BIOBJECTIVE OPTIMIZATION OVER THE EFFICIENT SET METHODOLOGY FOR PARETO SET REDUCTION IN MULTIOBJECTIVE DECISION MAKING: THEORY AND APPLICATION

A Dissertation

by

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ABSTRACT

A large number of available solutions to choose from poses a significant challenge for multiple criteria decision making. This research develops a methodology that reduces the set of efficient solutions under consideration.

This dissertation is composed of three major parts: (*i*) the formalization of a theoretical framework; (*ii*) the development of a solution approach; and (*iii*) a case study application of the methodology.

In the first part, the problem is posed as a multiobjective optimization over the efficient set and considers secondary robustness criteria when the exact values of decision variables are subjected to uncertainties during implementation. The contributions are centered at the modeling of uncertainty directly affecting decision variables, the use of robustness to provide additional trade-off analysis, the study of theoretical bounds on the measures of robustness, and properties to ensure that fewer solutions are identified.

In the second part, the problem is reformulated as a biobjective mixed binary program and the secondary criteria are generalized to any convenient linear functions. A solution approach is devised in which an auxiliary mixed binary program searches for unsupported Pareto outcomes and a novel linear programming filtering excludes any dominated solutions in the space of the secondary criteria. Experiments show that the algorithm tends to run faster than existing approaches for mixed binary programs. The algorithm enables dealing with continuous Pareto sets, avoiding discretization procedures common to the related literature.

In the last part, the methodology is applied in a case study regarding the electricity generation capacity expansion problem in Texas. While water and energy are interconnected issues, to the best of our knowledge, this is the first study to consider both water and cost objectives. Experiments illustrate how the methodology can facilitate decision making and be used to answer strategic questions pertaining to the trade-off among different generation technologies, power plant locations, and the effect of uncertainty. A simulation shows that robust solutions tend to maintain feasibility and stability of objective values when power plant design capacity values are perturbed.

To my wife, Carolina, and my kids, Laura and Javier.

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NOMENCLATURE

BB	Branch and bound
BO-CEP	Biobjective electricity generation capacity expansion problem
BOLP	Biobjective linear program
BOMBLP	Biobjective mixed binary linear program
ERCOT	Electric Reliability Council of Texas, Inc.
LCOE	Levelized cost of electricity
LACE	Levelized avoided cost of electricity
LMA	Land market areas
LS-LPF	Line search and linear programming filtering
MCDM	Multiple criteria decision making
MIP	Mixed integer program
MOP	Multiobjective program
MOLP	Multiobjective linear program
PDSI	Palmer Drought Severity Index
PSR	Pareto set reduction
RPSR	Robust Pareto set reduction problem
RMOLP	Robust counterpart of the MOLP
TS	Triangle splitting

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1. INTRODUCTION: THE IMPORTANCE OF THE RESEARCH

This section discusses the motivations and contributions of the research, and describes the research questions focus of this dissertation. In addition, it addresses related literature and introduces basic notation common to subsequent sections. The section ends by detailing the organization of this dissertation.

1.1 Motivation

Most real-life decision problems require taking into account multiple and often conflicting objectives. For instance, when formulating new regulations for the energy industry, policy makers must consider economic, environmental and social aspects, among other factors. Even in our personal lives, decision problems are usually of a multiobjective nature. For example, when deciding a career path, one should consider not only his/her financial aspirations, but also other important factors such as personal satisfaction, job location and stability, quality of life, opportunities for career growth, etc.

One alternative that is optimal in view of one of the objectives will often perform poorly in view of the remaining objectives. In the former example, posing strict environmental regulations will likely reduce the environmental impact associated with power generation at the expense of increased electricity costs. Likewise, in the later, deciding for a career in academia may require compromising one's income aspirations in order to engage in a personally rewarding path to research and teaching.

Because there are usually a large (if not infinite) number of alternatives to choose from, decision making under multiple objectives becomes a non-trivial process. The development of effective methodologies to aid decision making would expand the applicability of multiobjective optimization, hence benefiting society in important sectors, including health care, national security, finance and manufacturing industries, among many others.

The field of multiple criteria decision making (MCDM) deals with methodologies by which multiple objectives can be incorporated into a structured process in support of decision making (Steuer, 1986). The first step towards decision making would be eliminating all inferior solutions, i.e. those alternatives where there exists another solution that is strictly better in view of one of the objectives and not worse in view of the remaining objectives. For decision making purposes, non-inferiority is a highly desirable characteristic associated with a solution, otherwise implementing an inferior solution would imply a suboptimal performance across all criteria in the multiobjective problem (Zeleny, 1982).

This leads decision making towards focusing on the set of non-inferior solutions, termed efficient set (or its image on the space of the objective functions, termed Pareto set). However, the efficient set (as well as the Pareto set) can still be very large (if not infinite), posing significant challenges for decision making. For instance, it is well-known that the size of the efficient set of can grow exponentially with the number of objective functions in the problem (Deb and Saxena, 2005). Therefore, the second step towards decision making would be to identify from the efficient set fewer solutions having some desirable characteristics and restrict trade-off analysis to this reduced subset of promising solutions.

Motivated by the aforementioned challenges associated with MCDM, the idea of identifying a subset of promising efficient solutions to ease decision making lies in the heart of this dissertation. This text formalizes theory and application of a methodology aimed at the identification of such solutions.

1.2 Research objective and contribution

This section describes the main research objective and the contributions of the dissertation, where the research questions are presented.

1.2.1 Research objective

The main objective of this dissertation is to develop a mathematical programming methodology to aid decision making under multiple objective by introducing secondary criteria to further breaking ties among the many efficient solutions in the multiobjective program (MOP). In particular, this research is aimed to (*i*) formalize theoretical properties of the resulting optimization problem that considers measures of robustness as the secondary criteria to protect against uncertainties; (*ii*) develop solution algorithms to solve a special case of the problem when the efficient set is associated with a multiobjective linear program (MOLP); and (*iii*) illustrate the advantages of implementing the proposed methodology to aid decision making. The detailed contributions of this dissertation are discussed next.

1.2.2 Contribution of the research

The main contribution of this dissertation to the academic body of knowledge is the formalization of an optimization methodology to aid decision making under multiple objectives, where secondary criteria are introduced to restrict trade-off analysis to a subset of the efficient solutions having desirable characteristics. The use of secondary criteria provides further opportunities for trade-off analysis in the space of these secondary objective functions, hence allowing for additional modeling information to be taken into consideration in the selection of a most preferred solution. Although the methodology may consider any convenient secondary objective functions, particular focus was given to two robustness criteria when the actual values of the efficient solution selected for implementation are affected by uncertainties. While trade-off analysis in the objective space of the MOP may be challenging due to its possibly high dimensionality, decision making may be further facilitated with trade-off analysis in the space associated with the robustness criteria because of its reduced dimensionality.

The problem studied in this dissertation is posed as an optimization over the efficient set. Since the efficient set is usually non-convex, the resulting problem falls within the class of non-convex optimization. To address the challenges associated with solving such a non-convex problem and applying the methodology to effectively break ties among efficient solutions, the following research questions are studied in this dissertation:

(a) Using mathematical programming, can a methodology be developed that may ease decision making by restricting trade-off analysis to a subset of efficient solutions that are robust to uncertainties?

(b) What are the theoretical properties of the problem and how does it relate to a classical robust optimization approach?

(c) How effective is the methodology in breaking ties among the efficient solutions of an MOP?

(d) Can a solution algorithm be developed which benefits from problem structure and does not require discretization of the efficient set?

(e) How may the methodology be applied in a real-world problem and what would be the advantages of using the proposed measures of robustness for MCDM?

Question (a) is associated with the main contribution of this dissertation and is introduced in Section 2. Two robustness measures are proposed and allow for explicit trade-off analysis between objective function values loss of optimality and solution feasibility when subjected to uncertainty. The robustness measures provide complementary information to help in the selection of a solution for implementation.

Question (b) is dealt with in Section 2, where the findings allow for positioning the methodology within the context of robust optimization. In particular, it shows that, under certain conditions, the set of robust solutions obtained from the methodology is insensitive to the level of uncertainty; i.e. sufficient conditions where the methodology will identify the same robust solutions, regardless of the decision maker's knowledge about the actual uncertainty bounds. This result expands the applicability of the field of robustness assessment in situations where the lack of data would otherwise prevent the use of a classical robust optimization approach.

Question (c) is aimed at assessing the effectiveness of the methodology to restrict trade-off analysis to a smaller subset of solutions. In Section 2, a theoretical property shows that, under certain conditions, only one solution will be identified by the methodology, hence breaking ties among all efficient solutions. In addition, Section 4 illustrates the effectiveness of the methodology in a case study.

Question (d) is the focus of Section 3, where the secondary objective functions are generalized to consider any linear functions with nonnegative coefficients. Additional secondary decision variables and constraints are introduced. The problem is posed as a biobjective optimization over the efficient set and a solution approach is presented. Existing algorithm for optimization problems over the efficient set typically focus on optimizing a single objective function. Therefore, the algorithm proposed in Section 3 expands the applicability of the field of optimization over the efficient set.

Question (e) is addressed in Section 4, where the methodology is applied in an electricity generation capacity expansion problem to minimize cost and water withdrawal. A case study application illustrates the methodology in the context of capacity planning for the State of Texas. The results show the effectiveness of the approach in protecting against uncertainty, while providing the decision maker with a reduced subset of solutions to trade-off.

1.3 Literature review

This section provides an overview of the broader literature on multiobjective optimization methodologies related to this dissertation. The section also discusses the limitations on the existing literature.

1.3.1 Background on multiobjective optimization

MCDM problems are broadly dealt with in three distinct categories. In the *a priori* category, the decision maker expresses his/her preferences before the Pareto set is generated. This assumes that the decision maker has full knowledge about his/her preferences, expressed via a value function which aggregates all the objective functions of the MOP. The corresponding single objective problem is solved to directly find a most preferred solution, hence without the need to generate the entire Pareto set. One special case of value function is the linear weighted sum of objective functions, where each objective is multiplied by a positive scalar weight. One issue associated with this approach is that non-convex regions of the Pareto set cannot be achieved via a positive (convex) combination of the objective functions (Das and Dennis, 1997). Other approaches within the *a priori* category include the lexicographic ordering (Fishburn, 1974) and goal programming (Charnes et al., 1955). Although these approaches are computationally attractive (as they prevent the generation of the entire Pareto set), the main challenge relies on the specification of the value function, as it may be impossible to mathematically encode the decision maker's preferences reliably (Miettinen, 1999).

In the *interactive* category, few initial solutions are generated so that the decision maker is progressively asked to input his/her partial trade-offs in order to guide the search procedure for yet unknown solutions until a final most preferred alternative is achieved. For instance, in Engau and Wiecek (2007) and Engau and Wiecek (2008) interactive methods based on an objective space decomposition procedure are applied to achieve a most preferred solution. One limitation of interactive methods is that they rely on the progressive articulation of the decision maker's preferences in the search for a most preferred solution (Alves and Clímaco, 2007), hence potentially requiring extensive human intervention and leaving unexplored regions of the Pareto set (Benson and Sayin, 1997). The interested reader may refer to Steuer and Choo (1983), Luque et al. (2010) and Luque et al. (2011) for further applications of interactive methods.

In the *a posteriori* category, the decision maker intervention only occurs after the Pareto set is generated. The literature on generating the Pareto set is vast. This includes the development of exact algorithms for linear/non-linear problems (e.g. Yu and Zeleny, 1975; Chankong and Haimes, 1983; Das and Dennis, 1998; Ehrgott et al., 2007), approximating algorithms (e.g. Masin and Bukchin, 2008; Karasakal and Köksalan, 2009), solution procedures for specific discrete problems (e.g. Van Wassenhove and Gelders, 1980; Warburton, 1987; Chen and Bulfin, 1993; Steiner and Radzik, 2008; Bérubé et al., 2009; Rong et al., 2015) and meta-heuristics applications (e.g. Zitzler et al., 2001; Deb et al., 2002; Laumanns et al., 2006; Smith et al., 2008). While the main advantages of *a posteriori* methods are that they provide a range of solutions to choose from and don't require human intervention in the optimization phase, the main drawback is that they may generate a large number of solutions that are of no real interest to the decision maker, hence raising additional difficulties to the selection of a most preferred solution (Benson and Sayin, 1997; Alves and Clímaco, 2007). In addition, selecting a single solution in a high dimensional objective space can lead to challenging trade-off analysis (Engau and Wiecek, 2007). This problem is augmented when considering the incommensurability of the objectives in the MOP.

As the number of objective functions, variables and constraints increase, not only is the computational burden of generating the Pareto set increased, but also the issue of trading-off among all alternatives. Pareto set reduction (PSR) methods are available in the literature to alleviate the problem of having to trade off among several alternatives. In PSR methods, the Pareto set is assumed to be known and the goal is to find a subset of solutions, which is typically based on ranking and clustering methodologies.

Ranking methods attempt to prioritize solutions based on some function of their objective values (e.g. Das, 1999a; Branke et al., 2004; Vafaeyan and Thibault, 2009; Carrillo and Taboada, 2012), or rank solutions based on a degree to which each objective function affects efficiency (e.g. Das, 1999b; Venkat et al., 2004; Kao and Jacobson, 2008). Das (1999b) approached the PSR problem by proposing an ordering of efficient solutions based on a measure of non-dominance when one or more objective functions are disregarded. Similarly, Venkat et al. (2004) proposed a greedy heuristic to maximize a percentile function, which encapsulates the percentile ordinal rankings of efficient solutions with respect to each objective function. The percentiles values are computed with respect to the individual optimum for each criterion. In Kao and Jacobson (2008), an exact method to solve the percentile ordinal ranking was proposed and the problem was shown to be NP-complete.

Clustering methods aim at finding a representative subset of efficient solutions (e.g. Steuer and Harris, 1980; Morse, 1980; Rosenman and Gero, 1985; Taboada and Coit, 2008; Aguirre et al., 2011; Eusébio et al., 2014; Vaz et al., 2015). In these methods, a cluster groups solutions with similar objective values; the reduced subset includes a representative solution from each cluster.

In a recent work, Vaz et al. (2015) formulate the PSR as a facility location problem to find a representative set based on clustering solutions in the objective space of the MOP. The model proposed by the authors assumes that the Pareto set is discrete and is applied to the case of a biobjective program. Although similar, the problem dealt with here does not attempt to find a representative subset from the Pareto set, but rather, to provide additional trade-off context by introducing secondary criteria related to both the objective and the decision spaces of the MOP in order to break ties among the Pareto outcomes.

1.3.2 Summary of limitations from the existing literature

Considering the observations from the literature review, most PSR methods are aimed at finding a representative subset of efficient solutions based on measures of their proximities in the original objective space of the MOP, hence, without taking into account further information regarding their values in the decision space. Nonetheless, the values of decision variables should be considered when selecting a most preferred solution as they often represent important structural properties, including the design or physical characteristics of devices. In spite of reducing the size of the Pareto set, the existing methodologies do not provide further trade-off information that could ease the selection of a most preferred solution.

Because existing methods will lead to trade-off only in the objective space, as the number of objective functions increases, the analysis becomes cumbersome. Even plotting the Pareto set is challenging for problems with more than 3 objective functions.

Finally, the literature is limited to the case where the Pareto set is assumed to be discrete. This poses a clear limitation for decision making. Especially for continuous problems, discretization procedures may exclude from further consideration potentially promising solutions.

This dissertation will address the aforementioned issues in the development of a novel methodology for PSR.

1.4 Preliminaries and notation

This subsection gives preliminaries and introduces basic notation that is common to all sections in this dissertation.

Let $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^k$ be finite dimensional Euclidean vector spaces. For any two sets $Y \subseteq \mathbb{R}^k$ and $Z \subseteq \mathbb{R}^k$, $(Y + Z)$ denotes the Minkowski sum, i.e.:

$$(Y + Z) = \{y + z | y \in Y, z \in Z\} \text{ and}$$
$$(Y + \emptyset) = \emptyset$$

Let $f = (f_1, ..., f_k)^T$ denote a vector-valued function of objective functions $f_i : \mathbb{R}^n \to \mathbb{R}$, for $i = 1, ..., k$, where each f_i assigns to each decision vector $x \in \mathbb{R}^n$ a real number $f_i(x)$, such that $f(x) = (f_1(x), ..., f_k(x))^T$.

Definition 1. *(Stoer et al., 1970) A function $f : \mathbb{R}^n \mapsto \mathbb{R}^k$ is said to be positively homogeneous if $f(\lambda x) = \lambda f(x)$ for all $x \in \mathbb{R}^n$ and $\lambda > 0$.*

Let X be the feasible set defined on the decision space $\mathbb{R}^n$, such that $X = \{x : h_i(x) \geq b_i, i = 1, ..., m\}$, where $b_i \in \mathbb{R}$ is the right-hand-side parameter of the i-th constraint and the mapping $h_i : \mathbb{R}^n \to \mathbb{R}$ assigns to each $x \in \mathbb{R}^n$ a real number $h_i(x)$ defining the left-hand-side of the i-th constraint, for $i = 1, ..., m$. Let $Y = \{y \in \mathbb{R}^k : y = f(x), \text{ for } x \in X\}$ be the corresponding outcome set in the objective space $\mathbb{R}^k$. Then, the MOP is formulated as follows.

$$\begin{aligned} &\text{Min } f(x) \\ &\text{s.t. } x \in X \end{aligned} \tag{MOP}$$

For any x and $x' \in \mathbb{R}^n$, the following notation will be adopted: $f(x) < f(x^{'})$ if $f_i(x) < f_i(x^{'})$ for all $i = 1, ..., k$; $f(x) \leqq f(x^{'})$ if $f_i(x) \leq f_i(x^{'})$ for all $i = 1, ..., k$; and $f(x) \leq f(x^{'})$ if $f(x) \leqq f(x^{'})$ but $f(x) \neq f(x^{'})$. Let $\mathbb{R}^k_{\geqq} = \{y \in \mathbb{R}^k : y \geqq 0\}$ be the nonnegative orthant of $\mathbb{R}^k$. The sets $\mathbb{R}^k_{\geq}$ and $\mathbb{R}^k_{>}$ are defined similarly. Let int Y and bd Y denote the interior and the boundary of the set Y, respectively.

To solve the above MOP is understood as to find its set of efficient solutions, defined next.

Definition 2. *(Yu and Zeleny, 1975) A solution $x \in X$ to the MOP is called efficient if and only if there does not exist another $x' \in X$ such that $f(x') \leq f(x)$.*

Any solution $x \in X$ that does not satisfy efficiency is called a dominated solution. A solution $x' \in X$ is said to dominate another solution $x \in X$ if $f(x') \leq f(x)$.

Definition 3. *(Lowe et al., 1984) If x is an efficient solution and there does not exist another $x' \in X$ such that $f(x) = f(x')$, then x is said to be strictly efficient.*

Throughout this dissertation, the image of an efficient solution in the objective space is referred to as a Pareto outcome. The set of all efficient solutions in the decision space $\mathbb{R}^n$ is termed efficient set $\mathcal{X}$, whereas the set of all Pareto outcomes in the objective space $\mathbb{R}^k$ is referred to as the Pareto set $\mathcal{Y}$. Here, we assume that $\mathcal{X}$ and $\mathcal{Y}$ are non-empty compact sets. For details on the existence of the Pareto and the efficient sets, see Sawaragi et al. (1985).

Let $\underline{f}_i = \min\{f_i(x) : x \in \mathcal{X}\}$ and let $\overline{f}_i = \max\{f_i(x) : x \in \mathcal{X}\}$ denote the minimum and the maximum of the i-th objective function over the efficient set, respectively. It is assumed that $\underline{f}_i \neq \overline{f}_i$ or else objective function f_i would be non-conflicting in the MOP. The outcome $\underline{f} = (\underline{f}_1, ..., \underline{f}_k)^T$ is referred as the ideal point and $\overline{f} = (\overline{f}_1, ..., \overline{f}_k)^T$ is referred as the Nadir point. In general, computing the Nadir point can be computationally challenging. However, an estimate may be obtained from a payoff table (Miettinen, 1999). For further details on computing the Nadir point, the reader may refer to Ehrgott and Tenfelde-Podehl (2003) and Alves and Costa (2009). In this dissertation, it is assumed that $\mathcal{X}$, $\mathcal{Y}$, $\overline{f}$ and $\underline{f}$ are known and given as input.

1.5 Organization of the dissertation

This dissertation is organized in five sections, detailed as follows.

Section 2 formalizes the PSR problem, in which secondary robustness criteria are introduced to further break ties among the many Pareto outcomes associated with an MOP. The desired reduction is achieved by identifying the solutions that are also efficient in view of the secondary robustness criteria. Structural properties are uncovered, in particular when the MOP is an MOLP. These properties include the study of theoretical bounds and sufficient conditions to guarantee that fewer solutions are identified. Examples illustrate the characteristics of solutions obtained from the methodology.

Section 3 focuses on the case of MOLPs and generalizes the PSR problem from Section 2 to linear objective functions, where additional secondary decision variables and constraints are considered. It is shown that the problem can be reformulated as a biobjective mixed binary linear program when the efficient set is assumed to be given as the union of the maximal efficient faces of the MOLP. The solution algorithm recursively solves an auxiliary mixed binary linear program to find in the search for unsupported Pareto outcomes; in turn, a novel linear programming filtering excludes any solutions that are found to be dominated. The algorithm allows for solving the problem without discretization procedures common to the existing literature on PSR methods. The section ends by providing numerical experiments, suggesting that the algorithm runs faster than existing algorithms for solving biobjective mixed binary programs.

Section 4 illustrates the application of the methodology for assessing robustness of solutions from the biobjective electricity generation capacity expansion problem to minimize cost and water withdrawal. Each solution prescribes locations and technologies for new power plants, as well as their designed capacities. The problem is modeled as a two-stage optimization process. The first stage finds the set of solutions that are efficient in view of cost and water withdrawal objectives. The second stage finds the subset of first-stage solutions that are robust when the designed capacities of power plants are subjected to uncertainties at the time of their construction. The methodology is illustrated in a case study with data from Texas, USA. We demonstrate how the methodology can aid multiobjective decision making and be used to answer strategic questions in expansion planning related to the trade-off among the different technologies and locations, and the effect of uncertainty.

Finally, Section 5 presents summary of conclusions and contributions of this dissertation, as well as paths for the related future research.

2. BIOBJECTIVE ROBUST OPTIMIZATION OVER THE EFFICIENT SET FOR PARETO SET REDUCTION

2.1 Introduction

This section formalizes a methodology for identifying a subset of robust solutions from the Pareto set of an MOP. We term this problem the robust Pareto set reduction (PSR) problem.

The Pareto set of an MOP contains solutions that are indistinguishable (i.e., non-dominated) with respect to the objective functions for optimization. The size of the Pareto set can be very large and possibly infinite, posing a significant challenge to a decision maker needing to select a single solution for implementation.

PSR methods discussed in Section 1.3 are available to alleviate the problem of having to trade off among several alternatives. Briefly, this section models the PSR as the second stage in a two-stage optimization process. The first stage is the problem of generating the Pareto set (or an approximation) corresponding to an MOP; the second stage (i.e. the PSR problem) is an optimization problem over the efficient set of the MOP to identify solutions that are also efficient with respect to the aforementioned secondary objectives. It must be noted that the literature on optimization over the efficient set typically focuses on algorithms to find an optimal solution to a single objective (e.g Benson, 1984; Horst and Thoai, 1999; Sayin, 2000; Yamamoto, 2002; Jorge, 2009; Thang, 2015). In contrast, this section focuses on a new formulation and theoretical properties of the Pareto set reduction problem when the first stage problem is an MOLP. It is implied that the secondary objectives, although meaningful (i.e. they capture desirable characteristics of a solution), are not as important as the ones in the first stage MOP; otherwise, they should be considered together with the objectives in the MOP typically resulting in a larger, rather than reduced, Pareto set.

The consideration of robustness criteria to aid decision making is central to this section. Kouvelis and Yu (1997) describes that decision problems can be broadly classified into three categories: certainty, risk and uncertainty. Certainty category is the case where no element of chance affects the realization of the selected decision. Optimization problems that falls within this category can be solved via deterministic optimization techniques (e.g. simplex method and specialized network optimization algorithms). Risk category is the case where the connection between the selected decision and its realization is given by known probabilities. In this case, optimization problems are usually solved via stochastic optimization techniques. Uncertainty category is the case where it is not possible to attribute probabilities to the possible realizations of selected decisions. For instance, this may arise from the consideration of future events of non-repeatable nature for which the quantification of probabilities would yield unreliable information. One such example is the capacity expansion problem dealt with in Section 4. Optimization problems in this case are usually solved via robust optimization techniques.

Here a solution is deemed robust if it remains "close" to its optimal objective values and/or "almost" feasible when affected by uncertainty. The notions of what is "close" and "almost" are made precise later in the section. Mulvey et al. (1995) termed the robustness with respect to objective value loss as *solution robustness*, and robustness with respect to feasibility as *model robustness*. While it is unlikely that a solution remains at the same time optimal and feasible when affected by uncertainty, the robust optimization literature often focuses on finding solutions that will remain feasible for all possible realizations within the uncertainty set (e.g. Ben-Tal and Nemirovski, 2002; Bertsimas and Sim, 2004), hence not explicitly providing the decision maker with trade-off information between model and solution robustness; see Bertsimas et al. (2011) for a survey on the extensive work in single objective robust optimization, and Gunawan and Azarm (2005), Deb and Gupta (2006), Ehrgott et al. (2014), Mavrotas et al. (2015b) and Goberna et al. (2015) for literature on the less studied multiobjective case. The choice of the uncertainty set plays an important role in tractability of robust counterpart problems. For examples of different structures of uncertainty sets see Bertsimas and Brown (2009) and Bertsimas and Thiele (2006a). Here, we focus on the box and cardinality-constrained uncertainty sets. A major advantage from these cases is that they lead to tractable robust counterpart problems in the linear case (Bertsimas and Sim, 2004).

A few recent papers deal with problems similar to the one presented here. A methodology for identifying efficient solutions within a set of robust solutions was proposed by Iancu and Trichakis (2013). The authors assessed efficiency of a solution in terms of performance for all scenarios in the uncertainty set. Here our methodology seeks the opposite, i.e. finding robust solutions within the set of efficient solutions of the MOP. Finally, Mavrotas et al. (2015a) used Monte Carlo simulation to assess the robustness of efficient solutions of an MOP using nominal values of the problem parameters. A simulation trial consists of solving the MOP with an instantiation of the uncertain problem parameters; in turn, the robustness of each nominal efficient solution is evaluated by a ratio that reflects the number of times it belongs to the efficient set across all Monte Carlo trials. Important differences are that the proposed methodology is based on mathematical programming rather than simulation on a finite efficient set, and that it explicitly considers model and solution robustness as opposed to a unique robustness ratio.

The remainder of the section is organized as follows. Section 2.2 formalizes the model, introducing notation and mathematical definitions used in the section. Section 2.3 describes the robust Pareto set reduction problem formulation and analysis. Section 2.4 develops properties when the first stage problem is an MOLP. Section 2.5 provides numerical examples to illustrate the methodology and the characteristics of robust solutions. Section 2.6 presents concluding remarks for the section.

2.2 Model description

In the modeling of uncertainty, this research is motivated by cases where the prescribed values of the decision variables cannot be implemented exactly as computed. These implementation errors, referred here to as implementation uncertainty, will occur due to lack of model fidelity resulting from practical issues such as insufficient modeling time or unavailable knowledge during model building. Ben-Tal et al. (2009) state that implementation uncertainty may arise due to the inherent characteristics of some physical devices (e.g. antenna design) and that these implementation uncertainties are equivalent to "appropriate artificial data uncertainties". Here, uncertainty is modeled directly affecting the decision variables; the results, however, are applicable to both data and implementation uncertainty because the focus here is on MOLPs. It must be noted that in other cases the equivalence is not as straightforward; e.g. in conic quadratic programs under implementation uncertainty with ellipsoidal uncertainty set (Ben-Tal and Den Hertog, 2011).

Next, we proceed by formalizing the general representation of the perturbation on a solution values.

Definition 4. *For some given uncertainty level $\alpha > 0$, the perturbation factor $\tilde{\beta} = (\tilde{\beta}_1, ..., \tilde{\beta}_n)^T$ is a random vector such that $1 - \alpha \leq \tilde{\beta}_j \leq 1 + \alpha$, for all $j = 1, ..., n$. Given a realization β of $\tilde{\beta}$, $\beta x = (\beta_1 x_1, ..., \beta_j x_j)^T$ denotes a perturbation on the solution values of x.*

Although this paper restricts the analysis to the case of a multiplicative pertubation factor, the model may alternatively be defined by additive perturbations, which would be equivalent to perturbing the right-hand-side of the constraint set. Let $\mathcal{U}$ denote the uncertainty set of all possible realizations of the perturbation factor $\tilde{\beta}$. At this point, no further structure is imposed on $\mathcal{U}$ other than being bounded.

To measure the degree of model robustness, let the function δ: $\mathbb{R}^n \mapsto \mathbb{R}_{\geqq}$ be a mapping that assigns to each $x \in \mathbb{R}^n$ a non-negative real number, denoted by $\delta(x) \in \mathbb{R}_{\geqq}$, that measures the degree of constraint violation due to perturbations on the values of x, given as follows.

Definition 5. *Let $x \in \mathbb{R}^n$. Then,*

$$\delta(x) = \sup_{\substack{\beta \in \mathcal{U} \\ i=1,\ldots,m}} \left\{ \max \left\{ \left(b_i - h_i(\beta x) \right) / \left| b_i \right|, 0 \right\} \right\}$$

is called the infeasibility level *associated with x.*

It is further assumed that efficiency of a solution with respect to the MOP has priority over hedging against the uncertainty. Hence, $\mathcal{X}$ restricts the feasible set for the subsequent problem of finding those efficient solutions that satisfy model and/or solution robustness.

Definition 5 requires that $b_i \neq 0$; if some $b_i = 0$, an alternative would be to replace the denominator $|b_i|$ for $|b_i| + 1$. The case when $\delta(x) = 0$ implies that the solution will remain feasible for any perturbation within the uncertainty set.

Similarly, in order to measure solution robustness let the function γ: $\mathbb{R}^n \mapsto \mathbb{R}_{\geqq}$ be a mapping that assigns to each $x \in \mathbb{R}^n$ a non-negative real number, denoted by $\gamma(x) \in \mathbb{R}_{\geqq}$, that measures the degree of objective function values loss due to perturbations on the values of x, given as follows.

Definition 6. *Let $x \in \mathbb{R}^n$. Then,*

$$\gamma(x) = \sup_{\substack{\beta \in \mathcal{U} \\ i=1,\ldots,k}} \left\{ \max \left\{ \frac{f_i(\beta x) - f_i(x)}{\overline{f}_i - \underline{f}_i}, 0 \right\} \right\}$$

is called the outcome degradation level *associated with x.*

In Definition 6, a normalization of objective function values is used due to the possibly different ranges of the objective functions in the MOP. The case when $\gamma(x) = 0$ implies that the perturbations on x within the uncertainty set will not worsen any objective function value.

Considering both model and solution robustness, Definition 7 introduces the robustness projection set and the robustness space.

Definition 7. *The* robustness projection set $\mathcal{D}$ *defined on the robustness space $\mathbb{R}^2_{\geqq}$ contains the set of points*

$$\mathcal{D} = \left\{ (\delta(x), \gamma(x))^T, \ \forall x \in \mathcal{X} \right\}$$

2.3 Problem formulation and analysis

Given the efficient set $\mathcal{X}$ of an MOP and the uncertainty set $\mathcal{U}$, the robust Pareto set reduction problem (RPSR) is a bicriterion program to minimize the infeasibility and the outcome degradation levels over the efficient set of the MOP. The RPSR is formulated as follows:

$$
\begin{aligned}
\text{Min } & (\delta, \gamma)^T \\
\text{s.t. } & h_i(\beta x) + |b_i|\delta \geq b_i && \forall i = 1, ..., m,\ \beta \in \mathcal{U} \\
& f_i(x) - f_i(\beta x) + (\overline{f}_i - \underline{f}_i)\gamma \geq 0 && \forall i = 1, ..., k,\ \beta \in \mathcal{U} \\
& x \in \mathcal{X} \\
& \delta, \gamma \geq 0
\end{aligned}
\tag{RPSR}
$$

The formulation of the RPSR, in its general form, may contain infinitely many constraints because $\mathcal{U}$ is not explicitly specified. As in a classical robust optimization approach, this issue is overcome by considering uncertainty sets with certain geometries; in particular, the box and the cardinality-constrained uncertainty sets will be used in the following sections.

Let $\mathcal{X}'$ denote the subset of efficient solutions to the RPSR, termed the robust efficient set. Let $\mathcal{Y}'$ denote its image in the objective space, the robust Pareto set. The set $\mathcal{D}' = \{(\delta(x), \gamma(x)) : x \in \mathcal{X}'\}$ is the image of the robust efficient set in the robustness space. In general, $\mathcal{Y}'$ can still possess an infinite number of outcomes when the MOP is a continuous problem. In this case, the term "reduction" is used in a broader sense to denote that the RPSR identifies only the subset of $\mathcal{Y}$ that contains robust solutions.

The model is illustrated in Figure 2.1. In the first stage, we assume that the MOP is solved to generate the efficient set $\mathcal{X}$ and the Pareto set $\mathcal{Y}$. In the second stage, i.e. the PSR problem focus of this section, the efficient set $\mathcal{X}$ and the uncertainty set $\mathcal{U}$ are taken as input to find the subset of efficient solutions that remain Pareto optimal in the robustness space. The model defines a map $\mathbb{R}^k \to \mathbb{R}^2$ from the k-dimensional objective space of the MOP onto a 2-dimensional robustness space (i.e. the objective space of the RPSR).

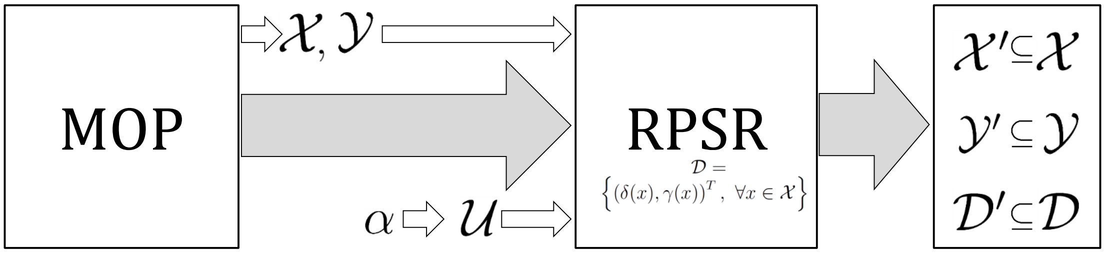


Figure 2.1: Robust optimization model for Pareto set reduction

Solving the RPSR is not equivalent to solving a robust counterpart of the MOP, i.e. when uncertainty is considered directly when solving the MOP from the first stage. This could result in solutions dominated by the ones in $\mathcal{X}$, as $\mathcal{Y}$ lies on the boundary of Y. The RPSR formulation is neither analogue to introducing the infeasibility and outcome degradation criteria directly into the MOP, i.e.:

$$
\begin{aligned}
\text{Min}\ & (f_1(x), ..., f_k(x), \delta, \gamma)^T \\
\text{s.t.}\ & h_i(\beta x) + |b_i|\delta \geq b_i && \forall i = 1, ..., m,\ \beta \in \mathcal{U} \\
& f_i(x) - f_i(\beta x) + (\overline{f}_i - \underline{f}_i)\gamma \geq 0 && \forall i = 1, ..., k,\ \beta \in \mathcal{U} \\
& x \in X \\
& \delta, \gamma \geq 0
\end{aligned}
\tag{P2}
$$

Let $\mathcal{X}^{P2}$ be the efficient set of P2. Then, the following relates P2 and the RPSR.

Theorem 2.3.1. *If all $x \in \mathcal{X}$ are strictly efficient, then $\mathcal{X}' \subseteq \mathcal{X}^{P2}$.*

The proof of Theorem 2.3.1 is straightforward by noting that $\mathcal{X}' \subseteq \mathcal{X}$ and that, by strict efficiency, $\mathcal{X} \subseteq \mathcal{X}^{P2}$. Hence, adding the measures of robustness directly when solving the MOP would not yield the desired set reduction.

Next, limit conditions are presented for the infeasibility and the outcome degradation levels.

