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MODELS FOR CHANGE IN QUANTITATIVE VARIABLES, PART I

DETERMINISTIC MODELS*

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This report considers a variety of models for changes to quantitative variables such as wealth, academic achievement, organizational size, intensity of intergroup hostility, etc. By quantitative we mean variables that may take on a continuum of values--usually the real numbers but sometimes only the non-negative real numbers. When this is so, no real interest attaches to any particular level as was true in the qualitative case just discussed. Still, sociologists following Lazarsfeld's lead (see also Davis [1971]) have tended to collapse information on a continuum into a few broad categories, e.g., break the wealth distribution at the median. In recent years, under the influence of econometric methods this tendency has waned. Sociologists are now more prone to use more of the information contained in the distributions of such variables, that is, to analyze the joint distributions of quantitative variables. The so-called structural equation approach has concentrated almost completely on such analysis (see Duncan (1975) for an overview of the principles involved).

There is nothing inherently static in the use of structural equation methods. In fact, in the fields in which they were developed -- biometrics and macro-economics -- they are routinely used to test dynamic hypotheses (though usually in discrete time formulations). Nonetheless, sociological usage of such methods has been almost wholly static. Even when data over time are analyzed, e.g., the pioneering study of status attainment by Blau and Duncan (1967), inferences do not concern the otherwise deterministic model.

It might seem a simple matter to rectify this difference: formulate probabilistic models of change in quantitative variables. But this task is far from simple. The stochastic differential equations that result demand very delicate handling. Even an elementary treatment requires considerable mathematical sophistication. So we find ourselves on the horns of a dilemma. Our interest in synthesizing qualitative and quantitative analysis suggests that we use stochastic differential equations. But the shift to such models introduces a quantum leap in mathematical and statistical complexity. And we cannot guarantee that the additional complexity will pay off in terms of deeper insight into social process. Coleman (1964, 1968) apparently takes the view that it will not; he treats qualitative analysis probabilistically and quantitative analysis deterministically.

One might argue that information about the sizes of changes may compensate for some lack of realism concerning randomness in the process. Moreover, if we keep a deterministic perspective, we can estimate models with widely available tools. In other words we find ourselves in a situation in which the likely costs of retaining a stochastic perspective are high and the convention wisdom holds that the gains are likely to be small. However, we are not convinced that the conventional cost and benefit calculations have much merit. We will argue the case somewhat differently.

The overriding issue concerns logical consistency in the handling of quantitative and qualitative outcomes. Consider studies of changes in socioeconomic status. Sociologists sometimes conceptualize and measure SES as a quantitative variable (see, for example, Blau and Duncan 1967). Other times they think only of ordered status categories (see, for example, Duncan 1979). And surely the two conceptions are related. Suppose there is some underlying status continuum as in Figure 1. Then the discrete state approach involves making cuts at various points on the continuum (say between "lower" blue collar and "upper" blue collar). Then status categories may be considered internes on the status dimension. And we simply name or number these categories and typically study transitions among them (e.g., father to son mobility). In such studies, randomness plays an essential role. Mobility among categories is almost always viewed as a stochastic process.

Suppose one were to make successively finer cuts as in Figure 1b, producing more and more status categories. Certainly if transitions among course categories are governed by a stochastic process, moves among finer categories must also be stochastic. But the limit of this refinement procedure gives the continuous status variable. So by the above argument, transitions from one "level" of SES to another must also be governed by a stochastic process. Nothing in the "disaggregation" of status categories eliminates randomness. Thus as long as we retain the view that transitions among discrete states in social structure are stochastic, it is difficult to avoid the implication that changes in levels in a social structure are also stochastic.

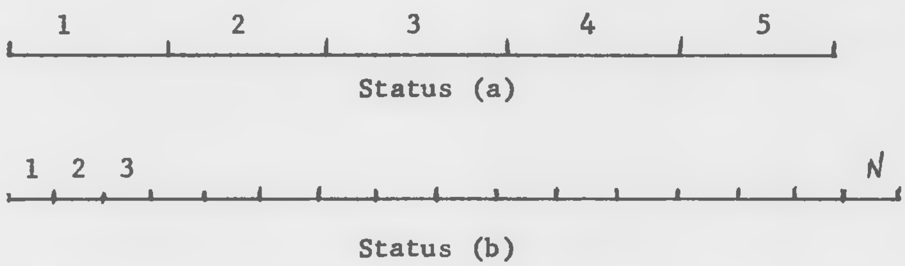

Figure 1. Alternative category systems in a status metric.

This concern for consistency is rather abstract in the context of current sociological practice and appears unlikely to sway opinions. There is at least one circumstance when it may bear directly on practice. We seek to create a framework for building and testing models for _systems_ of qualitative and quantitative outcomes, e.g., changes in marital status and changes in levels of earnings. But how can we defend a model that combines stochastic equations for change in discrete outcomes with deterministic equations for quantitative outcomes? Obviously we cannot.

To this point our argument has the flavor of an exhortation to pursue some difficult and joyless strategy because it is somehow the correct way. But there are positive benefits to be gained from pursuing this line. Foremost among them is possible added leverage in testing certain types of arguments about deep properties of social structure. It is often noted (see, for example, Stinchcombe 1968) that social structure affects the variance of behaviors and outcomes as well as the mean and that some processes may be seen more clearly in variances. Most social scientists find the shape of the income distribution (e.g., inequality) more interesting than its mean. We have argued (Hannan and Freeman 1977) that the evolution of size distributions of organizations tells much about the competitive nature of the niche structure that may not be observed directly. We suspect that it is often the case that theoretically important structural properties that are difficult to observe directly have implications regarding the _distribution_ of some outcome.

It is with regard to these sorts of issues that a focus on randomness pays off. Deterministic models cannot explain distributions (except in the weak sense that given some assumed initial distribution, a deterministic model can explain changes in the distribution). In the case of stochastic models for changes in quantitative variables, the fundamental equations concern the evolution of probability distributions. Thus they provide a natural context in which to pursue the study of distributional properties of social structure.

