Thus, if the initial arrangement of the books was $B_1 B_2 \cdots B_n$, and book B_j was selected, the new arrangement of the books would be (from left to right)$B_j B_1 B_2 \cdots B_{j-1} B_{j+1} \cdots B_n$. This process defines a finite, aperiodic Markov chain known as the Tsetlin library (the term library comes from this image of placing and arranging books on a shelf), and the operation of selecting a book and placing it at the leftmost spot on the shelf is known as the *move-to-front rule*. More formally, given a permutation σ (which we understand in this context as a bijective function from positions in the permutation to the actual placeholders of the elements), we can more formally define the Markov chain as follows:

Definition 2. *For each $j \in [n]$, let O_j denote the operation which selects j from a given permutation σ and such that $O_j(\sigma) = \tau$. Then, τ satisfies the following:*

$$\tau^{-1}(i) = \begin{cases} \sigma^{-1}(i) + 1, & \text{if } \sigma^{-1}(i) < \sigma^{-1}(j) \\ 1, & \text{if } i = j \\ \sigma^{-1}(i), & \text{if } \sigma^{-1}(i) > \sigma^{-1}(j) \end{cases} .$$

The Tsetlin library may be generalized in many ways - one particular way would be to be look at other classes of permutation operations, which takes as inputs permutations of size n, and outputs another permutation of size n depending on which element of the permutation was chosen. We will instead consider a generalization of the Tsetlin library to Ferrers boards.

1.3.2 The Tsetlin Library on Ferrers Boards

Let B be a Ferrers board, and let c_i denote the ith column from the left of B. Then there is a lowest row r_{β_i} such that the square $(r_{\beta_i}, c_i) \in B$. Consider a permutation σ on B, and suppose that initially σ contains the square (r_j, c_i) (where the indexing of rows is ascending and the indexing of columns is from left-to-right). The Tsetlin library defined on B will be viewed as choosing the column c_i (denoted by operation O_i) so that in $O_i(\sigma)$, the token in column c_i will move to the lowest row in c_i, and all rows in between the token"s original and final positions will be moved up by one. Formally:

Definition 3. *For each $j \in [n]$, let O_i denote the operation which selects a column c_i, and suppose that σ is a permutation such that (r_j, c_i) is in σ. Then, the permutation $O_j(\sigma)$ satisfies the following properties:*

1. $(r_{\beta_i}, c_i) \in O_i(\sigma)$
2. $(r_{m+1}, c_k) \in O_i(\sigma)$ *if* $(r_m, c_k) \in \sigma$ *and* $\beta_i < m < i$
3. $(r_b, c_l \in O_i(\sigma)$ *if* $(r_b, c_l) \in \sigma$ *and* $b < \beta_i$ *or* $b > i$

We show an example in Figure using the pictorial representation of a Ferrers board below.

We can equivalently define the Tsetlin library on permutation using the traditional element approach on $[n]$ with a few modifications. First, by identifying r_{β_i} with the number β_i, we obtain a nondecreasing sequence of numbers $\{\beta_j\}_{j=1}^{n}$. It is not difficult to see that given any board B, the sequence $\{\beta_j\}_{j=1}$ can be defined, and conversely given such a nondecreasing sequence, with the condition that $\beta_i \leq i$ for each $i \in [n]$, a unique Ferrers board B is defined. We can now define the Tsetlin library on permutations as classically understood:

Definition 4. *For each $j \in [n]$, let O_j denote the operation which selects j from a given permutation σ and such that $O_j(\sigma) = \tau$. Then, τ satisfies the following:*

$$\tau^{-1}(i) = \begin{cases} \sigma^{-1}(i)+1, & \textit{if } \beta_j \leq \sigma^{-1}(i) < \sigma^{-1}(j) \\ \beta_j, & \textit{if } i = j \\ \sigma^{-1}(i), & \textit{if } \sigma^{-1}(i) > \sigma^{-1}(j) \textit{ or } \sigma^{-1}(i) < \beta_j \end{cases}.$$

The third formulation of the Tsetlin library on Ferrers boards will be on patterns and heir associated matchings. Given L_i in some pattern P, let $J_i = \{j | R_j > L_i\}$ and set $\beta_i = \min(J_i)$. Let M be any initial matching on P, and let O_i denote the operation transforming M via L_i. If we set $M' = O_i(M)$, then the following holds:

1. $(L_i, R_{\beta_i}) \in M'$
2. $(L_j, R_{m+1}) \in M'$ if $(L_j, R_m) \in M$ and $\beta_i < m < i$
3. $(L_b, R_k) \in M'$ if $(L_b, R_k) \in M$ and $k < \beta_i$ or $k > i$

By the earlier equivalences of permutations as elements to permutations on Ferrers boards, and permutation on Ferrers boards to matchings on patterns, it is clear that all three of these formulations of the Tsetlin library are equivalent. In any case, the formulation preserves the essential viewpoint of a move-to-front rule, but defines the "front" for a given permutation element based on the structure of an underlying Ferrers board.

2. LITERATURE REVIEW

We now briefly state the main results relating to the study of the classical Tsetlin library and also briefly consider applications of the Tsetlin library and different generalizations that have been studied.