Theorem 2.3.2. *When $h_i : \mathbb{R}^n \mapsto \mathbb{R}$, for $i = 1, ..., m$, and $f_i : \mathbb{R}^n \mapsto \mathbb{R}$, for $i = 1, ..., k$, are continuous functions, then for every $\xi > 0$ there exists $\nu > 0$ such that*

$$0 < \alpha < \nu \implies 0 \leq \delta(x) < \xi \text{ and } 0 \leq \gamma(x) < \xi, \text{ for all } x \in \mathcal{X}.$$

The proof of Theorem 2.3.2 follows straightforward analysis of limits. As a result, as α approaches zero, the values of infeasibility and outcome degradation levels of all efficient solutions tend to zero.

2.4 Pareto set reduction in the linear case

Let $C^i = (c_1^i, ..., c_n^i)^T$, $i = 1, ..., k$, be the cost vector for the i-th objective function. Let $X = \{x : Ax \geqq b, x \geqq 0\}$, where $A \in \mathbb{R}^{m \times n}$ is a matrix of elements a_{ij} for $i = 1, ..., m$ and $j = 1, ..., n$. Then, the RPSR associated with an MOLP will have $h_i(x) = \sum_{j=1}^{n} a_{ij} x_j$, for $i = 1, ..., m$, and $f_i(x) = \sum_{j=1}^{n} c_j^i x_j$, for $i = 1, ..., k$.

The following subsections describe several structural properties of the problem resulting from the linearity condition.

2.4.1 Upper bounds on the infeasibility and outcome degradation levels

The bounds presented next are developed as a function of the uncertainty level α; hence if the value of α can be adjusted, the decision maker gains additional control over the robustness of the solution selected for implementation. For instance, in engineering design problems (e.g. the antenna design in Ben-Tal et al. (2009)), improving the precision of the manufacturing system would allow for a lower value of α, hence decreasing the value of the upper bounds. Conversely, given a maximum infeasibility and outcome degradation levels that are acceptable to the decision maker, the bounds may be useful for finding a target value for α.

Theorem 2.4.1. *For all $x \in \mathcal{X}$, the infeasibility level is bounded by*

$$\delta^u(x) = \alpha \max_{i=1,..,m} \left\{ \sum_{j=1}^{n} \left| \frac{a_{ij}x_j}{b_i} \right| \right\}$$

Proof. Let J_i^+ be the columns of row i in constraint matrix A where $a_{ij} \geq 0$ and let J_i^- be the columns where $a_{ij} < 0$. Then, for $a_{ij} \in J_i^+$, the upper bound on the constraint violation will be when $\beta_j = 1 - \alpha$. Similarly, for $a_{ij} \in J_i^-$, the upper bound will be when $\beta_j = 1 + \alpha$. Therefore:

$$\begin{aligned}
\delta(x) &\leq \max_{i=1,...,m} \left\{ \max \left\{ \frac{b_i - \left(\sum_{j \in J_i^+} a_{ij}(1-\alpha)x_j + \sum_{j \in J_i^-} a_{ij}(1+\alpha)x_j \right)}{|b_i|}, 0 \right\} \right\} \\
&= \max_{i=1,...,m} \left\{ \max \left\{ \frac{b_i - \sum_{j=1}^n a_{ij}x_j}{|b_i|} - \alpha \frac{\sum_{j \in J_i^-} a_{ij}x_j - \sum_{j \in J_i^+} a_{ij}x_j}{|b_i|}, 0 \right\} \right\} \\
&\leq \max_{i=1,...,m} \left\{ \max \left\{ -\alpha \frac{\sum_{j \in J_i^-} a_{ij}x_j - \sum_{j \in J_i^+} a_{ij}x_j}{|b_i|}, 0 \right\} \right\} \\
&= \alpha \max_{i=1,...,m} \left\{ \sum_{j=1}^{n} \left| \frac{a_{ij}x_j}{b_i} \right| \right\}
\end{aligned}$$

where the last inequality follows from the fact that $b_i - \sum_{j=1}^n a_{ij}x_j \leq 0, \forall i = 1, ..., m$. $\square$

If parameters a_{ij} and b are non-negative, then the following corollary follows:

Corollary 2.4.2. *If $b_i > 0$ and $a_{ij} \geq 0$ for all $i = 1, ..., m$ and $j = 1, ..., n$, then for all $x \in \mathcal{X}$, the infeasibility level is bounded by $\delta^u(x) = \alpha$.*

Proof. Because $b_i > 0$, the modulus in the denominator of $\delta(x)$ can be dropped. The upper bound on the constraint will be when $\beta_j = 1 - \alpha$ for all $j = 1, ..., n$. Then, it becomes:

$$\begin{aligned}\delta(x) &\leq \max_{i=1,\dots,m}\left\{\max\left\{\frac{b_i - \sum_{j=1}^{n} a_{ij}(1-\alpha)x_j}{b_i}, 0\right\}\right\} \\ &\leq \max_{i=1,\dots,m}\{\max\{1-(1-\alpha),0\}\} = \alpha\end{aligned}$$

where the last inequality follows from the fact that $\sum_{j=1}^{n} a_{ij}x_j/b_i \geq 1, \forall i = 1, ..., m.$ $\square$

Upper bounds for the outcome degradation level are developed next.

Theorem 2.4.3. *For all $x \in \mathcal{X}$, the outcome degradation level is bounded by*

$$\gamma^u(x) = \alpha \max_{i=1,\dots,k}\left\{\frac{\sum_{j=1}^{n}\left|c_j^i\right| x_j}{\overline{f}_i - \underline{f}_i}\right\}$$

Proof. Let J_i^+ be the columns j where, for objective function $i = 1, ..., k$, $c_j^i \geq 0$ and let J_i^- be the columns where $c_j^i < 0$. Then, for $c_j^i \in J_i^+$, the upper bound on $\gamma(x)$ will be when $\beta_j = 1 + \alpha$. Similarly, for $c_j^i \in J_i^-$, the upper bound will be when $\beta_j = 1 - \alpha$. Then, it follows that

$$\begin{aligned}\gamma(x) &\leq \max_{i=1,\dots,k}\left\{\frac{\sum_{j\in J_i^+} c_j^i(1+\alpha)x_j + \sum_{j\in J_i^-} c_j^i(1-\alpha)x_j - \sum_{j=1}^{n} c_j^i x_j}{\overline{f}_i - \underline{f}_i}\right\} \\ &= \alpha \max_{i=1,\dots,k}\left\{\frac{\sum_{j=1}^{n}\left|c_j^i\right| x_j}{\overline{f}_i - \underline{f}_i)}\right\} \quad \square\end{aligned}$$

Corollary 2.4.4. *If $c_j^i \geq 0$ for all $i = 1, ..., k$ and $j = 1, ..., n$, then for all $x \in \mathcal{X}$, the outcome degradation level is bounded by*

$$\gamma^u(x) = \alpha \max_{i=1,\dots,k}\left\{\frac{\overline{f}_i}{\overline{f}_i - \underline{f}_i}\right\}$$

Proof. Because $\overline{f}_i \geq \sum_{j=1}^{n} c_j^i x_j \, \forall x \in \mathcal{X}$:

$$\gamma(x) \leq \alpha \max_{i=1,\dots,k} \left\{ \frac{\sum_{j=1}^{n} c_j^i x_j}{\overline{f}_i - \underline{f}_i} \right\} \leq \alpha \max_{i=1,\dots,k} \left\{ \frac{\overline{f}_i}{\overline{f}_i - \underline{f}_i} \right\} \quad \square$$

While Theorem 2.4.3 provides a tighter bound on the outcome degradation level, the bound in Corollary 2.4.4 is independent from the value of the solution x.

2.4.2 Properties using the box uncertainty set

In this subsection, we assume that the uncertainty set is given as the box

$$\mathcal{U} = \{\beta \in \mathbb{R}^n : 1 - \alpha \leq \beta_j \leq 1 + \alpha, \; j = 1, ..., n\}$$

for some $\alpha > 0$. Then, the worst case violation for the i-th constraint will have $\beta_j = 1 + \alpha$, if $a_{ij} < 0$, and $\beta_j = 1 - \alpha$, if $a_{ij} > 0$. Similarly, the worst case degradation for the i-th objective function value will have $\beta_j = 1 + \alpha$, if $c_j^i > 0$, and $\beta_j = 1 - \alpha$, if $c_j^i < 0$. Therefore, the RPSR formulation becomes:

$$\begin{aligned}
\text{Min} \;\; & (\delta, \gamma)^T \\
\text{s.t.} \;\; & \sum_{j=1}^{n} (a_{ij} - |a_{ij}|\alpha) x_j + |b_i|\delta \geq b_i \quad \forall i = 1, ..., m \\
& (\overline{f}_i - \underline{f}_i)\gamma - \sum_{j=1}^{n} |c_j^i| \alpha x_j \geq 0 \qquad \forall i = 1, ..., k \\
& x \in \mathcal{X} \\
& \delta, \gamma \geq 0
\end{aligned} \tag{B-RPSR}$$

If constraints $x \in \mathcal{X}$ are relaxed to $x \in X$, B-RPSR becomes a biobjective linear programming problem. Since the efficient set can be represented by the corresponding maximal efficient faces of X, this suggests that B-RPSR may be solved by an iterative procedure: in each iteration, a different maximal efficient face is fixed and the corresponding constrained version of the B-RPSR is solved. After all maximal efficient faces have been examined, a filtering would be necessary to eliminate all obtained solutions that are dominated in the robustness space. In the worst case, there might be exponentially many maximal efficient faces, and for each of these faces there might be exponentially many extreme points, so that the problem grows exponentially. A solution a approach that does not rely on exhaustive search over the maximal efficient faces will be presented in Section 3.

Next, we show special properties of the B-RPSR and discuss their implication in support of decision making.

Theorem 2.4.5. *If all $x \in \mathcal{X}$ are strictly efficient, $c_j^i \geq 0$, $j = 1, ..., n$, $i = 1, 2$, $b_i > 0$ and $a_{ij} \geq 0$, $i = 1, ..., m$, $j = 1, ..., n$, then solving the B-RPSR will result in $|\mathcal{X}'| = 1$.*

Proof. Because all perturbations $\tilde{\beta}_j$ can take the worst case simultaneously, and all parameters are non-negative, the bounds from Corollary 2.4.2 and Theorem 2.4.3 will be tight, hence for $x \in \mathcal{X}$:

$$\delta(x) = \alpha$$

$$\gamma(x) = \alpha \max \left\{ \frac{f_1(x)}{\overline{f}_1 - \underline{f}_1}, \frac{f_2(x)}{\overline{f}_2 - \underline{f}_2} \right\}$$

Therefore, the set of robust solutions will be given by:

$$\mathcal{X}' = \arg\min_{x \in \mathcal{X}} \gamma(x)$$

Because of efficiency, there does not exist x and $x' \in \mathcal{X} : f_1(x') \leq f_1(x)$ and $f_2(x') \leq$

$f_2(x)$ with at least one inequality holding strictly. Hence, it holds that:

$$\mathcal{X}' = \arg\min_{x \in \mathcal{X}} \gamma(x) = \left\{ x \in \mathcal{X} : \frac{f_1(x)}{\overline{f}_1 - \underline{f}_1} = \frac{f_2(x)}{\overline{f}_2 - \underline{f}_2} \right\}$$

From strict efficiency, only one solution $x \in \mathcal{X}$ can satisfy the above condition, hence $|\mathcal{X}'| = 1$. $\square$

Theorem 2.4.6. *Let $\mathcal{X}'(\alpha')$ be the set of robust solutions of B-RPSR for a given value of $\alpha = \alpha' > 0$. If $b_i > 0$ and $a_{ij} \geq 0$, $i = 1, ..., m$, $j = 1, ..., n$, then $\mathcal{X}'(\alpha') = \mathcal{X}'(\alpha'')$ for all $\alpha'' > 0$.*

Proof. Since the box uncertainty set is assumed and constraint parameters are non-negative, the bounds from Corollary 2.4.2 and Theorem 2.4.3 will be tight, hence for and some $\alpha' > 0$:

$$\delta(x) = \alpha'$$

$$\gamma(x) = \alpha' \max_{i=1,\ldots,k} \left\{ \frac{|f_i(x)|}{\overline{f}^i - \underline{f}^i} \right\}$$

The definition of Pareto optimality induces a binary relation among outcomes that is compatible with scalar multiplication; that is, for any x and $x^* \in \mathcal{X}$ such that $\delta(x) \leq \delta(x^*)$ and $\gamma(x) \leq \gamma(x^*)$ with at least one inequality holding strictly, it follows that $t\delta(x) \leq t\delta(x^*)$ and $t\gamma(x) \leq t\gamma(x^*)$ for any $t > 0$, with at least one inequality holding strictly. Let $\delta^{\alpha'}(x)$ and $\gamma^{\alpha'}(x)$ denote the values of the the functions $\delta(x)$ and $\gamma(x)$ when $\alpha = \alpha'$. Given $\alpha', t > 0$, then for all $x^* \in \mathcal{X}$ such that $x^* \notin \mathcal{X}'(\alpha')$, there exists some $x \in \mathcal{X}'(\alpha')$ such that, with at least one inequality holding strictly:

$$\begin{aligned} \delta^{\alpha'}(x) \leq \delta^{\alpha'}(x^*) &\implies t\delta^{\alpha'}(x) \leq t\delta^{\alpha'}(x^*) \\ &\implies \delta^{t\alpha'}(x) \leq \delta^{t\alpha'}(x^*) \end{aligned}$$

and

$$\gamma^{\alpha'}(x) \leq \gamma^{\alpha'}(x^*) \implies t\gamma^{\alpha'}(x) \leq t\gamma^{\alpha'}(x^*)$$
$$\implies \gamma^{t\alpha'}(x) \leq \gamma^{t\alpha'}(x^*)$$

Taking $t = \alpha''/\alpha'$ results in $x^* \notin \mathcal{X}'(\alpha'')$ and $x \in \mathcal{X}'(\alpha'')$, hence $\mathcal{X}'(\alpha'') = \mathcal{X}'(\alpha')$. $\square$

The practical implications of Theorems 2.4.5 and 2.4.6 are significant. When assuming the worst case robustness with the box uncertainty set, Theorem 2.4.5 provides a condition where the RPSR methodology effectively breaks ties among efficient solutions, leading the decision maker to a unique solution for implementation. Moreover, Theorem 2.4.6 provides conditions where the set of robust solutions will remain the same, regardless of the level of uncertainty α; in turn, the practitioner may apply the proposed methodology even in situations when there is no knowledge about the uncertainty bounds, circumventing limitations that would otherwise prevent the application of a classical robust optimization approach (e.g. Ben-Tal and Nemirovski, 2002; Bertsimas and Sim, 2004).

Of notice is that the result from Theorem 2.4.6 may be easily extended to uncertainty sets with different geometries as long as the corresponding $\delta(x)$ and $\gamma(x)$ are positively homogeneous functions with respect to α.

2.4.3 Properties using the cardinality-constrained uncertainty set

The cardinality-constrained uncertainty set arises when the number of decision variables that are simultaneously affected by uncertainty is limited by some given non-negative integer Γ. As in Bertsimas and Sim (2003), the value of Γ can be adjusted to control the level of conservatism when finding robust efficient solutions. In this case, $\mathcal{U}$ is given by:

$$\mathcal{U} = \left\{ \beta \in \mathbb{R}^n : (1 - \alpha y_j) \leq \beta_j \leq (1 + \alpha y_j),\ \sum_{j=1}^{n} y_j \leq \Gamma,\ y \in \{0,1\}^n \right\} \tag{2.1}$$

Let δ^c and γ^c denote the infeasibility and outcome degradation levels under the cardinality-constrained uncertainty set. Then, the cardinality-constrained RPSR, denoted as C-RPSR, becomes:

$$\begin{aligned}
\text{Min}\ & (\delta^c, \gamma^c)^T \\
\text{s.t.}\ & \sum_{j=1}^{n} a_{ij} x_j + |b_i|\delta^c - \alpha \max_{y_j \in \{0,1\}} \left\{ \sum_{j=1}^{n} |a_{ij}|\, x_j y_j : \sum_{j=1}^{n} y_j \leq \Gamma \right\} \geq b_i \quad \forall i = 1, ..., m \\
& (\overline{f}_i - \underline{f}_i)\gamma^c - \alpha \max_{v_j \in \{0,1\}} \left\{ \sum_{j=1}^{n} \left|c_j^i\right| x_j v_j : \sum_{j=1}^{n} v_j \leq \Gamma \right\} \geq 0 \qquad \forall i = 1, ..., k \\
& x \in \mathcal{X} \\
& \delta^c, \gamma^c \geq 0
\end{aligned} \tag{C-RPSR}$$

where the first and second sets of constraints come from the application of the uncertainty set (2.1) to Definitions 5 and 6, respectively.

In order to see that the first set of constraints follows, let J_i^- and J_i^+ denote the set of columns in row i where coefficients a_{ij} are negative and positive, respectively. With the cardinality restriction, the constraints $\sum_{j=1}^{n} a_{ij}\beta_j x_j + |b_i|\delta \geq b_i$ in the RPSR formulation become:

$$\max_{y_j \in \{0,1\}} \left\{ \sum_{j \in J_i^+} a_{ij}(1 - \alpha y_j) x_j + \sum_{j \in J_i^-} a_{ij}(1 + \alpha y_j) x_j : \sum_{j=1}^{n} y_j \leq \Gamma \right\} + |b_i|\delta^c(x) \geq b_i$$

which yields the first set of constraints in the C-RPSR. Similarly, let J_i^- and J_i^+ denote the set of columns in the i-th objective function where coefficients c_j^i are negative and positive, respectively. The cardinality-constrained outcome degradation level from constraints $\sum_{j=1}^{n} c_j^i x_j - \sum_{j=1}^{n} c_j^i \beta_j x_j + (\overline{f}_i - \underline{f}_i)\gamma \geq 0$ in the RPSR formulation becomes:

$$\sum_{j=1}^{n} c_j^i x_j - \max_{v_j \in \{0,1\}} \left\{ \sum_{j \in J_i^+} c_j^i (1 + \alpha v_j) x_j + \sum_{j \in J_i^-} c_j^i (1 - \alpha v_j) x_j : \sum_{j=1}^{n} v_j \leq \Gamma \right\} + (\overline{f}_i - \underline{f}_i)\gamma^c \geq 0$$

which yields the second set of constraints in C-RPSR.

When $\mathcal{X}$ is approximated by a finite number of solutions (e.g. by solving the MOLP using the ϵ-constraint method or some metaheuristic), x_j's can be regarded as constants in the C-RPSR so that the values of infeasibility and outcome degradation levels can be computed for each efficient solution. Then, the inner maxima in the first and second sets of constraints in the C-RPSR consist of knapsack problems with equal weights. Hence, the solution for a given x will be simply to set $y_j = 1$ and $v_j = 1$ for the columns with the Γ largest absolute values of $a_{ij} x_j$ and $c_j^i x_j$, respectively. In the worst case, computing $\delta(x)$ and $\gamma(x)$ for each solution can be done in $O((m+k)n)$ time, since: (i) finding the Γ largest $|a_{ij} x_j|$ values for each constraint $i = 1, ..., m$, and the Γ largest $|c_j^i x_j|$ values for each objective function $i = 1, ..., k$, are selection problems and can be done in $O(n)$ time in the worst case; (ii) taking the maximum of the m constraint violations and the maximum of the k objective function values degradations associated with the previous step require $O(m)$ and $O(k)$ time, respectively. Filtering dominated solutions in the robustness space requires pairwise comparisons of all solutions in the worst case. Hence, the computational complexity associated with solving the C-RPSR is $O(|\hat{\mathcal{X}}|^2 + |\hat{\mathcal{X}}|(m + k)n)$, where $\hat{\mathcal{X}}$ is the approximation of $\mathcal{X}$ by a finite number of solutions.

In the C-RPSR formulation, the first and the second constraint sets are non-linear due to the inner maxima in each of them. Similar to Bertsimas and Sim (2004), next we show that using an LP relaxation of the binary variables and invoking strong duality principles in the inner maxima yields an equivalent formulation of the C-RPSR where the constraints related to the infeasibility and outcome degradation levels are convex.

Theorem 2.4.7. *The C-RPSR can be reformulated as:*

$$\begin{aligned}
&Min \;\; (\delta^c, \gamma^c)^T \\
&s.t. \;\; \sum_{j=1}^{n} a_{ij}x_j + |b_i|\delta^c - \alpha(\Gamma\omega_i + \sum_{j=1}^{n}\pi_{ij}) \geq b_i && \forall i = 1, ..., m \\
&\quad (\overline{f}_i - \underline{f}_i)\gamma^c - \alpha(\Gamma\rho_i + \sum_{j=1}^{n}\xi_{ij}) \geq 0 && \forall i = 1, ..., k \\
&\quad \omega_i + \pi_{ij} - |a_{ij}|x_j \geq 0 && \forall = 1, ..., m; j = 1, ...n \\
&\quad \rho_i + \xi_{ij} - |c^i_j|x_j \geq 0 && \forall = 1, ..., k; j = 1, ...n \\
&\quad \omega_i \geq 0 && \forall = 1, ..., m \\
&\quad \rho_i \geq 0 && \forall = 1, ..., k \\
&\quad \pi_{ij} \geq 0 && \forall = 1, ..., m; j = 1, ...n \\
&\quad \xi_{ij} \geq 0 && \forall = 1, ..., k; j = 1, ...n \\
&\quad \delta^c, \gamma^c \geq 0 \\
&\quad x \in \mathcal{X}
\end{aligned} \tag{C-D}$$

Proof. Notice that in the inner maxima from the C-RPSR formulation the only variables are y_j and v_j, and that the corresponding constraints in the inner maxima show the total unimodularity property (Heller and Tompkins, 1956). Hence the binary requirements on these variables can be dropped. Furthermore, in the first set of constraints in the C-RPSR formulation, the dual of the inner maximum for the i-th constraint becomes:

$$\begin{aligned} \text{Min } & \Gamma\omega_i + \sum_{j=1}^{n} \pi_{ij} \\ \text{s.t. } & \omega_i + \pi_{ij} \geq |a_{ij}|x_j \quad \forall j = 1, ..., n \\ & \pi_{ij} \geq 0 \qquad\qquad\qquad \forall j = 1, ..., n \\ & \omega_i \geq 0 \end{aligned}$$

In the first set of constraints in the C-RPSR formulation, the inner maximum for the i-th constraint will be always feasible and bounded for any $0 \leq \Gamma \leq n$, so will be the dual; hence by strong duality, their optimal objective values coincide. Applying the same procedure to the second set of constraints in the C-RPSR formulation, and substituting the duals for the maxima in the C-RPSR formulation yields the desired result. $\square$

Therefore, as in the case with the box uncertainty set, the C-D formulation becomes a biobjective linear program when constraint $x \in \mathcal{X}$ is replaced by a maximal efficient face of X. Hence, the solution procedure presented in Section 3 may be also applied to solve the case with the cardinality-constrained uncertainty set.

In the formulation of the C-RPSR, if a low value of Γ is used, the actual realization of the infeasibility and the outcome degradation levels might be higher than the ones obtained from the C-RPSR. This is because, in reality, more than Γ decision variables might be simultaneously affected by uncertainty. Indeed, in the worst case, all columns would be simultaneously affected by uncertainty. Theorems 2.4.8 and 2.4.9 determine upper bounds on the probabilities of constraint violations and objective function degradations that exceed the calculated values using the cardinality-constrained uncertainty set. These theorems are extensions of Bertsimas and Sim (2004) in the sense that the probability bounds are developed for i.i.d. and bounded random variables with unknown distributions. However, in Bertsimas and Sim (2004) the bounds were derived for the probability of constraint violations of any size, whereas here the bounds are for the excess probability.

For $i = 1, ..., m$, let $\tilde{\delta}^i(x) = \max\left\{\left(b_i - \sum_{j=1}^n a_{ij}\tilde{\beta}_j x_j\right) / |b_i|, 0\right\}$ and denote the cardinality-constrained infeasibility level for the i-th constraint, $\delta_i^c(x)$, as:

$$\begin{aligned}
\delta_i^c(x) &= \max\left\{\frac{b_i - \sum_{j=1}^n a_{ij}x_j + \alpha \max_{y\in\{0,1\}^n}\left\{\sum_{j=1}^n |a_{ij}|\, x_j y_j \,:\, \sum_{j=1}^n y_j \le \Gamma\right\}}{|b_i|}, 0\right\} \\
&= \frac{b_i - \sum_{j=1}^n a_{ij}x_j + u_i}{|b_i|} \qquad (2.2)
\end{aligned}$$

where $u_i = \max\left\{\alpha \max_{y\in\{0,1\}^n}\left\{\sum_{j=1}^n |a_{ij}|\, x_j y_j \,:\, \sum_{j=1}^n y_j \le \Gamma\right\}, \sum_{j=1}^n a_{ij}x_j - b_i\right\}$.

For $i = 1, ..., k$ let $\tilde{\gamma}^i(x) = \max\left\{\sum_{j=1}^n c_j^i x_j(\tilde{\beta}_j - 1)/(\overline{f}_i - \underline{f}_i), 0\right\}$ and denote the cardinality-constrained outcome degradation level for the i-th objective function, $\gamma_i^c(x)$, as:

$$\gamma_i^c(x) = \left(\alpha \max_{v\in\{0,1\}^n}\left\{\sum_{j=1}^n |c_j^i| x_j v_j \,:\, \sum_{j=1}^n v_j \le \Gamma\right\}\right) / (\overline{f}_i - \underline{f}_i) \qquad (2.3)$$

The bounds using the cardinality-constrained uncertainty set are presented next.

Theorem 2.4.8. *Let $\mathcal{U}$ be as in (2.1) and let $\tilde{\beta}_j$ from Definition 4 be symmetric i.i.d. random variables centered at 1. For $i = 1, ..., m$, the probability that the i-th constraint violation exceeds the value of δ_i^c is given by:*

$$\Pr\left(\tilde{\delta}^i(x) > \delta_i^c(x)\right) \le \exp\left(-\frac{\tau_i^2}{2\sum_{j=1}^n (a_{ij}x_j)^2}\right)$$

where $\tau_i = \max\left\{\max\limits_{y\in\{0,1\}^n}\left\{\sum_{j=1}^n |a_{ij}|\, x_j y_j : \sum_{j=1}^n y_j \le \Gamma\right\}, \frac{\sum_{j=1}^n a_{ij}x_j - b_i}{\alpha}\right\}$

Proof. Let $\tilde{\xi}_j = (1-\tilde{\beta}_j)/\alpha$, such that $\tilde{\xi}_j \in [-1,1]$. Let $\tilde{S}_n^i = \alpha\sum_{j=1}^n a_{ij}x_j\tilde{\xi}_j$. Then, it follows that:

$$
\begin{aligned}
\Pr\left(\tilde{\delta}^i(x) > \delta_i^c(x)\right) &= \Pr\left(\max\left\{\frac{\tilde{S}_n^i + b_i - \sum_{j=1}^n a_{ij}x_j}{|b_i|}, 0\right\} > \delta_i^c(x)\right) \\
&= \Pr\left(\tilde{S}_n^i > |b_i|\delta_i^c(x) + \sum_{j=1}^n a_{ij}x_j - b_i\right) \\
&\le \min_{\theta>0}\left\{\frac{\prod_{j=1}^n \mathbb{E}[\exp(\theta\alpha a_{ij}x_j\tilde{\xi}_j)]}{\exp(\theta(|b_i|\delta_i^c(x) + \sum_{j=1}^n a_{ij}x_j - b_i))}\right\} \quad (2.4) \\
&\le \min_{\theta>0}\left\{\frac{\prod_{j=1}^n \exp((\theta\alpha a_{ij}x_j)^2/2)}{\exp(\theta(|b_i|\delta_i^c(x) + \sum_{j=1}^n a_{ij}x_j - b_i))}\right\} \quad (2.5) \\
&= \min_{\theta>0}\left\{\exp(\theta^2\alpha^2/2\sum_{j=1}^n (a_{ij}x_j)^2 - \theta(|b_i|\delta_i^c(x) + \sum_{j=1}^n a_{ij}x_j - b_i))\right\} \\
&\qquad (2.6) \\
&= \exp\left(-\frac{\left(|b_i|\delta_i^c(x) + \sum_{j=1}^n a_{ij}x_j - b_i\right)^2}{2\alpha^2\sum_{j=1}^n (a_{ij}x_j)^2}\right) \quad (2.7)
\end{aligned}
$$

Because it holds that $|b_i|\delta_i^c(x) + \sum_{j=1}^n a_{ij}x_j - b_i > 0\ \forall i = 1,...,m$, inequality (2.4) is a result of the Chernoff bound, following from independence of $\tilde{\xi}_j$ (Chernoff, 1952). Because $\mathbb{E}[\tilde{\xi}_j] = 0$ and $-1 \le \tilde{\xi}_j \le 1$, inequality (2.5) follows from Hoeffding's lemma (Hoeffding, 1963). In (2.6), selecting $\theta = (|b_i|\delta_i^c(x) + \sum_{j=1}^n a_{ij}x_j - b_i)/(\alpha^2\sum_{j=1}^n (a_{ij}x_j)^2)$, yields (2.7). Substituting for $\delta_i^c(x)$ in (2.7) results in the desired bound. □

Theorem 2.4.9. *Let $\mathcal{U}$ be as in (2.1) and let $\tilde{\beta}_j$ from Definition 4 be symmetric i.i.d. random variables centered at 1. For $i = 1,...,k$, the probability of outcome degradation for the i-th objective function that exceeds the value of γ_i^c is given by:*

$$\Pr\left(\tilde{\gamma}^i(x) > \gamma_i^c(x)\right) \leq \exp\left(-\frac{\left(\max\limits_{v\in\{0,1\}^n}\left\{\sum_{j=1}^n |c_j^i| x_j v_j \,:\, \sum_{j=1}^n v_j \leq \Gamma\right\}\right)^2}{2\sum_{j=1}^n (c_j^i x_j)^2}\right)$$

Proof. The proof proceeds similar to the one from Theorem 2.4.8. Let $\tilde{\xi}_j = (\tilde{\beta}_j - 1)/\alpha$, such that $\tilde{\xi}_j \in [-1, 1]$. Then:

$$\begin{aligned}
\Pr\left(\tilde{\gamma}^i(x) > \gamma_i^c(x)\right) &= \Pr\left(\alpha \sum_{j=1}^n c_j^i x_j \tilde{\xi}_j > \gamma_i^c(x)(\overline{f}_i - \underline{f}_i)\right) \\
&\leq \min_{\theta>0}\left\{\frac{\prod_{j=1}^n \mathbb{E}[\exp(\theta\alpha c_j^i x_j \tilde{\xi}_j)]}{\exp(\theta\gamma_i^c(x)(\overline{f}_i - \underline{f}_i))}\right\} \qquad (2.8) \\
&\leq \min_{\theta>0}\left\{\frac{\prod_{j=1}^n \exp((\theta\alpha c_j^i x_j)^2/2)}{\exp(\theta\gamma_i^c(x)(\overline{f}_i - \underline{f}_i))}\right\} \qquad (2.9) \\
&= \min_{\theta>0}\left\{\exp(\theta^2\alpha^2/2\sum_{j=1}^n (c_j^i x_j)^2 - \theta\gamma_i^c(x)(\overline{f}_i - \underline{f}_i))\right\} \qquad (2.10) \\
&= \exp\left(-\frac{(\gamma_i^c(x)(\overline{f}_i - \underline{f}_i))^2}{2\alpha^2\sum_{j=1}^n (c_j^i x_j)^2}\right) \qquad (2.11)
\end{aligned}$$

Since $\gamma_i^c(x) > 0\ \forall i = 1, ..., k$ and by independence of $\tilde{\xi}_j$, the Chernoff bound yields inequality (2.8). Because $\mathbb{E}[\tilde{\xi}_j] = 0$ and $-1 \leq \tilde{\xi}_j \leq 1$, inequality (2.9) follows from Hoeffding's lemma. In (2.10), selecting $\theta = (\gamma_i^c(x)(\overline{f}_i - \underline{f}_i))/(\alpha^2\sum_{j=1}^n (c_j^i x_j)^2)$, yields (2.11). Substituting for $\gamma_i^c(x)$ in (2.11) yields the bound. $\square$

For any realization of $\tilde{\beta}$ within the cardinality-constrained uncertainty set, the formulation of the C-RPSR provides a deterministic guarantee that the actual infeasibility and outcome degradation levels will be less than or equal to $\delta^c(x) = \max\{\delta_1^c(x), ..., \delta_m^c(x)\}$ and $\gamma^c(x) = \max\{\gamma_1^c(x), ..., \gamma_k^c(x)\}$, respectively. Moreover, even if the realization of $\tilde{\beta}$ falls outside the cardinality-constrained uncertainty set, Theorems 2.4.8 and 2.4.9 still provide probabilistic guarantees that the infeasibility level associated with the i-th constraint, for $i = 1, ..., m$, and outcome degradation levels associated with the i-th objective function, for $i = 1, ..., k$, will be less than or equal to the values calculated from (2.2) and (2.3) using the cardinality-constrained uncertainty set. Hence, Theorems 2.4.8 and 2.4.9 may be utilized to determine an appropriate value for Γ according to the decision maker's level of conservatism.

The bounds in Theorems 2.4.8 and 2.4.9 were developed for the general case of bounded random variables. Next, we derive a simple-to-compute bound that is independent of the solution value when perturbations on each x_j are a composite random variable $\tilde{\beta}_j \circ \tilde{\nu}_j$ where $\tilde{\nu}_j$ are i.i.d. Bernoulli trials. Let $\tilde{\delta}(x) = \max\{\tilde{\delta}^1(x), ..., \tilde{\delta}^m(x)\}$ and let $\tilde{\gamma}(x) = \max\{\tilde{\gamma}^1(x), ..., \tilde{\gamma}^k(x)\}$. The bound is shown in Theorem 2.4.10.

Theorem 2.4.10. *Let $1 - \alpha\tilde{\nu}_j \leq \tilde{\beta}_j \leq 1 + \alpha\tilde{\nu}_j$, where $\tilde{\nu}_j$ are i.i.d. Bernoulli trials with probability of success $0 < p < 1$. If $\Gamma \geq pn - 1$, the following upper bound holds:*

$$\Pr\left(\tilde{\delta}(x) > \delta^c(x)\right) < \exp\left(-2n\left(\frac{\Gamma+1}{n} - p\right)^2\right)$$

and

$$\Pr\left(\tilde{\gamma}(x) > \gamma^c(x)\right) < \exp\left(-2n\left(\frac{\Gamma+1}{n} - p\right)^2\right)$$

Proof. We will proceed by showing the probability bound for $\tilde{\delta}(x) > \delta^c(x)$. The proof procedure for $\tilde{\gamma}(x) > \gamma^c(x)$ follows analogously. It holds that $\Pr\left(\tilde{\delta}(x) > \delta^c(x)\right) = 1 - \Pr\left(\tilde{\delta}(x) \leq \delta^c(x)\right) = 1 - \Pr\left(\tilde{\delta}^1(x) \leq \delta^c(x), ..., \tilde{\delta}^m(x) \leq \delta^c(x)\right)$. Let $\tilde{T} = \sum_j^n \tilde{\nu}_j$. Because $1 - \alpha\tilde{\nu}_j \leq \tilde{\beta}_j \leq 1 + \alpha\tilde{\nu}_j$, it follows from the C-RPSR formulation that $\Pr\left(\tilde{\delta}^1(x) \leq \delta^c(x), ..., \tilde{\delta}^m(x) \leq \delta^c(x) | \tilde{T} \leq \Gamma\right) = 1$. From Bayes' Rule, it follows that:

$$
\begin{aligned}
&\Pr\left(\tilde{\delta}^1(x) \le \delta^c(x), ..., \tilde{\delta}^m(x) \le \delta^c(x)\right) \\
&\quad = \Pr\left(\tilde{T} \le \Gamma\right) \frac{\Pr\left(\tilde{\delta}^1(x) \le \delta^c(x), ..., \tilde{\delta}^m(x) \le \delta^c(x) | \tilde{T} \le \Gamma\right)}{\Pr\left(\tilde{T} \le \Gamma | \tilde{\delta}^1(x) \le \delta^c(x), ..., \tilde{\delta}^m(x) \le \delta^c(x)\right)} \\
&\quad \ge \Pr\left(\tilde{T} \le \Gamma\right)
\end{aligned}
$$

Therefore $\Pr\left(\tilde{\delta}(x) > \delta^c(x)\right) \le 1 - \Pr\left(\tilde{T} \le \Gamma\right) = \Pr\left(\tilde{T} \ge \Gamma + 1\right)$. Given that $\tilde{T} \sim B(n, p)$, from Okamoto's inequality (Okamoto, 1959), we have that for any $c \ge 0$ and $0 < p < 1$, $\Pr\left(\tilde{T} \ge n(c + p)\right) < \exp(-2nc^2)$. Selecting $c = (\Gamma + 1)/n - p$ for any $\Gamma \ge pn - 1$, yields the desired result. $\square$

Clearly p, Γ, $\tilde{\delta}(x)$, $\tilde{\gamma}(x)$ and the values of $\delta^c(x)$ and $\gamma^c(x)$ are related. While Γ controls the degree of conservatism of the formulation of the C-RPSR and hence affects the calculated values of $\delta^c(x)$ and $\gamma^c(x)$, the realized values associated with $\tilde{\delta}(x)$ and $\tilde{\gamma}(x)$ increase with p.