For these reasons we choose to venture into the hazardous terrain of stochastic models for changes in quantitative variables. But we will keep our discussion at a very elementary level. And, we begin with deterministic models so that we may fix the general strategy in a simpler and more traditional framework.

2. Linear Models for Rates of Change

Sociologists usually model the effects of variables on the levels of other variables. Coleman (1968) proposed that we follow the lead of physical and biological sciences and model effects on rates of change. In this perspective the behavioral or fundamental relations are differential equations. In this section we explore possible sociological interpretations of differential equation models for quantiative variables.

Since we wish to emphasize the relations between dynamic and static models, we direct attention first to dynamic models that imply the usual structural equation models as steady-state outcomes. We start with single equation models. In empirical work, the typical structural model has the form (excluding the disturbance term for the moment):

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_J X_J \quad (1)$$

The "parent" dynamic model is

$$\frac{dY(t)}{dt} = a + bY(t) + c_1X_1(t) + c_2X_2(t) + \ldots + c_JX_J(t), \quad (2)$$

which we can see by setting (2) equal to zero, the condition that holds in equilibrium. This gives:

$$Y_e = -\frac{a}{b} - \frac{c_1}{b} X_1(t) - \frac{c_2}{b} X_2(t) - \ldots - \frac{c_J}{b} X_J(t) \quad (3)$$

Comparing (3) with (1) we see that the parameters of the static model may be thought of as composites of the parameters of an underlying dynamic model in much the manner that reduced-form parameters of a system of structural equations are composites of structural parameters.

As we work extensively with models of the form of (2), it is important to explore the model in some depth. It holds that the rate of change in some outcomes depends linearly on its own level at the same moment and the levels of a set of exogenous variables also at the same moment. We could introduce some explicit lags in these effects.[1] However, as the resulting differential-difference equations are more cumbersome, we will not so as to keep the exposition simple. Although we will pay particular attention to linear models such as (2) because of their tractability, we will also consider below in Section 6 some important non-linear models.

How does one motivate such a model for the study of social process? We will consider two different approaches: negative feedback and partial adjustment. Coleman (1968) motivates linear negative feedback models as follows. It is commonly found in repeated measurements of the same unit that those who were far above (or below) the mean on the first measurement tend to be closer to the mean on the second. Such a result, called regression towards the mean, may be an artifact of random measurement errors (see Lord and Novick, 1966).

But the phenomenon also occurs in situations where measurement accuracy is very high. Is there any more fundamental principle involved? Consider equation (2). If b is positive, any system that begins above the equilibrium level will grow indefinitely; any that begins below equilibrium will decay to zero. That is, systems in which the feedback is positive are unstable. And while many social processes may be unstable, surely some are stable. Stability requires that the feedback be negative. And negative feedback produces regression towards some criterion--perhaps reflected in the mean.

Does negative feedback have any unambiguous sociological interpretation? Coleman offers two related interpretations. First, we may interpret negative feedback as characteristic of equilibrating systems. In particular we may consider it a defining attribute of "functional" systems in which elements of social structure are retained through their beneficial consequences. Stinchcombe (1968) pursues this line of reasoning in depth. Second, we may treat the existence of negative feedback as evidence that we have omitted cycles of causation from the model. That is, negative feedback might be considered the consequence of effects of Y on say W which in turn affects Y. Coleman (1968: 440-1) argues:

> ...the variable acts as a surrogate for all the variables involved in cycles leading back to itself...this approach does not aid much in the development of theory, because it obscures the relationships of which the system is composed.... As the formal system becomes more complete, this [negative feedback] coefficient should approach zero. Thus the size of the coefficient allows a way of evaluating the completeness

of any representation of the empirical system by a system of differential equations.

So we may take negative feedback as either a measure of ignorance or a systemic property of an equilibrating system.

Other researchers offer direct substantive interpretations of negative feedback effects. For example, Sørensen (1977) and Hallinan and Sørensen (1977) focus on the equilibrium relationship, (10.3), and adopt the following input-output imagery. If the X's are the input that persons bring to, say, the status attainment process or the learning process and the c's are fixed, variations in b will affect the outputs associated with any given level of inputs. So for example, if c_1 is the effect of ability on the rate of learning in school, the payoff to ability varies as an inverse function of b. If, moreover, b varies among schools, these variations may be interpreted as structural effects on the opportunities for learning -- those with b close to zero provide the most favorable opportunity structure for learning. In this view, b is interpreted as an index of opportunity, a property of the structure.

One might still argue, with Coleman, that opportunity connotes a set of unanalyzed micro-processes within structures. Our point is not to contend this issue but merely to show that, depending on one's substantive focus, the negative feedback effect may be interpreted positively as an interesting property of social structure. The latter view leads one to study variations from structure to structure. So, for example, Freeman and Hannan (1975) used such an argument to motivate the comparison of negative feedback effects in growth rates for numbers of administrators in growing and declining organizations.

There is a second broad approach to motivating linear differential equation models of social process: partial adjustment models. Suppose that the outcome of interest adjusts each period to the gap between its current level and some criterion. Denote the criterion by $Y^*(t)$. Then full adjustment occurs when:

$$Y(t+\Delta t) - Y(t) = [Y^*(t) - Y(t)]\ \Delta t$$

or, letting $\Delta t \to 0$:

$$\frac{dY(t)}{dt} = Y^*(t) - Y(t)\ .$$

Social systems rarely adjust fully in any short period. So we generalize the adjustment model by introducing a parameter that indicates the fraction of the gap that is closed in each period. This gives the simplest partial adjustment model:

$$\frac{dY(t)}{dt} = k[Y^*(t) - Y(t)] \qquad 0 < k \leq 1 \tag{4}$$

So far the model has two parameters, the adjustment parameter and the criterion, but no causal effects. However, the criterion generally depends on environmental conditions, that is on levels of exogenous variables. That is, in general:

$$Y^*(t) = f(X_1(t), \ldots . X_J(t), t)$$

To obtain a specification that gets us back to (2), assume that this dependence is linear and time-homogenous.

$$Y^*(t) = a^* + c_1^*\, X_1(t) + \ldots c_J^*\, X_J(t) \tag{5}$$

Then by substituting (5) into (4), we obtain

$$\frac{dY(t)}{dt} = k[a^* + c_1^* X_1(t) + \ldots + c_J^* X_J(t) - Y(t)] \qquad (6)$$

$$= a + b\, Y(t) + c_1\, X_1(t) + \ldots + c_J X_{J(t)}$$

where

$$a = -ka^*$$

$$b = -k \qquad (7)$$

$$c_j = -kc_j^* \qquad j = 1, \ldots, J$$

Thus the negative feedback model may also be viewed as a partial adjustment model where the criterion is a linear function of exogenous variables.

