INVENTORY CONTROL FOR REMANUFACTURING WITH BATCH PROCESSING, SEED STOCK PLANNING, AND COORDINATION CONSIDERATIONS

A Dissertation

by

YI ZHANG

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Chair of Committee,	Sıla Çetinkaya
Co-Chair of Committee,	Guy L. Curry
Committee Members,	Georgia-Ann Klutke
	Daren B.H. Cline
Head of Department,	Cesar O. Malave

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ABSTRACT

In recent years, the general area of remanufacturing has received significant attention both in academia and practice. While there is a growing body of literature in production planning models for remanufacturing, there is still a need for analytical decision-making tools considering general cost/revenue structures, stochastic demands, stochastic returns, and multiple agents/decision makers. Of particular interest in this dissertation are inventory control models with batch processing, seed stock planning, and coordination considerations for efficient inventory control practices.

More specifically we investigate three distinct, yet related, inventory control problems: (1) a fundamental inventory and production planning problem arising in a batch processing environment for a third party remanufacturer, which is characterized by a stochastic used-item return process along with a stochastic remanufactured-item demand process; (2) a seed stock planning problem in a batch processing environment with two agents including an original equipment manufacturer (OEM) and a remanufacturing supplier (RS), for which three game-theoretic scenarios and two types of controls are investigated; (3) a channel coordination problem in the reverse supply chain, which generalizes the above two problems in the sense that the stochastic nature of returns is modeled in a batch processing environment for channel coordination purposes.

Our analytical decision-making models contribute to the existing literature in the following ways: (1) we investigate the impact of more general cost structures (including both fixed operational costs and inventory-related costs) and disposal options in a batch processing environment with stochastic demand and return; (2) we

systematically study seed stock planning issues in a batch processing environment for remanufacturing using the game-theoretic framework; and (3) we build an analytical framework for channel coordination mechanism design for the reverse supply chain in a stochastic environment.

DEDICATION

To my parents, my husband, and my unborn son

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1. INTRODUCTION

In a recent paper, Pishchulov et al. (2014) argue that "collection of used products and their reuse has become in recent years the subject of increasing attention from both industrial practice and academic research due to important economic considerations". Thierry et al. (1995) identify five recovery options (repair, refurbishing, remanufacturing, cannibalization and recycling), and they define remanufacturing as the process to "bring used products up to quality standards that are as rigorous as those for new products". As noted by Thierry et al. (1995), the main advantage of remanufacturing over other recovery options is that it recovers the value of used products more efficiently. However, as pointed by Guide (2000), remanufacturing is more complex and difficult to manage than traditional manufacturing due to uncertainties about time, amount and quality in return flows. Guide (2000) outlines and discusses the complicated characteristics of production planning and control activities in remanufacturing. Much progress has been made in this area, especially in the last two decades. There is now a growing body of literature in production planning models in remanufacturing.

For a comprehensive review of the existing literature in this area, the reader is referred to Akçalı and Çetinkaya (2011) who discuss the following limitations and/or gaps in the literature:

- "Rather simplistic (i.e. *linear*) approximations of *cost* and/or *revenue structures* are used in the current literature." For example, see Cohen et al. (1980); Fleischmann and Kuik (2003); Heyman (1977); Whisler (1967); Yuan and Cheung (1998).

- Existing literature considering *stochastic demand* and *stochastic return* ignores *disposal* options.

  - For *continuous models*, see Fleischmann et al. (2002); Heyman (1977); Toktay et al. (2000); Van der Laan (2003); Yuan and Cheung (1998);
  - For *periodic models*, see Buchanan and Abad (1998); Cohen et al. (1980); Fleischmann and Kuik (2003); Kelle and Silver (1989); Mostard and Teunter (2006); Whisler (1967).

- Very little research in remanufacturing addresses *seed stock* considerations. Seed stock is defined as "the quantity of new products that are released" (Akçalı and Morse (2004)). The existing literature with seed stock considerations either
  - describes a specific case study (Linton and Johnston (2000)),
  - or focuses on the simulation approach (Akçalı and Morse (2004)).

- No *game theory* model (multi-agent model) has been used for analyzing seed stock considerations.

- Existing remanufacturing literature considering *channel coordination* issues
  - either focuses on the integration between *forward* and *reverse flows* (Ketzenberg et al. (2003) and Nativi and Lee (2012));
  - or on the coordination strategies that actually focus on operational or pricing decisions for the forward flows rather than for the reverse flows (Bhattacharya et al. (2006); Vorasayan and Ryan (2006); Liu et al. (2009); Dobos et al. (2013); Pishchulov et al. (2014)).

In the above, the key words that identify the limitations and/or gaps of the existing literature are highlighted in italics, and they are related to the current dissertation. More specifically, three distinct, yet related, inventory control problems are of interest in the current dissertation while addressing these limitations and gaps:

- **Alternative Batching Policies for Remanufacturing under Stochastic Demand and Return**: The first problem focuses on an analytical investigation of alternative batching policies for remanufacturing under *stochastic demand* and *stochastic return*, along with *disposal options* and *fixed operational cost* considerations.

- **Seed Stock Planning Strategies with Multiple Agents**: The second problem focuses on *seed stock* planning with *multi-agents* for which *game theory* approach is used.

- **Channel Coordination Strategies in the Reverse Supply Chain**: The third problem deals with *channel coordination* in the *reverse supply chain* in a *stochastic setting*.

The first problem is investigated in Section 2, and it deals with a fundamental inventory and production planning setting characterized by a *stochastic used-item return process* along with a *stochastic remanufactured-item demand process* faced by a remanufacturer. We investigate five *batching policies* inspired by the previous literature in shipment consolidation (Çetinkaya (2005)) (three *periodic policies* and two *threshold policies*) in the make-to-order environment. Under each policy, we explicitly take into account all relevant costs, including the *fixed operational costs* (associated with remanufacturing of used-items and dispatching of remanufactured-item orders in batches) and inventory-related costs (associated with remanufactured-item order *waiting costs* and used-item inventory holding costs). We develop analytical models with the objective of minimizing the long-run average expected total cost of the remanufacturer for computing the policy parameters of interest. Since the exact optimal policy parameters are not analytically tractable, we propose analytically tractable approximations on the cost functions for the policies. Through numerical investigation, we demonstrate that the approximate policy parameters work impressively well for all practical purposes in terms of the actual cost performance. Then, we extend the five policies by considering disposal options when needed. For this extension, an effective parameter-based approximation is developed for estimating the policy parameters. Numerical experiments demonstrate the effectiveness of the proposed approximation approach.

The second problem is investigated in Section 3, and it deals with a basic *game-theoretic setting* for *seed stock* planning in remanufacturing. The problem can be characterized as a *finite horizon* inventory control problem with *multiple agents* including an OEM, a new part supplier (NPS), and a RS. The OEM provides a particular type of replacement part for a product it sells. The demand of the replacement parts throughout the whole planning horizon T can be satisfied by using new-items procured from the NPS at the beginning of T, as well as remanufactured-items provided by the RS until the end of T. The initial inventory, i.e., seed stock, is treated as an operational decision variable along with other decisions. Since only a fraction of used-items can be remanufactured, seed stock is crucial to guarantee enough supply of returns for remanufacturing as well as to satisfy the demand during the initial phase of the planning horizon. The objective is to maximize the total profit by optimizing the seed stock level of new-items, initial lot size and exchange lot size of used-items. Seed stock optimization may or may not be controlled by the OEM due to the interactions between multiple agents. We investigate three scenarios and two types of controls, leading to several different system settings. We are interested in the interactions between the agents, and the impacts of the interactions on strategy performance. We aim to identify the system setting that performs best through our analytical models and numerical experiments.

The third problem is investigated in Section 4, and it deals with *coordination strategies* for the OEM and a collection center (CC). This problem may be referred as a *reverse channel coordination* problem due to its relationship with the traditional channel coordination problem (Toptal and Çetinkaya (2008)). Used-items arrive to the CC according to a stochastic process which is referred as the return process. The CC consolidates used-items using a threshold policy, and then sends them to the OEM in a large lot. Since the OEM and the CC have different cost considerations and make decisions individually, coordination mechanisms are useful such that the system-wide total profit is maximized. First, the return process is modeled as a general renewal process, and we prove that an all-unit-premium mechanism is able to coordinate the system. We develop analytical expressions for deriving the parameters representing the coordination mechanism. We find conditions under which these analytical expressions lead to closed-form solutions. Then we apply our results considering several special cases including the cases of deterministic return process, renewal return process with unit load, and renewal return process with exponentially distributed loads. For these special cases, we also extend out results to the situation that the return rate depends on the collection price. When the return rate depends on the collection price, we prove that all-unit-premium mechanism cannot guarantee the centralized optimal profit, i.e., channel coordination. However, by employing all-unit-premium and franchise fee mechanisms together, the channel coordination objective can be achieved. Analytical and numerical examples are provided to illustrate the profit improvement due to coordination.

As we have discussed above, although there is a large body of literature on inventory and production planning problems in remanufacturing, there is a need for analytical models with batch processing, seed stock planning, and channel coordination considerations. Of particular interest in this dissertation is the explicit modeling of fixed operation costs, stochastic nature of demands, stochastic nature of returns, disposal options, seed stock quantities, and channel coordination issues. Hence, our contributions include:

- building analytical remanufacturing models with stochastic demand and stochastic return while considering more general cost structures as well as disposal options;
- analyzing the interactions between different agents for seed stock planning using game theory;
- applying channel coordination strategies for the collection channel specifically in a stochastic environment.

The remainder of the dissertation is organized as follows. In Sections 2, 3 and 4, we investigate the three problems of interest as described above. In each section, we present a detailed discussion of relevant literature for the specific problem of interest. Also, each section is concluded with a discussion of our findings and potential future research directions.

2. ALTERNATIVE BATCHING POLICIES FOR REMANUFACTURING UNDER STOCHASTIC DEMAND AND RETURN

2.1 Overview of Section 2

This section deals with a fundamental inventory and production planning setting characterized by a stochastic used-item return process along with a stochastic remanufactured-item demand process faced by a remanufacturer. We investigate five batching policies inspired by the previous literature in shipment consolidation (Çetinkaya (2005)) (two periodic policies and three threshold policies) in the make-to-order environment. Under each policy, we explicitly take into account for all relevant costs, including the fixed operational costs (associated with remanufacturing of used-items and dispatching of remanufactured-item orders in batches) and inventory-related costs (associated with remanufactured-item order waiting costs and used-item inventory holding costs). We develop analytical models with the objective of minimizing the long-run average expected total cost of the remanufacturer for computing the policy parameters of interest. Since the exact optimal policy parameters are not analytically tractable, we propose analytically tractable approximations on the cost functions for the policies. Through numerical investigation, we demonstrate that the approximate policy parameters work impressively well for all practical purposes in terms of the actual cost performance. Then, we extend the five policies by considering disposal options when needed. For this extension, an effective parameter-based approximation is developed for estimating the policy parameters. Numerical experiments demonstrate the effectiveness of the proposed approximation approach.

2.2 Problem Motivations and Related Literature

As noted above, we consider a fundamental inventory and production planning problem characterized by a *stochastic* used-item return process along with a *stochastic* remanufactured-item demand process. The problem of interest arises in the context of *valuable* discrete parts remanufacturing, such as engines or transmissions in the automotive industry and cellular phones in the consumer electronics industry. That is, the used-items are valuable and remanufacturable and are returned to a third-party remanufacturer according to a Poisson arrival stream representing the stochastic return process. Likewise, the remanufactured items are valuable and in short supply and are ordered from the remanufacturer according to a Poisson arrival stream representing the stochastic demand process.

For example, the return process is generated by a large base of used-item suppliers (i.e., insurance companies and automotive repair shops in the automotive industry; and cellular network providers and retailers in the consumer electronics industry) and the demand process is driven by a different market consisting remanufactured-item buyers (i.e., automotive part sellers and automotive repair shops in the automotive industry; and secondary market sellers in consumer electronics industry). Due to the nature of the applications of interest and the involvement of a remanufacturer, the stochastic return and demand processes are treated as independent (as in Buchanan and Abad (1998); Fleischmann and Kuik (2003); Heyman (1977); Muckstadt and Isaac (1981); Whisler (1967)).

For the applications of interest here, due to the labor-intensive nature of remanufacturing activity and the valuable nature of remanufactured items, both **fixed operational costs** and **inventory-related costs** are significant. Hence, the remanufacturer operates in a *batch processing* mode by first observing and then satisfying realized demands in a *make-to-order* fashion, i.e., the remanufacturer does not carry any remanufactured items but accumulates used-items as dictated by the return process. This, in turn, implies that the remanufactured-item buyers are willing to place orders in ahead of time, while the remanufacturer has to bear **order waiting costs** and **used-item holding costs**. The order waiting cost is due to the make-to-order environment (i.e., the intentional avoidance of expensive remanufactured-item inventories). That is, the only inventory holding cost is due to used-item inventories held in stock.

For each batch processing run, the fundamental difficulty is due to the mismatch of the so-called supply and demand, e.g., the used-item inventories may or may not be sufficient to satisfy the remanufactured-item orders to be delivered once the batch is processed. As we have noted earlier, used-item returns are in short supply relative to remanufactured-item orders, i.e., arrival rate of the return process is typically smaller than arrival rate of the demand process in most practical applications. Hence, an agent that has access to returns and the technological know-how on how to remanufacture the returns is in a lucrative opportunity to capture the financial benefits associated with matching the supply and demand. When the returns fall short of the demands, the opportunity to satisfy the entire demand via remanufacturing is lost and the cost of obtaining an alternative source to satisfy the excess demand needs to be accommodated. When the returns exceed the demands, it is crucial to make the best of excess returns to hedge against future demand uncertainty and to avoid excess remanufactured-item inventories. Hence, it is worthwhile to have a closer examination of the efficiency of clearing policies that rely on the clearance of used-item inventories, remanufactured-item orders or perhaps clearance on a periodic basis.

To this end, we propose alternative operating policies tackling inventory and production planning problem of the remanufacturer. These policies are inspired by stochastic clearing models applicable in the context of outbound shipment consolidation practices and vendor-managed inventory systems (Çetinkaya (2005); Çetinkaya and Bookbinder (2003); Çetinkaya and Lee (2000); Çetinkaya et al. (2008)). Of particular interest are two classes of policies referred as periodic policies and threshold policies. The former class includes (i) the fixed period policy, (ii) the demand-driven periodic policy and (iii) the return-driven periodic policy. The latter class includes (i) the demand-driven threshold policy and (ii) the return-driven threshold policy.

Then, the remanufacturer either executes a batch run (1) at regular intervals or (2) when the remanufactured-item orders waiting to be released dictated by the demand process or (3) when the used-item inventories dictated by the return process reaches a particular threshold value. The duration between two consecutive batch runs is then referred as a *remanufacturing cycle*. More specifically,

- When a *fixed period policy* is in effect, a batch processing run is executed on a periodic basis, i.e., every T_F time units, leading to a fixed remanufacturing cycle length of T_F. Hence, it is referred as the T_F*-policy*. On the contrary, for the remaining policies, the remanufacturing cycle length is a random variable.

- Under a *demand-driven periodic policy*, a batch processing run is executed after a particular duration of time, denoted by T_D, of time elapses beyond the arrival of the first demand. Hence, it is referred as the T_D*-policy*.

- Under a *return-driven periodic policy*, a batch processing run is executed after a particular duration of time, denoted by T_R, elapses beyond the arrival of the first return. Hence, it is referred as the T_R*-policy*.

- Under a *demand-driven threshold policy*, a batch processing run is executed after the remanufactured-item orders accumulated during the cycle (accumulated demand) reaches a particular level, denoted by Q_D. Hence, it is referred as the Q_D-*policy*.

- Under a *return-driven threshold policy*, a batch processing run is executed once the used-item inventories (available returns) reach a particular level, denoted by Q_R. Hence, it is referred as the Q_R-*policy*.

Under each policy, we explicitly take into account for all relevant costs, including the fixed operational costs (associated with remanufacturing of used-items and dispatching of remanufactured-item orders in batches) and inventory-related costs (associated with remanufactured-item order waiting costs and used-item inventory holding costs). Our goal is to develop an analytical model with the objective of minimizing the long-run average expected total cost of the remanufacturer for computing the policy parameter of interest. Despite the seemingly simple nature of these policies, we demonstrate that the resulting used-item inventory profile of the remanufacturer is more complicated than in the case of many of the traditional stochastic inventory problems, deeming the analytical derivation of inventory-related costs difficult, if not impossible. As a result, the operational cost minimization problem faced by the remanufacturer represents a practical and technical challenge that deserves further academic attention.

Pertinent details of the sequence of events for each remanufacturing cycle is characterized as follows:

- At the time when a batch run is to be executed, i.e., at the end of a remanufacturing cycle, if the existing used-item inventories exceed remanufactured-item orders accumulated during the cycle then (i) a sufficient number of used-items are processed as a batch; (ii) the entire demand is satisfied; and (iii) the excess quantity of used-items can be kept in inventory until the next batch run. This case is referred as the case of **supply overage**.

- At the time when a batch run is to be executed, if the used-item inventories fall short of the remanufactured-item orders accumulated during the cycle then (i) the remanufacturer procures additional used-items from a spot market (i.e., vehicle salvage yards in the automotive industry and cellular phone brokers in the consumer electronics industry); (ii) all of the available used-items (dictated by the return process and procured from the spot market) are processed as a batch; and (iii) the entire demand is satisfied, i.e., all outstanding remanufactured-item orders waiting to be released are cleared. This case is referred as the case of **supply underage**.

- Both the spot market procurement lead time as well as the batch processing lead time are negligible relative to the length of a remanufacturing cycle.

While the assumption regarding the availability of a spot market with ample supply simplifies the underlying stochastic return-item inventory and remanufactured-item order profiles, it is also well justified as argued in the previous literature (Atasu et al. (2013); Savaskan et al. (1999, 2004)) and exemplified in several contemporary applications. Of the practical applications considered here, in the automotive industry, for example, a typical remanufacturer works with a network of insurance companies whose decisions dictate the stochastic return process modeled in this section. However, the remanufacturer is also connected to a network of vehicle salvage yards with virtually ample supply of returns.

Although closed-form expressions of the long-run average expected total cost functions under the policies of interest are very hard to derive, we demonstrate that the used-item inventory position can be treated as a $G/G/1$ queue regardless of the policy under consideration. With this observation, we derive analytically tractable approximations on the cost functions for the policies. The approximations are then utilized to compute the cost-effective policy parameters considering the long-run average expected total costs for the remanufacturer. Despite the fact that the exact optimal policy parameters are not analytically tractable and can only be obtained via computationally intensive simulation approaches, the approximations lead to superb near-optimal operating parameters in **closed-form** for all of the policies.

A diligent numerical investigation demonstrates that despite the deviation between the proposed approximations and the exact cost functions, the resulting policy parameters would work impressively well for all practical purposes in terms of the actual cost performance. Hence, our contribution lies in providing a systematic and comprehensive analysis of cost performance of periodic and threshold policies for the remanufacturer and determining analytically-tractable and practically-effective operating parameters. That is, the exact cost penalty of using the approximate policy parameters is negligible in most cases as demonstrated by a careful numerical study with 48 instances for each of the five policies of interest leading to $48 \times 5 = 240$ problem settings. More specifically, our numerical results reveal that the cost penalty associated with using the approximate policy parameters is 0.02% on average and is less than 1% in the worst case. Remarkably, the ideal performance with 0% cost penalty is achievable in many cases.

The operational cost minimization problem introduced and examined here is closely related to two streams of previous research. The first stream of research deals with inventory and production planning models for remanufacturing, while the second stream deals with stochastic clearing applications related to shipment consolidation.

For a comprehensive review of the existing literature in the first stream of closely related research, we refer the readers to Akçalı and Çetinkaya (2011). According to the classification framework in Akçalı and Çetinkaya (2011), the problem setting analyzed in this section is a *single stock-point system with stochastic return and demand processes*. Both continuous (Heyman (1977); Yuan and Cheung (1998)) and periodic (Buchanan and Abad (1998); Cohen et al. (1980); Fleischmann and Kuik (2003); Kelle and Silver (1989); Muckstadt and Isaac (1981); Whisler (1967)) review inventory and production planning models have been investigated previously while making strong assumptions leading to a need for analytical models with explicit consideration of fixed operational costs and make-to-order environments with explicit order waiting costs. We examine both types of review; T_F-policy is concerned with a periodic-review scheme whereas T_D-policy, T_R-policy, Q_D-policy, and Q_R-policy follow a continuous-review scheme. The general contribution of this section is then two-fold. The modeling contribution is in the explicit consideration of a make-to-order environment with both stochastic return and stochastic demand processes.The technical contribution is in the development of simple closed-form expressions for computing cost-effective near-optimal policy parameters with superior performance when benchmarked against the computationally demanding exact optimal policy parameters. While the specific technical contribution relies on standard approaches in stochastic clearing and queuing theory and the idea of developing cost minimization models draws from stochastic inventory theory, the simple closed-form expressions derived here remedy the computational burden associated with an exact optimization approach in a remarkable fashion.

As noted earlier, the alternative operating policies tackling the operational cost minimization problem faced by the remanufacturer are inspired by stochastic clearing models applicable in the context of outbound shipment consolidation practices and vendor-managed inventory systems (Çetinkaya (2005); Çetinkaya and Bookbinder (2003); Çetinkaya and Lee (2000); Çetinkaya et al. (2008)). Temporal shipment consolidation refers to active intervention by management to combine several small orders arriving over time into a single shipment achieving high truck utilization and realizing scale economies associated with transportation. Hence, temporal shipment consolidation problems have been treated by considering time-based and quantity-based policies that operate in a fashion similar to the alternative policies of interest introduced in this section. As in the case of the operational cost minimization problem faced by the remanufacturer here, a deliberate temporal shipment consolidation policy also leads to order waiting costs. However, the existing body of work in this area is largely motivated by inbound or outbound distribution applications (Çetinkaya and Bookbinder (2003)) arising in the context of traditional forward supply chains where return and reuse opportunities do not exist. Explicit consideration of the stochastic nature of both return and demand processes, however, complicate the implied inventory and order profiles. Hence, it is worthwhile to have a closer examination of the efficiency of similar policies in the context of remanufacturing and reverse supply chains.

In the traditional shipment consolidation literature, typically a single process that models the demand is taken into account so that quantitative approaches that rely on Renewal Theory (Çetinkaya (2005); Çetinkaya and Bookbinder (2003); Çetinkaya and Lee (2000); Çetinkaya et al. (2008)), Markov decision processes (Higginson and Bookbinder (1995)), and matrix-geometric methods (Bookbinder et al. (2011)) have been useful. In our work however, two processes that model the returns and demands need to be considered. Consequently, while existing temporal shipment models deal with time- or quantity-based decision parameters for handling the demand process, the operational remanufacturing decisions can be based on time- or quantity-based batching of the return process or the demand process in our setting. Due to the potential valuable nature of returns, a remanufacturing cycle is *not* a simple inventory clearing cycle, regenerative process as in many of the traditional stochastic inventory problems, deeming the analytical derivation of inventory-related costs difficult, if not impossible as we have noted earlier. As a result, the operational cost minimization problem faced by the remanufacturer is new. To address this new problem, we utilize existing approaches in stochastic clearing and queuing theory (Stidham Jr (1974, 1977); Kingman (1962)) along with some of the properties of Poisson processes (Ross (1996), Page 59) as they relate to the stochastic return and demand processes and properties of Normal distribution (Barlow (1989), Page 40) as they related to the accumulated returns and demands in each remanufacturing cycle.

The remainder of the section is organized as follows. In the next section, we discuss our modeling assumptions, introduce our notation, and our demand and return process modeling approach. Section 2.3 describes the basic model and the underlying processes. Section 2.4 derives the total cost functions for each of the policies and derives the approximate minimizers of those cost functions. Section 2.5 summarizes and compares the properties of the cost functions and the optimal results. In Section 2.6, we present some results from a numerical experimentation to investigate the quality of our approximations and illustrate their practical relevance. Section 2.7 considered the situation that the return rate is greater than or equal to the demand rate, and extend the policies to include disposal options. Approximation approaches are proposed and numerical tests are used to check their performances. Section 2.8 summarizes the results of the section and provides future research directions.

2.3 Modeling Basics

The problem setting as described in Section 2.2 is illustrated in Figure 2.1, where some of modeling parameters are also introduced. See Table 2.1 for a summary of notation introduced so far along with additional essential notation used in the remainder of the section. This section is aimed at a closer examination of the fundamental stochastic/random components of the inventory and production planning system under consideration that are essential for the development of an operational cost minimization approach. Hence, we proceed with a detailed discussion of the stochastic demand and return processes (Subsection 2.3.1) along with the random natures of remanufacturing cycles under alternative policies (Subsection 2.3.2), underlying used-item inventory profiles (Subsection 2.3.3.1), and spot market procurements (Subsection 2.3.3.2). This section concludes with the illustration of a realization of the used-item inventory profile and outstanding remanufactured-item order profiles for all of the policies (Subsection 2.3.4).

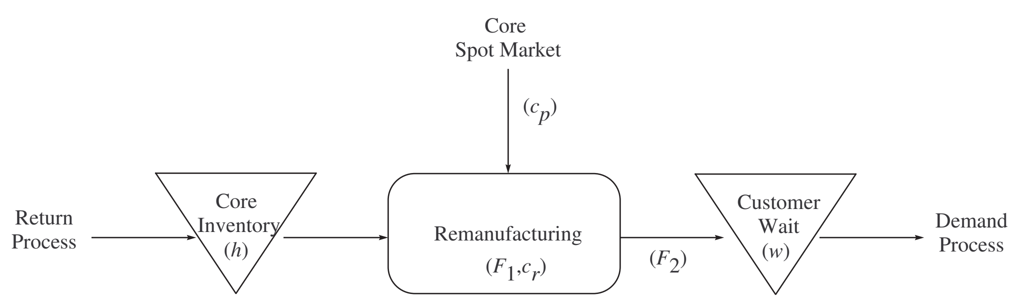

Figure 2.1: An illustration of the remanufacturing system for the batch processing problem.

2.3.1 Stochastic Demand and Return Processes

As we have noted earlier the used-items are valuable and remanufacturable and are returned to a third-party remanufacturer according to a Poisson stream $\{W(t), t > 0\}$ over time t with an arrival rate of r, and the remanufactured items are valuable

Table 2.1: Notation for batch processing problem.

T_F	Time-based operating parameter under fixed period policy
T_D	Time-based operating parameter under demand-driven periodic policy
T_R	Time-based operating parameter under return-driven periodic policy
Q_D	Quantity-based operating parameter under demand-driven threshold policy
Q_R	Quantity-based operating parameter under return-driven threshold policy
U	Threshold operating parameter incorporating the disposal option
$W(t)$	Number of returns by time t
r	Return rate (units/unit time)
$N(t)$	Number of demands by time t
a	Demand rate (units/unit time)
R_n	Number of returns generated specifically in remanufacturing cycle n
D_n	Number of demands received in remanufacturing cycle n
B_n	Number of used-items procured from the spot market in remanufacturing cycle n
I_n	Number of used-items in inventory at the end of remanufacturing cycle n
$CL(\cdot)$	Remanufacturing cycle length as a function of the policy parameter of interest (i.e., T_F, T_D, T_R, Q_D, or Q_R)
X_1	the time of arrival for the first unit of demand in a remanufacturing cycle
Y_1	the time of arrival for the first unit of return in a remanufacturing cycle
S_{Q_D}	the time that $N(t)$ reaches Q_D
Z_{Q_R}	the time that $W(t)$ reaches Q_R
h	Used-item inventory holding cost ($/unit/unit time)
w	Order waiting cost ($/unit/unit time)
c	Variable cost of remanufacturing ($/unit)
c^d	Unit disposal cost ($/unit)
p	Variable spot market procurement cost ($/unit)
K	Fixed operational cost associated with remanufacturing of used-items and dispatching of remanufactured-items ($/cycle)
$TC(\cdot)$	Long-run average expected total cost per unit time as a function of the policy parameter of interest (i.e., T_F, T_D, T_R, Q_D, or Q_R)

and are in short supply and are ordered from the remanufacturer according to a Poisson stream $\{N(t), t > 0\}$ over time t with an arrival rate a.

We denote the inter-arrival times of returns by Y_i, $i = 1, 2, \ldots$, so that Y_i's are exponentially distributed with rate r, i.e., $Y_i \sim \exp(r)$. Let $Z_0 = 0$ and $Z_i = \sum_{j=1}^{i} Y_j$ so that $Z_i \sim \text{Gamma}\,(i, r)$ is the arrival time of the i^{th} return. Hence,

$$W(t) = \sup\{i : Z_i \leq t\}$$

is the number of returns by time t and, by definition, $W(t) \sim \text{Poisson}(rt)$.

We denote the inter-arrival times of demands by X_i, $i = 1, 2, \ldots$, so that X_i's are exponentially distributed with rate a, i.e., $X_i \sim \exp(a)$. Let $S_0 = 0$ and $S_i = \sum_{j=1}^{i} X_j$ so that $S_i \sim \text{Gamma}\,(i, a)$ is the arrival time of the i^{th} demand. Hence,

$$N(t) = \sup\{i : S_i \leq t\}$$

is the number of demands by time t and, by definition, $N(t) \sim \text{Poisson}(at)$.

Initially, we assume that the return rate r is smaller than the demand rate a, i.e., $r < a$. Also, as we have already justified in the spirit of previous literature and in the context of practical motivations of interest, we consider the case where return and demand processes are independent of each other (as in Buchanan and Abad (1998); Fleischmann and Kuik (2003); Heyman (1977); Muckstadt and Isaac (1981); Whisler (1967)).

2.3.2 Remanufacturing Cycles under Alternative Policies

We let $CL_n(\cdot)$ denote the length of remanufacturing cycle n as a function of the policy parameter of interest, i.e., T_F, T_D, T_R, Q_D, or Q_R, and, we take the liberty of dropping the index n for obvious reasons and use $CL(\cdot)$ in the remainder of the section. Recall that by definition of the policies, we have

$$CL(T_F) = T_F$$

while $CL(T_D)$, $CL(T_R)$, $CL(Q_D)$, and $CL(Q_R)$ are random variables whose characteristics are presented momentarily in Properties 1 through 5.

Now, for all policies, let R_n and D_n denote the returns generated and demands received, respectively, during the course of the n^{th} remanufacturing cycle. Hence, by definition, random variables D_n, $n = 1, 2, \ldots$, are independent and identically distributed (i.i.d.) as well as random variables R_n, $n = 1, 2, \ldots$. Clearly, the underlying distributions of these random variables depend on the policy type as demonstrated momentarily in Properties 1 through 5.

Recalling that $N(t) \sim \text{Poisson}(at)$ and $W(t) \sim \text{Poisson}(rt)$, utilizing the properties of random variables X_1 and S_{Q_D} (note that X_1 and S_{Q_D} are stopping times* for $\{N(t), t > 0\}$), and the properties of random variables Y_1 and Z_{Q_R} (note that Y_1 and Z_{Q_D} are stopping times for $\{W(t), t > 0\}$), and considering the definitions of the policies, we have the following results.

Property 1 *Under T_F-policy, $CL(T_F) = T_F$ while $D_n = N(T_F)$ and $R_n = W(T_F)$. It then follows that $E[D_n] = Var(D_n) = aT_F$ and $E[R_n] = Var(R_n) = rT_F$.*

Property 2 *Under T_D-policy, $CL(T_D) = X_1 + T_D$ while $D_n = 1 + N(T_D)$ by the Strong Markov Property† and, given $X_1 = x$, $R_n \sim \text{Poisson}(r(x + T_D))$. It then follows that*

*A random variable, e.g., X_1, is a stopping time with respect to the process $\{N(t), t > 0\}$ if for every $t \geq 0$, the event $[X_1 \leq t]$ is determined by the process up to time t (Resnick (2013), Page 504).

†Given that X_1 is a stopping time with respect to the process $\{N(t), t > 0\}$, for every $t \geq 0$ and given $N(X_1)$, $N(X_1 + t)$ is independent of the events up to X_1 (Resnick (2013), Page 162).

$$E\left[CL(T_D)\right] = \frac{1}{a} + T_D, \quad \text{Var}\left(CL(T_D)\right) = \frac{1}{a^2},$$

$$E[D_n] = 1 + aT_D, \quad \text{Var}(D_n) = aT_D,$$

$$E[R_n] = E[E[R_n|X_1]] = E[r(X_1 + T_D)] = \frac{r}{a}(1 + aT_D), \quad \textit{and}$$

$$\begin{aligned}\text{Var}(R_n) &= E[\text{Var}(R_n|X_1)] + \text{Var}(E[R_n|X_1]) = E[r(X_1 + T_D)] + \text{Var}(r(X_1 + T_D)) \\ &= \frac{r}{a} + rT_D + \frac{r^2}{a^2}.\end{aligned}$$

Notation $|$ indicates a conditioning argument.

Property 3 *Under T_R-policy, $CL(T_R) = Y_1 + T_R$ while $R_n = 1 + W(T_R)$ by the Strong Markov Property and, given $Y_1 = y$, $D_n \sim \text{Poisson}(a(y+T_R))$. It then follows that*

$$E\left[CL(T_R)\right] = \frac{1}{r} + T_R, \quad \text{Var}\left(CL(T_R)\right) = \frac{1}{r^2},$$

$$E[R_n] = 1 + rT_R, \quad \text{Var}(R_n) = rT_R,$$

$$E[D_n] = E[E[D_n|Y_1]] = E[a(Y_1 + T_R)] = \frac{a}{r}(1 + rT_R), \quad \textit{and}$$

$$\begin{aligned}\text{Var}(D_n) &= E[\text{Var}(D_n|Y_1)] + \text{Var}(E[D_n|Y_1]) = E[a(Y_1 + T_R)] + \text{Var}(a(Y_1 + T_R)) \\ &= \frac{a}{r} + aT_R + \frac{a^2}{r^2}.\end{aligned}$$

Property 4 *Under Q_D-policy, $CL(Q_D) = S_{Q_D}$ while $D_n = Q_D$ and, given $S_{Q_D} = s$, $R_n \sim \text{Poisson}(rs)$. It then follows that*

$$
\begin{aligned}
&E\left[CL(Q_D)\right] = \frac{Q_D}{a}, \quad \mathrm{Var}\left(CL(Q_D)\right) = \frac{Q_D}{a^2}, \\
&E[R_n] = E[E[R_n|S_{Q_D}]] = E[rS_{Q_D}] = \frac{rQ_D}{a}, \quad \textit{and} \\
&\mathrm{Var}(R_n) = E[\mathrm{Var}(R_n|S_{Q_D})] + \mathrm{Var}(E[R_n|S_{Q_D}]) = E[rS_{Q_D}] + \mathrm{Var}(rS_{Q_D}) \\
&\qquad\qquad = \frac{rQ_D}{a} + \frac{r^2 Q_D}{a^2}.
\end{aligned}
$$

Property 5 *Under Q_R-policy, $CL(Q_R) = Z_{Q_R}$ while $R_n = Q_R$ and, given $Z_{Q_R} = z$, $D_n \sim \mathrm{Poisson}(az)$. It then follows that*

$$
\begin{aligned}
&E\left[CL(Q_R)\right] = \frac{Q_R}{r}, \quad \mathrm{Var}\left(CL(Q_R)\right) = \frac{Q_R}{r^2}, \\
&E[D_n] = E[E[D_n|Z_{Q_R}]] = E[aZ_{Q_R}] = \frac{aQ_R}{r}, \quad \textit{and} \\
&\mathrm{Var}(D_n) = E[\mathrm{Var}(D_n|Z_{Q_R})] + \mathrm{Var}(E[D_n|Z_{Q_R}]) = E[aZ_{Q_R}] + \mathrm{Var}(aZ_{Q_R}) \\
&\qquad\qquad = \frac{aQ_R}{r} + \frac{a^2 Q_R}{r^2}.
\end{aligned}
$$

2.3.3 Matching Supply with Demand over Remanufacturing Cycles

Now that we have complete results characterizing the stochastic nature of remanufacturing cycles under alternative policies let us recall the details of the sequence of events associated with matching supply and demand over remanufacturing cycles. First, we consider the case where $\rho < 1$, i.e., $r < a$. As noted earlier, we have two possibilities referred to as the cases of supply overage and supply underage. The first case leads to the discussion in Section 2.3.3.1 and the second case leads to the discussion in Section 2.3.3.2. Before we proceed to the detailed discussions of the supply overage and supply underage, we summarize the sequence of events for remanufacturing cycle n:

- the remanufacturer measures the initial inventory of used-items, which is the inventory of used-items at the end of the $n-1^{st}$ remanufacturing cycle, i.e., I_{n-1};
- the remanufacturer consolidates returns during cycle n, and the return amount is denoted by R_n;
- the remanufacturer observes the demand in cycle n, and the demand received is denoted by D_n;
- a batch processing run is triggered according to the policy of interest, and all the demands D_n are cleared:
  - if the used-item inventory before batch processing, i.e., $I_{n-1}+R_n$, is larger than or equal to D_n, then after the batch run, $I_{n-1}+R_n-D_n$ units of used-items are left and retained to the next cycle;
  - otherwise, the remanufacturer purchases $D_n-(I_{n-1}+R_n)$ units of used-items from the spot market, and clears all the demand with the used-items on hand as well as the used-items purchased from the spot market.

2.3.3.1 Supply Overage: Excess Used-item Inventories

Let I_n denote the number of used-items in inventory at the end of the remanufacturing cycle n, and recall that the batch processing lead time is negligible. It then follows that the used-item inventory levels in successive remanufacturing cycles can be characterized by a material flow equation of the form

$$I_n = \begin{cases} I_{n-1}+R_n-D_n, & D_n \le I_{n-1}+R_n, \\ 0, & D_n > I_{n-1}+R_n. \end{cases} \tag{2.1}$$

As we have noted earlier, despite the seemingly simple nature of the inventory and production planning problem at hand, we now demonstrate that the resulting used-item inventory profile is more complicated than in the case of traditional stochastic inventory problems arising in the context of forward supply chains, deeming the analytical derivation of $E[I_n]$ challenging. To this end, an analogy to a queueing system is useful:

- Let us interpret I_{n-1} in (2.1) as the waiting time of the n^{th} customer, in front of whom there are $n-1$ customers, in an arbitrary single-server queue.
- Then, interpreting R_n as the service time of the n^{th} customer and interpreting D_n as the inter-arrival time between the n^{th} customer and the $n+1^{\text{st}}$ customer, we have the waiting time I_n of the next customer.
- Clearly, $I_n = 0$ if the $n+1^{\text{st}}$ customer arrives after the previous customer leaves, i.e., if $D_n \geq I_{n-1} + R_n$.
- Considering the various distributions of R_n and D_n in Properties 1 through 5 , we then conclude that the used-item inventory profile is dictated by a $G/G/1$ queue.

While there does not exist an exact method to compute the steady-state distribution of a $G/G/1$ queue and, hence, $E[I_n]$, we rely on a fundamental result by Kingman (1962):

$$E[I_n] \leq \frac{E[D_n](C_a^2 + \rho^2 C_r^2)}{2(1-\rho)} \tag{2.2}$$

where

$$C_a^2 = \frac{\mathrm{Var}(D_n)}{(E[D_n])^2}, \quad C_r^2 = \frac{\mathrm{Var}(R_n)}{(E[R_n])^2}, \quad \text{and} \quad \rho = \frac{E[D_n]}{E[R_n]}. \tag{2.3}$$

As we demonstrate momentarily in Section 2.4, this result is directly applicable for our goal of developing an analytical model with the objective of minimizing the long-run average expected total cost of the remanufacturer for computing the policy parameter of interest.

There are many ways to approximate the average waiting time in queueing theory as summarized by Table 2.2 in Myskja (1990), on Page 290. The bound by Kingman (1962) and is given by (2.2) is applicable for any stable G/G/1 queue with $\rho < 1$, as proved in Medhi (2002), Page 360. Our methodology here is applicable using other approximations summarized by Table 2.2 (Myskja (1990)). Here we focus on the approximation by (2.2).

Table 2.2: A collection of approximation formulas for the GI/GI/1 queue

Kingman	
(upper limit)	$W \geq \frac{v_a+v_s}{2(\lambda^{-1}-\mu^{-1})} = \frac{\rho(c_a^2/\rho^2+c_s^2)}{2\mu(1-\rho)}$
Kobayashi	$W \approx \frac{\hat{\rho}}{\mu(1-\hat{\rho})}, \quad \hat{\rho} = \exp\{-2(1-\rho)/(\rho(c_a^2 + c_s^2/\rho^2))\}$
Heyman	
(heavy load)	$W \approx \frac{\mu}{2} \cdot \frac{v_a\rho^2+v_s\rho^{-1}}{\lambda^{-1}-\mu^{-1}} = \frac{\rho(c_a^2+c_s^2/\rho^2)}{2\mu(1-\rho)}$
Marchal	$W \approx \frac{\lambda(1+c_s^2)}{1/\rho^2+c_s^2} \cdot \frac{v_a+v_s}{2(1-\rho)} = \frac{\rho(1+c_s^2)}{2\mu(1-\rho)} \cdot \frac{c_a^2+\rho^2 c_s^2}{1+\rho^2 c_s^2}$
Gelenbe	$W \approx \frac{\rho(c_a^2+c_s^2)}{2\mu(1-\rho)}$
Krämer	
/Langenbach-Belz	$W \approx \frac{\rho(c_a^2+c_s^2)}{2\mu(1-\rho)} \cdot \begin{cases} \exp\{-2(1-\rho)(1-c_a^2)^2/3\rho(c_a^2+c_s^2)\} \\ \exp\{-(1-\rho)(c_a^2-1)/(c_a^2+4c_s^2)\} \end{cases}$
Kimura	$W \approx \frac{\sigma(c_a^2+c_s^2)}{\mu(1-\sigma)(c_a^2+1)}$

2.3.3.2 Supply Underage: Spot Market Procurement

Now, let us consider the case of supply underage leading to a spot market procurement in a remanufacturing cycle. In this case, under the assumption that additional used-items can be obtained with negligible delivery lead time, let

$$B_n = \max\{0, D_n - (I_{n-1} + R_n)\} = \begin{cases} 0, & D_n \leq I_{n-1} + R_n, \\ I_{n-1} + R_n - D_n, & D_n > I_{n-1} + R_n. \end{cases} \tag{2.4}$$

That is, B_n is the spot market procurement quantity associated with remanufacturing cycle n. Clearly, B_n depends on I_n complicating the analytical derivation of $E\left[B_n\right]$. In order to overcome this difficulty, let $B(t)$ denote the total number of used-items procured from the spot market during $[0, t]$ so that

$$B(t) = \left(N(t) - W(t)\right)^+ .$$

Then, the long-run average expected spot market procurements per unit time is given by

$$\lim_{t\to\infty} \frac{E\left[B(t)\right]}{t} = \lim_{t\to\infty} \frac{E\left[\left(N(t) - W(t)\right)^+\right]}{t}.$$

Recalling that $N(t) \sim \text{Poisson}(at)$ and $W(t) \sim \text{Poisson}(rt)$ and considering that a Poisson distribution with a large arrival rate can be effectively approximated by a Normal distribution (Barlow (1989), Page 40), we can approximate $N(t) \sim \text{Normal}(at, \sqrt{at})$ and $W(t) \sim \text{Normal}(rt, \sqrt{rt})$. Then $N(t) - W(t) \sim \text{Normal}((a-r)t, \sqrt{(a+r)t})$ (Ross (2010), Page 280). Relying on that, we can prove the following property.

Property 6 *The long-run average expected spot market procurements per unit time is given by*

$$\lim_{t\to\infty}\frac{E\left[B(t)\right]}{t}=\lim_{t\to\infty}\frac{E\left[(N(t)-W(t))^{+}\right]}{t}=a-r. \tag{2.5}$$

Proof. Since $N(t)-W(t)\sim \text{Normal}((a-r)t,\sqrt{(a+r)t})$, then we have

$$\lim_{t\to\infty}E\left[\frac{B(t)}{t}\right]=\lim_{t\to\infty}\frac{E\left[(N(t)-W(t))^{+}\right]}{t}=\lim_{t\to\infty}\frac{\int_0^\infty \frac{z}{\sqrt{2\pi}\sqrt{(a+r)t}}e^{-\frac{(z-(a-r)t)^2}{2(a+r)t}}dz}{t}. \tag{2.6}$$

Letting $v=\frac{z-(a-r)t}{\sqrt{(a+r)t}}$ and rewriting (2.6), we have

$$\begin{aligned}
\lim_{t\to\infty}E\left[\frac{B(t)}{t}\right] &= \lim_{t\to\infty}\frac{E\left[(N(t)-W(t))^{+}(t)\right]}{t}\\
&= \lim_{t\to\infty}\frac{\int_{-\frac{(a-r)\sqrt{t}}{\sqrt{a+r}}}^{\infty}\frac{\sqrt{(a+r)t}v+(a-r)t}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}dv}{t}\\
&= \lim_{t\to\infty}\frac{\sqrt{(a+r)t}\int_{-\frac{(a-r)\sqrt{t}}{\sqrt{a+r}}}^{\infty}\frac{v}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}dv}{t}\\
&\quad +\lim_{t\to\infty}\frac{(a-r)t\int_{-\frac{(a-r)\sqrt{t}}{\sqrt{a+r}}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}dv}{t}\\
&= \lim_{t\to\infty}\frac{\sqrt{(a+r)}\int_{-\frac{(a-r)\sqrt{t}}{\sqrt{a+r}}}^{\infty}\frac{v}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}dv}{\sqrt{t}}\\
&\quad +\lim_{t\to\infty}(a-r)\int_{-\frac{(a-r)\sqrt{t}}{\sqrt{a+r}}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}dv\\
&= 0\cdot\int_{-\infty}^{\infty}\frac{v}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}dv+(a-r)\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}dv\\
&= a-r.
\end{aligned}$$

□

Observe that (2.5) is directly applicable for our goal of developing an analytical model with the objective of minimizing the long-run average expected total cost of the remanufacturer for computing the policy parameter of interest.

2.3.4 A Realization: An Illustration of Alternative Policies

For illustrative purposes, realizations of $W(t)$ and $N(t)$ along with the corresponding realizations of the used-item inventory profile and outstanding remanufactured-item order profiles are depicted in Figures 2.2 through 2.6 for all of the policies. As noted earlier, in the description of pertinent details of the sequence of events for each remanufacturing cycle (see Section 2.2), we have two possible cases as implied by (2.1): The case when the available returns fall short of the accumulated demands is illustrated in the first remanufacturing cycle, while the case when available returns exceed the accumulated demands is illustrated in following cycle.

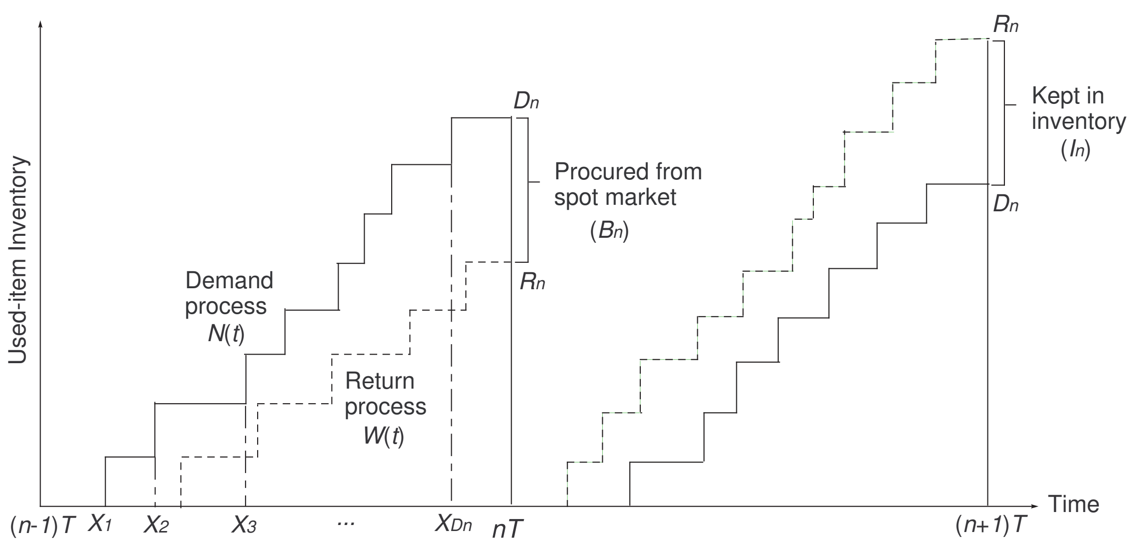

Figure 2.2: A realization under T_F-policy for two successive cycles, i.e. cycle $n-1$ and cycle n.

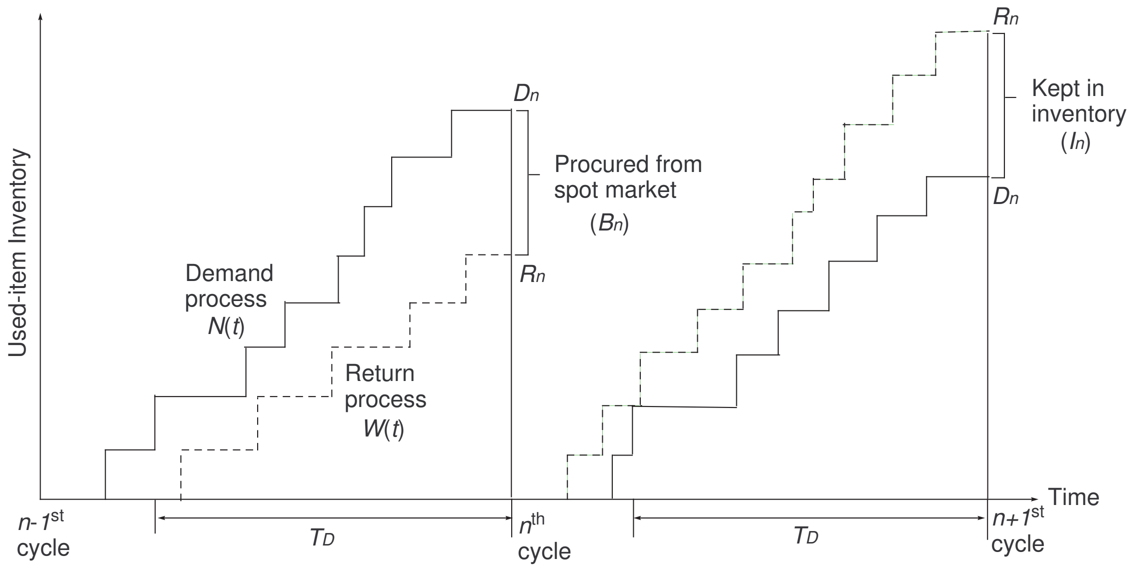

Figure 2.3: A realization under T_D-policy for two successive cycles, i.e. cycle $n-1$ and cycle n.