2.4.4 Relations with a classical robust optimization approach

The robust counterpart of the MOLP, represented via the minimax absolute robustness approach of Kouvelis and Yu (1997), introduces uncertainty directly when solving the first-stage MOLP. In this section we assume the box uncertainty set, and formulate the robust counterpart of the MOLP, denoted as RMOLP, as follows.

$$
\begin{aligned}
\text{Min} \quad & \left(\sum_{j=1}^{n}(c_j^1 + |c_j^1|\alpha)x_j, ..., \sum_{j=1}^{n}(c_j^k + |c_j^k|\alpha)x_j\right)^T \\
\text{s.t.} \quad & \sum_{j=1}^{n}(a_{ij} - |a_{ij}|\alpha)x_j \ge b_i && \forall i = 1, ..., m \\
& x_j \ge 0 && \forall j = 1, ..., n
\end{aligned}
\tag{RMOLP}
$$

Let $\mathcal{X}^{RMOLP}$ denote the set of efficient solutions to the RMOLP. From Definition 5, it is straightforward to see that the RMOLP guarantees model robustness, i.e. $\delta(x^*) = 0$ for all $x^* \in \mathcal{X}^{RMOLP}$. Using Definitions 5 and 6 to project $x^* \in \mathcal{X}^{RMOLP}$ in the robustness space, the following relates $\mathcal{X}^{RMOLP}$ and $\mathcal{X}'$.

Theorem 2.4.11. *For the box uncertainty set, if* $\mathcal{X} \subseteq \mathbb{R}^n_> \bigcap \text{bd } X$, *the following statements are true:*

(a) For all $x^* \in \mathcal{X}^{RMOLP}$ $\nexists x \in \mathcal{X}' : \delta(x) \leq \delta(x^*)$ *and* $\gamma(x) \leq \gamma(x^*)$ *with at least one inequality holding strictly.*

(b) If $c^i_j \geq 0,\ j = 1, ..., n,\ i = 1, ..., k,$ *it follows that for some* $x \in \mathcal{X}'$ $\nexists x^* \in \mathcal{X}^{RMOLP} : \delta(x^*) \leq \delta(x)$ *and* $\gamma(x^*) \leq \gamma(x)$ *with at least one inequality holding strictly.*

Proof. (a) follows trivially since $\delta(x^*) = 0$ for all $x^* \in \mathcal{X}^{RMOLP}$ and $\delta(x) > 0$ for all $x \in \mathcal{X}'$ as $\mathcal{X}' \subseteq \mathbb{R}^n_> \bigcap \text{bd } X$.

(b) Assume that for all $x \in \mathcal{X}'$, $\exists x^* \in \mathcal{X}^{RMOLP} : \delta(x^*) \leq \delta(x)$ and $\gamma(x^*) \leq \gamma(x)$ with at least one inequality holding strictly. Let $x^{'}$ be an optimal solution to $\operatorname{lex\,min}\{\gamma(x), \delta(x) | x \in \mathcal{X}\}$, i.e. the solution with smallest value $\delta(x)$ among all solutions with smallest value of $\gamma(x)$. Notice that $x^{'} \in \mathcal{X}'$. Because all perturbations $\tilde{\beta}_j$ can take the worst case simultaneously, the bound from Theorem 2.4.3 will be tight and since $c^i_j \geq 0$ we have that $\forall x \in \mathcal{X}', \exists x^* \in \mathcal{X}^{RMOLP}$:

$$\gamma(x^*) \leq \gamma(x) \implies \max_{i=1,\ldots,k} \frac{f_i(x^*)}{\overline{f}_i - \underline{f}_i} \leq \max_{i=1,\ldots,k} \frac{f_i(x)}{\overline{f}_i - \underline{f}_i}$$

which implies that $\exists x^* \in \mathcal{X}^{RMOLP}$:

$$\max_{i=1,\ldots,k} \frac{f_i(x^*)}{\overline{f}_i - \underline{f}_i} \leq \max_{i=1,\ldots,k} \frac{f_i(x^{'})}{\overline{f}_i - \underline{f}_i}$$

Because $\mathcal{X} \subseteq \mathbb{R}^n_> \bigcap \text{bd } X$ and $c^i_j \geq 0$, $\exists x \in \mathcal{X}$ such that $f_i(x) < f_i(x^*)$, $\forall i = 1, ..., k$. Hence, $\exists x \in \mathcal{X}$: $\max_{i=1,\dots,k} f_i(x)/(\overline{f}_i - \underline{f}_i) < \max_{i=1,\dots,k} f_i(x^*)/(\overline{f}_i - \underline{f}_i)$. Given that $x^{'}$ is optimal to $\min_{x\in\mathcal{X}}\{\max_{i=1,\dots,k} f_i(x)/(\overline{f}_i - \underline{f}_i)\}$, it follows that

$$\max_{i=1,\dots,k} \frac{f_i(x^{'})}{\overline{f}_i - \underline{f}_i} < \max_{i=1,\dots,k} \frac{f_i(x^*)}{\overline{f}_i - \underline{f}_i}$$

which yields a contradiction. $\square$

Part (a) of Theorem 2.4.11 shows a case where the solutions obtained from introducing robustness directly when solving the MOLP (i.e. the RMOLP) will not be dominated in the robustness space by any solution obtained from the RPSR. Part (b) provides a condition where at least one solution obtained from the RPSR will not be dominated in the robustness space by any solution from the RMOLP. While the former is due to the fact that solutions from the RMOLP have zero infeasibility level, the later shows that solutions from the proposed methodology may have lower outcome degradation levels. This will be illustrated in Example 2 in Section 2.5.

The theoretical properties show that the RPSR may find fewer robust solutions than a classical robust optimization approach, and at least one solution that is not dominated in the robust space by any solution found by a classical robust optimization approach. For the purpose of protecting against worst case model robustness, one could argue that a classical robust optimization approach would be more appropriate since all solutions would remain feasible for all possible realizations within the uncertainty set. On the other hand, in a less conservative case of model robustness or when robustness is considered as secondary criteria, one could argue that the PSR methodology would be more appropriate to aid decision making since it allows the trade-off analysis between objective function values losses and constraint violations, while providing the decision maker with a reduced subset of solutions.

2.5 Numerical examples

The examples in this section are aimed to illustrate the proposed methodology in the linear case previously discussed. In addition, the first example compares the solution sets obtained via different PSR methods. The second example shows how the methodology compares to a classical robust optimization approach, i.e. the RMOLP. Moreover, it demonstrates that applying an approach often utilized to select a most preferred alternative based on knee solutions (e.g. Das, 1999a; Branke et al., 2004) may lead to the selection of a non-robust solution. While the first two examples assume a box uncertainty set, the third one focuses on the cardinality-constrained uncertainty set and illustrates how solutions vary with different values of Γ. For simplicity, we use $\alpha = 0.10$ in all instances.

The first stage MOLPs were solved via the ϵ-constraint method (Haimes et al., 1971), programmed in C++ and CPLEX 12.6 was used as the solver. The assessment of infeasibility and outcome degradation levels, as well as the filtering of non-robust solutions, was coded in C++.

Example 1. The first instance is an MOLP with $n = 3$ decision variables, $m = 6$ constraints and $k = 3$ objective functions. The formulation is as follows:

$$
\begin{aligned}
\text{Min } & (3x_1 + x_2, -x_1 + 3x_2 + 10x_3, -x_3)^T \\
\text{s.t. } & x_1 + x_2 - 2x_3 \geq 8 \\
& -x_1 + x_2 \geq -9 \\
& -2x_1 - 3x_2 + x_3 \geq -36 \\
& 2x_1 - x_2 \geq 12 \\
& 3x_1 + 2x_2 + 5x_3 \geq 35 \\
& x_1 + 11x_2 \geq 32 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
\tag{N3M6O3}
$$

The efficient set $\mathcal{X}$ of N3M6O3 is formed by all solutions lying on $X \cap (\{x \in \mathbb{R}^3 : x_1 + x_2 - 2x_3 = 8\} \cup \{x \in \mathbb{R}^3 : x_1 + 11x_2 = 32\})$. The red points in Figure 2.2 show 1,407 efficient solutions obtained via the ϵ-constraint method. The RPSR formulation of N3M6O3 is shown next.

$$
\begin{aligned}
\text{Min} \;\; & (\delta, \gamma)^T \\
\text{s.t.} \;\; & 0.9x_1 + 0.9x_2 - 2.2x_3 + 8\delta \geq 8 \\
& -1.1x_1 + 0.9x_2 + 9\delta \geq -9 \\
& -2.2x_1 - 3.3x_2 + 0.9x_3 + 36\delta \geq -36 \\
& 1.8x_1 - 1.1x_2 + 12\delta \geq 12 \\
& 2.7x_1 + 1.8x_2 + 4.5x_3 + 35\delta \geq 35 \\
& 0.9x_1 + 9.9x_2 + 32\delta \geq 32 \\
& -0.3x_1 - 0.1x_2 + 19.91\gamma \geq 0 \\
& -0.1x_1 - 0.3x_2 - 1x_3 + 56.67\gamma \geq 0 \\
& -0.1x_3 + 5.13\gamma \geq 0 \\
& x \in \mathcal{X} \\
& \delta, \gamma \geq 0
\end{aligned}
$$

The robust efficient set was obtained by computing $\delta(x)$ and $\gamma(x)$ for each solution x found by the ϵ-constraint method and filtering out the ones that were dominated in the robustness space. The PSR methods considered in this instance were the forward filtering clustering (FC) (Steuer and Harris, 1980) and the greedy reduction for percentile maximization (GRP) (Venkat et al., 2004). In addition, the SEABOE method (Das, 1999b) was applied, but no solutions of order of efficiency $k = 2$ were found for the problem (i.e. no solutions in $\mathcal{X}$ would remain efficient if one of the objective functions was dropped from the MOLP).

The FC algorithm was implemented using Minkowski's distance metric and the objectives were normalized to a [0,1] scale with indifference distance value set to $d = \alpha$. The GRP algorithm was implemented using equal weights in the percentile function. In order to have a more comprehensive analysis, different runs of the GRP algorithm were performed with subset sizes from $N' = 1$ to 15 solutions. The run with $N' = 15$ results in a superset of solutions from the other runs. Figures 2.2-2.4 show the solutions obtained in the decision, robustness and objective spaces, respectively.

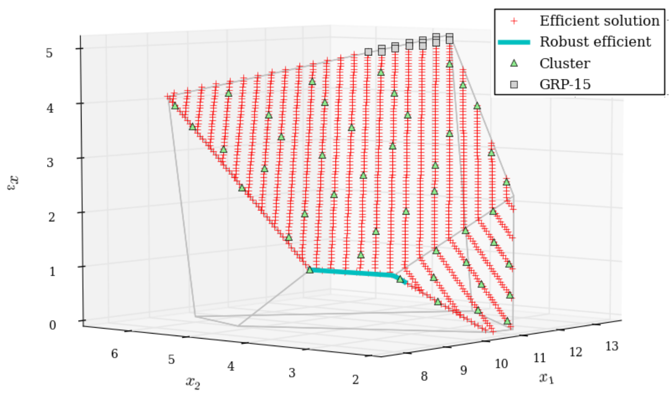


Figure 2.2: Decision space - N3M6O3

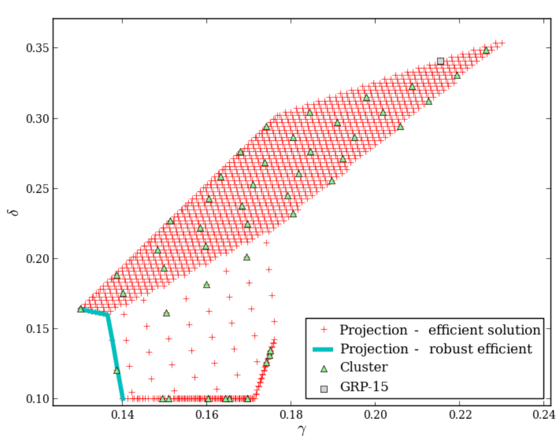


Figure 2.3: Robustness space - N3M6O3

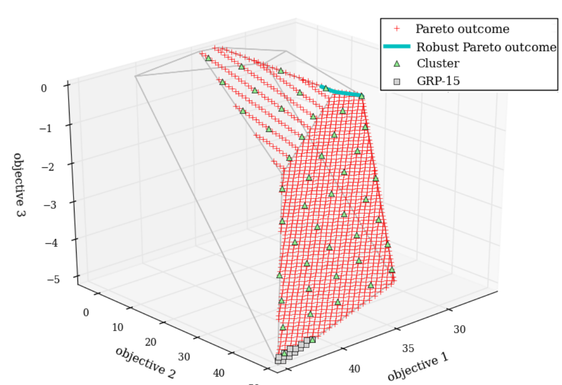


Figure 2.4: Objective space - N3M6O3

The plot in Figure 2.2 illustrates that the PSR methods can lead to distinct regions of the decision space, capturing different aspects of the goodness of the solutions. All solutions of the GRP algorithm were mapped to the same point in the robustness space, having both a high outcome degradation and infeasibility levels. The FC algorithm generated the entire representation of the Pareto set in a reduced number of outcomes. Of notice is that the clustering technique is particularly useful to understand the shape of the Pareto set while also achieving the desired size reduction. This suggests that clustering solutions may be used as input for the RPSR in larger instances.

Alike other PSR methods, the RPSR was effective in reducing the size of the Pareto set. However, the proposed methodology focuses only on the regions of the efficient set that maintain efficiency of solutions in the robustness space (cf. Figure 2.3). In this example, the size of the robust Pareto set was reduced to 12 outcomes, a 99.1% reduction compared to the discrete sample of the Pareto set of the MOLP.

Example 2. Instance N2M6O2 is a bicriterion program with $n = 2$ decision variables and $m = 6$ constraints. The formulation is presented next.

$$
\begin{aligned}
\text{Min } & (3x_1 + x_2, -x_1 + 3x_2)^T \\
\text{s.t. } & x_1 + x_2 \geq 12 \\
& -x_1 + x_2 \geq -9 \\
& -2x_1 - 3x_2 \geq -36 \\
& 2x_1 - x_2 \geq 12 \\
& 3x_1 + 2x_2 \geq 33 \\
& x_1 + 11x_2 \geq 32 \\
& x_1, x_2 \geq 0
\end{aligned}
\tag{N2M6O2}
$$

The efficient set of N2M6O2 is given by the union of three maximal efficient faces, $\mathcal{X} = X \cap (\{x \in \mathbb{R}^2 : 3x_1 + 2x_2 = 33\} \cup \{x \in \mathbb{R}^2 : x_1 + x_2 = 12\} \cup \{x \in \mathbb{R}^2 : x_1 + 11x_2 = 32\})$ (cf. red dotted lines in Figure 2.7). The RPSR becomes:

$$
\begin{aligned}
\text{Min } & (\delta, \gamma)^T \\
\text{s.t. } & 0.9x_1 + 0.9x_2 + 12\delta \geq 12 \\
& -1.1x_1 + 0.9x_2 + 9\delta \geq -9 \\
& -2.2x_1 - 3.3x_2 + 36\delta \geq -36 \\
& 1.8x_1 - 1.1x_2 + 12\delta \geq 12 \\
& 2.7x_1 + 1.8x_2 + 33\delta \geq 33 \\
& -1.1x_2 + 32\delta \geq 32 \\
& -0.3x_1 - 0.1x_2 + 5.95\gamma \geq 0 \\
& -0.1x_1 - 0.3x_2 + 9.88\gamma \geq 0 \\
& x \in \mathcal{X} \\
& \delta, \gamma \geq 0
\end{aligned}
$$

To illustrate how the RPSR differs from a classical robust optimization approach, robustness was incorporated directly in the MOLP by solving the RMOLP:

$$
\begin{aligned}
\text{Min } & (3.3x_1 + 1.1x_2, -0.9x_1 + 3.3x_2)^T \\
\text{s.t. } & 0.9x_1 + 0.9x_2 \geq 12 \\
& -1.1x_1 + 0.9x_2 \geq -9 \\
& -2.2x_1 - 3.3x_2 \geq -36 \\
& 1.8x_1 - 1.1x_2 \geq 12 \\
& 2.7x_1 + 1.8x_2 \geq 33 \\
& 0.9x_1 + 9.9x_2 \geq 32 \\
& x_1, x_2 \geq 0
\end{aligned}
\qquad \text{(N2M6O2-RMOLP)}
$$

Results are depicted in Figures 2.5-2.7.

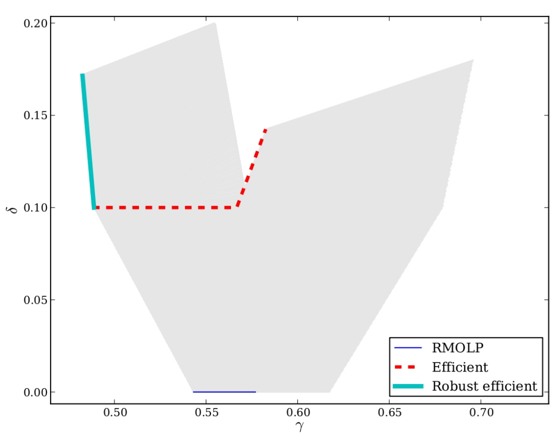


Figure 2.5: Robustness space - N2M6O2

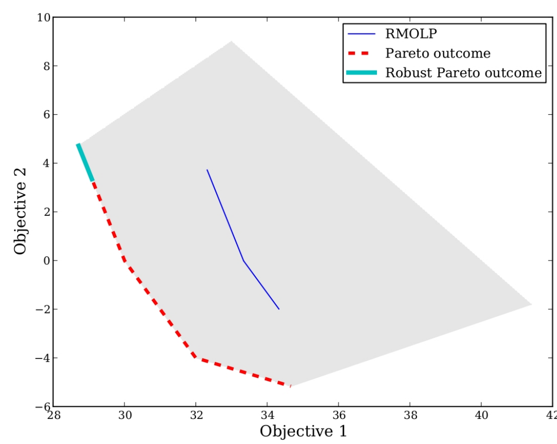


Figure 2.6: Objective space - N2M6O2

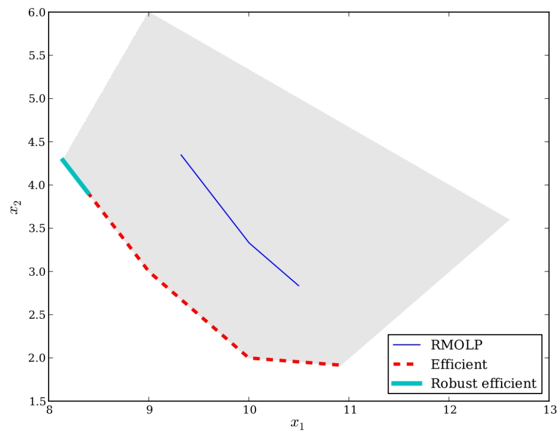


Figure 2.7: Decision space - N2M6O2

The shaded areas in Figures 2.6 and 2.7 denote the outcome set Y and the feasible set X, respectively. The red dotted lines in Figures 2.5-2.7 denote $\mathcal{D}$, $\mathcal{Y}$ and $\mathcal{X}$, while the blue lines are the subsets $\mathcal{D}'$, $\mathcal{Y}'$ and $\mathcal{X}'$, respectively.

As shown in Figure 2.6, the efficient solutions obtained from N2M6O2-RMOLP are dominated by the efficient solutions of N2M6O2 in the objective space. Although the conditions from Theorem 2.4.11 part (b) are not met (i.e. $c_j^i \geq 0$), Figure 2.5 shows that the robust efficient solutions obtained by the RPSR are non-dominated in the robustness space with respect to the solutions obtained from the RMOLP. While the solutions from the RMOLP remain always feasible, their outcome degradation levels are higher than the ones from solutions in $\mathcal{X}'$. In fact, the robust efficient solutions obtained by the RPSR are non-dominated in the robustness space with respect to any feasible solution in problem N2M6O2 (cf. Figure 2.5). Although this case is also observed in Example 1, it does not always hold true and is problem dependent. In addition, during our experimentation, we have found cases where the solutions from the RPSR dominate solutions from the RMOLP, and vice-versa.

When compared to the knee approach, selecting solutions with maximum convex bulge on the Pareto curve (e.g. solutions with objective values in the region of $f = (30, 0)^T$ and $(32, -4)^T$ in Figure 2.6) would yield non-robust solutions. This illustrates that an approach often utilized to select a most preferred alternative may lead to a solution that is highly affected by uncertainty. Although the knee solutions in Figure 2.6 show a low infeasibility level ($\delta(x) = 0.1$ for all knee solutions), they are all dominated in the robustness space by solutions in $\mathcal{X}'$, which maintain lower outcome degradation levels.

The RPSR was able to reduce the number of maximal efficient faces that remained robust efficient. In addition, only 11.6% of the length covered by the line segments forming the Pareto set of N2M6O2 remained Pareto optimal in view of the RPSR.

Example 3. Instance N10M5O2 is a problem with $n = 10$ decision variables, $m = 5$ constraints and $k = 2$ objective functions. The formulation is presented next:

$$
\begin{aligned}
\text{Min } & (10x_1 + 5x_2 + 3x_3 + 2x_4 + 2x_5 + x6 + x_7 + x_8 + x_9 + x_{10}, \\
& - x_1 - x_2 - x_3 - x_4 + 4x_5 + 5x_6 + 6x_7 + 7x_8 + 8x_9 + 9x_{10})^T \\
\text{s.t. } & x_1 - 2x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} \geq 5 \\
& - 2x_1 - 2x_2 - 3x_3 - 5x_4 - 10x_5 + 20x_6 + 3x_7 + 2x_8 + 5x_9 + x_{10} \geq 3 \\
& - x_1 - x_6 + 2x_{10} \geq 1 \\
& 6x_1 + x_2 - 2x_8 \geq 7 \\
& 2x_2 + 4x_7 + x_8 \geq -1 \\
& x \geq 0
\end{aligned}
\tag{N10M5O2}
$$

The RPSR was formulated using the cardinality-constrained uncertainty set with

Γ varying from 1 to 10. Figure 2.8 shows $\mathcal{D}'$ for the various values of Γ.

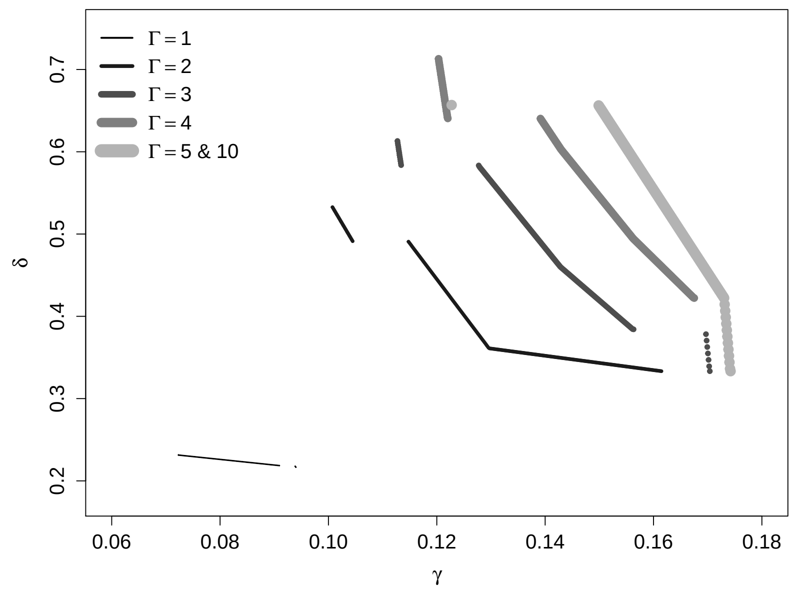


Figure 2.8: Robustness space - N10M5O2 - with varying Γ

From Figure 2.8, the values of $\delta(x)$ and $\gamma(x)$ increase with Γ. When the most optimistic case of robustness is assumed, i.e. $\Gamma = 1$, the infeasibility and outcome degradation levels are lower because only one decision variable is assumed to be affected by uncertainty at one time. As the decision maker increases his/her level of conservatism, i.e. by increasing Γ, the infeasibility and outcome degradation levels also increase since more decision variables are assumed to be simultaneously affected by uncertainty. As shown in Figure 2.8, the most conservative case, i.e. $\Gamma = 10$, yields the same results as the case with $\Gamma = 5$. This is because the efficient solutions have decision variables at zero, which suggests that using a value of $\Gamma < n$ may still provide solutions that are highly protected against uncertainties.

If the perturbations on each x_j are a composite random variable $\tilde{\beta}_j \circ \tilde{\nu}_j$ where $\tilde{\nu}_j$ are i.i.d. Bernoulli trials with probability p, then Theorem 2.4.10 may be used to guide the selection of the appropriate Γ according to the decision maker's level of conservatism. For instance, if $p = 0.1$, then the selection of $\Gamma = 4$ would guarantee that the probability of constraint violations or outcome degradation levels exceeding the calculated values from the C-RPSR formulation would be less than 0.04; similarly, using a less conservative value of $\Gamma = 3$ would guarantee a probability bound from Theorem 2.4.10 of 0.165.

2.6 Summary and conclusions

The large number of Pareto outcomes from an MOP makes it difficult for a decision maker to select a particular solution for implementation; this issue hinders the applicability of multiobjective optimization in practice. To the best of our knowledge, PSR methods mainly rely on clustering solutions according to their similarity in the objective space of the MOP. This section poses the PSR as an optimization problem over the efficient set based on the idea of incorporating secondary criteria related to both the objective and the decision spaces of the MOP in order to break ties among the Pareto outcomes, providing the decision maker with additional trade-off information in the space of the secondary criteria.

Although the methodology can be used with any secondary criteria of interest, in this section we use model and solution robustness such that solutions in the reduced set are less sensitive to uncertainties, in addition to being efficient with respect to the original MOP. Hence, the proposed model integrates aspects from the areas of multiobjective optimization and robustness to aid decision making.

The proposed formulation allows dealing with continuous Pareto sets circumventing the need for discretization procedures that characterize existing PSR methodologies. However, since the PSR is a non-convex problem, of interest becomes the development of an effective solution procedure. This leads us to the next section in this dissertation.

3. LINE SEARCH AND LINEAR PROGRAMMING FILTERING ALGORITHM FOR A CLASS OF BIOBJECTIVE OPTIMIZATION PROBLEMS OVER THE EFFICIENT SET

3.1 Introduction

This section presents an algorithm for solving a special case of a biobjective optimization problem over the efficient set of an MOLP. It is assumed that the efficient set of the MOLP is known and given as the union of the maximal efficient faces. The two objectives for optimization over the efficient set, termed secondary objective functions, are linear; and in addition to the efficient set, the feasible region is complemented by linear constraints associated with continuous variables, termed secondary decision variables, also not present in the MOLP. As in bilevel programming (Fülöp, 1993), this allows for a hierarchical decision process: first, the set of efficient solutions to the MOLP is determined; then two secondary objective functions are optimized over the efficient set with additional constraints to determine the subset of efficient solutions and secondary decision variable values that are efficient in this subsequent problem.

The literature on optimization over the efficient set is considerably vast, but as pointed out in Section 2, it is limited to the optimization of a single objective function (e.g. Benson, 1984; Horst and Thoai, 1999; Sayin, 2000; Yamamoto, 2002; Jorge, 2009; Thang, 2015). Fülöp (1993) showed that the bilevel linear program can be equivalently formulated as an optimization problem over the efficient set of an MOLP. In bilevel programing, a mixed binary linear reformulation was proposed by Fortuny-Amat and McCarl (1981) when both upper and lower level problems are single objectives. Here we show that the problem dealt with in this section can be reformulated as a biobjective mixed binary linear program (BOMBLP). Each binary variable in the reformulated problem is associated with a maximal efficient face of the original MOLP and represents whether, or not, there exist solutions on the given face that are also efficient in view of the secondary objective functions.

BOMBLPs are a class of multiobjective programs where both binary and continuous variables are present. While it is well-known that the Pareto set of an MOLP is connected (e.g. Ehrgott, 2005), this is not the case in multiobjective binary linear programs. The Pareto set of a BOMBLP is formed by the union of line segments and discrete outcomes, which may have connected as well as disconnected regions in the objective space (Vincent et al., 2013). In BOMBLPs, fixing all values of the binary variables to one or zero reduces the problem to a biobjective linear program (BOLP). This suggests that the BOMBLP can be solved by taking the union of the Pareto sets of BOLPs corresponding to all possible combinations of binary variables values, and filtering out any dominated segments from the resulting set. Major drawbacks from this approach are that the number of BOLPs to be explored increases exponentially with problem size, and the filtering steps may be computationally challenging (Boland et al., 2014).

Although there are only a few studies proposing algorithms to solve BOMBLPs, the subject has been receiving growing attention in literature. This includes heuristic approaches (e.g. Masin and Bukchin, 2008; Soylu, 2015), branch and bound (BB) algorithms, which are based on solving a sequence of BOLP relaxations (e.g. Mavrotas and Diakoulaki, 1998, 2005; Vincent et al., 2013; Stidsen et al., 2014), and objective space search algorithms, which rely on solving mixed integer programs (MIPs) while searching in the space of the outcomes (e.g. Boland et al., 2014; Rong et al., 2015).

It has been suggested that objective space search procedures are likely more successful than decision space algorithms for solving multiobjective optimization problems (e.g. Benson and Sun, 2000, 2002). In the case of BOMBLPs, experiments have shown that objective space search algorithms tend to outperform runtimes of BB algorithms (Boland et al., 2014). One reason for that is attributed to the fact that the bounds in BB algorithms may not be as tight as the ones in single objective optimization problems.

Objective space search algorithms follow a general two-phase approach (Stidsen et al., 2014), which has been applied predominantly in pure integer multiobjective optimization problems (e.g. Visée et al., 1998; Przybylski et al., 2008; Steiner and Radzik, 2008), and more recently to BOMBLPs (e.g. Boland et al., 2014; Rong et al., 2015). In the two-phase approach, phase 1 searches for extreme supported Pareto outcomes, and is typically carried out by solving a scalirizing version of the BOMBLP via the parametric weighted sum of objective functions. Phase 2 consists of searching for unsupported Pareto outcomes (i.e. outcomes not found during phase 1) within upper triangles formed by two successive extreme supported Pareto outcomes. The search during phase 2 can be carried out in different ways and is often problem specific (Stidsen et al., 2014). In the case of the triangle splitting (TS) algorithm (Boland et al., 2014), the procedure checks whether all outcomes lying on the hypotenuse of the upper triangle are Pareto optimal by solving an auxiliary mixed integer program. If no Pareto outcome can be found on the hypotenuse, the triangle is split horizontally or vertically in half, and the test is repeated in the new intervals until no further Pareto outcomes can be found.

This section develops an objective space search algorithm, termed line search and linear programming filtering (LS-LPF), for the BOMBLP formulation of the biobjective optimization problem over the efficient set. In phase 1, the algorithm searches for supported Pareto outcomes using the parametric weighted sum method. In phase 2, the algorithm searches for unsupported Pareto outcomes taking advantage of a single-choice constraint in problem structure to solve a sequence of mixed binary linear programs with decreasing number of free binary variables (i.e. variables not already fixed to zero or one) at each iteration. In addition, auxiliary linear programs (LPs) enables the implementation of filtering steps.

This section is organized as follows. Section 3.2 introduces the model and preliminaries. Section 3.3 shows the problem formulation and its reformulation as a BOMBLP. Section 3.4 describes the LS-LPF algorithm and provides an illustrative example. Section 3.5 shows the results of an experimental analysis, while Section 3.6 presents concluding remarks.

3.2 Model and and preliminaries

Let $X = \{x : Ax \geqq b, x \geqq 0\}$ be compact, where $A \in \mathbb{R}^{m\times n}$ is a constraint matrix of elements a_{ij} for $i = 1, ..., m$ and $j = 1, .., n$ and $b \in \mathbb{R}^m$ is the right-hand-side vector. Let $C \in \mathbb{R}^{k\times n}$ be a cost matrix of elements c^i_j for $i = 1, ..., k$ and $j = 1, .., n$, where $C^i = (c^i_1, ..., c^i_n)$ denotes the cost vector for the i-th objective function. Then, the multiobjective linear program (MOLP) can be expressed as:

$$\begin{aligned} &\text{Min } Cx \\ &\text{s.t. } x \in X \end{aligned} \tag{MOLP}$$

Proposition 3.2.1. *(Zadeh, 1963) If there exists some $w \in \mathbb{R}^k_>$ such that x' is an optimal solution to $\min_{x\in X} w^T Cx$, then $x' \in \mathcal{X}$.*

A solution $x' \in \mathcal{X}$ to the program in Proposition 3.2.1 is called a supported efficient solution and $y' = Cx'$ is termed a supported Pareto outcome. Likewise, an efficient solution that does not satisfy Proposition 3.2.1 is called an unsupported efficient solution and its image in the objective space is termed as an unsupported Pareto outcome. In the case of MOLPs, it is well-known that all efficient solutions are supported efficient solutions (Steuer, 1986). For non-convex problems, it is also well-known that there may exist unsupported efficient solutions (e.g. see Das and Dennis, 1997).

If x is a supported efficient solution and $y = Cx$ is an extreme point of $conv(Y)$, then x is called an extreme supported efficient solution and y is called an extreme supported Pareto outcome.

Given that X is assumed to be compact, the parametric program in Proposition 3.2.1 can be solved in a finite number of steps to find the set of extreme supported efficient solutions and the corresponding extreme supported Pareto outcomes (Gass and Saaty, 1955; Steuer, 1986).

The set of all efficient solutions $\mathcal{X}$ of an MOLP is connected; consequently, the set $\mathcal{Y}$ is connected; moreover, $\mathcal{Y}$ lies on the boundary of Y (Yu and Zeleny, 1975; Naccache, 1978; Steuer, 1986). In the BOLP, since $\mathcal{Y}$ is assumed to be compact, $\mathcal{Y}$ can be characterized by the union of finitely many line segments, where each endpoint of a line segment corresponds to an extreme supported Pareto outcome of the BOLP (Boland et al., 2014).

Let F denote a face of X, i.e. a convex subset of X such that every line segment in X with relative interior in F has both endpoints in F (Rockafellar, 1970). A face F is called efficient if, for all $x \in F$, $x \in \mathcal{X}$ (Yu and Zeleny, 1975). F is termed a maximal efficient face if F is an efficient face and there does not exists another efficient face F' of X such that $F \subset F'$ (Yu and Zeleny, 1975).

As in Sayin (1996), a maximal efficient face of X is here represented by a collection of indices corresponding to the constraints holding at equality at that face. Define $\hat{A} = \begin{bmatrix} A \\ I \end{bmatrix}$ where I is an $n \times n$ identity matrix and let and $\hat{b}$ be the corresponding right-hand-side vector augmented by n zero elements. Let $M = \{1, ..., m+n\}$. Then, for some subset $P \subseteq M$, let $\hat{A}^P$ and $\hat{b}^P$ denote the matrix and vector, respectively, obtained by excluding from $\hat{A}$ and $\hat{b}$ the rows whose indices are not in P. Hence, $F(P) = \{x \in X | \hat{A}^P x = \hat{b}^P\}$ denotes the face of X formed by the constraint indices in P holding at equality. Let S denote the number of maximal efficient faces of the MOLP and let $\{P_1, ..., P_S\}$ be the collection of subsets of M that represent these faces. Then, the efficient set of the MOLP is here assumed to be given by $\mathcal{X} = \bigcup_{s=1}^{S} F(P_s)$ (Yu and Zeleny, 1975). For further details on finding maximal efficient faces of an MOLP, see the review by Ehrgott (2005) and the references therein.