In this framework the parameter associated with the dependent variable, earlier called the negative feedback coefficient, has an important substantive meaning. It conveys the <u>speed of adjustment</u> of the system to exogenous changes. When k is close to zero (but positive) the system adjusts very slowly; it moves only a small fraction of the distance to the criterion in Δt. Larger k's imply faster adjustment in the time scale chosen for the analysis (years, days, etc.) We argued in Chapter 3 that speed of adjustment depends on properties of structure, e.g., complexity of internal structure and density of connections with other structures, etc. And one can often gain substantive insight by separating the effects of internal structure from effects of environmental properties on speed of adjustment. Such separation can be achieved by designing research that permits only one dimension (internal or external) to vary and estimating partial adjustment models for various conditions. For example, Nielsen and Hannan (1977) argued that educational organizations would adjust to changes in population and in levels of economic production more rapidly in wealthy nations than in poor nations. A comparison of estimates of k for rich and poor nations confirmed this hypothsis. We also exploited differences in complexity among levels of educational systems, primary, secondary, and university systems, to test for effects of structural complexity on speed of adjustment. Within either generalized environment (rich or poor), the more complex systems adjusted more slowly to exogenous changes that affect the long run levels of enrollments, as we hypothesized. This research, like that of Hallinan and Sørensen (1977) discussed above, gives direct substantive interpretation to the effects of levels of a variable on rates of change in the same variable.

Nielsen (1977) and Rosenfeld and Nielsen (1978) stress an implication of the partial adjustment interpretation of negative feedback. Consider the case in which the exogenous variables are constant over the history of the process, and individuals enter a system at the bottom at some initial time (t = 0) and then rise in the system in a manner that depends on their initial attributes, the X's. For example, we might consider the levels of earnings or status achieved by individuals in some social system in which individuals enter at different levels. Among other things we would be interested in how the parameters of the dynamic model determine the endurance of initial conditions, e.g., point of initial entry. To do this, solve (5) over the period (0,t) to obtain:

$$Y(t) = -a^*(e^{-kt}-1) + e^{-kt}Y(0) - c_1^*X_1(e^{-kt}-1) - \ldots - c_J^*X_J(e^{-kt}-1) \quad (8)$$

$$= e^{-kt}\,Y(0) - [a^* + c_1^*X_1 + \ldots + c_J^*\,X_J]\,(e^{-kt}-1)$$

but the quantity in brackets is just the equilibrium level of Y(t), Y_e. So we can write (8) as

$$Y(t) = e^{-kt}\,Y(0) - (e^{-kt}-1)\,Y_e \quad (9)$$

$$= e^{-kt}\,Y(0) + (1 - e^{-kt})\,Y_e$$

So the level of Y at any time is a weighted average of the starting level and the steady state. The weight given to history, that is to Y(0), goes to zero as $t \to \infty$. But notice that the weight also depends on k, the speed of adjustment parameter. For k close to unity, the effects of history recede quickly. For k close to zero, the effects of history hold over much longer periods.

Consider what this implies for mobility through status structures. If two individuals with identical fixed characteristics enter the opportunity structure at different levels -- due to discrimination, luck, etc. -- this initial difference will persist longer in systems that have high "opportunity" in Sørensen and Hallinan's usage.

Of course most work with partial adjustment models gives priority to the causal effects of exogenous variables. And in the partial adjustment model consideration of such effects requires that we clarify the interpretation of what we have called the criterion, $Y^*(t)$. This is sometimes equated with the equilibrium of the system (see Land 1970; Hummon, Teuter, and Dorien 1975). From (4) it is clear that this interpretation fits the model. That is, setting (4) equal to zero gives $Y(t) = Y^*(t)$ as the equilibrium relationship. Nonetheless.we judge that this interpretation is not helpful more generally. As we see below, both for many systems models and also for many nonlinear single equation models, no equilibrium exists, or it is at least problematic whether or not a system will reach equilibrium. In such cases, it is not useful to conceptualize causal effects in terms of equilibria. The treatment of the single equation case should be consistent with that of systems; therefore, we argue that $Y^*(t)$ in (4) should not be defined as the equilibrium level of Y.

The alternative is to define $Y^*(t)$ as a property of the structure -- more properly of the interaction of the structure with a particular environment. Then the c_j are to be thought of as a set of parameters of the process, not an outcome of the process.

For concreteness, consider the modern formalization of the concept of the niche of a species in some environment. If the reproductive success of some population is constrained by, say, N environmental factors (e.g., climate, food supply, density of various predators and competitors, etc.), then the set of points in this N-dimensional space within which reproductive success exceeds some minimum value is called the niche (Hutchinson 1957). We usually wish some compact representation of the niche and thus formulate functional representations of the dependence of reproductive success -- and thus population growth -- on the levels of environmental factors. Then the parameters that relate levels of environmental variables to fitness or reproductive success are called the parameters of the niche. In the model we outlined, the c_j serve the same role as niche parameters.

And, the $Y^*(t)$ obtained given some realized levels of the set of X_j would be called the carrying capacity of the environment for the particular species. It is important to see that the niche parameters and the carrying capacity are substantively interpretable even in conditions under which the population will not reach the carrying capacity and the system will not hit the equilibrium.