2.1 Review of Basic Results in the Existing Literature

The Tsetlin library is named after the Soviet mathematician Tsetlin, who studied some simple examples of self-organizing systems in his work on automata [12]. Hendricks defined the Markov chain on permutations now known as the Tsetlin library, and computed its stationary distribution in two different ways (one by induction and using the defining equations of a stationary distribution [5], and one by more combinatorial reasoning[7]). Hendricks was able to prove that for the permutation $\alpha = 123 \cdots N$, the stationary probability μ_α is given by

$$\mu_\alpha = \prod_{i=1}^{N} \left(\frac{p_i}{\sum_{j=1}^{N} p_j} \right),$$

and similar results apply for other permutations. Hendricks also considered basic rules similar to the move-to-front rule, such as a transposition rule (where when i is selected, i is moved one to the left, instead of to the front of the line)[6][7]. Much of the work surrounding these questions has focused on the eigenvalues of the chain's transition matrix, or on the average search cost for the items. Rivest [11] was able to show, for instance, that the long term average search cost for the transposition rule is less than for the move-to-front rule. However, Bitner[1] was able to show that, on the contrary, certain distributions yielded quicker convergence to stationary for the move-to-front rule than for the stationary rule. As the Tsetlin library models a relatively simple system for searching a collection of files, the area continues to be of interest to mathematicians and computer scientists, with Fill[4] noting applications to binary search trees, VLSI circuit simulation, data compression and communications networks.

2.2 Relation to Previous Work by the Author

The eigenvalues for the classical Tsetlin library have been computed independently by Donnelly[2] and Phatarfod[10]. It is known that the eigenvalues are sums of the form $\sum p_i$ where the sum may be taken over any number of terms except for $n-1$. 0 and 1 are also clearly eigenvalues (and it is known that in general 1 is the maximal eigenvalue). The multiplicities of the eigenvalues correspond to certain permutations called derangements, which are permutations for which no element is in its "proper place" (in particular, a derangement is a permutation σ with no fixed points). Thus, if an eigenvalue is a sum taken over m elements of the permutation, then its multiplicity is D_{n-m}, the number of derangements on $n-m$ elements.

The author's undergraduate thesis [9] investigated properties of derangements on Ferrers boards. Given any permutation σ, one can in general define a derangement to be any permutation which has no elements in the same place as σ. On a Ferrers board, this is equivalent to stipulating that the derangement have no tokens overlapping with one of the initial permutation. It is clear in the classical case that the number of derangements is independent of the initial choice of permutation. The author investigated the conditions under which derangment number is still independent of the initial permutation in the Ferrers board case. The answer is that the derangement number is independent of the initial permutation only in the case that the Ferrers board is rectangular. Hence, it has been conjectured that the eigenvalues of the Tsetlin library defined on the Ferrers board should have a connection to derangements on the Ferrers board in a similar way to that observed in the classical case. At present, we have only computations and conjectures - no actual formula for the eigenvalues or their multiplicities in even simple cases of the Ferrers boards.

3. COMBINATORIAL AND ALGEBRAIC PROPERTIES OF THE TSETLIN LIBRARY

In the classical setting of the Tsetlin library, the shape of the board on which the permutations exist is highly symmetric, so certain combinatorial properties of the Tsetlin library (that in some sense describe how the operations act on the permutations) are not very interesting. However, in the more general case of a Ferrers Board B, these operations are much more nontrivial and interesting. In the classical case, not only is it clear that the Tsetlin library is transitive, but it is very easy to see that transitivity can be accomplished in at most n moves. We will in fact prove that, as in the classical case, not only is the Tsetlin library transitive, but also any permutation can be reached from any other permutation by a sequence of at most n operations (and further, this sequence may be chosen independently of the initial permutation). We also investigate some results related to permutation inversions under the operation of the Tsetlin library, and consider some algebraic properties such as commutativity of the various operations on the Tsetlin library.

3.1 Transitivity of the Tsetlin Library

Transitivity is clear in the classical case for the full set of permutations. Suppose that one wishes to reach the permutation $i_1 i_2 \cdots i_n$. Then, the sequence of operations $O_{i_n} O_{i_{n-1}} \cdots O_{i_1}$ will yield the desired permutation. Note that this sequence of operations will give $i_1 i_2 \cdots i_n$ regardless of what the initial permutation is. This procedure will not in general work for an arbitrary Ferrers board. In fact we can describe precisely the permutations on an arbitrary Ferrers board B for which this procedure is satisfactory.

Theorem 1. *Suppose that there are k distinct levels of B (k distinct values of β_i in* *the sequence defining B). Then the sequence of operations described above will yield the permutation $i_1 i_2 \cdots i_n$ if and only if $\beta_{i_k} \leq \beta_{i_{k+1}}$ for each $k \in [n-1]$.*

One can induct on the number of levels of B to obtain this theorem, or simply note that by the conditions the highest level will be set first (in relation to each other), than the second-highest level, and so on, and there is no possibility of distortion in the values.

We now consider transitivity for an arbitrary Ferrers board, for which we will prove the following result:

Theorem 2. *Given any two states i and j of the Tsetlin library on an arbitrary Ferrers board B, there is a sequence of n operations that will transform state i into state j. Furthermore, this sequence of operations is independent of the initial state i.*

From the discussion of the classical case, we can determine the appropriate way to order the elements within each level in the permutation (and it is clear that this is necessary to obtain the permutation). From this observation, with an induction and some small manipulations, we can determine that for each σ on B, there is a sequence of operations which will yield σ starting from any permutation. Furthermore, each operation is selected only once, so transitivity can always be achieved in n moves or less (as in the classical case).