3.3 Problem formulation

Given the efficient faces of the MOLP, we consider secondary decision variables π_j (i.e. variables not included in the formulation of the MOLP) and nonnegative parameters d_j^1 and d_j^2, for $j = 1, ..., n_2$. The functions $\phi(\pi) = \sum_{j=1}^{n_2} d_j^1 \pi_j$ and $\psi(\pi) = \sum_{j=1}^{n_2} d_j^2 \pi_j$ to be minimized over the efficient set are termed secondary objective functions. The biobjective optimization over the efficient set is formulated as:

$$
\begin{aligned}
\text{Min} \quad & \left(\phi(\pi) = \sum_{j=1}^{n_2} d_j^1 \pi_j, \ \psi(\pi) = \sum_{j=1}^{n_2} d_j^2 \pi_j \right)^T \\
\text{s.t.} \quad & \sum_{j=1}^{n_2} h_j^i \pi_j + \sum_{j=1}^{n} g_j^i x_j \geq u_i && \forall i = 1, ..., m_2 \\
& \pi_j \geq 0 && \forall j = 1, ..., n_2 \\
& x \in \mathcal{X}
\end{aligned}
\tag{P1}
$$

where h_j^i, g_j^i and u_i are given parameters for the additional constraints relating the decision variables $x \in \mathbb{R}^n$ of the MOLP and the secondary decision variables $\pi \in \mathbb{R}^{n_2}$.

Let $\mathcal{X}'$ denote the subset of solutions that are efficient to problem P1 and let $\mathcal{D}'$ denote its image under the mappings of the secondary objective functions $\phi(\pi)$ and $\psi(\pi)$. Because $\mathcal{X}$ is usually non-convex even in the case of MOLPs, P1 is a non-convex problem. Here, since $\mathcal{X}$ is assumed to be known and given as the union of the maximal efficient faces of the MOLP, one way of solving P1 would be to find the projection of each efficient face onto the space of the secondary objective functions, then filter out the dominated portions of the corresponding line segments, as illustrated by the bold line segments in Figure 3.1.

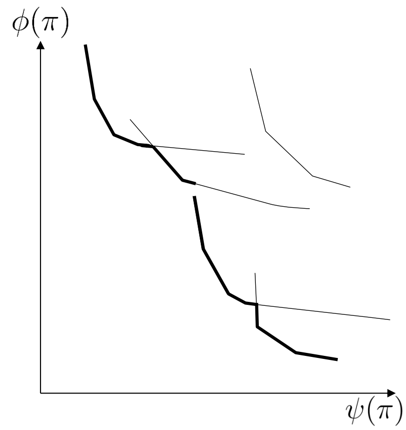


Figure 3.1: Pareto outcomes to problem P1

For larger instances, however, the aforementioned approach is not practical since it would require exhaustive search over all efficient faces. Next we show a reformulation of the problem that allows for a different solution approach.

3.3.1 Mixed binary linear reformulation of problem P1

This subsection introduces a reformulation of P1 as a mixed binary linear program by replacing the restriction $x \in \mathcal{X}$ with a series of constraints involving additional auxiliary continuous and binary variables. Let z_s denote a binary variable associated with a efficient face $F(P_s)$ of the MOLP, for $s = 1, ..., S$. Let $z_s = 1$ if $x \in F(P_s)$, and $z_s = 0$ otherwise. In addition, for each decision variable x_j, $j = 1, ..., n$, let x'_{js}

denote a nonnegative continuous variables, for $s = 1, ..., S$, such that:

$$x_j - \sum_{s=1}^{S} x'_{js} = 0 \qquad \forall j = 1, ..., n$$

Disjunctive constraints are introduced to ensure that either a solution belongs to the face $F(P_s)$, hence forcing $\hat{A}^{P_s}x = \hat{b}^{P_s}$, or simply $Ax \geqq b$ otherwise. This is achieved with the additional constraints:

$$\begin{aligned}
&\sum_{j=1}^{n} a_{ij}x_j \geq b_i && \forall i = 1, ..., m\\
&\sum_{j=1}^{n} \hat{a}_{ij}x'_{js} - \hat{b}_i z_s = 0 && \forall i \in P_s : 1 \leq i \leq m,\ s = 1, ..., S\\
&\sum_{j=1}^{n} \hat{a}_{ij}x'_{js} - M(1 - z_s) \leq 0 && \forall i \in P_s : m + 1 \leq i \leq m + n,\ s = 1, ..., S\\
&\sum_{j=1}^{n} x'_{js} - Mz_s \leq 0 && \forall s = 1, ..., S
\end{aligned}$$

where $\hat{a}_{ij}$ and $\hat{b}_i$ are elements of $\hat{A}$ and $\hat{b}$, respectively, and M is a large enough real.

Finally, a constraint is added to ensure with the previous disjunctions that the solution will be related to an efficient face of the MOLP:

$$\sum_{s=1}^{S} z_s = 1$$

If $z_s = 1$, for some $s = 1, ..., S$, then $x_j = x'_{js}$ and $\sum_{j=1}^{n} \hat{a}_{ij}x'_{js} = \hat{b}_i$, for all $i \in P_s : 1 \leq i \leq m$, and $x'_{(i-m)s} = 0$, for all $i \in P_s : m + 1 \leq i \leq m + n$. Otherwise, if $z_s = 0$ then $x'_{js} = 0$ for all $j = 1, ..., n$.

The previous constraints lead to the following reformulation of P1 as a biobjective mixed binary linear program:

$$
\begin{aligned}
\text{Min} \quad & \left(\phi(\pi) = \sum_{j=1}^{n_2} d_j^1 \pi_j, \; \psi(\pi) = \sum_{j=1}^{n_2} d_j^2 \pi_j \right)^T \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \geq b_i && \forall i = 1, ..., m \\
& \sum_{j=1}^{n_2} h_j^i \pi_j + \sum_{j=1}^{n} g_j^i x_j \geq u_i && \forall i = 1, ..., m_2 \\
& \sum_{j=1}^{n} \hat{a}_{ij} x'_{js} - \hat{b}_i z_s = 0 && \forall i \in P_s : 1 \leq i \leq m, \; s = 1, ..., S \\
& \sum_{j=1}^{n} \hat{a}_{ij} x'_{js} - M(1 - z_s) \leq 0 && \forall i \in P_s : m+1 \leq i \leq m+n, \; s = 1, ..., S \\
& \sum_{j=1}^{n} x'_{js} - M z_s \leq 0 && \forall s = 1, ..., S \\
& x_j - \sum_{s=1}^{S} x'_{js} = 0 && \forall j = 1, ..., n \\
& \sum_{s=1}^{S} z_s = 1 \\
& \pi_j, x_j, x'_{js} \geq 0 \\
& z_s \in \{0, 1\}
\end{aligned} \tag{MB-P1}
$$

The MB-P1 formulation has at most $m(S+1)+n(S+1)+m_2+S+1$ constraints, $n(S+1)+n_2$ continuous variables and S binary variables. Of notice is that the formulation is polynomial in size with respect to the number of efficient faces of the original MOLP. Even when the MOLP possesses a large number of efficient faces, and consequently a large number of binary variables in the MB-P1 formulation, the single-choice constraint $\sum_{s=1}^{S} z_s = 1$ guarantees that the number of branches in the search tree only increases polynomially with the number of efficient faces.

3.3.2 Linear programming formulation for filtering line segments

Since $\mathcal{X}$ is here assumed to be compact and P1 can be formulated as a BOMBLP, $\mathcal{D}'$ will be given as the union of finitely many line segments; each line segment will be formed by solutions that are non-dominated in view of the secondary objective functions. Next, we show that two auxiliary linear programming formulations can be used to detect the segments of each line that correspond to non-dominated solutions.

Let L_i and L_j be two closed line segments in the space $\mathbb{R}^2_{\geqq}$ of the secondary objective functions. Define $\underline{\phi}^{L_i} = \min_{\phi \in L_i}\{\phi : \phi = \sum_{j=1}^{n_2} d_j^1 \pi_j\}$ and $\overline{\phi}^{L_i} = \max_{\phi \in L_i}\{\phi : \phi = \sum_{j=1}^{n_2} d_j^1 \pi_j\}$. Similarly, let $\underline{\psi}^{L_i} = \min_{\psi \in L_i}\{\psi : \psi = \sum_{j=1}^{n_2} d_j^2 \pi_j\}$ and $\overline{\psi}^{L_i} = \max_{\psi \in L_i}\{\psi : \psi = \sum_{j=1}^{n_2} d_j^2 \pi_j\}$. We represent a closed line segment by its two endpoints, e.g. $L_i = [(\overline{\phi}^{L_i}, \underline{\psi}^{L_i})^T; (\underline{\phi}^{L_i}, \overline{\psi}^{L_i})^T]$. Without loss of generality, we treat open and half-opened line segments as closed line segments by subtracting a small enough quantity Δ from their endpoints. Similarly, L_i represents a singleton when $\overline{\phi}^{L_i} = \underline{\phi}^{L_i}$ and $\overline{\psi}^{L_i} = \underline{\psi}^{L_i}$. Define the following programs:

$$Z_{LP1} = \min\{\psi : \psi - a^{L_i}\phi \geq b^{L_i}, \psi \geq \underline{\psi}^{L_i}, \phi \geq \underline{\phi}^{L_i}, (\phi, \psi)^T \in L_j\} \qquad (LP1(L_i, L_j))$$

$$Z_{LP2} = \min\{\phi : \psi - a^{L_i}\phi \geq b^{L_i}, \psi \geq \underline{\psi}^{L_i}, \phi \geq \underline{\phi}^{L_i}, (\phi, \psi)^T \in L_j\} \qquad (LP2(L_i, L_j))$$

where a^{L_i} and b^{L_i} are the slope and intercept of L_i and $\phi = \sum_{j=1}^{n_2} d_j^1 \pi_j$ and $\psi = \sum_{j=1}^{n_2} d_j^2 \pi_j$. For convenience, when L_i is a singleton, define $a^{L_i} = b^{L_i} = 0$. Let the optimal solutions to $LP1(L_i, L_j)$ and $LP2(L_i, L_j)$ be $(\phi, \psi)^T = (\phi_{L_j}^{LP1}, \psi_{L_j}^{LP1})^T$ and $(\phi_{L_j}^{LP2}, \psi_{L_j}^{LP2})^T$, respectively. We denote by $+\infty$ the objective value of an infeasible program.

Programs $LP1(L_i, L_j)$ and $LP2(L_i, L_j)$ can be used to check whether L_j contains solutions that are dominated by solutions in L_i. Theorem 3.3.1 formalizes the conclusions that can be drawn from solving $LP1(L_i, L_j)$ and $LP2(L_i, L_j)$.

Theorem 3.3.1. *Given two line segments L_i and L_j in the space of the secondary objective functions, it holds that:*

(a) $Z_{LP1} = +\infty \iff$ *all solutions in L_j are non-dominated with respect to solutions in L_i.*

(b) $\psi_{L_j}^{LP1} = \underline{\psi}^{L_j}$ *and* $\phi_{L_j}^{LP2} = \underline{\phi}^{L_j} \implies$ *all solutions in L_j are dominated by solutions in L_i.*

(c) $\underline{\psi}^{L_j} < \psi_{L_j}^{LP1} \leq \overline{\psi}^{L_j}$ *or* $\underline{\phi}^{L_j} < \phi_{L_j}^{LP2} \leq \overline{\phi}^{L_j} \implies$ *only solutions in the segment* $L = [(\phi_{L_j}^{LP1}, \psi_{L_j}^{LP1})^T; (\phi_{L_j}^{LP2}, \psi_{L_j}^{LP2})^T] \subset L_j$ *are dominated by solutions in L_i.*

The proof of Theorem 3.3.1 is straightforward. For part (a), the feasible sets of $LP1(L_i, L_j)$ and $LP2(L_i, L_j)$ are the same and are given by $(\phi, \psi)^T \in L = L_j \bigcap (L_i + \mathbb{R}^2_{\geq})$. Hence, $Z_{LP1} = +\infty \iff L = \emptyset \iff$ for all $(\phi, \psi)^T \in L_j$, there does not exist $(\phi', \psi')^T \in L_i$ such that $\phi' \leq \phi$ and $\psi' \leq \psi$ with at least one inequality holding strictly. In part (b), notice that $L_j \bigcap (L_i + \mathbb{R}^2_{\geq}) = L_j$, so the result follows. Part (c) follows similarly.

Figure 3.2 illustrates the different cases arising from Theorem 3.3.1. In case (a), all solutions in L_j are non-dominated with respect to solutions in L_i; in case (b) all solutions in L_j are dominated by solutions in L_i; in case (c), example 1, all solutions in L_j except the endpoint $(\underline{\psi}^{L_j}, \overline{\phi}^{L_j})^T$ are non-dominated with respect to solutions in L_i, while in example 2 all solutions in the line segment $[(\phi_{L_j}^{LP1}, \psi_{L_j}^{LP1})^T; (\phi_{L_j}^{LP2}, \psi_{L_j}^{LP2})^T] \subset L_j$ are dominated by solutions in L_i.

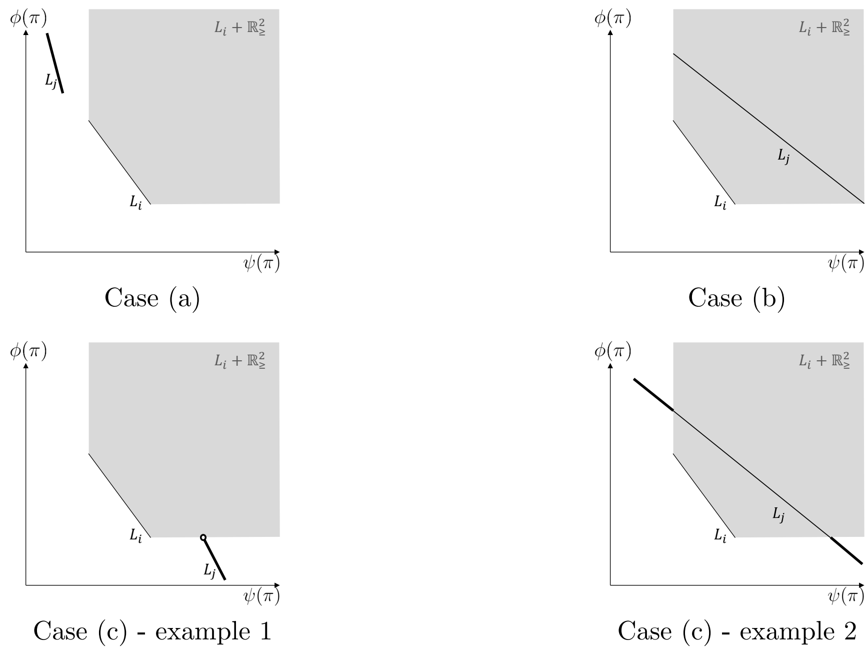


Figure 3.2: Different cases from Theorem 3.3.1

3.4 Line search and linear programming filtering (LS-LPF) algorithm

This section presents the LS-LPF algorithm, which finds the set $\mathcal{D}'$ of Pareto outcomes of the mixed binary program MB-P1. The section ends by providing a short illustrative example of the algorithm.

3.4.1 The algorithm

As a two-phase algorithm, the search begins by finding all extreme supported Pareto outcomes via the weighted sum approach and proceeds by searching for unsupported Pareto outcomes within upper triangles formed by successive pairs of extreme supported Pareto outcomes that are found to be disconnected.

Let $\mathcal{S} = \{1, ..., S\}$ and let $\mathcal{Q} \subset \mathcal{S}$. In order to find the extreme supported Pareto outcomes of MB-P1, the parametric weighted sum program using weights w_ϕ, $w_\psi > 0$ is defined next.

$$
\begin{aligned}
\text{Min } & w_\phi \phi + w_\psi \psi \\
\text{s.t. } & \sum_{j=1}^{n} a_{ij} x_j \geq b_i && \forall i = 1, ..., m \\
& \sum_{j=1}^{n_2} h_j^i \pi_j + \sum_{j=1}^{n} g_j^i x_j \geq u_i && \forall i = 1, ..., m_2 \\
& \sum_{j=1}^{n} \hat{a}_{ij} x'_{js} - \hat{b}_i z_s = 0 && \forall i \in P_s : 1 \leq i \leq m,\ s = 1, ..., S \\
& \sum_{j=1}^{n} \hat{a}_{ij} x'_{js} - M(1 - z_s) \leq 0 && \forall i \in P_s : m+1 \leq i \leq m+n, s = 1, ..., S \\
& \sum_{j=1}^{n} x'_{js} - M z_s \leq 0 && \forall s = 1, ..., S \\
& x_j - \sum_{s=1}^{S} x'_{js} = 0 && \forall j = 1, ..., n \\
& \sum_{s=1}^{S} z_s = 1 \\
& \sum_{q \in \mathcal{Q}} z_q = 0 \\
& \phi - \sum_{j=1}^{n_2} d_j^1 \pi_j \geq 0 \\
& \psi - \sum_{j=1}^{n_2} d_j^2 \pi_j \geq 0 \\
& \underline{\phi} \leq \phi \leq \overline{\phi} \\
& \underline{\psi} \leq \psi \leq \overline{\psi} \\
& x_j, x'_{js} \geq 0 \\
& z_s \in \{0, 1\}
\end{aligned}
$$

$$(\text{WS}(w_\phi, w_\psi, \underline{\phi}, \overline{\phi}, \underline{\psi}, \overline{\psi}, \mathcal{Q}))$$

Formulation WS$(w_\phi, w_\psi, \underline{\phi}, \overline{\phi}, \underline{\psi}, \overline{\psi}, \mathcal{Q})$ defines a single objective program with specified bounds on the objective values of the secondary objective functions. In addition, the program defines a subset of binary variables z_q, for $q \in \mathcal{Q}$, that are set to zero.

In the LS-LPF algorithm, the parametric program WS$(w_\phi, w_\psi, \underline{\phi}, \overline{\phi}, \underline{\psi}, \overline{\psi}, \mathcal{Q})$ is solved in phase 1 with $\mathcal{Q} = \emptyset$ and varying weights w_ϕ and w_ψ until all extreme supported Pareto outcomes of MB-P1 have been identified. In addition, WS$(w_\phi, w_\psi, \underline{\phi}, \overline{\phi}, \underline{\psi}, \overline{\psi}, \mathcal{Q})$ is successively solved in phase 2 of the algorithm with varying parameters $\underline{\phi}$, $\overline{\phi}$, $\underline{\psi}$, $\overline{\psi}$ and $\mathcal{Q}$ to find unsupported Pareto outcomes. Theorem 3.4.1 provides a core property in the search for unsupported Pareto outcomes.

Theorem 3.4.1. *Let $(\phi^1, \psi^1)^T$ and $(\phi^2, \psi^2)^T$ be two successive extreme supported Pareto outcomes from program MB-P1 such that $\phi^1 < \phi^2$ and $\psi^2 < \psi^1$. Let $s \neq q$ be the indices of the binary variables associated with $(\phi^1, \psi^1)^T$ and $(\phi^2, \phi^2)^T$, such that $z_s^1 = 1$ and $z_q^2 = 1$, respectively. If program WS$(1 - \Delta, \Delta, \phi^1, \phi^2, \psi^2, \psi^1, \{s, q\})$ is infeasible, then there does not exist another outcome $(\phi^*, \psi^*)^T$, such that $\phi^1 < \phi^* < \phi^2$ and $\psi^2 < \psi^* < \psi^1$, with index r, for $s \neq r \neq q$, such that $z_r^* = 1$.*

Proof. Because $\mathcal{Q} = \{s, q\}$, we have that the feasible set for the binary variables in program WS$(1 - \Delta, \Delta, \phi^1, \phi^2, \psi^2, \psi^1, \mathcal{Q})$ is given by $\{z_s = z_q = 0, \quad z_i \in \{0, 1\} \; \forall i \in \mathcal{S} \setminus \mathcal{Q}\}$. Since the feasible set is empty, there cannot exist a solution with $z_r = 1$ for $r \in \mathcal{S} \setminus \mathcal{Q}$. $\square$

Theorem 3.4.1 provides a condition when searching for unsupported Pareto outcomes in the upper triangle defined by $(\phi^1, \psi^1)^T$ and $(\phi^2, \psi^2)^T$. If the program in Theorem 3.4.1 is infeasible, there will be no new Pareto outcomes in the interval having a different binary variable at one. This provides a sufficient condition for stopping the search within the upper triangle and will be later detailed in the LS-LPF algorithm.

Next, we present the procedure to solve the parametric program WS($w_\phi, w_\psi, \underline{\phi}, \overline{\phi}, \underline{\psi}, \overline{\psi}, \mathcal{Q}$). As in Boland et al. (2014), the weights are successively calculated by the slope of the imaginary line connecting two outcomes in the space of the secondary objective functions.

Algorithm 1 Weighted sum algorithm for finding extreme supported Pareto outcomes

1: **procedure** WeightedSum($\underline{\phi}, \overline{\phi}, \underline{\psi}, \overline{\psi}, \mathcal{Q}$)
2:     **initialize:** $LIST \leftarrow \emptyset$; $EO \leftarrow \emptyset$
3:     solve WS($1-\Delta, \Delta, \underline{\phi}, \overline{\phi}, \underline{\psi}, \overline{\psi}, \mathcal{Q}$)
4:     let $(\phi, \psi)^T = (\phi^1, \psi^1)^T$ be its optimal solution
5:     let $Z^1 = (1-\Delta)\phi^1 + \Delta\psi^1$ be the corresponding optimal objective value
6:     **if** $Z^1 = +\infty$ **then**: STOP; **return** EO
7:     solve WS($\Delta, 1-\Delta, \underline{\phi}, \overline{\phi}, \underline{\psi}, \overline{\psi}, \mathcal{Q}$)
8:     let $(\phi, \psi)^T = (\phi^2, \psi^2)^T$ be its optimal solution
9:     add element $E^1 = \{\phi^1, \psi^1, \phi^2, \psi^2\}$ to $LIST$
10:     **if** $(\phi^1, \psi^1)^T \neq (\phi^2, \psi^2)^T$ **then**:
11:         add outcomes $(\phi^1, \psi^1)^T$ and $(\phi^2, \psi^2)^T$ to EO
12:         **while** $LIST \neq \emptyset$ **do**:
13:             remove last element $E^{|LIST|} = \{\phi^1, \psi^1, \phi^2, \psi^2\}$ from $LIST$
14:             solve WS($\psi^1 - \psi^2, \phi^2 - \phi^1, \underline{\phi}, \overline{\phi}, \underline{\psi}, \overline{\psi}, \mathcal{Q}$)
15:             let $(\phi, \psi)^T = (\phi^*, \psi^*)^T$ be its optimal solution
16:             **if** $\phi^1 < \phi^* < \phi^2$ and $\psi^2 < \psi^* < \psi^1$ **then**:
17:                 add element $E^{|LIST|+1} = \{\phi^1, \psi^1, \phi^*, \psi^*\}$ to $LIST$
18:                 add element $E^{|LIST|+1} = \{\phi^*, \psi^*, \phi^2, \psi^2\}$ to $LIST$
19:                 add outcome $(\phi^*, \psi^*)^T$ to EO
20:     **else**:
21:         add outcome $(\phi^1, \psi^1)^T$ to EO
22:     **return** EO, with outcomes sorted by nondecreasing values of ϕ.

Procedure WeightedSum($\underline{\phi}, \overline{\phi}, \underline{\psi}, \overline{\psi}, \mathcal{Q}$) begins in lines 3-5 by finding the two extreme supported Pareto outcomes, $(\phi^1, \psi^1)^T$ and $(\phi^2, \psi^2)^T$, within the region defined by $\{\underline{\phi}, \overline{\phi}, \underline{\psi}, \overline{\psi}\}$. If the program is infeasible, the search ends in line 6. Otherwise, the weights are parametrically changed according to the slope of successive outcomes in $LIST$. If the optimal solution $(\phi^*, \psi^*)^T$ in line 15 is different from the outcomes $(\phi^1, \psi^1)^T$ and $(\phi^2, \psi^2)^T$ that defined the current weights, then lines 17 and 18 add to $LIST$ the new intervals of outcomes to be explored. This is illustrated in Figure 3.3, where the new outcome $(\phi^*, \psi^*)^T$ found by the algorithm defines two new intervals given by $\{\phi^1, \psi^1, \phi^*, \psi^*\}$ and $\{\phi^*, \psi^*, \phi^2, \psi^2\}$ to be further explored. The algorithm ends when there are no more intervals to be explored and $LIST$ is empty, returning the extreme supported Pareto outcomes, EO.

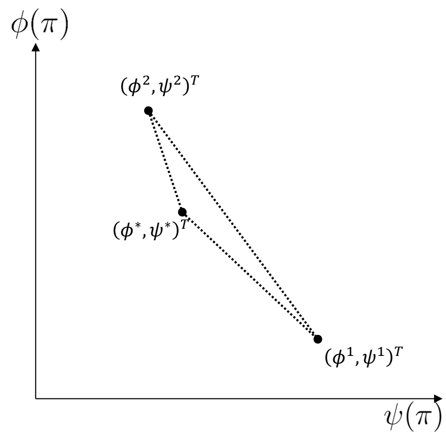


Figure 3.3: Finding new extreme supported Pareto outcome $(\phi^*, \psi^*)^T$ in Algorithm 1

The outcomes found by Procedure WeightedSum$(\underline{\phi}, \overline{\phi}, \underline{\psi}, \overline{\psi}, \mathcal{Q})$ are only guaranteed to be locally Pareto optimal within the given region defined by $\{\underline{\phi}, \overline{\phi}, \underline{\psi}, \overline{\psi}\}$, and considering the binary variables in $\mathcal{Q}$ that are fixed to zero; i.e. EO might contain outcomes that are dominated in view of MB-P1. Therefore, another key component of the LS-LPF algorithm is a pairwise comparison of line segments associated with outcomes in the space of the secondary objective functions. Procedure Filter($LINES, \mathcal{D}'$) takes a set of line segments, $LINES = \{L_1, ..., L_{|LINES|}\}$, and carries out the pairwise comparison to update the set $\mathcal{D}'$ with only those segments that correspond to non-dominated solutions. The filtering steps are shown in Algorithm 2.

Algorithm 2 Filtering of non-dominated solutions from line segments

---

1: **procedure** Filter($LINES, \mathcal{D}'$)
2:     **initialize:** $LIST \leftarrow \emptyset$; $DLINES \leftarrow \emptyset$
3:     **for** $i = 1, ..., |LINES| - 1$ **do**:
4:         **for** $k = i + 1, ..., LINES$ **do**:
5:             let L_i and L_k denote the i-th and the k-th lines in the set $LINES$
6:             add the pair $\{L_i, L_k\}$ to $LIST$
7:     **while** $LIST \neq \emptyset$ **do**:
8:         remove the last pair of lines $\{L_i, L_k\}$ from $LIST$
9:         **if** L_i and $L_k \notin DLINES$ **then**:
10:             let $L_A = L_i$ and let $L_B = L_k$
11:             solve $LP1(L_A, L_B)$ and $LP2(L_A, L_B)$; let their optimal solutions be $(\phi, \psi)^T = (\phi_{L_B}^{LP1}, \psi_{L_B}^{LP1})^T$ and $(\phi_{L_B}^{LP2}, \psi_{L_B}^{LP2})^T$, respectively.
12:             **if** $\psi_{L_B}^{LP1} < +\infty$ **then**:
13:                 add L_B to $DLINES$
14:                 **if** $\underline{\psi}^{L_B} < \psi_{L_B}^{LP1} \leq \overline{\psi}^{L_B}$ **then**:
15:                     define $L_1 = [(\overline{\phi}^{L_B}, \underline{\psi}^{L_B})^T; (\phi_{L_B}^{LP1}, \psi_{L_B}^{LP1})^T)$; add L_1 to $LINES$
16:                     **for** all pair of lines $\{L, L'\} \in LIST$ **do**:
17:                         **if** $L = L_B$ **then** add the pair of lines $\{L_1, L'\}$ to $LIST$
18:                         **if** $L' = L_B$ **then** add the pair of lines $\{L, L_1\}$ to $LIST$
19:                 **if** $\underline{\phi}^{L_B} < \phi_{L_B}^{LP2} \leq \overline{\phi}^{L_B}$ **then**:
20:                     define $L_2 = ((\phi_{L_B}^{LP2}, \psi_{L_B}^{LP2})^T; (\underline{\phi}^{L_B}, \overline{\psi}^{L_B})^T]$; add L_2 to $LINES$
21:                     **for** all pair of lines $\{L, L'\} \in LIST$ **do**:
22:                         **if** $L = L_B$ **then** add the pair of lines $\{L_2, L'\}$ to $LIST$
23:                         **if** $L' = L_B$ **then** add the pair of lines $\{L, L_2\}$ to $LIST$
24:         let $L_A = L_k$ and $L_B = L_i$; go back to line 10
25:     **return** $\mathcal{D}' \leftarrow \mathcal{D}' \bigcup LINES \setminus DLINES$

Procedure $\textsc{Filter}(LINES, \mathcal{D}')$ maintains an archive, $LIST$, of pairs of lines, $\{L_i, L_k\}$, to be checked for non-dominance, and an archive, $DLINES \subseteq LINES$, of lines that contain dominated solutions. The procedure iteratively solves programs $LP1(L_i, L_k)$ and $LP2(L_i, L_K)$ in line 11 (and later, via line 24, $LP1(L_k, L_i)$ and $LP2(L_k, L_i)$) until $LIST$ is empty. The validity of Algorithm 2 relies on Theorem 3.3.1. In line 12, if $\psi_{L_k}^{LP1} < +\infty$, from Theorem 3.3.1, there exist solutions in L_k that are dominated by solutions in L_i, hence L_k is added to the archive $DLINES$. If $\psi_{L_k}^{LP1} = \underline{\psi}^{L_k}$ and $\phi_{L_k}^{LP2} = \underline{\phi}^{L_k}$, then, according to Theorem 3.3.1, case (b), all solutions in L_k are dominated by solutions in L_i.

In line 14, if $\underline{\psi}^{L_k} < \psi_{L_k}^{LP1} \leq \overline{\psi}^{L_k}$, then, by Theorem 3.3.1, case (c), all solutions in segment $L_1 = [(\overline{\phi}^{L_B}, \underline{\psi}^{L_B})^T; (\phi_{L_B}^{LP1}, \psi_{L_B}^{LP1})^T) \subset L_k$ are non-dominated with respect to solutions from L_i; hence, L_1 is added to $LINES$ for further exploration. The loop in lines 16-18 updates the search archive $LIST$ with the newly added segment L_1. Likewise, in line 19, if $\underline{\phi}^{L_k} < \phi_{L_k}^{LP2} \leq \overline{\phi}^{L_k}$, then all solutions in segment $L_2 = ((\phi_{L_k}^{LP2}, \psi_{L_k}^{LP2})^T; (\underline{\phi}^{L_k}, \overline{\psi}^{L_k})^T]$ are non-dominated with respect to solutions from L_i; hence L_2 is added to $LINES$ for further exploration. Lines 21-23 update the search archive $LIST$ with the newly added segment L_2. At termination, the algorithm outputs the segments in $LINES$, removing the ones in $DLINES$, which contain dominated solutions.

Having defined the procedures to obtain extreme supported Pareto outcomes and to filter out line segments corresponding to dominated solutions, we show next the steps of the LS-LPF algorithm.

Algorithm 3 Line search and linear programming filtering (LS-LPF) algorithm

1: **initialize:** $\mathcal{D}' \leftarrow \emptyset$
2: $EO \leftarrow$ WeightedSum$(0, \infty, 0, \infty, \emptyset)$ ▷ (phase 1)
3: **if** $|EO| \leq 1$ **then** $\mathcal{D}' \leftarrow EO$; stop and **return** $\mathcal{D}'$
4: **else**:
5:     **for** $i = 1, ..., |EO| - 1$ **do**:
6:         **if** $(z_1^i, ..., z_S^i)^T = (z_1^{i+1}, ..., z_S^{i+1})^T$ **then**:
7:             add line $L_i = [(\phi^{i+1}, \psi^{i+1})^T;\ (\phi^i, \psi^i)^T]$ to $\mathcal{D}'$; **next** i
8:         $CONNECTED \leftarrow FALSE$
9:         **for** $s = 1, ..., S$ **do**:
10:             **if** WS$(1, 0, \phi^{i+1}, \phi^{i+1}, \psi^{i+1}, \psi^{i+1}, \mathcal{S}\backslash\{s\})$, WS$(1, 0, \phi^i, \phi^i, \psi^i, \psi^i, \mathcal{S}\backslash\{s\})$ are feasible **then**:
11:                 $CONNECTED \leftarrow TRUE$
12:                 add line $L_i = [(\phi^{i+1}, \psi^{i+1})^T;\ (\phi^i, \psi^i)^T]$ to $\mathcal{D}'$; **break for**
13:         **if** $CONNECTED = FALSE$ **then**: ▷ (phase 2)
14:             $SP \leftarrow \emptyset$; $LINES \leftarrow \emptyset$; $POINTS \leftarrow \emptyset$
15:             **loop**:
16:                 solve WS$(1 - \Delta, \Delta, \phi^i, \phi^{i+1}, \psi^{i+1}, \psi^i, SP)$; let $(\phi, \psi)^T = (\phi^*, \psi^*)^T$ be optimal solution values and Z^* its objective value
17:                 **if** $Z^* < +\infty$ **then**:
18:                     let q be the binary variable index in the optimal solution of WS$(1 - \Delta, \Delta, \phi^i, \phi^{i+1}, \psi^{i+1}, \psi^i, SP)$ such that $z_q^* = 1$
19:                     $POINTS \leftarrow$ WeightedSum$(\phi^i, \phi^{i+1}, \psi^{i+1}, \psi^i, \mathcal{S} \setminus \{q\})$
20:                     **if** $|POINTS| = 1$ **then**:
21:                         $L = [(\phi^*, \psi^*)^T; (\phi^*, \psi^*)^T]$ is a singleton
22:                         **if** $\phi^* \neq \phi^i$ and $\phi^* \neq \phi^{i+1}$ **then**:
23:                             add L to $LINES$
24:                     **else**:
25:                         **for** $j = 1, ..., |POINTS| - 1$ **do**:
26:                             add line $L_j = [(\phi^{j+1}, \psi^{j+1})^T;\ (\phi^j, \psi^j)^T]$ to $LINES$
27:                     $SP \leftarrow SP \bigcup \{q\}$
28:                 **else break loop**
29:             **if** $|LINES| > 1$ **then** $\mathcal{D}' \leftarrow$ Filter$(LINES, \mathcal{D}')$
30:             **else** $\mathcal{D}' \leftarrow \mathcal{D}' \bigcup LINES$
31: **return** $\mathcal{D}'$

The LS-LPF algorithm begins phase 1 in line 2 by finding all extreme supported Pareto outcomes of MB-P1 via procedure WeightedSum$(0, \infty, 0, \infty, \emptyset)$. In line 3, if only one outcome is found, then the ideal point is attainable and the search ends. Likewise, if the program is infeasible, EO is empty and the search ends. Else, the algorithm loops through all extreme supported Pareto outcomes in EO. In line 6, if the values of the binary variables in the current extreme supported Pareto outcome $(\phi^i, \psi^i)^T$ are the same as in the subsequent outcome $(\phi^{i+1}, \psi^{i+1})^T$, then the outcomes are connected and a line segment joining them is added to $\mathcal{D}'$. Else, the loop in lines 9-12 solves a sequence of LPs, as each time a different binary variable is set to one. If both programs WS$(1, 0, \phi^{i+1}, \phi^{i+1}, \psi^{i+1}, \psi^{i+1}, \mathcal{S} \setminus \{p\})$ and WS$(1, 0, \phi^i, \phi^i, \psi^i, \psi^i, \mathcal{S} \setminus \{p\})$ are feasible, then the outcomes $(\phi^i, \psi^i)^T$ and $(\phi^{i+1}, \psi^{i+1})^T$ are connected as they share the same binary variable with value equal to one (cf. line 11 of the algorithm).