Of course, there are other ways to interpret $Y^*(t)$ without relying on an equilibrium interpretation. One generic approach is to introduce the notion of the goal of a system. If we are considering a formal organization, $Y^*(t)$ may be the objective to which the organization is committed. Alternatively, if we wish to adopt rational utility maximization models, we might define $Y^*(t)$ as the utility maximizing level of Y given preferences and objective constraints (prices, etc.) In either case, we assume that purposeful actors or organizations run by purposeful ruling coalitions will seek to adjust outcomes to close the gap between the objective, Y^*, and reality, Y. Again, we stress that it is meaningful to use this conceptualization even when the objective is unreachable and no equilibrium exists.

10.3 Time Paths of Changes: Integral Equations

In a continuous-time formulation, rates of change are not observable. Thus the differential equations do not have direct empirical implications. To work towards empirical implications we must solve the differential equations subject to some boundary conditions to obtain the more complicated integral equations. The latter describe the time paths of changes in observable quantities implied by the model. So an intermediate step in empirical work always involves obtaining such integral equations.

The solution to the linear differential equation in (2) when the exogenous variables are constant over the entire period of analysis has already been displayed in (8). Here we consider the general case. Let us write the model more generally as

$$\frac{dY(t)}{dt} = a + bY(t) + f(t); \tag{10}$$

and where f(t) is some function of time and the initial condition and $Y(t_0) = Y_0$.

The solution of (10) obtained by integrating from t_0 to t is

$$Y(t) = \frac{a}{b}(e^{b(t-t_0)}-1) + e^{b(t-t_0)}Y_0 + \int_{t_0}^{t} e^{b(s-t_0)}f(s)ds \tag{11}$$

Depending on the functional form chosen for f(t), this equation may be simplified further. For example, in the case in which the causal factor is constant over the period of interest, f(t) = X for all t, then

$$Y(t) = \frac{a}{b}(e^{b\Delta t}-1) + e^{b\Delta t}Y_0 + \frac{c}{b}(e^{b\Delta t}-1)X \tag{12}$$

where we let Δt denote $t-t_0$ as noted earlier. Notice that Y(t) is a linear function of lagged Y and of X, but that the coefficients are complicated functions of the dynamic parameters and of elapsed time. This suggests that we treat (12) as an estimation equation, that is estimate:

$$Y(t) = \beta_0 + \beta_1 Y_0 + \beta_2 X \tag{13}$$

and use estimate of the β's to recover estimates of the dynamic parameters (see Coleman 1968).

This is a good opportunity to demonstrate the advantages of continuous-time models for processes in which there is no inherent lag structure. Only in a continuous-time framework can one meaningfully compare estimates from studies that employ different time lags--due usually to differences in availability of data. For example, suppose one researcher analyzes data on earnings at points spaced one year apart and another researcher uses a data set in which the observations are spaced three years apart. If there is no natural preference for any particular time lag in an analysis of growth in earnings we would want to convert the two analyses into the same metric. Such a conversion would be necessary if we wished to analysis of the factors affecting changes in earnings we would want to convert analyses with three year lags into the same metric as analyses with one year lags. This would be necessary if we wished to contrast the process in the two populations studied. Such comparisons are possible for the model we are considering -- as well as for the remainder of the continuous-time models we consider. Note in comparing (12) and (13) that $\ln\beta_1 = b\Delta t$. So if the same process holds in both systems studied, the natural logarithm of the autoregression term for a three year lag will be three times that for a one year lag. If this ratio does not hold (within some sampling limits presumably), we would conclude that the parameters of the process differ across systems.

Alternatively, we can exploit the relations between (12) and (13) to use data with different lag structures to estimate a single dynamic model. We treat this important problem in Part III.

An important complication in estimating integral equations is that the causal factors of interest are rarely constant over the study period. However, as long as we can represent the time-varying behavior of these factors by some reasonably simple function of time, we can move from (11) to some form suitable for empirical analysis. Coleman (1968) suggests that it is often reasonable to approximate the behavior of the causal variables as changing <u>linearly</u> from $X(t_0)$ to $X(t)$. That is

$$\frac{dX(t)}{dt} = g \; ; \; X(t_0) = X_0$$

or

$$X(t) = X_0 + g(t-t_0)$$

Then the solution of the basic model is slightly more complicated:

$$Y(t) = \frac{a}{b}\left(e^{b\Delta t}-1\right) + e^{b\Delta t}Y(t_0) + \frac{c}{b}\left(e^{b\Delta t}-1\right)X(t_0) + \frac{c}{b}\left(\frac{e^{b\Delta t}-1}{b\Delta t}-1\right)\Delta X(t) \tag{14}$$

where $\Delta X(t) = X(t) - X_0$.

Note again that this model has a general form suitable for regression[2] analysis:

$$Y(t) = \beta_0 + \beta_1 Y(t_0) + \beta_2 X(t_0) + \beta_3 \Delta X(t) \; . \tag{15}$$

4. Linear Systems

Theoretical and empirical work often concerns systems of coupled processes. Consider a two equation model with negative feedback:

$$\frac{dY_1(t)}{dt} = a_1 + b_{11}Y_1(t) + b_{12}\, Y_2(t) + c_1X(t) \tag{16}$$

$$\frac{dY_2(t)}{dt} = a_2 + b_{21}Y_1(t) + b_{22}Y_2(t) + c_2X(t) \tag{17}$$

The only change from the model considered earlier is the presence of what might be called cross-effect or coupling parameters, b_{12} and b_{21}. In this model the level of $Y_1(t)$ affects $\frac{dY_1(t)}{dt}$ both directly (through negative feedback) and indirectly by affecting $\frac{dY_2(t)}{dt}$ and thus $Y_2(t)$, which in turn affects $\frac{dY_1(t)}{dt}$. Consequently, the issue of stability is more complex in such models. It is not enough that feedback be negative as it was in the single equation case. The system in (16) and (17) has a stable equilibrium if and only if the sum of both adjustment parameters is negative and the cycle of f e e d b a c k is larger than the cycle of cross-effects: $b_{11}b_{22} > b_{12}b_{21}$ (see Blalock 1969) for an introductory treatment of stability conditions). If this condition holds changes dampen over time. Otherwise changes are amplified, and the system evolves towards zero or infinity.