Proof. We first consider a rectangular board $B = (n \times n) - (a \times b)$. Let A_1 and A_2 be the distinct levels of B. If $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ is a desired permutation on B, we can describe the sequence of operations necessary to achieve σ as follows:

1. The elements of σ in the two levels A_1 and A_2 can be properly sorted (*e.g.* by restriction)

2. If the topmost elements of σ are in A_2, their operations may come first until a first element in A_1 is reached.

3. For all other σ_i:

   (a) If σ_i is in level A_1, do nothing.

   (b) If σ_i is in level A_2, then interlace σ_i by moving σ_i to the right past a elements of the permutation in level A_1 (starting initially from the classical permutation construction and keeping the elements of A_1 and A_2 sorted in their correct order).

The statement in the third part has an equivalent formulation: suppose that σ_i is destined to be in row r_k, where $k \geq a$. Then, the operation O_{σ_i} will be the $(n - (|r_k| - a)$th operation performed (equivalently, the $(|r_k| - b + 1)$th operation from the end).

It is clear that this interlacing can always be done: by construction, each of the elements of A_2 has at least a elements of A_1 below it in B, so this movement to the left by a spaces is always possible.

If the topmost m elements of σ are in A_2, selecting these elements first followed by some sequence including all elements above it after the first m movements will push these elements to the top. We may therefore suppose that some element j of A_1 precedes the elements of A_2. Then, j will be selected first by the algorithm, and there must be a elements of A_1 that can push j forward enough so that the next element of A_2 can follow. But we have seen that such elements always exist, so this interlacing will yield the desired sequence σ.

Now, suppose that we know how to order the operations for a board with $k - 1$ levels, and consider a board B with k levels $A_1, A_2, \ldots, A_k$. By the inductive hypothesis, the order of operations for the restriction to the first $k-1$ levels is known. To achieve any σ on B, we can order as follows:

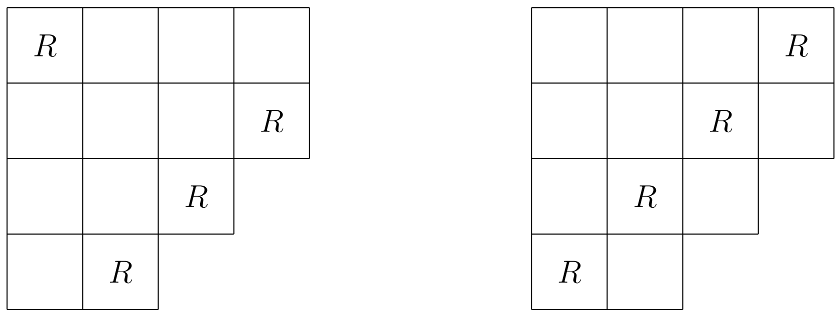
1. If the topmost c elements of σ are on level A_k, these can be ordered first (by the rule for permutation ordering).

2. After the first element of σ not on level A_k is chosen, for all subsequent elements in level A_k, let the operations proceed as they normally would if A_k were not present. Then, if $\epsilon_i \in A_k$, perform operation O_{ϵ_i} after the element in the desired permutation immediately following ϵ_i moves to β_{ϵ_i} (which is the lowest row to which the elements in level A_k can go).

By following the order given by the inductive hypothesis, one can always push up the necessary elements to follow ϵ_i (since indeed, if they are m elements over level A_k, there must be l rows at the top above β_{ϵ_i}, where $l \geq m$). Therefore, if it is possible for there to be a permutation with elements not from A_k in the topmost l rows, then there are $l-m$ such elements in the topmost rows in any permutation. Thus, when these elements drop down, if the order given by the inductive hypothesis is followed, the necessary elements in the desired σ will be eventually pushed to the top, and must eventually be in one of the topmost l rows. But, they must find β_{ϵ_i} first, so the ϵ_i can be interlaced as necessary to yield the desired sigma. We thus conclude that given any permutation σ on arbitrary B, there is a sequence of moves which will achieve σ from any starting point. $\square$

We have shown that at most n moves are necessary, although in most particular cases fewer than n moves will suffice. Further, for any particular case, the order of operations certainly need not be unique. From the proof above, one can in fact derive an order that will work (and an algorithm to produce the order), but both are very messy to describe in the general case.

We can also deduce some additional nice results in particular cases of desired permutations, namely in the case of noncrossing and nonnesting permutations. We will show what these permutations are by example (see Figure 3.1). They are equivalent to the case of noncrossing and nonnesting matchings on a given pattern P.



(a) A Noncrossing Permutation (b) A Nonnesting Permutation

Figure 3.1: Examples of noncrossing and nonnesting permutations

Proposition 3. *Let P be any pattern and M a given matching on P. Then the noncrossing matching can be reached by the sequence of operations $O_1O_2\cdots O_n$ and the nonnesting matching by a sequence of operations $O_nO_{n-1}\cdots O_1$.*

Proof. In this proof, we will abuse notation slightly as follows: if M is a matching, and $(L_i, R_j) \in M$, then we will define ρ to be the function such that $\rho(L_i) = R_j$. The notation of ρ is meant to invoke the equivalence of permutations on Ferrers boards and matchings.

The nonnesting permutation (which is the permutation on the northwest-southeast diagonal) is a special case of Theorem 1, since $\{\beta_i\}_{i=1}$ is nondecreasing.