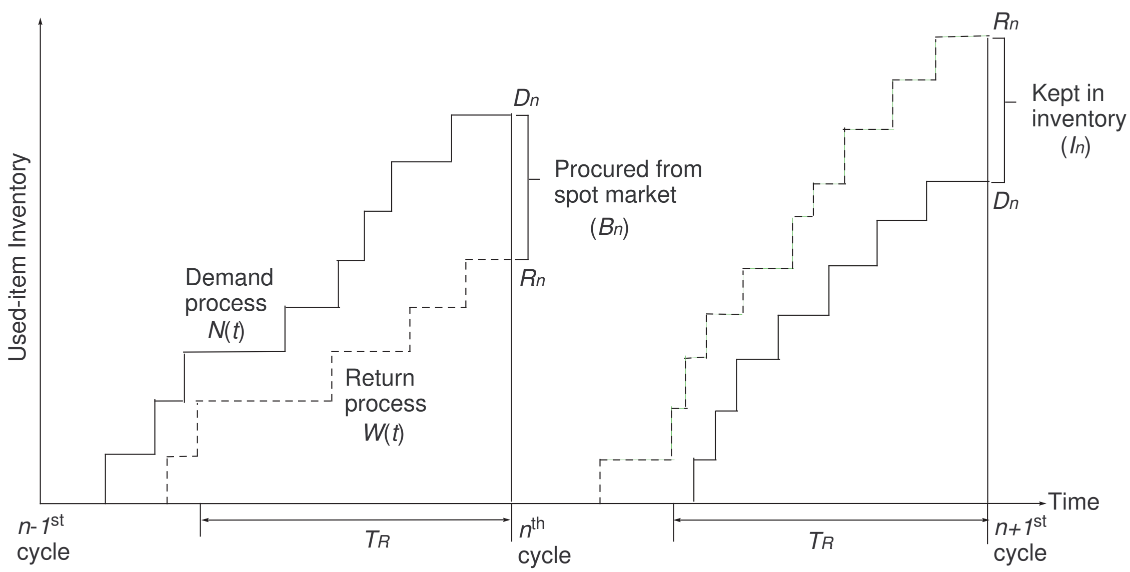

Figure 2.4: A realization under T_R-policy for two successive cycles, i.e. cycle $n-1$ and cycle n.

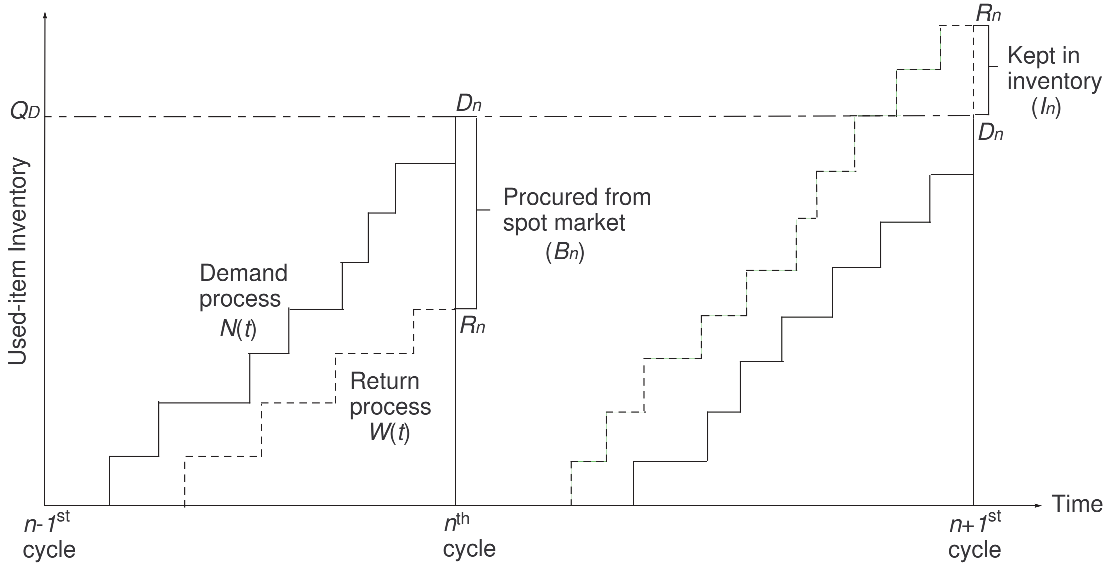

Figure 2.5: A realization under Q_D-policy for two successive cycles, i.e. cycle $n-1$ and cycle n.

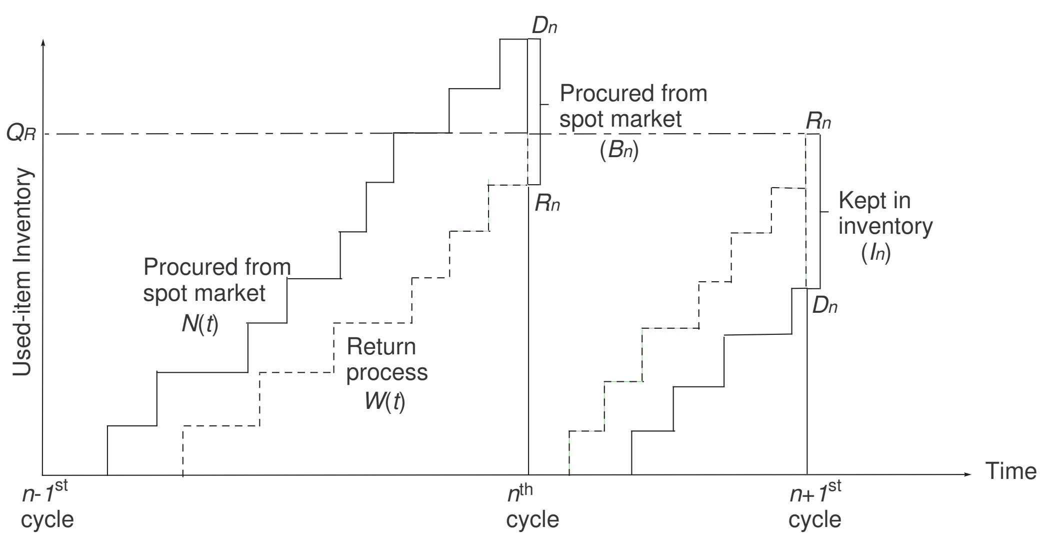

Figure 2.6: A realization under Q_R-policy for two successive cycles, i.e. cycle $n-1$ and cycle n.

We have now set the stage to derive the long-run average expected total cost functions under policies of interest. To this end, we consider the case where the remanufacturer has already operated for a sufficiently long time. Consequently, (used-item) inventory and (remanufactured-item) order profiles associated with the stochastic return and demand processes are in steady-state so that we can work with steady-state distributions.

2.4 Long-run Average Expected Total Cost Functions

Let $TC(\cdot)$ denote the long-run average expected total cost per unit time as a function of the policy parameter, e.g., as a function of T_F, T_D, T_R, Q_D or Q_R. Each of these functions consists of five main components representing the relevant terms associated with the cost parameters, h, w, K along with c and p introduced in Table 2.1:

- the used-item inventory holding cost is accrued at rate h (\$/unit/unit time);
- the remanufactured-item order waiting cost is accrued at rate w (\$/unit/unit time);
- the fixed operational cost K (\$/cycle) is incurred in each remanufacturing cycle;
- the variable cost c is incurred for each remanufactured-item; and
- the variable cost p is incurred for each used-item procured from the spot market.

It then follows that $TC(\cdot)$ is given by computing the individual terms

1. long-run average expected used-item inventory carrying cost per unit time,
2. long-run average expected remanufactured-item order waiting cost per unit time,

3. long-run average expected fixed operational cost per unit time,

4. long-run average expected variable remanufacturing cost per unit time, and

5. long-run average expected variable spot market procurement cost per unit time.

While terms 2, 3, and 4 can be evaluated by a straightforward application of the Renewal Reward Theorem (Ross (1996), Page 133), i.e.,

$$\text{Long-run average expected cost per unit time} = \frac{E\left[\text{Cycle cost}\right]}{E\left[\text{Cycle length}\right]},$$

exact expressions for terms 1 and 5 are difficult to obtain as we have already demonstrated in the previous section so that

$$\begin{aligned} TC(\cdot) = h &\left(E\left[I_n\right] + \frac{E\left[\text{Cumulative returns received in } CL(\cdot)\right]}{E\left[CL(\cdot)\right]} \right) \\ &+ \frac{wE\left[\text{Cumulative demands waiting in } CL(\cdot)\right]}{E\left[CL(\cdot)\right]} \\ &+ \frac{E\left[\text{Fixed operational cost in } CL(\cdot)\right]}{E\left[CL(\cdot)\right]} \\ &+ \frac{cE\left[\text{Remanufacturing quantity in } CL(\cdot)\right]}{E\left[CL(\cdot)\right]} \\ &+ p \lim_{t\to\infty} E\left[\frac{B(t)}{t}\right]. \end{aligned}$$

Recalling (2.2) and (2.5), under each policy[‡] the following quantities can be evaluated using Properties 1 through 5 along with (2.3):

[‡]Observe that $E\left[\text{Fixed operational cost in } CL(\cdot)\right] = K$ for all policies except for the T_F-policy. That is, under this policy, if an empty batch is not allowed then $E\left[\text{Fixed operational cost in } CL(\cdot)\right] = K\left(1 - e^{-aT_F}\right)$. One can argue that the treatment provided in this section, however, allows empty dispatches so that $E\left[\text{Fixed operational cost in } CL(\cdot)\right] = K$. Equivalently, one can argue that the demand rate a is large enough so that aT_F is also sufficiently large. Hence, the probability that no demand arrives in a cycle is nearly zero. This, in turn, implies that $K\left(1 - e^{-aT_F}\right) \approx K$.

$$
\begin{aligned}
E[I_n] &\leq \frac{E[D_n](C_a^2 + \rho^2 C_r^2)}{2(1-\rho)}, \\
E\left[\text{Cumulative returns received in } CL(\cdot)\right] &= E\left[\int_0^{CL(\cdot)} W(t)dt\right], \qquad (2.7) \\
E\left[\text{Cumulative demands waiting in } CL(\cdot)\right] &= E\left[\int_0^{CL(\cdot)} N(t)dt\right], \qquad (2.8) \\
E\left[\text{Fixed operational cost in } CL(\cdot)\right] &= K, \\
E\left[\text{Remanufacturing quantity in } CL(\cdot)\right] &= E\left[D_n\right], \quad \text{and} \\
\lim_{t\to\infty} E\left[\frac{B(t)}{t}\right] &= a - r.
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
TC(\cdot) = & hE[I_n] + \frac{hE\left[\int_0^{CL(\cdot)} W(t)dt\right]}{E\left[CL(\cdot)\right]} + \frac{wE\left[\int_0^{CL(\cdot)} N(t)dt\right]}{E\left[CL(\cdot)\right]} + \frac{K}{E\left[CL(\cdot)\right]} \\
& + \frac{cE\left[D_n\right]}{E\left[CL(\cdot)\right]} + p(a-r), \qquad (2.9)
\end{aligned}
$$

and one can approximate $TC(\cdot)$ in (2.9) using

$$
\begin{aligned}
\overline{TC}(\cdot) = & \frac{hE[D_n](C_a^2 + \rho^2 C_r^2)}{2(1-\rho)} + \frac{hE\left[\int_0^{CL(\cdot)} W(t)dt\right]}{E\left[CL(\cdot)\right]} + \frac{wE\left[\int_0^{CL(\cdot)} N(t)dt\right]}{E\left[CL(\cdot)\right]} \\
& + \frac{K}{E\left[CL(\cdot)\right]} + \frac{cE\left[D_n\right]}{E\left[CL(\cdot)\right]} + p(a-r). \qquad (2.10)
\end{aligned}
$$

2.4.1 Cost Function under T_F-policy

Under T_F-policy, let us recall Property 1 and then use (2.3) in conjunction with (2.2) so that we have

$$E\left[I_n\right] \leq \frac{a+r}{2(a-r)}. \tag{2.11}$$

Also, evaluating (2.7) and (2.8) we have

$$\begin{aligned} E\left[\int_0^{T_F} W(t)dt\right] &= \int_0^{T_F} E[W(t)]dt = \int_0^{T_F} rtdt = \frac{r{T_F}^2}{2}, \quad \text{and} \\ E\left[\int_0^{T_F} N(t)dt\right] &= \int_0^{T_F} E[N(t)]dt = \int_0^{T_F} atdt = \frac{a{T_F}^2}{2}. \end{aligned}$$

Using (2.10), it is then easy to verify that

$$\overline{TC}(T_F) = \frac{h(a+r)}{2(a-r)} + \frac{hrT_F}{2} + \frac{waT_F}{2} + \frac{K}{T_F} + ca + p(a-r) \tag{2.12}$$

which is an economic order frequency type convex function of T_F whose unique minimizer is given by

$$\hat{T}_F = \sqrt{\frac{2K}{wa+hr}}. \tag{2.13}$$

Substituting (2.13) in (2.12), we have

$$\overline{TC}(\hat{T}_F) = \frac{h(a+r)}{2(a-r)} + \sqrt{2(wa+hr)K} + ca + p(a-r).$$

While the ideal performance of the T_F-policy is difficult to benchmark and estimate in terms of problem parameters, the above result regarding $\overline{TC}(\hat{T}_F)$ provides an easily computable proxy in closed-form for the total cost which would otherwise require a computationally intensive simulation approach.

After a closer examination of the impact of utilizing (2.11) in the right hand side of (2.12) under T_F-policy, it is easy to verify that $\hat{T}_F$ is independent of the first term of (2.12). Hence, if one can identify the conditions under which $E[I_n] \approx 0$ then it can easily be argued that $\hat{T}_F$ is a superb near-optimal policy parameter under those conditions. Observations 1 and 2 examine such conditions.

Observation 1 If $r < a/3$ then $E[I_n] < 1$.

Proof. Substituting $r/a < 1/3$ in the right hand side of (2.11), we obtain

$$E[I_n] \leq \frac{1 + \frac{r}{a}}{2(1 - \frac{r}{a})} < \frac{1 + \frac{1}{3}}{2(1 - \frac{1}{3})} = 1.$$

□

By Observation 1, if r is less than one third of a, i.e., the return rate is truly less than the demand rate, then the expected number of used-items in inventory at the end of each remanufacturing cycle is less than 1.

Observation 2 If $r \leq a\left(1 - \frac{3}{\sqrt{aT_F}}\right)^2$ then $P(R_n \geq D_n) \approx 0$.

Proof. Considering that a Poisson distribution with a large arrival rate can be effectively approximated by a Normal distribution (Barlow (1989), Page 40), let us approximate the distributions of D_n and R_n under T_F-policy so that $D_n \sim \text{Normal}(aT_F, \sqrt{aT_F})$ and$R_n \sim \text{Normal}(rT_F, \sqrt{rT_F})$, respectively.

Now, recalling the well-known property of a Normal random variable which implies that about 99.7% of its possible values lie within three standard deviations of the mean, we argue that $P(R_n \geq D_n) \approx 0$ when the difference between $E[D_n]$ and

$E[R_n]$ is more than three times the sum of $\sqrt{\mathrm{Var}(D_n)}$ and $\sqrt{\mathrm{Var}(R_n)}$, as illustrated in Figure 2.7. That is, if

$$aT_F - rT_F \geq 3\left(\sqrt{aT_F} + \sqrt{rT_F}\right)$$

then $P(R_n \geq D_n) \approx 0$. Rearranging the terms of the above inequality completes the proof. $\square$

By Observation 2, if r is less than $a\left(1 - \frac{3}{\sqrt{aT_F}}\right)^2$, then the probability that the number of returns generated exceed the number of demands received in each remanufacturing cycle is approximately equal to zero.

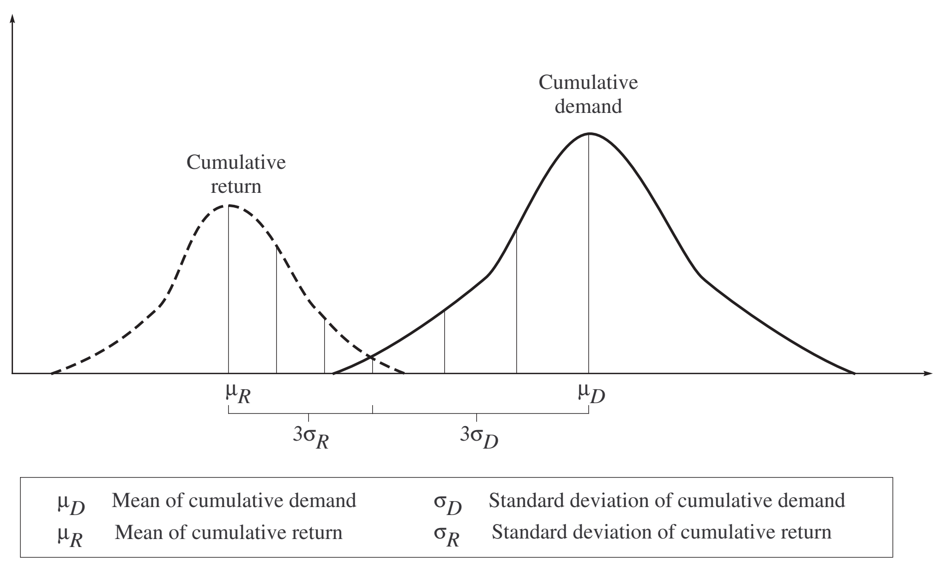

Figure 2.7: Normal distribution approximations for cumulative return and cumulative demand.

Finally, observe that under the potentially practical conditions of Observations

1 or 2, not only the unique minimizer $\hat{T}_F$ of the approximate cost function $\overline{TC}(T_F)$ in (2.12) provides an effective parameter for the T_F-policy but also the cost proxy obtained by

$$\sqrt{2(wa+hr)K} + ca + p(a-r)$$

is a superb estimate of the ideal performance. This result can be easily verified by utilizing the fact that $E[I_n]$ is negligible under the conditions of Observations 1 and 2 along with (2.9), (2.10), (2.12), and (2.13).

Now that we have established formal analytical conditions demonstrating the performance of the approximation approach proposed here, we conclude with referring the reader to the impressive numerical results in Section 2.6 examining the performance when these conditions are violated.

2.4.2 Cost Function under T_D-policy

Under T_F-policy, it is possible to observe no demand arrivals in a remanufacturing cycle. In order to avoid this situation, we consider T_D-policy. Under T_D-policy, let us recall Property 2 and then use (2.3) in conjunction with (2.2) so that we have

$$E\left[I_n\right] \leq \frac{(a+r)}{2(a-r)} \frac{(\frac{r}{a}+aT_D)}{(1+aT_D)}. \tag{2.14}$$

Also, evaluating (2.7) and (2.8) we have

$$\begin{aligned}
E\left[\int_0^{X_1+T_D} W(t)dt\right] &= E\left[\int_0^{X_1} W(t)dt\right] + E\left[\int_0^{T_D} W(t)dt\right] \\
&= E\left[E\left[\int_0^{X_1} W(t)dt \middle| X_1\right]\right] + \frac{r{T_D}^2}{2} \\
&= E\left[\int_0^{X_1} E[W(t)]dt \middle| X_1\right] + \frac{r{T_D}^2}{2} \\
&= E\left[\int_0^{X_1} rtdt \middle| X_1\right] + \frac{r{T_D}^2}{2} \\
&= E\left[\frac{rX_1^2}{2}\right] + \frac{r{T_D}^2}{2} = r\left(\frac{1}{a^2} + \frac{{T_D}^2}{2}\right) \quad \text{and} \\
E\left[\int_0^{X_1+T_D} N(t)dt\right] &= E\left[\int_0^{X_1} N(t)dt\right] + E\left[\int_0^{T_D} N(t)dt\right] \\
&= E\left[\int_0^{X_1} 0dt\right] + \int_0^{T_D} E[N(t)]dt \\
&= \int_0^{T_D} atdt = \frac{a{T_D}^2}{2}.
\end{aligned}$$

Using (2.10), it is then easy to verify that

$$\begin{aligned}
\overline{TC}(T_D) =& \frac{h(a+r)\left(\frac{r}{a} + aT_D\right)}{2(a-r)(1+aT_D)} + \frac{har}{1+aT_D}\left(\frac{1}{a^2} + \frac{{T_D}^2}{2}\right) + \frac{wa^2{T_D}^2}{2(1+aT_D)} + \frac{aK}{1+aT_D} \\
&+ ca + p(a-r). \qquad (2.15)
\end{aligned}$$

It can be proved that (2.15) is a convex function of T_D, and its minimizer is given by

$$\hat{T}_D = \sqrt{\frac{2}{a^2} + \frac{2K - \frac{w+h}{a}}{wa+hr}} - \frac{1}{a}. \qquad (2.16)$$

Subsequently, the corresponding remanufacturing cycle length is

$$\hat{T}_D + \frac{1}{a} = \sqrt{\frac{2}{a^2} + \frac{2K - \frac{w+h}{a}}{wa+hr}}, \qquad (2.17)$$

and substituting (2.16) in (2.15), one can compute $\overline{TC}(\hat{T}_D)$ as an easily computable proxy for the total cost which would otherwise require a computationally intensive simulation approach, although in this case a convenient closed-form proxy does not exist.

2.4.3 Cost Function under T_R-policy

To ensure that at least one unit of used-item return is present in each remanufacturing cycle, we consider T_R policy. Under T_R-policy, let us recall Property 3 and then use (2.3) in conjunction with (2.2) so that we have

$$E\left[I_n\right] \leq \frac{(a+r)}{2(a-r)} \frac{\left(\frac{a}{r}+rT_R\right)}{(1+rT_R)}. \tag{2.18}$$

Also, evaluating (2.7) and (2.8) we have

$$E\left[\int_0^{Y_1+T_R} W(t)dt\right] = E\left[\int_0^{Y_1} W(t)dt\right] + E\left[\int_0^{T_R} W(t)dt\right] = \frac{r{T_R}^2}{2}, \quad \text{and}$$
$$E\left[\int_0^{Y_1+T_R} N(t)dt\right] = E\left[\int_0^{Y_1} N(t)dt\right] + E\left[\int_0^{T_R} N(t)dt\right] = a\left(\frac{1}{r^2}+\frac{{T_R}^2}{2}\right).$$

Using (2.10), it is then easy to verify that

$$\begin{aligned}\overline{TC}(T_R) =& \frac{h(a+r)\left(\frac{a}{r}+rT_R\right)}{2(a-r)(1+rT_R)} + \frac{hr^2{T_R}^2}{2(1+rT_R)} + \frac{wra}{1+rT_R}\left(\frac{1}{r^2}+\frac{{T_R}^2}{2}\right) + \frac{rK}{1+rT_R} \\ &+ ca + p(a-r).\end{aligned} \tag{2.19}$$

It can be proved that (2.19) is a convex function of T_R, and its minimizer is given by

$$\hat{T}_R = \sqrt{\frac{2}{r^2} + \frac{2K + \left(\frac{a}{r}\right)^2 \frac{w+h}{a}}{wa+hr}} - \frac{1}{r}. \tag{2.20}$$

Subsequently, the corresponding remanufacturing cycle length is

$$\hat{T}_R + \frac{1}{r} = \sqrt{\frac{2}{r^2} + \frac{2K + (\frac{a}{r})^2 \frac{w+h}{a}}{wa + hr}}. \tag{2.21}$$

and substituting (2.20) in (2.19), one can compute $\overline{TC}(\hat{T}_R)$ as an easily computable proxy for the total cost which would otherwise require a computationally intensive simulation approach, although in this case a convenient closed-form proxy does not exist.

2.4.4 Cost Function under Q_D-policy

When Q_D-policy is in effect, the cumulative demand in each remanufacturing cycle is a constant with value Q_D but the remanufacturing cycle length is a random variable denoted by S_{Q_D}.

Under Q_D-policy, let us recall Property 4 and then use (2.3) in conjunction with (2.2) so that we have

$$E\left[I_n\right] \leq \frac{r(a+r)}{2a(a-r)}. \tag{2.22}$$

Also, evaluating (2.7) and (2.8) we have

$$E\left[\int_0^{S_{Q_D}} W(t)dt\right] = E\left[\int_0^{S_{Q_D}} rtdt\right] = \frac{r}{2}E\left[S_{Q_D}^2\right] = \frac{rQ_D(Q_D+1)}{2a^2}, \quad \text{and}$$
$$E\left[\int_0^{S_{Q_D}} N(t)dt\right] = E\left[\sum_{i=1}^{Q_D} iX_{i+1}\right] = \sum_{i=1}^{Q_D} iE\left[X_{i+1}\right] = \sum_{i=1}^{Q_D} \frac{i}{a} = \frac{Q_D(Q_D-1)}{2a}.$$

Using (2.10), it is then easy to verify that

$$\overline{TC}(Q_D) = \frac{hr(a+r)}{2a(a-r)} + \frac{hr(Q_D+1)}{2a} + \frac{w(Q_D-1)}{2} + \frac{aK}{Q_D} + ca + p(a-r). \tag{2.23}$$

It can be shown that (2.23) is convex with respect to Q_D, and its minimizer is given by

$$\hat{Q}_D = \sqrt{\frac{2aK}{w + h\frac{r}{a}}}. \tag{2.24}$$

Substituting (2.24) in (2.23), we have

$$\overline{TC}(\hat{Q}_D) = \frac{r(a+r)}{2a(a-r)} + \sqrt{2(wa+hr)K} + \frac{hr - wa}{2a} + ca + p(a-r).$$

Then, $\overline{TC}(Q_D)$ provides an easily computable proxy in closed-form for the total cost function which would otherwise require a computationally intensive simulation approach.

After a closer examination of the impact of utilizing (2.22) in the right hand side of (2.23) under Q_D-policy, it is easy to verify that $\hat{Q}_D$ is independent of the first term of (2.23). Hence, if one can identify the conditions under which $E[I_n] \approx 0$ then it can easily be argued that $\hat{Q}_D$ is a superb near-optimal policy parameter under those conditions. Observations 3 and 4 examine such conditions.

Observation 3 If $r < a\left(\sqrt{17} - 3\right)/2$ then $E[I_n] < 1$.

The proof is straightforward and similar to the proof of Observation 1, and, hence, it is omitted.

By Observation 3, when the return rate is less than $\left(\sqrt{17} - 3\right)/2$ times the demand rate, the expected number of used-items in inventory at the end of each remanufacturing cycle is less than 1.

Observation 4 If $Q_D > \frac{9r(a+r)}{(a-r)^2}$ then $P(R_n \geq Q_D) \approx 0$.

Proof. Similar to the Proof of Observation 2, if the difference between Q_D and $E[R_n]$ is more than three times the sum of their standard deviations, i.e.,

$$Q_D - \frac{rQ_D}{a} > 3\sqrt{\left(\frac{rQ_D}{a} + \frac{r^2Q_D}{a^2}\right)} \tag{2.25}$$

then $P(R_n \geq Q_D) \approx 0$. Note that (2.25) is equivalent to $Q_D > \frac{9r(a+r)}{(a-r)^2}$. □

By Observation 4, if the threshold value Q_D in Q_D-policy is larger than $\frac{9r(a+r)}{(a-r)^2}$, then the probability that the number of returns generated, i.e., R_n, exceed the number of demands received, i.e., Q_D, in each remanufacturing cycle is approximately zero.

Finally, observe that under the potentially practical conditions of Observations 3 or 4, not only the unique minimizer $\hat{Q}_D$ of the approximate cost function $\overline{TC}(Q_D)$ in (2.23) provides an effective parameter for the Q_D-policy but also the cost proxy obtained by

$$\sqrt{2(wa + hr)K} + \frac{hr - wa}{2} + ca + p(a - r)$$

is a superb estimate of the ideal performance. This result can be easily verified by utilizing the fact that $E[I_n]$ is negligible under the conditions of Observations 3 and 4 along with (2.9), (2.10), (2.23), and (2.24).

Now that we have established formal analytical conditions demonstrating the performance of the approximation approach proposed here, we conclude with referring the reader to the impressive numerical results in Section 2.6 where we examine the performance when these conditions are violated.

2.4.5 Cost Function under Q_R-policy

When Q_R-policy is in effect, the cumulative return in each remanufacturing cycle is a constant with value Q_R but the remanufacturing cycle length is a random variable denoted by Z_{Q_R}.

Under Q_R-policy, let us recall Property 5 and then use (2.3) in conjunction with (2.2) so that we have

$$E[I_n] \leq \frac{E[D_n](C_a^2 + \rho^2 C_s^2)}{2(1-\rho)} = \frac{(a+r)}{2(a-r)}\frac{a}{r}. \tag{2.26}$$

Also, evaluating (2.7) and (2.8) we have

$$\begin{aligned} E\left[\int_0^{Z_{Q_R}} W(t)dt\right] &= \frac{Q_R(Q_R-1)}{2r} \quad \text{and} \\ E\left[\int_0^{Z_{Q_R}} N(t)dt\right] &= \frac{aE[Z_{Q_R}^2]}{2} = \frac{aQ_R(Q_R+1)}{2r^2}. \end{aligned}$$

Using (2.10), it is then easy to verify that

$$\overline{TC}(Q_R) = \frac{ha(a+r)}{2r(a-r)} + \frac{h(Q_R-1)}{2} + \frac{wa(Q_R+1)}{2r} + \frac{rK}{Q_R} + ca + p(a-r). \tag{2.27}$$

It can be shown that (2.27) is convex with respect to Q_R, and its minimizer is given by

$$\hat{Q}_R = \sqrt{\frac{2rK}{\frac{wa}{r}+h}}. \tag{2.28}$$

Substituting (2.28) in (2.27), we have

$$\overline{TC}(\hat{Q}_R) = \frac{ha(a+r)}{2r(a-r)} + \sqrt{2(wa+hr)K} + \frac{wa-hr}{2r} + ca + p(a-r).$$

Then, $\overline{TC}(\hat{Q}_R)$ provides an easily computable proxy for the total cost function which would otherwise require a computationally intensive simulation approach.

After a closer examination of the impact of utilizing (2.26) in the right hand side of (2.27) under Q_R-policy, it is easy to verify that $\hat{Q}_R$ is independent of the first term of (2.27). Hence, if one can identify the conditions under which $E[I_n] \approx 0$ then it can easily be argued that $\hat{Q}_R$ is a superb near-optimal policy parameter under those conditions. Observation 5 examines such conditions.

Observation 5 If $Q_R > \frac{9a(a+r)}{(a-r)^2}$ then $P(Q_R \geq D_n) \approx 0$.

The proof is straightforward and similar to the proof of Observation 4, and, hence, it is omitted.

By Observation 5, if the threshold value Q_R in Q_R-policy is larger than $\frac{9a(a+r)}{(a-r)^2}$, then the probability that the number of returns generated, i.e., Q_R, exceed the number of demands received, i.e., D_n, in each remanufacturing cycle is approximately equal to zero.

Finally, observe that under the potentially practical condition of Observation 5, not only the unique minimizer $\hat{Q}_R$ of the approximate cost function $\overline{TC}(Q_R)$ in (2.27) provides an effective parameter for the Q_R-policy but also the cost proxy obtained by

$$\sqrt{2(wa+hr)K} + \frac{wa-hr}{2r} + ca + p(a-r)$$

is a superb estimate of the ideal performance. This result can be easily verified by utilizing the fact that $E[I_n]$ is negligible under the condition of Observation 5 along with (2.9), (2.10), (2.27), and (2.28).

Now that we have established formal analytical conditions demonstrating the performance of the approximation approach proposed here, we conclude with referring the reader to the impressive numerical results in Section 2.6 examining the performance when these conditions are violated.

2.5 Comparisons and Insights

In this section, we provide an overview of our results for a comparative analysis. To this end, Table 2.3 summarizes the results in Property 1 though Property 5.

Next, we summarize the exact and approximate cost functions for our five policies in Table 2.4. The minimizers of the approximate cost functions in Table 2.4 are summarized in Table 2.5. By comparing the results under our five policies, we have some additional observations.

Observation 6 The expected optimal remanufacturing cycle length under T_R-policy is longer than the expected optimal remanufacturing cycle length under T_D-policy. That is, we have

$$\hat{T}_R + \frac{1}{r} > \hat{T}_D + \frac{1}{a}.$$

Observation 7 When $w > h(1 - \frac{2r}{a})$, the expected optimal remanufacturing cycle length under T_F-policy is the shortest one among all the three periodic policies. That is, we have

$$\hat{T}_R + \frac{1}{r} > \hat{T}_D + \frac{1}{a} > \hat{T}_F.$$

Observation 8 The optimal threshold value $\hat{Q}_R$ under Q_R-policy is smaller than the approximate optimal threshold value $\hat{Q}_D$ under Q_D-policy, and they have the following relationship:

$$\hat{Q}_R = \frac{r}{a}\hat{Q}_D.$$

Observation 9 The expected optimal remanufacturing cycle length when Q_D-policy or Q_R-policy is in effect is same as the expected optimal remanufacturing cycle length under T_F-policy, i.e.,

$$E\left[S_{\hat{Q}_D}\right] = E\left[Z_{\hat{Q}_R}\right] = \hat{T}_F.$$

Table 2.3: Expected value and variance of the number of returns accumulated in a remanufacturing cycle; expected value and variance of the number of demands in a remanufacturing cycle, squared coefficient of variation of the number of demands in a remanufacturing cycle; squared coefficient of variation of the number of returns in a remanufacturing cycle; and traffic intensity.

Policy	$(E[R_n])$	$(\mathrm{Var}(R_n))$	$(E[D_n])$	$(\mathrm{Var}(D_n))$	(C_a^2)	(C_s^2)	(ρ)
T_F-policy	rT_F	rT_F	aT_F	aT_F	$\frac{1}{aT_F}$	$\frac{1}{rT_F}$	$\frac{r}{a}$
T_D-policy	$\frac{r}{a}(1+aT_D)$	$\frac{r}{a}+rT_D+\frac{r^2}{a^2}$	$1+aT_D$	aT_D	$\frac{aT_D}{(1+aT_D)^2}$	$\frac{1+\frac{a}{r}(1+aT_{DP})}{(1+aT_D)^2}$	$\frac{r}{a}$
T_R-policy	$1+rT_R$	rT_R	$\frac{a}{r}(1+rT_R)$	$\frac{a}{r}+aT_R+\frac{a^2}{r^2}$	$\frac{1+\frac{r}{a}(1+rT_R)}{(1+rT_R)^2}$	$\frac{rT_R}{(1+rT_R)^2}$	$\frac{r}{a}$
Q_D-policy	$\frac{rQ_D}{a}$	$\frac{rQ_D}{a}+\frac{r^2Q_D}{a^2}$	Q_D	0	0	$\frac{1}{Q_D}\left(\frac{a}{r}+1\right)$	$\frac{r}{a}$
Q_R-policy	Q_R	0	$\frac{aQ_R}{r}$	$\frac{aQ_R}{r}+\frac{a^2Q_R}{r^2}$	$\frac{1}{Q_R}\left(\frac{r}{a}+1\right)$	0	$\frac{r}{a}$

Table 2.4: Exact cost functions and approximate cost functions.

Policy	Exact Cost Function	Approximate Cost Function
T_F-policy	$hE[I_n] + \frac{(wa+hr)T_F}{2} + \frac{K}{T_F} + ca + p(a-r)$	$\frac{h(a+r)}{2(a-r)} + \frac{(wa+hr)T_F}{2} + \frac{K}{T_F} + ca + p(a-r)$
T_D-policy	$hE[I_n] + \frac{wa^2{T_D}^2}{2(1+aT_D)} + \frac{hra}{1+aT_D}\left(\frac{1}{a^2} + \frac{{T_D}^2}{2}\right) + \frac{aK}{1+aT_D} + ca + p(a-r)$	$\frac{h\left(\frac{r}{a}+aT_D\right)(a+r)}{2(1+aT_D)(a-r)} + \frac{wa^2{T_D}^2}{2(1+aT_D)} + \frac{hra}{1+aT_D}\left(\frac{1}{a^2} + \frac{{T_D}^2}{2}\right) + \frac{aK}{1+aT_D} + ca + p(a-r)$
T_R-policy	$hE[I_n] + \frac{wra}{1+rT_R}\left(\frac{1}{r^2} + \frac{{T_R}^2}{2}\right) + \frac{hr^2{T_R}^2}{2(1+rT_R)} + \frac{rK}{1+rT_R} + ca + p(a-r)$	$\frac{h\left(\frac{a}{r}+rT_R\right)(a+r)}{2(1+rT_R)(a-r)} + \frac{wra}{1+rT_R}\left(\frac{1}{r^2} + \frac{{T_R}^2}{2}\right) + \frac{hr^2{T_R}^2}{2(1+rT_R)} + \frac{rK}{1+rT_R} + ca + p(a-r)$
Q_D-policy	$hE[I_n] + \frac{w(Q_D-1)}{2} + \frac{hr(Q_D+1)}{2a} + \frac{aK}{Q_D} + ca + p(a-r)$	$\frac{hr(a+r)}{2a(a-r)} + \frac{w(Q_D-1)}{2} + \frac{hr(Q_D+1)}{2a} + \frac{aK}{Q_D} + ca + p(a-r)$
Q_R-policy	$hE[I_n] + \frac{wa(Q_R+1)}{2r} + \frac{h(Q_R-1)}{2} + \frac{rK}{Q_R} + ca + p(a-r)$	$\frac{ha(a+r)}{2r(a-r)} + \frac{wa(Q_R+1)}{2r} + \frac{h(Q_R-1)}{2} + \frac{rK}{Q_R} + ca + p(a-r)$

Table 2.5: Near-optimal policy parameter and expected cycle length.

Policy	Policy Parameter	Cycle Length
$T_F - policy$	$\hat{T}_F = \sqrt{\frac{2K}{wa+hr}}$	$\sqrt{\frac{2K}{wa+hr}}$
T_D-policy	$\hat{T}_D = \sqrt{\frac{2}{a^2} + \frac{2K - \frac{w+h}{a}}{wa+hr}} - \frac{1}{a}$	$\sqrt{\frac{2}{a^2} + \frac{2K - \frac{w+h}{a}}{wa+hr}}$
T_R-policy	$\hat{T}_R = \sqrt{\frac{2}{r^2} + \frac{2K + \left(\frac{a}{r}\right)^2 \frac{w+h}{a}}{wa+hr}} - \frac{1}{r}$	$\sqrt{\frac{2}{r^2} + \frac{2K + \left(\frac{a}{r}\right)^2 \frac{w+h}{a}}{wa+hr}}$
Q_D-policy	$\hat{Q}_D = \sqrt{\frac{2aK}{w + h\frac{r}{a}}}$	$\sqrt{\frac{2K}{wa+hr}}$
Q_R-policy	$\hat{Q}_R = \sqrt{\frac{2rK}{\frac{wa}{r} + h}}$	$\sqrt{\frac{2K}{wa+hr}}$

Observations 6 to 9 can be obtained directly by checking the policy parameters summarized in Table 2.5. By Observations 6 to 9, in general, the expected optimal remanufacturing cycle lengthes under T_F-policy and threshold policies are the same. Under T_D-policy, the expected optimal remanufacturing cycle length is longer, compared with T_F-policy. The expected optimal remanufacturing cycle under T_R-policy is even longer, compared with T_D-policy. That is because the remanufacturer needs to wait at least one unit of demand arrives under T_D-policy, or to wait at least one unit of return arrives under T_R-policy, which is not required in T_F-policy; and the mean arrival time of the first return is longer than the mean arrival time of the first demand, since the return rate is less than the demand rate.

The above four observations compared the expected optimal cycle lengthes under different policies. The following observations investigate the dependence of policy parameters in Table 2.5 on the model parameters.

Observation 10 *All the policy parameters in Table 2.5, i.e.,* $\hat{T}_F$, $\hat{T}_D$, $\hat{T}_R$, $\hat{Q}_D$ *and* $\hat{Q}_R$, *are increasing in* K, *whereas they are decreasing in* w.

Proof. We only provide the proof for that $\hat{T}_R$ is decreasing in w, since the other part of the observation is obvious by checking the expressions of the policy parameters in Table 2.5.

Since $\hat{T}_R = \sqrt{\frac{2}{r^2} + A} - \frac{1}{r}$, where $A = \frac{2K + \left(\frac{a}{r}\right)^2 \frac{w+h}{a}}{wa+hr}$, that to prove $\hat{T}_R$ is decreasing in w is equivalent to prove A is decreasing in w. This can be done by checking the first derivative of A with respect to w:

$$\frac{\partial A}{\partial w} = \frac{\frac{a}{r^2}(wa+hr) - \left(2K + \frac{a}{r^2}(w+h)\right)a}{(wa+hr)^2} = \frac{-2Ka - \frac{a}{r}h\left(\frac{a}{r} - 1\right)}{(wa+hr)^2} < 0.$$

Hence, A is decreasing in w, which indicates that $\hat{T}_R$ is decreasing in w. $\square$

Observation 10 indicates that the remanufacturer needs to do remanufacturing and then satisfies the cumulative demands less frequently as the fixed cost K increases, whereas it needs to remanufacture and then satisfies the demands more frequently as the waiting cost w increases. This is intuitive: when the fixed cost is high, the remanufacturer needs to prolong the remanufacturing cycle, i.e., to keep low processing frequency, in order to avoid high fixed cost; when the waiting cost is high, the remanufacturer needs to shorten the remanufacturing cycle, i.e., to keep high processing frequency, in order to reduce the total waiting time of the demands.

Observation 11 $\hat{T}_F$, $\hat{T}_D$, $\hat{Q}_D$ and $\hat{Q}_R$ are decreasing in h; $\hat{T}_R$ is decreasing in h if $2K \geq \frac{a}{r^2}\left(\frac{a}{r} - 1\right)w$.

Proof. By checking the expressions of $\hat{T}_F$, $\hat{T}_D$, $\hat{Q}_D$ and $\hat{Q}_R$ in Table 2.5, it is obvious that these policy parameters are decreasing in h. Thus we only need to check the monotonicity of $\hat{T}_R$ in h, which is equivalent to check the monotonicity of A in h, where $A = \frac{2K + \left(\frac{a}{r}\right)^2 \frac{w+h}{a}}{wa+hr}$. This can be done by checking the first derivative of A

with respective to h:

$$\frac{\partial A}{\partial h} = \frac{\frac{a}{r^2}(wa+hr) - \left(2K + \frac{a}{r^2}(w+h)\right) r}{(wa+hr)^2} = \frac{-2Kr + \frac{a}{r} w \left(\frac{a}{r} - 1\right)}{(wa+hr)^2},$$

which is less than 0 if $2K \geq \frac{a}{r^2}\left(\frac{a}{r} - 1\right) w$. □

Observation 11 indicates that, in general, the remanufacturer needs to do remanufacturing and then satisfies the cumulative demands more frequently as the holding h increases. That means when the holding cost is high, the remanufacturer needs to shorten the remanufacturing cycle, i.e., to keep high processing frequency, in order to reduce the total inventory of used-items in each cycle, and thus to reduce the total inventory holding cost.

Observation 12 For T_F-policy, $\hat{T}_F$ is decreasing in a and r.

This observation is obvious by checking the expression of $\hat{T}_F$ in Table 2.5, and hence, the proof is omitted.

By Observations 9 and 12, the expected optimal cycle lengths under Q_D-policy and Q_R-policy are also decreasing in a and r. This indicates that, under T_F-policy, Q_D-policy and Q_R-policy, the remanufacturer needs to do remanufacturing and then satisfies the cumulative demands more frequently as the demand rate and/or the return rate increase. This is because when the demand rate is high, the waiting cost of the demands in each cycle is significant, and shorter cycle length helps to reduce the total waiting time of the demands in each cycle, and hence, helps to reduce the waiting cost. Similarly, when the return rate is high, the inventory cost of used-items in each cycle is significant, and shorter cycle length helps to reduce the total inventory of used-items, and hence, helps to reduce the inventory holding cost.

Observation 13 Under T_D-policy, $\hat{T}_D$ is decreasing in r.

This observation is obvious by checking the expression of $\hat{T}_D$ in Table 2.5, and hence, the proof is omitted.

Observations 13 indicates that the remanufacturer needs to shorten the remanufacturing cycle as the return rate increases. This is because when the return rate is high, the inventory cost of used-items in each cycle is significant, and shorter cycle length helps to reduce the total inventory of used-items, and hence, helps to reduce the inventory holding cost.

Observation 14 Under T_R-policy, $\hat{T}_R$ is decreasing in a if $2K \geq \frac{h}{r}\left(1 + \frac{h}{w}\right)$.

Proof. That to check the monotonicity of $\hat{T}_R$ in a is equivalent to check the monotonicity of A in a, where $A = \frac{2K + \left(\frac{a}{r}\right)^2 \frac{w+h}{a}}{wa+hr}$. This can be done by checking the first derivative of A with respective to a:

$$\frac{\partial A}{\partial a} = \frac{\frac{w+h}{r^2}(wa+hr) - \left(2K + \frac{a}{r^2}(w+h)\right) w}{(wa+hr)^2} = \frac{-2Kw + \frac{(w+h)h}{r}}{(wa+hr)^2},$$

which is less than 0 if $2K \geq \frac{h}{r}\left(1 + \frac{h}{w}\right)$. □

Observations 14 indicates that, in general, the remanufacturer needs to shorten the remanufacturing cycle as the demand rate increases. This is because when the demand rate is high, the waiting cost of the demands in each cycle is significant, and shorter cycle length helps to reduce the total waiting time of the demands in each cycle, and hence, helps to reduce the waiting cost.

Observation 15 Under Q_D-policy, $\hat{Q}_D$ is decreasing in r, whereas it is increasing in a.

Proof. By checking the expression of $\hat{Q}_D$ in Table 2.5, it is obvious that $\hat{Q}_D$ is decreasing in r. To prove that $\hat{Q}_D$ is increasing in a, we only need to prove that

$\frac{2aK}{w+h\frac{r}{a}}$ is increasing in a. Let us denote $\frac{2aK}{w+h\frac{r}{a}}$ by B, then we have

$$\frac{\partial B}{\partial a} = \frac{4Ka(wa+hr) - 2Kwa^2}{(wa+hr)^2} = \frac{2Kwa^2 + 4Khar}{(wa+hr)^2} > 0.$$

Hence, B is increasing in a, which indicates that $\hat{Q}_D$ is increasing in a. $\square$

Observations 15 indicates that the remanufacturer needs to decrease the threshold value Q_D as the return rate increases. That means the remanufacturer will shorten the remanufacturing cycle in order to reduce the inventory cost which is significant when the return rate is high. Meanwhile, large demand rate means a large amount of demand can be accumulated in a short time, and thus, the threshold value Q_D can be large.

Observation 16 Under Q_R-policy, $\hat{Q}_R$ is decreasing in a, whereas it is increasing in r.

The proof is similar with the proof of Observation 16, and hence, is omitted.

Observations 16 indicates that the remanufacturer needs to decrease the threshold value Q_R as the demand rate increases. That means the remanufacturer will shorten the remanufacturing cycle in order to reduce the waiting cost which is significant when the demand rate is high. Meanwhile, large return rate means a large amount of returns can be accumulated in a short time, and thus, the threshold value Q_R can be large.

2.6 Numerical Experiments

A diligent numerical investigation demonstrates that although the difference between our approximate and exact cost functions can be substantial in some cases, the use of the approximate policy parameters we propose would work well in practice. Hence, our contribution to the literature lies in providing a systematic and comprehensive analysis of cost performance of periodic and threshold policies for the remanufacturer and determining analytically tractable and practically effective approximate operating parameters.

2.6.1 Objective of Experimentation

Recall that the exact total cost function and the approximation on the total cost function are denoted by $TC(\cdot)$ and $\overline{TC}(\cdot)$, respectively, for policies characterized by parameters T_F, T_D, T_R, Q_D, and Q_R. Also, we let T_F^*, T_D^*, T_R^*, Q_D^*, and Q_R^* denote the optimal values of these parameters, and recall that $\hat{T}_F$, $\hat{T}_D$, $\hat{T}_R$, $\hat{Q}_D$, and $\hat{Q}_R$ denote the near-optimal values of these parameters.

The goal of our numerical experimentation is two-fold. First, we want to assess the *quality* of our approximations. To this end, we examine the performance implication of using the approximation (the minimizer of which can be evaluated analytically) rather than the exact function itself (the minimizer of which can only be evaluated numerically) to specify the operating parameter of the system. For this purpose, we use the following metric:

$$\frac{\overline{TC}(\hat{\cdot}) - TC(\cdot^*)}{TC(\cdot^*)}.$$

Second, we want to test the *effectiveness* of our approximation. To this end, we examine the performance implication of using the minimizer of the approximation (which can be evaluated analytically) as an *approximate* minimizer for the exact cost function. For this purpose, we use the following metric:

$$\frac{TC(\hat{\cdot}) - TC(\cdot_i^*)}{TC(\cdot^*)}.$$

It is possible to make a distinction between problem settings where our approximation approaches would perform well or poorly. Specifically, we would expect our approaches to perform well, when the contribution of the used-item inventory holding costs to the total cost is negligible. For this to happen, one or more of the following factors should be in effect: (i) the policy parameter (cycle length or threshold value) is sufficiently large and a small change in the policy parameter does not lead to a substantial change in the value of the total cost; (ii) the return rate is sufficiently low in comparison to the demand rate and the initial inventory of returned used-items is nearly zero in each remanufacturing cycle; (iii) the unit used-item inventory holding cost is sufficiently low and the used-item inventory holding cost accounts for a small fraction of the total cost. In settings, where the contribution of the used-item inventory holding cost to the total cost is much more substantial, we expect our approaches to perform poorly. Therefore, in choosing parameter sets for our numerical experiments, we include a broad set of parameter values that would lead to settings where our approximations would perform well or poorly.

2.6.2 Parameter Settings

In our experiments, we consider three levels for the demand rate a (50, 12, and 3), two levels for the fixed cost K (125 and 25), and two levels for the unit used-item procurement cost p (10 and 80). To set the value of unit used-item inventory holding cost, we use $h = 0.10p$. A careful examination of our analytical results show that the optimal policy parameters depend on the ratios K/w and h/w. Hence, we fix the unit customer waiting cost at $w = 10$. We also use a fixed value for the unit reprocessing cost at $c = 20$. For a given level of the demand rate, we specify the return rate such that $r = \alpha a$. For our numerical experiments, we consider four levels of α (0.1, 0.2, 0.4, 0.8). As a result, we consider a total of 48 problem instances and analyze each instance under each of the five policies.

2.6.3 Experimentation

Given a problem instance and a particular policy, we first obtain the optimal policy parameter using simulation as follows. We generate return and demand amounts independently for 10,000 consecutive remanufacturing cycles and determine the used-item purchase quantities for each of the remanufacturing cycles. Since our analysis is based on the assumption that the system reaches steady state, we discard the data that correspond to the first 50,000 remanufacturing cycles as warm-up and use the data from the remaining 50,000 remanufacturing cycles to evaluate the long-run average expected total cost. We begin by verifying the convexity of the cost function using a plot over the search region. Using the plot, we reduce the length of the search region and step size until we determine the optimal value of the minimizer for the exact cost function for the policy. We then determine optimal value of the minimizer of the approximation for the policy and evaluate the performance metrics.