The two-equation coupled partial adjustment model is:

$$\frac{dY_1(t)}{dt} = k_1[Y_1^*(t) - Y_1(t)] \quad (18)$$

$$\frac{dY_2(t)}{dt} = k_2[Y_2^*(t) - Y_2(t)] \quad (19)$$

$$Y_1^*(t) = f_1(Y_2(t),X,t) \quad (20)$$

$$Y_2^*(t) = f_2(Y_1(t),X,t) \quad (21)$$

If the dependence of the criterion on observable variables is linear and time-homogenous as we assumed above, i.e.,

$$Y_1^*(t) = a_1^* + b_1^*Y_2(t) + c_1^*X(t) \quad (22)$$

$$Y_2^*(t) = a_2^* + b_2^*Y_1(t) + c_2^*X(t) \quad (23)$$

then by substituting (22) and (23) into the partial adjustment model in (18) and (19) we obtain equations with the same form as the coupled feedback system in (16 and 17).

Again the only difference from the single equation case discussed earlier is the effect of levels of endogenous or dependent variables on the criterion of every other dependent variable. Such effects have straightforward interpretation in a variety of conceptual schemes. Two of the most famous applications of such models in the social sciences are Simon's (1957: Ch. 3) formalization of Homans' (1950) account of small group process and Richardson's (1960) model of arms races. These models have been much discussed in the sociological literature -- see Blalock (1970), for example.

These systems models also fit the types of interpretations we have considered above. Suppose, as mentioned earlier, that the criteria are set by rational utility maximization. Then this model holds that the optional level of investment in some quantity Y_1, say, depends on the current level of investment in Y_2. For example, consider the allocation of time between work in the market and other activities. Let $Y_1(t)$ and $Y_2(t)$ be the hours per week of work of female and male heads of the family. Then the model holds that under some form of utility maximizing, the optimal labor supply of each spouse depends in part on the current labor supply of the other.

Or, suppose Y_1 and Y_2 refer to two goals of some organization (e.g., quality of medical care and quality of scientific production in a university hospital). Then the model holds that the target on each dimension shifts according to current outcomes on the other dimension. Thus even this simple linear model may induce a rather complicated dynamic interdependence among goals and outcomes. Though we suspect that real organizations use even more complex decision-structures, this is a potentially useful starting point for analysis of the behavior of goal seeking structures with multiple goals. This strategy has the particular advantage of leaving goals unmeasured and thus avoids serious methodological difficulties that beset comparative studies of measured deviations from goals (see Hannan and Freeman 1977b).

The situation is more interesting when the model is applied to interacting systems or subsystems. For example, let the Y's denote levels of success (e.g., size of organizations, profits, etc.) of several potentially interacting systems or subsystems such as firms in a market, occupational classes in an organization, etc. Then the b^*'s record the intensity and direction of the consequences of the interactions. The pattern of these coefficients is most important. When b_1^* and b_2^* are both negative, then systems are said to compete; this is the case of pure competition. When both are negative, we refer to the pattern of interaction as mutualism. When one is positive and the other negative we have the sort of relationship that characterizes predator-prey and host-parasite interactions. This latter case typically gives rise to cycles of success. Wilson and Bossert (1971: 129-36) provide a lucid elementary treatment of the dynamics of such interactions. Hannan and Freeman (1978) analyze the interactions of growth in the sizes of personnel components in organizations, interpreted from the perspective of competition theory.

5. Integral Equations for Linear Systems

As before we must integrate over some period (that corresponds to observation times) to obtain an equation with all observable variables. For the system (or multiple equation) case we must employ matrix notation.

Let $y(t)$ be the vector $[(Y_1(t), . . . , Y_N(t)]'$ and A be the N by N matrix whose ijth entry is the effect of $Y_j(t)$ on $dY_i(t)/dt$. Then a general model parallel to that used for the single equation case is

$$\frac{dy(t)}{dt} = A\, y(t) + f(t) \qquad (24)$$

As before we solve the initial value problem with $y(t_0) = y_0$. The solution (see Braun 1975: 484) is

$$y(t) = e^{A(t-t_0)}\, y_0 + \int_{t_0}^{t} e^{A(s-t_0)} f(s)ds \qquad (25)$$

This has the same general form as (11) but now we have to evaluate the anti-log of a matrix: $e^{A\Delta t}$. The quantity is defined as

$$e^{A\Delta t} = I + A\Delta t + \frac{A^2(\Delta t)^2}{2} + \cdot\cdot\cdot \qquad (26)$$

However, only in exceptional cases can $e^{A\Delta t}$ be expressed in closed form.

There is nonetheless a feasible strategy for estimating a system of linear equations in observables and using estimates of parameters to recover estimates of dynamic parameters. For simplicity we consider the case in which there is only one fixed exogenous variable, i.e., f(t) = X. Then the relevant equations in observables are

$$Y_1(t) = \beta_{01} + \beta_{11} Y_1(0) + \ldots + \beta_{1N} Y_N(0) + \gamma_1 X$$
$$\vdots$$
$$Y_N(t) = \beta_{0N} + \beta_{1N} Y_1(0) + \ldots + \beta_{NN} Y_N(0) + \gamma_N X$$

or in matrix form:

$$y(t) = \beta y(0) + \gamma X \tag{27}$$

Now the real problem is to take estimates of β and γ and estimate $\underline{A}$ (estimation of causal effects is straightforward once we get $\underline{A}$). We will only sketch the general strategy here. Readers who have not encountered these materials previously are advised to consult a text on differential equations. We find Braun (1975: Chapter 3) particularly lucid. Bellman (1970: Chapters 10-11) presents in compact form the necessary results for the simple case we consider as well as for less well behaved cases.