For the noncrossing permutation, we will use one of its properties: if $i, j \in [n]$, $L_i < L_j$, then it is not the case that $L_i < L_j < \rho(L_i) < \rho(L_j)$. We claim that, beginning from any arbitrary matching, selecting the vertices in the left part of the matching in the sequence O_1, O_2, O_3, $\ldots$, O_n will give the noncrossing matching. Suppose that τ is the resulting matching after this sequence of n moves is performed. Let i, j be arbitrary elements of $[n]$ as above. We have two cases: $\beta_i = \beta_j$ and $\beta_i < \beta_j$.

If $\beta_i = \beta_j$, if ρ' is the new matching after L_j is selected, then since L_i has already been selected $\rho'(L_i) > \beta_i = \beta_j$, hence $L_i < L_j < \rho'(L_j) < \rho'(L_i)$. Since L_i and L_j are not selected in any subsequent matching, this ordering relation is preserved, and $L_i < L_j < \tau(L_j) < \tau(L_i)$.

If $\beta_i < \beta_j$, let ρ_1 be the matching before L_j is selected. There are then two subcases: $\rho_1(L_i) \geq \beta_j$ and $\rho_1(L_i) < \beta_j$. The first subcase is equivalent to the case above. In the second subcase, selecting L_j will leave $\rho_1(i)$ fixed, so we will subsequently have $L_i < \rho(L_i) < L_j < \rho(L_j)$ for all matching states σ following ρ_1. In particular, $L_i < \tau(L_i) < L_j < \tau(L_j)$. It is thus never the case that $L_i < L_j < \tau(L_i) < \tau(L_j)$, so we conclude that τ is the noncrossing matching.

□

3.2 Combinatorial Properties of the Tsetlin Library

We now investigate some combinatorial properties of the Tsetlin library, mostly in relation to inversions of permutations, commutativity, and relation to noncrossing and nonnesting permutations.

Noncrossing and nonnesting permutations are particularly nice cases of permutations on Ferrers boards that correspond in some sense to opposite extremal cases (in the full board case, the nonnesting permutation would be the diagonal from northwest to southeast).

3.2.1 Commutativity and Fixed Points of the Tsetlin Library

We will assume that a board B is given (but place no special restrictions on the board). We now make a few observations about fixed points of an operation O_i being performed on σ.

Proposition 4. *The operation O_i fixes σ if and only if $(r_{\beta_i}, c_i) \in \sigma$.*

Proof. This follows from $O_i((r_k, c_i)) = (r_{\beta_i}, c_i)$, and simply noting that if (r_{β_i}, c_i), then there are no squares to move. □

By considering the square token that is located in the bottom row, it is easily seen that every σ admits at least one operation which fixes σ. In a similarly easy way, it can be shown that the only board on which every permutation is fixed by every operation is the case when $\beta_i = i$ for all i (this is in some sense the trivial case of a Ferrers board).

We now state a few results related to commutativity on the board. For these results, we will use the language of patterns and matchings. We may suppose a pattern P is given, on which a matching M is placed.

Proposition 5. *Suppose that L_i and L_j are two left arcs with $L_i < L_j$. Then, if $(L_i, R_a), (L_j, R_b) \in M$ and $L_i < R_a < L_j < R_b$, we have that $O_iO_j(M) = O_jO_i(M)$.*

Proof. The only elements of P on which O_i operates are those in between L_i and R_a, while the only elements on which O_j operates are those in between L_j and R_b. These two sets are distinct. □

Corollary 6. *Apart from the classical permutation matching pattern, on any pattern P there is a matching M and operations O_j and O_k for which $O_jO_k(M) = O_kO_j(M)$.*

Proof. The classical permutation case is the only one in which there are no L_i, L_j, R_a, R_b such that $L_i < R_a < L_j < R_b$. On any other pattern, such pairs exist, so select M containing those pairs. ☐

In a similar way, it can be shown that an operation O_j which commutes with every other operation for every matching is possible if and only if the pair (L_i, R_i) exists in every matching on P, which happens if and only if L_i is isolated in some sense (that is, if every matching consists of a matching on the first $2i-2$ arcs, the pair (L_i, R_i) and a matching on the final $2n-2i$ arcs).

We thus observe that commutativity of the operations is closely related to the pairs $(L_i, R_a), (L_j, R_b)$ in a matching M for which $L_i < R_a < L_j < R_b$. This number is known as the alignment number of the matching. We will investigate these pairs more closely in the subsequent section, along with the crossing pairs (where $L_i < L_j < R_a < R_b$) and nesting pairs (where $L_i < L_j < R_b < R_a$).

3.2.2 Inversions

We first note that for any matching M on any given pattern P, alignment number+ crossing number + nesting number = $\binom{n}{2}$, since each pair of matchings must satisfy one of the three cases, and there are $\binom{n}{2}$ such pairs. The trivial pattern is the pattern for which there is maximal alignment, while the classical pattern is the pattern for which there is minimal alignment. In fact, alignment is a property of the pattern independent of the matching placed on it, and crossing number + nesting number is a constant for any matching M on a given pattern P. We now relate some of these quantities to the inversion number of a permutation on a given Ferrers board B.

Definition 5. *Suppose that we have a permutation ρ (with no restrictions). Then, for each $i \in [n]$, we define the inversion number of i to be $inv_\rho(i) = \{j | \rho^{-1}(i) > \rho^{-1}(j), i < j\}$. By computing the inversion number for each i, we can write out the* *inversion table (which is a sequence of n numbers cataloging the number of inversions each number possesses). The sum of all inversion numbers of σ gives the number of inversions of σ.*

We now briefly describe what crossings, nestings and alignments look like on a Ferrers board. Nestings correspond to pairs of tokens for which one is down and to the right from the other (*i.e.* $i < j$, $\sigma(i) > \sigma(j)$). Alignments are pairs of tokens where one is up and to the right, but the column of the token up and to the right does not extend to the row of the other one. Crossings are all other pairs of tokens.