When the outcomes $(\phi^i, \psi^i)^T$ and $(\phi^{i+1}, \psi^{i+1})^T$ are not connected, line 13 starts phase 2 and the algorithm searches for line segments in the upper triangle defined by $(\phi^i, \psi^i)^T$ and $(\phi^{i+1}, \psi^{i+1})^T$. At the beginning, no binary variables are restricted, so the archive of indices of binary variables, SP, is empty. The archives of line segments, $LINES$, and potential Pareto outcomes in the region, $POINTS$, are initialized in line 14. At each iteration of the loop, line 16 solves an MIP with decreasing number of free binary variables, i.e. variables that are not set to zero. If the MIP is feasible with optimal value of binary variable $z_q = 1$, then Procedure WeightedSum$(\phi^i, \phi^{i+1}, \psi^{i+1}, \psi^i, \mathcal{S} \setminus \{q\})$ in line 19 finds all locally extreme supported Pareto outcomes of the BOLP corresponding to fixing the binary variable z_q to one. If only one locally extreme supported Pareto outcomes is found, L is a singleton and is added to $LINES$ in line 23. Otherwise, all pairs of successive locally extreme supported Pareto outcomes of the BOLP are connected in line 26 to form line segments, which are added to $LINES$. The binary variable index q is then added in line 27 to the archive of indices of binary variables set to zero, SP, and the loop proceeds to the next iteration in line 15. If the MIP from line 16 becomes infeasible, by Theorem 3.4.1, no further Pareto outcomes can be found in the interval between $(\phi^i, \psi^i)^T$ and $(\phi^{i+1}, \psi^{i+1})^T$ with different binary variables at one; hence line 28 breaks the search for new line segments in the interval. Finally, all lines identified in the upper triangle defined by $(\phi^i, \psi^i)^T$ and $(\phi^{i+1}, \psi^{i+1})^T$ are filtered in line 29 via Procedure $\textsc{Filter}(LINES, \mathcal{D}')$ and the set $\mathcal{D}'$ is updated accordingly. Of notice is that the filtering steps are only necessary if the search has identified more than one line segment in the interval. The search then proceeds to the next pair of extreme supported Pareto outcomes $(\phi^{i+1}, \psi^{i+1})^T$ and $(\phi^{i+2}, \psi^{i+2})^T$, until all the $|EO| - 1$ regions have been explored.

The validity of the LS-LPF algorithm relies on identifying all extreme supported Pareto outcomes of MB-P1 in phase 1, connecting those that share the same values of the binary variables, and finding all line segments in the upper triangle formed by pairs of successive extreme supported Pareto outcomes that are not connected. Phase 1 exploits Proposition 3.2.1, so that all extreme supported Pareto outcomes can be found by a positive combination of weights in the weighted sum problem $\text{WS}(w_\phi, w_\psi, 0, \infty, 0, \infty, \emptyset)$. Phase 2 exploits Theorem 3.4.1 and finds all line segments by searching for BOLPs that are feasible in each upper triangle; each BOLP solved during phase 2 corresponds to fixing a single binary variable to one and restricting the values $\phi^i \leq \phi \leq \phi^{i+1}$ and $\psi^{i+1} \leq \psi \leq \psi^i$ in MB-P1. Finally, Procedure $\textsc{Filter}(LINES, \mathcal{D}')$ is used to remove the segments containing dominated solutions, exploiting conditions from Theorem 3.3.1.

Even though the efficient set is assumed to be known and given as input, of notice is that in the worst case there might be exponentially many maximal efficient faces, so that the computational complexity of generating $\mathcal{X}$, i.e. solving the first stage problem, may grow exponentially. Empirical analysis have shown that the number of maximal efficient faces in an MOLP, in general, does not increase exponentially with problem size (Sayin, 1996). However, for each of these faces, in the worst case there might be exponentially many extreme points, resulting in an exponential number of line segments in the LS-LPF algorithm. Let $(L_i : L_j)$ denote the verification of whether line L_i contains solutions dominated by the ones in L_j. Let us consider the example shown in Figure 3.4 and restrict our attention to L_1. When comparing $(L_1 : L_3)$, we find the two bold segments corresponding to the solutions in L_1 that are not dominated by solutions from L_3. Hence, L_1 becomes the collection of the two bold segments denoted by $L_{1.1}$ and $L_{1.2}$. Similarly, because L_2 also contains solutions dominated by the ones in L_3, line L_2 becomes the collection of $L_{2.1}$ and $L_{2.2}$. Therefore, $(L_1 : L_2)$ implies performing $(L_{1.1} : L_{2.1})$, $(L_{1.2} : L_{2.1})$, $(L_{1.2} : L_{2.2})$ and $(L_{1.1} : L_{2.2})$, resulting in a total of 5 comparisons associated with the filtering of L_1. Hence, in line 29 of the LS-LPF algorithm, if all N segments in the set $LINES$ are parallel in such a way that each will split another line segment in half (as in the example from Figure 3.4), the number of filtering problems to be solved will be bounded by $N \sum_{i=1}^{N-1} 2^{2(i-1)}$, hence growing exponentially in the worst case. This illustrates the challenges associated with solving such a non-convex problem. However, the empirical study in Section 3.5 suggests that the LS-LPF algorithm tends to run faster than existing algorithm for biobjective mixed binary programs, i.e. the TS method. One reason for that, as discussed in Section 3.5, may be attributed to the fact that phase 2 of the LS-LPF algorithm involves solving a sequence of MIPs with decreasing number of free binary variables and takes advantage of the early stopping condition from Theorem 3.4.1.

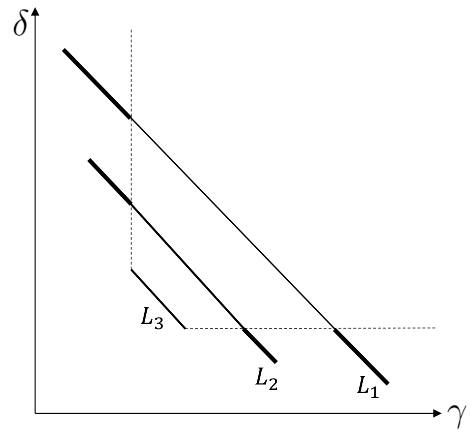


Figure 3.4: Example of worst case scenario for filtering steps from the LS-LPF Algorithm

3.4.2 Illustration of the algorithm

To illustrate the algorithm, consider the following biobjective linear program, N6M5O2:

$$
\begin{aligned}
\text{Min } & (5x_1 + 8x_2 + 12x_3 + 4x_4 - x_5 - x_6, -5x_1 + 2x_2 + 14x_3 - 4x_4 + 5x_5 + 5x_6)^T \\
\text{s.t. } & -2x_1 - 3x_2 - 5x_3 - 3x_4 - 4x_5 - 4x_6 \geq -105 \\
& 2x_1 - x_2 - 3x_3 + x_4 - x_5 + 4x_6 \geq 10 \\
& -4x_3 + 3x_4 + 3x_6 \geq 10 \\
& 2x_2 - 5x_3 - 3x_4 - 4x_5 \geq -50 \\
& -4x_1 + 3x_2 + 4x_3 - x_4 + 2x_5 \geq 20 \\
& x \in \mathbb{R}^6_{\geqq}
\end{aligned}
\tag{N6M5O2}
$$

Applying the procedure from Sayin (1996), the following three maximal efficient faces are obtained:

$$F(P_1) = \left\{ \begin{array}{ll} x \in X: & 2x_1 - 1x_2 - 3x_3 + x_4 - x_5 + 4x_6 = 10; \\ & -4x_1 + 3x_2 + 4x_3 - x_4 + 2x_5 = 20; \\ & x_1 = x_3 = x_4 = 0 \end{array} \right\}$$

$$F(P_2) = \left\{ \begin{array}{ll} x \in X: & 2x_1 - 1x_2 - 3x_3 + x_4 - x_5 + 4x_6 = 10; \\ & -4x_1 + 3x_2 + 4x_3 - x_4 + 2x_5 = 20; \\ & x_1 = x_3 = x_5 = 0 \end{array} \right\}$$

$$F(P_3) = \left\{ \; x \in X: \;\; x_1 = x_2 = x_3 = x_4 = 0 \; \right\}$$

with $\overline{f} = (189.25, 131.25)^T$ and $\underline{f} = (-26.25 - 51.25)^T$.

Consider the following nonnegative secondary decision variables δ and γ, and the additional constraints:

$$\sum_{j=1}^{n} (a_{ij} - |a_{ij}|\alpha)x_j + |b_i|\delta \geq b_i, \; \forall i = 1, ..., m \tag{3.1}$$

$$(\overline{f}_i - \underline{f}_i)\gamma - \sum_{j=1}^{n} \alpha |c_j^i| x_j \geq 0, \; \forall i = 1, ..., k \tag{3.2}$$

for some $\alpha > 0$ and $x \in \mathcal{X}$. Given constraints (3.1) and (3.2), the variables δ and γ represent measures of robustness of an efficient solution associated with maximum relative constraint violations and objective values losses, respectively, for a given uncertainty level $\alpha > 0$ (cf. Definitions from 5 and 6 introduced in Section 2).

The minimization of δ and γ will be used as the secondary objective functions. For $\alpha = 0.1$, the biobjective optimization problem over the efficient set of N6M5O2 is formulated as the B-RPSR introduced in Section 2, where parameters a_{ij}, b_i and c_j^i are given from the formulation N6M5O2. The mixed binary reformulation of from instance N6M5O2 contains 36 constraints, 26 continuous variables and 3 binary variables, and is shown in Appendix A.

In phase 1, the LS-LPF algorithm solves the weighted sum problem to find the two extreme supported Pareto outcomes A and B shown in Figure 3.5. Outcome A is given by $(\delta, \gamma)^T = (0, 0.0474)^T$, and is associated with face $F(P_3)$, while outcome B is given by $(\delta, \gamma)^T = (0.2459, 0.0230)^T$, and is associated with both faces $F(P_1)$ and $F(P_2)$.

Because outcomes A and B don't share the same values for the binary variables in the MB-P1 reformulation (i.e. $z^A = (0, 0, 3)^T$ and $z^B = (1, 0, 0)^T$ and $(0, 1, 0)^T$), phase 2 is necessary in order to search for unsupported Pareto outcomes in the upper triangle defined by $(\delta, \gamma)^T = (0, 0.0474)^T$ and $(0.2459, 0.0230)^T$. From the loop in lines 15-28 of the LS-LPF algorithm, one line segment is found associated with faces $F(P_1)$ and $F(P_2)$ and two line segments are found associated with face $F(P_3)$. Table 3.1 summarizes these steps.

Efficient face	**Line segments by the LS-LPF algorithm**	
	Before filtering	**After filtering**
$F(P_3)$	$L_1 = [(0.1, 0.043)^T; (0, 0.047)^T]$	$[(0.1, 0.043)^T; (0, 0.047)^T]$
	$L_2 = [(0.246, 0.041)^T; (0.1, 0.043)^T]$	$((0.233, 0.042)^T; (0.1, 0.043)^T]$
$F(P_1)$,$F(P_2)$	$L_3 = [(0.246, 0.023)^T; (0.233, 0.03)^T]$	$[(0.246, 0.023)^T; (0.233, 0.03)^T]$

Table 3.1: Line segments found during phase 2 of the LS-LPF algorithm - instance N6M5O2

As shown in Figure 3.6, L_2 contains solutions that are dominated by solutions from L_3. In the last step of the LS-LPF algorithm, Procedure FILTER($LINES, \mathcal{D}'$) removes the segment of line L_2 that corresponds to the dominated solutions. Figure 3.7 displays the final set $\mathcal{D}'$.

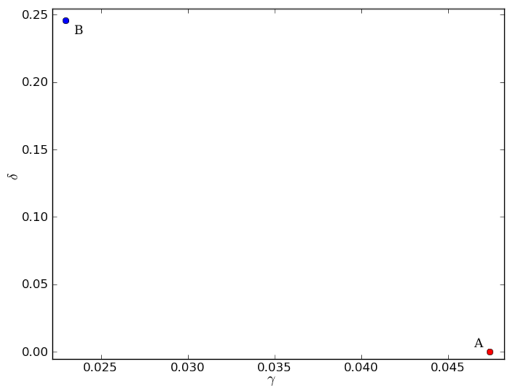


Figure 3.5: Phase 1 - instance N6M5O2

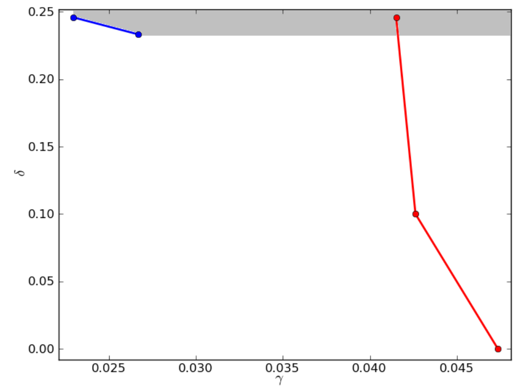


Figure 3.6: Phase 2 - search for line segments

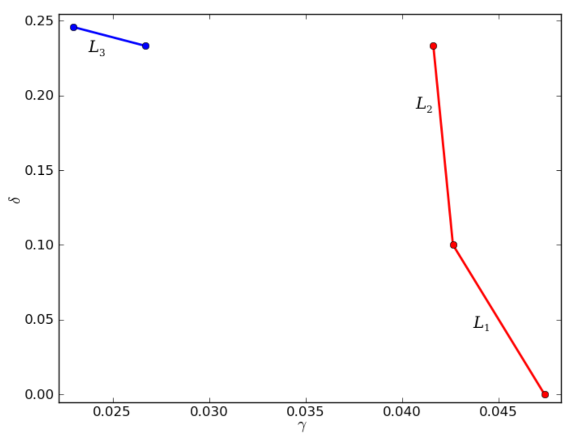


Figure 3.7: Final Pareto set - instance N6M5O2

3.5 Experimental analysis

This section provides an experimental analysis of the LS-LPF algorithm applied to randomly generated instances. The secondary objective functions are the minimization of the infeasibility and outcome degradation levels using the box uncertainty set. The LS-LPF algorithm was coded in C++ and CPLEX 12.6 was used as the solver, with tolerance set to 10^{-5}. All experiments were run on a 2.8GHz 8-CPU CentOS 6.6 64-bits system with 16MB RAM.

The MOLPs were generated based on Steuer (1994), with $k \in \{2, 5\}$ objective functions. Problem parameters were randomly generated with $c_j^i \in [-5, 15]$, $a_{ij} \in [-5, 5]$, $b_i \in [-5, 5]$ and density of zeros in the LHS matrix set to 0.20. The efficient faces of each MOLP were found using the algorithm proposed by Sayin (1996). In a preliminary analysis, only smaller instances were solved due to the exponential nature of the algorithm from Sayin (1996) for finding the efficient faces of the MOLPs. These instances were generated with a number of 5 to 10 decision variables and 5 to 15 constraints. For each problem, 5 different instances were generated.

Table 3.2 shows the results of only those instances where $\mathcal{D}'$ contains outcomes from more than one efficient face of the MOLP, i.e. excluding the trivial instances where the weighted sum approach would suffice to find $\mathcal{D}'$. In Table 3.2, NCTR denotes the number of constraints in the MB-P1 formulation, while NCV and NBV are the number of continuous and binary variables, respectively; NFACES is the number of faces of the MOLP containing solutions that are also efficient in view of the secondary objective functions; the last column denotes the total solution time of the LS-LPF algorithm.

Instance	NCTR	NCV	NBV	NFACES	LS-LPF Runtime (secs.)
1	45	22	3	2	0.02
2	65	22	3	2	0.02
3	51	26	3	2	0.04
4	53	27	4	2	0.03
5	81	42	4	2	0.04
6	131	58	6	2	0.1
7	110	50	7	3	0.15
8	111	58	7	2	0.08
9	176	82	7	3	0.22
10	168	83	8	2	0.23
11	168	98	11	3	0.21
12	253	146	15	3	0.26
13	257	146	15	2	0.31
14	486	262	25	3	1.04
15	405	234	28	2	1.00

Table 3.2: Experiments with the LS-LPF algorithm on randomly generated instances

Table 3.2 shows that the LS-LPF algorithm can solve relatively small instances in a short computational time. In order to have a more comprehensive analysis and a comparison with the TS algorithm, larger instances of MOLPs were randomly generated with the number of decision variables ranging from 40 to 80 and the number of constraints set to twice the number of decision variables. For these larger instances, a simple heuristic was devised to generate efficient faces of the MOLPs. The heuristic procedure is a modification of the algorithm of Sayin (1996) and consists of generating $n + m$ faces of the MOLP; each face is denoted by a binary string corresponding to the indices of the constraints of the problem that are holding at equality at the face. The heuristic retains only those faces that satisfy efficiency, however it does not guarantee that the face is maximally efficient.

Table 3.3 compares the runtimes between the LS-LPF and the TS algorithms.

The last column in Table 3.3 denotes the relative runtime reduction, calculated as 1 - LS-LPF Runtime/TS Runtime. The TS algorithm was also coded in C++ with the same CPLEX tolerance setting.

Results from Table 3.3 show that the LS-LPF algorithm achieves an average runtime reduction of 25.8%, with a p-value of 8.8×10^{-6}. As before, only those instances where NFACES > 1 were considered. Of notice is that NFACES is an upper bound on the actual number of distinct efficient faces of the MOLP that are associated with solutions that are efficient to MB-P1; this is due to the fact that the representation of a face by a subset of constraint indices is not necessarily unique (Sayin, 1996). Although the number of variables and constraints greatly vary in these instances (e.g. from 346 to 3,420 constraints), NFACES only varies from 2 to 4 efficient faces. This observation is in line with empirical analysis of Sayin (1996) in the context of MOLP, suggesting that the number of efficient faces typically does not increase exponentially with the problem size.

Instance	NCTR	NCV	NBV	NFACES	TS Runtime (secs.)	LS-LPF Runtime (secs.)	Improvement
1	368	152	2	2	14.4	10.5	26.8%
2	346	162	3	2	12.7	8.9	29.6%
3	394	202	4	2	11.5	6.3	44.9%
4	441	242	5	2	12.3	9.0	26.3%
5	546	302	5	2	37.4	20.7	44.7%
6	482	282	6	2	18.1	15.0	17.0%
7	482	282	6	2	12.9	8.9	30.9%
8	721	452	8	2	48.6	37.5	22.8%
9	1149	722	8	4	343.9	263.2	23.5%
10	801	502	9	2	64.9	44.3	31.8%
11	727	482	11	3	66.6	44.4	33.4%
12	1616	1122	13	2	322.0	334.8	-4.0%
13	1830	1282	15	3	791.5	725.9	8.3%
14	1005	722	17	4	99.3	41.0	58.7%
15	1130	842	20	3	155.9	77.3	50.4%
16	1489	1102	21	3	87.8	78.3	10.8%
17	2762	2082	25	3	2299.2	2448.6	-6.5%
18	3420	2642	32	3	1294.9	1101.0	15.0%

Table 3.3: Runtime analysis of TS and LS-LPF algorithms on randomly generated instances

In order to demonstrate that the runtime reduction from Table 3.3 is not simply due to improvements on the implementation of the weighted sum method in phase 1 of the algorithm, Table 3.4 compares the runtimes of the LS-LPF and TS algorithms on instances where NFACES = 1.

Instance	NCTR	NCV	NBV	TS Runtime (secs.)	LS-LPF Runtime (secs.)	Improvement
1	1209	952	18	2.19	2.07	5.48%
2	2537	2082	25	2.33	2.56	-9.87%
3	1147	962	31	1.16	1.12	3.45%
4	4224	3682	45	9.04	11.57	-27.99%
5	2739	2402	47	5.82	5.7	2.06%
6	1770	1532	50	1.8	1.71	5.00%
7	2940	2552	50	3.77	3.67	2.65%
8	3063	2802	53	3.77	3.41	9.55%
9	5099	4552	64	4.01	4.66	-16.21%
10	3580	3202	79	8.75	9.53	-8.91%
11	7135	6402	79	4.6	6.99	-51.96%
12	3631	3322	82	2.85	3.17	-11.23%
13	4016	3682	91	10.82	9.79	9.52%
14	5359	4852	96	2.16	2.41	-11.57%
15	4515	4082	101	5.29	4.72	10.78%
16	4785	4322	107	11.45	12.63	-10.31%
17	4907	4442	110	17.14	12.56	26.72%
18	15725	14562	207	193.14	202.5	-4.85%

Table 3.4: Experiments with problems with NFACES = 1

Results from Table 3.4 indicate that the difference between runtimes of the weighted sum steps in the LS-LPF and TS algorithms are not significant (p-value of 0.30). Hence, the average runtime improvement of the LS-LPF algorithm on instances in Table 3.3 is likely due to the steps involving phase 2 of the algorithm, i.e. the search for unsupported Pareto outcomes. While the search in phase 2 of the TS algorithm can be expensive because they involve a sequence of MIPs with equal combinatorial complexity, the LS-LPF algorithm solves MIPs with decreasing number of free binary variables. The approach is possible because the single-choice constraint $\sum_{s=1}^{S} z_s = 1$ in MB-P1 limits the maximum number of MIPs to S in each upper triangle of phase 2, while Theorem 3.4.1 defines an early stopping condition.

Otherwise, the possible exponential number of MIPs to be solved would likely hinder the applicability of the approach.

3.6 Summary and conclusions

This section presented a biobjective mixed binary linear programming formulation to optimize two linear functions, termed secondary objective functions, over the given efficient set of an MOLP with additional secondary decision variables and constraints. The formulation connects the areas of biobjective mixed binary linear programming and biobjective optimization over the efficient set.

The LS-LPF algorithm proposed in this section uses problem structure, which contains a single-choice constraint, to carry out a search over the efficient faces of the MOLP and is amenable to parallelization. The procedure can be parallelized since the search for unsupported Pareto outcomes in different upper triangles is independent.

Experimental analysis showed runtime improvement with respect to general purpose algorithm for biobjective mixed binary linear programs. The algorithm was illustrated in the context of finding efficient solutions that are robust to perturbations, using measures of maximum relative constraint violation and objective value losses as the secondary objective functions.

4. CASE STUDY APPLICATION IN THE ELECTRICITY GENERATION CAPACITY EXPANSION PROBLEM TO MINIMIZE COST AND WATER WITHDRAWAL

4.1 Introduction

Several studies have shown the degree of dependency of electricity generation on water availability (e.g. Gleick, 1994; Kenny et al., 2009; Blackhurst et al., 2010; Stillwell et al., 2011; Macknick et al., 2012; Scanlon et al., 2013). Water is intensively used for cooling processes in thermal power plants, such as in natural gas and coal-fired facilities. Approximately 40% of total freshwater withdrawals in the U.S. are used in thermoelectric power plants and water scarcity is a cause of concern for power grid operators. During the 2011 drought in Texas, a heat wave caused demand for electricity to reach historic levels, while less than half of the water supply was available due to the drought. In some cases, water had to be diverted from farm areas. Some facilities had to reduce operations during night so that the necessary water would be available for operation during the day, when demand would reach its peak (Galbraith, 2011; Faeth, 2013; Scanlon et al., 2013). Despite the connections between water usage and electricity generation, few studies have considered water requirements when addressing the capacity expansion problem (e.g. Stults, 2015). The literature is mainly focused on models to optimize cost (e.g. Teghem and Kunsch, 1985; Rabensteiner, 1987; Malcolm and Zenios, 1994; Rentizelas and Tatsiopoulos, 2010; Chaudry et al., 2014) and, more recently, to minimize greenhouse gas emissions (e.g. Tekiner et al., 2010; Stoyan and Dessouky, 2012; Zhang et al., 2013; Walmsley et al., 2014; Chen et al., 2015a). This section deals with the case where both cost and water withdrawal requirements are included in the expansion planning. While fossil-fueled technologies are typically price-competitive, their intensive use of water poses a risk for electricity generation, as illustrated in the previous example. On the other hand, renewable technologies (e.g. wind and solar power plants) require minimal use of water. Therefore, the different technologies for capacity expansion must be analyzed in light of cost and water objectives.

The consideration of uncertainty when addressing the capacity expansion problem is fundamental to this section; for instance, at the time of building the power plants, uncertain events or situations not considered in the optimization model may occur (e.g. changes in regulation and environmental policies, access to limited financing resources, unexpected land availability or environmental changes, etc.) forcing those power plants to be built with capacities different to the ones prescribed by the optimization model, which in turn force design revisions to make the altered system operable under the new capacities. This example illustrates how systems may be implemented somewhat different to how they were originally prescribed by an optimization model. While robust optimization typically focus on data uncertainty, such as the variability on electricity demand and fuel prices (e.g. Murphy et al., 1982; Malcolm and Zenios, 1994; Jin et al., 2011; Tolis and Rentizelas, 2011; Feng and Ryan, 2013; Dehghan et al., 2014), our model considers the uncertainty affecting the designed capacities of the power plants at the time of implementation; we refer to this as "implementation uncertainty". In the previous example, one would consider a solution to be robust if the final power plant capacities yield a total cost and water withdrawal close to the values previously prescribed by the optimization model; furthermore, assuming that small constraint violations associated with a solution not meeting the total demand require minor system design modifications, the solution could also be considered robust if it is almost feasible under the final off-target capacities. Arguably, this issue may be avoided by increasing the fidelity of the model;

however, in most real-life problems model fidelity will be limited by the complexity of the mathematical formulation, time constraints, availability and accuracy of modeling information, among other factors.

Motivated by the need to protect against implementation uncertainty, this section applies the two-stage methodology introduced in Section 2 for assessing robustness of solutions from the electricity generation capacity expansion problem. The first stage considers the biobjective optimization problem to minimize cost and water withdrawal and finds the corresponding set of efficient solutions. Each solution prescribes a set of locations and technologies for new power plants, as well as their designed capacities. The second stage considers the problem of finding the subset of efficient solutions that tend to maintain feasibility and their prescribed cost and water withdrawal objective values in the face of unforeseen deviations of their specified capacities at the time of implementation. A novelty of the proposed methodology is that the modeling of uncertainty is not conflictive with models considering problem data uncertainty when solving the first stage problem, which is illustrated in the numerical experiments. In the two-stage methodology, hedging against implementation uncertainty would be considered hierarchically secondary with respect to solving the first stage problem, hence providing the decision maker with secondary criteria to break ties among technologies and locations for the new power plants.

The applicability of the methodology is illustrated in the context of Texas, USA, to demonstrate how it could be used to answer strategic questions concerning long-term capacity expansion planning. Strategic questions considered include: how does the trade-off among cost, water withdrawal and uncertainty affect technology and location selection required to meet demand for 2040; how robust solutions compare to other efficient solutions when subjected to implementation uncertainty; the locations that should be prioritized in order to minimize cost and water requirements;

the minimum number of power plants that should be built; the effect of demand uncertainty; and the impact of varying dispatchability requirements. The experiments demonstrate how it might be advantageous to use robustness as secondary criteria to break ties among technologies and locations in the selection of power plants.

The remainder of this section is organized as follows. Section 4.2 introduces the model and formulation of the biobjective electricity generation capacity expansion problem. Section 4.3 describes the robustness assessment methodology. Section 4.4 shows the experimental analysis applied in the case of Texas, while section 4.5 provides concluding remarks.

4.2 Definition and formulation of the electricity generation capacity expansion problem

The electricity generation capacity expansion problem dealt with in this section is defined as the optimization problem to find locations, technology types and design capacities for new power plants required for meeting additional forecasted demand for electricity. The decisions variables are to determine the locations and technologies of new power plants, as well as their designed capacities, and the objective functions minimize cost and water withdrawal, while satisfying the additional demand for electricity generation.

The set of available technologies include both dispatchable and non-dispatchable alternatives. A dispatchable technology, such as nuclear and natural gas-fired power plants, is one that can be turned on and off as needed with relative ease, allowing the operator to control the level of supply based on the economic attractiveness of the technology and on demand variability. While wind and other forms of renewable energy might be advantageous for minimizing environmental impact, such as greenhouse gas emissions, the electricity generated by these technologies is intermittent due to weather conditions, hence cannot be controlled by the operator in the same way that dispatchable technologies are (Joskow, 2011). Therefore, we model the capacity expansion problem such that at least a given p % of the additional electricity generated by new facilities is required to rely on dispatchable technologies. Here, p is termed the dispatchability requirement.

A cost parameter relates the \$/MWh of generating electricity from a new power plant in a specific location, for a given technology. It is assumed that the parameter includes all annualized costs necessary for generating electricity, such as capital costs, operating expenses, emissions, maintenance and financial costs.

A water withdrawal parameter relates the amount of water required to generate 1 MWh of electricity utilizing a specific technology. A weight factor penalizes water withdrawals in those power plants that are prescribed in locations where water resources are more scarce. Therefore, the problem not only prescribes the technologies that minimize water withdrawal, but also considers their locations in order to minimize the impact on areas most affected by droughts.

The design capacities of the prescribed power plants are required to comply with given upper and lower bounds. These bounds typically represent engineering design limitations associated with each technology.

The problem is modeled considering assumptions that are frequent in the related literature (e.g. Tekiner et al., 2010; Mustakerov and Borissova, 2011; Stults, 2015): (1) the additional electricity demand should be met from new power plant capacity (i.e. as opposed to meeting the demand from reserve levels); (2) at most one new power plant is allowed to open at each potential location; (3) problem parameters are known (this assumption will be partially relaxed in the experimental analysis, where we will demonstrate how implementation uncertainty may be dealt with in the case where electricity demand is also assumed to be uncertain); (4) the model does not account for transmissions losses. Transmission losses may be incorporated into the model by further considering the available transmission lines and their distances to each location; (5) capacity factors are given for each technology, independently of the location of a power plant. While the model may be further extended to consider data at a location-level (such as wind speed at each location to determine a more accurate capacity factor dependent on the region), we utilize a constant capacity factor across locations for the sake of illustrating the proposed methodology for long-term planning.

Symbol	**Description**
Sets:	
J	Set of available technologies
J_d	Subset of technologies that are dispatchable, $J_d \subseteq J$
I	Set of potential locations for power plants
Parameters:	
c_{ij}	Cost of electricity generation using technology $j \in J$ at location $i \in I$ in \$/MWh
w_j	Water withdrawal requirement for technology $j \in J$ in gal/MWh
s_i	Water scarcity factor at location $i \in I$ (dimensionless)
g_j	Capacity factor for technology $j \in J$ (dimensionless)
E	Energy generation requirement in MWh to satisfy additional demand
p	Minimum proportion of energy generation requirement to be met from dispatchable technologies
L_j	Minimum capacity (MW) required from a new power plant using technology $j \in J$
U_j	Maximum capacity (MW) required from a new power plant using technology $j \in J$
Decision variables:	
x_{ij}	Designed capacity (MW) of a power plant at location $i \in I$ using technology $j \in J$
y_{ij}	Equals 1 if technology $j \in J$ is selected at location $i \in I$ and 0 otherwise

Table 4.1: Notation of the BO-CEP

The notation in the formulation of the problem is presented in Table 4.1. The biobjective electricity generation capacity expansion problem (BO-CEP) is formulated as a bicriterion program to minimize total cost and water withdrawal. The formulation is shown next:

$$\text{Min } \left(f_1(x) = \sum_{i \in I} \sum_{j \in J} 8760 c_{ij} g_j x_{ij},\ f_2(x) = \sum_{i \in I} \sum_{j \in J} 8760 s_i w_j g_j x_{ij} \right)^T \tag{4.1}$$

$$\text{s.t. } 8760 \sum_{i \in I} \sum_{j \in J} g_j x_{ij} \geq E \tag{4.2}$$

$$8760 \sum_{i \in I} \sum_{j \in J_d} g_j x_{ij} \geq pE \tag{4.3}$$

$$\sum_{j \in J} y_{ij} \leq 1 \qquad \forall i \in I \tag{4.4}$$

$$U_j y_{ij} - x_{ij} \geq 0 \qquad \forall j \in J;\ i \in I \tag{4.5}$$

$$x_{ij} - L_j y_{ij} \geq 0 \qquad \forall j \in J;\ i \in I \tag{4.6}$$

$$x_{ij} \geq 0 \qquad \forall j \in J;\ i \in I \tag{4.7}$$

$$y_{ij} \in \{0, 1\} \qquad \forall j \in J;\ i \in I \tag{4.8}$$

The objective functions in (4.1) minimize total annual cost for generating the required additional electricity and total annual water withdrawals, weighted by a scarcity factor s_i associated with the extent to which water resources are available at each location $i \in I$. Constraint (4.2) ensures that the capacity expansion will meet the additional demand for electricity while (4.3) guarantees that p % of the energy generated will come from dispatchable technologies. Constraints (4.4) allow at most one new power plant per location. Constraints (4.5) and (4.6) ensure that the capacity of each new power plant will fall within the given minimum and maximum capacity bounds associated with each technology. Nonnegativity and binary requirements are given by (4.7) and (4.8).

The BO-CEP is a mixed binary linear program and the efficient set can be disconnected in the general case (Vincent et al., 2013). In this section, we solve the BO-CEP via the ϵ-constraint method (Haimes et al., 1971) in order to obtain a discrete set of efficient solutions. In the ϵ-constraint method, one of the objectives is selected as the single objective function to be minimized, whiled the remaining objectives are converted into constraints, where their upper bounds are systematically modified. In Miettinen (1999), it is shown that every solution obtained from altering the upper bounds in the ϵ-constraint approach is efficient.

4.3 Methodology for robustness assessment in the BO-CEP

This section describes how the methodology to assess robustness of efficient solutions previously introduced in Section 2 may be applied to the BO-CEP when the prescribed capacities of the power plants are subjected to implementation uncertainty. The uncertainty is given by a multiplicative perturbation factor on the prescribed capacities of power plants. Definition 8 is an adaptation of Definition 4 to the case of the BO-CEP and is formalized next.

Definition 8. *For some level of uncertainty $\alpha > 0$, the perturbation factor $\tilde{\beta} = (\tilde{\beta}_{11}, ..., \tilde{\beta}_{|I||J|})^T$ is a random vector such that $1-\alpha \leq \tilde{\beta}_{ij} \leq 1+\alpha$, for all $i \in I$, $j \in J$. Given a realization β of $\tilde{\beta}$, $\beta x = (\beta_{11}x_{11}, ..., \beta_{|I||J|}x_{|I||J|})^T$ denotes a perturbation on* *the prescribed power plant capacity values.*

Let $\mathcal{U}$ denote the uncertainty set of all possible realizations of the perturbation factor $\tilde{\beta}$. In this section, two particular uncertainty sets are considered: the box and the cardinality-constrained uncertainty sets. The former represents the most conservative case where it is assumed that all power plant design capacities can be simultaneously affected by uncertainty. The later allows the decision maker to control the level of conservatism when finding robust solutions, and arises when the number of power plant capacities that are simultaneously affected by uncertainty is limited by some non-negative integer parameter Γ. The most optimistic case is represented by letting $\Gamma = 1$, implying that only 1 power plant will have its design capacity affected by perturbations; the worst case, i.e. when the decision maker assumes that all n power plant capacities can be simultaneously affected by perturbations, is represented by letting $\Gamma = n$. Of notice is that $\Gamma = 0$ reduces to the deterministic problem, where all values of the variables are assumed to be known with certainty. The box uncertainty set is the special case of the cardinality constrained uncertainty set with $\Gamma = n$.

Considering the box and the cardinality-constrained geometries, the uncertainty sets in the BO-CEP will given by, respectively:

$$\mathcal{U}_{box} = \{\beta \in \mathbb{R}^{|I||J|} : (1-\alpha) \leq \beta_{ij} \leq (1+\alpha)\ \forall i \in I, j \in J\} \tag{4.9}$$

$$\mathcal{U}_{card} = \{\beta \in \mathbb{R}^{|I||J|} : (1-\alpha)z_{ij} \leq \beta_{ij} \leq (1+\alpha)z_{ij}\ \forall i \in I, j \in J;\ \sum_{i\in I}\sum_{j\in J} z_{ij} \leq \Gamma;\ z \in \{0,1\}^{|I||J|}\} \tag{4.10}$$

Given the demand satisfaction constraint (4.2) and the dispatchability requirements (4.3) in the BO-CEP formulation, the worst case constraint violation will be when the perturbation factor takes the value $\beta_{ij} = 1 - \alpha$, for all locations $i \in I$ and technologies $j \in J$; in contrast, the worst case objective value loss will be when $\beta_{ij} = 1 + \alpha$, for all $i \in I$ and $j \in J$.

In this section, we consider the degree of infeasibility with respect to constraints (4.2) and (4.3) in the BO-CEP. It is implied that violating capacity bounds constraints (4.5) and (4.6) are not as significant as violating the constraints associated with the demand for electricity generation from constraints (4.2) and (4.3). Hence, the infeasibility level from Definition 5 becomes:

$$\delta(x) = \max_{\beta \in \mathcal{U}} \left\{ \frac{E - 8760 \sum_{j \in J} \sum_{i \in I} \beta_{ij} g_j x_{ij}}{E}, \frac{pE - 8760 \sum_{j \in J_d} \sum_{i \in I} \beta_{ij} g_j x_{ij}}{pE} \right\} \tag{4.11}$$

In order to assess solution robustness, the outcome degradation level from Definition 6 becomes:

$$\gamma(x) = \max_{\beta \in \mathcal{U}} \left\{ \frac{f_1(\beta x) - f_1(x)}{\overline{f}_1 - \underline{f}_1}, \frac{f_2(\beta x) - f_2(x)}{\overline{f}_2 - \underline{f}_2} \right\} \tag{4.12}$$

where the values of $\overline{f}_1$, $\underline{f}_1$, $\overline{f}_2$ and $\underline{f}_2$ are the individual maximum and minimum of cost and water objectives, respectively, over the set of solutions in $\mathcal{X}$.