2.6.4 Quality of the Bounds

In Tables 2.6 and 2.7, we report the average-case and worst-case performance for the quality of our bounds. In particular, the value reported in each cell of Table 2.6 (Table 2.7) is the average (maximum) value for the performance metric that we use to assess the quality of the bounds over 12 instances considered for the corresponding level of α when a particular policy is in effect. The cells in the last row of these Tables report the average (maximum) values for the performance metric over all of the problem instances considered.

Based on the results summarized in Tables 2.6 and 2.7, we can make a number of observations on the quality of our approximations . First and foremost, it can be observed that our approximations perform well on average under all of the policies. For each of the policies, the performance deteriorates as the ratio of the return rate to

Table 2.6: Quality of the approximations (2.12) for T_F-policy, (2.15) for T_D-policy, (2.19) for T_R-policy, (2.23) for Q_D-policy, and (2.27) for Q_R-policy: Average-case performance.

α	T_F-policy	T_D-policy	T_R-policy	Q_D-policy	Q_R-policy
0.1	0.42	0.37	2.17	0.03	2.86
0.2	0.53	0.49	1.52	0.11	2.08
0.4	0.83	0.87	1.50	0.30	1.79
0.8	2.01	4.00	4.52	1.32	2.60
Overall	0.95	1.43	2.43	0.46	2.33

Table 2.7: Quality of the approximations (2.12) for T_F-policy, (2.15) for T_D-policy, (2.19) for T_R-policy, (2.23) for Q_D-policy, and (2.27) for Q_R-policy: Worst-case performance.

α	T_F-policy	T_D-policy	T_R-policy	Q_D-policy	Q_R-policy
0.1	1.50	1.24	8.12	0.12	11.22
0.2	1.89	1.66	6.23	0.35	8.07
0.4	2.87	3.08	6.39	1.09	6.53
0.8	6.34	15.42	18.29	4.26	8.10
Overall	6.34	15.42	18.29	4.26	8.10

the demand rate increases. This is not surprising. A higher return rate increases the probability of having initial used-item inventory, which, in turn, might have a notable impact on the expected inventory carrying cost. Consequently, the deviation between the approximation and the exact cost function increases, reducing the quality of our approximations . Furthermore, among the periodic policies, the approximation for T_F-policy exhibits the best average- and worst-case performance. Between the two threshold policies, the approximation for Q_D-policy performs better than the one for Q_R-policy both in terms of average- and worst-case performance. Last but not least, the best threshold policy, i.e., Q_D-policy, is better than the best periodic policy, i.e., T_F-policy, in terms of both the average- and worst-case performance.

We also observe that return-based policies perform worse than their demand-based counterparts both for periodic and threshold policies. This, in fact, is not surprising as it is a result of our approximation approach: The approximations of the initial used-item inventory for demand-driven policies is influenced by the ratio r/a, whereas for return-driven polices, the approximations are influenced by a/r. Since $r < a$, the demand-driven policies lend themselves into tighter approximations.

2.6.5 Effectiveness of the Bounds

Our numerical results summarized in Tables 2.8 and provide some convincing evidence on the practical relevance of our approximations . As before, the value reported in each cell of Table 2.8 (Table 2.9) is the average (maximum) value for the performance metric that we use to assess the effectiveness of the approximations over 12 instances considered for the corresponding level of α when a particular policy is in effect.

Table 2.8: Effectiveness of the approximations (2.12) for T_F-policy, (2.15) for T_D-policy, (2.19) for T_R-policy, (2.23) for Q_D-policy, and (2.27) for Q_R-policy: Average-case performance.

α	T_F-policy	T_D-policy	T_R-policy	Q_D-policy	Q_R-policy
0.1	0.00	0.00	0.05	0.00	0.00
0.2	0.00	0.00	0.03	0.00	0.01
0.4	0.01	0.00	0.01	0.00	0.00
0.8	0.13	0.06	0.05	0.01	0.08
Overall	0.04	0.02	0.04	0.00	0.02

Recall that this performance metric quantifies the benefit of using the approximation as an approximate minimizer for the exact cost function itself. By Tables 2.8 and , it can be observed that both the average- and worst-case results are within

Table 2.9: Effectiveness of the approximations (2.12) for T_F-policy, (2.15) for T_D-policy, (2.19) for T_R-policy, (2.23) for Q_D-policy, and (2.27) for Q_R-policy: Worst-case performance.

α	T_F-policy	T_D-policy	T_R-policy	Q_D-policy	Q_R-policy
0.1	0.00	0.01	0.38	0.00	0.00
0.2	0.02	0.01	0.23	0.00	0.17
0.4	0.05	0.02	0.09	0.01	0.00
0.8	0.97	0.43	0.30	0.06	0.90
Overall	0.97	0.43	0.38	0.06	0.90

1%, i.e., if the minimizer of the approximations were to be used as the approximate policy parameter, the deviation in the exact total cost function would not be larger than 1% in the worst case across all test parameters and under any of the policies we consider. Consequently, the minimizers of the approximations that can be obtained numerically can be used as high-quality approximate minimizers of the exact total cost functions for each of the corresponding policies.

2.7 The Case Where $r \geq a$

In previous sections, by using Kingman's approximation in (2.2), we analyze the case where $r < a$. Now, let us turn our attention to the case where $r \geq a$. Obviously, we need to avoid excessive amount of used-items by considering the disposal option explicitly. For this reason, we need to extend the definitions of the five policies. To this end, we introduce a new parameter U for incorporating the disposal option, and our five new policies are:

- T_F (fixed period) with disposal option: decision is (T_F, U_{TF});
- T_D (demand-driven periodic) with disposal option: decision is (T_D, U_{TD});
- T_R (return-driven periodic) with disposal option: decision is (T_R, U_{TR});

- Q_D (demand-driven threshold) with disposal option: decision is (Q_D, U_{QD});
- Q_R (return-driven threshold) with disposal option: decision is (Q_R, U_{QR}).

We denote I'_n as the used-item inventory level at the end of the n_{th} remanufacturing cycle after disposal, to differentiate it with I_n. B'_n denotes the spot market procurement quantity associated with remanufacturing cycle n, to be differentiated with B_n.

The sequence of events for remanufacturing cycle n is as follows:

- the remanufacturer measures the initial inventory of used-items, which is the inventory of used-items at the end of the $n-1$st remanufacturing cycle after disposal, which is denoted by I'_{n-1};
- the remanufacturer consolidates returns during cycle n, and the return amount is denoted by R_n;
- the remanufacturer observes the demand in cycle n, and the demand received is denoted by D_n;
- a batch processing is triggered according to the policy of interest, and all the demand D_n are cleared:
  - if the used-item inventory before batch processing, i.e., $I'_{n-1}+R_n$, is larger than or equal to D_n, then after the batch run, $I'_{n-1}+R_n-D_n$ units of used-items are left:
    * if $I'_{n-1}+R_n-D_n$ is above U, then the remanufacturer disposes $I'_{n-1}+R_n-D_n-U$ units, and keeps U units of used-items to the next cycle,
    * otherwise, the remanufacturer keeps $I'_{n-1}+R_n-D_n$ units of used-items to the next cycle;

– if $I'_{n-1}+R_n$ is less than D_n, the remanufacturer purchases $D_n-(I'_{n-1}+R_n)$ units of used-items from the spot market, and clears all the demand with the used-items on hand as well as the used-items purchased from the spot market.

Thus, the used-item inventory at the end of remanufacturing cycle n after disposal, i.e., I'_n, is given by:

$$I'_n = \begin{cases} \min\{U, I'_{n-1} + R_n - D_n\}, & D_n \leq I'_{n-1} + R_n, \\ 0, & D_n > I'_{n-1} + R_n. \end{cases} \tag{2.29}$$

and the procurement from spot market in remanufacturing cycle n, denoted by B'_n, is given by:

$$B'_n = \max\{0, D_n - \left(I'_{n-1} + R_n\right)\} = \begin{cases} 0, & D_n \leq I'_{n-1} + R_n, \\ I'_{n-1} + R_n - D_n, & D_n > I'_{n-1} + R_n. \end{cases} \tag{2.30}$$

From (2.29) and (2.30), we know that the exact analytical closed-form expressions of $E[I'_n]$ and $E[B'_n]$ are hard to obtain, if not impossible, because of the underliniing G/G/1 queue being controlled by two policy parameters under the new five policies. The exact analytical closed-form expression for the expected disposal amount in remanufacturing cycle n, which is given by $E[(I'_{n-1} + R_n - D_n - U)^+]$, is also hard to derive.

All the cost components are the same as stated in Section 2.4, except the disposal cost incurred under the new policies. By using $E[I'_n]$ and $E[B'_n]$, and considering the disposal cost, the long-run average expected total cost function under our new policies are given by:

$$
\begin{aligned}
TC(\cdot) &= hE[I'_n] + \frac{hE\left[\int_0^{CL(\cdot)} W(t)dt\right]}{E[CL(\cdot)]} + \frac{wE\left[\int_0^{CL(\cdot)} N(t)dt\right]}{E[CL(\cdot)]} + \frac{K}{E[CL(\cdot)]} \\
&\quad + \frac{cE[D_n]}{E[CL(\cdot)]} + \frac{pE[B'_n]}{E[CL(\cdot)]} + \frac{c^d E[(I'_{n-1} + R_n - D_n - U)^+]}{E[CL(\cdot)]}, \qquad (2.31)
\end{aligned}
$$

where I'_n and B'_n are given by (2.29) and (2.30), respectively. Note that, in (2.31), except the first item and the last two items, all the other items are same as in (2.9). Let us recall Property 1 to Property 5, and then use the results for (2.7) and (2.8) in Section 2.4, so that we can obtain the cost functions under the five new policies which are respectively given by

- T_F-policy with disposal option:

$$
\begin{aligned}
TC(T_F, U_{TF}) &= hE[I'_n] + \frac{hrT_F}{2} + \frac{waT_F}{2} + \frac{K}{T_F} + ca \\
&\quad + \frac{pE[B'_n]}{T_F} + \frac{c^d E[(I'_{n-1} + R_n - D_n - U)^+]}{T_F}; \qquad (2.32)
\end{aligned}
$$

- T_D-policy with disposal option:

$$
\begin{aligned}
TC(T_D, U_{TD}) &= hE[I'_n] + \frac{hra}{1+aT_D}\left(\frac{1}{a^2} + \frac{{T_D}^2}{2}\right) + \frac{wa^2{T_D}^2}{2(1+aT_D)} + \frac{aK}{1+aT_D} \\
&\quad + ca + \frac{apE[B'_n]}{1+aT_D} + \frac{ac^d E[(I'_{n-1} + R_n - D_n - U)^+]}{1+aT_D}; \qquad (2.33)
\end{aligned}
$$

- T_R-policy with disposal option:

$$
\begin{aligned}
TC(T_R, U_{TR}) &= hE[I'_n] + \frac{hr^2{T_R}^2}{2(1+rT_R)} + \frac{wra}{1+rT_R}\left(\frac{1}{r^2} + \frac{{T_R}^2}{2}\right) + \frac{rK}{1+rT_R} \\
&\quad + ca + \frac{rpE[B'_n]}{1+rT_R} + \frac{rc^d E[(I'_{n-1} + R_n - D_n - U)^+]}{1+rT_R}; \qquad (2.34)
\end{aligned}
$$

- Q_D-policy with disposal option:

$$\begin{aligned} TC(Q_D, U_{QD}) &= hE[I'_n] + \frac{w(Q_D-1)}{2} + \frac{hr(Q_D+1)}{2a} + \frac{aK}{Q_D} + ca \\ &+ \frac{apE[B'_n]}{Q_D} + \frac{ac^d E[(I'_{n-1}+R_n-D_n-U)^+]}{Q_D}; \end{aligned} \quad (2.35)$$

- Q_R-policy with disposal option:

$$\begin{aligned} TC(Q_R, U_{QR}) &= hE[I'_n] + \frac{wa(Q_R+1)}{2r} + \frac{h(Q_R-1)}{2} + \frac{rK}{Q_R} + ca \\ &+ \frac{rpE[B'_n]}{Q_R} + \frac{rc^d E[(I'_{n-1}+R_n-D_n-U)^+]}{Q_R}. \end{aligned} \quad (2.36)$$

We let (T_F^*, U_{TF}^*), (T_D^*, U_{TD}^*), (T_R^*, U_{TR}^*), (Q_D^*, U_{QD}^*) and $(Q_R^*, , U_{QR}^*)$ denote the minimizers of the above five cost functions, respectively. Since the analytical closed-form expressions of $E[I'_n]$, $E[B'_n]$ and the expected disposal amount are hard to obtain, equations (2.32) to (2.36) cannot be minimized analytically. Thus, we find approximations for the cost functions, and use the minimizers of those approximate cost functions as approximations for optimal policy parameters. For this purpose, we propose three approximation approaches: (1) myopic approximation; (2) simulation-based approach and (3) parameter-based approximation. We will explain these three approaches in details in the following. Also we will check the performance of each approximation approach numerically.

2.7.1 Myopic Approximation

The myopic approximation is intuitive and easy to implement. We let $U = 0$, which means that all the extra used-items are disposed. It is intuitive because that when $r > a$, there is a high chance that the return amount is larger than the demand amount in each cycle. By (2.29) and (2.30), it is easy to verify that $I'_{n-1} = I'_n = 0$,

$B'_n = (D_n - R_n)^+$, and $(I'_{n-1} + R_n - D_n - U)^+ = (R_n - D_n)^+$, given $U = 0$. Since $r > a$, we roughly assume that $E[B'_n] \approx 0$ and $E[(R_n - D_n)^+] \approx (r-a)E[CL(\cdot)]$. Then we can obtain the approximate function for (2.31), denoted by $\overline{TC}^{(m)}(\cdot)$, which are given by:

$$\begin{aligned}\overline{TC}^{(m)}(\cdot) &= \frac{hE\left[\int_0^{CL(\cdot)} W(t)dt\right]}{E[CL(\cdot)]} + \frac{wE\left[\int_0^{CL(\cdot)} N(t)dt\right]}{E[CL(\cdot)]} \\ &\quad + \frac{K}{E[CL(\cdot)]} + \frac{cE[D_n]}{E[CL(\cdot)]} + c^d(r-a).\end{aligned} \tag{2.37}$$

Then, (2.32) to (2.36) can be approximated by the following equations:

$$\overline{TC}^{(m)}(T_F, U_{TF}) = \frac{hrT_F}{2} + \frac{waT_F}{2} + \frac{K}{T_F} + ca + c^d(r-a), \tag{2.38}$$

$$\begin{aligned}\overline{TC}^{(m)}(T_D, U_{TD}) &= \frac{hra}{1+aT_D}\left(\frac{1}{a^2} + \frac{{T_D}^2}{2}\right) + \frac{wa^2{T_D}^2}{2(1+aT_D)} + \frac{aK}{1+aT_D} \\ &\quad + ca + c^d(r-a),\end{aligned} \tag{2.39}$$

$$\begin{aligned}\overline{TC}^{(m)}(T_R, U_{TR}) &= \frac{hr^2{T_R}^2}{2(1+rT_R)} + \frac{wra}{1+rT_R}\left(\frac{1}{r^2} + \frac{{T_R}^2}{2}\right) + \frac{rK}{1+rT_R} \\ &\quad + ca + c^d(r-a),\end{aligned} \tag{2.40}$$

$$\begin{aligned}\overline{TC}^{(m)}(Q_D, U_{QD}) &= \frac{w(Q_D - 1)}{2} + \frac{hr(Q_D+1)}{2a} + \frac{aK}{Q_D} \\ &\quad + ca + c^d(r-a), \quad \text{and}\end{aligned} \tag{2.41}$$

$$\begin{aligned}\overline{TC}^{(m)}(Q_R, U_{QR}) &= \frac{wa(Q_R+1)}{2r} + \frac{h(Q_R-1)}{2} + \frac{rK}{Q_R} \\ &\quad + ca + c^d(r-a),\end{aligned} \tag{2.42}$$

respectively.

Compare equations (2.38) to (2.42) with equations in the last column in Table 2.4, respectively, we observe that the difference is just constant. Due to the similarity, the minimizers of (2.38) to (2.42) are given by $\hat{T_F}$, $\hat{T_D}$, $\hat{T_R}$, $\hat{Q_D}$ and $\hat{Q_R}$, respectively, which are the minimizers of the approximate cost functions in previous situation and are summarized in Table 2.5. Thus, using myopic approximation approach, the approximations for optimal policy parameters under the five new policies are given by ($\hat{T_F}$,0), ($\hat{T_D}, 0$), ($\hat{T_R}, 0$), ($\hat{Q_D}, 0$) and ($\hat{Q_R}, 0$), respectively, where $\hat{T_F}$, $\hat{T_D}$, $\hat{T_R}$, $\hat{Q_D}$ and $\hat{Q_R}$ are as in Table 2.5. The myopic approximations for the optimal policy parameters under the five new policies are summarized in Table 2.10.

Table 2.10: Myopic approximations for optimal policy parameters.

Policy	Policy Parameters
T_F-policy with disposal option	$(\hat{T}_F, \hat{U}_{TF}) = \left(\sqrt{\frac{2K}{wa+hr}}, 0\right)$,
T_D-policy with disposal option	$(\hat{T}_D, \hat{U}_{TD}) = \left(\sqrt{\frac{2}{a^2} + \frac{2K - \frac{w+h}{a}}{wa+hr}} - \frac{1}{a}, 0\right)$
T_R-policy with disposal option	$(\hat{T}_R, \hat{U}_{TR}) = \left(\sqrt{\frac{2}{r^2} + \frac{2K + \left(\frac{a}{r}\right)^2 \frac{w+h}{a}}{wa+hr}} - \frac{1}{r}, 0\right)$
Q_D-policy with disposal option	$(\hat{Q}_D, \hat{U}_{QD}) = \left(\sqrt{\frac{2aK}{w+h\frac{r}{a}}}, 0\right)$
Q_R-policy with disposal option	$(\hat{Q}_R, \hat{U}_{QR}) = \left(\sqrt{\frac{2rK}{\frac{wa}{r}+h}}, 0\right)$

The myopic approximation actually implies that the remanufacturer disposes all extra used-items at the end of each remanufacturing cycle, i.e., $U = 0$. Meanwhile, $\hat{T_F}$, $\hat{T_D}$, $\hat{T_R}$, $\hat{Q_D}$ and $\hat{Q_R}$ are used as approximations for the optimal values of T_F, T_D, T_R, Q_D and Q_R, respectively. Next, we will check numerically whether this approximation approach works well, i.e., whether ($\hat{T_F}$,0), ($\hat{T_D}, 0$), ($\hat{T_R}, 0$), ($\hat{Q_D}, 0$) and ($\hat{Q_R}, 0$) can be used as approximations for (T_F^*, U_{TF}^*), (T_D^*, U_{TD}^*), (T_R^*, U_{TR}^*),

(Q_D^*, U_{QD}^*) and $(Q_R^*, , U_{QR}^*)$, respectively. For this purpose, we consider six levels of α (1.0, 1.2, 1.4, 1.6, 1.8, 2.0) and use the same parameter settings in Section 2.6. Currently, we let $c^d = 0$, and check the following metric for each instance

$$\frac{TC(\hat{\cdot}, 0) - TC(\cdot^*, \cdot^*)}{TC(\cdot^*, \cdot^*)}.$$

This performance metric quantifies the benefit of using the minimizers of those approximate cost functions as approximations for optimal policy parameters. By Table 2.11, it can be observed that the average-case results are within 5%. By Table 2.12, the worst-case results are within 30%. Based on the results in Tables 2.11 and 2.12, we can conclude that the performance getting better as the ratio of the return rate to the demand rate increases, when the disposal cost is zero. This is not surprising. A higher return rate decreases the probability of needing procurement from the spot market. Thus, there is no need to keep left-over used-items.

Table 2.11: Effectiveness of the myopic approximation: Average-case performance.

α	T_F with dis.	T_D with dis.	T_R with dis.	Q_D with dis.	Q_R with dis.
1	9.43%	9.62%	9.2%	9.18%	8.36%
1.2	6.35%	6.491%	5.93%	6.25%	4.55%
1.4	3.6%	3.74%	3.36%	4.241%	2.03%
1.6	2.09%	2.34%	1.91%	2.48%	0.65%
1.8	1.27%	1.37%	1.10%	1.85%	0.34%
2	0.81%	1.01	0.56%	1.27%	0.22%
Overall	3.92%	4.09%	3.68%	4.26%	2.69%

A careful examination of the data sets reveals that the worst case for each α value happens when the shortage cost, i.e., unit purchase cost of used-item, has major impacts on the total cost. That is in the case that the fixed cost K is low, and

Table 2.12: Effectiveness of the myopic approximation: Worst-case performance.

α	T_F with dis.	T_D with dis.	T_R with dis.	Q_D with dis.	Q_R with dis.
1	29.43%	29.85%	29.36%	28.51%	28.86%
1.2	21.88%	21.47%	19.88%	21.22%	17.01%
1.4	13.83%	14.9%	13.54%	14.95%	9.47%
1.6	8.98%	10.46%	8.42%	9.92%	3.63%
1.8	7.29%	7.07%	5.52%	9.64%	1.89%
2	4.5 %	6.78%	3.02%	7.05%	1.69%
Overall	29.43 %	29.85%	29.36%	28.51%	28.86%

the purchase price p of use-item is high. Then, setting $U = 0$ implies high chance of stocking out, especially for the case that α is not large. Thus, under the situation that the shortage cost has major impacts, the dispose-all policy might result high cost for small α.

2.7.2 Simulation-based Approach

The above section proposed a myopic approximation that assuming $U = 0$. The numerical results show that this approximation approach does not work well in general, especially in the situation that the ratio of the return rate to the demand rate is not large enough. Next we will provide an accurate approximation approach which is based on computationally intensive simulations.

We will take advantage of the results obtained previously. To be more specific, we will still use $\hat{T_F}$, $\hat{T_D}$, $\hat{T_R}$, $\hat{Q_D}$ and $\hat{Q_R}$ in Table 2.5, as approximations for T_F^*, T_D^*, T_R^*, Q_D^* and Q_R^*, respectively. Then, we search for the minimizer of the approximate cost function, which can be used as the approximation for the optimal U, numerically. Thus, using simulation-based approach, the approximations for optimal policy parameters under the five new policies are given by $(\hat{T_F}, U^*(\hat{T_F}))$, $(\hat{T_D}, U^*(\hat{T_D}))$, $(\hat{T_R}, U^*(\hat{T_R}))$, $(\hat{Q_D}, U^*(\hat{Q_D}))$ and $(\hat{Q_R}, U^*(\hat{Q_R}))$, respectively, where $\hat{T_F}$, $\hat{T_D}$, $\hat{T_R}$, $\hat{Q_D}$

and $\hat{Q_R}$ are as in Table 2.5, and $U^*(\hat{\cdot})$ can be obtained numerically. $U^*(\hat{\cdot})$ is the approximation for the optimal U for the given $\hat{T_F}$, $\hat{T_D}$, $\hat{T_R}$, $\hat{Q_D}$ or $\hat{Q_R}$ value. We will check the following metric to evaluate the effectiveness of this approximation approach:

$$\frac{TC(\hat{\cdot}, U^*(\hat{\cdot})) - TC(\cdot^*, \cdot^*)}{TC(\cdot^*, \cdot^*)}.$$

The numerical results are summarized in Table 2.13 and Table 2.14 (we use the same parameter settings as in Myopic approximation approach). By Table 2.13, it can be observed that the average-case results are within 0.5%. By Table 2.14, it can be observed that the worst-case results are within 3%. The numerical investigation demonstrates that this approximation approach works well in general for our new policies when the disposal cost is zero.

Table 2.13: Effectiveness of the simulation-based approximation: Average-case performance.

α	T_F with dis.	T_D with dis.	T_R with dis.	Q_D with dis.	Q_R with dis.
1	0.27%	0.12%	0.17%	0.02%	0%
1.2	0.16%	0.22%	0.16%	0%	0.04%
1.4	0.22%	0.2%	0.26%	0.04%	0.05%
1.6	0.25%	0.3%	0.24%	0.04%	0.04%
1.8	0.22%	0.3%	0.23%	0.01%	0.12%
2	0.27%	0.37	0.22%	0.02%	0.23%
Overall	0.23%	0.25%	0.21%	0.02%	0.08%

2.7.3 Parameter-based Approximation

The above two sections proposed two approximation approaches for estimating the total cost functions under our five new policies. The myopic approximation is

Table 2.14: Effectiveness of the simulation-based approximation: Worst-case performance.

α	T_F with dis.	T_D with dis.	T_R with dis.	Q_D with dis.	Q_R with dis.
1	1.01%	0.59%	0.62%	0.3%	0%
1.2	0.6%	0.99%	0.74%	0%	0.19%
1.4	0.99%	0.73%	1.11%	0.18%	0.25%
1.6	0.9%	1.16%	0.78%	0.22%	0.21%
1.8	0.91%	1.55%	1.14%	0.05%	0.76%
2	1.6 %	2.37%	0.91%	0.15%	1.83%
Overall	1.6 %	2.37%	1.14%	0.3%	1.83%

easy to implement, but is not accurate in general. The simulation approach can guarantee the accuracy, but is computationally intensive. In this section we will propose a parameter-based approximation approach which can provide easy ways to compute the approximations for the optimal policy parameters while guarantee the accuracy in general.

If we assume $I'_n = U$, and solve B'_n from (2.30), then we have $B'_n = \max\{0, D_n - R_n - U\}$. The disposal amount in remanufacturing cycle n is given by $(U + R_n - D_n - U)^+ = (R_n - D_n)^+$. Recalling Properties 1 to 5, if we use Normal distributions to approximate the Poisson distributions for D_n and R_n, then $D_n - R_n$ is also normal distributed. For each policy we can approximate $D_n - R_n$ as follows:

- T_F-policy with disposal option:

$$D_n - R_n \sim Norm\left((a-r)T_F, \sqrt{(a+r)T_F}\right); \tag{2.43}$$

- T_D-policy with disposal option:

$$D_n - R_n \sim Norm\left((a-r)T_D + 1 - \frac{r}{a}, \sqrt{(a+r)T_D + \frac{r}{a} + \frac{r^2}{a^2}}\right); \tag{2.44}$$

- T_R-policy with disposal option:

$$D_n - R_n \sim Norm\left((a-r)T_R - 1 + \frac{a}{r}, \sqrt{(a+r)T_R + \frac{a}{r} + \frac{a^2}{r^2}}\right); \quad (2.45)$$

- Q_D-policy with disposal option:

$$D_n - R_n \sim Norm\left(\left(1 - \frac{r}{a}\right) Q_D, \sqrt{\left(\frac{r}{a} + \frac{r^2}{a^2}\right) Q_D}\right); \quad (2.46)$$

- Q_R-policy with disposal option:

$$D_n - R_n \sim Norm\left(\left(\frac{a}{r} - 1\right) Q_R, \sqrt{\left(\frac{a}{r} + \frac{a^2}{r^2}\right) Q_R}\right). \quad (2.47)$$

We denote the CDFs of $(D_n - R_n)$ by $F_{TF}(\cdot)$, $F_{TD}(\cdot)$, $F_{TR}(\cdot)$, $F_{QD}(\cdot)$, and $F_{QR}(\cdot)$, respectively for the five policies. The corresponding pdfs are denoted by $f_{TF}(\cdot)$, $f_{TD}(\cdot)$, $f_{TR}(\cdot)$, $f_{QD}(\cdot)$, and $f_{QR}(\cdot)$, respectively. Then we have

$$E[B'_n] = \int_U^{\infty} (x-U) f_i(x) dx, \quad i \in \{TF, TD, TR, QD, QR\}, \quad \text{and} \quad (2.48)$$

$$E[(R_n - D_n)^+)] = \int_{-\infty}^{0} -x f_i(x) dx, \quad i \in \{TF, TD, TR, QD, QR\}. \quad (2.49)$$

Substituting $I'_n = U$, (2.48), and (2.49) in cost functions (2.32) to (2.36), we obtain the following approximations for (2.32) to (2.36) respectively:

$$\begin{aligned} \overline{TC}^{(p)}(T_F, U_{TF}) &= hU + \frac{hrT_F}{2} + \frac{waT_F}{2} + \frac{K}{T_F} + ca \\ &\quad + \frac{p\int_U^{\infty}(x-U)f_{TF}(x)dx}{T_F} + \frac{c^d \int_{-\infty}^{0} -x f_{TF}(x)dx}{T_F}, \quad (2.50) \end{aligned}$$

$$\overline{TC}^{(p)}(T_D, U_{TD}) = hU + \frac{hra}{1+aT_D}\left(\frac{1}{a^2} + \frac{{T_D}^2}{2}\right) + \frac{wa^2{T_D}^2}{2(1+aT_D)} + \frac{aK}{1+aT_D} + ca + \frac{ap\int_U^\infty (x-U)f_{TD}(x)dx}{1+aT_D} + \frac{ac^d\int_{-\infty}^0 -xf_{TD}(x)dx}{1+aT_D}, \tag{2.51}$$

$$\overline{TC}^{(p)}(T_R, U_{TR}) = hU + \frac{hr^2{T_R}^2}{2(1+rT_R)} + \frac{wra}{1+rT_R}\left(\frac{1}{r^2} + \frac{{T_R}^2}{2}\right) + \frac{rK}{1+rT_R} + ca + \frac{rp\int_U^\infty (x-U)f_{TR}(x)dx}{1+rT_R} + \frac{rc^d\int_{-\infty}^0 -xf_{TR}(x)dx}{1+rT_R}, \tag{2.52}$$

$$\overline{TC}^{(p)}(Q_D, U_{QD}) = hU + \frac{w(Q_D-1)}{2} + \frac{hr(Q_D+1)}{2a} + \frac{aK}{Q_D} + ca + \frac{ap\int_U^\infty (x-U)f_{QD}(x)dx}{Q_D} + \frac{ac^d\int_{-\infty}^0 -xf_{QD}(x)dx}{Q_D}, \quad \text{and} \tag{2.53}$$

$$\overline{TC}^{(p)}(Q_R, U_{QR}) = hU + \frac{wa(Q_R+1)}{2r} + \frac{h(Q_R-1)}{2} + \frac{rK}{Q_R} + ca + \frac{rp\int_U^\infty (x-U)f_{QR}(x)dx}{Q_R} + \frac{rc^d\int_{-\infty}^0 -xf_{QR}(x)dx}{Q_R}. \tag{2.54}$$

By (2.50) to (2.54), it can be easily proved that: for any given T_F, $\overline{TC}^{(p)}(T_F, U_{TF})$ is convex in U; for any given T_D, $\overline{TC}^{(p)}(T_D, U_{TD})$ is convex in U; for any given T_R, $\overline{TC}^{(p)}(T_R, U_{TR})$ is convex in U; for any given Q_D, $\overline{TC}^{(p)}(Q_D, U_{QD})$ is convex in U; and for any given Q_R, $\overline{TC}^{(p)}(Q_R, U_{QR})$ is convex in U. Therefore, we use $\hat{T_F}$, $\hat{T_D}$, $\hat{T_R}$, $\hat{Q_D}$ and $\hat{Q_R}$ in Table 2.5, as approximations for T_F^*, T_D^*, T_R^*, Q_D^* and Q_R^*, respectively, and approximate the optimal U value by setting the first derivative of the above five cost functions with respect to U equal to zero. Then we obtain the approximation for the optimal U as follows:

- T_F-policy with disposal option:

$$\hat{U}_{TF} = F_{TF}^{-1}\left(1 - \frac{h\hat{T}_F}{P}\right); \tag{2.55}$$

- T_D-policy with disposal option:

$$\hat{U}_{TD} = F_{TD}^{-1}\left(1 - \frac{h(1 + a\hat{T}_D)}{aP}\right); \tag{2.56}$$

- T_R-policy with disposal option:

$$\hat{U}_{TR} = F_{TR}^{-1}\left(1 - \frac{h(1 + r\hat{T}_R)}{rP}\right); \tag{2.57}$$

- Q_D-policy with disposal option:

$$\hat{U}_{QD} = F_{QD}^{-1}\left(1 - \frac{h\hat{Q}_D}{aP}\right); \tag{2.58}$$

- Q_R-policy with disposal option:

$$\hat{U}_{QR} = F_{QR}^{-1}\left(1 - \frac{h\hat{Q}_R}{rP}\right). \tag{2.59}$$

By (2.55) to (2.59), together with (2.43) to (2.47), we can obtain $\hat{U}$ values which can be used as approximations for the real optimal U values, i.e., U^*. Thus, using parameter-based approach, the approximations for optimal policy parameters under the five new policies are given by $(\hat{T_F}, \hat{U}_{TF})$, $(\hat{T_D}, \hat{U}_{TD})$, $(\hat{T_R}, \hat{U}_{TR})$, $(\hat{Q_D}, \hat{U}_{QD})$ and $(\hat{Q_R}, \hat{U}_{QR})$, respectively, where $\hat{T_F}$, $\hat{T_D}$, $\hat{T_R}$, $\hat{Q_D}$ and $\hat{Q_R}$ are as in Table 2.5, and $\hat{U}_{TF}$, $\hat{U}_{TD}$, $\hat{U}_{TR}$, $\hat{U}_{QD}$ and $\hat{U}_{QR}$ are given by (2.55) to (2.59), respectively. The parameter-based approximations for the optimal policy parameters under the five new policies are summarized in Table 2.15. Next we will check the effectiveness of

Table 2.15: Parameter-based approximations for optimal policy parameters.

Policy	Policy Parameters
T_F-policy with disposal option	$(\hat{T}_F, \hat{U}_{TF}) = \left(\sqrt{\frac{2K}{wa+hr}}, F_{TF}^{-1}\left(1-\frac{h\hat{T}_F}{P}\right)\right)$
T_D-policy with disposal option	$(\hat{T}_D, \hat{U}_{TD}) = \left(\sqrt{\frac{2}{a^2}+\frac{2K-\frac{w+h}{a}}{wa+hr}}-\frac{1}{a}, F_{TD}^{-1}\left(1-\frac{h(1+a\hat{T}_D)}{aP}\right)\right)$
T_R-policy with disposal option	$(\hat{T}_R, \hat{U}_{TR}) = \left(\sqrt{\frac{2}{r^2}+\frac{2K+\left(\frac{a}{r}\right)^2\frac{w+h}{a}}{wa+hr}}-\frac{1}{r}, F_{TR}^{-1}\left(1-\frac{h(1+r\hat{T}_R)}{rP}\right)\right)$
Q_D-policy with disposal option	$(\hat{Q}_D, \hat{U}_{QD}) = \left(\sqrt{\frac{2aK}{w+h\frac{r}{a}}}, F_{QD}^{-1}\left(1-\frac{h\hat{Q}_D}{aP}\right)\right)$
Q_R-policy with disposal option	$(\hat{Q}_R, \hat{U}_{QR}) = \left(\sqrt{\frac{2rK}{\frac{wa}{r}+h}}, F_{QR}^{-1}\left(1-\frac{h\hat{Q}_R}{rP}\right)\right)$

this approximation using the following metric

$$\frac{TC(\hat{\cdot},\hat{\cdot})-TC(\cdot^*,\cdot^*)}{TC(\cdot^*,\cdot^*)}.$$

The results are summarized in Tables 2.16 and 2.17 (we use the same parameter settings as in Myopic approximation approach). By Table 2.16, it can be observed that the average-case results are within 1%. By Table 2.17, it can be observed that the worst cases are around 5% over all policies. Thus, we can conclude that the parameter-based approximation works well in general when the disposal cost is zero. Recall that $\hat{T_F}$, $\hat{T_D}$, $\hat{T_R}$, $\hat{Q_D}$ and $\hat{Q_R}$ are as in Table 3, and $\hat{U_i}$, where $i \in \{TF, TD, TR, QD, QR\}$, can be obtained by using (2.55) to (2.59). Thus, this approximation approach is easy to implement and well-performed in the sense of providing accurate policy parameters.

Table 2.16: Effectiveness of parameter-based approximation : Average-case performance.

α	T_F with dis.	T_D with dis.	T_R with dis.	Q_D with dis.	Q_R with dis.
1	1.11%	0.99%	0.99%	0.9%	0.78%
1.2	0.31%	0.36%	0.27%	0.06%	0.22%
1.4	0.27%	0.23%	0.29%	0.11%	0.13%
1.6	0.27%	0.45%	0.28%	0.26%	0.12%
1.8	0.22%	0.53%	0.27%	0.45%	0.12%
2	0.33%	0.48%	0.38%	0.58%	0.26%
Overall	0.42%	0.5 %	0.41%	0.39%	0.27%

Table 2.17: Effectiveness of parameter-based approximation: Worst-case performance.

α	T_F with dis.	T_D with dis.	T_R with dis.	Q_D with dis.	Q_R with dis.
1	4.45%	4.89%	5.27%	4.72%	4.47%
1.2	1.08%	0.99%	0.83%	0.37%	1.12%
1.4	0.99%	0.73%	1.14%	0.45%	0.71%
1.6	0.9%	2.14%	0.78%	0.77%	0.86%
1.8	0.91%	3.85%	1.41%	1.19%	0.76%
2	1.97 %	2.89%	2.42%	2.21%	2.19%
Overall	4.45 %	4.89%	5.27%	4.72%	4.47%

All the above numerical results are assuming zero disposal cost, and in this situation we showed that parameter-based approximation is easy to implement and works well in general. Next we will verify the effectiveness of the parameter-based approach considering positive disposal cost explicitly. We consider two levels of the unit disposal cost for a given level of used-item inventory holding cost: $c_d = 1.5h$ and $c_d = 3h$. The numerical results with positive disposal cost are summarized in Tables 2.18 and 2.19. It can be observed that the average-case results are within 1.7% and the worst cases are around 7.6% over all policies. Thus, the parameter-based approximation still works well with positive disposal cost.

Table 2.18: Effectiveness of parameter-based approximation considering positive disposal cost: Average-case performance.

α	T_F with dis.	T_D with dis.	T_R with dis.	Q_D with dis.	Q_R with dis.
1	1.6%	1.63%	1.57%	1.44%	1.36%
1.2	0.49%	0.51%	0.44%	0.15%	0.32%
1.4	0.28%	0.2%	0.23%	0.04%	0.14%
1.6	0.28%	0.33%	0.22%	0.1%	0.07%
1.8	0.2%	0.38%	0.18%	0.2%	0.12%
2	0.22%	0.35%	0.24%	0.3%	0.17%
Overall	0.51%	0.57 %	0.48%	0.37%	0.36%

2.8 Conclusions

In this section, considering a fundamental inventory and production planning problem characterized by a batch processing environment with stochastic demands and stochastic returns along with fixed operational costs and disposal opportunities, we propose a comprehensive set of periodic and threshold batching policies.

We aim to derive analytical expressions for the long-run average expected total cost functions under the proposed policies to determine the optimal policy parame-

Table 2.19: Effectiveness of parameter-based approximation considering positive disposal cost: Worst-case performance.

α	T_F with dis.	T_D with dis.	T_R with dis.	Q_D with dis.	Q_R with dis.
1	7.47%	7.19%	7.63%	7.03%	7.16%
1.2	2.31%	1.99%	1.98%	1.12%	2.13%
1.4	1%	0.89%	0.75%	0.21%	0.52%
1.6	1.13%	1.5%	0.93%	0.45%	0.4%
1.8	0.77%	2.55%	0.74%	0.7%	0.68%
2	1.31 %	2.05%	1.53%	1.31%	1.62%
Overall	7.47 %	7.19%	7.63%	7.03%	7.16%

ters. We demonstrate that the mismatch of the return and the demand leads to the fundamental difficulty in obtaining exact closed-form expressions of the cost functions. Therefore, we develop analytically tractable approximations, and we report numerical results demonstrating the quality and effectiveness of these approximations. Last by not least, we observe that the demand-driven threshold policy performs the best on the average in term of the resulting expected cost for the case when the return rate is less than the demand rate.

For the case when the return rate exceeds the demand rate so that we execute the disposal option, the relative cost performances of the alternative batching policies considered depend on the approximation approach associated with the disposal decision. More specifically, when we use the myopic approach for the disposal decision, the return-driven threshold policy is superior on the average in terms of the resulting expected cost; when we use the simulation-based approach for the disposal decision, the demand-driven threshold policy is superior; and when we use the parameter-based approach for the disposal decision, the return-driven threshold policy is superior.

An important extension of our work should explore the potential benefit of building a reprocessed device buffer in the system to reduce the impact of customer waiting costs for settings where this cost can be substantial. Clearly, this extension requires a thorough investigation of a two stock-point system, where both returned used-items and remanufactured-items can be kept in inventory. Another important extension is to investigate the structure of the exact optimal batching policies using stochastic dynamic programming or Markov decision processes. Other interesting extensions include explicit modeling of multiple decentralized agents, e.g., remanufacturer, collection center, and retailer and/or considering more general return process.

3. SEED STOCK PLANNING STRATEGIES WITH MULTIPLE AGENTS

3.1 Overview of Section 3

We consider a basic game-theoretic setting for the seed stock planning problem in remanufacturing. The problem can be characterized as a finite horizon inventory control problem with multiple agents including an OEM, a NPS, and a RS. The OEM provides a particular type of replacement part for a product it sells. The demand of the replacement parts throughout the whole planning horizon T can be satisfied by using new-items procured from the NPS at the beginning of T, as well as remanufactured-items provided by the RS until the end of T. The initial inventory, i.e., seed stock, is treated as an operational decision variable along with other decision variables. Since only a fraction of used-items can be remanufactured, seed stock is crucial to guarantee enough supply of used-items for remanufacturing as well as to satisfy the demand during the initial phase of the planning horizon. The objective is to maximize the total profit by optimizing the seed stock level of new-items, initial lot size and exchange lot size of used-items. Seed stock optimization may or may not be controlled by the OEM due to the interactions between multiple agents. We investigate three scenarios and two types of controls, leading to several different system settings. We are interested in the interactions between the agents, and the impacts of the interactions on strategy performance. We aim to identify the system setting that performs best through our analytical models and numerical experiments.

3.2 Problem Motivations and Related Literature

As noted above, we consider a basic game-theoretic setting for seed stock planning problem in remanufacturing with multiple agents including an OEM, a NPS, and a RS. In automotive industry and electronic industry, the OEM often establishes remanufacturing programs to recover used products. Kodak's remanufacturing program for single-use-cameras is successful, and Xerox recycles and remanufactures photocopiers and print toner cartridges (Daniel et al. (2002)). Cell phone companies often establish remanufacturing programs to recycle used phones and re-market the remanufactured products. As noted by Akçalı and Morse (2004), to initiate the remanufacturing, the OEM needs to collect a certain amount of used-items, and these used-items are from the returns of previously sold new products. Akçalı and Morse (2004) define the amount of new products released as the seed stock.

Considering the practical importance of seed stock planning, we focus on a new seeding problem, i.e., the problem of determining the optimal seed stock level for the OEM with explicitly modeling the NPS and RS. The goal is to analyze multi-agent model with seed stock planning by applying game theory. First we look at deterministic environment for the sake of characterizing the fundamental coordination issues arising in the forward (new-items) and reverse (used-items) flows.

The problem studied in the current section is related to the previous research in deterministic lot sizing models in remanufacturing (Schrady (1967); Teunter (2001); Dobos and Richter (2004); Atasu and Çetinkaya (2006)). However, this line of research does not address seed stock planning.

Another line of closely related research focuses on the application of game theory in modeling remanufacturing decisions. We refer the reader to Souza (2013) for a critical review of recent work. One stream of research work within this line studies competition between new and remanufactured products when remanufacturing cannibalizes the demand for manufacturing (Ferrer (1996); Majumder and Groenevelt (2001); Debo et al. (2005); Ferrer and Swaminathan (2006); Ferguson and Toktay (2006); Ferrer and Swaminathan (2010); Oraiopoulos et al. (2012); Heese et al. (2005); Atasu et al. (2008)). Another stream of research work in game-theoretic models deals with selection problems for reverse channel structures (Savaskan et al. (2004); Savaskan and Van Wassenhove (2006); Atasu et al. (2013); Karakayali et al. (2007); Choi et al. (2013); De Giovanni and Zaccour (2014)). A third stream of research work addresses the operational and coordination aspects of remanufacturing practices together (Bhattacharya et al. (2006); Vorasayan and Ryan (2006); Liu et al. (2009); Dobos et al. (2013); Pishchulov et al. (2014)). None of these papers in the application of game theory in modeling remanufacturing decisions considers seed stock planning. They also do not focus on how the decision domain structure impacts the performance. Here, we consider a basic seed stock planning problem in remanufacturing, focusing on the interaction between OEM and RS. We are interested in evaluating different OEM-lead Stackelberg settings and compare them with the centralized setting. Hence, our focus is on examining how the decision domain structure impacts the total profit and identifying the most efficient one.

We analyze three different scenarios. For each scenario, we consider both centralized control and decentralized control. Under centralized control, we consider three strategies:

- Centralized control strategy under which the system-wide total profit is maximized;
- OEM-centric control strategy under which only the OEM's profit is maximized;
- RE-centric control strategy under which only the RS's profit is maximized.

Under decentralized control, we consider the OEM-lead strategies under which the OEM makes decisions on some variables, and then the RS makes decisions accordingly on the remaining undecided variables.

Hence, we provide a systematic and thorough analysis of decentralized and centralized control strategies for seed stock planning. Our results offer managerial insights for both the OEM and RS in making decisions on seed stock level, initial batch size for remanufacturing, exchange lot size and remanufacturing frequency, under different technological or operational conditions.

The remainder of this section is organized as follows. In the next section, we introduce the system setting and notation, and we derive the profit functions for the OEM, RS, and system. Section 3.4 describes different decision-making scenarios and different strategies in each scenario and formulates the corresponding optimization problems. Section 3.5 examines the structural properties of the profit functions. Sections 3.6, 3.7 and 3.8 analyze the problems considering three different scenarios, separately: Section 3.6 focuses on the scenario that the quantity of used-items shipped from the OEM to the RS is exogenous, Section 3.7 focuses on the scenario that the quantity of used-items shipped to the RS to initiate the remanufacturing program is exogenous, and Section 3.8 focuses on the scenario in a Stackelberg setting with three decision variables. In Section 3.9, we discuss numerical results. Section 3.10 summarizes the results of this section and provide future research directions.

3.3 System Setting and Profit Functions

We consider a setting with an OEM, a NPS, and a RS. The OEM sells a product for which it has to provide a particular type of replacement part throughout the life cycle of the product, i.e., until the end of the planning horizon denoted by T. This type of replacement parts is remanufacturable, and hence, can be sent to the RS for remanufacturing. The problem setting is illustrated in Figure 3.1.

Throughout the planning horizon T, there is a known constant demand rate for the replacement part, denoted by a. Since each unit demand of the replacement part generates a unit of used-item, the return rate for the used-item is also a. In order to satisfy the demand, the OEM can either use new-items procured from the NPS

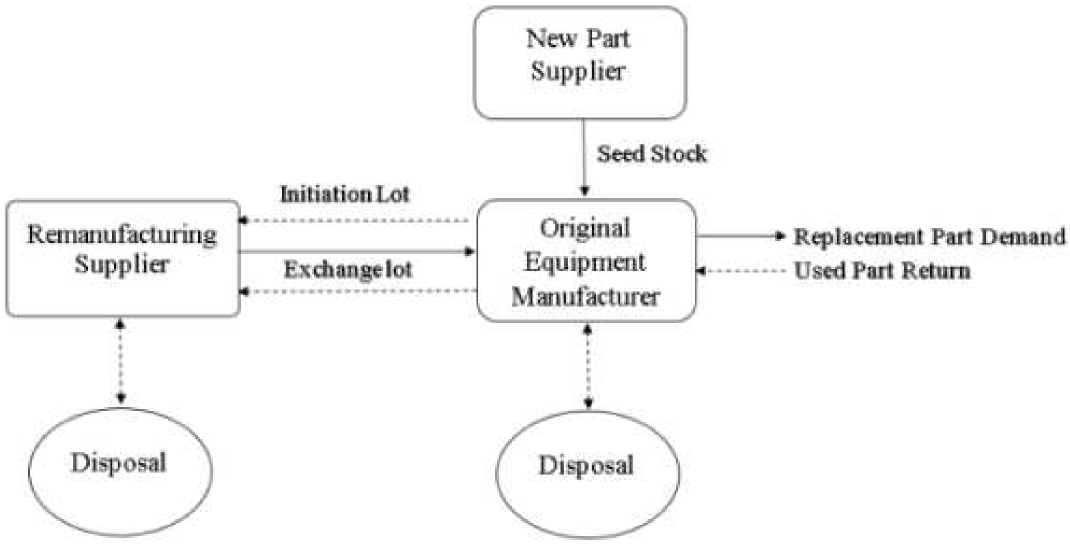

Figure 3.1: An illustration of the seed stock planning problem.

at the beginning of the planning horizon, or use remanufactured-items provided by the RS. The unit price for the serviceable part is π. Meanwhile, the OEM collects used-items from customers and sends them in batches to the RS for remanufacturing.

The sequence of events are as follows: At the beginning of the planning horizon, the OEM procures new-items at per-unit cost of c^{np} from the NPS and places them in inventory as the seed stock. The seed stock with lot size Q_s is depleted by the demand. We define the time period during which the inventory level of new-items drops to 0 as the *seed stock cycle* and denote it by CL_s. At time CL_0, the OEM begins to collect used-items (which are removed from products that are brought in warranty service) for the program initiation lot with size Q_i. Before CL_0, all the returned used-items are disposed. The time period during which Q_i units of used-items are collected is defined as the *remanufacturing initiation cycle*, denoted by CL_i. The initiation lot sent to the RS allows the RS to have a buffer of used-items on hand to hedge against potential remanufacturing yield loss. After the release of the initiation lot to start the remanufacturing program, a *lot-for-lot policy* goes into effect until the last but one shipment. That is, whenever the OEM sends a batch of used-items to the RS, the RS is required to send a batch of the same size of remanufactured-items to the OEM in exchange. The exchange batch size is denoted by Q_r. The time period between two successive lot exchanges is defined as the *collection cycle*, denoted by CL_r. The OEM will pay the RS c^{rp} for each unit remanufactured-item. The per-unit remanufacturing cost for the RS is c. The remanufacturing program stops when the RS sends the last batch of remanufactured-items to the OEM without getting any used-items. Then, the RS disposes the remaining unremanufactured-items, and the OEM disposes what is returned. We assume that unit disposal cost is same for both OEM and RS, which is denoted by c^d.

Only a fraction of used-items can be remanufactured. This fraction is denoted by γ. The remanufacturing rate, denoted by m, is known. Furthermore, $m > a$. The RS inspects each coming batch of used-items with infinite rate. All the unremanufacturable items are disposed immediately after the inspection.