We now note the following: in the Tsetlin library formulation, we utilize σ^{-1}, to be able to describe the operation in a similar way to the move-to-front rule. We find that $\text{inv}_{\sigma^{-1}}(i) = \{j | i < j, \sigma(i) > \sigma(j)\}$, which is the number of inversions in which i is up and to the left. Thus, the inversion number of a permutation is related to its inverse permutation description, which is transformed via the different operations to form a Markov chain.

4. PROBABILISTIC PROPERTIES OF THE TSETLIN LIBRARY

We investigate some of the probabilistic properties of the Tsetlin library, in particular looking at the chain's stationary distribution and eigenvalues. Although these questions have been well-studied and analyzed in the classical case, in the case where the library is defined on a Ferrers Board B we mostly have only partial results and computations of specific examples. These computations do serve to demonstrate that one of the obvious conjectures for a generalization to the Ferrers board case is invalid, and that some deeper properties of the Ferrers board or the operations of the Tsetlin library on the permutations of B may be needed to give a complete analysis of the Tsetlin library on Ferrers boards.

4.1 Stationary Distribution Computations

In section 2, we gave Hendricks'[5] result for the stationary distribution in the classical permutation case. Computations were carried out in Maple for boards of size 4 with λ variously 1×1, 2×1, 1×2 and the single nonrectangular case (the case 2×2 was not considered since it reduces to a product of two permutations of size 2, for which the result is already known). The computations did not reveal any sort of recognizable pattern for the non-permutation case, and were different from Hendricks' result in the 2×1 case (which is 2 squares missing from the last column). Thus, for even relatively simple configurations, there are extra terms which need to be accounted for in a complete description of the stationary distribution, and for which we are at present unable to provide an adequate explanation. However, for the case of $\lambda = 1 \times k$, we are able to give a complete description of the stationary distribution, and find that it is similar to Hendricks' result in the classical case. We will first show the case when $B = (n \times n) - (1 \times 1)$ and then proceed to $\lambda = 1 \times k$.

The argument is essentially an adaptation of Hendricks' original argument[5], so it will give us the ability to see how the distribution is proven in the classical case as well.

Consider the Ferrers Board B with one missing square in the lower right corner (thus if $|r_i|$ represents the number of squares in row r_i, we have $|r_1| = n-1$, $|r_j| = n$ for $2 \leq j \leq n$). We denote the probability that column c_i is selected by p_i.

To prove his claimed stationary distribution, Hendricks defined a family of functions as follows: Let $\{x_i\}_{i=1}^N$ be any set of N distinct positive real numbers, and let X be the collection of permutations of $\{x_i\}_{i=1}$. Then, we can define the real-valued function ϕ_N on X by

$$\phi_N(x) = \prod_{n=1}^{N} \left(\frac{x_{i_n}}{\sum_{j=n}^{N} x_{i_j}} \right), \text{where } x = (x_{i_1}, \ldots, x_{i_N}) \in X.$$

We can define a similar function under the same setup to that which Hendricks described as follows:

Let ψ_N be a real-valued function on X defined as follows:

Definition 6. *If $x = (x_{i_1}, \ldots, x_{i_N}) \in X$, then set*

$$\psi_N(x) = \frac{x_{i_1}}{\sum_{j=1}^{n} x_{i_j} - x_N} \prod_{n=2}^{N} \left(\frac{x_{i_n}}{\sum_{j=n}^{N} x_{i_j}} \right).$$

Note that under this definition, $\psi_N(x) = \frac{x_{i_1}}{\sum_{j=1}^{N} x_{i_j} - x_N} \phi_{N-1}(x_{i_2}, \ldots, x_{i_N})$.

We are now in position to state the proposition for the stationary distribution:

Proposition 7. *Suppose that the permutations on the Ferrers Board $B = (n \times n) - \{(r_1, c_n)\}$ are enumerated by the integers $1, \ldots, n! - (n-1)!$. Let $\tau_k = (i_1, \ldots, i_n)$ be one such permutation, and let $B(\tau_k)$ be the corresponding state of the Markov chain.*

If μ_k represents the stationary probability of $B(\tau_k)$, then we have that

$$\mu_k = \frac{p_{i_1}}{\sum_{j=1} p_{i_j} - p_n} \prod_{m=2}^{n} \left(\frac{p_{i_m}}{\sum_{j=m}^{n} p_{i_j}} \right).$$

Proof. Since $p_{i_1} + \cdots + p_{i_n} = 1$, we can write that

$$\psi_n(p_{i_1}, \cdots, p_{i_n}) = \frac{1}{1 - p_n} \phi_n(p_{i_1}, \ldots, p_{i_n}).$$

Thus, since the function ψ_n is equivalent to ϕ_n multiplied by a constant factor, we can freely use many of the results that Hendricks utilizes in his calculations.

We proceed by induction on n, noting that the result is easy to see when $n = 2$ and can be verified computationally for $n = 3$. To achieve the permutation $i_1 i_2 \cdots i_n$ there are two separate cases to consider: the case where n is not in the second position of the permutation, and the case where n is in the second position of the permutation.