The set of robust solutions is given by $\mathcal{X}' = \{x \in \mathcal{X} : \nexists x' \in \mathcal{X}$ such that $\delta(x') \leq \delta(x)$ and $\gamma(x') \leq \gamma(x)$ with at least one inequality holding strictly$\}$, and denotes the subset of efficient solutions from the BO-CEP that are non-dominated with respect to the measures of infeasibility and outcome degradation levels. Given that $\mathcal{X}$ is assumed to be represented by a discrete set of solutions in this section, $\mathcal{X}'$ may be obtained by computing the values of the functions in $\delta(x)$ and $\gamma(x)$ for each solution $x \in \mathcal{X}$ and filtering out the ones that are dominated in view of the robustness measures. The filter in this case, simply consist of pairwise dominance comparisons of the efficient solutions identified by the ϵ-constrained method. Figure 4.1 summarizes the application of the methodology in the BO-CEP.

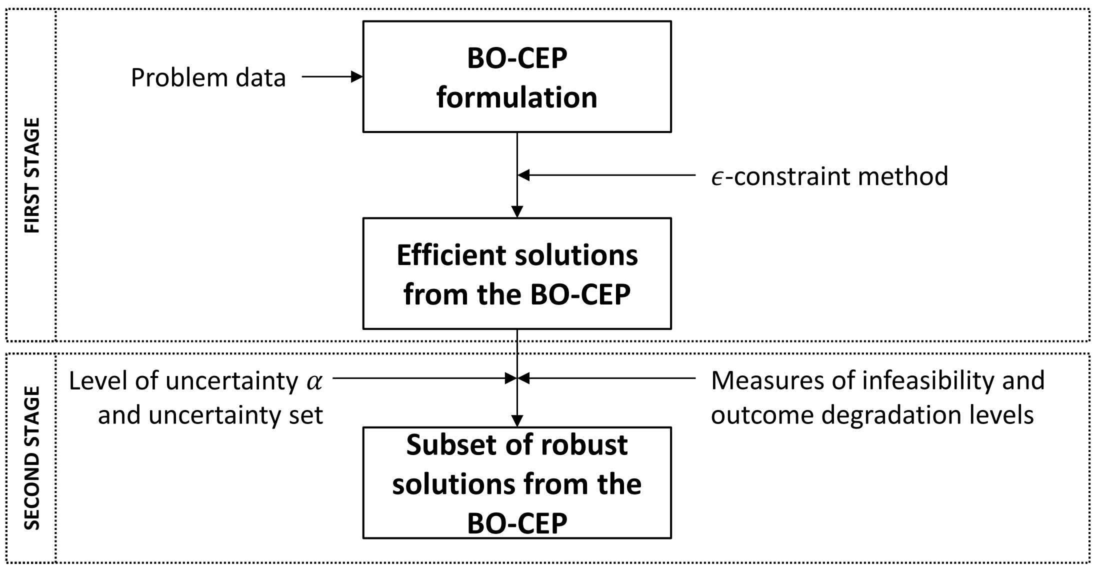


Figure 4.1: Robustness methodology applied to the BO-CEP

As will be illustrated in our experimental study in Section 4.4, the set of efficient solutions $\mathcal{X}$ in the BO-CEP may comprise a large number of solutions. The robustness measures as given in (4.11) and (4.12) may aid decision making for capacity expansion planning by providing secondary criteria to break ties among those efficient solutions that are more stable in face of implementation uncertainty.

4.4 Electricity generation capacity expansion in the case of Texas

The State of Texas withdraws an average of 23,600 million gallons of freshwater per day, of which 9,680 million gallons are dedicated to power plant cooling, the second highest averages across all States in the U.S. (Kenny et al., 2009). With demand for electricity expected to grow by an average of 0.7% per year for the next decades (EIA, 2015a), water requirements for cooling processes may introduce vulnerability into the system, i.e. if the capacity expansion mainly relies on thermoelectric power plants and the severity of droughts in Texas increases, water shortages may hamper operations of the new power plants. Motivated by these circumstances, this section shows the application of the proposed biobjective formulation and robustness methodology to discuss key strategic questions for long-term capacity expansion planning for meeting the projected electricity demand for year 2040 in Texas. In particular, numerical experiments are developed and the following issues are discussed:

(Q1) How does the trade-off among cost, water withdrawal and implementation uncertainty affect the technology and location selection of power plants required to meet projected electricity demand?

(Q2) How do robust solutions compare to other efficient solutions when subjected to implementation uncertainty?

(Q3) What are the locations that should be prioritized for deploying the new power plants in order to minimize cost and water withdrawal? What is the minimum number of new power plants that should be built in order to meet the projected electricity demand?

(Q4) What is the impact of demand uncertainty on the robust technology-location solutions?

(Q5) What is the impact of changing the dispatchability requirement of new power plants on the resulting technology-location solutions, as well as on cost and water withdrawal?

If the focus for capacity expansion is on reducing water withdrawal, renewable technologies would be probably prioritized as their water requirements are marginal. On the other hand, solutions that focus on reducing cost would likely rely on fossil-fueled technologies. Therefore, it must be considered how cost and water withdrawal are affected by different combinations of technologies and locations, while meeting the necessary demand and dispatchability requirements. Since the number of efficient solutions in the BO-CEP may be large, of interest is to consider how effective is the methodology in breaking ties among solutions, to provide the decision maker with a reduced set of technologies and locations to choose from. Hence, of interest is to study the trade-off among cost, water withdrawal and robustness of solutions.

The advantages of implementing second-stage robust solutions, rather than other first-stage non-robust solutions, must be evaluated to demonstrate the benefit of the proposed methodology. In addition, if the level of uncertainty is unknown, it becomes necessary to assess the sensitivity of robust solutions with respect to changes α. This would allow the decision maker to understand how different levels of uncertainty may affect the selection of a solution for implementation.

The locations of the new power plants must also be considered because the performance in terms of cost and water withdrawal are likely dependent on the prescribed regions for these facilities. Technologies that require a more intensive use of land, such as in the case of biomass power plants, may need to focus on less expensive land areas. Similarly, droughts can vary significantly across different regions; thermoelectric power plants that require more extensive water withdrawals would probably need to focus on areas where water resources are less scarce. Deploying new power plants across several regions would likely require challenging coordination within governmental agencies, public organizations and the private sector. Hence, it becomes necessary to consider what would the minimum number of new power plants that should be built and what locations should be prioritized in order to minimize cost and water withdrawal, while being robust to implementation uncertainty.

One of the major difficulties facing decision makers in long-term capacity expansion planning is demand uncertainty. Although the proposed methodology is intended to hedge against implementation uncertainty, it is non-conflictive with models that take into account uncertainty in problem data. Therefore, of interest becomes analyzing how electricity demand uncertainty can affect the set of solutions that are robust to implementation uncertainty.

Another important factor for deploying the new power plants is related to the amount of electricity generated from dispatchable technologies since they provide means to satisfy peak demand. While renewable sources of energy require minimal use of water, such as in wind and solar power plants, these are non-dispatchable technologies, leaving the operator with less flexibility to adapt to demand variability. Changing the dispatchability requirement p can affect the prescribed technologies and locations for the new power plants. Therefore, it becomes necessary to understand how different levels of dispatchability requirements may affect the resulting set of solutions, as well as their impact on total cost and water withdrawal requirements.

4.4.1 Problem data

In Texas, the total required electricity generation E to meet additional demand for year 2040 was estimated as 67,539 GWh. The value of E was obtained by considering the net generated load of 339,643 GWh during 2014 on the ERCOT grid (ERCOT,

2014) and the expected annual increase of 0.7% in electricity demand (EIA, 2015a).

The available technologies and their cooling systems, where applicable, were the following: (1) conventional coal, cooling tower; (2) advanced coal, cooling tower; (3) natural gas conventional combined cycle, pond cooling; (4) advanced natural gas combined cycle, open loop cooling; (5) nuclear, pond cooling; (6) biomass from energy crop, pond cooling; (7) wind onshore; (8) solar PV flat paneled. Geothermal and hydro power plants were not considered in this case study due to the nature of geological formation and geography of Texas. Technologies (1)-(6) are considered dispatchable, while (7)-(8) are non-dispatchable. Water withdrawal requirement parameters w_j were taken from Macknick et al. (2011) for biomass technology, and from Meldrum et al. (2013) for the remaining technologies. Their values are presented in Table 4.2. The use of water in wind and solar power plants are minimal and are required for periodic cleaning.

The upper and lower bounds on the design capacity for power plants from each technology were defined by considering current minimum and maximum nameplate capacities of power plants in operation in the U.S. (EIA, 2013). The capacity factors, g_j, were assumed from EIA (2015b).

The potential locations considered for the new power plants were taken from TAMU (2015) and are given by 33 land market areas (LMA) in Texas, as illustrated in Figure 4.4.1.

The scarcity factor, s_i, was obtained by considering the Palmer Drought Severity Index (PDSI) at each location, which reflects long-term drought and considers temperature, and precipitation data (TWDB, 2015b). In the experimental analysis, the median monthly PDSI values between 2013 and 2014 were used, normalized to a scale from zero to one. Table 4.3 shows the values of water scarcity factor for each location.

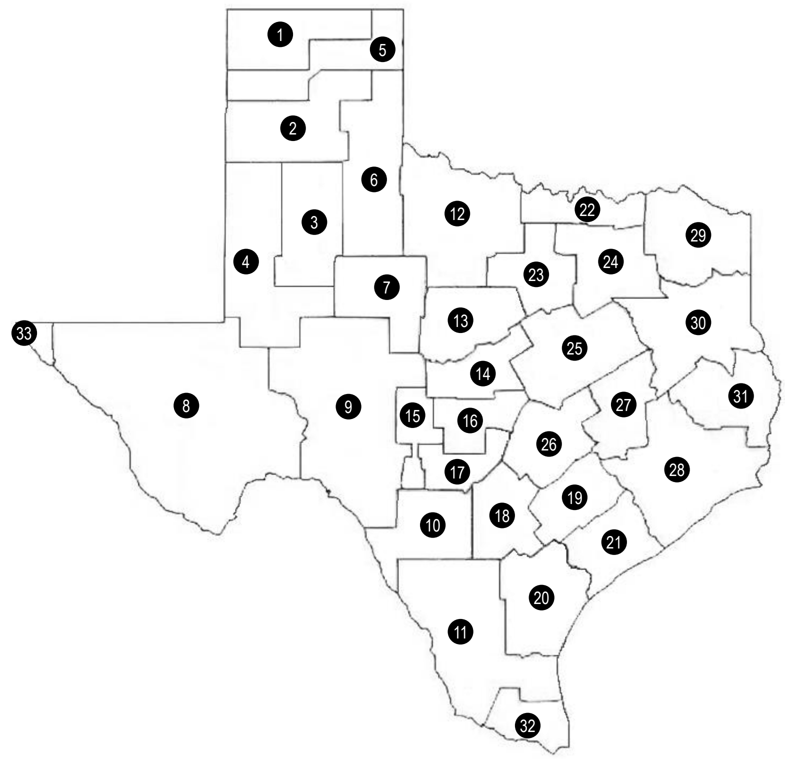


Figure 4.2: Potential locations for new power plants

Adapted from: TAMU (2015)

In order to estimate the cost parameters, c_{ij}, the values from EIA (2015b) of levelized cost of electricity (LCOE) and levelized avoided cost of electricity (LACE) for year 2040 were considered for each technology. As suggested by EIA (2015b), comparing each technology's LACE to its LCOE may be used in order to determine which set of technologies provide the best net economic value for the capacity expansion planning, and is appropriate for situations where both dispatchable and non-dispatchable technology options are considered. Subtracting LACE from LCOE provides a surrogate measure associated with the generating cost per MWh of the corresponding technology.

The cost parameters will be likely dependent on land prices, and the impact of land price on cost is typically higher on those technologies requiring more extensive use of land. In order to adjust the cost parameter to a specific technology, the required land area to generate 1 MWh of electricity was taken from McDonald et al. (2009). To further adjust the cost to a specific location, the 2014 median price per acre at each potential location was taken from TAMU (2015). Mone et al. (2015) estimate that the contribution of land price on the LCOE of onshore wind technology is approximately 3%, so the variation of the cost for wind technology among all locations was restricted to 3% of the total range of variability on the cost from EIA (2015b) for wind. Taking wind as a reference case, the variations of the cost parameter for the remaining technologies were restricted from 1% to 15% of the corresponding total ranges of variability in EIA (2015b), proportionally to the land use of each technology. This is shown in Table 4.2.

Technology	**Cost parameter ($/MWh)**			**Water withdrawal**
	Lower limit	**Median**	**Upper limit**	**(gal/MWh)**
Conv. coal	12.62	12.8	12.98	660
Adv. coal	26.44	26.6	26.76	600
Conv. natural gas	3.24	3.3	3.36	6,000
Adv. natural gas	0.05	0.1	0.15	9,000
Nuclear	10.20	10.3	10.40	1,100
Biomass	11.47	13.9	16.33	450
Wind - onshore	2.61	3.4	4.19	1
Solar PV	15.41	16.1	16.79	6

Table 4.2: Water withdrawal requirements and range of variation for cost parameters

From Table 4.2, the cost associated with each technology at each location was adjusted by land prices. Table 4.3 shows the resulting cost parameters, c_{ij}, for each location and technology used in the experiments. Appendix B provides further details on the formula used to adjust the cost parameters.

LMA	Conv. Coal	Adv. Coal	Conv. N.G.	Adv. N.G.	Nu-clear	Bio-mass	Wind	Solar PV	Water scarcity
1	12.67	26.49	3.26	0.06	10.23	12.23	2.85	15.62	0.7508
2	12.65	26.47	3.25	0.05	10.22	11.87	2.74	15.52	0.7508
3	12.66	26.48	3.25	0.06	10.23	12.06	2.80	15.58	0.7508
4	12.64	26.46	3.25	0.05	10.22	11.77	2.71	15.49	0.7508
5	12.65	26.47	3.25	0.06	10.22	11.94	2.76	15.54	0.7508
6	12.64	26.47	3.25	0.05	10.22	11.80	2.72	15.50	0.7542
7	12.65	26.48	3.25	0.06	10.22	11.96	2.77	15.55	0.7542
8	12.62	26.44	3.24	0.05	10.20	11.47	2.61	15.41	0.7096
9	12.67	26.49	3.26	0.06	10.23	12.18	2.84	15.61	0.7100
10	12.78	26.58	3.29	0.09	10.29	13.64	3.32	16.03	0.7058
11	12.69	26.51	3.26	0.07	10.24	12.44	2.93	15.69	0.7058
12	12.67	26.49	3.26	0.06	10.24	12.25	2.86	15.63	0.6749
13	12.72	26.53	3.27	0.08	10.26	12.82	3.05	15.79	0.6749
14	12.75	26.55	3.28	0.08	10.27	13.19	3.17	15.90	0.7100
15	12.72	26.54	3.27	0.08	10.26	12.89	3.07	15.81	0.7100
16	12.86	26.65	3.32	0.12	10.33	14.74	3.67	16.34	0.7100
17	12.98	26.76	3.36	0.15	10.40	16.33	4.19	16.79	0.7100
18	12.83	26.62	3.31	0.11	10.31	14.26	3.52	16.20	0.7492
19	12.87	26.66	3.32	0.12	10.34	14.85	3.71	16.37	0.7492
20	12.75	26.56	3.28	0.09	10.28	13.27	3.19	15.92	0.7492
21	12.77	26.57	3.29	0.09	10.28	13.44	3.25	15.97	0.6892
22	12.79	26.59	3.30	0.10	10.30	13.78	3.36	16.06	0.6749
23	12.88	26.66	3.33	0.12	10.34	14.89	3.72	16.38	0.6749
24	12.76	26.57	3.29	0.09	10.28	13.43	3.25	15.97	0.6749
25	12.73	26.54	3.28	0.08	10.26	12.95	3.09	15.83	0.6749
26	12.84	26.64	3.31	0.11	10.32	14.49	3.59	16.27	0.7492
27	12.86	26.65	3.32	0.12	10.33	14.65	3.65	16.31	0.5379
28	12.96	26.74	3.35	0.15	10.38	16.02	4.09	16.70	0.6892
29	12.70	26.52	3.27	0.07	10.25	12.62	2.98	15.74	0.5379
30	12.73	26.54	3.28	0.08	10.27	13.03	3.12	15.85	0.5379
31	12.73	26.54	3.28	0.08	10.26	12.97	3.10	15.84	0.5379
32	12.75	26.56	3.28	0.09	10.28	13.27	3.19	15.92	0.5488
33	12.92	26.70	3.34	0.14	10.36	15.53	3.93	16.57	0.7096

Table 4.3: Cost ($/MWh) and water scarcity parameters used in the experimental analysis

In this study, the base case scenario for dispatchability requirement was set to

$p = 50\%$. This would enable the State of Texas to reach up to 17 % of its electricity generated from renewable sources, close to the leading States in the U.S. in the use of such technologies (EIA, 2011). For the sake of illustration, the level of uncertainty was assumed as $\alpha = 0.05$. Where indicated, different values for both the level of uncertainty and the dispatchability requirements were also tested. The BO-CEP was solved via the ϵ-constraint method using a step increment of 500 Kgal for the water objective. The ϵ-constraint method was programmed in C++ and CPLEX 12.6 was used as the solver. The robustness methodology was applied using the cardinality constrained uncertainty set with $\Gamma = 1$, as well as the box uncertainty set, in order to show the behavior of robust solutions in the most optimistic and the most conservative cases of robustness. The assessment of infeasibility and outcome degradation levels, as well as the filtering of non-robust solutions, was coded in C++.

4.4.2 Strategic analysis

This section presents the results on the experimental study and the analysis in light of the strategic questions mentioned before.

4.4.2.1 How does the trade-off among cost, water withdrawal and implementation uncertainty affect the technology and location selection of power plants?

The trade-off between cost and water withdrawal is illustrated in Figure 4.3, corresponding to 100 efficient solutions obtained via the ϵ-constraint method. The solutions that are robust to implementation uncertainty, as well as solutions that are optimal to cost and water objectives, are labeled with their corresponding solution indices.

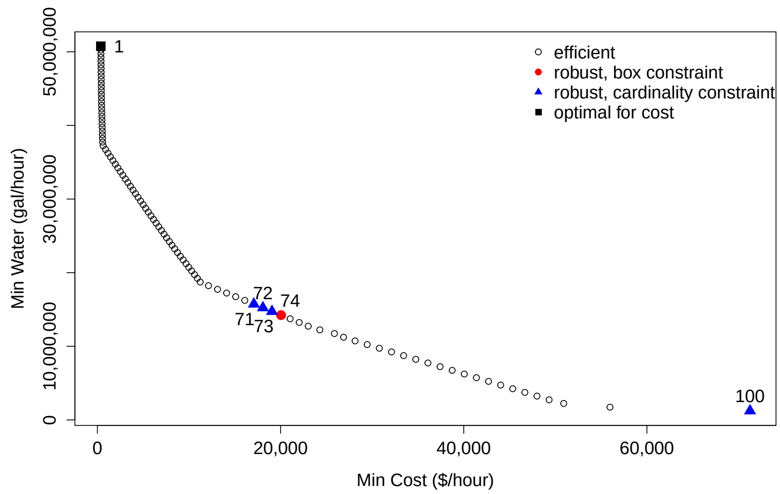
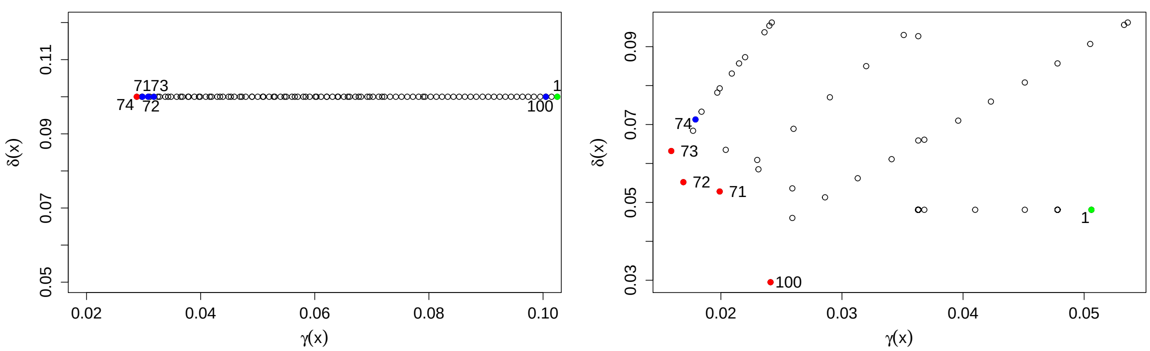


Figure 4.3: Trade-off among cost, water and robustness

The cost objective values in Figure 4.3 range from \$377/hour to \$71,259/hour, while water withdrawal objective values range from 1,232,059 gal/hour to 50,732,059 gal/hour. While there were 100 efficient solutions obtained via the ϵ-constraint method, the set of robust solutions contains only 1 solution in the case with the box uncertainty set and 4 solutions in the case with the cardinality constraint; this clearly illustrates how the proposed methodology helps the decision maker select promising solutions from a large set of efficient solutions. In the case with the box uncertainty set, only Solution 74 is found to be robust to implementation uncertainty. As per Theorem 2.4.5, this will be always the case. Solution 74 is a compromise between water and cost objectives lying close to the middle of the ranges of both objectives. In the case with the cardinality constraint, the set of robust solutions is given by: Solutions 71-73, which are a compromise between the two objective functions (similar to Solution 74); and Solution 100, which is also the optimal solution for water withdrawal. Figure 4.4 shows the trade-offs between infeasibility and outcome degradation levels.



Conservative case

(box uncertainty set)

Optimistic case

(cardinality constraint, $\Gamma = 1$)

Figure 4.4: Trade-off between infeasibility and outcome degradation levels

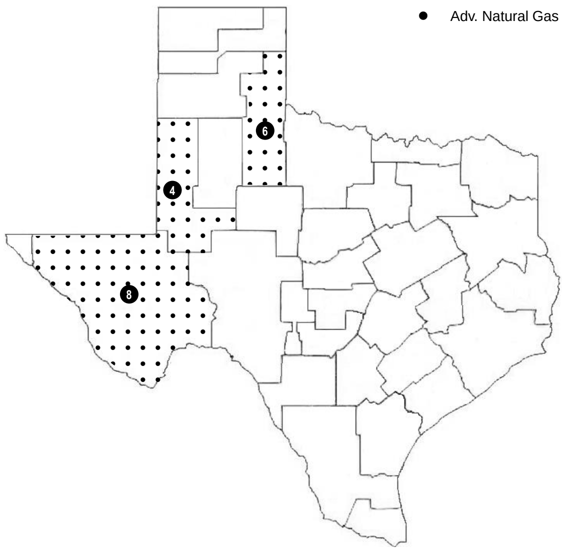
While the values of infeasibility and outcome degradation levels for Solutions 71-74 remain somewhat close, Solution 100 has a higher outcome degradation and a lower infeasibility level in the box uncertainty set case. For the purpose of illustration, Solution 1, which is optimal for cost, is also shown in Figure 4.4. Solution 1 is not robust neither in the box nor in the cardinality constrained cases; in fact, Solution 1 has the worst outcome degradation level when considering the case with the box uncertainty set. This suggests that selecting a solution purely on cost objective may yield an alternative that is highly sensitive to implementation uncertainty.

Figure 4.5 shows the locations and technologies for those solutions that are optimal for cost (Solution 1) and water (Solution 100), as well as the unique solution that is robust in the case with the box uncertainty (Solution 74). Solutions 71-73 (robust in the case with the cardinality constraint) prescribe the same locations and technologies as Solution 74, with the only difference being the designed capacities associated with the power plants. Table 4.4 presents the capacity prescribed by each solution for each technology and the associated objective values.

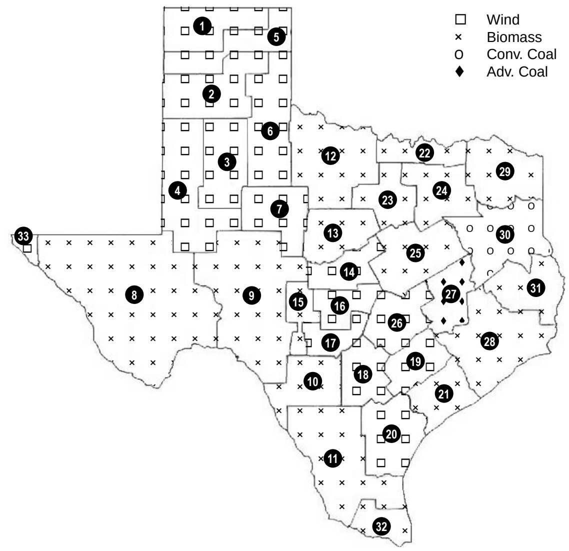
Capacity	**Sol.1**	**Sol. 71**	**Sol. 72**	**Sol. 73**	**Sol. 74**	**Sol. 100**
Conv. Coal (MW)	-	-	-	-	-	1,337
Adv. Coal (MW)	-	-	-	-	-	753
Conv. N.G. (MW)	-	2,090	2,446	2,802	3,158	-
Adv. N.G. (MW)	8,862	2,341	1,985	1,629	1,273	-
Biomass (MW)	-	-	-	-	-	2,504
Wind (MW)	-	10,708	10,708	10,708	10,708	10,708
Object. values						
$f_1(x)$ (\$/hour)	377	17,084	18,074	19,064	20,055	71,259
$f_2(x)$ (gal/hour)	50,732,059	15,732,059	15,232,059	14,732,059	14,232,059	1,232,059

Table 4.4: Additional capacity prescribed by solutions from the BO-CEP

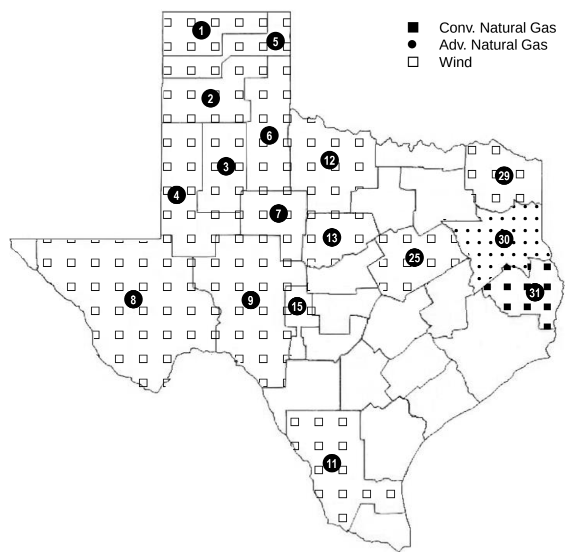
The cost-optimal Solution 1 prescribes the deployment of 3 advanced natural gas power plants. Although Solution 1 is optimal for cost, it is the worst efficient solution in terms of water withdrawal. The prescribed locations are within the regions most affected by droughts in Texas: the high plains and the low rolling plains, in the western half of Texas. In addition, Solution 1 is highly sensitive to uncertainty, as shown in Figure 4.4. These reasons would likely hinder its selection for implementation.



Solution 1: non-robust, cost-optimal



Solution 100: water-optimal, robust in the optimistic case (cardinality constr., $\Gamma = 1$)



Solutions 71-74: robust in the optimistic and conservative cases

Figure 4.5: Technologies and locations of solutions

The water-optimal Solution 100 prescribes the deployment of power plants in all of the 33 locations. This would likely require challenging coordination among several entities (public/private sector) for implementation. Although Solution 100 is optimal for water withdrawal and robust in the optimistic case, it is the worst efficient solution in terms of cost. In addition, it prescribes deploying new coal-fired power plants. Although not considered in the model, environmental regulations tend to limit the construction of new coal-fired facilities (e.g. Venkatesh et al., 2012); hence, the selection of such a solution would be unlikely.

The robust Solutions 71-74 prescribe the deployment of wind power plants across several locations in Texas, mostly clustered in the high plains and low rolling plains, hence avoiding any additional water withdrawal in areas where droughts occur more often. They also prescribe natural gas power plants in East Texas, where water resources tend to be less scarce.

Considering the different technologies among efficient solutions, Figure 4.6 shows that power plants prescribed with biomass, wind and advanced coal technologies tend to have lower design capacity values; this is mainly due to the associated design upper bound constraints (4.5) in the BO-CEP formulation. In contrast, advanced natural gas power plants are prescribed with a higher average capacity.

The experiments show that the total capacity from wind and advanced natural gas technologies influence the cost and water withdrawal associated with a solution. These are the predominant technologies within the set of efficient solutions; other technologies, such as nuclear and coal, are not as often prescribed. Figure 4.7 shows the effect of wind and advanced natural gas technologies on cost and water objectives. As the contribution of advanced natural gas on total capacity increases, the associated cost of the solution decreases, at the expense of higher water withdrawals. As wind capacity is increased, the associated cost of a solution is also increased, while

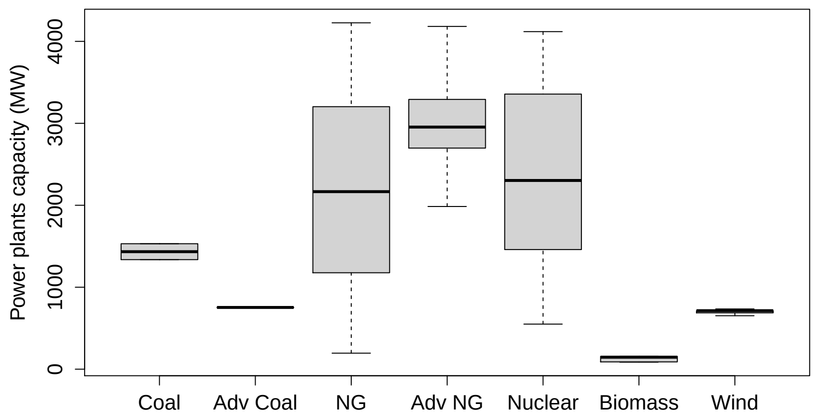
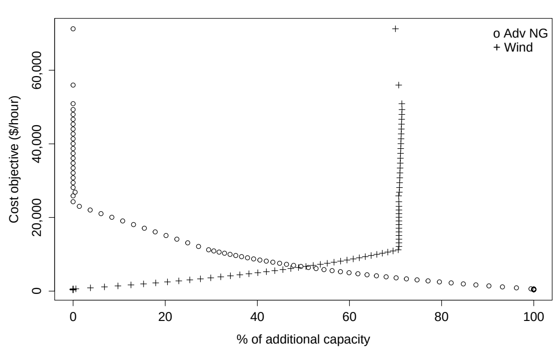


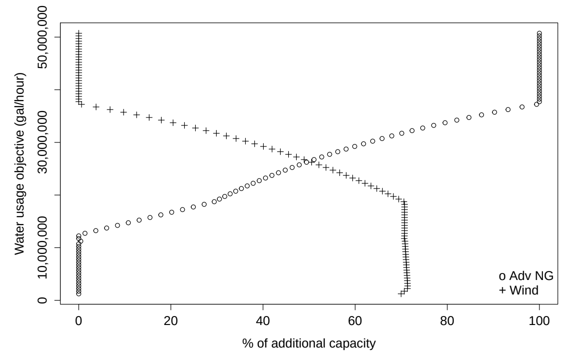
Figure 4.6: Power capacity per technology

water withdrawal is reduced. Of notice is that there are no cases where solar technology is prescribed. Although solar only requires marginal water withdrawals, its cost is higher than the one for wind technology (EIA, 2015b); arguably, this scenario could change if the associated costs are affected.

The solutions indicate different regions for deployment of the necessary power plants. Figure 4.8 shows the average capacity prescribed at each location, across all solutions. Figure 4.9 presents the allocation of capacity for each technology at the top 10 locations with higher average capacities from Figure 4.8. Solutions that select locations 29-31, in East Texas, predominantly rely on natural gas and other fossil-fueled technologies. In contrast, solutions that select locations 2-8, in West Texas, rely mainly on wind technology. While fossil-fueled power plants are prescribed with larger designed capacities and contribute towards minimizing cost and satisfying dispatchability requirements, wind power plants are allocated in regions where water resources are more scarce, contributing towards reducing total water withdrawal.



Effect on cost



Effect on water withdrawal

Figure 4.7: Effect of wind and advanced natural gas on cost and water

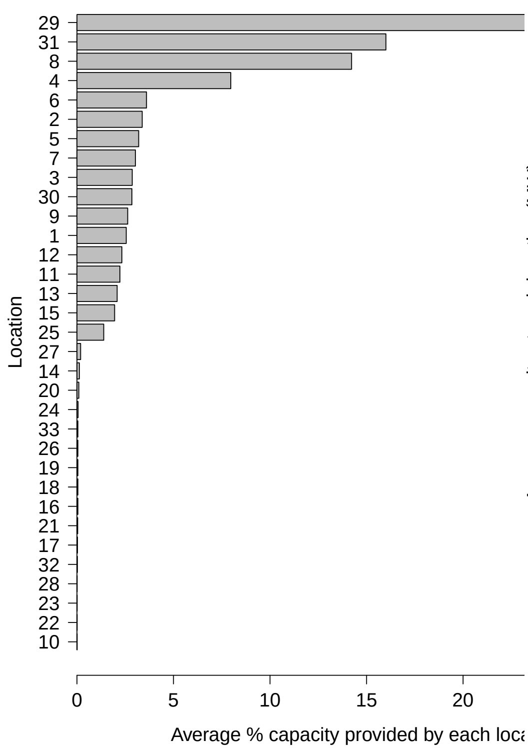


Figure 4.8: % total capacity per location

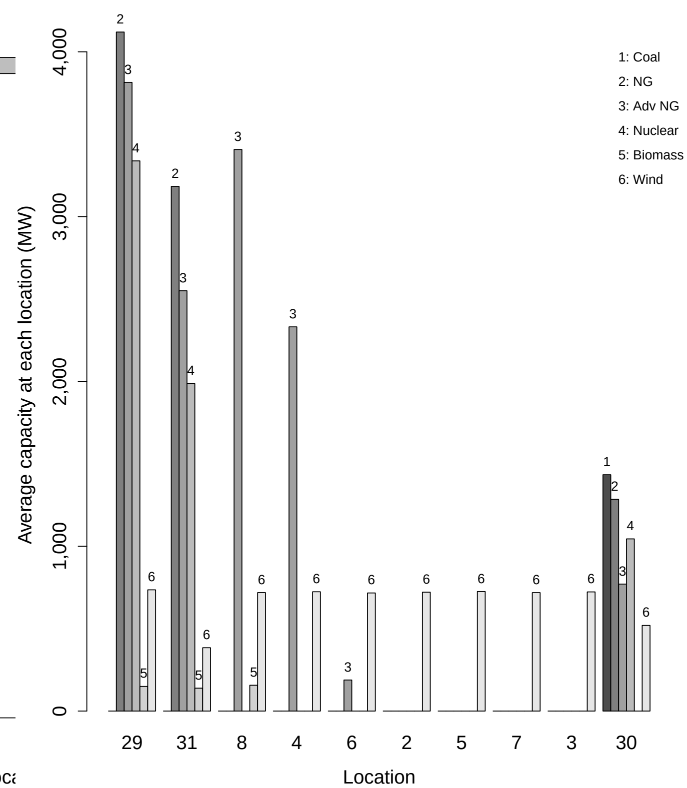


Figure 4.9: Capacity per location

4.4.2.2 *How do robust solutions compare to other efficient solutions when subjected to implementation uncertainty?*

To demonstrate advantages of using robustness measures as secondary criteria to further break ties among the many efficient solutions from the first-stage MOP, a simulation model was devised to randomly perturb the designed capacities of the power plants and assessed the actual infeasibility and outcome degradation levels. For each of the efficient solution previously found, a simulation trial consisted of randomly perturbing the designed capacity associated with each power plant by a multiplicative factor $\tilde{\beta} \sim U(1-\alpha, 1+\alpha)$ and computing the corresponding infeasibility and outcome degradation levels associated with the perturbed values. A total of 10,000 simulation trials were generated. Figure 4.10 presents the average simulated values of infeasibility and outcome degradation levels for each efficient solution.