A summary of notation is provided in Table 3.1. The inventory profiles for the OEM and the RS are depicted in Figure 3.2. Since $n = 0$ implies that remanufacturing program is not executed, the inventory profiles in Figure 3.2 are actually meaningful for $n \geq 1$. Observe that, when $n \geq 1$, we have the following:

(i) The entire demand throughout the planning horizon must be satisfied:

$$aT = Q_s + nQ_r,$$

and hence, Q_s can be written as a function of n and Q_r, which is given by

$$Q_s = aT - nQ_r. \tag{3.1}$$

(ii) Seed stock quantity must be sufficient to initiate the remanufacturing program and to satisfy the demand before obtaining the first batch of remanufactured-items from the RS:

$$Q_s \geq Q_i + Q_r. \tag{3.2}$$

Substituting (3.1) in (3.2), we have

$$aT \geq Q_i + (n+1)Q_r, \tag{3.3}$$

and, hence, the time CL_0 at which the OEM starts to collect used-items is given by

$$CL_0 = \frac{aT - Q_i - (n+1)Q_r}{a}. \tag{3.4}$$

(iii) The total amount of used-items that are remanufacturable should be sufficient to satisfy the entire remanufacturing requirement throughout the planning horizon:

$$\gamma Q_i \geq (n-1)(1-\gamma)Q_r + Q_r. \tag{3.5}$$

In our analysis, we will treat Q_s as a dependent variable. That is, we will calculate the optimal values of n and Q_r, and then, use (3.1) to obtain the optimal value of

Q_s. Thus, we actually have two inventory conservation constraints:

- inventory conservation constraint for the OEM which is given by (3.3);
- inventory conservation constraint for the RS which is given by (3.5).

Hence, the OEM's decisions are restricted by (3.3), while the RS's decisions are restricted by (3.5).

3.3.1 OEM's Total Profit Function

First, we derive the OEM's profit function which consists of seven components:

(O.i) total revenue of selling serviceable parts to customers: πaT;

(O.ii) total procurement cost for new-items at the beginning of T: $c^{np}aT$ if $n = 0$, and $c^{np}(aT - nQ_r)$ otherwise;

(O.iii) total fixed shipment cost for exchanging used-items with remanufactured-items throughout T: 0 if $n = 0$, and nK_s otherwise;

(O.iv) total procurement cost for remanufactured-items throughout T: 0 if $n = 0$, and $nc^{rp}Q_r$ otherwise;

(O.v) cumulative inventory carrying cost for used-items throughout T: 0 if $n = 0$, and $h^u_M I_1$ otherwise, where I_1 is the cumulative used-item inventory held by the OEM throughout T;

(O.vi) cumulative inventory carrying cost for new-items throughout T: $h^n_M \frac{aT^2}{2}$ if $n = 0$, and $h^n_M A_3$ otherwise, where A_3 is the cumulative new-item inventory held by the OEM during the seed stock cycle;

Table 3.1: Notation for the seed stock planning problem.

T	Finite planning horizon
a	Finite demand rate
γ	Fraction of used-items that can be remanufactured
m	Finite remanufacturing rate
Q_s	The quantity of new-items procured from the NPS (i.e., seed stock lot size)
Q_r	The quantity of used-items shipped from the OEM to the RS, which is also equal to the quantity of remanufactured-items shipped from the RS to the OEM (i.e., exchange lot size)
Q_i	The quantity of used-items shipped from the OEM to the RS to initiate the remanufacturing program (i.e., initiation lot size)
n	The number of consecutive remanufacturing replenishments throughout the planning horizon
CL_0	The time at which the OEM starts collecting used-items
CL_i	The length of the *remanufacturing initiation cycle*, i.e., $CL_i = Q_i/a$
CL_r	The length of a used-item *collection cycle*, i.e., $CL_r = Q_r/a$
CL_s	The length of the *seed stock cycle*, i.e., $CL_s = Q_s/a$
CL_R	The length of a *remanufacturing run* for processing a batch of remanufacturable items, i.e., $CL_m = Q_r/m$
K_s	Fixed shipment cost from the OEM to the RS
K_r	Fixed remanufacturing setup cost incurred by the RS
π	Unit price of serviceable part
c^{np}	Unit new-item procurement cost
c^{rp}	Unit remanufactured-item procurement cost, $c^{rp} < c^{np}$
c	Unit remanufacturing cost, $c < c^{rp}$
c^d	Unit used-item disposal cost
h_M^n	Unit inventory holding cost per new-item incurred by the OEM
h_M^r	Unit inventory holding cost per remanufactured-item incurred by the OEM, $h_M^r < h_M^n$
h_M^u	Unit inventory holding cost per used-item incurred by the OEM, $h_M^u < h_M^r$
h_R^r	Unit inventory holding cost per remanufactured-item incurred by the RS
h_R^u	Unit inventory holding cost per used-item incurred by the RS, $h_R^u < h_R^r$
A_1	Cumulative used-item inventory held at the OEM during the remanufacturing initiation cycle
A_2	Cumulative used-item inventory held at the OEM during each collection cycle
A_3	Cumulative new-item inventory held at the OEM during the seed stock cycle
I_1	Cumulative used-item inventory held at the OEM throughout the planning horizon
I_2	Cumulative remanufactured-item inventory held at the OEM throughout the planning horizon
I_3	Cumulative remanufacturable item inventory held at the RS throughout the planning horizon
I_4	Cumulative remanufactured-item inventory held at the RS throughout the planning horizon
I_R^E	Cumulative echelon inventory held at the RS throughout the planning horizon
I_M^d	Total quantity of units disposed by the OEM throughout the planning horizon
I_R^d	Total quantity of units disposed by the RS throughout the planning horizon
Π_{OEM}	The OEM's total profit throughout the planning horizon
Π_{RS}	The RS's total profit throughout the planning horizon
Π	The system-wide total profit throughout the planning horizon

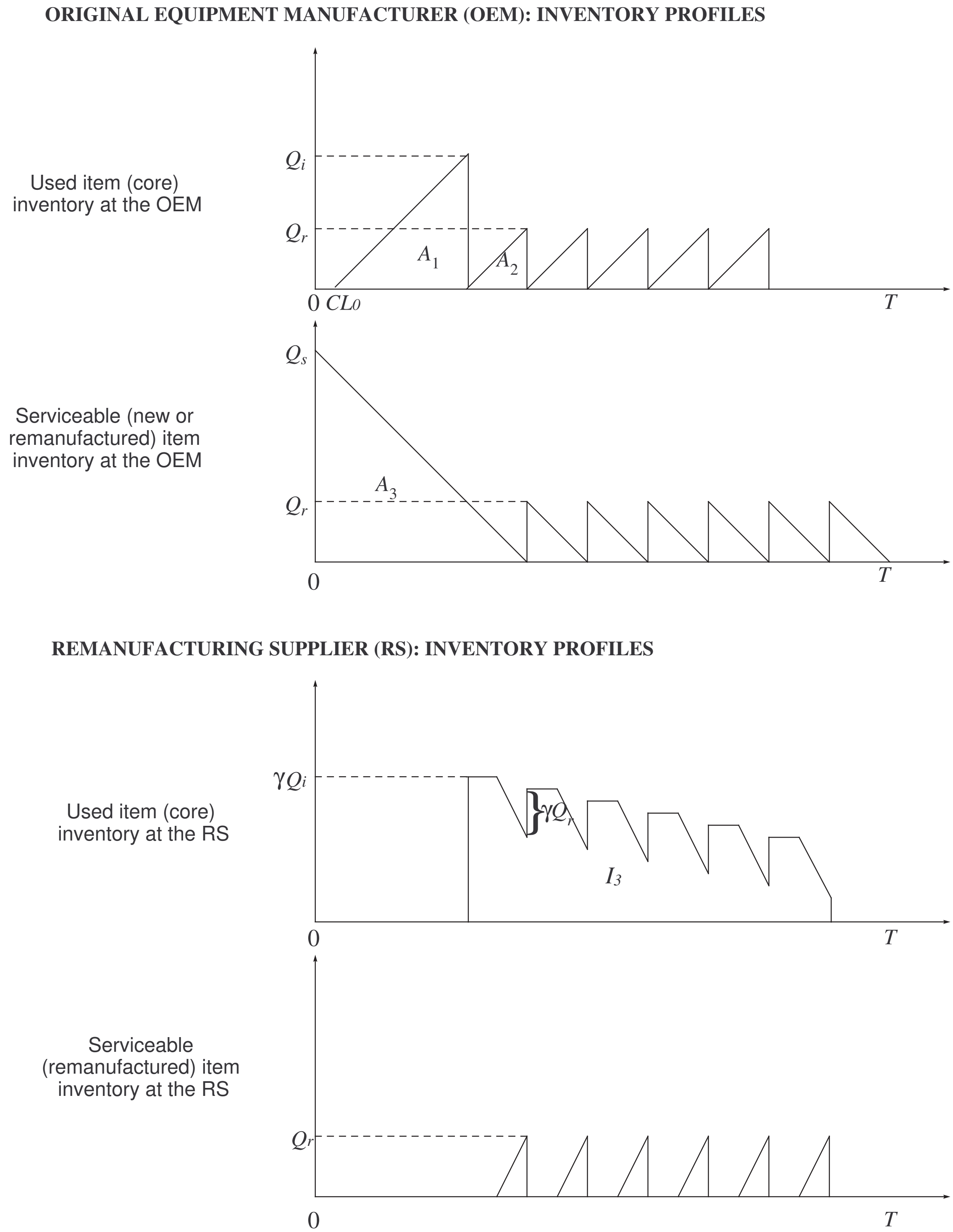

Figure 3.2: A realization of inventory profiles for the seed stock planning problem.

(O.vii) cumulative inventory carrying cost for remanufactured-items throughout T: 0 if $n = 0$, and $h_M^r I_2$ otherwise, where I_2 is the cumulative remanufactured-item inventory held by the OEM throughout T;

(O.viii) total disposal cost throughout T: $c^d aT$ if $n = 0$, and $c^d I_M^d$ otherwise, where I_M^d is the total units disposed throughout T.

The OEM's total profit function is given by:

$$\Pi_{OEM} = \begin{cases} \pi aT - \left[c^{np}aT + h_M^n \frac{aT^2}{2} + c^d aT\right] & n = 0 \\ \pi aT - [c^{np}(aT - nQ_r) + (K_s + c^{rp}Q_r)n & \\ \qquad\qquad + h_M^u I_1 + h_M^n A_3 + h_M^r I_2 + c^d I_M^d] & n = 1, 2, \ldots \end{cases} \tag{3.6}$$

Among the above components, (O.v)–(O.viii) depend on the values of I_1, A_3, I_2 and I_M^d, respectively, which can be derived as follows:

- Cumulative used-item inventory held during the remanufacturing initiation cycle:

$$A_1 = \frac{Q_i}{2} CL_i = \frac{Q_i^2}{2a}. \tag{3.7}$$

- Cumulative used-item inventory held during each collection cycle:

$$A_2 = \frac{Q_r}{2} CL_r = \frac{Q_r^2}{2a}. \tag{3.8}$$

- By (3.7) and (3.8), we can calculate the cumulative used-item inventory held throughout the planning horizon which is given by:

$$I_1 = A_1 + (n-1)A_2 = \frac{Q_i^2 + (n-1)Q_r^2}{2a}. \tag{3.9}$$

- Cumulative new-item inventory held during the seed stock cycle:

$$A_3 = \frac{Q_s}{2} CL_s = \frac{Q_s^2}{2a} = \frac{(aT - nQ_r)^2}{2a}. \tag{3.10}$$

- By (3.8), we can calculate the cumulative remanufactured-item inventory held throughout the planning horizon which is given by:

$$I_2 = nA_2 = n\frac{Q_r^2}{2a}. \tag{3.11}$$

- Total quantity of used-items disposed by the OEM throughout the planning horizon is given by:

$$I_M^d = Q_s - (Q_i + Q_r) + 2Q_r = aT - nQ_r - (Q_i + Q_r) + 2Q_r. \tag{3.12}$$

Substituting I_1, A_3, I_2 and I_M^d, given by (3.9), (3.10), (3.11) and (3.12), respectively, in (3.6), the OEM's total profit is given by

$$\Pi_{OEM}(n, Q_r, Q_i) = \begin{cases} (\pi - c^{np} - c^d)aT - h_M^n \frac{aT^2}{2} & n = 0 \\ \pi aT - \Big[c^{np}(aT - nQ_r) + (K_s + c^{rp}Q_r)\, n & \\ +h_M^u \frac{Q_i^2}{2a} + (h_M^u + h_M^r)\, \frac{Q_r^2}{2a} n - h_M^u \frac{Q_r^2}{2a} & \\ +h_M^n \frac{(aT - nQ_r)^2}{2a} + c^d(aT - Q_i - (n-1)Q_r)\Big] & n = 1, 2, \ldots \end{cases} \tag{3.13}$$

3.3.2 RS's Total Profit Function

When $n = 0$, i.e., the remanufacturing program is not executed, the RS's total proft is zero. Otherwise, the RS's total profit consists of a total of six components:

(R.i) total revenue of selling remanufactured-items to the OEM throughout T, i.e., $nc^{rp}Q_r$,

(R.ii) total fixed remanufacturing setup cost throughout T, i.e., nK_r,

(R.iii) total remanufacturing cost throughout T, i.e., ncQ_r,

(R.iv) cumulative inventory carrying cost for (remanufacturable) used-items throughout T, i.e., $h_R^u I_3$,

(R.v) cumulative inventory carrying cost for remanufactured-items throughout T, i.e., $h_R^r I_4$, and

(R.vi) total disposal cost of used-items throughout T, i.e., $c^d I_R^d$.

Thus, the RS's total profit function is given by

$$\Pi_{RS} = \begin{cases} 0 & n = 0 \\ ((c^{rp} - c)Q_r - K_r)n - h_R^u I_3 - h_R^r I_4 - c^d I_R^d & n = 1, 2, \dots \end{cases} \tag{3.14}$$

Among the above components, (R.iv)–(R.vi) depend on the values of I_3, I_4 and I_R^d, respectively, which can be derived as follows:

- Cumulative echelon inventory of remanufacturable items and remanufactured-items held by the RS throughout T is depicted in Figure 3.3, and can be calculated as follows:

$$I_R^E = \gamma Q_i nCL_r - (1-\gamma)Q_r CL_r \sum_{j=1}^{n-1} j = \frac{\gamma n Q_i Q_r}{a} - \frac{n(n-1)}{2}(1-\gamma)\frac{Q_r^2}{a}. \tag{3.15}$$

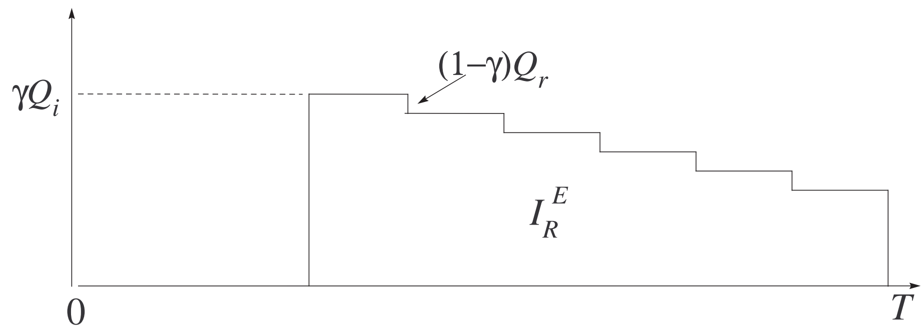

Figure 3.3: Echelon inventory of remanufacturable items and remanufactured-items at the RS.

- Cumulative used-items inventory held:

$$I_3 = I_R^E - I_4. \tag{3.16}$$

- Cumulative remanufactured-items inventory held:

$$I_4 = \frac{nQ_r^2}{2m}. \tag{3.17}$$

- Total quantity of used-items disposed by the RS throughout T:

$$\begin{aligned} I_R^d &= (1-\gamma)Q_i + (1-\gamma)Q_r(n-1) + (\gamma Q_i - Q_r - (n-1)(1-\gamma)Q_r) \\ &= Q_i - Q_r. \end{aligned} \tag{3.18}$$

Substituting I_R^E, I_3, I_4 and I_R^d, given by (3.15), (3.16), (3.17) and (3.18), respectively, in (3.14), the RS's total profit is given by

$$\Pi_{RS}(n,Q_r,Q_i)=\begin{cases}0 & n=0\\ \left(\left(c^{rp}-c\right)Q_r-K_r\right)n & \\ -h_R^u\left(\frac{\gamma Q_iQ_r}{a}n-(1-\gamma)\frac{Q_r^2}{2a}n(n-1)\right) & \\ -\left(h_R^r-h_R^u\right)\frac{Q_r^2}{2m}n-c^d\left(Q_i-Q_r\right) & n=1,2,\ldots\end{cases} \tag{3.19}$$

3.3.3 System-wide Total Profit Function

Given the OEM's proft function in (3.13) and the RS's proft function in (3.19), the total system-wide total profit during planning horizon T, denoted by Π, is given by

$$\Pi(n,Q_r,Q_i)=\Pi_{OEM}(n,Q_r,Q_i)+\Pi_{RS}(n,Q_r,Q_i). \tag{3.20}$$

The decision variables of interest are n, Q_r and Q_i. Using (3.1) and (3.4), Q_s and CL_0 can be computed, respectively. The remainder of this section focuses on the analysis of three different scenarios: (1) computing n and Q_i for an exogenous Q_r, (2) computing n and Q_r for an exogenous Q_i and (3) computing n, Q_r and Q_i in a Stackelberg setting. Next, we present the problem formulations.

3.4 Problem Formulation

As mentioned earlier, we consider three different decision-making scenarios: (1) computing n and Q_i for an exogenous Q_r, (2) computing n and Q_r for an exogenous Q_i and (3) computing n, Q_r and Q_i in a Stackelberg setting where OEM is the leader and RS is the follower. Furthermore, we consider both centralized and decentralized settings. Although we are primarily interested in the analysis of system performance under a decentralized control setting (where OEM is the leader and RS the follower), we consider several centralized control settings to serve as benchmarks. Specifically, we consider three centralized and one decentralized setting. Under *centralized control*, the optimal values of the decision variables of interest are determined to maximize system-wide total profit. Under *OEM-centric* control, the optimal values of the decision variables of interest are determined to maximize OEM's total profit. Under *RS-centric* control, the optimal values of the decision variables of interest are determined to maximize RS's total profit. Under *OEM-lead* control, given RS's decisions on variables that maximize RS's total profit, OEM makes decisions on the other decision variables so as to maximize OEM's total profit. Consequently, for Scenario 1, we consider three centralized and two decentralized strategies:

P1.1 The values of n and Q_i are identified so as to maximize system-wide total profit.

P1.2 The values of n and Q_i are identified so as to maximize OEM's total profit.

P1.3 The values of n and Q_i are identified so as to maximize RS's total profit.

P1.4 Given RS's decision on Q_i that maximizes RS's total profit, OEM chooses the value of n that maximizes OEM's total profit.

P1.5 Given RS's decision on n that maximizes RS's total profit, OEM chooses the value of Q_i that maximizes OEM's total profit.

For Scenario 2, we again consider three centralized and two decentralized strategies:

P2.1 The values of n and Q_r are identified so as to maximize system-wide total profit.

P2.2 The values of n and Q_r are identified so as to maximize OEM's total profit.

P2.3 The values of n and Q_r are identified so as to maximize RS's total profit.

P2.4 Given RS's decision on Q_r that maximizes RS's total profit, OEM chooses the value of n that maximizes OEM's total profit.

P2.5 Given RS's decision on n that maximizes RS's total profit, OEM chooses the value of Q_r that maximizes OEM's total profit.

For Scenario 3, we consider one centralized and six decentralized strategies:

P3.1 The values of n, Q_r and Q_i are identified so as to maximize system-wide total profit.

P3.2 Given RS's decision on n and Q_i that maximize RS's total profit, OEM chooses the value of Q_r that maximizes OEM's total profit.

P3.3 Given RS's decision on n and Q_r that maximize RS's total profit, OEM chooses the value of Q_i that maximizes OEM's total profit.

P3.4 Given RS's decision on Q_r and Q_i that maximize RS's total profit, OEM chooses the value of n that maximizes OEM's total profit.

P3.5 Given RS's decision on Q_i that maximizes RS's total profit, OEM chooses the value of n and Q_r that maximize OEM's total profit.

P3.6 Given RS's decision on Q_r that maximizes RS's total profit, OEM chooses the value of n and Q_i that maximize OEM's total profit.

P3.7 Given RS's decision on n that maximizes RS's total profit, OEM chooses the value of Q_r and Q_i that maximize OEM's total profit.

Table 3.2 summarizes the control problems for each scenario.

Table 3.2: Problem configurations.

Scenario	Setting	Objective Function	Decision Variables			Problem (Reference)
			n	Q_i	Q_r	
1	1. Centralized Control	Π	OEM-RS	OEM-RS	Given	P1.1 (3.21)
	2. OEM-centric Control	Π_{OEM}	OEM	OEM	Given	P1.2 (3.22)
	3. RS-centric Control	Π_{RS}	RS	RS	Given	P1.3 (3.23)
	4. OEM-lead Control	Π_{RS}, Π_{OEM}	OEM (lead)	RS (follow)	Given	P1.4.1, P1.4.2 (3.24), (3.25)
	5. OEM-lead Control	Π_{RS}, Π_{OEM}	RS (follow)	OEM (lead)	Given	P1.5.1, P1.5.2 (3.26) (3.27)
2	1. Centralized Control	Π	OEM-RS	Given	OEM-RS	P2.1 (3.28)
	2. OEM-centric Control	Π_{OEM}	OEM	Given	OEM	P2.2 (3.29)
	3. RS-centric Control	Π_{RS}	RS	Given	RS	P2.3 (3.30)
	4. OEM-lead Control	Π_{RS}, Π_{OEM}	OEM (lead)	Given	RS (follow)	P2.4.1, P2.4.2 (3.31), (3.32)
	5.OEM-lead Control	Π_{RS}, Π_{OEM}	RS (follow)	Given	OEM (lead)	P2.5.1, P2.5.2 (3.26) (3.33)
3	1. Centralized Control	Π	OEM-RS	OEM-RS	OEM-RS	P3.1 (3.34)
	2. OEM-lead Control	Π_{RS}, Π_{OEM}	RS (follow)	RS (follow)	OEM (lead)	P3.2.1, P3.2.2 (3.23), (3.35)
	3. OEM-lead Control	Π_{RS}, Π_{OEM}	RS (follow)	OEM (lead)	RS (follow)	P3.3.1, P3.3.2 (3.30), (3.36)
	4. OEM-lead Control	Π_{RS}, Π_{OEM}	OEM (lead)	RS (follow)	RS (follow)	P3.4.1, P3.4.2 (3.37), (3.38)
	5. OEM-lead Control	Π_{RS}, Π_{OEM}	OEM (lead)	RS (follow)	OEM (lead)	P3.5.1, P3.5.2 (3.24), (3.39)
	6. OEM-lead Control	Π_{RS}, Π_{OEM}	OEM (lead)	OEM (lead)	RS (follow)	P3.6.1, P3.6.2 (3.31), (3.40)
	7. OEM-lead Control	Π_{RS}, Π_{OEM}	RS (follow)	OEM (lead)	OEM (lead)	P3.7.1, P3.7.2 (3.26), (3.41)

Thus, although the profit functions of the OEM, the RS, and the system are functions of n, Q_r, and Q_i, some of these decision variables might be given, or decided by the other agent, under some circumstances. We use $\Pi(\cdot \mid\mid *)$ (or $\Pi_{OEM}(\cdot \mid\mid *)/\Pi_{RS}(\cdot \mid\mid *)$) to denote the profit function as the function of $\cdot$ when $*$ is given, i.e., when $*$ is treated as fixed value rather than a variable.

3.4.1 Scenario 1: Computing n and Q_i for an Exogenous Q_r

Under this scenario, Q_r is dictated by some technological or operational constraint. We are interested in the following problems:

1. ***Centralized Control*:** When centralized control is in effect, the decision variables n and Q_i are specified by maximizing the total system-wide total profit. Thus, the centralized problem, P1.1, is

$$\max_{n \in Z^*, Q_i \geq 0} \Pi(n, Q_i \mid\mid Q_r) \qquad (3.21)$$
$$\text{s.t.} \quad (3.3) \text{ and } (3.5),$$

   where $\Pi(n, Q_i \mid\mid Q_r)$ is as in (3.20).

2. ***OEM-centric Control*:** When OEM-centric control is in effect, the decision variables n and Q_i are specified by maximizing the OEM's total profit. Thus, the OEM-centric problem, P1.2, is

$$\max_{n \in Z^*,\ Q_i \geq 0} \quad \Pi_{OEM}(n, Q_i \mid\mid Q_r) \qquad (3.22)$$
$$\text{s.t.} \quad (3.3),$$

   where $\Pi_{OEM}(n, Q_i \mid\mid Q_r)$ is as in (3.13).

3. ***RS-centric Control*:** When RS-centric control is in effect, the decision variables n and Q_i are specified by maximizing the RS's total profit. Thus, the RS-centric problem, P1.3, is

$$\begin{aligned} \max_{n\in Z^*,\ Q_i\geq 0} \quad & \Pi_{RS}(n, Q_i \mid\mid Q_r) \\ \text{s.t.} \quad & (3.5), \end{aligned} \tag{3.23}$$

where $\Pi_{RS}(n, Q_i \mid\mid Q_r)$ is as in (3.19).

4. ***OEM-lead Control: OEM decides* n *and RS decides* Q_i:** In this case, for each n given by the OEM, the RS determines Q_i that maximizes its total profit. With this information, the OEM then specifies n. In this case, the RS's problem, P1.4.1, is

$$\begin{aligned} \max_{Q_i\geq 0} \quad & \Pi_{RS}(Q_i \mid\mid (n, Q_r)) \\ \text{s.t.} \quad & (3.5), \end{aligned} \tag{3.24}$$

where $\Pi_{RS}(Q_i \mid\mid (n, Q_r))$ is as in (3.19). Given RS's optimal decision for Q_i, denoted by $Q^*_{i,1.4}(n, Q_r)$, the OEM's problem, P1.4.2, is

$$\begin{aligned} \max_{n\in Z^*} \quad & \Pi_{OEM}(n, Q^*_{i,1.4}(n, Q_r) \mid\mid Q_r) \\ \text{s.t.} \quad & (3.3), \end{aligned} \tag{3.25}$$

where $\Pi_{OEM}(n, Q_i \mid\mid Q_r)$ is as in (3.13).

5. ***OEM-lead Control: OEM decides* Q_i *and RS decides* n:** In this case, for each Q_i given by the OEM, the RS determines n that maximizes its total

profit. With this information, the OEM then specifies Q_i. In this case, the RS's problem, P1.5.1, is

$$\begin{aligned} \max_{n \in Z^*} \quad & \Pi_{RS}(n \mid\mid (Q_i, Q_r)) \\ \text{s.t.} \quad & (3.5), \end{aligned} \tag{3.26}$$

where $\Pi_{RS}(n \mid\mid (Q_i, Q_r))$ is as in (3.19). Given RS's optimal decision for n, denoted by $n^*_{1.5}(Q_i, Q_r)$, the OEM's problem, P1.5.2, is

$$\begin{aligned} \max_{Q_i \geq 0} \quad & \Pi_{OEM}(n^*_{1.5}(Q_i, Q_r), Q_i) \mid\mid Q_r) \\ \text{s.t.} \quad & (3.3), \end{aligned} \tag{3.27}$$

where $\Pi_{OEM}(n, Q_i \mid\mid Q_r)$ is as in (3.13).

3.4.2 Scenario 2: Computing n and Q_r for an Exogenous Q_i

Under this scenario, Q_i is dictated by some technological or operational constraint. We are interested in the following problems:

1. ***Centralized Control*:** When centralized control is in effect, the decision variables n and Q_r are specified by maximizing the total system-wide total profit. Thus, the centralized problem, P2.1, is

$$\begin{aligned} \max_{n \in Z^*,\ Q_r \geq 0} \quad & \Pi(n, Q_r \mid\mid Q_i) \\ \text{s.t.} \quad & (3.3) \text{ and } (3.5), \end{aligned} \tag{3.28}$$

   where $\Pi(n, Q_r \mid\mid Q_i)$ is as in (3.20).

2. ***OEM-centric Control*:** When OEM-centric control is in effect, the decision

variables n and Q_r are specified by maximizing the OEM's total profit. Thus, the OEM-centric problem, P2.2, is

$$\begin{aligned} \max_{n \in Z^*,\ Q_r \geq 0} \quad & \Pi_{OEM}(n, Q_r \mid\mid Q_i) \\ \text{s.t.} \quad & (3.3), \end{aligned} \tag{3.29}$$

where $\Pi_{OEM}(n, Q_r \mid\mid Q_i)$ is as in (3.13).

3. ***RS-centric Control*:** When RS-centric control is in effect, the decision variables n and Q_r are specified by maximizing the RS's total profit. Thus, the RS-centric problem, P2.3, is

$$\begin{aligned} \max_{n \in Z^*,\ Q_r \geq 0} \quad & \Pi_{RS}(n, Q_r \mid\mid Q_i) \\ \text{s.t.} \quad & (3.5), \end{aligned} \tag{3.30}$$

where $\Pi_{RS}(n, Q_r \mid\mid Q_i)$ is as in (3.19).

4. ***OEM-lead Control: OEM decides* n *and RS decides* Q_r:** In this case, for each n given by the OEM, the RS determines Q_r that maximizes its total profit. With this information, the OEM then specifies n. In this case, the RS's problem, P2.4.1, is

$$\begin{aligned} \max_{Q_r \geq 0} \quad & \Pi_{RS}(Q_r \mid\mid (n, Q_i)) \\ \text{s.t.} \quad & (3.5), \end{aligned} \tag{3.31}$$

where $\Pi_{RS}(Q_r \mid\mid (n, Q_i))$ is as in (3.19). Given RS's optimal decision for Q_r, denoted by $Q^*_{r,2.4}(n, Q_i)$, the OEM's problem, P2.4.2, is

$$\begin{aligned} \max_{n \in Z^*} \quad & \Pi_{OEM}(n, Q^*_{r,2.4}(n, Q_i) \mid\mid Q_i) \\ \text{s.t.} \quad & (3.3), \end{aligned} \tag{3.32}$$

where $\Pi_{OEM}(n, Q_r \mid\mid Q_i)$ is as in (3.13).

5. ***OEM-lead Control: OEM decides Q_r and RS decides n*:** In this case, for each Q_r given by the OEM, the RS determines n that maximizes its total profit. With this information, the OEM then specifies Q_r. In this case, the RS's problem, P2.5.1, is as in (3.26), which is equivalent to P1.5.1.

   Given RS's optimal decision for n, denoted by $n^*_{1.5}(Q_r, Q_i)$, the OEM's problem, P2.5.2, is

$$\begin{aligned} \max_{Q_i \geq 0} \quad & \Pi_{OEM}(n^*_{1.5}(Q_r, Q_i), Q_r) \mid\mid Q_i) \\ \text{s.t.} \quad & (3.3), \end{aligned} \tag{3.33}$$

   where $\Pi_{OEM}(n, Q_r \mid\mid Q_i)$ is as in (3.13).

3.4.3 Scenario 3: Computing n, Q_r and Q_i in a Stackelberg Setting

Under this scenario, n, Q_r and Q_i are determined under the OEM-lead Stackelberg settings. We are interested in the following problems:

1. ***Centralized Control*:** When centralized control is in effect, the variables n, Q_r and Q_i are specified to maximize the total system-wide total profit. Thus,

the centralized problem, P3.1, is

$$\begin{aligned} \max_{n \in Z^*,\ Q_r \geq 0,\ Q_i \geq 0} \quad & \Pi(n, Q_r, Q_i) \\ \text{s.t.} \quad & (3.3) \quad \text{and} \quad (3.5), \end{aligned} \tag{3.34}$$

where $\Pi(n, Q_r, Q_i)$ is as in (3.20).

2. ***OEM-lead Control: OEM decides Q_r and RS decides n and Q_i*:** In this case, for each Q_r given by the OEM, the RS determines n and Q_i that maximize its total profit. With this information, the OEM then determines Q_r. In this case, the RS's problem, P3.2.1, is as in (3.23), which is equivalent to P1.3.

   Given RS's optimal decisions for n and Q_i, denoted by $n^*_{1.3}(Q_r)$ and $Q^*_{i,1.3}(Q_r)$, respectively, the OEM's problem, P3.2.2, is

$$\begin{aligned} \max_{Q_r \geq 0} \quad & \Pi_{OEM}(n^*_{1.3}(Q_r), Q_r, Q^*_{i,1.3}(Q_r)) \\ \text{s.t.} \quad & (3.3), \end{aligned} \tag{3.35}$$

   where $\Pi_{OEM}(n, Q_r, Q_i)$ is as in (3.13).

3. ***OEM-lead Control: OEM decides Q_i and RS decides n and Q_r*:** In this case, for each Q_i given by the OEM, the RS determines n and Q_r that maximize its total profit. With this information, the OEM then determines Q_i. In this case, the RS's problem, P3.3.1, is as in (3.30), which is equivalent to P2.3.

   Given RS's optimal decisions for n and Q_r, denoted by $n^*_{2.3}(Q_i)$ and $Q^*_{r,2.3}(Q_i)$, respectively, the OEM's problem, P3.3.2, is

$$\begin{aligned} \max_{Q_i \geq 0} \quad & \Pi_{OEM}(n^*_{2.3}(Q_i), Q^*_{r,2.3}(Q_i), Q_i) \\ \text{s.t.} \quad & (3.3), \end{aligned} \tag{3.36}$$

where $\Pi_{OEM}(n, Q_r, Q_i)$ is as in (3.13).

4. ***OEM-lead Control: OEM decides n and RS decides Q_r and Q_i:*** In this case, for each n given by the OEM, the RS determines Q_r and Q_i that maximize its total profit. With this information, the OEM then determines n. In this case, the RS's problem, P3.4.1, is

$$\begin{aligned} \max_{Q_r \geq 0,\ Q_i \geq 0} \quad & \Pi_{RS}(Q_r, Q_i \mid\mid n) \\ \text{s.t.} \quad & (3.5), \end{aligned} \tag{3.37}$$

where $\Pi_{RS}(Q_r, Q_i \mid\mid n)$ is as in (3.19).

Given RS's optimal decisions for Q_r and Q_i, denoted by $Q^*_{r,3.4}(n)$ and $Q^*_{i,3.4}(n)$, respectively, the OEM's problem, P.3.4.2, is

$$\begin{aligned} \max_{n \in Z^*} \quad & \Pi_{OEM}(n, Q^*_{r,3.4}(n), Q^*_{i,3.4}(n)) \\ \text{s.t.} \quad & (3.3), \end{aligned} \tag{3.38}$$

where $\Pi_{OEM}(n, Q_r, Q_i)$ is as in (3.13).

5. ***OEM-lead Control: OEM decides n and Q_r, and RS decides Q_i:*** In this case, for each (n, Q_r) pair given by the OEM, the RS determines Q_i that maximizes its total profit. With this information, the OEM then determines

n and Q_r. In the case, the RS's problem, P3.5.1, is as in (3.24), which is equivalent to P1.4.1.

Given RS's optimal decision for Q_i, denoted by $Q^*_{i,1.4}(n, Q_r)$, the OEM's problem, P3.5.2, is

$$\begin{aligned} \max_{n \in Z^*,\ Q_r \geq 0} \quad & \Pi_{OEM}(n, Q_r, Q^*_{i,1.4}(n, Q_r)) \\ \text{s.t.} \quad & (3.3), \end{aligned} \tag{3.39}$$

where $\Pi_{OEM}(n, Q_r, Q_i)$ is as in (3.13).

6. ***OEM-lead Control: OEM decides n and Q_i, and RS decides Q_r*:** In this case, for each (n, Q_i) pair given by the OEM, the RS determines Q_r that maximizes its total profit. With this information, the OEM then determines n and Q_i. In this case, the RS's problem, P3.6.1, is as in (3.31), which is equivalent to P2.4.1.

   Given RS's optimal decision for Q_r, denoted by $Q^*_{r,2.4}(n, Q_i)$, the OEM's problem, P3.6.2, is

$$\begin{aligned} \max_{n \in Z^*,\ Q_i \geq 0} \quad & \Pi_{OEM}(n, Q^*_{r,2.4}(n, Q_i), Q_i) \\ \text{s.t.} \quad & (3.3), \end{aligned} \tag{3.40}$$

   where $\Pi_{OEM}(n, Q_r, Q_i)$ is as in (3.13).

7. ***OEM-lead Control: OEM decides Q_r and Q_i, and RS decides n*:** In this case, for each (Q_r, Q_i) pair given by the OEM, the RS determines n that maximizes its total profit. With this information, the OEM then determines Q_r and Q_i. In this case, the RS's problem, P3.7.1, is as in (3.26), which is

equivalent to P1.5.1 and P2.5.1.

Given RS's optimal decision for n, denoted by $n^*_{1.5}(Q_r, Q_i)$, the OEM's problem, P.3.7.2, is

$$\begin{aligned} \max_{Q_r \geq 0,\ Q_i \geq 0} \quad & \Pi_{OEM}(n^*_{1.5}(Q_r, Q_i), Q_r, Q_i) \\ \text{s.t.} \quad & (3.3), \end{aligned} \tag{3.41}$$

where $\Pi_{OEM}(n, Q_r, Q_i)$ is as in (3.13).

3.5 Structural Properties of Cost Functions

Before we proceed to calculate the optimal solutions of the problems formulated in Section 3.4, we will examine some structural properties of Π_{OEM}, Π_{RS} and Π that can be utilized to develop approaches to compute the optimal solutions.

Property 7 *The following are structural properties of* Π_{OEM}, Π_{RS} *and* Π *for* $n = 0$.

1. *When* $n = 0$, $\Pi_{OEM}(0, Q_r, Q_i) = \left(\pi - c^{np} - c^d\right) aT - h^n_M \frac{aT^2}{2} \equiv \Pi_{OEM}(n = 0)$.

2. *When* $n = 0$, $\Pi_{RS}(0, Q_r, Q_i) = 0 \equiv \Pi_{RS}(n = 0)$.

3. *When* $n = 0$, $\Pi(0, Q_r, Q_i) = \left(\pi - c^{np} - c^d\right) aT - h^n_M \frac{aT^2}{2} \equiv \Pi(n = 0)$.

Property 8 *The following relate to the structural properties of* Π_{OEM}, Π_{RS} *and* Π *for any given* (n, Q_r) *pair, where* $n \in Z^+$ *and* $Q_r \geq 0$.

1. *For any given* (n, Q_r) *pair, where* $n \in Z^+$ *and* $Q_r \geq 0$, $\Pi_{OEM}(n, Q_r, Q_i)$ *is concave in* Q_i *with the unique maximizer given by*

$$Q_{i,OEM}(n, Q_r) = \frac{ac^d}{h^u_M}. \tag{3.42}$$

2. *For any given (n, Q_r) pair, where $n \in Z^+$ and $Q_r \geq 0$, $\Pi_{RS}(n, Q_r, Q_i)$ is a linearly decreasing function of Q_i for any Q_i.*

3. *For any given (n, Q_r) pair, where $n \in Z^+$ and $Q_r \geq 0$, $\Pi(n, Q_r, Q_i)$ is a linearly decreasing function of Q_i for $Q_i > 0$.*

Proof.

1. For any given (n, Q_r) pair, where $n \in Z^+$ and $Q_r \geq 0$, it can be easily shown that

$$\begin{aligned} \frac{\partial \Pi_{OEM}}{\partial Q_i} &= -h_M^u \frac{Q_i}{a} + c^d, \quad \text{and} \\ \frac{\partial^2 \Pi_{OEM}}{\partial Q_i^2} &= -\frac{h_M^u}{a} < 0. \end{aligned}$$

   Hence, $\Pi_{OEM}(n, Q_r, Q_i)$ is concave in Q_i for any given (n, Q_r) pair with the unique maximizer given by (3.42).

2. For any given (n, Q_r) pair, where $n \in Z^+$ and $Q_r \geq 0$, it can be easily shown that

$$\frac{\partial \Pi_{RS}}{\partial Q_i} = -h_R^u \frac{\gamma Q_r n}{a} - c^d < 0.$$

   Thus, $\Pi_{RS}(n, Q_r, Q_i)$ is linearly decreasing in Q_i, for any given (n, Q_r) pair, where $n \in Z^+$ and $Q_r \geq 0$.

3. It can be shown that, for any given (n, Q_r) pair, where $n \in Z^+$ and $Q_r \geq 0$, we have

$$\frac{\partial \Pi}{\partial Q_i} = -h_M^u \frac{Q_i}{a} - h_R^u \frac{\gamma Q_r n}{a} < 0,$$

for any $Q_i > 0$. Thus, $\Pi(n, Q_r, Q_i)$ is linearly decreasing in $Q_i > 0$, for any given (n, Q_r) pair, where $n \in Z^+$ and $Q_r \geq 0$.

$\square$

Property 9 *The following relate to the structural properties of Π_{OEM}, Π_{RS} and Π for any given (n, Q_r) pair, where $n \in Z^+$ and $Q_i \geq 0$.*

1. *For any given (n, Q_i) pair, where $n \in Z^+$ and $Q_i \geq 0$, $\Pi_{OEM}(n, Q_r, Q_i)$ is concave in Q_r with the unique maximizer given by*

$$Q_{r,OEM}(n, Q_i) = \frac{(c^{np} - c^{rp} + h_M^n T)n + (n-1)c^d}{h_M^n n^2 + h_M^r n + h_M^u (n-1)} a. \tag{3.43}$$

2. *For any given (n, Q_i) pair, where $n \in Z^+$ and $Q_i \geq 0$, $\Pi_{RS}(n, Q_r, Q_i)$ is not concave in Q_r in general:*

   - *For any given $Q_i \geq 0$, and n in the region $1 \leq n \leq \frac{(h_R^r - h_R^u)a}{h_R^u (1-\gamma) m} + 1$, $\Pi_{RS}(n, Q_r, Q_i)$ is concave in Q_r with the unique maximizer given by*

$$Q_{r,RS}(n, Q_i) = \frac{(c^{rp} - c)na + c^d a - h_R^u \gamma Q_i n}{\frac{(h_R^r - h_R^u)a}{m} n - h_R^u (1-\gamma) n(n-1)}. \tag{3.44}$$

   - *For any given $Q_i \geq 0$, and n in the region $n > \frac{(h_R^r - h_R^u)a}{h_R^u (1-\gamma) m} + 1$, $\Pi_{RS}(n, Q_r, Q_i)$ is convex in Q_r with the unique minimizer given by (3.44).*

3. *For any given (n, Q_i) pair, where $n \in Z^+$ and $Q_i \geq 0$, $\Pi(n, Q_r, Q_i)$ is concave in Q_r with the unique maximizer given by*

$$Q_r(n, Q_i) = \frac{(c^{np} + c^d - c + h_M^n T)a - h_R^u \gamma Q_i}{h_M^n n + h_M^r + h_M^u (1 - \frac{1}{n}) - h_R^u (1-\gamma)(n-1) + \frac{(h_R^r - h_R^u)a}{m}}. \tag{3.45}$$

Proof.

1. For any given (n, Q_i) pair, where $n \in Z^+$ and $Q_i \geq 0$, it is easy to show that

$$\begin{aligned}
\frac{\partial \Pi_{OEM}}{\partial Q_r} &= (c^{np} + c^d - c^{rp} + h_M^n T)n - c^d \\
&\quad - \left(h_M^n n^2 + h_M^r n + h_M^u (n-1)\right) \frac{Q_r}{a}, \quad \text{and} \\
\frac{\partial^2 \Pi_{OEM}}{\partial Q_r^2} &= -\frac{h_M^n n^2 + h_M^r n + h_M^u (n-1)}{a} < 0.
\end{aligned}$$

Hence, $\Pi_{OEM}(n, Q_r, Q_i)$ is concave in Q_r for any given (n, Q_i) pair, where $n \in Z^+$ and $Q_i \geq 0$, with the unique maximizer given by (3.43).

2. For any given (n, Q_i) pair, where $n \in Z^+$ and $Q_i \geq 0$, it is easy to show that

$$\begin{aligned}
\frac{\partial \Pi_{RS}}{\partial Q_r} &= (c^{rp} - c)n + c^d - \frac{h_R^u \gamma Q_i n}{a} \\
&\quad - \left(\frac{(h_R^r - h_R^u)n}{m} - \frac{h_R^u (1-\gamma) n(n-1)}{a}\right) Q_r, \quad \text{and} \\
\frac{\partial^2 \Pi_{RS}}{\partial Q_r^2} &= -\frac{(h_R^r - h_R^u)n}{m} + \frac{h_R^u (1-\gamma) n(n-1)}{a}.
\end{aligned}$$

Hence, either of the following two cases is true:

- If n is in the region $1 \leq n \leq \frac{(h_R^r - h_R^u)a}{h_R^u (1-\gamma) m} + 1$ then $\frac{\partial^2 \Pi_{RS}}{\partial Q_r^2} \leq 0$, and hence, $\Pi_{RS}(n, Q_r, Q_i)$ is concave in Q_r with the unique maximizer given by (3.44)
- If n is in the region $n > \frac{(h_R^r - h_R^u)a}{h_R^u (1-\gamma) m} + 1$ then $\frac{\partial^2 \Pi_{RS}}{\partial Q_r^2} > 0$, and hence, $\Pi_{RS}(n, Q_r, Q_i)$ is convex in Q_r with the unique minimizer given by (3.44).

3. For any given (n, Q_i) pair, where $n \in Z^+$, and $Q_i \geq 0$, it is easy to show that

$$\begin{aligned}
\frac{\partial \Pi}{\partial Q_r} &= (c^{np} + c^d - c + h_M^n T)n - \frac{h_R^u \gamma Q_i n}{a} - \Bigg(h_M^n n^2 + h_M^r n + h_M^u (n-1) \\
&\quad - h_R^u (1-\gamma) n(n-1) + \frac{(h_R^r - h_R^u) n a}{m} \Bigg) \frac{Q_r}{a}, \quad \text{and} \\
\frac{\partial^2 \Pi}{\partial Q_r^2} &= -\left(\frac{h_M^n n^2 + h_M^r n + h_M^u (n-1) - h_R^u (1-\gamma) n(n-1)}{a} + \frac{(h_R^r - h_R^u) n}{m} \right).
\end{aligned}$$

Since $h_M^n > h_R^u$, $\gamma \leq 1$ and $n \geq 1$, it can be easily shown that $\frac{\partial^2 \Pi}{\partial Q_r^2} < 0$, and hence, $\Pi(n, Q_r, Q_i)$ is concave in Q_r, and $\frac{\partial \Pi}{\partial Q_r} = 0$ yields its unique maximizer as in (3.45).

□

Property 10 *Treating n as a positive continuous variable momentarily, for any given (Q_r, Q_i) pair, $\Pi_{RS}(n, Q_r, Q_i)$ is convex in n.*

Proof. For any given (Q_r, Q_i) pair, it is easy to show that

$$\begin{aligned}
\frac{\partial \Pi_{RS}}{\partial n} &= -\Bigg[K_r - (c^{rp} - c) Q_r + \frac{h_R^u \gamma Q_i Q_r}{a} + \frac{h_R^u (1-\gamma) Q_r^2}{2a} \\
&\quad + \frac{(h_R^r - h_R^u) Q_r^2}{2m} - \frac{h_R^u (1-\gamma) Q_r^2}{a} n \Bigg], \quad \text{and} \\
\frac{\partial^2 \Pi_{RS}}{\partial n^2} &= \frac{h_R^u (1-\gamma) Q_r^2}{a} \geq 0.
\end{aligned}$$

Hence, $\Pi_{RS}(n, Q_r, Q_i)$ is convex in n, for any given (Q_r, Q_i) pair. □

3.6 Scenario 1: Computing n and Q_i for an Exogenous Q_r

We begin with analyzing the problems formulated in Section 3.4.1 where Q_r is exogenous. We will develop formal approach for computing the optimal n value and Q_i value for each of the following five problem settings: (1) centralized control;

(2) OEM-centric control; (3) RS-centric control; (4) OEM-lead control where OEM decides n and RS decides Q_i, and (5) OEM-lead control where OEM decides Q_i and RS decides n.

3.6.1 Scenario 1: Centralized Control

In this section, we analyze P1.1 in (3.21) to determine the values of n and Q_i that maximize the system-wide total profit. Using (3.3) and (3.5), we can derive an upper bound on the value of n.

Property 11 *When Q_r is exogenous, under centralized control, the feasible region for n is given by $0 \leq n \leq (\frac{aT}{Q_r} - 2)\gamma$.*

Proof. By definition, n is non-negative, i.e., $n \geq 0$. (3.3) and (3.5) together imply

$$aT - (n+1)Q_r \geq Q_i \geq \frac{1}{\gamma}\left((n-1)(1-\gamma)Q_r + Q_r\right), \tag{3.46}$$

and, hence, we have $aT - (n+1)Q_r \geq \frac{1}{\gamma}\left((n-1)(1-\gamma)Q_r + Q_r\right)$, which is equivalent to $n \leq (\frac{aT}{Q_r} - 2)\gamma$. □

Recall that $\Pi(n, Q_i \mid\mid Q_r) = \Pi(n, Q_r, Q_i)$, and is as in (3.20). The following property is helpful for computing the optimal solution of P1.1 in (3.21).

Property 12 *When Q_r is exogenous, under centralized control, for any given $n \geq 1$, the optimal value of Q_i is given by*

$$Q^*_{i,1.1}(n, Q_r) = \left(\frac{n}{\gamma} - n + 1\right) Q_r. \tag{3.47}$$

Proof. By (3.46), for any given (n, Q_r) pair, the smallest value of Q_i is

$$\frac{1}{\gamma}\left((n-1)(1-\gamma)Q_r + Q_r\right),$$

which is equivalent to $\left(\frac{n}{\gamma} - n + 1\right)Q_r$. By Property 8.3, $\Pi(n, Q_i \mid\mid Q_r)$ is non-increasing in Q_i for any given (n, Q_r) pair, where $n \geq 1$ and $Q_r \geq 0$. Hence, the optimal value of Q_i is given by (3.47). $\square$

By Property 12, we obtain the upper bound of $\Pi(n, Q_r \mid\mid Q_i)$ for $n \geq 1$, which is given by $\Pi(n, Q^*_{i,1.1}(n, Q_r) \mid\mid Q_r)$. Remember that the profit function $\Pi(n, Q_r \mid\mid Q_i)$ for $n \geq 1$ is different with the profit function $\Pi(n = 0)$, however, by substituting $n = 0$ in $\Pi(n, Q^*_{i,1.1}(n, Q_r) \mid\mid Q_r)$, we obtain

$$\Pi(0, Q^*_{i,1.1}(0, Q_r) \mid\mid Q_r) = (\pi - c^{np} - c^d)aT - h^n_M \frac{aT^2}{2} = \Pi(n = 0),$$

by Property 7.3. Thus, $\Pi(n, Q^*_{i,1.1}(n, Q_r) \mid\mid Q_r)$ can be treated as the upper bound of $\Pi(n, Q_r \mid\mid Q_i)$ for $n \geq 0$, i.e., the whole feasible region of n. Then, for computing the optimal value of n for P1.1 in (3.21), it suffices to find $n^*_{1.1}(Q_r)$ such that

$$n^*_{1.1}(Q_r) = \underset{0 \leq n \leq (\frac{aT}{Q_r} - 2)\gamma}{\arg\max} \{\Pi(n, Q^*_{i,1.1}(n, Q_r) \mid\mid Q_r)\},$$

where $Q^*_{i,1.1}(n, Q_r)$ is given by (3.47). The following property of $\Pi(n, Q^*_{i,1.1}(n, Q_r) \mid\mid Q_r)$ is sufficient for computing $n^*_{1.1}(Q_r)$.