For the first case, we consider the permutation $i_1 i_2 \cdots i_n$. Now, using our inductive hypothesis, we can borrow a result from Hendricks and have the following statement:

$$\phi_n(p_{i_1}, p_{i_3}, \ldots, p_{i_{n+1}}) + \phi_n(p_{i_3}, p_{i_1}, p_{i_4}, \ldots, p_{i_{n+1}}) + \cdots + \phi_n(p_{i_3}, p_{i_4}, \ldots, p_{i_{n+1}}, p_{i_1})$$

$$= \frac{1 - p_{i_2}}{p_{i_1}} \phi_n(p_{i_1}, p_{i_3}, \ldots, p_{i_{n+1}})$$

The stationary distribution μ_k is the unique solution of the system of equations

$$\mu_k = \sum_{j=1}^{n! - (n-1)!} \mu_j p_{jk} \text{ for } k = 1, 2, \ldots, N! - (N-1)!$$

For the permutation $i_1 i_2 \cdots i_n$, we can only reach this state of the Markov Chain by passing i_1 from some permutation of $i_2 \cdots i_n$ for which $i_2 \neq n$. Thus,

$$\begin{aligned}
\sum_{j=1} \mu_j p_{jk} &= p_{i_1}[\psi_{n+1}(p_{i_1}, p_{i_2}, \ldots, p_{i_{n+1}}) + \psi_{n+1}(p_{i_2}, p_{i_1}, \ldots, p_{i_{n+1}}) + \cdots + \\
&\quad +\psi_{n+1}(p_{i_2}, p_{i_3}, \ldots, p_{i_{n+1}}, p_{i_1})] \\
&= p_{i_1}[\frac{p_{i_1}}{1-p_{n+1}}\phi_n(p_{i_2}, \ldots, p_{i_n}) + \frac{p_{i_2}}{1-p_{n+1}}\phi_n(p_{i_1}, p_{i_3}, \ldots, p_{i_{n+1}}) + \\
&\quad + \cdots + \frac{p_{i_2}}{1-p_{n+1}}\phi_n(p_{i_3}, p_{i_4}, \ldots, p_{i_{n+1}}, p_{i_1})] \\
&= p_{i_1}\mu_k + \frac{p_{i_2}}{1-p_{n+1}}[\phi_n(p_{i_1}, p_{i_3}, \ldots, p_{i_{n+1}}) + \cdots + \phi_n(p_{i_3}, \ldots, p_{i_{n+1}}, p_{i_1})] \\
&= p_{i_1}\mu_k + \frac{(1-p_{i_2})}{1-p_{n+1}}\phi_n(p_{i_1}, p_{i_3}, \ldots, p_{i_{n+1}}) \\
&= p_{i_1}\mu_k + (1-p_{i_1})\mu_k = \mu_k.
\end{aligned}$$

Note that the above proof is essentially equivalent to Hendricks' original proof.

For the second case, we can consider the permutation $i_1\, n+1\, i_2 \cdots i_n$ for ease of notation, and let the corresponding state be $B(\tau_{k'})$. There are two separate ways that the permutation can reach $B(\tau_{k'})$: one can be in the state already and select i_1, or one can be in a permutation of the form $i_1 i_2 i_3 \cdots\ n+1 \cdots i_n$ and select $n+1$. Thus,

$$\begin{aligned}
\sum_{j=1} \mu_j p_{jk'} &= p_{i_1}\psi_{n+1}(p_{i_1}, p_{n+1}, p_{i_2}, \ldots, p_{i_n}) + p_{n+1}(\psi_{n+1}(p_{i_1}, p_{n+1}, p_{i_2}, \ldots, p_{i_n}) + \\
&\quad + \cdots \psi_{n+1}(p_{i_1}, p_{i_2}, \ldots, p_{i_n}, p_{n+1})) \\
&= p_{i_1}\mu_{k'} + \frac{p_{n+1}p_{i_1}}{1-p_{n+1}}(\phi_n(p_{n+1}, p_{i_2}, \ldots, p_{i_n}) + \\
&\quad + \cdots + \phi_n(p_{i_2}, p_{i_3}, \cdots, p_{n+1})) \\
&= p_{i_1}\mu_{k'} + \frac{p_{n+1}p_{i_1}}{1-p_{n+1}}\left(\frac{1-p_{i_1}}{p_{n+1}}\right)\phi_n(p_{n+1}, \ldots, p_{i_n}) \\
&= p_{i_1}\mu_{k'} + (1-p_{i_1})\psi_{n+1}(p_{i_1}, p_{n+1}, \ldots, p_{i_n}) = \mu_{k'}
\end{aligned}$$

□

This same method works when the missing section is of the form $1 \times k$.

Proposition 8. *Suppose that the permutations on the Ferrers Board $B = n \times n - \{(r_1, c_{n-k+1}), \ldots, (r_1, c_n)\}$ are enumerated by the integers $1, \ldots, (n-k)(n-1)!$. Let $\tau_m = (i_1, \ldots, i_n)$ be one such permutation, and let $B(\tau_m)$ be the corresponding state of the Markov chain. If μ_m represents the stationary probability of $B(\tau_m)$, then we have that*

$$\mu_m = \frac{p_{i_1}}{\sum_{j=1} p_{i_j} - (p_{n-k+1} + \cdots + p_n)} \prod_{m=2}^{n} \left(\frac{p_{i_m}}{\sum_{j=m}^{n} p_{i_j}}\right).$$

Proof. We can define the corresponding function

$$\psi_{k,n}(p_{i_1}, \ldots, p_{i_n}) = \frac{1}{1-(p_{n-k+1} + \cdots + p_n)}\phi_n(p_{i_1}, \ldots, p_{i_n}).$$

The base case $n = k+1$ is satisfied, since then the Ferrers Board can be thought of as the disjoint union of one square (r_1, c_1) with a full $k \times k$ board, and it is easy to see that the claimed stationary distribution matches the computed one for the $k \times k$ library computed by Hendricks.