Suppose that the endogenous portion of the system

$$\frac{dy(t)}{dt} = Ay(t) \quad ; \tag{28}$$

where $\quad y(t_0) = y_0$,

has distinct roots (N independent solutions). Denote the characteristic roots or eigenvalues of A by $\lambda_1, \ldots, \lambda_N$. Now make a change of variable, Z(t) = Ly(t) where L is a constant nonsingular matrix. Then the equation for Z(t) is

$$\frac{dZ(t)}{dt} = L^{-1}AL\, Z(t)\,; \qquad Z(t_0) = L^{-1}y_0 \tag{29}$$

Our objective is to chose L such that the system of equations will break into N independent equations of the type that we know how to handle. That is we need to find L such that

$$L^{-1}AL = \begin{bmatrix} \mu_1 & & & & \\ & \mu_2 & & 0 & \\ & & \cdot & & \\ & 0 & & \cdot & \\ & & & & \mu_N \end{bmatrix}$$

because then (29) decomposes into N independent equations of the form:

$$\frac{dZ_i(t)}{dt} = \mu_i\, Z_i(t)\,; \qquad i = 1, \ldots, N$$

Each of these equations has solution: $Z_i(t) = e^{\mu_i (t-t_0)} Z_0$.

But we know that $\mu_i = \lambda_i$ since the roots of $L^{-1}AL$ are the same as those of A. It then follows that the columns of L must be the characteristic vectors or eigenvectors of A. It is then easy to show that

$$e^{At} = L \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & 0 & \\ & \cdot & & \\ & 0 & \cdot & \\ & & & e^{\lambda_N t} \end{bmatrix} L^{-1} \tag{30}$$

So the strategy is clear. Comparing (27) with (25) we see that $B = e^{A(t-t_0)}$. So we estimate B and solve for the eigenvalues of B. If they are all distinct, the strategy just outlined goes through. We can calculate the elements on the main diagonal of $e^{A(t-t_0)}$ as $e^{\hat{\lambda}_j(t-t_0)}$ where $\hat{\lambda}_j$ is the jth root of $\hat{B}$. Then by finding the eigenvectors of $\hat{B}$ we can use (30) to solve for the off-diagonal elements.

If the roots are not distinct, we must use a more complex procedure. Braun (1975: 466-7) outlines the procedure by which we can usually form N independent solutions to (28) from $j < N$ distinct eigenvalues. Thus the general strategy may still be applied.

Finally there is the case of complex roots. For each complex root we obtain two solutions to (28). However, as long as A is real, these complex roots must appear in conjugate pairs. In this case we can always construct another fundamental set of solutions to (28), all of which are real-valued. The method is outlined in Boyce and Di Prima (1969: 7.8). Thus, again the general strategy may also be applied to this case, after some manipulation. Readers wishing to handle the more complex possibilities mentioned in previous paragraphs should consult the references cited.

6. Comparisons With Some Widely Used Alternative Models

In this section we contrast the linear models just discussed, particularly the partial adjustment model, with some models that are widely used in the social and biological sciences. Such comparisons afford a deeper understanding of the utility of linear models[3] as well as the need to consider nonlinear generalizations.

We begin with the simplest model for the diffusion of some item (information, a disease bearing organism, a cultural trait, etc.) through a fixed population. Suppose that the item diffuses from a fixed source and that individual carriers cannot transmit it. Then the usual model for the rate of diffusion is (Coleman 1964):

$$\frac{dX(t)}{dt} = v[N - X(t)] \tag{31}$$

where X(t) is the number of carriers at time t and N is the (fixed) size of the population at risk of acquiring the item. The model holds that in each period of fixed length the same fraction v of those still at risk will acquire the item.

This model of diffusion from a source in a fixed population bears a striking similarity to the partial adjustment model. However, the latter is more general in two important respects. First, in the diffusion model the ceiling N is a fixed parameter. In the partial adjustment model, the criterion or target may be treated as a variable affected by environmental parameters and is subject to its own dynamics. Only when the environmental parameters are fixed is the criterion also a fixed parameter in the partial adjustment model. The second difference concerns applicability to decline processes. In the diffusion model, negative growth is not defined; by definition the number acquiring the item cannot exceed the population size. In the partial adjustment model decline is well defined. Environmental variations may drive down the criterion in any period. Then the partial adjustment model implies adjustment down towards the new lower criterion.

While the model of diffusion from a constant source sometimes fits well (Coleman, Katz and Menzel 1966), time paths of diffusion often exhibit an S-shape. That is, the initial rate of diffusion is small, then speeds up at some point, and finally approaches some ceiling asymptotically. A simple process that generates such dynamics can be formed by combining diffusion from a constant source with transmission between individuals (see, for example, Bartholomew 1973: 298-307). To include transmission between individuals in the model under discussion, define w as the intensity of transmission between individuals or the strength of the inter-individual transmission process. At any time t there are N-X(t) individuals who have not yet acquired the item and X(t) who have. Of the N(N-1)/2 pairs of individuals that might be formed, X(t)[N-X(t)] consists of one bearer and one non-bearer. If the pairs form at random in the population, the effect of transmission between individuals on the rate of transmission will be equal to wX(t)[N-X(t)]. Thus a model that combines the two processes has the form:

$$\frac{dX(t)}{dt} = [v + wX(t)][N - X(t)] \tag{32}$$

And this is simply a form of the well known <u>logistic model</u>.

In line with our previous discussion it is natural to generalize this model to the case in which the criterion depends on exogenous variables. This gives a logistic model with

$$\frac{dX(t)}{dt} = [v + wX(t)][X^*(t) - X(t)] \tag{33}$$

The most important thing to notice about this model is the manner in which it generalizes the adjustment process. In the linear partial adjustment model, the speed of adjustment is constant. In the logistic model, it is <u>state-dependent</u>. That is, the speed of adjustment, $v + w(t)$ rises from approximately v when $X(t)$ is very small to $v + wX^*(t)$, as long as w is positive.