Property 13 *Treating n as a continuous variable momentarily, for any given $Q_r > 0$, $\Pi(n, Q^*_{i,1.1}(n, Q_r) \mid\mid Q_r)$ is concave in n, and its unique maximizer, denoted by $n^0_{1.1}$, is given by*

$$n^0_{1.1}(Q_r) = \frac{\left\{\begin{array}{c} (c^{np} + c^d - c + h^n_M T)aQ_r - (K_s + K_r)a \\ -\frac{(h^r_R - h^u_R)aQ_r^2}{2m} - \frac{\left(h^r_M + \left(\frac{2}{\gamma}-1\right)h^u_M + (1-\gamma)h^u_R\right)Q_r^2}{2} \end{array}\right\}}{\left(h^u_M\left(\frac{1}{\gamma}-1\right)^2 + h^n_M + h^u_R(1-\gamma)\right)Q_r^2}. \tag{3.48}$$

Proof. Treating n as a continuous variable momentarily, it can be easily shown that

$$\begin{aligned} \frac{\partial \Pi(n, Q^*_{i,1.1}(n, Q_r) \mid\mid Q_r)}{\partial n} &= -\Bigg[\frac{\left(h^u_M\left(\frac{1}{\gamma}-1\right)^2 + h^n_M + h^u_R(1-\gamma)\right)Q_r^2}{a}n + \\ &\quad K_s + K_r - (c^{np} + c^d - c + h^n_M T)Q_r + \frac{(h^r_R - h^u_R)Q_r^2}{2m} + \\ &\quad \frac{\left(h^r_M + \left(\frac{2}{\gamma}-1\right)h^u_M + (1-\gamma)h^u_R\right)Q_r^2}{2a}\Bigg], \quad \text{and} \\ \frac{\partial^2 \Pi(n, Q^*_{i,1.1}(n, Q_r) \mid\mid Q_r)}{\partial n^2} &= -\frac{\left(h^u_M\left(\frac{1}{\gamma}-1\right)^2 + h^n_M + h^u_R(1-\gamma)\right)Q_r^2}{a} < 0. \end{aligned}$$

Hence, $\Pi(n, Q^*_{i,1.1}(n, Q_r) \mid\mid Q_r)$ is concave in n with a unique maximizer given by (3.48). □

Using Properties 11 and 13, we can characterize the optimal solution of P1.1.

Corollary 1 *The optimal solution under centralized control for Scenario 1, denoted by $(n^*_{1.1}(Q_r), Q^*_{i,1.1}(Q_r))$, is given by*

$$n^*_{1.1}(Q_r) = \begin{cases} 0\ , & n^0_{1.1}(Q_r) \le 0, \\ \arg\max\{\Pi(\lfloor n^0_{1.1}(Q_r)\rfloor, Q^*_{i,1.1}\left(\lfloor n^0_{1.1}(Q_r)\rfloor, Q_r\right) \mid\mid Q_r), & 0 < n^0_{1.1}(Q_r) \\ \qquad \Pi(\lceil n^0_{1.1}(Q_r)\rceil, Q^*_{i,1.1}\left(\lfloor n^0_{1.1}(Q_r)\rfloor, Q_r\right) \mid\mid Q_r)\}, & < \left(\frac{aT}{Q_r} - 2\right)\gamma, \\ \left\lfloor \left(\frac{aT}{Q_r} - 2\right)\gamma \right\rfloor, & n^0_{1.1}(Q_r) \ge \left(\frac{aT}{Q_r} - 2\right)\gamma, \end{cases} \tag{3.49}$$

$$Q^*_{i,1.1}(Q_r) = \begin{cases} 0, & if \quad n^*_{1.1}(Q_r) = 0, \\ Q^*_{i,1.1}(n^*_{1.1}(Q_r), Q_r), & otherwise, \end{cases} \tag{3.50}$$

where $n^0_{1.1}(Q_r)$ and $Q^*_{i,1.1}(n, Q_r)$ are given by (3.48) and (3.47).

3.6.2 Scenario 1: OEM-centric Control

In this section, we analyze P1.2 in (3.22) to determine the values of n and Q_i that maximize OEM's total profit. Using (3.3), we can derive an upper bound on the value of n, and for any given n, we can derive an upper bound on the value of Q_i.

Property 14 *When Q_r is exogenous, under OEM-centric control, the feasible region for n is given by $0 \le n \le \frac{aT}{Q_r} - 1$. For each n, the feasible region for Q_i is given by $0 \le Q_i \le aT - (n+1)Q_r$.*

Proof. By definition, n and Q_i are non-negative, i.e., $n \ge 0$ and $Q_i \ge 0$. (3.3) implies that $0 \le Q_i \le aT - (n+1)Q_r$. Consequently, we have $0 \le aT - (n+1)Q_r$, which is equivalent to $n \le \frac{aT}{Q_r} - 1$. □

Recall that $\Pi_{OEM}(n, Q_i \mid\mid Q_r) = \Pi_{OEM}(n, Q_r, Q_i)$, and is as in (3.13). The following property is helpful for computing the optimal solution of P1.2 in (3.22).

Property 15 *When Q_r is exogenous, under OEM-centric control, for any given $n \geq 1$, the optimal value of Q_i is given by*

$$Q^*_{i,1.2}(n, Q_r) = \begin{cases} \frac{ac^d}{h^u_M}, & \frac{ac^d}{h^u_M} < aT - (n+1)Q_r, \\ aT - (n+1)Q_r, & \frac{ac^d}{h^u_M} \geq aT - (n+1)Q_r. \end{cases} \tag{3.51}$$

Proof. By Property 14, for any given n, the feasible region of Q_i is given by $0 \leq Q_i \leq aT - (n+1)Q_r$. By Property 8.1, for any given (n, Q_r) pair, where $n \geq 1$ and $Q_r \geq 0$, $\Pi_{OEM}(n, Q_i \mid\mid Q_r)$ is concave in Q_i with a unique maximizer given by (3.42). Hence, the optimal value of Q_i is given by (3.51). □

By Property 15, we obtain the upper bound of $\Pi_{OEM}(n, Q_i \mid\mid Q_r)$ for $1 \leq n \leq \frac{aT}{Q_r} - 1$, which is given by $\Pi_{OEM}(n, Q^*_{i,1.2}(n, Q_r) \mid\mid Q_r)$, and can be written as follows

$$\Pi_{OEM}(n, Q^*_{i,1.2}(n, Q_r) \mid\mid Q_r) = \begin{cases} F^a_{1.2}(n \mid\mid Q_r), & 1 \leq n < \frac{a}{Q_r}\left(T - \frac{c^d}{h^u_M}\right) - 1, \\ F^b_{1.2}(n \mid\mid Q_r), & \frac{a}{Q_r}\left(T - \frac{c^d}{h^u_M}\right) - 1 \leq n \leq \frac{aT}{Q_r} - 1, \end{cases} \tag{3.52}$$

where, $F^a_{1.2}(n \mid\mid Q_r)$ and $F^a_{1.2}(n \mid\mid Q_r)$ are given by

$$F^a_{1.2}(n \mid\mid Q_r) = \Pi_{OEM}\left(n, \frac{ac^d}{h^u_M} \,\middle|\middle|\, Q_r\right) \quad \text{and} \tag{3.53}$$

$$F^b_{1.2}(n \mid\mid Q_r) = \Pi_{OEM}(n, aT - (n+1)Q_r \mid\mid Q_r), \tag{3.54}$$

respectively.

For computing the optimal value of n for P1.2 in (3.22), it suffices to find $n^*_{1.2}(Q_r)$ such that

$$n^*_{1.2}(Q_r) = \underset{0 \leq n \leq \frac{aT}{Q_r} - 1}{\arg\max} \{\Pi_{OEM}(n, Q^*_{i,1.2}(n, Q_r) \mid\mid Q_r), \Pi_{OEM}(n = 0)\},$$

where $Q^*_{i,1.2}(n, Q_r)$ is given by (3.51), $\Pi_{OEM}(n, Q^*_{i,1.2}(n, Q_r) \mid\mid Q_r)$ is given by (3.52), and $\Pi_{OEM}(n = 0)$ is as in Property 7.1. The following properties of $F^a_{1.2}(n \mid\mid Q_r)$ and $F^a_{1.2}(n \mid\mid Q_r)$ are sufficient for computing $n^*_{1.2}(Q_r)$.

Property 16 *Treating n as a continuous variable momentarily, for any given $Q_r > 0$, $\Pi_{OEM}(n, Q^*_{i,1.2}(n, Q_r) \mid\mid Q_r)$ in (3.52) is piecewise concave in n. To be more specific, we have*

- *For any given $Q_r > 0$, $F^a_{1.2}(n \mid\mid Q_r)$ in (3.53) is concave in n with a unique maximizer, denoted by $n^a_{1.1}(Q_r)$, which is given by*

$$n^a_{1.2}(Q_r) = \frac{(c^{np} + c^d - c^{rp} + h^n_M T)aQ_r - K_s a - \frac{(h^u_M + h^r_M)Q_r^2}{2}}{h^n_M Q_r^2}. \quad (3.55)$$

- *For any given $Q_r > 0$, $F^a_{1.2}(n \mid\mid Q_r)$ in (3.53) is concave in n with a unique maximizer, denoted by $n^a_{1.1}(Q_r)$, which is given by*

$$n^b_{1.2}(Q_r) = \frac{(c^{np} - c^{rp} + h^u_M T + h^n_M T)aQ_r - K_s a - \frac{(3h^u_M + h^r_M)Q_r^2}{2}}{(h^u_M + h^n_M)\, Q_r^2}. \quad (3.56)$$

Proof. Treating n as a continuous variable momentarily, it is easy to show that

$$\begin{aligned} \frac{\partial F^a_{1.2}(n \mid\mid Q_r)}{\partial n} &= (c^{np} + c^d - c^{rp} + h^n_M T)Q_r - K_s - \frac{(h^u_M + h^r_M)Q_r^2}{2a} - \frac{h^n_M Q_r^2}{a} n \quad \text{and} \\ \frac{\partial^2 F^a_{1.2}(n \mid\mid Q_r)}{\partial n^2} &= -\frac{h^n_M Q_r^2}{a} < 0. \end{aligned}$$

Hence, $F^a_{1.2}(n \mid\mid Q_r)$ is concave in n with a unique maximizer given by (3.55).

$$
\begin{aligned}
\frac{\partial F^b_{1.2}(n \mid\mid Q_r)}{\partial n} &= (c^{np} - c^{rp} + h^u_M T + h^n_M T)Q_r - K_s \\
&\quad - \frac{(3h^u_M + h^r_M)Q_r^2}{2a} - \frac{(h^u_M + h^n_M)\,Q_r^2}{a} n, \quad \text{and} \\
\frac{\partial^2 F^b_{1.2}(n \mid\mid Q_r)}{\partial n^2} &= -\frac{(h^u_M + h^n_M)\,Q_r^2}{a} < 0.
\end{aligned}
$$

Hence, $F^b_{1.2}(n \mid\mid Q_r)$ is concave in n with a unique maximizer given by (3.56). $\square$

Recalling (3.52), and by Property 16, we can derive the maximizer of $\Pi_{OEM}(n, Q^*_{i,1.2}(n, Q_r) \mid\mid Q_r)$ for $1 \leq n \leq \frac{aT}{Q_r} - 1$.

Corollary 2 *The maximizer of $\Pi_{OEM}(n, Q^*_{i,1.2}(n, Q_r) \mid\mid Q_r)$, where $1 \leq n < \frac{a}{Q_r}\left(T - \frac{c^d}{h^u_M}\right) - 1$, denoted by $n^{a*}_{1.2}(Q_r)$, is given by*

$$
n^{a*}_{1.2}(Q_r) = \begin{cases} 1, & n^a_{1.2}(Q_r) \leq 1, \\ \arg\max\{F^a_{1.2}(\lfloor n^a_{1.2}(Q_r)\rfloor \mid\mid Q_r), & 1 < n^a_{1.2}(Q_r) < \\ \qquad F^a_{1.2}(\lceil n^a_{1.2}(Q_r)\rceil \mid\mid Q_r)\}, & \quad \frac{a}{Q_r}\left(T - \frac{c^d}{h^u_M}\right) - 1, \\ \left\lfloor \frac{a}{Q_r}\left(T - \frac{c^d}{h^u_M}\right) - 1 \right\rfloor, & n^a_{1.2}(Q_r) \geq \frac{a}{Q_r}\left(T - \frac{c^d}{h^u_M}\right) - 1, \end{cases} \tag{3.57}
$$

where $F^a_{1.2}(n \mid\mid Q_r)$ and $n^a_{1.2}(Q_r)$ are given by (3.53) and (3.55), respectively.

The maximizer of $\Pi_{OEM}(n, Q^*_{i,1.2}(n, Q_r) \mid\mid Q_r)$, where $\frac{a}{Q_r}\left(T - \frac{c^d}{h^u_M}\right) - 1 \leq n \leq \frac{aT}{Q_r} - 1$, denoted by $n^{b*}_{1.2}(Q_r)$, is given by

$$n^{b*}_{1.2}(Q_r) = \begin{cases} \left\lfloor \frac{a}{Q_r}\left(T - \frac{c^d}{h^u_M}\right) - 1 \right\rfloor, & n^b_{1.2}(Q_r) < \frac{a}{Q_r}\left(T - \frac{c^d}{h^u_M}\right) - 1, \\ \arg\max\{F^b_{1.2}(\lfloor n^b_{1.2}(Q_r) \rfloor \mid\mid Q_r), & \frac{a}{Q_r}\left(T - \frac{c^d}{h^u_M}\right) - 1 \leq \\ \qquad F^b_{1.2}(\lceil n^b_{1.2} \rceil \mid\mid Q_r)\}, & \qquad n^b_{1.2}(Q_r) < \frac{aT}{Q_r} - 1, \\ \left\lfloor \frac{aT}{Q_r} - 1 \right\rfloor, & n^b_{1.2}(Q_r) \geq \frac{aT}{Q_r} - 1, \end{cases} \tag{3.58}$$

where $F^b_{1.2}(n \mid\mid Q_r)$ and $n^b_{1.2}(Q_r)$ are given by (3.54) and (3.56), respectively.

Proof. The proof is straightforward by using (3.52) and Property 16, and hence, is omitted. □

Recalling that $\Pi_{OEM}(n, Q^*_{i,1.2}(n, Q_r) \mid\mid Q_r)$ in (3.52) is the upper bound of $\Pi_{OEM}(n, Q_r \mid\mid Q_i)$, where $1 \leq n \leq \frac{aT}{Q_r} - 1$, and by Corollary 2, we can characterize the optimal solution of P1.2.

Corollary 3 *The optimal solution under OEM-centric control for Scenario 1, denoted by $(n^*_{1.2}(Q_r), Q^*_{i,1.2}(Q_r))$, is given by*

$$n^*_{1.2}(Q_r) = \arg\max\{F^a_{1.2}(n^{a*}_{1.2}(Q_r) \mid\mid Q_r), F^b_{1.2}(n^{b*}_{1.2}(Q_r) \mid\mid Q_r), \Pi_{OEM}(n = 0)\}, \tag{3.59}$$

$$Q^*_{i,1.2}(Q_r) = \begin{cases} 0, & if \quad n^*_{1.2}(Q_r) = 0, \\ Q^*_{i,1.2}(n^*_{1.2}(Q_r), Q_r), & otherwise. \end{cases} \tag{3.60}$$

Proof. The proof is straightforward by Corollary 2 and Property 15, and, hence, it is omitted. □

3.6.3 Scenario 1: RS-centric Control

In this section, we analyze P1.3 in (3.23) to determine the values of n and Q_i that maximize RS's total profit. Using (3.5), we can derive a lower bound on the value of Q_i, for any given $n \geq 1$.

Property 17 *When Q_r is exogenous, under RS-centric control, for any given $n \geq 1$, the feasible region for Q_i is given by $Q_i \geq \left(\frac{n}{\gamma} - n + 1\right) Q_r$.*

Proof. The proof is straightforward by rewriting (3.5). □

Recall that $\Pi_{RS}(n, Q_i \mid\mid Q_r) = \Pi_{RS}(n, Q_r, Q_i)$, and is as in (3.19). The following property is helpful for computing the optimal solution of P1.3 in (3.23).

Property 18 *When Q_r is exogenous, under RS-centric control, for any given $n \geq 1$, the optimal value of Q_i is given by*

$$Q^*_{i,1.3}(n, Q_r) = \left(\frac{n}{\gamma} - n + 1\right) Q_r. \tag{3.61}$$

Proof. By Property 8.2, for any given (n, Q_r) pair, where $n \geq 1$ and $Q_r \geq 0$, $\Pi_{RS}(n, Q_i \mid\mid Q_r)$ is linearly decreasing in Q_i. Thus, the optimal value of Q_i is given by the lower limit of the feasible region of Q_i, which is given by (3.61) as shown in Property 17. □

By Property 18, we obtain the upper bound of $\Pi_{RS}(n, Q_i \mid\mid Q_r)$ for $n \geq 1$, which is given by $\Pi_{RS}(n, Q^*_{i,1.3}(n, Q_r) \mid\mid Q_r)$. Remember that the profit function $\Pi_{RS}(n, Q_i \mid\mid Q_r)$ for $n \geq 1$ is different with the profit function $\Pi_{RS}(n = 0)$, however, by substituting $n = 0$ in $\Pi_{RS}(n, Q^*_{i,1.3}(n, Q_r) \mid\mid Q_r)$, we obtain

$$\Pi_{RS}(0, Q^*_{i,1.3}(0, Q_r) \mid\mid Q_r) = 0 = \Pi_{RS}(n = 0),$$

by Property 7.2. Thus, $\Pi_{RS}(n, Q^*_{i,1.3}(n, Q_r) \mid\mid Q_r)$ can be treated as the upper bound of $\Pi(n, Q_r \mid\mid Q_i)$ for $n \geq 0$, i.e., the whole feasible region of n. Then, for computing the optimal value of n for P1.3 in (3.23), it suffices to find $n^*_{1.3}(Q_r)$ such that

$$n^*_{1.3}(Q_r) = \arg\max_{n \in Z^*} \{\Pi_{RS}(n, Q^*_{i,1.3}(n, Q_r) \mid\mid Q_r)\}, \tag{3.62}$$

where $Q^*_{i,1.3}(n, Q_r)$ is given by (3.61). The following property of $\Pi(n, Q^*_{i,1.3}(n, Q_r) \mid\mid Q_r)$ is sufficient for computing $n^*_{1.3}(Q_r)$.

Property 19 *Treating n as a continuous variable momentarily, for any given $Q_r > 0$, $\Pi_{RS}(n, Q^*_{i,1.3}(n, Q_r) \mid\mid Q_r)$ in (3.62) is concave with a unique maximizer $n^0_{1.3}$ which is given by*

$$n^0_{1.3}(Q_r) = \frac{\left(c^{rp} - c - \left(\frac{1}{\gamma} - 1\right) c^d\right) aQ_r - K_r a - \left(\frac{h^u_R(1+\gamma)}{2} + \frac{(h^r_R - h^u_R)a}{2m}\right) Q^2_r}{h^u_R(1-\gamma)Q^2_r}. \tag{3.63}$$

Proof. Treating n as a continuous variable momentarily, it is easy to show that

$$\begin{aligned}\frac{\partial \Pi_{RS}(n, Q^*_{i,1.3}(n, Q_r) \mid\mid Q_r)}{\partial n} &= \left(c^{rp} - c - \left(\frac{1}{\gamma} - 1\right) c^d\right) Q_r - K_r \\ &- \left(\frac{h^u_R(1+\gamma)}{2a} + \frac{(h^r_R - h^u_R)}{2m}\right) Q^2_r - \frac{h^u_R(1-\gamma)Q^2_r}{a} n, \quad \text{and} \\ \frac{\partial^2 \Pi_{RS}(n, Q^*_{i,1.1}(n, Q_r) \mid\mid Q_r)}{\partial n^2} &= -\frac{h^u_R(1-\gamma)Q^2_r}{a} < 0.\end{aligned}$$

Hence, $\Pi_{RS}(n, Q^*_{i,1.3}(n, Q_r) \mid\mid Q_r)$ is concave in n with a unique maximizer given by (3.63). □

Using Property 19, and recalling (3.62), we can characterize the optimal solution of P1.3.

Corollary 4 *The optimal solution under RS-centric control for Scenario 1, de-*

noted by $(n^*_{1.3}(Q_r), Q^*_{i,1.3}(Q_r))$, *is given by*

$$
\begin{aligned}
n^*_{1.3}(Q_r) &= \begin{cases} 0, & n^0_{1.3}(Q_r) \le 0, \\ \arg\max\{\Pi_{RS}(\lfloor n^0_{1.3}(Q_r)\rfloor, Q^*_{i,1.3}(\lfloor n^0_{1.3}(Q_r)\rfloor, Q_r) \mid\mid Q_r), & \\ \qquad \Pi_{RS}(\lceil n^0_{1.3}(Q_r)\rceil, Q^*_{i,1.3}(\lfloor n^0_{1.3}(Q_r)\rfloor, Q_r) \mid\mid Q_r)\}, & n^0_{1.3}(Q_r) > 0, \end{cases} \\
Q^*_{i,1.3}(Q_r) &= \begin{cases} 0, & if \quad n^*_{1.3}(Q_r) = 0, \\ Q^*_{i,1.3}(n^*_{1.3}(Q_r), Q_r), & otherwise, \end{cases}
\end{aligned}
$$

where $n^0_{1.3}(Q_r)$ *and* $Q^*_{i,1.3}(n, Q_r)$ *are given by (3.63) and (3.61), respectively.*

3.6.4 Scenario 1: OEM-lead Control: OEM Decides n *and RS Decides* Q_i

In this section, we analyze P1.4.1 and P1.4.2 formulated by (3.24) and (3.25), respectively. The OEM determines the value of n, and then the RS determines the value of Q_i. To determine the value of n, the OEM will rely on the prediction of the RS's response for any given n value.

This prediction can be obtained by deriving the RS's optimal value of Q_i that maximizes the RS's total profit for any given (n, Q_r) pair. We denote the RS's optimal response by $Q^*_{i,1.4}(n, Q_r)$, which is the solution of P1.4.1 in (3.24). Recalling Property 18, for any given $n \ge 1$, and Q_r is exogenous, the optimal value of Q_i for the RS is given by $Q^*_{i,1.3}(n, Q_r)$. Thus, $Q^*_{i,1.4}(n, Q_r) = Q^*_{i,1.3}(n, Q_r)$ for any $n \ge 1$.

After predicting the RS's response, the OEM's decision is to determine the value of n that maximizes the OEM's total profit, i.e., the solution of P1.4.2 in (3.25). Substituting $Q^*_{i,1.4}(n, Q_r)$ in (3.25), the OEM's problem can be rewritten as

$$
\begin{aligned}
\max_{n \in Z^*} \quad & \Pi_{OEM}\left(n, \left(\frac{n}{\gamma} - n + 1\right) Q_r \,\middle|\middle|\, Q_r\right) \\
s.t. \quad & n \le \left(\frac{aT}{Q_r} - 2\right)\gamma.
\end{aligned} \tag{3.64}
$$

The following property of $\Pi_{OEM}\left(n, \left(\frac{n}{\gamma}-n+1\right) \middle\| Q_r\right)$ is sufficient for computing the optimal value of n for (3.64).

Property 20 *Treating n as a continuous variable momentarily, for any given $Q_r \geq 0$, $\Pi_{OEM}\left(n, \left(\frac{n}{\gamma}-n+1\right) Q_r \middle\| Q_r\right)$ in (3.64) is concave in n with a unique maximizer $n^0_{1.4}(Q_r)$ given by*

$$n^0_{1.4}(Q_r) = \frac{\left(c^{np}-c^{rp}+\frac{c^d}{\gamma}+h^n_M T\right) aQ_r - K_s a - \frac{\left(h^u_M\left(\frac{2}{\gamma}-1\right)+h^r_M\right)Q_r^2}{2}}{\left(h^n_M + h^u_M\left(\frac{1}{\gamma}-1\right)^2\right)Q_r^2}. \quad (3.65)$$

Proof. Treating n as a continuous variable momentarily, it can be easily shown that

$$\begin{aligned}
\frac{\partial \Pi_{OEM}\left(n, \left(\frac{n}{\gamma}-n+1\right) Q_r \middle\| Q_r\right)}{\partial n} &= \left(c^{np}-c^{rp}+\frac{c^d}{\gamma}+h^n_M T\right) Q_r - K_s \\
&\quad - \frac{\left(h^u_M\left(\frac{2}{\gamma}-1\right)+h^r_M\right)Q_r^2}{2a} \\
&\quad - \frac{\left(h^n_M + h^u_M\left(\frac{1}{\gamma}-1\right)^2\right)Q_r^2}{a} n, \quad \text{and} \\
\frac{\partial^2 \Pi_{OEM}\left(n, \left(\frac{n}{\gamma}-n+1\right) Q_r \middle\| Q_r\right)}{\partial n^2} &= -\frac{\left(h^n_M + h^u_M\left(\frac{1}{\gamma}-1\right)^2\right)Q_r^2}{a} < 0.
\end{aligned}$$

Hence, $\Pi_{OEM}\left(n, \left(\frac{n}{\gamma}-n+1\right) Q_r \middle\| Q_r\right)$ is concave in n with a unique maximizer given by (3.65). □

Using Property 20, we can characterize the optimal solutions of P1.4.1 in (3.24) and P1.4.2 in (3.25).

Corollary 5 *Under OEM-lead control where OEM decides n and RS decides Q_i*

for Scenario 1, the optimal value of n for the OEM, denoted by $n^*_{1.4}(Q_r)$, is given by

$$n^*_{1.4}(Q_r) = \begin{cases} 0, & n^0_{1.4}(Q_r) \leq 0, \\ \arg\max \Big\{ \Pi_{OEM}\left(\lfloor n^0_{1.4}(Q_r) \rfloor, \left(\frac{\lfloor n^0_{1.4}(Q_r) \rfloor}{\gamma} - \lfloor n^0_{1.4}(Q_r) \rfloor + 1 \right) Q_r \,\Big|\Big|\, Q_r \right), & \\ \qquad \Pi_{OEM}\left(\lceil n^0_{1.4}(Q_r) \rceil, \left(\frac{\lceil n^0_{1.4}(Q_r) \rceil}{\gamma} - \lceil n^0_{1.4}(Q_r) \rceil + 1 \right) Q_r \,\Big|\Big|\, Q_r \right) \Big\}, & \\ & 0 < n^0_{1.4}(Q_r) < \left(\frac{aT}{Q_r} - 2 \right) \gamma, \\ \left\lfloor \left(\frac{aT}{Q_r} - 2 \right) \gamma \right\rfloor & n^0_{1.4} \geq \left(\frac{aT}{Q_r} - 2 \right) \gamma, \end{cases} \tag{3.66}$$

where $n^0_{1.4}(Q_r)$ is given by (3.65).

The RS's optimal response is given by

$$Q^*_{i,1.4}(Q_r) = \begin{cases} 0, & if \quad n^*_{1.4(Q_r)} = 0, \\ \left(\frac{n^*_{1.4}(Q_r)}{\gamma} - n^*_{1.4}(Q_r) + 1 \right) Q_r, & otherwise. \end{cases}$$

3.6.5 Scenario 1: OEM-lead Control: OEM Decides Q_i and RS Decides n

In this section, we analyze P1.5.1 and P1.5.2 formulated by (3.26) and (3.27), respectively. The OEM determines the value of Q_i, and then the RS determines the value of n. To determine the value of Q_i, the OEM will rely on the prediction of the RS's response for any given Q_i value.

This prediction can be obtained by deriving the RS's optimal value of n that maximizes the RS's total profit for any given (Q_i, Q_r) pair. We denote the RS's optimal response by $n^*_{1.5}(Q_i, Q_r)$, which is the solution of P1.5.1 in (3.26). Recalling Property 10, for any given (Q_i, Q_r) pair, $\Pi_{RS}(n, Q_r, Q_i)$ is convex in n. Thus, the maximizer should be at one of the boundary constraints. Using (3.5), we can obtain the feasible region for n.

Property 21 *When Q_r is exogenous, under OEM-lead control where the OEM* *decides Q_i and the RS decides n, for any given Q_i, the feasible region for n is given by* $0 \leq n \leq \frac{\gamma(Q_i - Q_r)}{(1-\gamma)Q_r}$.

Proof. By definition, n is non-negative, i.e., $n \geq 0$. The proof is straightforward by rewriting (3.5), and, hence, it is omitted. □

Using Properties 10 and 21, we can specify the optimal value of n by checking the properties of the boundary points.

Property 22 *When Q_r is exogenous, under OEM-lead control where the OEM decides Q_i and the RS decides n, to determine the RS's optimal response to any Q_i value, i.e., to calculate $n^*_{1.5}(Q_i, Q_r)$, it suffices to consider the following cases:*

1. *When $Q_i < \frac{Q_r}{\gamma}$, $n^*_{1.5}(Q_i, Q_r) = 0$, i.e., no remanufacturing.*

2. *When $Q_i \geq \frac{Q_r}{\gamma}$, either of the following is true:*

   2.1. *If $\frac{\partial \Pi_{RS}}{\partial n}(1 \mid\mid (Q_i, Q_r)) \geq 0$ then we have*

$$n^*_{1.5}(Q_i, Q_r) = \arg\max \left\{ \Pi_{RS}(n = 0), \Pi_{RS}\left(\left\lfloor \frac{\gamma(Q_i - Q_r)}{(1-\gamma)Q_r} \right\rfloor \middle|\middle| (Q_i, Q_r) \right) \right\}.$$

   2.2. *If $\frac{\partial \Pi_{RS}}{\partial n}(1 \mid\mid (Q_i, Q_r)) < 0$ and*

   2.2.1. $\frac{\partial \Pi_{RS}}{\partial n}\left(\left\lfloor \frac{\gamma(Q_i - Q_r)}{(1-\gamma)Q_r} \right\rfloor \middle|\middle| (Q_i, Q_r) \right) \leq 0$ *then*

$$n^*_{1.5}(Q_i, Q_r) = \arg\max \left\{ \Pi_{RS}(n = 0), \Pi_{RS}(1 \mid\mid Q_i, Q_r) \right\}.$$

   2.2.2. $\frac{\partial \Pi_{RS}}{\partial n}\left(\left\lfloor \frac{\gamma(Q_i - Q_r)}{(1-\gamma)Q_r} \right\rfloor \middle|\middle| (Q_i, Q_r) \right) > 0$ *then*

$$\begin{aligned} n^*_{1.5}(Q_i, Q_r) = \arg\max \{ & \Pi_{RS}(n = 0), \Pi_{RS}(1 \mid\mid Q_i, Q_r), \\ & \Pi_{RS}\left(\left\lfloor \frac{\gamma(Q_i - Q_r)}{(1-\gamma)Q_r} \right\rfloor \middle|\middle| (Q_r, Q_r) \right) \}. \end{aligned}$$

where $\Pi_{RS}(n=0)=0$, $\frac{\partial \Pi_{RS}}{\partial n}$ *is given by (3.46).*

Proof. By Property 21, the feasible region of n is given by $0 \le n \le \frac{\gamma(Q_i-Q_r)}{(1-\gamma)Q_r}$. If $Q_i < \frac{Q_r}{\gamma}$, i.e., $\frac{\gamma(Q_i-Q_r)}{(1-\gamma)Q_r} < 1$, then the only possible value of n is 0. Otherwise, i.e., $\frac{\gamma(Q_i-Q_r)}{(1-\gamma)Q_r} \ge 1$, the feasible region of n contains positive value.

The proofs of parts 2.1 and 2.2 are straightforward by recalling Property 10, and, hence, they are omitted. □

After predicting the RS's response, the OEM's decision is to determine the value of Q_i that maximizes the OEM's total profit, i.e., solution of P1.5.2 in (3.27). By Property 22, in order to ensure that RS remanufactures, the lowest value of Q_i should be $\frac{Q_r}{\gamma}$. Moreover (3.3) sets an upper bound on the value of Q_i. The following property defines the search region for the optimal value of Q_i for P1.5.2.

Property 23 *When Q_r is exogenous, under OEM-lead control where the OEM decides Q_i and the RS decides n, the OEM's optimal decision for the value of Q_i, denoted by $Q^*_{i,1.5}(Q_r)$, should satisfy*

$$Q^*_{i,1.5}(Q_r) \in \{0\} \bigcup \left[\frac{Q_r}{\gamma}, \min\{aT-\gamma(aT-Q_r), aT-2Q_r\}\right].$$

Proof. By Property 22, in order to ensure that RS remanufactures, the lowest value of Q_i should be $\frac{Q_r}{\gamma}$, i.e., $Q_i \ge \frac{Q_r}{\gamma}$. Then, the RS may set n equal to 1 or $\left\lfloor \frac{\gamma(Q_i-Q_r)}{(1-\gamma)Q_r} \right\rfloor$. By (3.3), i.e., $aT \ge Q_i + (n^*_{1.5}(Q_i,Q_r)+1)Q_r$, either of the following is true:

- If $n^*_{1.5}(Q_i,Q_r)=1$ then $aT \ge Q_i + 2Q_r$, i.e., $Q_i \le aT - 2Q_r$.
- If $n^*_{1.5}(Q_i,Q_r) = \left\lfloor \frac{\gamma(Q_i-Q_r)}{(1-\gamma)Q_r} \right\rfloor$ then $aT \ge Q_i + \left(\left\lfloor \frac{\gamma(Q_i-Q_r)}{(1-\gamma)Q_r} \right\rfloor + 1\right) Q_r \ge Q_i + \frac{\gamma(Q_i-Q_r)}{(1-\gamma)}$, which is equivalent to $Q_i \le aT - \gamma(aT-Q_r)$.

Thus, either of the following is true:

- In the case RS remanufactures, $\frac{Q_r}{\gamma} \leq Q^*_{i,1.5}(Q_r) \leq \min\{aT - \gamma(aT - Q_r), aT - 2Q_r\}$.
- In the case RS does not remanufacture, $Q^*_{i,1.5}(Q_r) = 0$.

$\square$

Using Property 23, we can derive the optimal solutions of P1.5.1 in (3.26) and P1.5.2 in (3.27).

Corollary 6 *Under OEM-lead control where OEM decides Q_i and RS decides n for Scenario 1, the optimal value of Q_i for the OEM, denoted by $Q^*_{i,1.5}(Q_r)$, is given by*

$$Q^*_{i,1.5}(Q_r) = \begin{cases} 0, & \textit{if } \Pi_{OEM}(n^*_{1.5}(Q^0_{i,1.5}(Q_r), Q_r), Q^0_{i,1.5}(Q_r) \mid\mid Q_r) \\ & \quad \leq \Pi_{OEM}(n = 0), \\ Q^0_{i,1.5}(Q_r), & \textit{otherwise}, \end{cases} \tag{3.67}$$

where $n^*_{1.5}(Q_i, Q_r)$ is as in Property 22, $\Pi_{OEM}(n = 0)$ is as in Property 7, and $Q^0_{i,1.5}(Q_r)$ is given by

$$Q^0_{i,1.5}(Q_r) = \underset{\frac{Q_r}{\gamma} \leq Q_i \leq \min\{aT - \gamma(aT - Q_r), aT - 2Q_r\}}{\arg\max} \left\{\Pi_{OEM}(n^*_{1.5}(Q_i, Q_r), Q_i \mid\mid Q_r)\right\}.$$

The RS's optimal response is given by $n^*_{1.5}(Q^*_{i,1.5}(Q_r), Q_r)$.

3.7 Scenario 2: Computing n and Q_r for an Exogenous Q_i

In this section, problems formulated in Section 3.4.2, where Q_i is exogenous, are analyzed. We develop formal approach for computing the optimal n value and Q_r value for each of the following five problem settings: (1) centralized control;

(2) OEM-centric control; (3) RS-centric control; (4) OEM-lead control where OEM decides n and RS decides Q_r, and (5) OEM-lead control where OEM decides Q_r and RS decides n.

3.7.1 Scenario 2: Centralized Control

In this section, we analyze P2.1 in (3.28). The following property is helpful for computing the optimal solution of P2.1 in (3.28).

Property 24 *When Q_i is exogenous, under centralized control, for any given $n \geq 1$, the optimal value of Q_r is given by*

$$Q^*_{r,2.1}(n, Q_i) = \begin{cases} 0, & Q_r(n, Q_i) \leq 0, \\ Q_r(n, Q_i), & 0 < Q_r(n, Q_i) < \min\left\{\frac{\gamma Q_i}{n(1-\gamma)+\gamma}, \frac{aT-Q_i}{n+1}\right\}, \\ \min\left\{\frac{\gamma Q_i}{n(1-\gamma)+\gamma}, \frac{aT-Q_i}{n+1}\right\}, & Q_r(n, Q_i) \geq \min\left\{\frac{\gamma Q_i}{n(1-\gamma)+\gamma}, \frac{aT-Q_i}{n+1}\right\}, \end{cases} \tag{3.68}$$

where $Q_r(n, Q_i)$ is given by (3.45).

Proof. (3.3), (3.5) and the non-negative constraint of Q_r imply that

$$\begin{aligned} Q_r &\leq \frac{aT - Q_i}{n+1} \quad \text{and} \\ Q_r &\leq \frac{\gamma Q_i}{n(1-\gamma)+\gamma}. \end{aligned}$$

Hence, for any given $n \geq 1$, the feasible region of Q_i is given by

$$0 \leq Q_r \leq \min\left\{\frac{aT - Q_i}{n+1}, \frac{\gamma Q_i}{n(1-\gamma)+\gamma}\right\}. \tag{3.69}$$

Recalling Property 9.3 which states that, for any given $n \geq 1$, $\Pi(n, Q_r \mid\mid Q_i)$ is concave in Q_r with a unique maximizer given by (3.45). The proof immediately follows from Property 9.3 and (3.69), and, hence, it is omitted. □

By Property 24, we obtain the upper bound of $\Pi(n, Q_r \mid\mid Q_i)$ for $n \geq 1$, which is given by $\Pi(n, Q^*_{r,2.1}(n, Q_i) \mid\mid Q_i)$. For computing the optimal value of n for P2.1 in (3.28), it suffices to find $n^*_{2.1}(Q_i)$ such that

$$n^*_{2.1}(Q_i) = \arg\max_{n \in Z^*} \{\Pi(n, Q^*_{r,2.1}(n, Q_i) \mid\mid Q_i), \Pi(n = 0)\}, \tag{3.70}$$

where $Q^*_{r,2.1}(n, Q_i)$ is given by (3.68), and $\Pi(n = 0)$ is as in Property 7.3.

The following property defines an upper bound for the search region of n.

Property 25 *When Q_i is exogenous, under centralized control, the optimal value of n, denoted by $n^*_{2.1}(Q_i)$, satisfies*

$$0 \leq n^*_{2.1}(Q_i) \leq \frac{(c^{np} + c^d)aT + h^n_M \frac{aT^2}{2}}{K_s + K_r}. \tag{3.71}$$

Proof. Recalling Property 7.3, and using (3.70), it is easy to show that

$$\Pi(n = 0) \leq \Pi(n^*_{2.1}(Q_i), Q^*_{r,2.1}(n^*_{2.1}(Q_i), Q_i) \mid\mid Q_i) \leq \pi aT - (K_s + K_r)n^*_{2.1}(Q_i).$$

Hence, we have $n^*_{2.1}(Q_i) \leq \frac{(c^{np}+c^d)aT+h^n_M \frac{aT^2}{2}}{K_s+K_r}$, which together with $n \geq 0$ implies (3.71). □

Property 25 helps to limit the search region for the value of n:

$$n = 0, 1, \ldots, \left\lfloor \frac{(c^{np} + c^d)aT + h^n_M \frac{aT^2}{2}}{K_s + K_r} \right\rfloor.$$

Using Properties 24 and 25, we can derive the optimal solution of problem (3.28).

Corollary 7 *The optimal solution under centralized control for Scenario 2, de-*

noted by $(n^*_{2.1}(Q_i), Q^*_{r,2.1}(Q_i))$*, is given by*

$$n^*_{2.1}(Q_i) = \underset{n=0,1,\ldots,N_{2.1}}{\arg\max} \{\Pi(n, Q^*_{r,2.1}(n, Q_i) \mid\mid Q_i), \Pi(n=0)\},$$
$$Q^*_{r,2.1}(Q_i) = Q^*_{r,2.1}(n^*_{2.1}(Q_i), Q_i), \tag{3.72}$$

where $Q^*_{r,2.1}(n, Q_i)$ *is as in (3.68),* $\Pi(n=0)$ *is as in Property 7.3, and* $N_{2.1}$ *is given by*

$$N_{2.1} = \left\lfloor \frac{(c^{np} + c^d)aT + h^n_M \frac{aT^2}{2}}{K_s + K_r} \right\rfloor. \tag{3.73}$$

3.7.2 Scenario 2: OEM-centric Control

In this section, we analyze P2.2 in (3.29) to determine the values of n and Q_i that maximize OEM's total profit. The following property is helpful for computing the optimal solution of P2.2 in (3.29).

Property 26 *When* Q_i *is exogenous, under OEM-centric control, for any given* $n \geq 1$*, the maximizer of* $\Pi_{OEM}(n, Q_r \mid\mid Q_i)$ *is given by*

$$Q^*_{r,2.2}(n, Q_i) = \begin{cases} Q_{r,OEM}(n, Q_i), & 0 < Q_{r,OEM}(n, Q_i) < \frac{aT-Q_i}{n+1}, \\ \frac{aT-Q_i}{n+1}, & Q_{r,OEM}(n, Q_i) \geq \frac{aT-Q_i}{n+1}, \end{cases} \tag{3.74}$$

where $Q_{r,OEM}(n, Q_i)$ *is given by (3.43).*

Proof. (3.3) and the non-negative constraint for Q_r implies that, for any given $n \geq 1$, the feasible region of Q_i is given by

$$0 \leq Q_r \leq \frac{aT - Q_i}{n+1}. \tag{3.75}$$

Recalling Property 9.1 which states that, for any given $n \geq 1$, $\Pi_{OEM}(n, Q_r \mid\mid Q_i)$ is concave in Q_r with a unique maximizer given by (3.43). The proof immediately follows from Property 9.1 and (3.75), and, hence, it is omitted. $\square$

By Property 26, we obtain the upper bound of $\Pi_{OEM}(n, Q_r \mid\mid Q_i)$ for $n \geq 1$, which is given by $\Pi_{OEM}(n, Q^*_{r,2.2}(n, Q_i) \mid\mid Q_i)$. For computing the optimal value of n for P2.2 in (3.29), it suffices to find $n^*_{2.2}(Q_i)$ such that

$$n^*_{2.2}(Q_i) = \arg\max_{n \in Z^*} \{\Pi_{OEM}(n, Q^*_{r,2.2}(n, Q_i) \mid\mid Q_i), \Pi_{OEM}(n = 0)\}, \tag{3.76}$$

where $Q^*_{r,2.1}(n, Q_i)$ is given by (3.74), and $\Pi_{OEM}(n = 0)$ is as in Property 7.1.

The following property defines an upper bound for the search region of n.

Property 27 *When Q_i is exogenous, under OEM-centric control, the optimal value of n, denoted by $n^*_{2.2}(Q_i)$, satisfies*

$$0 \leq n^*_{2.2}(Q_i) \leq \frac{(c^{np} + c^d)aT + h^n_M \frac{aT^2}{2}}{K_s}. \tag{3.77}$$

Proof. Recalling Property 7.1, and using (3.76), it can be easily shown that

$$\Pi_{OEM}(n = 0) \leq \Pi_{OEM}(n^*_{2.2}(Q_i), Q^*_{r,2.2}(n^*_{2.2}(Q_i), Q_i) \mid\mid Q_i) \leq \pi aT - K_s n^*_{2.2}(Q_i).$$

Hence, we have $n^*_{2.2}(Q_i) \leq \frac{(c^{np}+c^d)aT+h^n_M \frac{aT^2}{2}}{K_s}$, which together with $n \geq 0$ implies (3.77). $\square$

Property 27 helps to limit the search region for the value of n:

$$n = 0, 1, \ldots, \left\lfloor \frac{(c^{np} + c^d)aT + h^n_M \frac{aT^2}{2}}{K_s} \right\rfloor.$$

Using Properties 26 and 27, we can derive the optimal solution of problem (3.29).

Corollary 8 *The optimal solution under OEM-centric control for Scenario 2, denoted by* $(n^*_{2.2}(Q_i), Q^*_{r,2.2}(Q_i))$*, is given by*

$$\begin{aligned} n^*_{2.2}(Q_i) &= \underset{n=0,1,\ldots,N_{2.2}}{\arg\max}\ \{\Pi_{OEM}(n, Q^*_{r,2.2}(n,Q_i) \mid\mid Q_i), \Pi_{OEM}(n=0)\}, \\ Q^*_{r,2.2}(Q_i) &= Q^*_{r,2.2}(n^*_{2.2}(Q_i), Q_i), \end{aligned} \tag{3.78}$$

where $Q^*_{r,2.2}(n, Q_i)$ *is as in (3.74),* $\Pi_{OEM}(n = 0)$ *is as in Property 7.1, and* $N_{2.2}$ *is given by*

$$N_{2.2} = \left\lfloor \frac{(c^{np} + c^d)aT + h^n_M \frac{aT^2}{2}}{K_s} \right\rfloor. \tag{3.79}$$

3.7.3 Scenario 2: RS-centric Control

In this section, we analyze the P2.3 in (3.30) to determine the values of n and Q_i that maximize RS's total profit. The following property is helpful for computing the optimal solution of P2.3 in (3.30).

Property 28 *When* Q_i *is exogenous, under RS-centric control, for any given* $n \geq 1$*, either of the following is true:*

- *For any given* $1 \leq n \leq \frac{(h^r_R - h^u_R)a}{h^u_R(1-\gamma)m} + 1$*, the maximizer of* $\Pi_{RS}(n, Q_r \mid\mid Q_i)$ *is given by*

$$Q^*_{r,2.3}(n, Q_i) = \begin{cases} 0, & Q_{r,RS}(n, Q_i) \leq 0, \\ Q_{r,RS}(n, Q_i), & 0 < Q_{r,RS}(n, Q_i) < \frac{\gamma Q_i}{n(1-\gamma)+\gamma}, \\ \frac{\gamma Q_i}{n(1-\gamma)+\gamma}, & Q_{r,RS}(n, Q_i) \geq \frac{\gamma Q_i}{n(1-\gamma)+\gamma}. \end{cases}$$

- *For any given $n > \frac{(h_R^r - h_R^u)a}{h_R^u(1-\gamma)m} + 1$, the maximizer of $\Pi_{RS}(n, Q_r \mid\mid Q_i)$ is given by*

$$Q^*_{r,2.3}(n, Q_i) = \arg\min_{Q_r} \left\{ \Pi_{RS}\left(n, Q_r = 0 \mid\mid Q_i\right), \Pi_{RS}\left(n, Q_r = \frac{\gamma Q_i}{n(1-\gamma)+\gamma} \mid\mid Q_i\right) \right\},$$

where $Q_{r,RS}(n, Q_i)$ is given by (3.44).

Proof. (3.5) and the non-negativity constraint for Q_r imply that, for any given $n \geq 1$, the feasible region of Q_r is given by

$$0 \leq Q_r \leq \frac{\gamma Q_i}{n(1-\gamma)+\gamma}. \tag{3.80}$$

Recalling Property 9.2 which states that $\Pi_{RS}(n, Q_r \mid\mid Q_i)$ is concave in Q_r for any n in the region $1 \leq n \leq \frac{(h_R^r - h_R^u)a}{h_R^u(1-\gamma)m} + 1$, while is convex in Q_r for any n in the region $n > \frac{(h_R^r - h_R^u)a}{h_R^u(1-\gamma)m} + 1$. The proof immediately follows from Property 9.2 and (3.80), and, hence, it is omitted. □

By Property 28, we obtain the upper bound of $\Pi_{RS}(n, Q_r \mid\mid Q_i)$ for $n \geq 1$, which is given by $\Pi_{RS}(n, Q^*_{r,2.3}(n, Q_i) \mid\mid Q_i)$. For computing the optimal value of n for P2.3 in (3.30), it suffices to find $n^*_{2.3}(Q_i)$ such that

$$n^*_{2.3}(Q_i) = \arg\max_{n \in Z^*} \{\Pi_{RS}(n, Q^*_{r,2.3}(n, Q_i) \mid\mid Q_i), \Pi_{RS}(n = 0)\}, \tag{3.81}$$

where $Q^*_{r,2.3}(n, Q_i)$ is as in Property 28, and $\Pi_{RS}(n = 0)$ is as in Property 7.2.

The following property defines an upper bound for the search region of n.

Property 29 *When Q_i is exogenous, under RS-centric control, the optimal value of n, denoted by $n^*_{2.3}(Q_i)$, satisfies*

$$0 \leq n^*_{2.3}(Q_i) \leq \frac{\gamma Q_i(c^{rp} - c) - \gamma K_r}{K_r(1-\gamma)}. \tag{3.82}$$

Proof. Recalling Property 7.2, and using (3.81), it is easy to show that

$$\begin{aligned} 0 = \Pi_{RS}(n=0) &\leq \Pi_{RS}(n^*_{2.3}(Q_i), Q^*_{r,2.3}(n^*_{2.3}(Q_i), Q_i) \mid\mid Q_i) \\ &\leq ((c^{rp} - c)Q^*_{r,2.3}(n^*_{2.3}(Q_i), Q_i) - K_r)n^*_{2.3}(Q_i). \end{aligned} \tag{3.83}$$

If $n^*_{2.3}(Q_i) = 0$ then $Q^*_{r,2.3}(n^*_{2.3}(Q_i), Q_i) = 0$. Otherwise, i.e., $n^*_{2.3}(Q_i) > 0$, (3.83) implies that

$$K_r + (c - c^{rp})Q^*_{r,2.3}(n^*_{2.3}(Q_i), Q_i) \leq 0,$$

which is equivalent to

$$Q^*_{r,2.3}(n^*_{2.3}(Q_i), Q_i) \geq \frac{K_r}{c^{rp} - c}. \tag{3.84}$$

Using (3.80) and (3.84), it can be easily shown that

$$\frac{K_r}{c^{rp} - c} \leq Q^*_{r,2.3}(n^*_{2.3}(Q_i), Q_i) \leq \frac{\gamma Q_i}{n^*_{2.3}(Q_i)(1-\gamma) + \gamma}, \tag{3.85}$$

which implies that $n^*_{2.3}(Q_i) \leq \frac{\gamma Q_i(c^{rp} - c) - \gamma K_r}{K_r(1-\gamma)}$. □

Property 29 helps to limit the search region for the value of n:

$$n = 0, 1, \ldots, \left\lfloor \frac{\gamma Q_i(c^{rp} - c) - \gamma K_r}{K_r(1-\gamma)} \right\rfloor.$$

Using Properties 28 and 29, we can derive the optimal solution of problem (3.30).