Now, for the inductive step, assume that the stationary distribution matches the claimed one for some $N \geq k+1$. We have two separate cases - those for which there no in the set $\{(r_2, c_{n-k+1}), \ldots, (r_2, c_n)\}$ are in the permutation, and those for which no square from the bottom row above the missing section λ are in the permutation. In the first case, the computation proceeds exactly as above in the 1×1 case.

In the second case, we may for ease of notation suppose that the given permutation is of the form $i_1 j i_2 i_3 \cdots$, where $j \geq n-k+1$. If $B(m)$ corresponds to this state in the Markov Chain, then we can compute

$$\begin{aligned}
\sum_{j=1} \mu_j p_{jm} &= p_{i_1}\psi_{k,n+1}(p_{i_1}, p_j, p_{i_2}, \ldots) + p_j(\psi_{n+1}(p_{i_1}, p_j, p_{i_2}, \ldots,) + \\
&\quad + \cdots \psi_{n+1}(p_{i_1}, p_{i_2}, \ldots, p_j)) \\
&= p_{i_1}\mu_m + \frac{p_j p_{i_1}}{1-(p_{n-k+1}+\cdots+p_{n+1})}(\phi_n(p_j, p_{i_2}, \ldots,) + \cdots + \\
&\quad + \phi_n(p_{i_2}, p_{i_3}, \cdots, p_j)) \\
&= p_{i_1}\mu_m + \frac{p_j p_{i_1}}{1-(p_{n-k+1}+\cdots+p_{n+1})}\left(\frac{1-p_{i_1}}{p_j}\right)\phi_n(p_{n+1}, \ldots, p_{i_n}) \\
&= p_{i_1}\mu_m + (1-p_{i_1})\psi_{n+1}(p_{i_1}, p_{i_{n+1}}, \ldots, p_{i_n}) = \mu_m
\end{aligned}$$

□

To show that is the stationary distribution, we must additionally show that $\sum \psi_{k,n}(x) = 1$. By induction, we know that $\phi_n(x) = 1$ for all n. Then, if $(x_{i_1}, x_{i_2}, \ldots, x_{i_n})$ is a n-tuple, we have that

$$\psi_{k,n}(x) = \frac{x_{i_1}}{\sum_{j=1} x_j - (x_{n-k+1} + \cdots + x_n)} \phi_{n-1}(x_{i_2}, \ldots, x_{i_n})$$

Summing over all permutations beginning with x_{i_1}, and noting that the first element of any permutation can only come from the set $\{1, \ldots, n-k\}$, we have that

$$\sum \psi_{k,n}(x) = \sum_{j=1}^{n-k} \frac{x_j}{\sum x_i - (x_{n-k+1} + \cdots + x_n)} = 1.$$

4.2 Eigenvalues and Other Computations

As mentioned previously, computations have been carried out for the case where the Ferrers board B is of size 4. In computing the eigenvalues for the rectangular cases, we find that the eigenvalues are always of the form that was found by Donnelly and Phatarfod (*e.g.* p_1, p_1+p_2, etc. are eigenvalues), and further no cases other than the summation of some of the p_i arise. However, even for the simplest case where $\lambda = 1 \times k$, we do not know what the multiplicities of the eigenvalues are, and we do not even have a proof that these are the only eigenvalues. We have some evidence that these should be the only eigenvalues, and we speculate that the multiplicities of the eigenvalues will have some relation to the derangement number (for rectangular Ferrers boards).

Phatarfod [10] derives the eigenvalues in the classical case partly from a consideration of time-dependent probabilities of the form $P(B_i < B_j|t)$, where the B_i notation is used to follow Phatarfod and is in reference to the book formulation of the Tsetlin library (for our purposes, $P(i < j|t)$ is an equivalent way to write the associated probability). The "<" means B_i is to the left of B_j (or on the Ferrers board, the token in column i is below the token in column j). Since terms of the form $1 - \sum p_i$ appear in the computation of the time-dependent probabilities, Phatarfod heuristically reasons that terms of that form should appear as eigenvalues of the Tsetlin library (and rigorously proves this assertion). The computation in the classical case is:

$$P(B_i < B_j|t) = \frac{p_i}{p_i + p_j} + \frac{(p_i - p_j)(1 - p_i - p_j)^t}{2(p_i + p_j)}$$

so that in the equilibrium case, as $t \to \infty$, $P(B_i < B_j) = \frac{p_i}{p_i+p_j}$.

We now compute $P(B_i < B_j|t)$ for $B = n \times n - (1 \times k)$. If B_i and B_j are on the same level ($\beta_i = \beta_j$), then the computation reduces to that above. We may therefore suppose that $\beta_i = 1$ and $\beta_j = k$ (we can compute $P(B_j < B_i|t) = 1 - P(B_i < B_j|t)$ for the other case). There are exactly two cases: either B_i was initially to the left of B_j and B_i was not subsequently selected, while B_j did not move ahead of it, or after B_i's last selection, B_j was not selected after another element on B_i's level was selected.