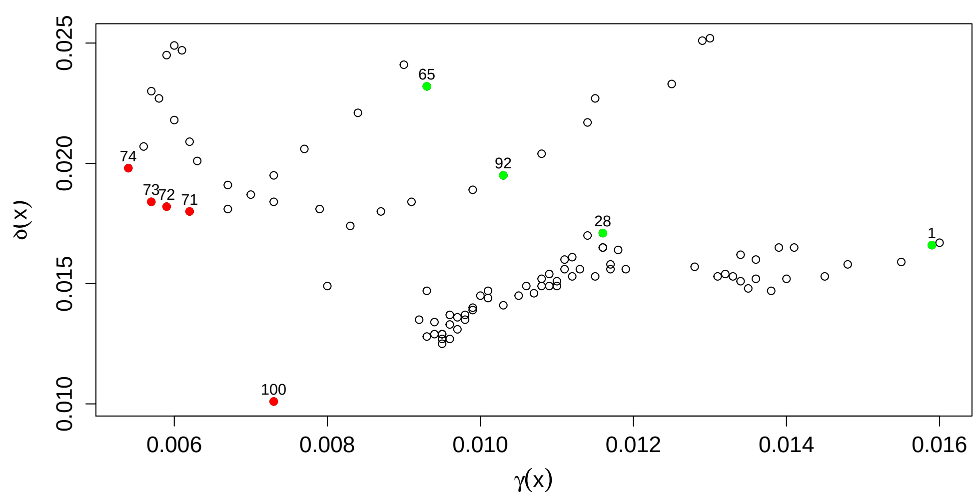
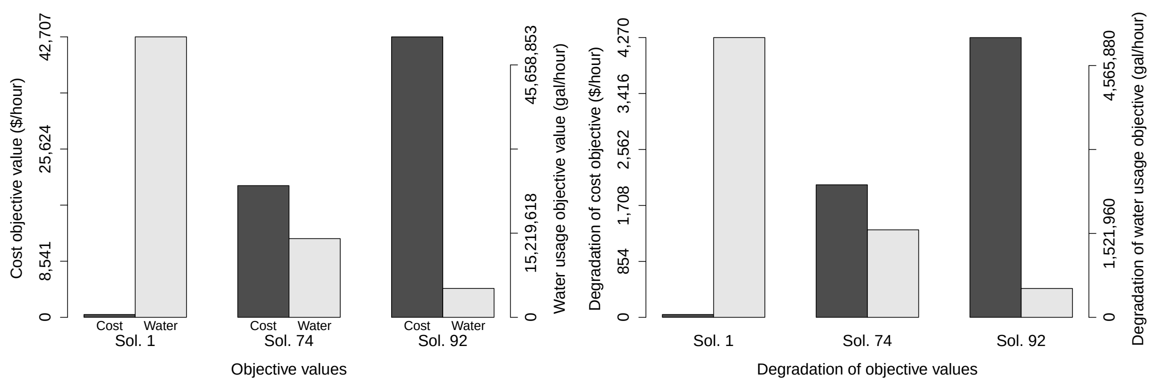


Figure 4.10: Average values for the simulation run

As shown in Figure 4.10, the set of solutions that are robust in the simulation run is given by Solutions 71-74 and 100. They are the union of robust solutions previously obtained from the different levels of decision maker's conservatism. This demonstrates that robust solutions tend to effectively maintain feasibility and stability of objective values. While selecting a most preferred solution out of the 100 efficient solutions may be challenging for decision making, the two-stage methodology narrowed the selection problem to only five solutions in the simulation run.

To emphasize the benefits of selecting from the set of robust solutions, consider two common sense approaches for finding a most preferred solution: the knee and the weighted sum approaches. In the knee approach (Branke et al., 2004), the decision maker selects from efficient solutions where a small improvement in one objective would lead to a large deterioration in at least on other objective. The two knee solutions in this example (Solutions 28 and 65) have their average simulated infeasibility and outcome degradation levels shown in Figure 4.10. These solutions tend to be highly sensitive to implementation uncertainties and are dominated by the robust Solution 100 in terms of infeasibility and outcome degradation levels in the simulation run. In the weighted sum approach (Steuer, 1986), the decision maker selects the efficient solution that minimizes the weighted sum of the objective functions, where the weights point to the direction of the gradient of the decision maker's utility function. The knee Solution 65 would also be the weighted sum optimal solution when assuming a weight factor of 0.5 for each objective function and normalizing the objective values to a range from zero to one. These examples show how common sense may lead decision making to solutions having high infeasibility and outcome degradation levels.

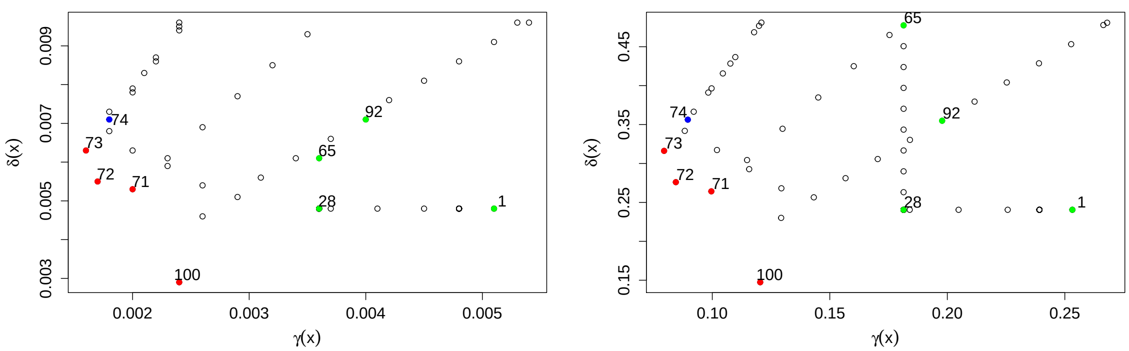
In order to compare the maximum degradation of cost and water objective values, consider the robust Solution 74, and the non-robust Solutions 1 and 92. Assuming the box uncertainty set, the cost and water objective values and their maximum degradation are shown in Figure 4.11. When subjected to implementation uncertainties, it is evident that the cost and water withdrawal are severely affected for Solution 92 and Solution 1, respectively. In contrast, the maximum degradation of objective values in robust Solution 74 is a balance between cost and water withdrawal, and so are the objective values of the solution. This suggests that using the proposed measures of robustness to break ties among technologies and locations tends to drive decision making in the conservative case towards solutions that are less affected by implementation uncertainty in a compromise between water and cost objectives.



Objective values Degradation of objective values

Figure 4.11: Cost and water objective values and their degradations

While using the proposed methodology to break ties among technologies and locations, of interest is the consideration of how sensitive the robust solutions are to changes in the level of uncertainty. In the case with the box uncertainty set, Theorem 2.4.6 guarantees that Solution 74 will remain robust for any value of α. In order to illustrate how the robust set is affected in the case with the cardinality constraint, the second stage problem was solved considering the cases of $\alpha = 0.01$ and $\alpha = 0.50$. The associated infeasibility and outcome degradation levels are shown in Figure 4.12. In both cases, the set of robust solutions is comprised by Solutions 71-73 and 100, the same as in the base case, when $\alpha = 0.05$ was assumed.



Low level of uncertainty ($\alpha = 0.01$) High level of uncertainty ($\alpha = 0.50$)

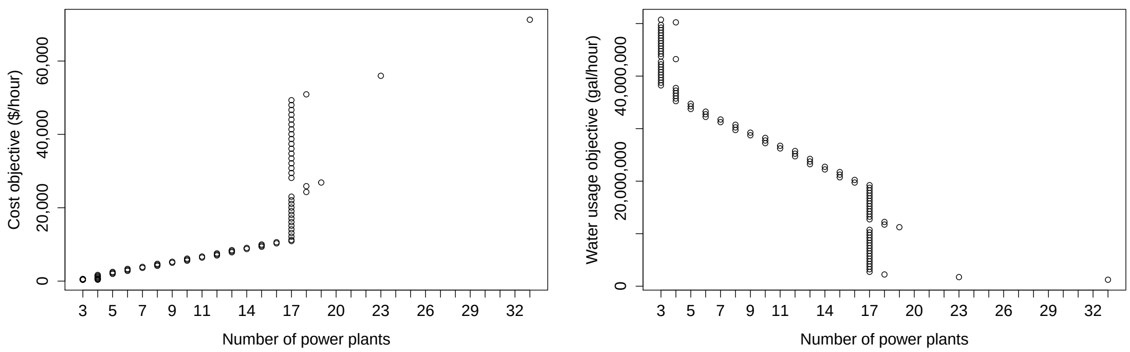
Figure 4.12: Robust solutions for different values α; optimistic case (cardinality constraint, $\Gamma = 1$)

In this experimental study, even when the cardinality-constrained uncertainty set is assumed, the corresponding set of robust solutions showed to be insensitive to implementation uncertainty. This suggests that the methodology may be effectively utilized even when the level of uncertainty, given by parameter α, is unknown.

4.4.2.3 What are the locations that should be prioritized for deploying the new power plants in order to minimize cost and water withdrawal? What is the minimum number of new power plants that should be built?

The analysis of the efficient solutions from the BO-CEP indicates that at least 3 power plants should be built in order to provide the additional capacity. Figure 4.13

shows that as the number of power plants increases, total cost increases, while water withdrawal tends to decrease.



Cost vs. number of power plants   Water withdrawal vs. number of power plants

Figure 4.13: Effect of the number of power plants on cost and water withdrawal

Figure 4.14 shows that the number of different technologies prescribed in a solution increases with the number of locations. When the number of locations is minimal, only one technology is prescribed: advanced natural gas; in this case, the solution will have a lower cost and a higher water withdrawal, as illustrated previously for Solution 1. Solutions that select a larger number of locations across Texas tend to rely on a larger number of technologies. For instance, this is the case of Solution 100, previously discussed.

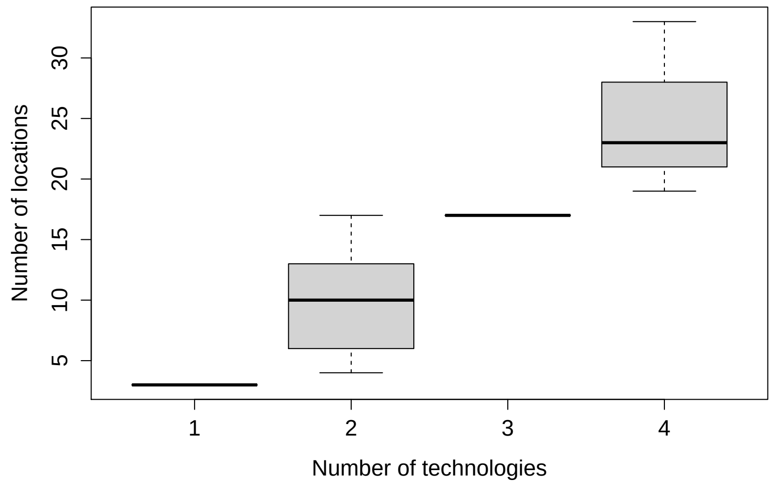


Figure 4.14: Number of different technologies and number of selected locations

The frequency that a location is selected varies across different regions, as shown in Figure 4.15. Regions having a higher land price, such as locations 17, 28 and 33, tend to be selected less frequently. The allocation of wind power plants tends to concentrate in the western areas of Texas, such as in locations 1-9. On the other hand, fossil-fueled technologies tend to cluster in locations 29-31, in East and Northeast Texas. This result is in agreement with the current distribution of power plants in Texas: the western half of the State, where droughts occur more often, rely more on wind technology, while eastern areas, which receive more rainfall, rely mostly on natural gas and coal-fired power plants (Stillwell et al., 2011).

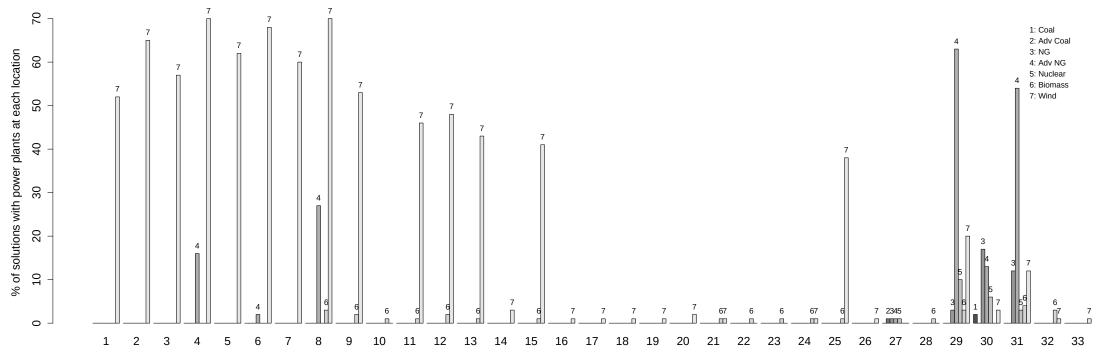


Figure 4.15: Frequency of selection of each location

4.4.2.4 What is the impact of demand uncertainty on the robust technology-location solutions?

A simulation-optimization model was developed in order to assess how electricity demand uncertainty can affect the set of solutions that are robust to implementation uncertainty. For the purpose of this study, it was assumed that demand is uniformly distributed centered at the nominal value $E = 67,539$ GWh, $\pm 20\%$. A simulation trial consisted of randomly generating a demand value from $U(0.8E, 1.2E)$, solving the associated instance of the BO-CEP and computing the infeasibility and outcome degradation levels of the corresponding efficient solutions. A total of 1,000 trials were generated.

Figures 4.16 and 4.17 show the frequency that each location and technology is prescribed in the set of robust solutions from the simulation run in the case with the box and the cardinality-constrained uncertainty sets, respectively.

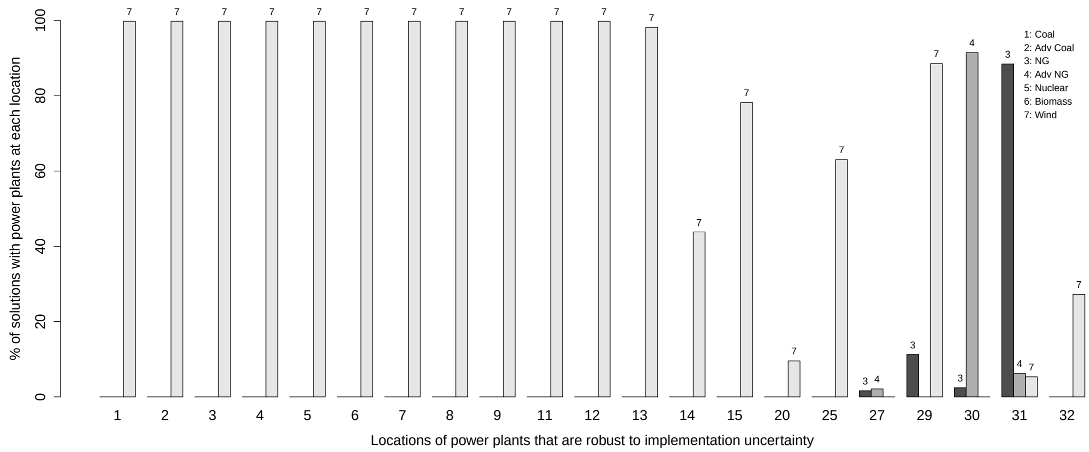


Figure 4.16: Robust solutions in the simulation run; conservative case (box uncertainty set)

In the box uncertainty set case, the robust solutions prescribe technologies and locations similar to the case when demand is assumed to be known: wind is assigned in locations 1-9, 11-15, 25 and 29, while natural gas is assigned in locations 30-31. There were cases where additional natural gas power plants were allocated in locations 27 and 29, when demand was higher than the average.

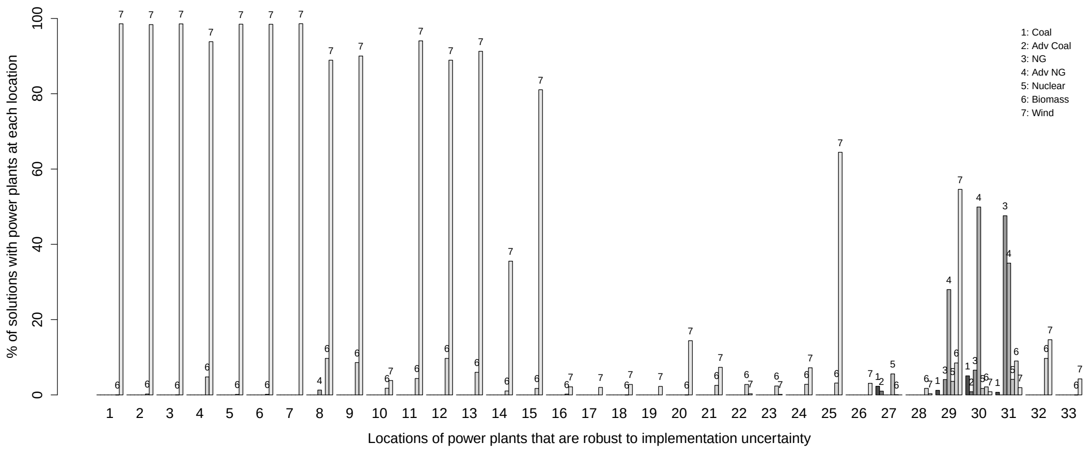


Figure 4.17: Robust solutions in the simulation run; optimistic case (cardinality constraint, $\Gamma = 1$)

In the cardinality constraint case, robust solutions are also similar to the ones when demand is assumed to be known: wind is prescribed across several regions, such as in locations 1-9 and 11-15, while natural gas is prescribed mainly in 29-31. A few cases of biomass are also prescribed, similar to Solution 100, when demand was assumed to be known. The main differences between the conservative and the optimistic cases in Figures 4.16 and 4.17, respectively, is that the former only contains solutions that are a compromise between water and cost objectives, while the later, in addition, contains solutions that are water-optimal, which prescribe power plants in all locations, as in the previous case with known demand.

The example illustrates how problem data uncertainty can be taken into account within the two-stage methodology, and suggests that the set of locations and technologies that are robust to implementation uncertainty tends to be insensitive to demand uncertainty.

4.4.2.5 *What is the impact of changing the dispatchability requirement of new power plants on the resulting technology-location solutions, as well as on cost and water withdrawal?*

In order to assess how dispatchability requirement can affect the allocation of technologies and locations, the BO-CEP was solved with $p = 0.3$ and $p = 0.7$, and the resulting solution sets were compared with the base case, when $p = 0.5$ was assumed.

Figure 4.18 shows the trade-off between cost and water withdrawal when the dispatchability requirement is set to $p = 0.3$ and $p = 0.7$. The effect of increasing the dispatchability requirement is higher on solutions having lower water withdrawals. This is because solutions that are close to the optimal cost mainly rely on dispatchable technologies; i.e., all power plants in Solution 1, previously discussed, are based on natural gas technology, hence are not affected by p.

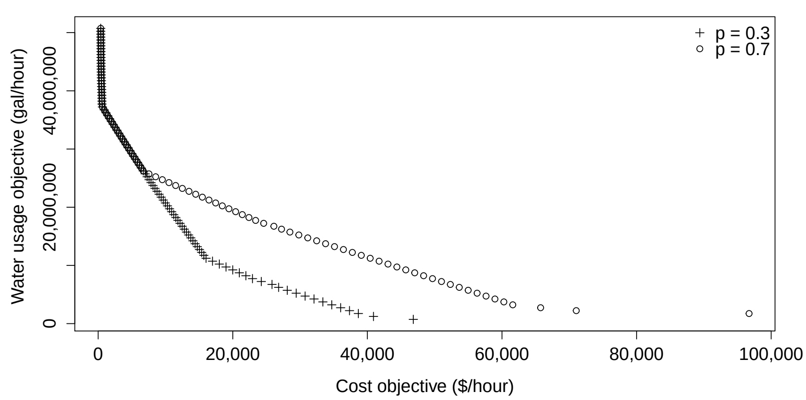
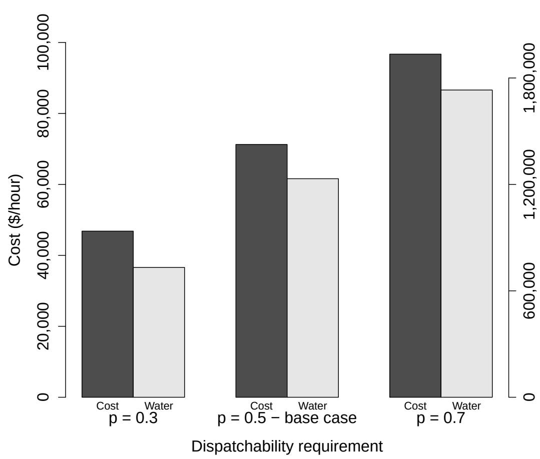
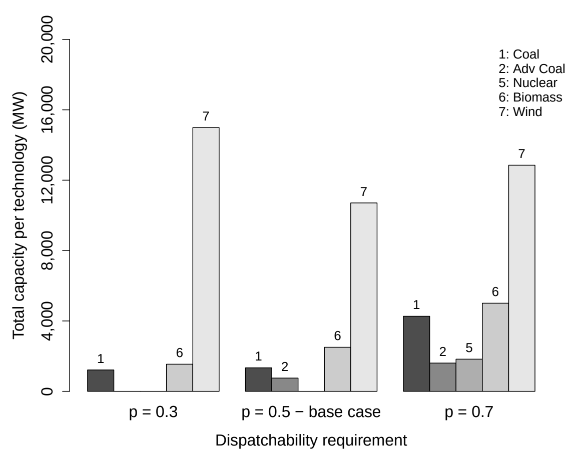


Figure 4.18: Trade-off between cost and water withdrawal for different values of p

Of notice is that the efficient solutions from the case where $p = 0.3$ tend to dominate the ones from the case where $p = 0.7$. In order to see why this is the case, Figure 4.19 examines the solutions having minimal water withdrawal for $p = 0.3$ and $p = 0.7$. For these solutions, cost and water withdrawal increase with p; in turn, a higher dispatchability requirement will lead to more capacity allocated to coal, natural gas, biomass and nuclear technologies, and lower capacity allocated to wind power plants. From Table 4.2, the cost parameters for coal, biomass and nuclear technologies are higher than the cost associated with wind technology. Therefore, the optimal solution for water objective in the case with $p = 0.7$ will be dominated by the optimal solution for water objective when $p = 0.3$.

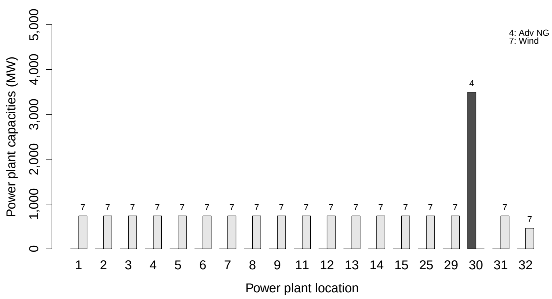




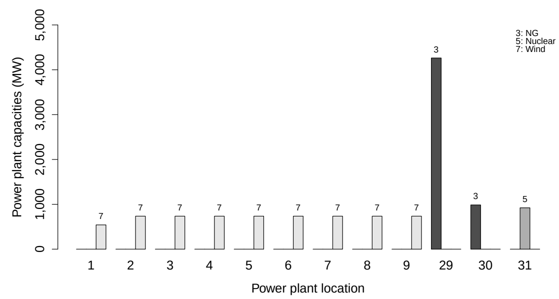
Cost and water withdrawal | Total capacity allocated for each technology

Figure 4.19: Solution with minimal water withdrawal for different values of p

Figure 4.20 shows how dispatchability requirement affect the locations of power plants associated with robust solutions in the case with the box uncertainty set. As in the base case (from robust Solution 74 previously discussed), wind power plants are prescribed in locations 1-9, even when p is changed. However, by increasing the dispatchability requirement, more capacity is allocated to natural gas facilities, and a nuclear power plant is prescribed in location 31.



Low dispatchability requirement ($p = 0.3$)



High dispatchability requirement ($p = 0.7$)

Figure 4.20: Robust solutions for different dispatchability requirements; conservative case (box uncertainty set)

Figures 4.21 and 4.22 summarize the set of robust solutions in the case with the cardinality-constrained uncertainty set, for $p = 0.3$ and $p = 0.7$, respectively. As before, wind power plants are prescribed in locations 1-9. By increasing dispatchability requirements, more capacity is allocated to biomass power plants, as well as nuclear facilities in locations 27 and 29. When $p = 0.3$, the robust solutions tend to allocate more capacity to wind power plants. In this case, the two natural gas facilities in locations 27 and 30 provide the necessary capacity from dispatchable technologies.

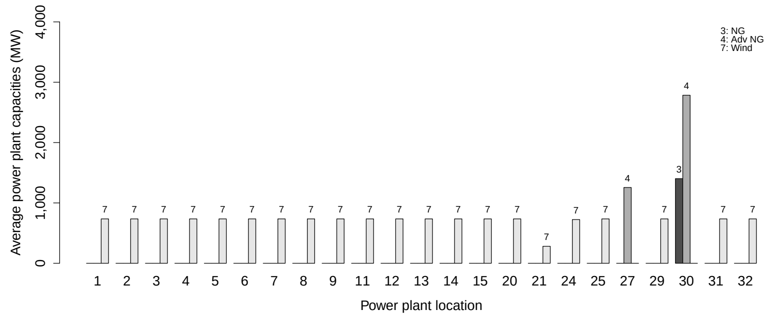


Figure 4.21: Robust solutions, low dispatchability ($p = 0.3$); optimistic case (cardinality constraint, $\Gamma = 1$)

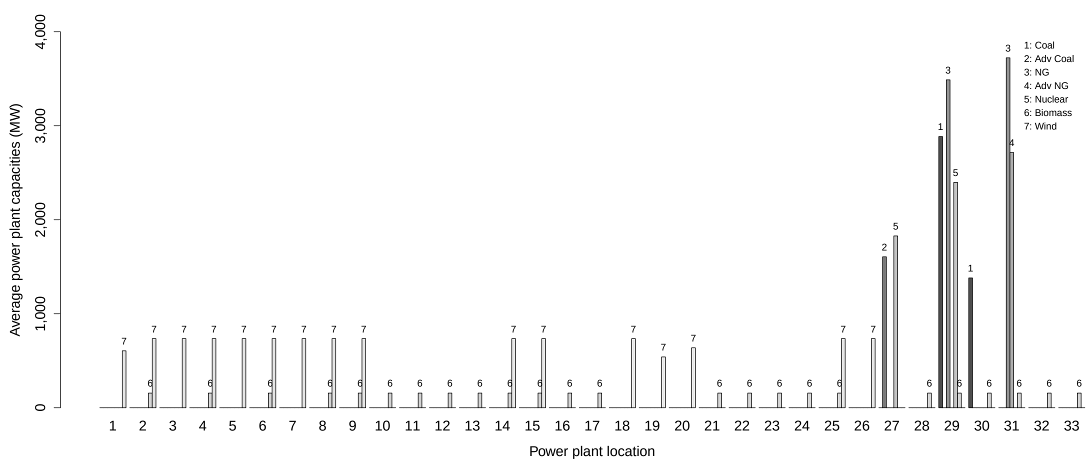


Figure 4.22: Robust solutions, high dispatchability ($p = 0.7$); optimistic case (cardinality constraint, $\Gamma = 1$)

4.5 Summary and conclusions

This section proposed a biobjective formulation for the electricity generation capacity expansion problem to minimize cost and water withdrawal. It showed how the methodology can be applied to find solutions that are robust when the designed capacities of the power plants are subjected to uncertainty at the time of their construction. In addition, the methodology mas applied to effectively break ties among efficient solutions by restricting decision making to the smaller subset of robust solutions.

The methodology was illustrated in a case study to analyze many strategic questions for capacity expansion in the State of Texas, USA. To the best of our knowledge, this was the first case study to consider both cost and water withdrawal objectives in capacity expansion planning. The results are encouraging for decision makers, as the solutions obtained from the methodology tend to maintain stability of objective values and remain feasible in both the optimistic and conservative cases of robustness. In addition, from Theorem 2.4.6, the methodology provides a set of solutions that is insensitive to the level of uncertainty when the worst case robustness is assumed (box uncertainty set). This result expands the applicability of robustness assessment in cases where the lack of information on the level of uncertainty would prevent the use of a classical robust optimization approach.

5. CONCLUSIONS AND FUTURE RESEARCH

This section begins by summarizing the objectives and main contributions from this dissertation. Finally, it provides directions on possible extensions and related future research.

5.1 Summary

Motivated by the difficulties associated with selecting a most preferred alternative from the many efficient solutions of an MOP, this dissertation aimed at developing a mathematical programming methodology to aid MCDM. The overarching contributions to the academic body of knowledge are as follows:

- The methodology provides rigorous foundations for the PSR problem, allowing for the consideration of additional secondary criteria to restrict trade-off analysis to a smaller set of the efficient solutions having desirable characteristics.

- The proposed secondary criteria consider characteristics of a solution associated with the objective and the decision spaces of the MOP, hence allowing for additional modeling information to be considered for trade-off analysis.

- The modeling of the PSR as an optimization problem over the efficient set allows for its reformulation as a mixed binary program. In turn, the developed solution procedure enables the handling of continuous efficient sets, circumventing a limitation that is common to the related PSR literature.

- The methodology was shown to effectively break-ties among efficient solutions in a case study applied to the electricity generation capacity expansion problem in the State of Texas, hence facilitating decision making.

Section 2 addressed research questions (a), (b) and (c) previously presented in Section 1.2.2. While it is unlikely that a solution remains both optimal and feasible when affected by uncertainty, the classical robust optimization approach focuses on finding solutions that remain feasible for all possible realizations within the uncertainty set. In contrast, the proposed methodology explicitly provides trade-off information between model and solution robustness, expanding the field of robustness assessment. While trade-off analysis in the original objective space of the MOP may be challenging due to the possibly high dimensionality, decision making can be further facilitated with trade-off analysis in the 2-dimensional robustness space. In addition, the section uncovered several theoretical properties of the problem in the case of MOLPs. The main implications from these properties are summarized in Table 5.1.

Section 3 focused on the research question (d) when the efficient set is associated with an MOLP. The main theoretical contribution in Section 3 is the reformulation of the PSR as a biobjective mixed binary linear program, which allowed for the development of a solution approach to solve the problem. The experimental study suggests that the proposed algorithm runs faster than existing methods for biobjective mixed binary problems, i.e. the TS algorithm. The main practical contribution in this section is that the solution approach allows the handling of continuous efficient sets, hence avoiding discretization procedures typically required in existing PSR literature. In addition, the solution approach allows for any linear secondary objective functions with nonnegative parameters to be considered.

Section 4 centered at the research question (e), and illustrated how the methodology may be applied in a real problem, such as in the electricity generation capacity expansion problem. While water and energy are interconnected issues, to the best of

Theoretical contribution	Practical contribution
Theorem 2.4.6: sufficient condition to guarantee that the set of robust solutions will be insensitive to the level of uncertainty	The methodology may be applied even when the practitioner has no knowledge about the uncertainty bounds
Theorem 2.4.5: sufficient condition to guarantee that the set of robust solutions will contain only one solution	The methodology effectively breaks ties among all solutions
Theorems 2.4.1 and 2.4.3: upper bounds on the values of $\delta(x)$ and $\gamma(x)$, regardless of the uncertainty set	The decision maker gains control over the robustness of solutions by adjusting the uncertainty level α. Conversely, given a maximum infeasibility and outcome degradation levels that are acceptable to the decision maker, the bounds ca be used to select a target value for α
Theorems 2.4.8−2.4.10: Probabilistic guarantees that the actual infeasibility and outcome degradation levels will be less than or equal to the values calculated using the cardinality-constrained formulation	The probabilistic guarantees may be utilized to determine an appropriate value for Γ according to the decision maker's level of conservatism

Table 5.1: Implications from main properties in Section 2

your knowledge, this was the first case study in this area to consider both the minimization of cost and water withdrawal. The study demonstrated how the methodology may aid MCDM by providing a reduced subset of robust solutions. In addition, experiments illustrated how the methodology may be utilized to aid in answering key strategic questions for expansion planning. The main findings show that:

- The proposed methodology was effective in identifying a smaller set of efficient solutions.
- Robust solutions identified by the methodology tend to effectively maintain feasibility and stability of objective values when compared to other solutions

in the efficient set.

- Using common sense approach for selecting a solution from the efficient set, such as the weighted sum or the knee approaches, may lead to a solution that is highly sensitive to uncertainties.

- A simulation-optimization model illustrates that the methodology can be integrated with a model that hedges against electricity demand uncertainty in the first stage MOP.

5.2 Future research

Two major areas of particular interest for future research are discussed next. The first area relates to the application of the proposed two-stage methodology. This would include different fields of application and the consideration of secondary criteria other than robustness to aid decision making. The second area relates to the development of solution algorithms and heuristic approaches for a variety of cases. Since the PSR is posed as an optimization problem over the efficient set, the problem is usually non-convex, hence specialized approaches become relevant for effective application of the methodology in large-scale problems. These opportunities are detailed next.

5.2.1 Applications

In the case of the electricity generation capacity expansion problem presented in Section 4, further analysis becomes necessary to enhance model fidelity and data accuracy to go beyond proof-of-concept. This would require detailed data at a location-level, such as:

- Sun radiation data: while the research assumed the national average values for power plant capacity factors, Texas is known for its high level of sun insolation, which may lead to higher-than-average capacity factors associated with solar power plants and potentially lower-than-average cost per MWh. The consideration of such data could make solar technologies part of the solution set in a detailed model. Sun radiation data could be considered from NREL (2015b).

- Wind speed data: the consideration of wind speed data at each location can be used to define a more precise capacity factor for wind power plants at each specific location. Wind speed data could be considered from EERE (2015).

- Energy crop availability: transportation of energy crops can be a large component of electricity production in biomass power plants. Hence, the cost parameter could be further enhanced by considering the distances from each power plant location to the available energy crop fields. Energy crop data could be considered from NREL (2015a).

- Groundwater availability: power plants may use groundwater reservoirs for cooling processes. The availability of such reservoirs should be taken into account in the computation of the scarcity factor in order to have a more precise evaluation of the limited availability of water resources at each location. Groundwater availability data could be considered from TWDB (2015a).

- Inclusion of other objective functions: while the case study focused on the minimization of water and cost, growing environmental concerns would require the additional consideration of greenhouse gas emissions in the expansion planning. Furthermore, in order to minimize risk associated with fluctuations on the cost per MWh, a technology diversity maximization objective function could be also considered. In this scenario, solar would likely become one of the prescribed technologies, as its water usage is minimal. Another advantage of considering such diversification of technologies is that wind speed may be lower during the day, hence the additional required electricity generation could be complemented by solar power plants.

- Geothermal resources: of interest is also the analysis of the geothermal potential from abandoned oil wells in the State of Texas for the purpose of electricity generation. As indicated in a study by Davis and Michaelides (2009), this may provide a sustainable source for capacity expansion in the future if the technology is further developed.

The aforementioned extensions should consider an interdisciplinary research team. For instance, this could involve collaboration with researchers from the Texas A&M Energy Institute, and different Departments, such as Geology, Civil Engineering and Petroleum Engineering. The detailed analysis could be valuable for governmental agencies in policy formation. Opportunities would include the DE-FOA-0001289 call for proposals from the Department of Energy Advanced Research Projects Agency-Energy (ARPA-E) in the area of network optimization for energy systems, and the PD 13-7607 call for proposals from the National Science Foundation (NSF) in the area of Energy, Power, Control and Networks (EPCN).

Although the proposed methodology was illustrated in the context of electricity generation capacity expansion planning, many other problems are also faced with implementation uncertainties and could benefit from the methodology. These include areas such as:

- Health-care: patient scheduling problems prescribe optimal time allocation for health-related procedures, but the actual arrival time of each patient may be subjected to uncertainty. The multiple objectives may include the maximization of room utilization (e.g. Silva et al., 2015) and the minimization of cost (e.g. Mak et al., 2015) and lead time between patient request and surgery date (e.g. Astaraky and Patrick, 2015), among others.

- Manufacturing: product design specifies physical characteristics for devices, but the actual values of the product characteristics are manufactured within a limited precision. Examples of such problems include the antenna and truss design (e.g. Ben-Tal and Nemirovski, 2002) and the vehicle design structure problem (e.g. Koch et al., 2004).

- Inventory control: inventory optimization problems specify optimal inventory levels, however unforeseen conditions not included in the model may force the decision makers to order a different replenishment quantity, affecting inventory levels. The multiple objectives may include profit and service level maximization (e.g. Chen et al., 2015b) and cost minimization (e.g. Bertsimas and Thiele, 2006b), among others.

While this research focused on robustness measures, other secondary criteria may be used in breaking ties among the efficient solution. In particular, the methodology may be applied in hierarchical decision making processes as it portrays a bilevel model, where the upper level decision maker requires efficiency of a solution, and the lower level introduces his/her additional secondary criteria in the selection of a most preferred alternative. In addition, the methodology may also be modified to the case of single objective optimization problems. The secondary criteria would provide means to break ties in the presence of alternative optimal solutions.