Corollary 9 *The optimal solution under RS-centric control for Scenario 2, de-*

noted by $(n^*_{2.3}(Q_i), Q^*_{r,2.3}(Q_i))$*, is given by*

$$\begin{aligned} n^*_{2.3}(Q_i) &= \underset{n=0,1,\ldots,N_{2.3}}{\arg\max}\ \{\Pi_{RS}(n, Q^*_{r,2.3}(n, Q_i) \mid\mid Q_i), \Pi_{RS}(n=0)\} \\ Q^*_{r,2.3}(Q_i) &= Q^*_{r,2.3}(n^*_{2.3}(Q_i), Q_i), \end{aligned} \tag{3.86}$$

where $Q^*_{r,2.3}(n, Q_i)$ *is as in Property 28,* $\Pi_{RS}(n=0)$ *is as in Property 7.2, and* $N_{2.3}$ *is given by*

$$N_{2.3} = \left\lfloor \frac{\gamma Q_i(c^{rp} - c) - \gamma K_r}{K_r(1-\gamma)} \right\rfloor.$$

3.7.4 Scenario 2: OEM-lead Control: OEM Decides n *and RS Decides* Q_r

In this section, we analyze P2.4.1 and P2.4.2 formulated by (3.31) and (3.32), respectively. The OEM determines the value of n, and then the RS determines the value of Q_r. To determine the value of n, the OEM will rely on the prediction of RS's response for any given n value.

This prediction can be obtained by deriving the RS's optimal value of Q_r that maximizes the RS's total profit for any given (n, Q_i) pair. We denote the RS's optimal response by $Q^*_{r,2.4}(n, Q_i)$, which is the solution of P2.4.1 in (3.31). Recalling Property 28, for any $n \geq 1$, and Q_i is exogenous, the optimal value of Q_r for the RS is given by $Q^*_{r,2.3}(n, Q_i)$. Thus, $Q^*_{r,2.4}(n, Q_i) = Q^*_{r,2.3}(n, Q_i)$ for any $n \geq 1$.

After predicting the RS's response, the OEM's decision is to determine the value of n that maximizes the OEM's total profit, i.e., the solution of P2.4.2 in (3.32). The following property defines an upper bound for optimal n value.

Property 30 *When* Q_i *is exogenous, under OEM-lead control where the OEM decides* n *and the RS decides* Q_r*, the optimal value of* n *for the OEM, denoted by*

$n^*_{2.4}$, *satisfies*

$$0 \leq n^*_{2.4} \leq N_{2.2}, \tag{3.87}$$

where $N_{2.2}$ *is given by (3.79).*

Proof. The proof is similar to the proof of Property 27, and, hence, it is omitted.

□

Property 30 helps to limits the searching region for the value of n: $n = 0, 1, \ldots, N_{2.2}$.

Using Properties 28 and 30, we can derive the optimal solutions of P2.4.1 in (3.31) and P2.4.2 in (3.32).

Corollary 10 *When Q_i is exogenous, under OEM-lead control where OEM decides n and RS decides Q_r, the OEM's optimal decision for the value of n, denoted by $n^*_{2.4}(Q_i)$, is given by*

$$n^*_{2.4}(Q_i) = \underset{n=0,1,\ldots,N_{2.2}}{\arg\max} \{\Pi_{OEM}(n, Q^*_{r,2.3}(n, Q_i) \mid\mid Q_i), \Pi_{RS}(n = 0)\}, \tag{3.88}$$

where $Q^*_{r,2.3}(n, Q_i)$ is as in Property 28, $\Pi_{RS}(n = 0)$ is as in Property 7.2, and $N_{2.2}$ is given by (3.79).

The RS's optimal response is given by $Q^*_{r,2.3}(n^*_{2.4}(Q_i), Q_i)$.

3.7.5 Scenario 2: OEM-lead Control: OEM Decides Q_r and RS Decides n

In this section, we analyze P2.5.1 and P2.5.2 formulated by (3.26) and (3.33), respectively. The OEM determines the value of Q_r, and then the RS determines the vlaue of n. To determine the value of Q_r, the OEM will rely on the prediction of the RS's response for any given Q_r value.

This prediction can be obtained by deriving the RS's optimal value of n that maximizes the RS's total profit for any given (Q_i, Q_r) pair. The RS's optimal response is given by $n^*_{2.5}(Q_i, Q_r)$ which is the solution of P2.5.1 in (3.26). Recalling Property 22, when (Q_i, Q_r) is given, the optimal value of n for the RS is given by $n^*_{1.5}(Q_i, Q_r)$. Thus, $n^*_{2.5}(Q_i, Q_r) = n^*_{1.5}(Q_i, Q_r)$ for any $Q_r \geq 0$.

After predicting the RS's response, the OEM's decision is to determine the value of Q_r that maximizes the OEM's total profit, i.e., the solution of P2.5.2 in (3.33). Recalling Property 22, in order to ensure that the RS remanufactures, the largest value of Q_r should be $Q_i\gamma$. The following property defines the search region for the optimal value of $Q_r \geq 0$.

Property 31 *When Q_i is exogenous, under OEM-lead control where the OEM decides Q_r and the RS decides n, the OEM's optimal decision for the value of Q_r, denoted by $Q^*_{r,2.5}(Q_i)$, satisfies $0 \leq Q_r \leq \min\{aT - Q_i, Q_i\gamma\}$.*

Proof. (3.3) and non-negativity constraints of Q_r and n imply that

$$0 \leq Q_r \leq aT - Q_i,$$

which together with Property 22.1 imply that $0 \leq Q_r \leq \min\{aT - Q_i, Q_i\gamma\}$. □

Using Property 31, we can derive the optimal solutions of P2.5.1 in (3.26) and P2.5.2 in (3.33).

Corollary 11 *Under OEM-lead control where OEM decides Q_r and RS decides n for Scenario 2, the optimal value of Q_r for the OEM, denoted by $Q^*_{r,2.5}(Q_i)$, is* *given by*

$$Q^*_{r,2.5}(Q_i) = \begin{cases} 0, & if \quad \Pi_{OEM}(n^*_{1.5}(Q^0_{r,2.5}(Q_i), Q_i), Q^0_{r,2.5}(Q_i) \mid\mid Q_i) \\ & \qquad \leq \Pi_{OEM}(n=0), \\ Q^0_{r,2.5}(Q_i), & otherwise, \end{cases} \tag{3.89}$$

where $n^*_{1.5}(Q_i, Q_r)$ *is as in Property 22,* $\Pi_{OEM}(n = 0)$ *is as in Property 7.1, and* $Q^0_{r,2.5}(Q_i)$ *is given by*

$$Q^0_{r,2.5}(Q_i) = \underset{0 \leq Q_r \leq \min\{aT - Q_i, Q_i\gamma\}}{\arg\max} \left\{\Pi_{OEM}(n^*_{1.5}(Q_i, Q_r), Q_r \mid\mid Q_i)\right\}.$$

The RS's optimal response is given by $n^*_{1.5}(Q^*_{r,2.5}(Q_i), Q_i)$*.*

3.8 Scenario 3: Computing n, Q_r and Q_i in a Stackelberg Setting

In this section, problems formulated in Section 3.4.3 are analyzed. We develop formal approach for computing the optimal values of n, Q_r and Q_i for each of the following seven problem settings: (1) centralized control; (2) OEM-lead control where OEM decides Q_r and RS decides n and Q_i; (3) OEM-lead control where OEM decides Q_i and RS decides n and Q_r; (4) OEM-lead control where OEM decides n and RS decides Q_r and Q_i; (5) OEM-lead control where OEM decides n and Q_r, and RS decides Q_i; (6) OEM-lead control where OEM decides n and Q_i, and RS decides Q_r; (6) OEM-lead control where OEM decides Q_r and Q_i, and RS decides n.

3.8.1 Scenario 3: Centralized Control

In this section, we analyze P3.1 in (3.34). Recalling Property 8.3, for any given (n, Q_r) pair, where $n \in Z^+$ and $Q_r \geq 0$, $\Pi(n, Q_r, Q_i)$ is a linearly non-increasing function of Q_i in $Q_i \geq 0$. Thus, we can use the result in Property 12, and conclude that for any given proper (n, Q_r) pair the optimal Q_i value under centralized control for Scenario 3, denoted by $Q^*_{i,3.1}(n, Q_r)$, is given by $Q^*_{i,3.1}(n, Q_r) = Q^*_{i,1.1}(n, Q_r) = \left(\frac{n}{\gamma} - n + 1\right) Q_r$. Hence, $\Pi\left(n, Q_r, \left(\frac{n}{\gamma} - n + 1\right) Q_r\right)$ provides an upper bound of $\Pi(n, Q_r, Q_i)$ for $n \geq 1$. We denote $\Pi\left(n, Q_r, \left(\frac{n}{\gamma} - n + 1\right) Q_r\right)$ by $F_{3.1}(n, Q_r)$. Note that, $F_{3.1}(0, Q_r) = \Pi(n = 0) = (\pi - c^{np} - c^d) - h^n_M \frac{aT^2}{2}$ by Property 7.3, which implies that $F_{3.1}(n, Q_r)$ can be treated as the upper bound of $\Pi(n, Q_r, Q_i)$ for $n \geq 0$, i.e., the whole feasible region of n. Then, to determine the optimal value of n for P3.1 in (3.34), it suffices to solve the following problem

$$\begin{aligned} &\min_{n \in Z^*, Q_r \geq 0} F_{3.1}(n, Q_r) \\ &\quad s.t. \quad aT \geq \left(\frac{n}{\gamma} - n + 1\right) Q_r + (n+1)Q_r = \left(\frac{n}{\gamma} + 2\right) Q_r \end{aligned} \tag{3.90}$$

where $F_{3.1}(n, Q_r) = \Pi\left(n, Q_r, \left(\frac{n}{\gamma} - n + 1\right) Q_r\right)$.

The following property of $F_{3.1}(n, Q_r)$ is helpful for calculating the optimal solution of problem (3.90).

Property 32 *For any given $n \geq 0$, $F_{3.1}(n, Q_r)$ is a concave function of Q_r with a unique positive maximizer given by*

$$Q^0_{r,3.1}(n) = \frac{(c^{np} + c^d - c + h^n_M T)a}{\left\{ \begin{array}{l} h^n_M n + h^r_M + h^u_M \left(\left(\frac{1}{\gamma} - 1\right)^2 n + \frac{2}{\gamma} - 1\right) \\ +h^u_R\left((1-\gamma)n + \gamma + 1\right) + \frac{(h^r_R - h^u_R)a}{m} \end{array} \right\}}. \tag{3.91}$$

Proof. Recalling that $F_{3.1}(n, Q_r) = \Pi\left(n, Q_r, \left(\frac{n}{\gamma} - n + 1\right) Q_r\right)$, it is easy to show that

$$\frac{\partial F_{3.1}(n,Q_r)}{\partial Q_r} = (c^{np}+c^d-c+h_M^n T)n - \left[\frac{h_M^n n}{a}+\frac{h_M^r}{a}+\frac{h_M^u}{a}\left(\left(\frac{1}{\gamma}-1\right)^2 n+\frac{2}{\gamma}-1\right)\right.$$
$$\left. +\frac{h_R^u}{a}\left((1-\gamma)n+\gamma+1\right)+\frac{h_R^r-h_R^u}{m}\right] nQ_r, \quad \text{and}$$
$$\frac{\partial^2 F_{3.1}(n,Q_r)}{\partial Q_r^2} = -\left[\frac{h_M^n n}{a}+\frac{h_M^r}{a}+\frac{h_M^u}{a}\left(\left(\frac{1}{\gamma}-1\right)^2 n+\frac{2}{\gamma}-1\right)\right.$$
$$\left. +\frac{h_R^u}{a}\left((1-\gamma)n+\gamma+1\right)+\frac{h_R^r-h_R^u}{m}\right] n \leq 0.$$

Hence, $F_{3.1}(n,Q_r)$ is concave in Q_r for any given $n \geq 0$ with a unique maximizer given by (3.91). Since $c^{np} \geq c$, $Q^*_{r,3.1}(n)$ is positive. □

Recalling the constraint in (3.90), and by Property 32, we can easily prove the following corollary.

Corollary 12 *For any given $n \geq 0$, the maximizer of $F_{3.1}(n,Q_r)$ is given by*

$$Q^*_{r,3.1}(n) = \begin{cases} 0, & n=0, \\ Q^0_{r,3.1}(n), & Q_{r,3.1}(n) < \frac{aT}{\frac{n}{\gamma}+2}, \\ \frac{aT}{\frac{n}{\gamma}+2}, & Q^0_{r,3.1}(n) \geq \frac{aT}{\frac{n}{\gamma}+2}, \end{cases} \tag{3.92}$$

where $Q^0_{r,3.1}(n)$ is given by (3.91).

Proof. When $n=0$, i.e., no remanufacturing happens, we have $Q_r = 0$. The following proof is straightforward by using Property 32 and the constraint in (3.90), and hence, it is omitted. □

By Corollary 12, $F_{3.1}(n,Q^*_{r,3.1}(n))$ is the upper bound of $F_{3.1}(n,Q_r)$ for $n \geq 0$, and hence, is the upper bound of $\Pi(n,Q_r,Q_i)$ for $n \geq 0$. Thus, to calculate the optimal value of n for P3.1 in (3.34), it suffices to find $n^*_{3.1}$ such that

$$n^*_{3.1} = \arg\max_{n \in Z^*} \{F_{3.1}(n, Q^*_{r,3.1}(n))\}, \tag{3.93}$$

where $F_{3.1}(n, Q^*_{r,3.1}(n)) = \Pi\left(n, Q^*_{r,3.1}(n), \left(\frac{n}{\gamma} - n + 1\right) Q^*_{r,3.1}(n)\right)$, and $Q^*_{r,3.1}(n)$ is given by (3.92).

The following property defines an upper bound for the search region of n.

Property 33 *The optimal value of n under centralized control for Scenario 3, denoted by $n^*_{3.1}$, satisfies*

$$0 \leq n^*_{3.1} \leq N_{2.1}, \tag{3.94}$$

where $N_{2.1}$ is given by (3.73).

Proof. By (3.93) and recalling that $F_{3.1}(n, Q_r) = \Pi\left(n, Q_r, \left(\frac{n}{\gamma} - n + 1\right) Q_r\right)$, it is easy to show that

$$F_{3.1}(0, Q^*_{r,3.1}(0)) \leq F_{3.1}(n^*_{3.1}, Q^*_{r,3.1}(n^*_{3.1})) \leq \pi a T - (K_s + K_r) n^*_{3.1}. \tag{3.95}$$

Hence, we have $n^*_{3.1} \leq \frac{(c^{np} + c^d) a T + h^n_M \frac{aT^2}{2}}{K_s + K_r} = N_{2.1}$, which together with $n \geq 0$ implies (3.94). $\square$

Using Property 33 and Corollary 12, we can derive the optimal solution of P3.1 in (3.34).

Corollary 13 *The optimal solution under centralized control for Scenario 3, denoted by $(n^*_{3.1}, Q^*_{r,3.1}, Q^*_{i,3.1})$, is given by*

$$
\begin{aligned}
n^*_{3.1} &= \underset{n=0,1,\ldots,N_{2.1}}{\arg\max} \{F_{3.1}(n, Q^*_{r,3.1}(n))\}, \quad Q^*_{r,3.1} = Q^*_{r,3.1}(n^*_{3.1}), \\
Q^*_{i,3.1} &= Q^*_{i,3.1}(Q^*_{r,3.1}, n^*_{3.1}),
\end{aligned} \tag{3.96}
$$

where $N_{2.1}$ *is given by (3.73),* $F_{3.1}(n, Q_r) = \Pi\left(n, Q_r, \left(\frac{n}{\gamma} - n + 1\right) Q_r\right)$*,* $Q^*_{r,3.1}(n)$ *is given by (3.92), and* $Q^*_{i,3.1}(n, Q_r) = \left(\frac{n}{\gamma} - n + 1\right) Q_r$.

3.8.2 Scenario 3: OEM-lead Control: OEM Decides Q_r and RS Decides n and Q_i

In this section, we analyze P3.2.1 and P3.2.2 formulated by (3.23) and (3.35), respectively. The OEM and the RS make decisions in a Stackelberg setting that the OEM decides the value of Q_r and then the RS decides the values of n and Q_i. To determine the value of Q_r, the OEM will reply on the prediction of RS's response for any given Q_r value.

This prediction is the solution of P1.3 in (3.23) which is given by $(n^*_{1.3}(Q_r), Q^*_{i,1.3}(Q_r))$ in Corollary 4. By observing the results in Corollary 4 and (3.63), the OEM can derive the conditions under which the RS agrees to start remanufacturing, as the following property shows.

Property 34 *The RS will agree to start remanufacturing only if all of the following inequalities hold*

$$
c^{rp} - c - \left(\frac{1}{\gamma} - 1\right) c^d \geq 0, \tag{3.97}
$$

$$
2\left(h^u_R(1-\gamma) + \frac{(h^r_R - h^u_R)a}{m}\right) K_r < \left(c^{rp} - c - \left(\frac{1}{\gamma} - 1\right) c^d\right)^2 a, \quad \text{and} \tag{3.98}
$$

$$\frac{\left\{\begin{array}{l}\left(c^{rp}-c-\left(\frac{1}{\gamma}-1\right)c^{d}\right)a\\-\sqrt{\left(c^{rp}-c-\left(\frac{1}{\gamma}-1\right)c^{d}\right)^{2}a^{2}-2\left(h_{R}^{u}(1-\gamma)+\frac{(h_{R}^{r}-h_{R}^{u})a}{m}\right)K_{r}a}\end{array}\right\}}{h_{R}^{u}(1-\gamma)+\frac{(h_{R}^{r}-h_{R}^{u})a}{m}}<Q_{r} \tag{3.99}$$
$$<\frac{\left\{\begin{array}{l}\left(c^{rp}-c-\left(\frac{1}{\gamma}-1\right)c^{d}\right)a\\+\sqrt{\left(c^{rp}-c-\left(\frac{1}{\gamma}-1\right)c^{d}\right)^{2}a^{2}-2\left(h_{R}^{u}(1-\gamma)+\frac{(h_{R}^{r}-h_{R}^{u})a}{m}\right)K_{r}a}\end{array}\right\}}{h_{R}^{u}(1-\gamma)+\frac{(h_{R}^{r}-h_{R}^{u})a}{m}}.$$

Proof. Recalling Corollary 4 and (3.63), it is easy to show that (3.97) is the necessary condition for $n^0_{1.3}(Q_r)$ to be positive, and then the necessary condition for $n^*_{1.3}(Q_r)$ to be positive.

By Corollary 4, $n^*_{1.3}(Q_r)$ can be positive only when $n^0_{1.3}(Q_r) > 0$ which is equivalent to

$$\left(\frac{h_{R}^{u}(1+\gamma)}{2}+\frac{(h_{R}^{r}-h_{R}^{u})a}{2m}\right)Q_{r}^{2}-\left(c^{rp}-c-\left(\frac{1}{\gamma}-1\right)c^{d}\right)aQ_{r}+K_{r}a<0. \tag{3.100}$$

(3.98) is the necessary condition for existing real Q_r such that (3.100) holds. Then, the real Q_r value that satisfies (3.100) is in the region determined by (3.99). □

After predicting the RS's response, the OEM's decision is to determine the value of Q_r that maximizes the OEM's total profit, i.e., the solution of P3.2.2 in (3.35). This can be done by search algorithm over the region determined by (3.99).

Corollary 14 *Under OEM-lead control where OEM decides Q_r and RS decides n and Q_i, the optimal value of Q_r for the OEM, denoted by $Q^*_{r,3.2}$, is given by*

$$Q^*_{r,3.2}=\begin{cases}0, & if \quad \Pi_{OEM}(n^*_{1.3}(Q^0_{r,3.2}),Q^*_{i,1.3}(Q^0_{r,3.2}),Q^0_{r,3.2})\le\Pi_{OEM}(n=0),\\ Q^0_{r,3.2}, & otherwise,\end{cases}$$

where $(n^*_{1.3}(Q_r), Q^*_{i,1.3}(Q_r))$ *is as in Corollary 4, and* $Q^0_{r,3.2}$ *is given by*

$$Q^0_{r,3.2} = \underset{Q_r \in (3.99)}{\arg\max} \left\{ \Pi_{OEM}(n^*_{1.3}(Q_r), Q^*_{i,1.3}(Q_r), Q_r) \right\}.$$

The RS's optimal response is given by $(n^*_{1.3}(Q^*_{r,3.2}), Q^*_{i,1.3}(Q^*_{r,3.2}))$.

3.8.3 Scenario 3: OEM-lead Control: OEM Decides Q_i and RS Decides n and Q_r

In this section, we analyze P3.3.1 and P3.3.2 formulated by (3.30) and (3.36), respectively. The OEM and the RS make decisions in a Stackelberg setting that the OEM decides the value of Q_i and then the RS decides the values of n and Q_r. To determine the value of Q_i, the OEM will reply on the prediction of RS's response for any given Q_i value.

This prediction is the solution of P2.3 in (3.30) which is given by $(n^*_{2.3}(Q_i), Q^*_{r,2.3}(Q_i))$ in Corollary 9.

After predicting the RS's response, the OEM's decision is to determine the value of Q_i that maximizes the OEM's total profit, i.e., the solution of P3.3.2 in (3.36). This can be done by search algorithm over the region c.

Corollary 15 *Under OEM-lead control where OEM decides Q_i and RS decides n and Q_r, the optimal value of Q_i for the OEM, denoted by $Q^*_{i,3.3}$, is given by*

$$Q^*_{i,3.3} = \begin{cases} 0, & if \quad \Pi_{OEM}(n^*_{2.3}(Q^0_{i,3.3}), Q^*_{r,2.3}(Q^0_{i,3.3}), Q^0_{i,3.3}) \le \Pi_{OEM}(n = 0), \\ Q^0_{i,3.3}, & otherwise, \end{cases}$$

where $(n^*_{2.3}(Q_i), Q^*_{r,2.3}(Q_i))$ *is as in Corollary 9, and* $Q^0_{i,3.3}$ *is given by*

$$Q^0_{i,3.3} = \underset{0 \le Q_i \le aT}{\arg\max} \left\{ \Pi_{OEM}(n^*_{2.3}(Q_i), Q^*_{r,2.3}(Q_i), Q_i) \right\}.$$

The RS's optimal response is given by $(n^*_{2.3}(Q^*_{i,3.3}), Q^*_{r,2.3}(Q^*_{i,3.3}))$.

3.8.4 Scenario 3: OEM-lead Control: OEM Decides n and RS Decides Q_r and Q_i

In this section, we analyze P3.4.1 and P3.4.2 formulated by (3.37) and (3.38), respectively. The OEM and the RS make decisions in a Stackelberg setting that the OEM decides the value of n, and then the RS decides the values of Q_r and Q_i. To determine the value of n, the OEM will rely on the prediction of RS's response for any given n value.

This prediction can be obtained by deriving the RS's optimal values of Q_r and Q_i that maximize the RS's total profit for any given n value. We denote the RS's optimal response by $(Q^*_{r,3.4}(n), Q^*_{i,3.4}(n))$, which is the solution of P3.4.1 in (3.37). If $n = 0$, i.e., no remanufacturing, $Q_r = Q_i = 0$ by Property 7. Thus we will focus on the case $n \geq 1$.

Recalling Property 8.2, when $n \geq 1$ is given, for any $Q_r \geq 0$, $\Pi_{RS}(Q_r, Q_i \mid\mid n)$ is a linearly decreasing function of Q_i. Thus, for any given proper (n, Q_r) pair, the optimal value of Q_i is given by $Q^*_{i,3.4}(Q_r, n) = \left(\frac{n}{\gamma} - n + 1\right) Q_r$ as proved in Property 18. Then, we obtain the upper bound of $\Pi_{RS}(Q_r, Q_i \mid\mid n)$ for $n \geq 1$, which is given by $\Pi_{RS}\left(Q_r, \left(\frac{n}{\gamma} - n + 1\right) Q_r \,\middle\|\, n\right)$. To compute the optimal solution for problem (3.37), it suffices to obtain $Q^*_{r,3.4}(n)$ such that

$$Q^*_{r,3.4}(n) = \arg\max_{Q_r \geq 0} \Pi_{RS}\left(Q_r, \left(\frac{n}{\gamma} - n + 1\right) Q_r \mid\mid n\right). \tag{3.101}$$

The following property is sufficient for computing $Q^*_{r,3.4}(n)$ in (3.101).

Property 35 *For any given $n > 0$, $\Pi_{RS}\left(Q_r, \left(\frac{n}{\gamma} - n + 1\right) Q_r \,\middle\|\, n\right)$ is concave in*

Q_r with a unique maximizer give by

$$Q^0_{r,3.4}(n) = \frac{c^{rp} - c^d\left(\frac{1}{\gamma} - 1\right) - c}{\frac{h^u_R(n(1-\gamma)+\gamma+1)}{a} + \frac{h^r_R - h^u_R}{m}}. \tag{3.102}$$

Proof. For any $n > 0$, it can be easily shown that

$$\begin{aligned}
\frac{\partial \Pi_{RS}\left(Q_r, \left(\frac{n}{\gamma} - n + 1\right) Q_r \,\Big|\Big|\, n\right)}{\partial Q_r} &= \left(c^{rp} - c^d\left(\frac{1}{\gamma} - 1\right) - c\right) n \\
&- \left(\frac{h^u_R\left(n(1-\gamma)+\gamma+1\right)}{a} + \frac{h^r_R - h^u_R}{m}\right) n Q_r, \quad \text{and} \\
\frac{\partial^2 \Pi_{RS}\left(Q_r, \left(\frac{n}{\gamma} - n + 1\right) Q_r \,\Big|\Big|\, n\right)}{\partial Q_r^2} &= -\left(\frac{h^u_R\left(n(1-\gamma)+\gamma+1\right)}{a} + \frac{h^r_R - h^u_R}{m}\right) n < 0.
\end{aligned}$$

Hence, $\Pi_{RS}\left(Q_r, \left(\frac{n}{\gamma} - n + 1\right) Q_r \,||\, n\right)$ is concave in Q_r with a unique maximizer give by (3.102). □

Using Property 35, we can derive the RS's optimal response for any given $n \geq 1$.

Corollary 16 *For any given $n \geq 1$, the maximizer of the RS's profit function is given by*

$$\begin{aligned}
Q^*_{r,3.4}(n) &= \begin{cases} 0 & if \quad c^{rp} \leq c + c^d\left(\frac{1}{\gamma} - 1\right), \\ Q^0_{r,3.4}(n) & if \quad c^{rp} > c + c^d\left(\frac{1}{\gamma} - 1\right), \end{cases} \qquad \text{and} \\
Q^*_{i,3.4}(n) &= \left(\frac{n}{\gamma} - n + 1\right) Q^*_{r,3.4}(n),
\end{aligned} \tag{3.103}$$

where $Q^0_{r,3.4}(n)$ is given by (3.102).

Proof. The proof is straightforward by using Property 35, $Q_r \geq 0$ and $Q^*_{i,3.4}(Q_r, n) = \left(\frac{n}{\gamma} - n + 1\right) Q_r$, and, hence, it is omitted. □

After predicting the RS's response, the OEM's decision is to determine the value of n, denoted by $n^*_{3.4}$, that maximizes the OEM's total profit, i.e., the solution of

P3.4.2 in (3.38). This can be done by searching over a finite region of n. The following property defines this searching region.

Property 36 *Under OEM-lead control where OEM decides n and the RS decides Q_r and Q_i, the optimal value of n for the OEM, denoted by $n^*_{3.4}$, should satisfy*

$$0 \leq n^*_{3.4} \leq N_{2.2},$$

where $N_{2.2}$ is given by (3.79).

Proof. The proof is similar with the proof for (3.77), and hence, it is omitted. □

Using Property 36 and Corollary 16, we can derive the optimal solutions of P3.4.1 in (3.37) and P3.4.2 in (3.38).

Corollary 17 *Under OEM-lead control where OEM decides n and the RS decides Q_r and Q_i, the optimal value of n for the OEM, denoted by $n^*_{3.4}$, is given by*

$$n^*_{3.4} = \underset{n=0,1,\ldots,N_{2.2}}{\arg\max} \{\Pi_{OEM}\left(n, Q^*_{r,3.4}(n), Q^*_{i,3.4}(n)\right)\}, \tag{3.104}$$

where $N_{2.2}$ is given by (3.79), $Q^*_{r,3.4}(n)$ and $Q^*_{i,3.4}(n)$ are as in Corollary 16.

The RS's optimal response, denoted by $(Q^*_{r,3.4}, Q^*_{i,3.4})$, is given by

$$Q^*_{r,3.4} = Q^*_{r,3.4}(n^*_{3.4}), \quad Q^*_{i,3.4} = Q^*_{i,3.4}(n^*_{3.4}). \tag{3.105}$$

3.8.5 Scenario 3: OEM-lead Control: OEM Decides n and Q_r and RS Decides Q_i

In this section, we analyze P3.5.1 and P3.5.2 formulated by (3.24) and (3.39), respectively. The OEM and the RS make decisions in a Stackelberg setting that the OEM decides the values of n and Q_r, and then the RS decides the value of Q_i. To determine the values of n and Q_r, the OEM will reply on the prediction of the RS's response for any given (n, Q_r) pair.

This prediction is the solution of P1.4.1 in (3.24) which is given by $Q^*_{i,1.4}(n, Q_r) = \left(\frac{n}{\gamma} - n + 1\right) Q_r$.

After predicting the RS's response, the OEM's decision is to determine the values of n and Q_r that maximize the OEM's total profit, i.e., the solution of P3.5.2 in (3.39), i.e., the OEM's problem is given by

$$\begin{aligned} \max_{n \in Z^*,\ Q_r \geq 0} \quad & \Pi_{OEM}\left(n, Q_r, \left(\frac{n}{\gamma} - n + 1\right) Q_r\right) \\ s.t. \quad & aT \geq \left(\frac{n}{\gamma} + 2\right) Q_r. \end{aligned} \tag{3.106}$$

The following property is helpful for calculating the optimal values of n and Q_r for the OEM.

Property 37 *For any $n \geq 1$, $\Pi_{OEM}\left(n, Q_r, \left(\frac{n}{\gamma} - n + 1\right) Q_r\right)$ is concave in Q_r with a unique maximizer given by*

$$Q^0_{r,3.5}(n) = \frac{c^{np} - c^{rp} + h^n_M T + \frac{c^d}{\gamma}}{h^u_M\left(n\left(\frac{1}{\gamma} - 1\right)^2 + \frac{2}{\gamma} - 1\right) + h^r_M + h^n_M n} a. \tag{3.107}$$

Proof. For any $n \geq 1$, it is easy to show that

$$\begin{aligned} \frac{\partial \Pi_{OEM}\left(n, Q_r, \left(\frac{n}{\gamma} - n + 1\right) Q_r\right)}{\partial Q_r} &= \left(c^{np} - c^{rp} + h^n_M T + \frac{c^d}{\gamma}\right) n \\ &- \frac{h^u_M\left(n\left(\frac{1}{\gamma} - 1\right)^2 + \frac{2}{\gamma} - 1\right) + h^r_M + h^n_M n}{a} n Q_r, \quad \text{and} \\ \frac{\partial^2 \Pi_{OEM}\left(n, Q_r, \left(\frac{n}{\gamma} - n + 1\right) Q_r\right)}{\partial Q_r^2} &= -\frac{h^u_M\left(n\left(\frac{1}{\gamma} - 1\right)^2 + \frac{2}{\gamma} - 1\right) + h^r_M + h^n_M n}{a} n < 0. \end{aligned}$$

Hence, $\Pi_{OEM}\left(n, Q_r, \left(\frac{n}{\gamma} - n + 1\right) Q_r\right)$ is concave in Q_r with a unique maximizer given by (3.107). $\square$

The constraint in problem (3.106) implies that, for any given $n \geq 1$, Q_r should satisfy $Q_r \leq \frac{aT}{\frac{n}{\gamma}+2}$. Hence, for any given $n \geq 0$, we can characterize the optimal value of Q_r for the OEM.

Corollary 18 *Under OEM-lead control where OEM decides n and Q_r and RS decides Q_i, for any $n \geq 0$, the optimal value of Q_r for OEM is given by*

$$Q^*_{r,3.5}(n) = \begin{cases} 0, & if \quad n = 0, \\ Q^0_{r,3.5}(n), & if \quad Q^0_{r,3.5}(n) < \frac{aT}{\frac{n}{\gamma}+2} \quad \text{and} \quad n \geq 1, \\ \frac{aT}{\frac{n}{\gamma}+2}, & if \quad Q^0_{r,3.5}(n) \geq \frac{aT}{\frac{n}{\gamma}+2} \quad \text{and} \quad n \geq 1. \end{cases} \tag{3.108}$$

Proof. The proof is straightforward by using (3.106) and Property 37, and hence, it is omitted. $\square$

By corollary 18, $\Pi_{OEM}\left(n, Q^*_{r,3.5}(n), \left(\frac{n}{\gamma} - n + 1\right) Q^*_{r,3.5}(n)\right)$ is the upper bound of

$\Pi_{OEM}\left(n, Q_r, \left(\frac{n}{\gamma} - n + 1\right) Q_r)\right)$ for any (n, Q_r) pair. To computing the optimal value of n for the OEM, it suffices to obtain $n^*_{3.5}$ such that

$$n^*_{3.5} = \arg\max_{n \in Z^*} \left\{ \Pi_{OEM}\left(n, Q^*_{r,3.5}(n), \left(\frac{n}{\gamma} - n + 1\right) Q^*_{r,3.5}(n)\right) \right\}. \tag{3.109}$$

The following property defines an upper bound for the search region of n.

Property 38 *Under OEM-lead control where OEM decides n and Q_r and RS decides Q_i, the optimal value of n for the OEM should satisfy*

$$0 \leq n^*_{3.5} \leq N_{2.2}, \tag{3.110}$$

where $N_{2.2}$ is given by (3.79).

Proof. The proof is similar with the proof for (3.77), and hence, it is omitted. $\square$

Property 38 helps to limit the search region of n value: $n = 0, 1, \ldots, N_{2.2}$.

Using Property 38 and Corollary 18, we can derive the optimal solutions of P3.5.1 in (3.24) and P3.5.2 in (3.39).

Corollary 19 *Under OEM-lead control where OEM decides n and Q_r and RS decides Q_i, the optimal values of n and Q_r for the OEM, denoted by $n^*_{3.5}$ and $Q^*_{r,3.5}$, respectively, are given by*

$$n^*_{3.5} = \underset{n=0,1,\ldots,N_{2.2}}{\arg\max} \left\{ \Pi_{OEM} \left(n, Q^*_{r,3.5}(n), \left(\frac{n}{\gamma} - n + 1 \right) Q^*_{r,3.5}(n) \right) \right\}, \tag{3.111}$$

where $N_{2.2}$ is given by (3.79), $Q^*_{r,3.5}(n)$ is as in Corollary 18.

The RS's optimal response, denoted by $Q^*_{i,3.5}$, is given by

$$Q^*_{i,3.5} = \left(\frac{n^*_{3.5}}{\gamma} - n^*_{3.5} + 1 \right) Q^*_{r,3.5}. \tag{3.112}$$

3.8.6 Scenario 3: OEM-lead Control: OEM Decides n and Q_i and RS Decides Q_r

In this section, we analyze P3.6.1 and P3.6.2 formulated by (3.31) and (3.40), respectively. The OEM and the RS make decisions in a Stackelberg setting that the OEM decides the values of n and Q_i, and then the RS decides the value of Q_r. To determine the values of n and Q_i, the OEM will reply on the RS's response for any given (n, Q_i) pair.

This prediction can be obtained by deriving the RS's optimal value of Q_r that maximizes the RS's total profit for any given (n, Q_i) pair, i.e., the solution of P2.4.1 in (3.31), and the solution is given by $Q^*_{r,2.3}(n, Q_i)$ in Property 28.

After predicting the RS's response, the OEM's decision is to determine the values of n and Q_i that maximize the OEM's total profit, i.e., the solution of P3.6.2 in (3.40), which can be obtained by searching over the region $0 \leq Q_i \leq aT$, $n = 0, 1, \ldots, N_{2.2}$, where $N_{2.2}$ is given by (3.79).

Corollary 20 *Under OEM-lead control where OEM decides n and Q_i and RS decides Q_r, the optimal (n, Q_i) pair for the OEM, denoted by $(n^*_{3.6}, Q^*_{i,3.6})$, is given by*

$$(n^*_{3.6}, Q^*_{i,3.6}) = \underset{n=0,1,\ldots,N_{2.2},\, 0\leq Q_i \leq aT}{\arg\max} \{\Pi_{OEM}\left(n, Q_i, Q^*_{r,2.3}(n, Q_i)\right)\}, \tag{3.113}$$

where $N_{2.2}$ is given by (3.79), $Q^*_{r,2.3}(n, Q_i)$ is as in Property 28.

The RS's optimal response, denoted by $Q^*_{r,3.6}$, is given by

$$Q^*_{r,3.6} = Q^*_{r,2.3}(n^*_{3.6}, Q^*_{i,3.6}). \tag{3.114}$$

3.8.7 Scenario 3: OEM-lead Control: OEM Decides Q_r and Q_i and RS Decides n

In this section, we analyze P3.7.1 and P3.7.2 formulated by (3.26) and (3.41), respectively. The OEM and the RS make decisions in a Stackelberg setting that the OEM determines the values of Q_r and Q_i, and then the RS determines the value of n. To determine the values of Q_r and Q_i, the OEM will reply on the RS's response for any given (Q_r, Q_i) pair.

This prediction can be obtained by deriving the RS's optimal value of n that maximizes the RS's total profit for any given (Q_r, Q_i) pair, i.e., the solution of P1.5.1 in (3.26), and the solution is given by $n^*_{1.5}(Q_i, Q_r)$ in Property 22.

After predicting the RS's response, the OEM's decision is to determine the (Q_i, Q_r) pair that maximizes the OEM's total profit, i.e., the solution of P3.7.2

in (3.41). Property 31 defines the search region for the optimal value of Q_r, for a given Q_i. The searching region for Q_i is from 0 to aT.

Corollary 21 *Under OEM-lead control where OEM decides Q_r and Q_i and RS decides n, the optimal (Q_r, Q_i) pair for the OEM, denoted by $(Q^*_{r,3.7}, Q^*_{i,3.7})$, is given by*

$$(Q^*_{r,3.7}, Q^*_{i,3.7}) = \arg\max_{0 \le Q_i \le aT}\{\arg\max_{0 \le Q_r \le \min\{aT - Q_i, Q_i\gamma\}} \Pi_{OEM}\left(n^*_{1.5}(Q_i, Q_r), Q_i, Q_r\right)\}, \quad (3.115)$$

where $n^*_{1.5}(Q_i, Q_r)$ is as in Property 22.

The RS's optimal response, denoted by $n^*_{3.7}$, is given by

$$n^*_{3.7} = n^*_{1.5}(Q^*_{r,3.7}, Q^*_{i,3.7}). \quad (3.116)$$

3.9 Numerical Experiments

In this section we provide numerical results for two types of remanufacturable automotive parts to demonstrate the performance of different system settings. These two types of automotive parts are engines and transmissions. Engines are difficult and expensive to remanufacture, whereas transmissions are easier and cheaper to remanufacture. For engines, the life cycle is short, and the demand is low. The difference between the unit cost for a new engine and the unit cost for a remanufactured engine is low. For transmissions, the life cycle is longer, and the demand is larger. The difference between the unit cost for a new transmission and the unit cost for a remanufactured transmission is substantial. The remanufacturing yield rate for engines is lower than that for transmissions. Thus the system benefit from remanufacturing transmissions is higher than that from remanufacturing engines. For both of these two different types of remanufacturable parts, we will demonstrate the best settings in different scenarios.

3.9.1 Parameter Settings

Our parameters are from an OEM in the automotive industry. We masked the numbers for confidential purpose. For the missing parameters, we use factorial design to consider different levels. The parameter settings are summarized in Table 3.3. We consider three levels for the fixed shipment cost from the OEM to the RS, i.e., K_s, (25, 50 and 100), and two levels for the fixed remanufacturing setup cost incurred by the RS, i.e., K_r, (100 and 200). For engines, we consider three levels of unit remanufacturing cost c (24, 26.25 and 28.125), and three levels of the remanufacturing rate m (600, 900 and 1350). For transmissions, we consider three levels of unit remanufacturing cost c (16, 17.5 and 18.75), and three levels of the remanufacturing rate m (1500, 2250 and 3375). As a result, we consider a total of 54 problem instances for each type of automotive parts.

Table 3.3: Parameter settings.

	π	c^{np}	c^{rp}	c^d	h_M^n	h_M^r	h_M^u	h_R^r	h_R^u	T	a	γ
Engine	77.91	38	37.5	4.125	$0.12 * c^{np}$	$0.12 * c^{rp}$	2.625	$0.12 * c$	2.625	10	600	0.8
Trans.	171.4125	58.5	25	3.125	$0.12 * c^{np}$	$0.12 * c^{rp}$	1.875	$0.12 * c$	1.875	15	1200	0.95

3.9.2 Experimentation

Given a problem instance, we solve the optimization problems for a total of 17 system settings summarized in Table 3.2, which belong to 3 different scenarios. For each scenario, we check the performance of each system setting by comparing its optimal profit with the optimal profit under centralized control. Let us denote the optimal profit of Setting j in Scenario i by $\Pi^*_{i,j}$, $i = 1, 2, 3$, $j = 1, 2, \cdots, 5$ for

Scenarios 1 and 2, and $j = 1, 2, \cdots, 7$ for Scenario 3. Then, we use the following metric to denote the performance of each setting:

$$\frac{\Pi^*_{i,1} - \Pi^*_{i,j}}{\Pi^*_{i,1}},$$

where $\Pi^*_{i,1}$ is the optimal profit of the centralized control setting in Scenario i, $i = 1, 2, 3$. Thus, the smaller value the above formula has, the better performance its corresponding setting has. To evaluate the impacts of different settings on the profits of the OEM and the RS, we use the following metrics

$$\frac{\Pi^*_{OEM,i,1} - \Pi^*_{OEM,i,j}}{\Pi^*_{OEM,i,1}} \quad \text{and} \quad \frac{\Pi^*_{RS,i,1} - \Pi^*_{RS,i,j}}{\Pi^*_{RS,i,1}},$$

respectively, where $\Pi^*_{OEM,i,1}$ ($\Pi^*_{RS,i,1}$) is the OEM's (RS's) profit under centralized control in Scenario i, $i = 1, 2, 3$, and $\Pi^*_{OEM,i,j}$ ($\Pi^*_{RS,i,j}$) is the OEM's (RS's) profit of Setting j in Scenario i, $i = 1, 2, 3$, $j = 1, 2, \cdots, 5$ for Scenarios 1 and 2, and $j = 1, 2, \cdots, 7$ for Scenario 3.

In Tables 3.4 and 3.5, we report the average performance of each setting for engines and transmissions, respectively. Based on Tables 3.4 and 3.5, we observe that the results for engines and transmissions are rather consistent:

- For Scenario 1 (Q_r is exogenous), we observe that
  - under Setting 2 (OEM-centric Control) and Setting 3 (RS- centric Control), at least one agent will not enter the market: the RS will not enter the market under Setting 2 (OEM-centric Control), while the OEM will not enter the market under Setting 3 (RS-centric Control). That is because when one agent has all the power on decision variables, the decentralized control will fail the market by neglecting the profit and requirement of

the other agent who does not have any power.

- under Settings 4 (OEM-lead control where OEM determines n) and 5 (OEM-lead control where OEM determines Q_i), both of which are OEM-lead controls where the power on decision variables is divided among the agents, the system can achieve channel coordination.

From the above observations, we can conclude that, for the situation that the exchange lot size Q_r is exogenous, channel coordination can be achieved under OEM-lead control strategies as long as the RS can determine one decision variable.

- For Scenario 2 (Q_i is exogenous), we observe that

  - under Setting 2 (OEM-centric Control), the RS will not enter the market. The reason is same as in Scenario 1, Setting 2.

  - under Setting 3 (RS-centric Control), the RS's profit is higher than the RS's profit under centralized control, while the OEM's profit is lower than the OEM's profit under centralized control. Although it is RS-centric control, the RS cannot take all the market profit. The reason is that, by the inventory conservation constraint for the RS in (3.5), Q_i restricts the values of n and Q_r, and thus limits the RS's power.

  - under Settings 4 (OEM-lead control where OEM determines n) and 5 (OEM-lead control where OEM determines Q_r), both of which are OEM-lead controls where the power on decision variables is divided among the agents, the OEM's profit is higher than the OEM's profit under centralized control, while the RS's profit is lower than the RS's profit under centralized control. That is because, under OEM-lead control strategies, the OEM can predict the RS's decisions and then lead the game, i.e., the OEM has more power on the decisions. However, The OEM cannot take all the market profit since the RS has some power to restrict the OEM's decisions.

  - under Setting 5 (OEM-lead control where OEM determines Q_r), the system-wide total profit is close to the optimal system-wide total profit under centralized control. The performance of Setting 5 is the **best** among all the OEM-lead decentralized control strategies.

- For Scenario 3, we observe that

  - Setting 3 (OEM lead control where OEM determines Q_i and then RS determines n and Q_r) has the **best** performance among all the OEM-lead decentralized control strategies in terms of resulting the system-wide total profit which is close to the optimal system-wide total profit under centralized control.
  - Settings 5 (OEM lead control where OEM determines n and Q_r and then RS determines Q_i), 6 (OEM lead control where OEM determines n and Q_i and then RS determines Q_r) and 7 (OEM lead control where OEM determines Q_i and Q_r and then RS determines n) also have good performances in terms of providing near centralized optimal solutions.
  - Settings 2 (OEM lead control where OEM determines Q_r and then RS determines n and Q_i) and 4 (OEM lead control where OEM determines n and then RS determines Q_i and Q_r) do not have good performances: Setting 4 is the worst, with 23.33% deterioration on average for engines or 13.83% deterioration on average for transmissions. Setting 2 is the second worst, with 10.93% deterioration on average for engines and 5.52% deterioration on average for transmissions.

From the above observations, it is better for the system to let the OEM determine Q_i. The explanation is as follows: Although the system-wide inventory related costs, shipment related cost and fixed cost are different with that incurred by the OEM, the system's main goal, i.e., to efficiently remanufacture used-items to satisfy the demand, is consistent with the main goal of the OEM. To achieve that goal, the initial lot size sent to the RS, i.e., Q_i, is an essential policy parameter. Indeed, by the inventory conservation constraint for the RS in (3.5), Q_i actually restricts the total amount of remanufactured-items sent back to the OEM by restricting the values of n and Q_r. The OEM has the intention to send large Q_i to the RS to obtain enough remanufactured-items as well as to get rid of used-items returned from customers. Once Q_i is decided, the RS will always utilize the initial batch efficiently due to profit consideration. Thus, all the settings in which the OEM can decide Q_i have fine performance, and the most efficient setting is that the OEM determines Q_i and then the RS determines n and Q_r. Another well-performed setting, in which the OEM cannot determine Q_i, is that the OEM can force the RS to order large Q_i, as in Setting 5. In Setting 5, the OEM determines n and Q_r, and thus determines the total amount of remanufacture-items that the RS has to provide. In this setting, the RS has to order large Q_i to satisfy the total demand of remanufactured-items.

In the settings with bad performance, the RS can decide Q_i as well as one of n and Q_r. Then the RS can actually determine the total amount of remanufactured-items that send to the OEM. Since the RS's profit is from selling remanufactured-items to the OEM, instead of satisfying the customer's demand directly, the RS will not take into consideration how to satisfy the demand efficiently by using remanufactured-items. Thus, the RS's decision might be far from what the system expects.

Table 3.4: Average performance of each setting for engine.

Scenario	Setting	$\frac{\Pi^*_{i,1}-\Pi^*_{i,j}}{\Pi^*_{i,1}}$	$\frac{\Pi^*_{OEM,i,1}-\Pi^*_{OEM,i,j}}{\Pi^*_{OEM,i,1}}$	$\frac{\Pi^*_{RS,i,1}-\Pi^*_{RS,i,j}}{\Pi^*_{RS,i,1}}$
1	2	\	−5.66%	\
	3	\	\	−46.13%
	4	0	0	0
	5	0	0	0
2	2	\	−57.57%	\
	3	2.48%	3.59%	−18.09%
	4	2.28%	−0.91%	63.56%
	5	1.41%	−0.85%	40.72%
3	2	10.93%	3.34%	66.29%
	3	0.21%	0.47%	−2.11%
	4	23.33%	20.12%	46.07%
	5	1.08%	−0.41%	13.81%
	6	1.07%	−0.41%	13.73%
	7	1.18%	−0.4%	14.65%

3.10 Conclusions

In this section, we consider a basic game-theoretic setting for seed stock planning in a batch remanufacturing environment with two agents including an OEM and a RS. The seed stock, batching decision, and the initial batch size for remanufacturing are characterized by variables Q_s, Q_r, and Q_i, respectively, along with the number of consecutive remanufacturing replenishments which is characterized by the variable n.

We consider three different scenarios that indicate three different practical situations: the exchange lot size Q_r is exogenous due to some technological or operational

Table 3.5: Average performance of each setting for transmission.

Scenario	Setting	$\frac{\Pi^*_{i,1}-\Pi^*_{i,j}}{\Pi^*_{i,1}}$	$\frac{\Pi^*_{OEM,i,1}-\Pi^*_{OEM,i,j}}{\Pi^*_{OEM,i,1}}$	$\frac{\Pi^*_{RS,i,1}-\Pi^*_{RS,i,j}}{\Pi^*_{RS,i,1}}$
1	2	\	-1.8%	\
	3	\	\	-172.94%
	4	0	0	0
	5	0	0	0
2	2	\	-1.95%	\
	3	2.2%	2.85%	-27.85%
	4	1.05%	-0.43%	72.56%
	5	0.74%	-0.39%	50.8%
3	2	5.52%	2.5%	88.7%
	3	0.03%	0.02%	0.11%
	4	13.83%	12.41%	52.32%
	5	0.33%	-0.13%	14.08%
	6	0.35%	-0.13%	14.51%
	7	0.36%	-0.12%	14.86%

constraint; the initial lot size Q_i is exogenous due to some technological or operational constraint; and both Q_r and Q_i are decision variables. For each of the above scenarios, we investigate both centralized control strategies and OEM-lead decentralized control strategies in the Stackelberg setting. We first evaluate the properties of the profit functions, and then we propose efficient methods for obtaining optimal solutions, i.e., optimal control strategies. We perform numerical experiments for two different types of automotive parts: engines and transmissions. The numerical investigation reveals the settings with good performance as well as the settings with bad performance. By analyzing our numerical results, we find that, in order to pursue high system-wide total profit, it is better to have the OEM determine the initial lot size Q_i directly or indirectly. Our work provides insights for how the decision domain structure impacts the system performance and helps to identify the efficient one. The results provide managerial insights for both the OEM and the RS in making decisions on seed stock level, initial batch size for remanufacturing, exchange lot size and remanufacturing frequency, under different technological or operational conditions.