It is instructive to build a logistic model from an alternative perspective. A somewhat simpler form of the logistic model is the standard elementary model for the growth of a closed population in a finite environment. The model is motivated as follows (for a fuller discussion, see Lotka 1925; Wilson and Bossert 1971: 16-19, 93-104). Let r denote the so-called intrinsic or natural rate of increase of a population. By definition r equals the difference between the birth and death rates when there are no environmental constraints (i.e., r reflects only physiological constraints). We write this as $r = b_0 - d_0$. In a period of length Δt, the increase (or decrease) in population size is then given by

$$X(t + \Delta t) - X(t) = rX(t)$$

or letting $\Delta t \rightarrow 0$

$$\frac{dX(t)}{dt} = rX(t). \tag{34}$$

That is, the per capita growth rate is constant:

$$\frac{1}{X}\frac{dX(t)}{dt} = r$$

This is just the usual compound interest model that generates exponential population growth. To see this integrate (34) with the initial condition $X(0) = X_0$ to obtain

$$X(t) = e^{rt} X_0 \tag{35}$$

Thus, the population either grows exponentially when r is positive or declines to zero when r is negative.

But when the environment contains finite resources, or the carrying capacity is finite, the population cannot expand exponentially for any extended period. New members of the population must compete with existing members for scarce resources and the rate of reproduction falls below the physiological maximum. For a number of reasons both birth rates and death rates ordinarily depend on the density of the population. More precisely, evolution tends to favor species with density-dependent vital rates. The rate of natural increase r introduced earlier is the difference between the physiological maximum birth rate b_0 and death rate d_0. Let us introduce the simplest form of density dependence. Let the birth rate be $b_0 - k_b X(t)$ and the death rate be $d_0 + k_d X(t)$. That is, the addition of each member of the population decreases the birth rate by k_b and increases the death rate by k_d. The growth model becomes:

$$\frac{dX(t)}{dt} = \left([b_0 - k_b X(t)] - [d_0 + k_d X(t)] \right) X(t)$$

As before, we let $b_0 - d_0 = r$. The steady-state population under this model is

$$\frac{b_0 - d_0}{k_b + k_d} = K$$

usually called the carrying capacity. Letting K denote the carrying capacity, the model may be written in its more common form:

$$\frac{dX(t)}{dt} = rX(t) \left[\frac{K - X(t)}{K} \right] \tag{36}$$

Alternatively, if one does not wish to define model parameters in terms of the steady-state, one may simply postulate the model in (36). The term in brackets varies between zero and one. It is zero when the population size hits the carrying capacity and population growth stops. If the carrying capacity falls below the population size, the term in brackets is negative and the growth rate is consequently negative. When the population is very small, the term in brackets is close to one and population growth is approximately exponential.

Note that the model for logistic population growth may be re-written in the same form as the model for diffusion with inter-individual transmission with $w = r/K$ (and, of course, $v = 0$). Clearly both models contain an element missing from the linear partial adjustment model, namely, interactions among units in the population. Below we consider this difference more thoroughly.

Logistic models may be analyzed by the methodology we propose. As usual we must form an integral equation, solving (36) subject to the initial condition $X(0) = X_0$. This gives:

$$X(t) = \frac{rX_0}{\frac{r}{K} X_0 + (r - \frac{rX_0}{K})\, e^{-r(t-t_0)}} \tag{37}$$

And (37) may be estimated by maximum likelihood, as we show in Chapter 1Ъ.

The logistic growth model differs from the linear partial adjustment model in that it contains the multiplier:

$$\frac{X(t)}{K}$$

Clearly as the population size approaches the carrying capacity the multiplier approaches unity and the two models converge. Thus they imply similar dynamics in the neighborhood of carrying capacities. But when the population is far from the carrying capacity, the growth rate of the logistic model is smaller than in the partial adjustment model (and, by implication, the model of diffusion from a constant source. Nonetheless, the linear and logistic models imply similar dynamics in decline (see Lotka 1925: 68). The relationships are sketched in Figure 2. We see that both models imply negative exponential decline to the carrying capacity of criterion. But the dynamics of growth differ. The logistic model has an S-shaped growth path with maximum rate of growth at K/2. The growth path for the partial adjustment model is concave (from the origin) with maximum rate of growth at the origin. Thus choice between the two models matters most in the study of systems far below their carrying capacities. For such systems, the logistic gives smaller growth rates than the linear -- see Figure 2 .

There is another useful approach to modeling processes that have S-shaped growth paths. Consider again the simple growth model of (34):

$$\frac{dX(t)}{dt} = r\,X(t).$$

We modified this model to obtain the logistic model by making r, the intrinsic rate of increase, dependent on the state of the process. Under some circumstances it may be substantively more meaningful to make r <u>time-dependent</u>. That is, assume that the growth "constant" evolves over the history of the process. One particular form of evolution of the growth "constant" gives analytically tractable results. Suppose the growth rate declines exponentially with time, i.e.

$$\frac{dr}{dt} = -\alpha r \quad ; \quad \alpha > 0 \tag{38}$$

Then, with initial condition $r(0) = r_0$, we have

$$r(t) = r_0\, e^{-\alpha t} \tag{39}$$

and substituting this in the growth model (34) gives

$$\frac{dX(t)}{dt} = r_0^{-\alpha t} X(t) \tag{40}$$

This has solution, with $X(0) = X_0$,

$$X(t) = X_0 \exp\left[\frac{r}{\alpha}0(1 - e^{-\alpha t})\right], \tag{41}$$

the so-called Gompertz growth law. This gives S-shaped growth to the ceiling, $X_0 e^{r_0/\alpha}$ (we see this by letting $t \to \infty$ in (41). However, unlike the logistic model, the process does not have a symmetric S-shape.

We can write the process model (34) and (38) in a form that shows more clearly its relation with models discussed previously. Let us see $\tilde{X}(t)$ to denote the carrying capacity under the Gompertz law, i.e., the population size at which the growth rate is zero. As noted above, $\tilde{X}(0) = X_0 e^{r_0/\alpha}$. Then it follows[4] that the Gompertz law is also the solution of:

$$\frac{dX(t)}{dt} = \alpha X(t) \log\left[\tilde{X}(t)/X(t)\right] \tag{42}$$

That is, it is the usual exponential growth model with a multiplier. When X(t) is small, the multiplier is large and positive. As the population approaches $\tilde{X}$, the multiplier approaches zero. Finally, in this formulation, decline is well defined. If the population exceeds $\tilde{X}$, the multiplier -- and thus the growth rate -- is negative.