5.2.2 Algorithms and heuristics development

The solution algorithm presented in Section 3 is restricted to the case where the efficient set is associated with an MOLP. Nonetheless, extensions in the realm of integer, binary and mixed-integer programming should also be explored. In these cases, not only the solution algorithm will require special consideration, but also the modeling of uncertainty. For instance, in the case of binary variables, one possibility would be to model the perturbation factor as a permutation. In turn, this would allow the methodology to be extended to the area of reliability modeling; i.e. a binary decision variable that might fail would be represented by its value being perturbed to zero.

Regarding the LS-LPF algorithm, an immediate extension would be to further explore the opportunities for parallelization. Since the algorithm performs a search over the different regions defined by pairs of supported Pareto outcomes that are found to be disconnected, the process may be parallelized. The effectiveness of such parallelization should be studied. In addition, it may be challenging to solve larger instances with the LS-LPF algorithm because the PSR is a non-convex problem. Hence, of interest becomes the study of heuristic procedures.

Another avenue for future research would be the development of solution algorithms to solve the PSR when the efficient set is assumed to be unknown. This would likely require the exploration of bilevel programming algorithms. In this case, the challenge arises from the fact that the solution to the lower level problem, i.e. the optimization of secondary objective functions, is not uniquely determined, preventing the upper level problem to compute its optimal solution. When addressing multiple objectives in the bilevel program, the literature mainly deals with the case of multiobjective functions on the upper level problem (e.g. Yin, 2002; Ye, 2011; Long et al., 2014). The fewer cases to deal with a bilevel multiobjective program in both levels usually consider a simplification at the lower level by assuming pessimistic and optimistic scenarios, or a combination of both (e.g. Eichfelder, 2010; Dedzo et al., 2012; Dempe et al., 2013). In the optimistic scenario, it is assumed that the lower level will choose the solution that will optimize the upper level problem, while the pessimistic assumes that the lower level will choose the solution that will be the worst possible for the upper level. Hence, extensions in that area would contribute towards the body of knowledge in bilevel programing.

REFERENCES

Aguirre, O., Taboada, H., Coit, D., and Wattanapongsakorn, N. (2011). Multiple objective system reliability post-Pareto optimality using self organizing trees. In *IEEE International Conference on Quality and Reliability (2011 IEEE ICQR)*, pages 225–229, Bangkok. IEEE.

Alves, M. and Clímaco, J. (2007). A review of interactive methods for multiobjective integer and mixed-integer programming. *European Journal of Operational Research*, 180(1):99–115.

Alves, M. J. and Costa, J. P. (2009). An exact method for computing the nadir values in multiple objective linear programming. *European Journal of Operational Research*, 198(2):637–646.

Astaraky, D. and Patrick, J. (2015). A simulation based approximate dynamic programming approach to multi-class, multi-resource surgical scheduling. *European Journal of Operational Research*, 245(1):309–319.

Ben-Tal, A. and Den Hertog, D. (2011). Immunizing conic quadratic optimization problems against implementation errors. Technical Report 2011-060, Tilburg University.

Ben-Tal, A., El Ghaoui, L., and Nemirovski, A. (2009). *Robust optimization*. Princeton University Press, Princeton, NJ.

Ben-Tal, A. and Nemirovski, A. (2002). Robust optimization-methodology and applications. *Mathematical Programming*, 92(3):453–480.

Benson, H. and Sun, E. (2000). Outcome space partition of the weight set in multiobjective linear programming. *Journal of Optimization Theory and Applications*, 105(1):17–36.

Benson, H. P. (1984). Optimization over the efficient set. *Journal of Mathematical Analysis and Applications*, 98(2):562–580.

Benson, H. P. and Sayin, S. (1997). Towards finding global representations of the efficient set in multiple objective mathematical programming. *Naval Research Logistics (NRL)*, 44(1):47–67.

Benson, H. P. and Sun, E. (2002). A weight set decomposition algorithm for finding all efficient extreme points in the outcome set of a multiple objective linear program. *European Journal of Operational Research*, 139(1):26–41.

Bertsimas, D. and Brown, D. B. (2009). Constructing uncertainty sets for robust linear optimization. *Operations Research*, 57(6):1483–1495.

Bertsimas, D., Brown, D. B., and Caramanis, C. (2011). Theory and applications of robust optimization. *SIAM Review*, 53(3):464–501.

Bertsimas, D. and Sim, M. (2003). Robust discrete optimization and network flows. *Mathematical Programming*, 98(1):49–71.

Bertsimas, D. and Sim, M. (2004). The price of robustness. *Operations Research*, 52(1):35–53.

Bertsimas, D. and Thiele, A. (2006a). Robust and data-driven optimization: Modern decision making under uncertainty. In *Models, Methods, and Applications for Innovative Decision Making*, Tutorials in Operations Research, pages 95–122. INFORMS.

Bertsimas, D. and Thiele, A. (2006b). A robust optimization approach to inventory theory. *Operations Research*, pages 150–168.

Bérubé, J.-F., Gendreau, M., and Potvin, J.-Y. (2009). An exact epsilon-constraint method for bi-objective combinatorial optimization problems: Application to the traveling salesman problem with profits. *European Journal of Operational Research*, 194(1):39–50.

Birge, J. R. and Louveaux, F. (2011). *Introduction to stochastic programming.* Springer Science & Business Media.

Blackhurst, B. M., Hendrickson, C., and Vidal, J. S. I. (2010). Direct and indirect water withdrawals for US industrial sectors. *Environmental Science & Technology*, 44(6):2126–2130.

Boland, N., Charkhgard, H., and Savelsbergh, M. (2014). The triangle splitting method for biobjective mixed integer programming. In Lee, J. and Vygen, J., editors, *Integer Programming and Combinatorial Optimization*, volume 8494 of *Lecture Notes in Computer Science*, pages 162–173. Springer International Publishing.

Branke, J., Deb, K., Dierolf, H., and Osswald, M. (2004). Finding knees in multi-objective optimization. In Yao, X., Burke, E., Lozano, J., Smith, J., Merelo-Guervs, J., Bullinaria, J., Rowe, J., Tio, P., Kabn, A., and Schwefel, H.-P., editors, *Parallel Problem Solving from Nature - PPSN VIII*, volume 3242 of *Lecture Notes in Computer Science*, pages 722–731. Springer Berlin Heidelberg.

Carrillo, V. M. and Taboada, H. (2012). A post-Pareto approach for multi-objective decision making using a non-uniform weight generator method. *Procedia Computer Science*, 12:116–121.

Chankong, V. and Haimes, Y. Y. (1983). *Multiobjective decision making: theory and methodology.* Number 8. Elsevier Science Publishing, New York.

Charnes, A., Cooper, W. W., and Ferguson, R. O. (1955). Optimal estimation of executive compensation by linear programming. *Management Science*, 1(2):138–151.

Chaudry, M., Jenkins, N., Qadrdan, M., and Wu, J. (2014). Combined gas and electricity network expansion planning. *Applied Energy*, 113:1171–1187.

Chen, C. and Bulfin, R. (1993). Complexity of single machine, multi-criteria scheduling problems. *European Journal of Operational Research*, 70(1):115–125.

Chen, F., Huang, G., and Fan, Y. (2015a). A linearization and parameterization approach to tri-objective linear programming problems for power generation expansion planning. *Energy*, 87:240–250.

Chen, X., Feng, Y., Keblis, M. F., and Xu, J. (2015b). Optimal inventory policy for two substitutable products with customer service objectives. *European Journal of Operational Research*, 246(1):76–85.

Chernoff, H. (1952). A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations. *The Annals of Mathematical Statistics*, 23(4):493–507.

Das, I. (1999a). On characterizing the "knee" of the Pareto curve based on normal-boundary intersection. *Structural Optimization*, 18(2-3):107–115.

Das, I. (1999b). A preference ordering among various Pareto optimal alternatives. *Structural and Multidisciplinary Optimization*, 18(1):30–35.

Das, I. and Dennis, J. (1997). A closer look at drawbacks of minimizing weighted sums of objectives for pareto set generation in multicriteria optimization problems. *Structural Optimization*, 14(1):63–69.

Das, I. and Dennis, J. E. (1998). Normal-boundary intersection: A new method for generating the pareto surface in nonlinear multicriteria optimization problems. *SIAM Journal on Optimization*, 8(3):631–657.

Davis, A. P. and Michaelides, E. E. (2009). Geothermal power production from abandoned oil wells. *Energy*, 34(7):866–872.

Deb, K. and Gupta, H. (2006). Introducing robustness in multi-objective optimization. *Evolutionary Computation*, 14(4):463–494.

Deb, K., Pratap, A., Agarwal, S., and Meyarivan, T. (2002). A fast and elitist multiobjective genetic algorithm: Nsga-ii. *Evolutionary Computation, IEEE Transactions on*, 6(2):182–197.

Deb, K. and Saxena, D. (2005). On finding Pareto-optimal solutions through dimensionality reduction for certain large-dimensional multi-objective optimization problems. Technical Report 2005011, Kanpur Genetic Algorithms Laboratory (KanGAL).

Dedzo, F. F., Fotso, L. P., and Pieume, C. O. (2012). Solution concepts and new optimality conditions in bilevel multiobjective programming. *Applied Mathematics*, 3(10):1395–1402.

Dehghan, S., Amjady, N., and Kazemi, A. (2014). Two-stage robust generation expansion planning: A mixed integer linear programming model. *IEEE Transactions on Power Systems*, 29(2):584–597.

Dempe, S., Gadhi, N., and Zemkoho, A. (2013). New optimality conditions for the semivectorial bilevel optimization problem. *Journal of Optimization Theory and Applications*, 157(1):54–74.

EERE (2015). Texas Wind Resource Map and Potential Wind Capacity. U.S. Department of Energy. Energy Efficiency & Renewable Energy (EERE). Retrieved from: http://apps2.eere.energy.gov/wind/windexchange/wind_resource_maps.asp?stateab=tx. Accessed on: 2015-09-09.

Ehrgott, M. (2005). *Multicriteria Optimization*. Springer Science & Business Media Berlin.

Ehrgott, M., Ide, J., and Schöbel, A. (2014). Minmax robustness for multi-objective optimization problems. *European Journal of Operational Research*, 239(1):17–31.

Ehrgott, M., Puerto, J., and Rodriguez-Chia, A. (2007). Primal-dual simplex method for multiobjective linear programming. *Journal of Optimization Theory and Applications*, 134(3):483–497.

Ehrgott, M. and Tenfelde-Podehl, D. (2003). Computation of ideal and Nadir values and implications for their use in MCDM methods. *European Journal of Operational*

Research, 151(1):119–139.

EIA (2011). Power Plant Operations Report EIA-923, 2011. U.S. Department of Energy, U.S. Energy Information Administration (EIA). Retrieved from: http://www.eia.gov/electricity/data/eia923/index.html. Accessed on: 2015-07-25.

EIA (2013). Electricity Generating Capacity, 2011. U.S. Department of Energy, U.S. Energy Information Administration (EIA). Retrieved from: http://www.eia.gov/electricity/capacity/. Accessed on: 2015-07-20.

EIA (2015a). Annual energy outlook with projections to 2040. Technical Report DOE/EIA-0383(2015), U.S. Department of Energy, U.S. Energy Information Administration (EIA), Washington, DC.

EIA (2015b). Levelized cost and levelized avoided cost of new generation resources in the annual energy outlook 2015. Technical report, U.S. Department of Energy, U.S. Energy Information Administration (EIA), Washington, DC.

Eichfelder, G. (2010). Multiobjective bilevel optimization. *Mathematical Programming*, 123(2):419–449.

Engau, A. and Wiecek, M. M. (2007). 2D decision-making for multicriteria design optimization. *Structural and Multidisciplinary Optimization*, 34(4):301–315.

Engau, A. and Wiecek, M. M. (2008). Interactive coordination of objective decompositions in multiobjective programming. *Management Science*, 54(7):1350–1363.

ERCOT (2014). Hourly Aggregate Wind Output Database. Electric Reliability Council of Texas (ERCOT). Retrieved from: http://www.ercot.com/gridinfo/generation/index.html. Accessed on: 2015-07-23.

Eusébio, A., Figueira, J., and Ehrgott, M. (2014). On finding representative non-dominated points for bi-objective integer network flow problems. *Computers & Operations Research*, 48:1–10.

Faeth, P. (2013). In drought-prone Texas, a threat to the energy supply. Retrieved from: http://www.dallasnews.com/opinion/sunday-commentary/20131220-in-drought-prone-texas-a-threat-to-the-energy-supply.ece/. Accessed on: 2015-07-23.

Feng, Y. and Ryan, S. M. (2013). Scenario construction and reduction applied to stochastic power generation expansion planning. *Computers & Operations Research*, 40(1):9–23.

Fishburn, P. C. (1974). Lexicographic orders, utilities and decision rules: A survey. *Management Science*, 20(11):1442–1471.

Fortuny-Amat, J. and McCarl, B. (1981). A representation and economic interpretation of a two-level programming problem. *The Journal of the Operational Research Society*, 32(9):783–792.

Fülöp, J. (1993). On the equivalence between a linear bilevel programming problem and linear optimization over the efficient set. Technical Report Technical Report WP 93-1, Laboratory of Operations Research and Decision Systems, Computer and Automation Institute, Hungarian Academy of Sciences.

Galbraith, K. (2011). Drought could pose problems for Texas power plants. Retrieved from: http://www.texastribune.org/2011/09/16/drought-could-post-problems-texas-power-plants/. Accessed on: 2015-07-23.

Gass, S. and Saaty, T. (1955). The computational algorithm for the parametric objective function. *Naval Research Logistics Quarterly*, 2(1-2):39–45.

Gleick, P. H. (1994). Water and energy. *Annual Review of Energy and the Environment*, 19(1):267–299.

Goberna, M., Jeyakumar, V., Li, G., and Vicente-Pérez, J. (2015). Robust solutions to multi-objective linear programs with uncertain data. *European Journal of Operational Research*, 242(3):730–743.

Gunawan, S. and Azarm, S. (2005). Multi-objective robust optimization using a sensitivity region concept. *Structural and Multidisciplinary Optimization*, 29(1):50–60.

Haimes, Y. Y., Lasdon, L. S., and Wismer, D. A. (1971). On a bicriterion formulation of the problems of integrated system identification and system optimization. *IEEE Transactions on Systems, Man and Cybernetics*, SMC-1(3):296–297.

Heller, I. and Tompkins, C. (1956). An extension of a theorem of Dantzig's. In Kuhn, H. W. and Tucker, A. W., editors, *Annals of Mathematics Studies*, volume 38 of *Linear Inequalities and Related System*, pages 247–254. Princeton: Princeton University Press, NJ.

Hoeffding, W. (1963). Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association*, 58(301):13–30.

Horst, R. and Thoai, N. V. (1999). Maximizing a concave function over the efficient or weakly-efficient set. *European Journal of Operational Research*, 117(2):239–252.

Iancu, D. A. and Trichakis, N. (2013). Pareto efficiency in robust optimization. *Management Science*, 60(1):130–147.

Jin, S., Ryan, S., Watson, J.-P., and Woodruff, D. (2011). Modeling and solving a large-scale generation expansion planning problem under uncertainty. *Energy Systems*, 2(3-4):209–242.

Jorge, J. M. (2009). An algorithm for optimizing a linear function over an integer efficient set. *European Journal of Operational Research*, 195(1):98–103.

Joskow, P. L. (2011). Comparing the costs of intermittent and dispatchable electricity generating technologies. *The American Economic Review*, 101(3):238–241.

Kao, G. K. and Jacobson, S. H. (2008). Finding preferred subsets of Pareto optimal solutions. *Computational Optimization and Applications*, 40(1):73–95.

Karasakal, E. and Köksalan, M. (2009). Generating a representative subset of the nondominated frontier in multiple criteria decision making. *Operations Research*, 57(1):187–199.

Kenny, J. F., Barber, N. L., Hutson, S. S., Linsey, K. S., Lovelace, J. K., and Maupin,

M. A. (2009). Estimated use of water in the United States in 2005. Technical Report Circular 1344, U.S. Geological Survey.

Koch, P., Yang, R.-J., and Gu, L. (2004). Design for six sigma through robust optimization. *Structural and Multidisciplinary Optimization*, 26(3-4):235–248.

Kouvelis, P. and Yu, G. (1997). *Robust discrete optimization and its applications*, volume 14 of *Nonconvex Optimization and Its Applications*. Kluwer Academic Publishers, Boston.

Laumanns, M., Thiele, L., and Zitzler, E. (2006). An efficient, adaptive parameter variation scheme for metaheuristics based on the epsilon-constraint method. *European Journal of Operational Research*, 169(3):932–942.

Long, J., Szeto, W., and Huang, H.-J. (2014). A bi-objective turning restriction design problem in urban road networks. *European Journal of Operational Research*, 237(2):426–439.

Lowe, T., Thisse, J.-F., Ward, J., and Wendell, R. (1984). On efficient solutions to multiple objective mathematical programs. *Management Science*, 30(11):1346–1349.

Luque, M., Ruiz, F., and Miettinen, K. (2011). Global formulation for interactive multiobjective optimization. *OR Spectrum*, 33(1):27–48.

Luque, M., Ruiz, F., and Steuer, R. E. (2010). Modified interactive chebyshev algorithm (MICA) for convex multiobjective programming. *European Journal of Operational Research*, 204(3):557–564.

Macknick, J., Newmark, R., Heath, G., and Hallett, K. (2011). A review of operational water consumption and withdrawal factors for electricity generating technologies. Technical Report NREL/TP-6A20-50900, National Renewable Energy Laboratory, Colorado.

Macknick, J., Newmark, R., Heath, G., and Hallett, K. C. (2012). Operational water consumption and withdrawal factors for electricity generating technologies: a review of existing literature. *Environmental Research Letters*, 7(4):045802.

Mak, H.-Y., Rong, Y., and Zhang, J. (2015). Appointment scheduling with limited distributional information. *Management Science*, 61(2):316–334.

Malcolm, S. A. and Zenios, S. A. (1994). Robust optimization for power systems capacity expansion under uncertainty. *The Journal of the Operational Research Society*, 45(9):pp. 1040–1049.

Masin, M. and Bukchin, Y. (2008). Diversity maximization approach for multiobjective optimization. *Operations Research*, 56(2):411–424.

Mavrotas, G. and Diakoulaki, D. (1998). A branch and bound algorithm for mixed zero-one multiple objective linear programming. *European Journal of Operational Research*, 107(3):530–541.

Mavrotas, G. and Diakoulaki, D. (2005). Multi-criteria branch and bound: A vector maximization algorithm for mixed 0-1 multiple objective linear programming. *Applied Mathematics and Computation*, 171(1):53–71.

Mavrotas, G., Figueira, J. R., and Siskos, E. (2015a). Robustness analysis methodology for multi-objective combinatorial optimization problems and application to project selection. *Omega*, 52(0):142–155.

Mavrotas, G., Pechak, O., Siskos, E., Doukas, H., and Psarras, J. (2015b). Robustness analysis in multi-objective mathematical programming using Monte Carlo simulation. *European Journal of Operational Research*, 240(1):193–201.

McDonald, R. I., Fargione, J., Kiesecker, J., Miller, W. M., Powell, J., et al. (2009). Energy sprawl or energy efficiency: climate policy impacts on natural habitat for the United States of America. *PLoS One*, 4(8):e6802.

Meldrum, J., Nettles-Anderson, S., Heath, G., and Macknick, J. (2013). Life cycle water use for electricity generation: a review and harmonization of literature estimates. *Environmental Research Letters*, 8(1):015031.

Miettinen, K. (1999). *Nonlinear multiobjective optimization*, volume 12 of *International Series in Operations Research & Management Science*. Kluwer Academic Publishers, Boston.

Mone, C., Smith, A., Maples, B., and Hand, M. (2015). 2013 cost of wind energy review. Technical Report NREL/TP-5000-63267, National Renewable Energy Laboratory (NREL), Golden, CO.

Morse, J. N. (1980). Reducing the size of the nondominated set: Pruning by clustering. *Computers & Operations Research*, 7(1):55–66.

Mulvey, J. M., Vanderbei, R. J., and Zenios, S. A. (1995). Robust optimization of large-scale systems. *Operations Research*, 43(2):264–281.

Murphy, F., Sen, S., and Soyster, A. (1982). Electric utility capacity expansion planning with uncertain load forecasts. *IIE Transactions*, 14(1):52–59.

Mustakerov, I. and Borissova, D. (2011). *Wind park layout design using combinatorial optimization*. INTECH Open Access Publisher.

Naccache, P. (1978). Connectedness of the set of nondominated outcomes in multicriteria optimization. *Journal of Optimization Theory and Applications*, 25(3):459–467.

NREL (2015a). Current Biomass Resource Supply. National Renewable Energy Laboratory (NREL). Retrieved from: http://www.nrel.gov/rredc/biomass_data.html. Accessed on: 2015-09-09.

NREL (2015b). National Solar Radiation Data Base. National Renewable Energy Laboratory (NREL). Retrieved from: http://rredc.nrel.gov/solar/old_data/nsrdb/. Accessed on: 2015-09-09.

Okamoto, M. (1959). Some inequalities relating to the partial sum of binomial probabilities. *Annals of the Institute of Statistical Mathematics*, 10(1):29–35.

Przybylski, A., Gandibleux, X., and Ehrgott, M. (2008). Two phase algorithms for the bi-objective assignment problem. *European Journal of Operational Research*, 185(2):509–533.

Rabensteiner, G. (1987). Sensitivity of economic objectives in strategic dynamic power system planning. *Engineering Costs and Production Economics*, 12(1-4):251–258.

Rentizelas, A. A. and Tatsiopoulos, I. P. (2010). Locating a bioenergy facility using a hybrid optimization method. *International Journal of Production Economics*, 123(1):196–209.

Rockafellar, R. T. (1970). Convex Analysis (Princeton Mathematical Series). *Princeton University Press*, 46:162.

Rong, A., Figueira, J. R., and Lahdelma, R. (2015). A two phase approach for the bi-objective non-convex combined heat and power production planning problem. *European Journal of Operational Research*, 245(1):296–308.

Rosenman, M. and Gero, J. (1985). Reducing the Pareto optimal set in multicriteria optimization (with applications to Pareto optimal dynamic programming). *Engineering Optimization*, 8(3):189–206.

Sawaragi, Y., Nakayam, H., and Tanino, T. (1985). *Theory of multiobjective optimization*. Mathematics in science and engineering: a series of monographs and texbooks. Elsevier, Burlington, MA.

Sayin, S. (1996). An algorithm based on facial decomposition for finding the efficient set in multiple objective linear programming. *Operations Research Letters*, 19(2):87–94.

Sayin, S. (2000). Optimizing over the efficient set using a top-down search of faces. *Operations Research*, 48(1):65–72.

Scanlon, B. R., Duncan, I., and Reedy, R. C. (2013). Drought and the water–energy nexus in Texas. *Environmental Research Letters*, 8(4):045033.

Silva, T. A., de Souza, M. C., Saldanha, R. R., and Burke, E. K. (2015). Surgical scheduling with simultaneous employment of specialised human resources. *European Journal of Operational Research*, 245(3):719–730.

Smith, K. I., Everson, R. M., Fieldsend, J. E., Murphy, C., and Misra, R. (2008). Dominance-based multiobjective simulated annealing. *IEEE Transactions on Evolutionary Computation*, 12(3):323–342.

Soylu, B. (2015). Heuristic approaches for biobjective mixed 0-1 integer linear programming problems. *European Journal of Operational Research*, 245(3):690–703.

Steiner, S. and Radzik, T. (2008). Computing all efficient solutions of the biobjective minimum spanning tree problem. *Computers & Operations Research*, 35(1):198–211. Part Special Issue: Applications of OR in Finance.

Steuer, R. (1986). *Multiple Criteria Optimization: Theory, Computation and Application*. Probability and Mathematical Statistics. John Wiley & Sons, New York.

Steuer, R. E. (1994). Random problem generation and the computation of efficient extreme points in multiple objective linear programming. *Computational Optimization and Applications*, 3(4):333–347.

Steuer, R. E. and Choo, E.-U. (1983). An interactive weighted tchebycheff procedure for multiple objective programming. *Mathematical Programming*, 26(3):326–344.

Steuer, R. E. and Harris, F. W. (1980). Intra-set point generation and filtering in decision and criterion space. *Computers & Operations Research*, 7(1):41–53.

Stidsen, T., Andersen, K. A., and Dammann, B. (2014). A branch and bound algorithm for a class of biobjective mixed integer programs. *Management Science*, 60(4):1009–1032.

Stillwell, A. S., King, C. W., Webber, M. E., Duncan, I. J., and Hardberger, A. (2011). The energy-water nexus in Texas. *Ecology and Society*, 16(1):2.

Stoer, J., Witzgall, C., Stoer, J., and Stoer, J. (1970). *Convexity and Optimization in Finite Dimensions*, volume 1. Springer-Verlag Berlin.

Stoyan, S. J. and Dessouky, M. M. (2012). A stochastic mixed-integer programming approach to the energy-technology management problem. *Computers & Industrial Engineering*, 63(3):594–606.

Stults, E. S. (2015). *Minimizing Water Requirements for Electricity Generation in Water Scarce Areas*. PhD thesis, Worcester Polytechnic Institute.

Taboada, H. A. and Coit, D. W. (2008). Multi-objective scheduling problems: Determination of pruned Pareto sets. *IIE Transactions*, 40(5):552–564.

TAMU (2015). Texas Rural Land. Real Estate Center. Texas A&M University (TAMU). Retrieved from: http://recenter.tamu.edu/data/rland/. Accessed on: 2015-07-02.

Teghem, J. and Kunsch, P. (1985). Application of multi-objective stochastic linear programming to power systems planning. *Engineering Costs and Production Economics*, 9(1-3):83–89.

Tekiner, H., Coit, D. W., and Felder, F. A. (2010). Multi-period multi-objective electricity generation expansion planning problem with Monte-Carlo simulation. *Electric Power Systems Research*, 80(12):1394–1405.

Thang, T. (2015). Outcome-based branch and bound algorithm for optimization over the efficient set and its application. In Dang, Q. A., Nguyen, X. H., Le, H. B., Nguyen, V. H., and Bao, V. N. Q., editors, *Some Current Advanced Researches on Information and Computer Science in Vietnam*, volume 341 of *Advances in Intelligent Systems and Computing*, pages 31–47. Springer International Publishing.

Tolis, A. I. and Rentizelas, A. A. (2011). An impact assessment of electricity and emission allowances pricing in optimised expansion planning of power sector portfolios. *Applied Energy*, 88(11):3791–3806.

TWDB (2015a). Currently Active Wells. Texas Water Development Board (TWDB). Retrieved from: http://www.waterdatafortexas.org/groundwater/. Accessed on: 2015-09-09.

TWDB (2015b). Palmer Drought Severity Index. Texas Water Development Board (TWDB). Retrieved from: http://waterdatafortexas.org/drought/pdsi/monthly?time=2015-06. Accessed on: 2015-07-10.

Vafaeyan, S. and Thibault, J. (2009). Selection of Pareto-optimal solutions for process optimization using rough set method: A new approach. *Computers & Chemical Engineering*, 33(11):1814–1825.

Van Wassenhove, L. and Gelders, L. (1980). Solving a bicriterion scheduling problem. *European Journal of Operational Research*, 4(1):42–48.

Vaz, D., Paquete, L., Fonseca, C. M., Klamroth, K., and Stiglmayr, M. (2015). Representation of the non-dominated set in biobjective discrete optimization. *Computers & Operations Research*, 63:172–186.

Venkat, V., Jacobson, S. H., and Stori, J. A. (2004). A post-optimality analysis algorithm for multi-objective optimization. *Computational Optimization and Applications*, 28(3):357–372.

Venkatesh, A., Jaramillo, P., Griffin, W. M., and Matthews, H. S. (2012). Implications of near-term coal power plant retirement for SO2 and NOX and life cycle GHG emissions. *Environmental Science & Technology*, 46(18):9838–9845.

Vincent, T., Seipp, F., Ruzika, S., Przybylski, A., and Gandibleux, X. (2013). Multiple objective branch and bound for mixed 0-1 linear programming: Corrections and improvements for the biobjective case. *Computers & Operations Research*, 40(1):498–509.

Visée, M., Teghem, J., Pirlot, M., and Ulungu, E. (1998). Two-phases method and branch and bound procedures to solve the bi-objective knapsack problem. *Journal of Global Optimization*, 12(2):139–155.

Walmsley, M. R., Walmsley, T. G., Atkins, M. J., Kamp, P. J., and Neale, J. R. (2014). Minimising carbon emissions and energy expended for electricity generation in New Zealand through to 2050. *Applied Energy*, 135:656–665.

Warburton, A. (1987). Approximation of pareto optima in multiple-objective, shortest-path problems. *Operations Research*, 35(1):70–79.

Yamamoto, Y. (2002). Optimization over the efficient set: overview. *Journal of Global Optimization*, 22(1-4):285–317.

Ye, J. J. (2011). Necessary optimality conditions for multiobjective bilevel programs. *Mathematics of Operations Research*, 36(1):165–184.

Yin, Y. (2002). Multiobjective bilevel optimization for transportation planning and management problems. *Journal of Advanced Transportation*, 36(1):93–105.

Yu, P. and Zeleny, M. (1975). The set of all nondominated solutions in linear cases and a multicriteria simplex method. *Journal of Mathematical Analysis and Applications*, 49(2):430–468.

Zadeh, L. (1963). Optimality and non-scalar-valued performance criteria. *IEEE Transactions on Automatic Control*, 8(1):59–60.

Zeleny, M. (1982). *Multiple criteria decision making*, volume 25. McGraw-Hill New York.

Zhang, Q., Mclellan, B. C., Tezuka, T., and Ishihara, K. N. (2013). An integrated model for long-term power generation planning toward future smart electricity systems. *Applied Energy*, 112:1424–1437.

Zitzler, E., Laumanns, M., Thiele, L., Zitzler, E., Zitzler, E., Thiele, L., and Thiele, L. (2001). Spea2: Improving the strength pareto evolutionary algorithm.

APPENDIX A

MB-P1 FORMULATION OF INSTANCE N6M5O2

$$
\begin{aligned}
\text{Min}\ \ & (\delta,\gamma)^T \\
\text{s.t.}\ \ & 2x_1+3x_2+5x_3+3x_4+4x_5+4x_6 <= 105 \\
& -2x_1+x_2+3x_3-x_4+x_5-4x_6 <= -10 \\
& 4x_3-3x_4-3x_6 <= -10 \\
& -2x_2+5x_3+3x_4+4x_5 <= 50 \\
& 4x_1-3x_2-4x_3+x_4-2x_5 <= -20 \\
& -105\delta+2.2x_1+3.3x_2+5.5x_3+3.3x_4+4.4x_5+4.4x_6 <= 105 \\
& -10\delta-1.8x_1+1.1x_2+3.3x_3-0.9x_4+1.1x_5-3.6x_6 <= -10 \\
& -10\delta+4.4x_3-2.7x_4-2.7x_6 <= -10 \\
& -50\delta-1.8x_2+5.5x_3+3.3x_4+4.4x_5 <= 50 \\
& -20\delta+4.4x_1-2.7x_2-3.6x_3+1.1x_4-1.8x_5 <= -20 \\
& -215.5\gamma+0.5x_1+0.8x_2+1.2x_3+0.4x_4+0.1x_5+0.1x_6 <= 0 \\
& -182.5\gamma+0.5x_1+0.2x_2+1.4x_3+0.4x_4+0.5x_5+0.5x_6 <= 0 \\
& -2x_{1,1}+x_{2,1}+3x_{3,1}-x_{4,1}+x_{5,1}-4x_{6,1}+10z_1 = 0 \\
& 4x_{1,1}-3x_{2,1}-4x_{3,1}+x_{4,1}-2x_{5,1}+20z_1 = 0 \\
& -2x_{1,2}+x_{2,2}+3x_{3,2}-x_{4,2}+x_{5,2}-4x_{6,2}+10z_2 = 0 \\
& 4x_{1,2}-3x_{2,2}-4x_{3,2}+x_{4,2}-2x_{5,2}+20z_2 = 0 \\
& x_{i,1}+1000000z_1 <= 1000000 & \forall i = 1,3,4 \\
& x_{i,2}+1000000z_2 <= 1000000 & \forall i = 1,3,5 \\
& x_{i,3}+1000000z_3 <= 1000000 & \forall i = 1,...,4 \\
& x_{1,i}+x_{2,i}+x_{3,i}+x_{4,i}+x_{5,i}+x_{6,i}-1000000z_i <= 0 & \forall i = 1,...,3 \\
& x_i-x_{i,1}-x_{i,2}-x_{i,3} = 0 & \forall i = 1,...,6 \\
& z_1+z_2+z_3 = 1 \\
& \delta,\gamma \geq 0,\ x \in \mathbb{R}^{26}_{\geqq},\ z \in \{0,1\}^3
\end{aligned}
$$

APPENDIX B

FORMULA TO ESTIMATE THE COST AT EACH LOCATION AND TECHNOLOGY

This appendix provides details on the formula used to estimate the cost parameter for each location.

EIA (2015b) provides the following cost estimates for each technology:

Technology	**Cost parameter ($/MWh)**		
	Minimum	**Average**	**Maximum**
Conv. coal	3.5	12.8	34.6
Adv. coal	17.1	26.6	43.3
Conv. natural gas	1.2	3.3	9.9
Adv. natural gas	-2.1	0.1	5.6
Nuclear	0.2	10.3	19.3
Biomass	1.6	13.9	34
Wind - onshore	-8.6	3.4	47.9
Solar PV	-3	16.1	70.1

Table B.1: Cost estimates from EIA (2015b)

Let t_j be the average generating cost associated with technology $j \in J$ on Table B.1, and let r_j be the corresponding range of variation of the cost associated with technology $j \in J$ (cf. columns Maximum $-$ Minimum on Table B.1).

From TAMU (2015), let ℓ_i be the median land price per acre at location $i \in I$ normalized to a scale $[-1, 1]$. The values of land price are shown in Table B.2.

Location i	**Median \$/acre**	**Normalized value** ℓ_i
1	275	-0.688
2	182	-0.837
3	231	-0.759
4	158	-0.875
5	200	-0.808
6	166	-0.863
7	206	-0.799
8	80	-1.000
9	262	-0.709
10	639	-0.107
11	331	-0.599
12	280	-0.681
13	429	-0.442
14	524	-0.291
15	447	-0.414
16	923	0.347
17	1,332	1.000
18	800	0.150
19	951	0.391
20	543	-0.260
21	588	-0.188
22	674	-0.051
23	962	0.409
24	586	-0.192
25	462	-0.390
26	858	0.243
27	900	0.310
28	1,251	0.871
29	376	-0.527
30	482	-0.358
31	467	-0.382
32	543	-0.260
33	1,127	0.673

Table B.2: Land prices (\$/acre) from TAMU (2015) and normalized values of land price

Let v_j denote the estimated contribution of land price on the generating cost for technology $j \in J$. The values of v_j were estimated as detailed in Section 4.4 and are given in Table B.3:

Technology	v_j
Conv. coal	0.012
Adv. coal	0.012
Conv. natural gas	0.014
Adv. natural gas	0.014
Nuclear	0.010
Biomass	0.150
Wind - onshore	0.028
Solar PV	0.019

Table B.3: Contribution of land price on cost

Then, the cost for generating 1 MWh of electricity at location $i \in I$ using technology $j \in J$ was estimated as:

$$c_{ij} = t_j + \ell_i r_j v_j / 2 \tag{B.1}$$

As an example, consider the average cost $t_{\text{biomass}} = \$13.9/\text{MWh}$ for biomass technology and the corresponding range of variation $r_{\text{biomass}} = \$32.4/\text{MWh}$ from Table B.1. From Table B.3, the estimated contribution of land price on the generating cost for biomass technology is $v_{\text{biomass}} = 15\%$. Consider locations $i = 8$ and $i = 17$, which have the lowest and highest median land prices, respectively. Then, the cost for generating 1 MWh from equation (B.1) will be:

$$c_{8,\text{biomass}} = t_{\text{biomass}} + \ell_8 r_{\text{biomass}} v_{\text{biomass}}/2 = 13.9 - 1 \times 32.4 \times 0.15/2 = 11.47$$

$$c_{17,\text{biomass}} = t_{\text{biomass}} + \ell_{17} r_{\text{biomass}} v_{\text{biomass}}/2 = 13.9 + 1 \times 32.4 \times 0.15/2 = 16.33$$

Table 4.3 in Subsection 4.4.1 shows the cost parameters for each location and technology resulting from the application of equation (B.1).