In the first case, there are $(n-k)(n-1)!$ permutations on B. If B_i is in the bottom row, then in each permutation $B_i < B_j$, and there are $(n-1)!$ such permutations. It does not matter if B_j is selected - since B_j cannot enter the first row, $B_i < B_j$ always, if none of $B_1, \ldots, B_{n-k}$ are selected. This occurs with probability $\frac{(1-(p_1+\cdots+p_{i-1}+p_{i+1}+\cdots+p_{n-k}))^t}{n-k}$.

If B_i is not in the first row, then there are $(n-k-1)(n-1)!$ such permutations, and half of them have $B_i < B_j$. Thus, the initial probability is

$$P(B_i < B_j|t = 0)_{B_i \text{above first row}} = \frac{\frac{1}{2}(n-k-1)(n-1)!}{(n-k)(n-1)!} = \frac{1}{2} - \frac{1}{2(n-k)}.$$

Hence, the probability that $B_i < B_j$ at time t in the first case is

$$\left((\frac{1}{2} - \frac{1}{2(n-k)})(1 - p_i - p_j)^t \right)$$

.

Combining this with the above calculation gives $P(B_i < B_j | t = 0)$.

In the second case, there are two subcases - after B_i is selected for the last time (or in the first row), B_j is not selected at all, or is not selected after one of B_1, …, B_{i-1}, B_{i+1}, …, B_{n-k} are selected. In the first case, the probability is relatively straightforward to compute. If the last time B_i is selected is at time m, then the probability is

$$P(B_i < B_j | t) = p_i \sum_{m=1}^{t} (1 - p_i - p_j)^{t-m} = p_i \left(\frac{1}{p_i + p_j} - \frac{(1 - p_i - p_j)^t}{p_i + p_j} \right)$$

In the second subcase, suppose that B_j is last selected at time l. Then, at times $m + 1$, …, $l - 1$, the only selections can have come from B_{n-k+1}, …, B_n. Therefore, we compute that in this case

$$P(B_i < B_j | t) = p_i \sum_{m=1}^{t} \sum_{l=m+1}^{t} (1 - (p_1 + \cdots + p_{n-k}))^{l-m-1} p_j (1 - p_i - p_j)^{t-l}$$

$$= p_i p_j \sum_{m=1}^{t} \frac{(1 - (p_1 + \cdots + p_{n-k}))^{t-m} - (1 - p_i - p_j)^{t-m}}{(1 - (p_1 + \cdots + p_{n-k})) - (1 - p_i - p_j))}$$

$$= \frac{p_i p_j}{(1-(p_1+\cdots+p_{n-k}))-(1-p_i-p_j))}\left(\frac{1-(1-(p_1+\cdots+p_{n-k}))^t}{p_1+\cdots+p_{n-k}} - \frac{1-(1-p_i-p_j)^t}{p_i+p_j}\right)$$

Combining the above three expressions together gives $P(B_i < B_j|t)$. In the equilibrium case, as $t \to \infty$, we have that

$$P_F(B_i < B_j) = \frac{p_i}{p_i+p_j} + \frac{p_i p_j}{(p_1+\cdots+p_{n-k})(p_i+p_j)},$$

where P_F is used to indicate that this is the associated probability for Ferrers boards of the form $B = (n \times n) - (1 \times k)$. Note that if the uniform distribution is assumed, then one obtains that $P(B_i < B_j) = \frac{1}{2} + \frac{1}{2(n-k)}$, which is expected.

One can compute in theory compute more expressions of this sort along the same lines, but the computations are not very elegant and become quite cumbersome. We hope that this gives heuristic evidence in support of the conjecture for the eigenvalues, but have not been able to prove a conjecture from this result.

5. SUMMARY OF RESULTS AND FUTURE DIRECTIONS

In the course of this investigation, we have been able to extend the classical notion of the Tsetlin library on permutations to the more general class of permutations on Ferrers boards. We have been able to rigorously define the Tsetlin library as a Markov chain in three equivalent ways, on equivalent combinatorial structures (as permutations on Ferrers boards; in the classical formulation of permutations as words; and as matchings on given patterns). We have been able to show that this new Tsetlin library retains several properties of the classical Tsetlin library; most prominently, that it is transitive, and that each permutation may be reached from any other permutation by a sequence of at most n steps. We have also been able to characterize the combinatorial action of the Tsetlin library on the permutations of a Ferrers boards, and relate these movements to the well-known combinatorial statistics of permutation inversions and nestings in matchings. We have also been able to compute the stationary distribution in the case that the missing section of the Ferrers board is of the form $1 \times k$, and we have computations for cases of other types. We also have computed specific expressions that we hope will be helpful in analyzing the stationary distribution in the outstanding cases, and to more generally compute the eigenvalues of the chain.

As there is a discrepancy between our actual computed stationary distribution for the Markov chain in higher cases (cases where $\lambda = a \times b$, with $a \geq 2$, $b \geq 1$), the next step will be to investigate the source of this discrepancy, and to see if we are able to assign a combinatorial interpretation for these new values. It is also hoped that from these investigations, we may be able to study the eigenvalues of the chain and their corresponding multiplicities in greater depth, and again assign a combinatorial interpretation to those values. Finally, as this research has applications in modeling database searches, it may be interesting to compute particular values for the search cost of the chain and other parameters that show how easy, efficient, or effective it is to search for a given file in a database whose search is defined by this extended Tsetlin library.

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