An important direction for future research is to investigate the seed stock planning problem in a stochastic environment with stochastic return process and/or random remanufacturing yield. Another important future research direction is to investigate potential channel coordination strategies for the settings in which system-wide profit maximization cannot be achieved in this section. Last but not least, it is worthwhile to investigate different types of decentralized control strategies including RS-lead decentralized control strategies or Nash setting.

4. CHANNEL COORDINATION STRATEGIES IN THE REVERSE SUPPLY CHAIN

4.1 Overview of Section 4

This section deals with coordination strategies for the OEM and CC. The problem studied here may be referred as reverse channel coordination problem due to its relationship with the traditional channel coordination problem (Toptal and Çetinkaya (2008)). Used-items arrive to the CC according to a stochastic process which is referred as the return process. The CC consolidates used-items using the return-driven threshold policy, and then sends them to the OEM in a large load. Since the OEM and CC have different cost considerations and make decisions individually, coordination mechanisms are useful such that the system-wide total profit is maximized. First, the return process is modeled as a general renewal process, and we prove that when the return flow is exogenous, an all-unit-premium mechanism is able to coordinate the system. We derive analytical expressions for calculating the parameters representing the coordination mechanism. We find conditions under which these analytical expressions lead to closed-form solutions. Then, we apply our results considering several special cases including the cases of deterministic return process, renewal return process with unit load, and renewal return process with exponentially distributed loads. For these special cases, we also extend our results to the situation that the return rate depends on the collection price. When the return rate depends on the collection price, we prove that all-unit-premium mechanism cannot guarantee the centralized optimal profit, i.e., channel coordination. However, by employing all-unit-premium and franchise fee mechanisms together, the channel coordination objective can be achieved. Analytical and numerical examples are provided to illustrate the profit improvement due to coordination.

4.2 Problem Motivations and Related Literature

Inspired by our results in Section 3, we now consider a fundamental coordination problem in the reverse supply chain. This problem is in fact a generalization of the problems studied in Sections 2 and 3 in the sense that a stochastic batch processing environment is modeled for channel coordination purposes. As defined by Toptal and Çetinkaya (2006), channel coordination is the approach to identify the inefficiencies in decentralized solutions for the purpose of aligning the individual incentives for multiple parties with those of the centralized solutions. That is, the decentralized solution may be improved such that: "(i) it results in the same values for the decision variables as the centralized solution; and (ii) it suggests a mutually agreeable way of sharing the resulting profits" (Toptal and Çetinkaya (2006); Toptal and Çetinkaya (2008)).

As we mentioned in the previous sections, the OEM often establishes a remanufacturing program to recover used-items. However, in general, the OEM does not necessarily collect used-items directly. Consumers often prefer the convenience of returning used-items to agents who are in close proximity (see Savaskan et al. (2004) for a systematic analysis of used-item return practices). Thus, retailers or third parties that are close to consumer markets usually act as CCs, where used-items are gathered, sorted and then sent in batches to OEMs.

For example, supermarkets, such as Walmart, pay customers for empty bottles; and mobile phone companies, like AT&T and TMobile, collect used iPhones or other Apple-brand products with attractive prices. The CC obtains revenue by selling remanufacturable used-items to the OEM. These used-items enter the production line to be remanufactured. Since the cost of remanufacturing of used-items, which are actually semi-manufactured-goods, is usually lower than the cost of manufacturing of raw materials, the OEM gains profits from remanufacturing.

Due to different cost considerations, the OEM and the CC prefer different batching strategies for sending used-items. Moreover, in many situations, the CC may decide the collection price of used-items, and then may manipulate the return flow in a way that might not be preferred by the OEM. Thus, effective coordination strategies are useful to achieve system-wide profit maximization.

In this context, we consider a channel coordination problem between the OEM and CC in a stochastic environment. The OEM is the leader who determines the purchase price of the used-item. After observing the purchase price of used-item, the CC decides the batching strategy and the collection price.

The coordination problem introduced and examined here is closely related to two streams of previous research. The first stream of research deals with channel coordination strategies in traditional (i.e., forward) supply chains, while the second stream deals with channel coordination strategies in closed-loop supply chains.

For a comprehensive review of the existing literature in the first stream, we refer the reader to several systematic literature reviews including Tsay et al. (1999), Cachon (2003), Arshinder et al. (2011) and Li and Wang (2007). It is worth noting that the coordination strategies investigated here are inspired by quantity discount pricing strategies investigated by Monahan (1984); Banerjee (1986); Lee and Rosenblatt (1986); Goyal (1987); Joglekar (1988); Monahan (1988); Weng and Wong (1993); Weng (1995a); and Weng (1995b) while considering deterministic demand settings in traditional supply chains. More recent research work on coordination strategies focuses on the traditional newsboy setting with stochastic demand (Cachon and Lariviere (2001, 2005); Gerchak and Wang (2004); Özer (2006); Özer et al. (2007, 2011); Taylor (2002); Taylor and Xiao (2010)). Inspired by the work on coordination strategies in traditional supply chains, we investigate how the coordination ideas can be applied in reverse supply chain with stochastic return flows.

For comprehensive reviews of the existing literature on channel coordination strategies in closed-loop supply chains, the reader is referred to Corbett and Savaskan (2002), Debo et al. (2004) and Govindan et al. (2013). Most of the current papers in this area focus on the coordination and integration of forward and reverse flows (Ketzenberg et al. (2003); Nativi and Lee (2012)); or on coordination strategies that focus on forward flows (Bhattacharya et al. (2006); Vorasayan and Ryan (2006); Liu et al. (2009); Dobos et al. (2013); Pishchulov et al. (2014)). That is, the proposed pricing strategies are applied for the remanufactured products, but the focus is on the resale channel, and the resulting price affects the demand instead of the return flow. Savaskan et al. (2004) focus on the collection channel selection problem and model a decentralized system considering three options: (1) the manufacturer collects returns, (2) the retailer collects returns, and (3) the third party collects returns. Option (2) is the most efficient one for which a two-part tariff mechanism is proposed to achieve channel coordination. However, they do not include operating costs, e.g., inventory holding cost, transportation cost etc., and they assume deterministic return flows. Also, all the other papers considering channel coordination problems in the reverse supply chain ignore inventory and transportation costs and only focus on the OEM's profit.

The coordination problem considered here focuses on the reverse channel consisting of the OEM and CC observing a general renewal return processes. The CC in charge of collection activity sends used-items to the OEM in batches and earns revenue for each unit of used-item delivered. In deciding the batch size of used-items sent to the OEM, the CC adopts the return-driven threshold policy introduced in Section 2. Each coming batch of used-items from the CC enters the OEM's production line immediately. Since the production rate is finite, each batch of used-items generates inventory cost. Thus, the OEM prefers fast delivery with small batches. However, every delivery is associated with a fixed cost for the CC, and, thus, the CC's optimal delivery batch size may be larger than what the OEM prefers. We also consider the situation in which the CC can determine the collection price and, in turn, can influence the return flow. From the perspective of the OEM, high return rate, which usually requires high collection price, means high opportunity for savings from remanufacturing of used-items. However, the CC is not always willing to increase the collection price. Hence, our goal is to design coordination mechanisms for a win-win solution for both the OEM and CC while maximizing the system-wide total profits.

The remainder of this section is organized as follows. In Section 4.3, we model the profit maximization problem for general return flows. In Section 4.4, we propose an effective coordination mechanism, and in Section 4.5, we investigate the coordination mechanism for a specific class of renewal return processes, and then extends the results to consider the case that the CC can determine the collection price. In Section 4.6, we examine some special cases to show how the coordination mechanism works. Section 4.7 investigates the cost saving due to coordination, and provides several numerical examples. Section 4.8 summarizes the results of this section.

4.3 Model Basics

We consider a single-OEM-single-CC system in a stochastic environment. The OEM produces single type product that can be produced from either remanufacturing of used-items or manufacturing of new materials. The cost of remanufacturing is lower than the cost of manufacturing. Thus, remanufacturing brings the OEM savings. The OEM pays the CC the unit price for each unit of used-item. Used-items are collected, sorted and cleaned by the CC. The CC incurs holding cost of used-items as well as fixed shipment cost for each delivery. Our model is similar to the model studied by Ruiz-Benitez et al. (2003), in which the model is under the control of a centralized processing center. We investigate the channel coordination problem where the OEM is the leader who determines the purchase price of the used-item. After observing the purchase price of used-item, the CC decides the batching strategy and the collection price. The system setting is illustrated in Figure 4.1.

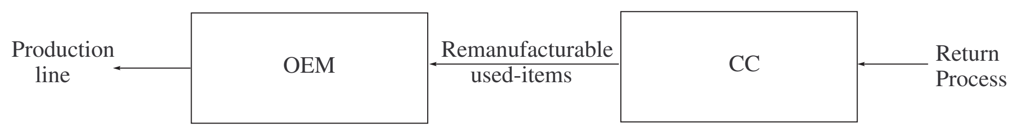

Figure 4.1: An illustration of the channel coordination problem.

The inter-arrival time between successive return loads is a random variable denoted by Y_i, $i = 1, \ldots, n$, where Y_i's are independent and identically distributed (i.i.d.) with $E[Y_i] = 1/\lambda$. We denote the arrival time of the i^{th} return load by S_i, $i = 1, \ldots, n$, and $S_i = \sum_{j=1}^{i} Y_j$, and hence, $W_1(t) = sup\{i : S_i \leq t\}$ is the number of return loads by time t. Each return load contains a random number of used-items L_i, $i = 1, \ldots, n$, where L_i's are independent and identically distributed (i.i.d.) with $E[L_i] = \mu$. Thus, the arrival rate of used-items are given by $r = \lambda\mu$. We denote the cumulative amount of used-items immediately after the i^{th} return load by R_i, $i = 1, \ldots, n$, and $R_i = \sum_{j=1}^{i} L_j$, and hence, $W_2(y) = sup\{i : R_i < y\}$ counts the maximum number of return loads consolidated up to y units. Hence, the cumulative amount of used-items up to time t is a renewal process denoted by $\{W(t), t > 0\}$, and $W(t) = \sum_{i=1}^{W_1(t)} L_i$. The CC adopts the return-driven threshold policy that sends all the accumulated used-items to the OEM whenever the on hand inventory level of used-items exceeds a threshold value Q. We consider threshold policy here because it has been proved that threshold policy superior to its alternatives (Çetinkaya et al. (2006)). The inventory profiles are depicted in Figure 4.2.

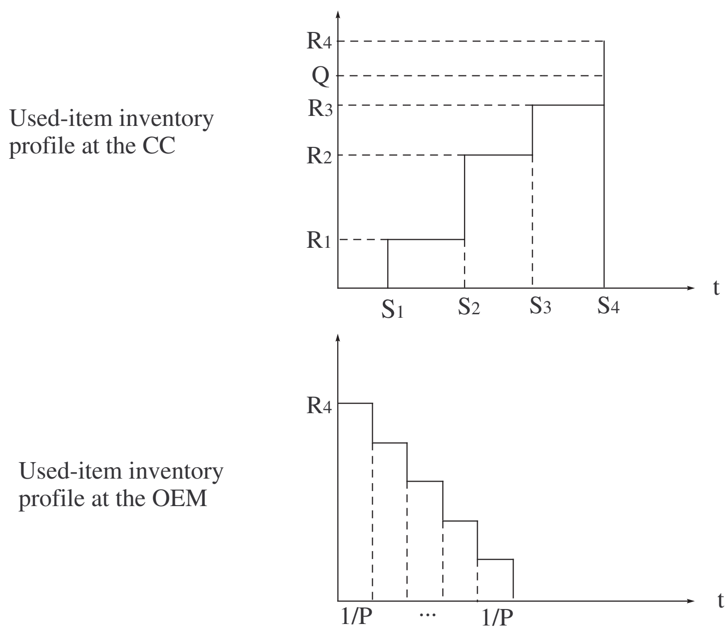

Figure 4.2: A realization of inventory profiles for the channel coordination problem.

The time between two successive batch deliveries is defined as a cycle. We assume that the demand rate of the products is much higher than the return rate of used-items. This is intuitive since not every sold products will be returned, and not all returned items are remanufacturable. Here, we do not consider the yield issue and call remanufacturable items as used-items. However, it is easy to include yield issue into our problem by timing the return rate by the yield rate. Our goal is to design effective coordination mechanisms in a stochastic environment. The notation is summarized in Table 4.1.

The inventory profile for the CC is actually same as that of the retailer in shipment consolidation problems (see Çetinkaya and Bookbinder (2003), Çetinkaya et al. (2008)). The return flow here is equivalent to the demand flow in shipment consolidation problems, and the inventory cost here is equivalent to the waiting cost. As shown in shipment consolidation literature, the expected cycle length is given by

$$E[S_{W_2(Q)+1}] = \frac{E[W_2(Q)+1]}{\lambda}, \tag{4.1}$$

and the expected batch size sent to the OEM is given by

$$E[R_{W_2(Q)+1}] = \mu E[W_2(Q)+1]. \tag{4.2}$$

Next we derive the profit functions for the OEM and CC, respectively.

4.3.1 Profit Function of the CC

Let $E[\Pi_{CC}(Q)]$ denote the CC's long-run average expected total profit per unit time, which is a function of the policy parameter Q. By renewal reward theory (Ross (1996), Page 133), we have

$$E[\Pi_{CC}(Q)] = \frac{E[\mathrm{CC's\quad Cycle\quad profit}]}{E[\mathrm{Cycle\quad length}]}. \tag{4.3}$$

For the CC, the expected cycle profit consists of three main components:

(i) expected revenue from selling used-items to the OEM, which is given by

$$E[(P_{OEM} - P_{CC} - v)R_{W_2(Q)+1}] = (P_{OEM} - P_{CC} - v)\mu E[W_2(Q)+1],$$

Table 4.1: Notation for channel coordination problem.

Q	Quantity-based operating parameter under return-driven threshold policy
P_{OEM}	Unit purchase price paid by the OEM to the CC for each unit of used-item
Δ	All-unit premium policy parameter, i.e., the premium that the OEM pays the CC
P_{CC}	Unit collection price
Y_i	Inter-arrival time between the $i-1^{st}$ return load and the i^{th} return load with $E[Y_i] = 1/\lambda$, and Y_1 is the time of arrival for the fist return load in a cycle
S_i	Arrival time of the i^{th} return load, $S_i = \sum_{j=1}^{i} Y_j$
$W_1(t)$	Number of return loads by time t, $W_1(t) = sup\{i : S_i \leq t\}$
L_i	Number of used-items in the i^{th} return load with $E[L_i] = \mu$ and $Var(L_i) = s^2$
R_i	Cumulative amount of used-items immediately after the i^{th} return load, $R_i = \sum_{j=1}^{i} L_j$
$W_2(y)$	Maximum number of return loads consolidated up to y units, $W_2(y) = sup\{i : R_i < y\}$
$W(t)$	Cumulative used-items up to time t, $W(t) = \sum_{i=1}^{W_1(t)} L_i$
r	Return rate of used-items (unit/unit time), $r = \lambda\mu$
m	Production rate of the OEM (unit/unit time)
π	Unit price of serviceable part ($/unit)
c	Unit remanufacturing cost ($/unit)
K	Fixed operational cost of each delivery incurred by the CC
v	Variable transportation cost ($/unit)
η	The CC's minimum profit per unit time allowed ($/unit time)
h_M^u	Used-item inventory holding cost incurred by the OEM ($/unit/unit time)
h_C^u	Used-item inventory holding cost incurred by the CC ($/unit/unit time)
Π_{OEM}	The OEM's total profit per unit-time
Π_{CC}	The CC's total profit per unit-time
Π	The system-wide total profit per unit-time

where the equation holds by (4.2);

(ii) fixed operational cost which is given by K;

(iii) expected holding cost which is given by

$$\begin{aligned}
h_C^u E\left[\sum_{i=1}^{W_2(Q)} R_i Y_{i+1}\right] &= h_C^u E\left[E\left[\sum_{i=1}^{W_2(Q)} R_i Y_{i+1}\middle| W_2(Q)\right]\right] \\
&= h_C^u E\left[\frac{1}{\lambda} E\left[R_1 + R_2 + \cdots + R_{W_2(Q)}\middle| W_2(Q)\right]\right] \\
&= \frac{h_C^u}{\lambda} E\left[\mu + 2\mu + \cdots + W_2(Q)\mu\right] \\
&= \frac{h_C^u \mu}{\lambda} E\left[\frac{W_2(Q)\left(W_2(Q)+1\right)}{2}\right] \\
&= \frac{h_C^u \mu}{2\lambda}\left(E\left[\left(W_2(Q)+1\right)^2\right] - E\left[W_2(Q)+1\right]\right).
\end{aligned}$$

The method used here to calculate the expected holding cost per collection cycle can also be used to calculate the expected waiting penalty cost per consolidation cycle considered by Çetinkaya et al. (2008). If we interpret our R_i as the cumulative demand after the i^{th} order, the expected cumulative inventory held per collection cycle in our problem is equivalent to the expected cumulative customer waiting per consolidation cycle considered by Çetinkaya et al. (2008). While Çetinkaya et al. (2008) rely on a renewal type equation to obtain the expression for expected waiting penalty cost, our method relies on the first and second moments of $W_2(Q)$. As we demonstrate momentarily, if the variance of $W_2(Q)$ is a linear function of the expectation of $W_2(Q)$, then we have analytical expressions for profit functions, their maximizers, and parameters for channel coordination mechanisms.

Recalling (4.1), and using (i), (ii), and (iii) in (4.3), the long-run average expected total profit per unit time for the CC is given by

$$E[\Pi_{CC}(Q)] = \lambda\mu(P_{OEM} - P_{CC} - v) - \frac{K\lambda}{E[W_2(Q)+1]} - \frac{h_C^u\mu}{2}\left(\frac{E[(W_2(Q)+1)^2]}{E[W_2(Q)+1]} - 1\right).$$

Since $E[(W_2(Q)+1)^2] = (E[W_2(Q)+1])^2 + Var(W_2(Q)+1)$, the above equation can be rewritten as follows

$$E[\Pi_{CC}(Q)] = \lambda\mu(P_{OEM} - P_{CC} - v) + \frac{h_C^u\mu}{2} - \frac{K\lambda}{E[W_2(Q)+1]} - \frac{h_C^u\mu}{2}E[W_2(Q)+1] - \frac{h_C^u\mu}{2}\frac{Var(W_2(Q)+1)}{E[W_2(Q)+1]}. \tag{4.4}$$

By (4.4), we observe that $E[\Pi_{CC}(Q)]$ only contains the first and second moments of $W_2(Q)$.

4.3.2 Profit Function of the OEM

Let $E[\Pi_{OEM}(P_{OEM})]$ denote the OEM's long-run average expected total profit per unit time, which is a function of the policy parameter P_{OEM}. By renewal reward theory (Ross (1996), Page 133), we have

$$E[\Pi_{OEM}(P_{OEM})] = \frac{E[\text{OEM's Cycle profit}]}{E[\text{Cycle length}]}. \tag{4.5}$$

For the OEM, the expected cycle profit consists of two main components:

(i) expected revenue from remanufacturing, which is given by

$$E\left[(\pi - c - P_{OEM})R_{W_2(Q)+1}\right] = (\pi - c - P_{OEM})\mu E\left[W_2(Q)+1\right],$$

where the equation holds by (4.2);

(ii) expected holding cost which is given by

$$h_M^u E\left[\frac{1}{m}\left(\sum_{i=1}^{R_{W_2(Q)+1}} i\right)\right] = \frac{h_M^u}{m}\frac{E\left[(R_{W_2(Q)+1}+1)R_{W_2(Q)+1}\right]}{2}$$
$$= \frac{h_M^u}{2m}\left(E\left[R^2_{W_2(Q)+1}\right]+E\left[R_{W_2(Q)+1}\right]\right). \quad (4.6)$$

In the above equation, the expression for $E[R_{W_2(Q)+1}]$ is given by (4.2), and we can calculate $E\left[R^2_{W_2(Q)+1}\right]$ as follows:

$$\begin{aligned} E\left[R^2_{W_2(Q)+1}\right] &= E\left[\left(\sum_{i=1}^{W_2(Q)+1} L_i\right)^2\right] = E\left[E\left[\left(\sum_{i=1}^{W_2(Q)+1} L_i\right)^2 \middle| W_2(Q)\right]\right] \\ &= E\left[E\left[\sum_{i=1}^{W_2(Q)+1} L_i^2 + 2\sum_{i\neq j, i,j=1,\cdots,W_2(Q)+1} L_iL_j \middle| W_2(Q)\right]\right] \\ &= E\left[(W_2(Q)+1)\left(\mu^2+s^2\right)+2C^2_{W_2(Q)+1}\mu^2\right] \\ &= E\left[(W_2(Q)+1)\left(\mu^2+s^2\right)+(W_2(Q)+1)W_2(Q)\mu^2\right] \\ &= E\left[(W_2(Q)+1)^2\mu^2+(W_2(Q)+1)s^2\right] \\ &= \mu^2E\left[(W_2(Q)+1)^2\right]+s^2E\left[(W_2(Q)+1)\right]. \end{aligned} \quad (4.7)$$

Substituting (4.2) and (4.7) in (4.6), the expected holding cost incurred by the OEM per cycle is given by

$$\frac{h_M^u}{2m}\left(\mu^2E\left[(W_2(Q)+1)^2\right]+\left(s^2+\mu\right)E\left[(W_2(Q)+1)\right]\right).$$

Recalling (4.1), and using (i) and (ii) in (4.5), the long-run average expected total profit per unit time for the OEM is given by

$$E[\Pi_{OEM}(P_{OEM})] = \lambda\mu\left(\pi-c-P_{OEM}-\frac{h_M^u}{2m}\left(1+\frac{s^2}{\mu}\right)\right)-\frac{h_M^u\lambda\mu^2}{2m}\frac{E[(W_2(Q)+1)^2]}{E[W_2(Q)+1]}.$$

Since $E[(W_2(Q)+1)^2] = (E[W_2(Q)+1])^2 + Var(W_2(Q)+1)$, the above equation can be rewritten as follows

$$\begin{aligned} E[\Pi_{OEM}(P_{OEM})] &= \lambda\mu\left(\pi - c - P_{OEM} - \frac{h_M^u}{2m}\left(1+\frac{s^2}{\mu}\right)\right) - \frac{h_M^u\lambda\mu^2}{2m}E[W_2(Q)+1] \\ &\quad -\frac{h_M^u\lambda\mu^2}{2m}\frac{Var(W_2(Q)+1)}{E[W_2(Q)+1]}. \end{aligned} \tag{4.8}$$

Observing the long-run average expected total profit per unit time for the CC in (4.4) and for the OEM in (4.8), both $E[\Pi_{CC}(Q)]$ and $E[\Pi_{OEM}(P_{OEM})]$ contain only $E[W_2(Q)+1]$ and $Var(W_2(Q)+1)$. For notation simplification purpose, we denote $E[W_2(Q)+1]$ by $f_1(Q)$ and denote $\frac{Var(W_2(Q)+1)}{E[W_2(Q)+1]}$ by $f_2(Q)$, and then rewrite $E[\Pi_{CC}(Q)]$ in (4.4) and $E[\Pi_{OEM}(P_{OEM})]$ in (4.8) as follows

$$\begin{aligned} E[\Pi_{CC}(Q)] &= \lambda\mu(P_{OEM} - P_{CC} - v) + \frac{h_C^u\mu}{2} - \frac{K\lambda}{f_1(Q)} - \frac{h_C^u\mu}{2}f_1(Q) \\ &\quad - \frac{h_C^u\mu}{2}f_2(Q), \quad \text{and} \end{aligned} \tag{4.9}$$

$$\begin{aligned} E[\Pi_{OEM}(P_{OEM})] &= \lambda\mu\left(\pi - c - P_{OEM} - \frac{h_M^u}{2m}\left(1+\frac{s^2}{\mu}\right)\right) - \frac{h_M^u\lambda\mu^2}{2m}f_1(Q) \\ &\quad - \frac{h_M^u\lambda\mu^2}{2m}f_2(Q). \end{aligned} \tag{4.10}$$

The long-run average expected system-wide total profit, denoted by $E[\Pi(Q)]$, is the summation of $E[\Pi_{CC}(Q)]$ and $E[\Pi_{OEM}(P_{OEM})]$:

$$\begin{aligned} E[\Pi(Q)] &= \lambda\mu\left(\pi - c - P_{CC} - v - \frac{h_M^u}{2m}\left(1+\frac{s^2}{\mu}\right)\right) - \frac{K\lambda}{f_1(Q)} \\ &\quad - \left(\frac{h_C^u\mu}{2} + \frac{h_M^u\lambda\mu^2}{2m}\right)f_1(Q) \\ &\quad - \left(\frac{h_C^u\mu}{2} + \frac{h_M^u\lambda\mu^2}{2m}\right)f_2(Q). \end{aligned} \tag{4.11}$$

In this section, we formulated the model and derived the profit functions for the

CC and OEM, respectively. Based on the profit functions, we will investigate the coordination mechanism in the next section. Similar to Section 3, we use $\Pi(\cdot \mid\mid *)$ (or $\Pi_{OEM}(\cdot \mid\mid *)/\Pi_{CC}(\cdot \mid\mid *)$) to denote the profit function as the function of the variable $\cdot$ when $*$ is given, i.e., when $*$ is treated as fixed value rather than a variable.

4.4 Coordination Mechanism for General Renewal Return Process

We denote the decentralized solutions of Q for the CC and OEM by Q^d_{CC} and Q^d_{OEM}, respectively, and denote the centralized solution of Q for the system by Q^*. Then for any given P_{OEM}, we have:

$$E[\Pi_{CC}(Q^*)] \leq E[\Pi_{CC}(Q^d_{CC})], \quad \text{and} \tag{4.12}$$

$$E[\Pi(Q^d_{CC})] \leq E[\Pi(Q^*)]. \tag{4.13}$$

Property 39 *For any given P_{OEM}, the OEM's profit is non-decreasing if centralized solution is adopted, compared with decentralized solution, i.e., $E[\Pi_{OEM}(P_{OEM}||Q^*)] - E[\Pi_{OEM}(P_{OEM}||Q^d_{CC})] \geq 0$.*

Proof. Since $E[\Pi(Q)] = E[\Pi_{OEM}(P_{OEM}||Q)] + E[\Pi_{CC}(Q)]$, then by (4.13) we have

$$E[\Pi_{OEM}(P_{OEM}||Q^d_{CC})] + E[\Pi_{CC}(Q^d_{CC})] \leq E[\Pi_{OEM}(P_{OEM}||Q^*)] + E[\Pi_{CC}(Q^*)],$$

which leads to

$$E[\Pi_{OEM}(P_{OEM}||Q^*)] - E[\Pi_{OEM}(P_{OEM}||Q^d_{CC})] \geq E[\Pi_{CC}(Q^d_{CC})] - E[\Pi_{CC}(Q^*)] \geq 0. \tag{4.14}$$

The second inequality holds because of (4.12). □

4.4.1 Decentralized Solution

Under decentralized control, the OEM determines the value of P_{OEM}, and then the CC determines the vale of Q. To determine the value of P_{OEM}, the OEM will rely on the prediction of the CC's response for any given P_{OEM} value.

This prediction can be obtained by deriving the CC's optimal value of Q that maximizes the CC's profit function for any given P_{OEM}. Recall the CC's profit function in (4.9). The CC's decentralized optimal value of Q, denoted by Q^d_{CC}, is the root of $dE[\Pi_{CC}(Q)]/dQ = 0$, i.e.,

$$\frac{K\lambda}{f_1^2(Q)} f_1'(Q) = \frac{h_C^u \mu}{2}(f_1'(Q) + f_2'(Q)).$$

Hence, Q^d_{CC} can be obtained by solving the following equation

$$f_1^2\left(Q^d_{CC}\right)\left(1 + \frac{f_2'(Q^d_{CC})}{f_1'(Q^d_{CC})}\right) = \frac{2K\lambda}{h_C^u \mu}. \tag{4.15}$$

By (4.15) we can observe that Q^d_{CC} does not depend on P_{OEM}. Then, by (4.10), we observe that, the OEM's profit is decreasing in P_{OEM}. However, if P_{OEM} is below the entry price for the CC, which is the price that guarantees a profit per unit time at least above η for the CC, i.e., $E[\Pi_{CC}(Q^d_{CC}||P_{OEM})] \geq \eta$, then the CC will not enter the market. Denote P^E_{OEM} as the entry price. The value of P_{OEM} has to satisfy

$$\begin{aligned} P_{OEM} &\geq P^E_{OEM}(Q^d_{CC}) \\ &= P_{CC} + v - \frac{h_C^u}{2\lambda} + \frac{K}{\mu f_1(Q^d_{CC})} + \frac{h_C^u}{2\lambda} f_1(Q^d_{CC}) - \frac{h_C^u}{2\lambda} f_2(Q^d_{CC}) + \frac{\eta}{\lambda\mu}. \end{aligned} \tag{4.16}$$

Thus, the OEM's optimal value of P_{OEM}, denoted by P^d_{OEM}, is given by $P^E_{OEM}(Q^d_{CC})$ in (4.16). Then, under the OEM-lead decentralized control, the CC can take its minimum allowed profit η, and the OEM can take all the rest profit in the system.

4.4.2 Coordination Mechanism

We let $I_{OEM}(P_{OEM}) = E[\Pi_{OEM}(P_{OEM}||Q^*)] - E[\Pi_{OEM}(P_{OEM}||Q^d_{CC})]$, then $I_{OEM}(P_{OEM})$ is the profit increment for a given P_{OEM} when centralized solution of Q, i.e., Q^*, is adopted. By Property 39, $I_{OEM}(P_{OEM}) \geq 0$ for any value of P_{OEM}. In order to induce the CC to adopt Q^*, the OEM needs to adjust P^d_{OEM}, i.e. pays the CC some extra money, i.e., premium for each unit of used-item, denoted by Δ, such that

$$E[\Pi_{CC}(Q^*||(P^d_{OEM} + \Delta))] \geq E[\Pi_{CC}(Q^d_{CC}||P^d_{OEM})].$$

Recalling the CC's profit function in (4.9), the above inequality is equivalent to

$$\lambda\mu\Delta \geq E[\Pi_{CC}(Q^d_{CC}||P^d_{OEM})] - E[\Pi_{CC}(Q^*||P^d_{OEM})].$$

We define the "break even premium", denoted by $\Delta(BE)$, as the minimum value of Δ such that the the CC agrees to adopt centralized solution. Then $\Delta(BE)$ is given by

$$\Delta(BE) = \frac{1}{\lambda\mu}\left(E[\Pi_{CC}(Q^d_{CC}||P^d_{OEM})] - E[\Pi_{CC}(Q^*||P^d_{OEM})]\right). \tag{4.17}$$

By (4.14) and (4.17), we have

$$E[\Pi_{OEM}(P^d_{OEM}||Q^*)] - E[\Pi_{OEM}(P^d_{OEM}||Q^d_{CC})] \geq \lambda\mu\Delta(BE).$$

By (4.10), the above inequality is equivalent to

$$E[\Pi_{OEM}(P^d_{OEM} + \Delta(BE)||Q^*)] \geq E[\Pi_{OEM}(P^d_{OEM}||Q^d_{CC})].$$

Thus, the OEM's profit increases under this all-unit-premium coordination mechanism. Actually, Δ is a pivot that decides how the profit is divided between the OEM and CC. $\Delta(BE)$ is the lower bound of Δ. The upper bound of Δ can be obtained by solving the following inequality

$$E[\Pi_{OEM}(P^d_{OEM} + \Delta||Q^*)] \geq E[\Pi_{OEM}(P^d_{OEM}||Q^d_{CC})],$$

which implies that the OEM will always set Δ at the value with which its profit is, at least, non-decreasing. Substituting the OEM's profit function in the above inequality, we have

$$\begin{aligned} \Delta &\leq \bar{\Delta} \\ &= \frac{1}{\lambda\mu}\left(E[\Pi_{OEM}(P^d_{OEM}||Q^*)] - E[\Pi_{OEM}(P^d_{OEM}||Q^d_{CC})]\right) \\ &= \frac{1}{\lambda\mu} I_{OEM}(P^d_{OEM}). \end{aligned} \tag{4.18}$$

We conclude this section by the following all-unit-premium mechanism that can coordinate the system:

All-unit-premium mechanism: the OEM pays the CC the premium of Δ for each unit of used-item if the CC sends used-items in a batch whenever the inventory of used-items exceeds Q^*, where $\Delta \in [\Delta(BE), \bar{\Delta}]$. $\Delta(BE)$ and $\bar{\Delta}$ are given by (4.17) and (4.18), respectively.

It is hard to obtain the closed-form expressions for Q^d_{CC} and Q^*, if not impossible, due to the complicated profit expressions. Thus, it is also hard to obtain the closed-form expression for the coordination parameter Δ. However, for a class of renewal return processes, we can obtain the closed-form expressions for the optimal solutions. We discuss that in the following section in which we also extend the model to consider price-dependent return flows.

4.5 Sufficient Conditions for Channel Coordination Mechanisms in Closed-form

In this section we will focus on a class of renewal return processes and derive closed-form optimal solutions. This class of renewal return processes satisfies the following two conditions:

Assumption 1 $Var(W_2(Q)+1) = \alpha E[W_2(Q)+1]+\beta$, where α and β are constants;

Assumption 2 $f_1(Q) = E[W_2(Q)+1]$ has inverse function. i.e., f_1^{-1} exists.

Note that $f_1(Q)$ is non-decreasing in Q. By Assumptions 1 and 2, we have $f_2 = \alpha + \beta/f_1(Q)$.

The renewal return processes with unit return load and with exponentially distributed loads are all belonging to this class.

For this class of return processes, the profit functions (4.9), (4.10), and (4.11) can be rewritten as follows:

$$\begin{aligned} E[\Pi_{CC}(Q)] = {}& \lambda\mu(P_{OEM} - P_{CC} - v) + \frac{h_C^u \mu}{2}(1-\alpha) \\ & - \left(K\lambda + \frac{h_C^u \mu \beta}{2}\right)\frac{1}{f_1(Q)} - \frac{h_C^u \mu}{2} f_1(Q), \end{aligned} \tag{4.19}$$

$$\begin{aligned} E[\Pi_{OEM}(P_{OEM})] = {}& \lambda\mu\left(\pi - c - P_{OEM} - \frac{h_M^u}{2m}\left(1 + \frac{s^2}{\mu} + \mu\alpha\right)\right) \\ & - \frac{h_M^u \lambda \mu^2}{2m}\left(f_1(Q) + \frac{\beta}{f_1(Q)}\right), \quad \text{and} \end{aligned} \tag{4.20}$$

$$
\begin{aligned}
E[\Pi(Q)] =& \lambda\mu\left(\pi - c - P_{CC} - v - \frac{h_M^u}{2m}\left(1 + \frac{s^2}{\mu} + \mu\alpha\right)\right) + \frac{h_C^u\mu}{2}(1-\alpha) \\
& - \left(K\lambda + \left(\frac{h_C^u\mu}{2} + \frac{h_M^u\lambda\mu^2}{2m}\right)\beta\right)\frac{1}{f_1(Q)} \\
& - \left(\frac{h_C^u\mu}{2} + \frac{h_M^u\lambda\mu^2}{2m}\right)f_1(Q). \qquad (4.21)
\end{aligned}
$$

4.5.1 Closed-form Expressions of the Parameters of Coordination Mechanism

Observing the profit functions in (4.19), (4.20), and (4.21), they are all functions of $\frac{1}{f_1(Q)}$ and $f_1(Q)$. The coefficients of $f_1(Q)$ are negative in all the three profit functions. By checking the coefficients of $\frac{1}{f_1(Q)}$, we have the following structural properties.

Property 40 *Consider the profit functions $E[\Pi_{CC}]$ in (4.19), $E[\Pi_{OEM}]$ in (4.20), and $E[\Pi]$ in (4.21) as functions of $f_1(Q)$, then the value of β determines the concavity of the above profit functions in the following way:*

1. *If $\beta \leq -\frac{2K\lambda}{h_C^u\mu}$ then $E[\Pi_{CC}]$, $E[\Pi_{OEM}]$, and $E[\Pi]$ are all decreasing in $f_1(Q)$.*

2. *If $-\frac{2K\lambda}{h_C^u\mu} < \beta \leq -\frac{2K\lambda m}{h_C^u\mu m + h_M^u\lambda\mu^2}$ then $E[\Pi_{OEM}]$ and $E[\Pi]$ are all decreasing in $f_1(Q)$, and $E[\Pi_{CC}]$ is concave in $f_1(Q)$ with the unique maximizer give by*

$$
f_{1,CC}^d = \sqrt{\frac{2K\lambda}{h_C^u\mu} + \beta}. \qquad (4.22)
$$

3. *If $-\frac{2K\lambda m}{h_C^u\mu m + h_M^u\lambda\mu^2} < \beta \leq 0$ then $E[\Pi_{OEM}]$ is decreasing in $f_1(Q)$, $E[\Pi_{CC}]$ is concave in $f_1(Q)$ with the unique maximizer give by (4.22), and $E[\Pi]$ is concave in $f_1(Q)$ with the unique maximizer give by*

$$
f_1^* = \sqrt{\frac{2K\lambda m}{h_C^u\mu m + h_M^u\lambda\mu^2} + \beta}. \qquad (4.23)
$$

4. If $\beta > 0$ then $E[\Pi_{CC}]$ and $E[\Pi]$ are all concave functions of $f_1(Q)$ with unique maximizers given by (4.22) and (4.23), respectively. $E[\Pi_{OEM}]$ is also concave function of $f_1(Q)$ with unique maximizers given by

$$f_{1,OEM}^d = \sqrt{\beta}. \tag{4.24}$$

Proof. By (4.19), (4.20), and (4.21), $E[\Pi_{CC}]$, $E[\Pi_{OEM}]$, and $E[\Pi]$ are functions of $f_1(Q)$, respectively.

Take the first and second derivatives of $E[\Pi_{CC}]$ with respect to $f_1(Q)$, and we have

$$\begin{aligned}
\frac{dE[\Pi_{CC}]}{df_1} &= \left(K\lambda + \frac{h_C^u \mu \beta}{2}\right)\frac{1}{f_1^2} - \frac{h_C^u \mu}{2}, \quad \text{and} \\
\frac{d^2 E[\Pi_{CC}]}{df_1^2} &= -2\left(K\lambda + \frac{h_C^u \mu \beta}{2}\right)\frac{1}{f_1^3}.
\end{aligned}$$

Recall that $f_1(Q)$ is positive and non-decreasing in Q. If $\beta \leq -\frac{2K\lambda}{h_C^u \mu}$ then $E[\Pi_{CC}]$ is decreasing in $f_1(Q)$, otherwise $E[\Pi_{CC}]$ is concave in $f_1(Q)$ with the unique maximizer given by (4.22). Similarly, take the first and second derivatives of $E[\Pi_{OEM}]$ and $E[\Pi]$ with respect to $f_1(Q)$, respectively, and we have

$$\begin{aligned}
\frac{dE[\Pi_{OEM}]}{df_1} &= -\frac{h_M^u \lambda \mu^2}{2m}\left(1 - \frac{\beta}{f_1^2}\right), \\
\frac{d^2 E[\Pi_{OEM}]}{df_1^2} &= -\frac{h_M^u \lambda \mu^2}{2m}\frac{2\beta}{f_1^3}, \\
\frac{dE[\Pi]}{df_1} &= \left(K\lambda + \left(\frac{h_C^u \mu}{2} + \frac{h_M^u \lambda \mu^2}{2m}\right)\beta\right)\frac{1}{f_1^2} - \left(\frac{h_C^u \mu}{2} + \frac{h_M^u \lambda \mu^2}{2m}\right), \quad \text{and} \\
\frac{d^2 E[\Pi]}{df_1^2} &= -2\left(K\lambda + \left(\frac{h_C^u \mu}{2} + \frac{h_M^u \lambda \mu^2}{2m}\right)\beta\right)\frac{1}{f_1^3}.
\end{aligned}$$

The following proof is similar as above, and hence, is omitted. □

By Property 40, the optimal values of Q for $E[\Pi_{CC}]$, $E[\Pi_{OEM}]$, and $E[\Pi]$ can be obtained, respectively, as the following corollary shows.

Corollary 22 *The values of Q that can maximize the cost functions $E[\Pi_{CC}]$ in (4.19), $E[\Pi_{OEM}]$ in (4.20), and $E[\Pi]$ in (4.21), denoted by Q^d_{CC}, Q^d_{OEM}, and Q^*, respectively, are given by*

1. *If $\beta \le -\frac{2K\lambda}{h^u_C\mu}$ then $Q^d_{CC} = Q^d_{OEM} = Q^* = 1$.*

2. *If $-\frac{2K\lambda}{h^u_C\mu} < \beta \le -\frac{2K\lambda m}{h^u_C\mu m + h^u_M\lambda\mu^2}$ then $Q^d_{OEM} = Q^* = 1$, and*

$$Q^d_{CC} = f_1^{-1}\left(\sqrt{\frac{2K\lambda}{h^u_C\mu} + \beta}\right). \tag{4.25}$$

3. *If $-\frac{2K\lambda m}{h^u_C\mu m + h^u_M\lambda\mu^2} < \beta \le 0$ then $Q^d_{OEM} = 1$, Q^d_{CC} is given by (4.25), and*

$$Q^* = f_1^{-1}\left(\sqrt{\frac{2K\lambda m}{h^u_C\mu m + h^u_M\lambda\mu^2} + \beta}\right). \tag{4.26}$$

4. *If $\beta > 0$ then Q^d_{CC} and Q^d_{CC} are given by (4.25) and (4.26), respectively, and*

$$Q^d_{OEM} = f_1^{-1}\left(\sqrt{\beta}\right). \tag{4.27}$$

Proof. The proof is straight-forward using (4.22), (4.23), and (4.24) in Property 40 and Assumption 2, and hence, is omitted. □

Corollary 23 *For any β, $Q^d_{OEM} \le Q^* \le Q^d_{CC}$.*

Proof. Recalling that $f_1(Q)$ is non-decreasing in Q, the proof is straight-forward by comparing Q^d_{OEM}, Q^*, and Q^d_{CC} in Corollary 22. □

By Corollary 23, the centralized optimal value of Q is no greater than the CC's decentralized optimal value of Q, and is no less than the OEM's decentralized optimal value of Q. Corollary 22 provides the closed-form expressions for the optimal values of Q. Using the results in Property 40 and Corollary 22, the all-unit-premium mechanism proposed in Section 4.4.2 can be modified as follows:

All-unit premium mechanism modified: the OEM pays the CC the premium of Δ for each unit of used-item if the CC lowers $f_{1,CC}^{d}(=f_1(Q_{CC}^{d}))$ by D factor, i.e., $f_1(Q) = Df_{1,CC}^{d}$, where $\Delta \in [\Delta(BE), \bar{\Delta}]$. $\Delta(BE)$ and $\bar{\Delta}$ are given by

$$\Delta(BE) = \frac{\sqrt{(2K\lambda + h_C^u\mu\beta)\, h_C^u\mu}}{2\lambda\mu}\frac{(1-D)^2}{D}, \quad \text{and}$$
$$\bar{\Delta} = \frac{h_M^u\mu}{2m}\frac{(2K\lambda + h_C^u\mu\beta)\, D - h_C^u\mu\beta}{\sqrt{(2K\lambda + h_C^u\mu\beta)\, h_C^u\mu}}\frac{1-D}{D},$$

respectively. The two equations above can be obtained by substituting $f_{1,CC}^{d}$ given by (4.22) in (4.17) and (4.18), respectively.

Note: in order to lower $f_{1,CC}^{d}$ by D, the CC needs to lower Q_{CC}^{d} by D', where

$$D' = \frac{f_1^{-1}\left(D\sqrt{\frac{2K\lambda}{h_C^u\mu} + \beta}\right)}{f_1^{-1}\left(\sqrt{\frac{2K\lambda}{h_C^u\mu} + \beta}\right)}. \tag{4.28}$$

4.5.2 Price-dependent Return Flows

In some situations, the CC can determine the collection price of used-items. The collection price usually can impact the return flow to some extend. Then the parameters of the return flow are price-dependent: $\lambda = \lambda(P_{CC})$, $\mu = \mu(P_{CC})$, and $s = s(P_{CC})$. We rewrite the profit functions in (4.19), (4.20), and (4.21) considering price factor as follows:

$$
\begin{aligned}
E[\Pi_{CC}(Q, P_{CC})] = {} & \lambda(P_{CC})\mu(P_{CC})(P_{OEM} - P_{CC} - v) + \frac{h_C^u \mu(P_{CC})}{2}(1-\alpha) \\
& - \left(K\lambda + \frac{h_C^u \mu(P_{CC})\beta}{2}\right) \frac{1}{f_1(Q||P_{CC})} - \frac{h_C^u \mu}{2} f_1(Q||P_{CC}), \qquad (4.29) \\
E[\Pi_{OEM}(P_{OEM})] = {} & \lambda(P_{CC})\mu(P_{CC}) \left(\pi - c - P_{OEM} - \frac{h_M^u}{2m}\left(1 + \frac{s^2(P_{CC})}{\mu(P_{CC})} + \mu(P_{CC})\alpha\right)\right) \\
& - \frac{h_M^u \lambda(P_{CC})\mu^2(P_{CC})}{2m}\left(f_1(Q||P_{CC}) + \frac{\beta}{f_1(Q||P_{CC})}\right), \quad \text{and} \qquad (4.30) \\
E[\Pi(Q, P_{CC})] = {} & \lambda(P_{CC})\mu(P_{CC}) \left(\pi - c - P_{CC} - v - \frac{h_M^u}{2m}\left(1 + \frac{s^2(P_{CC})}{\mu(P_{CC})} + \mu(P_{CC})\alpha\right)\right) \\
& + \frac{h_C^u \mu(P_{CC})}{2}(1-\alpha) \\
& - \left(K\lambda(P_{CC}) + \left(\frac{h_C^u \mu(P_{CC})}{2} + \frac{h_M^u \lambda(P_{CC})\mu^2(P_{CC})}{2m}\right)\beta\right)\frac{1}{f_1(Q||P_{CC})} \\
& - \left(\frac{h_C^u \mu(P_{CC})}{2} + \frac{h_M^u \lambda(P_{CC})\mu^2(P_{CC})}{2m}\right) f_1(Q||P_{CC}). \qquad (4.31)
\end{aligned}
$$

The following lemma shows that partial coordination is better than no coordination in the price-dependent return flow case.

Lemma 1 *If the OEM and CC adopt their decentralized optimal prices P_{OEM}^d and P_{CC}^d, respectively. Denote $Q^*(P_{CC}^d)$ as the jointly optimal threshold value. Then, the system-wide profit per unit time with $Q^*(P_{CC}^d)$, which is given by $\Pi(Q^*(P_{CC}^d), P_{CC}^d)$, is higher than $\Pi_{CC}^d + \Pi_{OEM}^d$, where $\Pi_{CC}^d = \Pi_{CC}(Q_{CC}^d(P_{CC}^d), P_{CC}^d,)$ and $\Pi_{OEM}^d = \Pi_{OEM}(P_{OEM}^d||(Q_{CC}^d(P_{CC}^d), P_{CC}^d))$, i.e., Π_{CC}^d and Π_{OEM}^d are the profit functions under decentralized control.*

Proof. By Corollary 22, when the collection price is P_{CC}^d, we have

$$
\begin{aligned}
Q_{CC}^d(P_{CC}^d) &= f_1^{-1}\left(\sqrt{\frac{2K\lambda(P_{CC}^d)}{h_C^u \mu(P_{CC}^d)} + \beta}\right), \quad \text{and} \\
Q^*(P_{CC}^d) &= f_1^{-1}\left(\sqrt{\frac{2K\lambda(P_{CC}^d)m}{h_C^u \mu(P_{CC}^d)m + h_M^u \lambda(P_{CC}^d)\mu^2(P_{CC}^d)} + \beta}\right).
\end{aligned}
$$

Then, by (4.29), (4.30), and (4.31), we have

$$
\begin{aligned}
&\Pi(Q^*(P^d_{CC}), P^d_{CC}) - (\Pi^d_{CC} + \Pi^d_{OEM}) \\
=&\Pi(Q^*(P^d_{CC}), P^d_{CC}) - \Big(\Pi_{CC}(Q^d_{CC}(P^d_{CC}), P^d_{CC}) + \Pi_{OEM}(P^d_{OEM}||(Q^d_{CC}(P^d_{CC}), P^d_{CC}))\Big) \\
=&\left(K\lambda(P^d_{CC}) + \left(\frac{h^u_C\mu(P^d_{CC})}{2} + \frac{h^u_M\lambda(P^d_{CC})\mu^2(P^d_{CC})}{2m}\right)\beta\right)\frac{\sqrt{h^u_C\mu(P^d_{CC})}}{2K\lambda(P^d_{CC}) + \beta h^u_C\mu(P^d_{CC})} \\
&+ \left(\frac{h^u_C\mu(P^d_{CC})}{2} + \frac{h^u_M\lambda(P^d_{CC})\mu^2(P^d_{CC})}{2m}\right)\frac{2K\lambda(P^d_{CC}) + \beta h^u_C\mu(P^d_{CC})}{\sqrt{h^u_C\mu(P^d_{CC})}}. \\
&- 2\sqrt{\begin{array}{l}\left(K\lambda(P^d_{CC}) + \left(\frac{h^u_C\mu(P^d_{CC})}{2} + \frac{h^u_M\lambda(P^d_{CC})\mu^2(P^d_{CC})}{2m}\right)\beta\right) \\ \times\left(\frac{h^u_C\mu(P^d_{CC})}{2} + \frac{h^u_M\lambda(P^d_{CC})\mu^2(P^d_{CC})}{2m}\right)\end{array}} \\
>&0.
\end{aligned}
$$

The last inequality holds since

$$
\begin{aligned}
&\Bigg\{\left(K\lambda(P^d_{CC}) + \left(\frac{h^u_C\mu(P^d_{CC})}{2} + \frac{h^u_M\lambda(P^d_{CC})\mu^2(P^d_{CC})}{2m}\right)\beta\right)\frac{\sqrt{h^u_C\mu(P^d_{CC})}}{2K\lambda(P^d_{CC}) + \beta h^u_C\mu(P^d_{CC})} \\
&+ \left(\frac{h^u_C\mu(P^d_{CC})}{2} + \frac{h^u_M\lambda(P^d_{CC})\mu^2(P^d_{CC})}{2m}\right)\frac{2K\lambda(P^d_{CC}) + \beta h^u_C\mu(P^d_{CC})}{\sqrt{h^u_C\mu(P^d_{CC})}}\Bigg\}^2 \\
&- 4\left(K\lambda(P^d_{CC}) + \left(\frac{h^u_C\mu(P^d_{CC})}{2} + \frac{h^u_M\lambda(P^d_{CC})\mu^2(P^d_{CC})}{2m}\right)\beta\right) \\
&\times\left(\frac{h^u_C\mu(P^d_{CC})}{2} + \frac{h^u_M\lambda(P^d_{CC})\mu^2(P^d_{CC})}{2m}\right) \\
=&\Bigg\{\left(K\lambda(P^d_{CC}) + \left(\frac{h^u_C\mu(P^d_{CC})}{2} + \frac{h^u_M\lambda(P^d_{CC})\mu^2(P^d_{CC})}{2m}\right)\beta\right)\frac{\sqrt{h^u_C\mu(P^d_{CC})}}{2K\lambda(P^d_{CC}) + \beta h^u_C\mu(P^d_{CC})} \\
&- \left(\frac{h^u_C\mu(P^d_{CC})}{2} + \frac{h^u_M\lambda(P^d_{CC})\mu^2(P^d_{CC})}{2m}\right)\frac{2K\lambda(P^d_{CC}) + \beta h^u_C\mu(P^d_{CC})}{\sqrt{h^u_C\mu(P^d_{CC})}}\Bigg\}^2 \geq 0.
\end{aligned}
$$

$\square$

Actually, for any given P_{CC}, in order to guarantee the agreement between the OEM and CC in adopting $Q^*(P_{CC})$, the following two equations need to be satisfied:

$$E[\Pi_{OEM}(P_{OEM}||(Q^*(P_{CC}), P_{CC}))] \geq E[\Pi^d_{OEM}], \quad \text{and} \tag{4.32}$$

$$E[\Pi_{CC}(Q^*(P_{CC}), P_{CC})] \geq E[\Pi^d_{CC}], \tag{4.33}$$

where $E[\Pi^d_{OEM}]$ and $E[\Pi^d_{CC}]$ are decentralized optimal profits for the OEM and CC, respectively.

By (4.32) we have

$$\begin{aligned} P_{OEM,max}(P_{CC}) =& \pi - c - \frac{h^u_M}{2m}\left(1 + \frac{s^2(P_{CC})}{\mu(P_{CC})} + \mu(P_{CC})\alpha\right) - \frac{E[\Pi^d_{OEM}]}{\lambda(P_{CC})\mu(P_{CC})} \\ & - \frac{h^u_M \mu(P_{CC})}{2m}\left(\sqrt{\frac{2K\lambda(P_{CC})m}{h^u_C\mu(P_{CC})m + h^u_M\lambda(P_{CC})\mu^2(P_{CC})} + \beta} \right. \\ & \left. + \frac{\beta}{\sqrt{\frac{2K\lambda(P_{CC})m}{h^u_C\mu(P_{CC})m + h^u_M\lambda(P_{CC})\mu^2(P_{CC})} + \beta}}\right). \end{aligned} \tag{4.34}$$

By (4.33) we have

$$\begin{aligned} P_{OEM,min}(P_{CC}) =& P_{CC} + v - \frac{h^u_C}{2\lambda(P_{CC})}(1-\alpha) \\ & + \left(\frac{K}{\mu(P_{CC})} + \frac{h^u_C\beta}{2\lambda(P_{CC})}\right)\frac{1}{\sqrt{\frac{2K\lambda(P_{CC})m}{h^u_C\mu(P_{CC})m + h^u_M\lambda(P_{CC})\mu^2(P_{CC})} + \beta}} \\ & + \frac{h^u_C}{2\lambda(P_{CC})}\sqrt{\frac{2K\lambda(P_{CC})m}{h^u_C\mu(P_{CC})m + h^u_M\lambda(P_{CC})\mu^2(P_{CC})} + \beta} \\ & + \frac{E[\Pi^d_{CC}]}{\lambda(P_{CC})\mu(P_{CC})}. \end{aligned} \tag{4.35}$$

Let $f(P_{CC}) = P_{OEM,max}(P_{CC}) - P_{OEM,min}(P_{CC})$, then

$$f(P_{CC}) = \frac{1}{\lambda(P_{CC})\mu(P_{CC})} \left(E[\Pi(Q^*(P_{CC}), P_{CC})] - \left(E[\Pi^d_{OEM}] + E[\Pi^d_{CC}] \right) \right). \quad (4.36)$$

Equation (4.36) can be rewritten as

$$E[\Pi(Q^*(P_{CC}), P_{CC})] = f(P_{CC})\lambda(P_{CC})\mu(P_{CC}) + E[\Pi^d_{OEM}] + E[\Pi^d_{CC}].$$

Hence, $f(P_{CC})$ represents the profit increase per unit used-item due to adopting the jointly optimal threshold value $Q^*(P_{CC})$ instead of the CC's decentralized optimal threshold value $Q^d_{CC}(P_{CC})$. The expression $f(P^*_{CC})\lambda(P^*_{CC})\mu(P^*_{CC})$ represents the profit increase per unit time by adopting the centralized control mechanism $(Q^*(P^*_{CC}), P^*_{CC})$.

Recall the profit functions of the OEM in (4.29) and the CC in (4.30), the OEM's unit purchase price P_{OEM} is a pivot that decides how the profit is divided between the two agents. According to Weng (1995b), a simple mechanism, which divides the profit increase between the two agents, is that let ϵ percentage of the profit increase goes into the OEM's pocket and $1-\epsilon$ percentage of the profit increase goes into the CC's pocket, where ϵ is determined by negotiation. This mechanism is also applicable in our model. The OEM's profit will be increased by $\epsilon f(P^*_{CC})\lambda(P^*_{CC})$ per unit time, while the CC's profit will be increased by $(1-\epsilon)f(P^*_{CC})\lambda(P^*_{CC}))$ per unit item. Then, the jointly optimal unit purchase price is given by

$$P^*_{OEM} = \epsilon P_{OEM,min}(P^*_{CC}) + (1-\epsilon)P_{OEM,max}(P^*_{CC}). \quad (4.37)$$

Next, we will check whether the all-unit-premium mechanism proposed in Section 4.4.2 can still coordinate the system in the case with price-dependent return flows.

Recall the mechanism that the OEM pays the CC P_{OEM}^* for per unit used-item and requires the CC deliver a batch when the on hand inventory of used-items exceeds Q^*. The profit function of the CC is given by:

$$\begin{aligned} E[\Pi_{CC}(P_{CC}||(P_{OEM}^*, Q^*))] =& \lambda(P_{CC})\mu(P_{CC})(P_{OEM}^* - P_{CC} - v) + \frac{h_C^u \mu(P_{CC})}{2}(1-\alpha) \\ & - \left(K\lambda(P_{CC}) + \frac{h_C^u \mu(P_{CC})}{2}\right) \frac{1}{f_1(Q^*||P_{CC})} \\ & - \frac{h_C^u \mu(P_{CC})}{2} f_1(Q^*||P_{CC}). \end{aligned}$$

The maximizer of the above equation might not be the centralized optimal collection price P_{CC}^* which can be obtained by solving $\frac{dE[\Pi(Q^*(P_{CC}),P_{CC})]}{dC} = 0$.

In order to induce the CC set the collection price at P_{CC}^*, the OEM has to set the unit purchase price at P_{OEM}^C such that

$$\frac{dE[\Pi_{CC}(P_{CC}^*||(P_{OEM}^C, Q^*))]}{dC} = 0. \tag{4.38}$$

We can solve P_{OEM}^C from the above equation. Meanwhile, in order to compensate the OEM for adopting P_{OEM}^C rather than P_{OEM}^*, the CC needs to pay the OEM a fixed payment $(P_{OEM}^C - P_{OEM}^*)\lambda(P_{CC}^*)\mu(P_{CC}^*)$ per unit time as the franchise fee.

The coordination mechanism is denoted by $\{P_{OEM}^C, Q^*, (P_{OEM}^C - P_{OEM}^*)\lambda(P_{CC}^*)\mu(P_{CC}^*)\}$: the OEM pays the unit purchase price P_{OEM}^C if the CC can deliver the used-items in a batch whenever the inventory exceeds Q^*. At the same time, the OEM charges the CC the franchise fee $(P_{OEM}^C - P_{OEM}^*)\lambda(P_{CC}^*)\mu(P_{CC}^*)$ per unit time, where P_{OEM}^C can be obtained by solving (4.38).

In this section, we focused on a class of stochastic process and investigated the coordination mechanisms. Closed-form expressions for coordination parameters were derived. Then the results were extended to the price-dependent return case where the CC determines the collection price. It showed that the all-unit-premium and franchise fee mechanism together can coordinate the system. In the following section, we will provide several examples to illustrate how our coordination mechanism is applied.

4.6 Special Cases

In this section, we provide some special cases to show how the coordination mechanism works. These special cases include deterministic return flow case, renewal return process with unit return load case, and renewal return process with exponentially distributed return loads case. Among these cases, renewal return process with unit return load includes the Poisson return process, and the renewal return process with exponentially distributed return loads includes the marked Poisson return process with exponentially distributed return loads. For these cases, the coordination mechanisms under return-driven threshold policy are provided for two situations: the collection price is exogenous, and the collection price is determined by the CC.

4.6.1 Deterministic Return Flow

We first evaluate the most basic case that the return flow is deterministic with rate $\lambda(P_{CC})$, and $\mu(P_{CC}) = 1$ and $s(P_{CC}) = 0$. Then, we have $f_1(Q) = E\left[W_2(Q) + 1\right] = Q$ and $f_2(Q) = \frac{Var(W_2(Q)+1)}{E[W_2(Q)+1]} = 0$ by the definition of $W_2(Q)$. Recalling Assumption 1, we have $\alpha = \beta = 0$. In this situation, the profit functions for the CC in (4.29), the OEM in (4.30), and the system in (4.31) can be rewritten as follows:

$$\Pi_{CC}(Q, P_{CC}) = \lambda(P_{CC})(P_{OEM} - P_{CC} - v) + \frac{h_C^u}{2} - \frac{K\lambda(P_{CC})}{Q} - \frac{h_C^u Q}{2}, \tag{4.39}$$

$$\Pi_{OEM}(Q, P_{CC}) = \lambda(P_{CC}) \left(\pi - c - P_{OEM} - \frac{h_M^u}{2m}\right) - \frac{h_M^u \lambda(P_{CC}) Q}{2m}, \quad \text{and} \tag{4.40}$$

$$\Pi(Q, P_{CC}) = \lambda(P_{CC}) \left(\pi - c - P_{CC} - v - \frac{h_M^u}{2m} \right) + \frac{h_C^u}{2} \\ - \frac{K\lambda(P_{CC})}{Q} - \frac{h_C^u m + h_M^u \lambda(P_{CC})}{2m} Q. \tag{4.41}$$

We first examine the situation that the collection price is exogenous. Recall the results in Section 4.5.1 (Corollary 22.3, Corollary 23). For any given P_{CC}, we have

$$Q_{CC}^d(P_{CC}) = \sqrt{\frac{2K\lambda(P_{CC})}{h_C^u}},$$
$$Q^*(P_{CC}) = \sqrt{\frac{2K\lambda(P_{CC})m}{h_C^u m + h_M^u \lambda(P_{CC})}}, \quad \text{and}$$
$$Q^*(P_{CC}) < Q_{CC}^d(P_{CC}).$$

The all-unit premium mechanism for deterministic return flow case is stated as follows:

The OEM pays the CC the premium of Δ for each unit of used-item if the CC lowers Q_{CC}^d by D factor, where

$$\sqrt{\frac{Kh_C^u}{2\lambda(P_{CC})}} \frac{(1-D)^2}{D} = \Delta(BE) \leq \Delta \leq \bar{\Delta} = \frac{h_M^u \lambda(P_{CC})}{m} \sqrt{\frac{K\lambda(P_{CC})}{2h_C^u}} (1-D) \, .$$

In order to lower Q_{CC}^d to Q^*, the factor D is given by

$$D^* = \frac{Q^*}{Q_{CC}^d} = \sqrt{\frac{h_C^u m}{h_C^u m + h_M^u \lambda(P_{CC})}}.$$

Recall that $Q^*(P_{CC})$ is the jointly optimal threshold value for a given P_{CC}. When P_{CC} is determined by the CC, in order to guarantee that both agents agree on $Q^*(P_{CC})$, the following two inequalities need to be satisfied:

$$\Pi_{OEM}(P_{OEM}||(Q^*(P_{CC}), P_{CC})) \geq \Pi^d_{OEM}, \quad \text{and}$$
$$\Pi_{CC}(Q^*(P_{CC}), P_{CC}) \geq \Pi^d_{CC},$$

as in (4.32) and (4.33), respectively. Substituting $\alpha = \beta = 0$, $\mu(P_{CC}) = 1$, and $s(P_{CC}) = 0$ in equations (4.34) and (4.35), we have

$$\begin{aligned} P_{OEM,max}(P_{CC}) =& \pi - c - \frac{h^u_M}{2m} - \frac{h^u_M}{m}\sqrt{\frac{K\lambda(P_{CC})}{2\left(h^u_C m + h^u_M \lambda(P_{CC})\right)}} - \frac{\Pi^d_{OEM}}{\lambda(P_{CC})}, \quad \text{and} \\ P_{OEM,min}(P_{CC}) =& P_{CC} + v - \frac{h^u_C}{2\lambda(P_{CC})} + \sqrt{\frac{K\left(h^u_C m + h^u_M\lambda(P_{CC})\right)}{2\lambda(P_{CC})m}} \\ & + \frac{h^u_C}{\lambda}\sqrt{\frac{K\lambda(P_{CC})m}{2\left(h^u_C m + h^u_M\lambda(P_{CC})\right)}} + \frac{\Pi^d_{CC}}{\lambda(P_{CC})}. \end{aligned}$$

Take the difference and we obtain that

$$\begin{aligned} f(P_{CC}) &= P_{OEM,max}(P_{CC}) - P_{OEM,min}(P_{CC}) \\ &= \frac{1}{\lambda(P_{CC})}\left(\Pi(Q^*(P_{CC}), P_{CC}) - \left(\Pi^d_{OEM} + \Pi^d_{CC}\right)\right) \\ &= \pi - c - P_{CC} - v - \frac{h^u_M}{2m} + \frac{h^u_C}{2\lambda(P_{CC})} - \sqrt{2K\left(\frac{h^u_C}{\lambda(P_{CC})} + \frac{h^u_M}{m}\right)} \\ &- \frac{\Pi^d_{OEM} + \Pi^d_{CC}}{\lambda(P_{CC})}. \end{aligned} \tag{4.42}$$

We know that $f(P_{CC})$ represents the profit increase per unit used-item due to adopting the jointly optimal threshold value instead of the CC's decentralized optimal threshold value, for any given P_{CC}. For deterministic return flow, $f(P_{CC})$ has the following property:

Property 41 *If $\lambda'(P_{CC}) > 0$ and $\lambda''(P_{CC}) < 0$, then $f(P_{CC})$ is a concave func-*

tion of P_{CC}. Moreover, If $\lambda(0)=0$ then $f(P_{CC})$ has two positive roots.

Proof. The second order condition of $f(P_{CC})$ is given by

$$\begin{aligned} f''(P_{CC}) &= -\frac{(\lambda'(P_{CC}))^2}{\lambda^3(P_{CC})}\frac{\sqrt{K}h_C^u}{\sqrt{\frac{h_M^u}{m}+\frac{h_C^u}{\lambda(P_{CC})}}}\left(\Pi_{OEM}^d+\Pi_{CC}^d+\sqrt{2}\left(1-\frac{1}{4}\frac{h_C^u}{h_M^u\frac{\lambda(P_{CC})}{m}+h_C^u}\right)\right) \\ &\quad +\frac{\lambda''(P_{CC})}{\lambda^2(P_{CC})}\left(\frac{\sqrt{K}h_C^u}{\sqrt{2\left(\frac{h_M^u}{m}+\frac{h_C^u}{\lambda(P_{CC})}\right)}}+\Pi_{OEM}^d+\Pi_{CC}^d\right). \end{aligned}$$

It is obvious that $\frac{h_C^u}{h_M^u\frac{\lambda(P_{CC})}{m}+h_C^u}<1$. Hence, if $\lambda''(P_{CC})<0$ then $f''(P_{CC})<0$, i.e., $f(P_{CC})$ is concave in P_{CC}.

To prove that $f(P_{CC})$ has two positive roots, i.e., $\exists \quad P_{CC}^U>P_{CC}^L>0$ such that $f(P_{CC}^L)=f(P_{CC}^U)=0$, it is sufficient to prove $\exists \quad P_{CC}^0>0$ such that $f(P_{CC}^0)>0$ and $f(0)=f(\infty)<0$. From Lemma 1, we know that $\Pi(Q^*(P_{CC}^d),P_{CC}^d)>\Pi_{CC}^d+\Pi_{OEM}^d$, thus $f(P_{CC}^d)>0$. Recalling $f(P_{CC})$ in (4.42), we have $f(0)=f(\infty)=-\infty$. □

By Property 41, we know that the jointly optimal collection price P_{CC}^* lies in (P_{CC}^L,P_{CC}^U). Then, the jointly optimal purchase price is given by (4.37).

As shown in Section 4.5.2, when the CC determines the collection price, the all-unit-premium mechanism might fail in inducing the CC to choose the jointly optimal collection price P_{CC}^*. To be more specific, by only adopting all-unit-premium mechanism, i.e., the OEM pays the CC P_{OEM}^* as in (4.37) for per unit used-item and requires the CC adopt the jointly optimal threshold value Q^*, the profit function of the CC is given by

$$\begin{aligned} \Pi_{CC}(P_{CC}||(P_{OEM}^*,Q^*)) &= \lambda(P_{CC})(P_{OEM}^*-P_{CC}-v)+\frac{h_C^u}{2} \\ &\quad -\frac{K\lambda(P_{CC})}{Q^*}-\frac{h_C^uQ^*}{2}. \end{aligned} \tag{4.43}$$

The maximizer of equation (4.43) might not be the centralized optimal collection price P^*_{CC}. Observe that the expression of $\Pi_{CC}(P_{CC}||(P_{OEM}, Q^*))$ depends on P_{OEM}. The first order and second order conditions for $\Pi_{CC}(P_{CC}||(P_{OEM}, Q^*))$ are given by

$$\frac{d\Pi_{CC}(P_{CC}||(P_{OEM}, Q^*))}{dC} = -\lambda(P_{CC}) + \lambda'(P_{CC})\left(P_{OEM} - P_{CC} - v - \frac{K}{Q^*}\right), \quad \text{and} \tag{4.44}$$

$$\frac{d^2\Pi_{CC}(P_{CC}||(P_{OEM}, Q^*))}{dC^2} = -2\lambda'(P_{CC}) + \lambda''(P_{CC})\left(P_{OEM} - P_{CC} - v - \frac{K}{Q^*}\right). \tag{4.45}$$

Since $\lambda'(P_{CC}) > 0$ and $\lambda''(P_{CC}) < 0$, $\frac{d^2\Pi_{CC}(P_{CC}||(P_{OEM},Q^*))}{dC^2} < 0$. Thus, $\Pi_{CC}(P_{CC}||(P_{OEM}, Q^*))$ is concave in P_{CC}, and its maximizer is the root of the equation $\frac{d\Pi_{CC}(P_{CC}||(P_{OEM},Q^*))}{dC} = 0$. In order to induce the CC to set the collection price at P^*_{CC}, the OEM has to set the unit purchase price at the value given by the following equation

$$P^C_{OEM} = P^*_{CC} + v + \frac{K}{Q^*} + \frac{\lambda(P^*_{CC})}{\lambda'(P^*_{CC})}. \tag{4.46}$$

In order to compensate the OEM for choosing P^C_{OEM} rather than P^*_{OEM}, the CC needs to pay the OEM a fixed payment $(P^C_{OEM} - P^*_{OEM})\lambda(P^*_{CC})$ per unit time as the franchise fee. Thus, we obtained all the optimal parameters for the coordination mechanism $\{P^C_{OEM}, Q^*, (P^C_{OEM} - P^*_{OEM})\lambda(P^*_{CC})\}$ in the deterministic return flow case.

4.6.2 Unit Return Load

In previous section, we evaluated the coordination mechanisms for deterministic case. Next, we focus on stochastic return flow. First we consider the case that the used-item is returned one by one, i.e., the return flow follows a renewal process. In this situation, we have $\mu(P_{CC}) = 1$ and $s^2(P_{CC}) = 0$, and thus, $f_1(Q) = E[W_2(Q)] + 1 = Q$ and $Var(W_2(Q)) = 0$. The profit functions (4.29) to (4.31) can be specifically written as

$$E[\Pi_{CC}(Q,P_{CC})] = \lambda(P_{CC})(P_{OEM} - P_{CC} - v) + \frac{h_C^u}{2} - \frac{K\lambda(P_{CC})}{Q} - \frac{h_C^u Q}{2}, \tag{4.47}$$

$$E[\Pi_{OEM}(P_{OEM})] = \lambda(P_{CC}) \left(\pi - c - P_{OEM} - \frac{h_M^u}{2m}\right) - \frac{h_M^u \lambda(P_{CC}) Q}{2m}, \quad \text{and} \tag{4.48}$$

$$\begin{aligned} E[\Pi(Q,P_{CC})] = \lambda(P_{CC}) \left(\pi - c - P_{CC} - v - \frac{h_M^u}{2m}\right) + \frac{h_C^u}{2} \\ - \frac{K\lambda(P_{CC})}{Q} - \frac{h_C^u m + h_M^u \lambda(P_{CC})}{2m} Q, \end{aligned} \tag{4.49}$$

respectively. Comparing (4.47) to (4.49) with (4.39) to (4.41), we can see that the profit functions with renewal return process are the same as the profit functions in the deterministic case. Thus, all the results in Section 4.6.1 can be carried over to the renewal return process directly.

4.6.3 Exponentially Distributed Return Loads

In this section, we consider the case that the return loads arrive according to a renewal process and each return load contains an exponentially distributed amount of used-items. In this situation, $s^2(P_{CC}) = \mu(P_{CC})$, $f_1(Q) = E[W_2(Q) + 1] = Q/\mu(P_{CC}) + 1$, and $Var(W_2(Q)) = Q/\mu(P_{CC}) = E[W_2(Q) + 1] - 1$, which leads to $f_2(Q) = 1 - 1/f_1(Q)$, i.e., $\alpha = 1, \beta = -1$. The profit functions (4.29) to (4.31) can be specifically written as

$$\begin{aligned} E[\Pi_{CC}(Q,P_{CC})] =& \lambda(P_{CC})\mu(P_{CC})(P_{OEM} - P_{CC} - v) \\ & - \left(K\lambda(P_{CC}) - \frac{h_C^u \mu(P_{CC})}{2}\right) \frac{1}{Q/\mu(P_{CC}) + 1} \\ & - \frac{h_C^u \mu(P_{CC})}{2} \left(Q/\mu(P_{CC}) + 1\right), \end{aligned} \tag{4.50}$$

$$E[\Pi_{OEM}(P_{OEM})] = \lambda(P_{CC})\mu(P_{CC})\left(\pi - c - P_{OEM} - \frac{h_M^u}{2m}\left(2 + \mu(P_{CC})\right)\right)$$
$$- \frac{h_M^u \lambda(P_{CC})\mu^2(P_{CC})}{2m}\left(Q/\mu(P_{CC}) + 1 - \frac{1}{Q/\mu(P_{CC}) + 1}\right), \quad \text{and} \tag{4.51}$$

$$E[\Pi(Q, P_{CC})] = \lambda(P_{CC})\mu(P_{CC})\left(\pi - c - P_{CC} - v - \frac{h_M^u}{2m}\left(2 + \mu(P_{CC})\right)\right)$$
$$- \left(K\lambda(P_{CC}) - \frac{h_C^u \mu(P_{CC})}{2} - \frac{h_M^u \lambda(P_{CC})\mu^2(P_{CC})}{2m}\right)\frac{1}{Q/\mu(P_{CC}) + 1}$$
$$- \left(\frac{h_C^u \mu(P_{CC})}{2} + \frac{h_M^u \lambda(P_{CC})\mu^2(P_{CC})}{2m}\right)\left(Q/\mu(P_{CC}) + 1\right), \tag{4.52}$$

respectively.

Recalling Corollary 22 and $\beta = -1$, for any given P_{CC}, the maximizers of the cost functions $E[\Pi_{CC}]$ in (4.19), $E[\Pi_{OEM}]$ in (4.20), and $E[\Pi]$ in (4.21) are given by

1. If $\frac{2K\lambda(P_{CC})}{h_C^u \mu(P_{CC})} \leq 1$ then $Q_{CC}^d(P_{CC}) = Q_{OEM}^d(P_{CC}) = Q^*(P_{CC}) = 1$.

2. If $\frac{2K\lambda(P_{CC})m}{h_C^u \mu(P_{CC})m + h_M^u \lambda(P_{CC})\mu^2(P_{CC})} \leq 1 \leq \frac{2K\lambda(P_{CC})}{h_C^u \mu(P_{CC})}$ then $Q_{OEM}^d(P_{CC}) = Q^*(P_{CC}) = 1$, and

$$Q_{CC}^d(P_{CC}) = \max\left(\left(\sqrt{\frac{2K\lambda(P_{CC})}{h_C^u \mu(P_{CC})} - 1} - 1\right)\mu(P_{CC}), 1\right). \tag{4.53}$$

3. If $1 \leq \frac{2K\lambda(P_{CC})m}{h_C^u \mu(P_{CC})m + h_M^u \lambda(P_{CC})\mu^2(P_{CC})}$ then $Q_{OEM}^d = 1$, Q_{CC}^d is given by (4.53), and

$$Q^*(P_{CC}) = \max\left(\left(\sqrt{\frac{2K\lambda(P_{CC})m}{h_C^u m + h_M^u \lambda(P_{CC})\mu^2(P_{CC})} - 1} - 1\right)\mu(P_{CC}), 1\right). \tag{4.54}$$

The all-unit premium mechanism that can coordinate the system when P_{CC} is exogenous is stated as follows: The OEM pays the CC the premium of Δ for each unit of used-item if the CC lowers Q^d_{CC} to Q^*, $\Delta \in [\Delta(BE), \bar{\Delta}]$, and

$$\Delta(BE) = \sqrt{\frac{(2K\lambda(P_{CC}) - h^u_C\mu(P_{CC}))\, h^u_C\mu(P_{CC})}{2\lambda(P_{CC})\mu(P_{CC})}\frac{(1-D)^2}{D}}, \quad \text{and}$$
$$\bar{\Delta} = \frac{h^u_M\mu(P_{CC})}{2m}\frac{2K\lambda(P_{CC}) + h^u_C\mu(P_{CC})(1-D)}{\sqrt{(2F\lambda(P_{CC}) - h^u_C\mu(P_{CC}))\, h^u_C\mu(P_{CC})}}\frac{1-D}{D},$$

where,

$$D = \frac{Q^*(P_{CC})}{Q^d_{CC}(P_{CC})}.$$

When P_{CC} is determined by the CC, the coordination mechanism $\{P^C_{OEM}, Q^*, (P^C_{OEM} - P^*_{OEM})\lambda(P^*_{CC})\mu(P^*_{CC})\}$ can be determined as in Section 4.5.2: P^*_{CC} can be obtained by solving

$$\frac{dE[\Pi(Q^*(P_{CC}), P_{CC})]}{dC} = 0,$$

and $Q^* = Q^*(P^*_{CC})$. Then P^C_{OEM} is determined by solving

$$\frac{dE[\Pi_{CC}(P^*_{CC}||(P^C_{OEM}, Q^*))]}{dC} = 0.$$

Substituting $\beta = -1$ in $P_{OEM,max}(P^*_{CC})$ given by (4.34) and in $P_{OEM,min}(P^*_{CC})$ given by (4.35), P^*_{OEM} can be determined by using (4.37).

In this section, we applied the coordination mechanisms proposed in Section 4.5 into three special cases: deterministic return flow, renewal return flow with unit return load, and renewal return flow with exponentially distributed return loads. The renewal type return flows with unit or exponentially distributed return loads are more general than the Poisson type return flows with unit or exponentially distributed return loads. For each case, we illustrated how to calculate the coordination parameters in two situations: (1) the collection price is exogenous; (2) the CC can determine the collection price.

4.7 Cost Saving Analysis

In Section 4.4, we investigated the coordination mechanism for the reverse supply chain in a stochastic environment. It has been proved that when the collection price is exogenous, the all-unit-premium mechanism can achieve channel coordination. Then, we provided the method to calculate coordination mechanism parameters for a special class of renewal return processes in Section 4.5. Section 4.6 applied the method to several specific examples. Next, focusing on the situation that the collection price is exogenous, we investigate the cost saving due to coordination.

We will provide some examples to show the circumstance where the cost saving is significant, as well as the circumstance where the cost saving is not obvious. Besides, we will illustrate the situations in which the deterministic results can and cannot be used as approximations for stochastic models.

We define the rate of improvement due to coordination by IR:

$$IR = \frac{\Pi(Q^*) - \Pi(Q^d_{CC})}{\Pi(Q^d_{CC})},$$

where Q^* is the centralized solution due to coordination, Q^d_{CC} is the decentralized solution. Then for deterministic return flow, we can measure the IR analytically.

Property 42 *For deterministic return flow, the rate of improvement due to coordination is given by:*

$$IR = \frac{\left(\sqrt{\frac{h_M^u \lambda}{h_C^u m} + 1} - 1\right)^2}{\left(\pi - c - P_{CC} - v - \frac{h_M^u}{2m}\right) \frac{2\lambda}{\sqrt{2K\lambda h_C^u}} + \sqrt{\frac{h_C^u}{2K\lambda}} - \frac{h_M^u \lambda}{h_C^u m} - 2}.$$

Proof.

$$\begin{aligned}
IR &= \frac{\Pi(Q^*) - \Pi(Q_{CC}^d)}{\Pi(Q_{CC}^d)} \\
&= \frac{\frac{K\lambda}{Q_{CC}^d} + \frac{h_C^u + h_M^u \lambda/m}{2} Q_{CC}^d - \frac{K\lambda}{Q^*} - \frac{h_C^u + h_M^u\lambda/m}{2} Q^*}{\left(\pi - c - P_{CC} - v - \frac{h_M^u}{2m}\right)\lambda + \frac{h_C^u}{2} - \frac{K\lambda}{Q_{CC}^d} - \frac{h_C^u + h_M^u\lambda/m}{2} Q_{CC}^d} \\
&= \frac{\sqrt{\frac{K\lambda h_C^u}{2}} + \frac{h_C^u + h_M^u\lambda/m}{2}\sqrt{\frac{K\lambda}{2h_C^u}} - \sqrt{2K\lambda\left(h_C^u + h_M^u\lambda/m\right)}}{\left(\pi - c - P_{CC} - v - \frac{h_M^u}{2m}\right)\lambda + \frac{h_C^u}{2} - \sqrt{\frac{K\lambda h_C^u}{2}} - \left(h_C^u + h_M^u\lambda/m\right)\sqrt{\frac{K\lambda}{2h_C^u}}} \\
&= \frac{h_M^u \frac{\lambda}{m}\sqrt{\frac{K\lambda}{2h_C^u}} - \sqrt{2K\lambda h_C^u}\left(\sqrt{1 + \frac{h_M^u\lambda}{h_C^u m}} - 1\right)}{\left(\pi - c - P_{CC} - v - \frac{h_M^u}{2m}\right)\lambda + \frac{h_C^u}{2} - \sqrt{2K\lambda h_C^u} - h_M^u\lambda/m\sqrt{\frac{K\lambda}{2h_C^u}}} \\
&= \frac{\frac{h_M^u\lambda}{m} - 2h_C^u\left(\sqrt{1 + \frac{h_M^u\lambda}{h_C^u m}} - 1\right)}{\left(\pi - c - P_{CC} - v - \frac{h_M^u}{2m}\right)\lambda\frac{\sqrt{2h_C^u}}{\sqrt{K\lambda}} + \frac{h_C^u}{2}\frac{\sqrt{2h_C^u}}{\sqrt{K\lambda}} - 2h_C^u - \frac{h_M^u\lambda}{m}} \\
&= \frac{\frac{h_M^u\lambda}{h_C^u m} - 2\sqrt{1 + \frac{h_M^u\lambda}{h_C^u m}} + 2}{\left(\pi - c - P_{CC} - v - \frac{h_M^u}{2m}\right)\lambda\frac{2}{\sqrt{2K\lambda h_C^u}} + \frac{\sqrt{h_C^u}}{\sqrt{2K\lambda}} - 2 - \frac{h_M^u\lambda}{h_C^u m}} \\
&= \frac{\left(\sqrt{1 + \frac{h_M^u\lambda}{h_C^u m}} - 1\right)^2}{\left(\pi - c - P_{CC} - v - \frac{h_M^u}{2m}\right)\lambda\frac{2}{\sqrt{2K\lambda h_C^u}} + \frac{\sqrt{h_C^u}}{\sqrt{2K\lambda}} - 2 - \frac{h_M^u\lambda}{h_C^u m}}.
\end{aligned}$$

□

Note that, Property 42 is also applicable for renewal type return flow with unit return load.

By Property 42, we have the following observations:

(O. 1) when $h_M^u = 0$, $IR = 0$;

(O. 2) IR increases if any of the following values increase: $\frac{h_M^u \lambda}{h_C^u m}$, K, and $P_{CC} + v$.

Thus, coordination will bring significant cost savings in the following situations: (1) the inventory holding cost is significant for the OEM; (2) the fixed cost is significant for the CC. Next, we will use an numerical example to illustrate this.

Example 1: The parameter set is given by: $m = 80$, $\pi - c = 20$, $P_{CC} = 16$, $r(= \lambda\mu) = 20$, $K = 200$, $v = 4$, $h_M^u = 4$, $h_C^u = 0.5$. Note that, for this data set $\frac{h_M^u \lambda}{h_C^u m} = 2$. Then, we have the following results:

- For Deterministic case: ($\mu = 1$, $s = 0$, $\lambda = 20$)

  $IR = 22.72\%$, $Q^* = 73$, $Q_{CC}^d = 126$.

- For renewal return process with unit return load: ($\mu = 1$, $s = 0$, $\lambda = 20$)

  $IR = 22.72\%$, $Q^* = 73$, $Q_{CC}^d = 126$.

- For renewal return process with exponentially distributed return loads:
  - ($\mu = s^2 = 1$, $\lambda = 20$): $IR = 23.19\%$, $Q^* = 72$, $Q_{CC}^d = 125$;
  - ($\mu = s^2 = 2$, $\lambda = 10$): $IR = 23.38\%$, $Q^* = 71$, $Q_{CC}^d = 124$;
  - ($\mu = s^2 = 4$, $\lambda = 50$): $IR = 23.73\%$, $Q^* = 69$, $Q_{CC}^d = 122$.

- For renewal return process with uniform return load:
  - $Y_i \sim U[0, 2]$, ($\mu = 1$, $s^2 = \frac{1}{3}$, $\lambda = 20$): $IR = 23.29\%$, $Q^* = 72$, $Q_{CC}^d = 126$;
  - $Y_i \sim U[0, 4]$, ($\mu = 2$, $s^2 = \frac{4}{3}$, $\lambda = 10$): $IR = 23.16\%$, $Q^* = 72$, $Q_{CC}^d = 125$;
  - $Y_i \sim U[0, 8]$, ($\mu = 4$, $s^2 = \frac{16}{3}$, $\lambda = 5$): $IR = 23.44\%$, $Q^* = 70$, $Q_{CC}^d = 124$.

In Example 1, where the two conditions for bringing large cost savings are satisfied, the rates of improvement IR's are high (around 23%) in all the listed cases.

The optimal values of Q in stochastic cases are close to that in deterministic case. If the two conditions are not satisfied, the profit improvement due to coordination might be trivial. Actually, when $h_M^u = 0$, there will be no profit improvement at all, by Property 42. Next, We provide an example in which h_M^u is same as in Example 1 and the profit improvement is trivial.

Example 2: In this data set, the values of m, $\pi - c$, P_{CC}, r, v, and h_M^u are as in Example 1. The fixed cost $K = 50$, and the CC's unit inventory cost per unit time $h_C^u = 4$. In this situation, the impact of fixed cost is less than that in Example 1. Note that, for this data set $\frac{h_M^u \lambda}{h_C^u m} = \frac{1}{4}$. Then, we have the following results:

- For Deterministic case: $(\mu = 1,\ s = 0,\ \lambda = 20)$

  $IR = 0.45\%,\ Q^* = 20,\ Q_{CC}^d = 22.$

- For renewal return process with unit return load: $(\mu = 1,\ s = 0,\ \lambda = 20)$

  $IR = 0.45\%,\ Q^* = 20,\ Q_{CC}^d = 22.$

- For renewal return process with exponentially distributed return loads:
  - $(\mu = s^2 = 1,\ \lambda = 20)$: $IR = 0.45\%,\ Q^* = 19,\ Q_{CC}^d = 21;$
  - $(\mu = s^2 = 2,\ \lambda = 10)$: $IR = 0.52\%,\ Q^* = 18,\ Q_{CC}^d = 20;$
  - $(\mu = s^2 = 4,\ \lambda = 50)$: $IR = 0.58\%,\ Q^* = 16,\ Q_{CC}^d = 18.$

- For renewal return process with uniform return load:
  - $Y_i \sim U[0,2]$, $(\mu = 1,\ s^2 = \frac{1}{3},\ \lambda = 20)$: $IR = 0.77\%,\ Q^* = 19,\ Q_{CC}^d = 22;$
  - $Y_i \sim U[0,4]$, $(\mu = 2,\ s^2 = \frac{4}{3},\ \lambda = 10)$: $IR = 0.56\%,\ Q^* = 19,\ Q_{CC}^d = 21;$
  - $Y_i \sim U[0,8]$, $(\mu = 4,\ s^2 = \frac{16}{3},\ \lambda = 5)$: $IR = 0.77\%,\ Q^* = 17,\ Q_{CC}^d = 20.$

In Example 2, the rates of improvement are less than 1% in all listed situations. It illustrates that when fixed cost is not the major cost for the CC, the coordination mechanism will not bring obvious cost saving.

In the above two examples, we observe that the optimal values of Q are large, and the results in stochastic cases are close to the results in deterministic case. Then, the solution for deterministic model can be used as the approximation for stochastic models. This might not be true in general. In the above examples, we also note that, the difference between the results in stochastic cases and the results in deterministic cases becomes more obvious when the value of μ increases. Actually, even with the same return rate r, the approximation might fail in some situations, especially when μ is large. Moreover, when Q^* is small, a slightly change of Q might cause large results difference.

We define the error of using results in deterministic case as approximations for stochastic situations by ER, and then

$$ER = \frac{E[\Pi^*] - E[\Pi(Q^*_{deterministic})]}{E[\Pi^*]} \cdot 100\%,$$

where $E[\Pi^*]$ is the optimal profit in stochastic case, and $E[\Pi(Q^*_{deterministic})]$ is the expected profit using the solution of deterministic model. Next, we provide an example in which the values of ER might be large, i.e., the solution of deterministic model cannot be treated as an approximation for stochastic models.

Example 3: When the parameter set is given by: $m = 10$, $\pi - c = 30$, $P_{CC} = 16$, $r(= \lambda\mu) = 4$, $K = 25$, $v = 4$, $h^u_M = 4$, $h^u_C = 0.5$, we have the following results:

- For Deterministic case: $Q^* = \frac{2K\lambda}{h^u_C} = 10$.

- For renewal return process with unit return load: ($\mu = 1$, $s = 0$, $\lambda = 20$)

$Q^* = 10$ and $ER = 0$.

- For renewal return process with exponentially distributed return loads:
  - ($\mu = s^2 = 1$, $\lambda = 4$): $Q^* = 9$ and $ER = 0.73\%$;
  - ($\mu = s^2 = 2$, $\lambda = 2$): $Q^* = 8$ and $ER = 3.08\%$;
  - ($\mu = s^2 = 4$, $\lambda = 1$): $Q^* = 5$ and $ER = \mathbf{11.79\%}$.
- For renewal return process with uniform return load:
  - $Y_i \sim U[0, 2]$, ($\mu = 1$, $s^2 = \frac{1}{3}$, $\lambda = 4$): $Q^* = 9$ and $ER = 0.45\%$;
  - $Y_i \sim U[0, 4]$, ($\mu = 2$, $s^2 = \frac{4}{3}$, $\lambda = 2$): $Q^* = 8$ and $ER = 1.27\%$;
  - $Y_i \sim U[0, 8]$, ($\mu = 4$, $s^2 = \frac{16}{3}$, $\lambda = 1$): $Q^* = 8$ and $ER = 4.54\%$.

From the above example, we observe that the value of ER increases in μ. In the situation that the variance of return load is large, e.g., exponentially distributed return loads, the ER is large. When $\mu = 4$, the variance $s^2 = 4$ in exponentially distributed return loads case, and the error is up to 11.79%. This example shows that the solution of deterministic model cannot always be used as the approximation for stochastic models in general. Thus, investigation of the coordination mechanism for general stochastic model is necessary.

4.8 Conclusions

This section extends the fundamental ideas of channel coordination in traditional supply chains to consider the collection channels in closed-loop supply chains. We consider an OEM-CC pair facing stochastic return flows. We derive analytical expressions for calculating the parameters representing the coordination mechanism. We find conditions under which these analytical expressions lead to closed-form solutions. Two situations are considered: the situation where the collection price is exogenous, and the situation where the CC can determine the collection price.

For the situation where the CC has no power on the collection price and the return flow is exogenous, we show that:

- For any given purchase price that the OEM pays the CC, the centralized control results in a higher profit for the OEM while lowering the CC's profit.
- The OEM can take all profits of the system by setting the purchase price as the entry price for the CC.
- By an all-unit-premium policy, the system can achieve coordination.

For the situation where the CC has power on the collection price and can influence the return flow, we show that:

- For any given collection price, the jointly optimal threshold value, i.e., collection quantity, is smaller than the CC's decentralized threshold value.
- Partial coordination is better than no coordination, i.e., even in the situation that the OEM adopts the decentralized purchase price and the CC adopts the decentralized collection price, the jointly optimal threshold value outperforms the decentralized threshold value.
- The all-unit-premium policy fails to induce the CC to choose the jointly optimal collection price. However, the all-unit-premium and franchise fee mechanisms together can coordinate the system.

Next, we consider several special cases including the cases of deterministic return flows, renewal type return flows with unit return load and exponentially distributed return loads. Then, we investigate the cost savings due to coordination in these special cases. Numerical examples are provided to illustrate the setting where coordination can or cannot improve profit significantly. We also provide examples to show that the solution of the deterministic model should not be used as an approximation. Thus, it is important to study the coordination mechanisms for stochastic model in reverse supply chains.

The contribution of this section is that we propose a basic framework for channel coordination mechanisms in reverse supply chains in a stochastic environment, building on which more complex systems and coordination mechanisms can be studied in the future. Some immediate extensions include: (1) considering yield issues of used-items so that the fraction of remanufacturable items is uncertain; (2) modeling competition, e.g., considering models with multiple OEMs and/or multiple CCs among which competition exists; and (3) integrating the reverse and forward channels via effective closed-loop channel coordination mechanisms.

5. CONCLUSIONS

This dissertation concentrates on inventory control models in remanufacturing with batch processing, seed stock planning, and coordination considerations. We investigate three distinct, yet related, inventory control problems in remanufacturing that aim at filling the gaps existing in current literature. Our contributions include

- Building analytical remanufacturing model with stochastic demand and stochastic return with disposal and fixed operational cost considerations;
- Analyzing seed stock models with multiple agents using game theory; and
- Applying channel coordination strategies for reverse supply chains under stochastic environment.

In Section 2, we consider a fundamental inventory and production planning problem characterized by stochastic demand and return along with fixed operational costs and disposal opportunities. By applying queueing theory and normal approximation, we develop effective and efficient approximations for optimal policy parameters under each proposed policy with or without disposal. We show that when the return rate is higher than the demand rate, a disposal option is a necessary decision variable to achieve cost minimization.

In Section 3, we consider a basic game-theoretic setting for seed stock planning problem in remanufacturing with two agents including an OEM and an RS. We investigate how decision domain structure impacts the system performance and provide managerial insights for both OEM and RS in making decisions.

In Section 4, we extend the fundamental ideas of channel coordination in traditional supply chains to consider the collection channels in closed-loop supply chains in a stochastic environment. A basic framework is proposed for channel coordination mechanisms in reverse supply chains and closed-form solutions are derived for models under mild conditions. We illustrate the situation when coordination can or cannot bring significant profit improvement, and demonstrate that the solution of a deterministic model cannot be used as the approximation in general.

Several extensions related to the presented work are proposed in each section. Besides, some other interesting extensions should explore more general and realistic models such as

- Considering more general stochastic process instead of Poisson process in Section 2, deterministic return process in Section 3, and renewal return process in Section 4;
- Integrating coordination mechanisms in reverse channels with forward channels under stochastic return and stochastic demand; and
- Considering finite horizon inventory control problems with multiple agents in the stochastic environment, and designing coordination mechanisms with batching and seed stock planning considerations.

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APPENDIX A

ALTERNATIVE APPROXIMATION FOR THE COST FUNCTION UNDER T_F POLICY IN SECTION 2

By taking advantage of the property of the Normal distribution, we can obtain another approximation for the cost function under T_F policy, i.e., $TC(T_F)$.

Property 43 *If* $r < a\left(1 - \frac{3}{\sqrt{maT_F}}\right)^2$ *then* $P(\sum_{i=1}^{m} R_i \geq \sum_{i=1}^{m} D_i) \approx 0$.

Proof. By the summation property of Poisson distribution, $\sum_{i=1}^{m} D_i$ is also a Poisson distributed random variable with mean maT_{PT} and variance maT. Similarly, $\sum_{i=1}^{m} R_i \sim Poisson(mrT_F)$. Using Normal distributions to approximate the distributions of D_n and R_n, we have that $(\sum_{i=1}^{m} R_i - \sum_{i=1}^{m} D_i) \sim Normal(mrT_F - maT_F, \sqrt{mrT_F + maT_F})$. Recalling the property of a Normal random variable that about 99.7% of its possible values lie within three standard deviations of the mean, we argue that $P(\sum_{i=1}^{m} R_i \geq \sum_{i=1}^{m} D_i) \approx 0$ if $maT_F - mrT_F > 3(\sqrt{maT_F} + \sqrt{mrT_F})$, which is equivalent to

$$r < a\left(1 - \frac{3}{\sqrt{maT_F}}\right)^2.$$

□

By Lemma 43, there is no more than m consecutive cycles hold ending inventories if we begin observing the system with no initial used-item inventory. Let us consider the case when $m = 2$ and the condition in Lemma 43 holds. Then we can conclude that if $I_{n-1} = 0$ then $I_n = (R_n - D_n)^+$, otherwise $I_n = 0$.

Recalling that I_{n-1} can be interpreted as the waiting time of the n^{th} customer, as stated in section 2.3.3.1, $P(I_{n-1} = 0)$ is the probability that the system is idle, i.e., $P(I_{n-1}=0)=(1-\frac{r}{a})$. Then, the expectation of I_n can be obtained as follows:

$$E[I_n]=\left(1-\frac{r}{a}\right)\int_0^\infty \frac{x}{\sqrt{2\pi(aT_F+rT_F)}}e^{-\frac{1}{2}\left(\frac{x-(r-a)T_F}{\sqrt{aT_F+rT_F}}\right)^2}dx. \tag{A.1}$$

Letting $y=\frac{x-(r-a)T_F}{\sqrt{aT_F+rT_F}}$, we can rewrite equation (A.1) as

$$E[I_n]=\frac{1-\frac{r}{a}}{\sqrt{2\pi}}\int_{\frac{(r-a)T_F}{\sqrt{(a+r)T_F}}}^\infty \left(\sqrt{(a+r)T_F}y+(r-a)T_F\right)e^{-\frac{1}{2}y^2}dy. \tag{A.2}$$

The integrand of (A.2) is positive. Hence, we have

$$\begin{aligned}
& \frac{1}{\sqrt{2\pi}}\int_{\frac{(r-a)T_F}{\sqrt{(a+r)T_F}}}^\infty (\sqrt{(a+r)T_F}y+(r-a)T_F)e^{-\frac{1}{2}y^2}dy \\
< \quad & \frac{1}{\sqrt{2\pi}}\int_0^\infty (\sqrt{(a+r)T_F}y+(r-a)T_F)e^{-\frac{1}{2}y^2}dy \\
= \quad & \sqrt{\frac{(a+r)T_F}{2\pi}}+\frac{(a-r)T_F}{2}.
\end{aligned}$$

Consequently, we have

$$E[I_n]<\left(1-\frac{r}{a}\right)\left(\sqrt{\frac{(a+r)T_F}{2\pi}}+\frac{(a-r)T_F}{2}\right). \tag{A.3}$$

using $E[I_n]\approx\left(1-\frac{r}{a}\right)\left(\sqrt{\frac{(a+r)T_F}{2\pi}}+\frac{(a-r)T_F}{2}\right)$, we can obtain another approximation for $TC(T_F)$ which is given by

$$\begin{aligned}
\overline{TC}'(T_F) \quad = \quad & h\left(1-\frac{r}{a}\right)\left(\sqrt{\frac{(a+r)T_F}{2\pi}}+\frac{(a-r)T_F}{2}\right)+\frac{(wa+hr)T_F}{2}+\frac{F}{T_F} \\
& +ca+p(a-r).
\end{aligned} \tag{A.4}$$

$\overline{TC}'(T_F)$ is not a convex function, however it can be proved that it has the unique local minimizer which is also the global minimizer.

Proposition 1 $\overline{TC}'(T_F)$ *has a unique global minimizer.*

Proof. By (A.4), if $T_F \to \infty$ then $\overline{TC}'(T_F) \to \infty$; if $T_F \to 0$ then $\overline{TC}'(T_F) \to \infty$. Thus, $\overline{TC}'(T_F)$ is a coercive continuous function and there is at least one global minimizer.

The first order condition for (A.4) is given by

$$\frac{wa + hr + h(1-\frac{r}{a})^2 a}{2} - \frac{F}{T_F^2} + \frac{h(1-\frac{r}{a})}{2}\sqrt{\frac{a+r}{2\pi}}\frac{1}{\sqrt{T_F}}. \tag{A.5}$$

The second order condition for (A.4) is given by

$$\left(\frac{2F}{T_F^2} - \frac{h(1-\frac{r}{a})}{4}\sqrt{\frac{a+r}{2\pi}}\frac{1}{\sqrt{T_F}}\right)\frac{1}{T_F}. \tag{A.6}$$

(A.5) has an unique root $T_F^{*'}$ which satisfies

$$\frac{wa + hr + h(1-\frac{r}{a})^2 a}{2} + \frac{h(1-\frac{r}{a})}{2}\sqrt{\frac{a+r}{2\pi}}\frac{1}{\sqrt{T_F^{*'}}} = \frac{F}{(T_F^{*'})^2}. \tag{A.7}$$

Substituting (A.7) into (A.6), we can obtain the following inequality

$$\left(wa + hr + h(1-\frac{r}{a})^2 a + \frac{3h(1-\frac{r}{a})}{4}\sqrt{\frac{a+r}{2\pi}}\frac{1}{\sqrt{T_F^{*'}}}\right)\frac{1}{T_F^{*'}} > 0.$$

Thus, $T_F^{*'}$ is the unique local minimizer.

Moreover, from (A.6) it is obvious that when $T_F < 4\sqrt[3]{\frac{2\pi F^2}{h^2(1-\frac{r}{a})^2(a+r)}}$, $\overline{TC}'(T_F)$ is convex, otherwise, it is concave. Then, $T_F^{*'}$ is also the unique global minimizer. $\square$