When X(t) takes on only positive values and the natural logarithm of X(t) is well defined, we can show the relationship of the Gompertz model to the linear partial adjustment model in still another way. Let $Y(t) = \log X(t)$. Then (42) becomes

$$\frac{de^{X(t)}}{dt} = \alpha e^{Y(t)} \log\, [e^{\tilde{Y}(t)}/e^{Y(t)}]$$

$$e^{Y(t)}\frac{dY}{dt} = \alpha e^{Y(t)}\, [\tilde{Y}(t) - Y(t))]$$

or

$$\frac{dY(t)}{dt} = \alpha\, [\tilde{Y}(t) - Y(t)]$$

So for positive variables, the Gompergz growth law expresses linear partial adjustment in the (natural) logarithmic scale.

So far we have considered three modeling strategies. The first, linear partial adjustment, assumes that adjustment to environmental conditions is independent of both the state of the system and of time (except, of course, as the environmental conditions themselves change over time). The first generalization of this model introduces an elementary form of state-dependence in the adjustment parameter. When the adjustment parameter is made to depend linearly on the state of the system we obtain a logistic growth model. The second generalization introduces time dependence, namely the growth constant is assumed to decline exponentially with time. Presumably this reflects unobserved causal processes. In fitting the Gompertz law to age at first marriage in a cohort, Hernes (1972) assumes that attractiveness as a mate declines exponentially with age. Pitcher, Hamblin and Miller (1978) in modeling the diffusion of violent events assume that the rate at which individuals become inhibited from engaging in violence declines exponentially -- as individuals learn of the costs incurred by those engaging in violence.[5] More generally, the rate at which violent acts are initiated by decline over time in some bounded system either because the technology of repression becomes more effective or because the state concedes the matter under dispute.

On this interpretation, time dependence summarizes the unobserved actions of the state. And it is then preferable to shift towards modeling the response to violence explicitly. This strategy leads to a system conception of the process. One of the main drawbacks of the Gompertz model is the difficulty in generalizing the model to handle systems of interacting units or populations. The logistic does not suffer such limitations. And we now turn attention to the system case for the logistic model.

The simplest possible extension of the logistic model, the so-called Lotka-Volterra equations, forms the basis of almost all theoretical work in population and community ecology. This model introduces interdependence in exactly the same manner as we did above for the linear partial adjustment model: the effect of the sizes of other systems (populations in this case) affects only the carrying capacity for a given system. Formally, let $X = (X_1 \ldots, X_N)'$ be the sizes of N interacting populations. For the ith population, assume that the growth rate has the form:

$$\frac{dX_i(t)}{dt} = r_i \left[\frac{X_i^*(t) - X_i(t)}{K_i(t)} \right] \quad (38)$$

and that the carrying capacity is given by:

$$X_i^*(t) = K_i(t) - \alpha_{i1} X_1(t) + \ldots + \alpha_{i,i-1} X_{i-1}(t) + \alpha_{i,i+1} X_{i+1}(t) + \ldots + \alpha_{iN} X_N(t). \quad (39)$$

Though this may appear a simple generalization, it is not. The system of equations is known to have a solution, but the solution has not been found, even for the case N = 2. Nevertheless we can derive a number of interesting and important qualitative conclusions from this model. Possible sociological applications of these qualitative results are explored in Hannan and Freeman (1977a) and Hannan (1979). However, we cannot employ the general empirical analysis strategy outlined to this point. Since we cannot write a closed solution to even a small Lotka-Volterra system, we cannot write direct estimation equations. Instead we show approximate the system with more tractable equations. We choose to begin with the linear partial adjustment model as an approximation since it may be analyzed by available methods. As we noted above, the approximation is reasonably good when systems are not very far below carrying capacity.

The foregoing analysis suggests that there is much merit in pursuing applications of linear charge models. Not only do linear models fit some general sociological perspectives, they also may approximate some interesting classes of nonlinear charge models. With this motivation, we henceforth restrict attention largely to linear models.

7. Conclusion

We have suggested that the linear structural equation systems so often analyzed by sociologists may profitably be viewed as steady state outcomes of continuous-time change models. Moreover, temporal analysis of systems out of equilibrium to estimate parameters of such change models affords deeper sociological insight into social structural processes than is given by conventional static structural equation analysis. For example, it permits separation of the effects of environmental variations on outcomes from the effects of internal structural arrangements on the speed of adjustment. More generally it permits us to relax or discard the assumption that social systems operate close to equilibria.

We concentrated on linear differential equation models as they give rise to simple estimation equations. We showed that such models have rich sociological grounding. In particular we reviewed two interpretations of such models, negative feedback and partial adjustment.

We then addressed the so-called system case, models for changes in levels of several interdependent variables. The "solutions" of such systems cannot be expressed in closed form. However, as long as the matrix of coefficients of the endogenous part of the system are distinct, we can form estimators of the parameters of the change model. The approach we outlined involves solving the characteristic equations and obtaining eigenvalues and eigenvectors of the endogenous portion of the system. We use this approach repeatedly in subsequent chapters.

We argue that sociologists not confine their attention to linear models for the study of change. And we treat the common non-linear generalizations of the negative feedback or partial adjustment models. In particular, we showed that the typical S-shaped path of changes in levels may be obtained by either the logistic model or the Gompertz model. The first generalizes the linear model by introducing state-dependence in parameters. The Gompertz model introduces a simple form of time-dependence in the parameters. Thus these two simple generalizations suggest a range of strategies for extending the simple models that occupy us in most of the remaining chapters. However, even these simple complications give very unwieldy integral equations that make estimation more difficult. In fact, the widely analyzed generalization of the logistic to the system case does not even have a known solution. Thus it cannot be estimated directly. More complex approxmation strategies, beyond the scope of this report, must be used to obtain empirical estimates of such systems. It is for this reason, and not because we think that linear models are natural, that we focus so much attention on the linear case.

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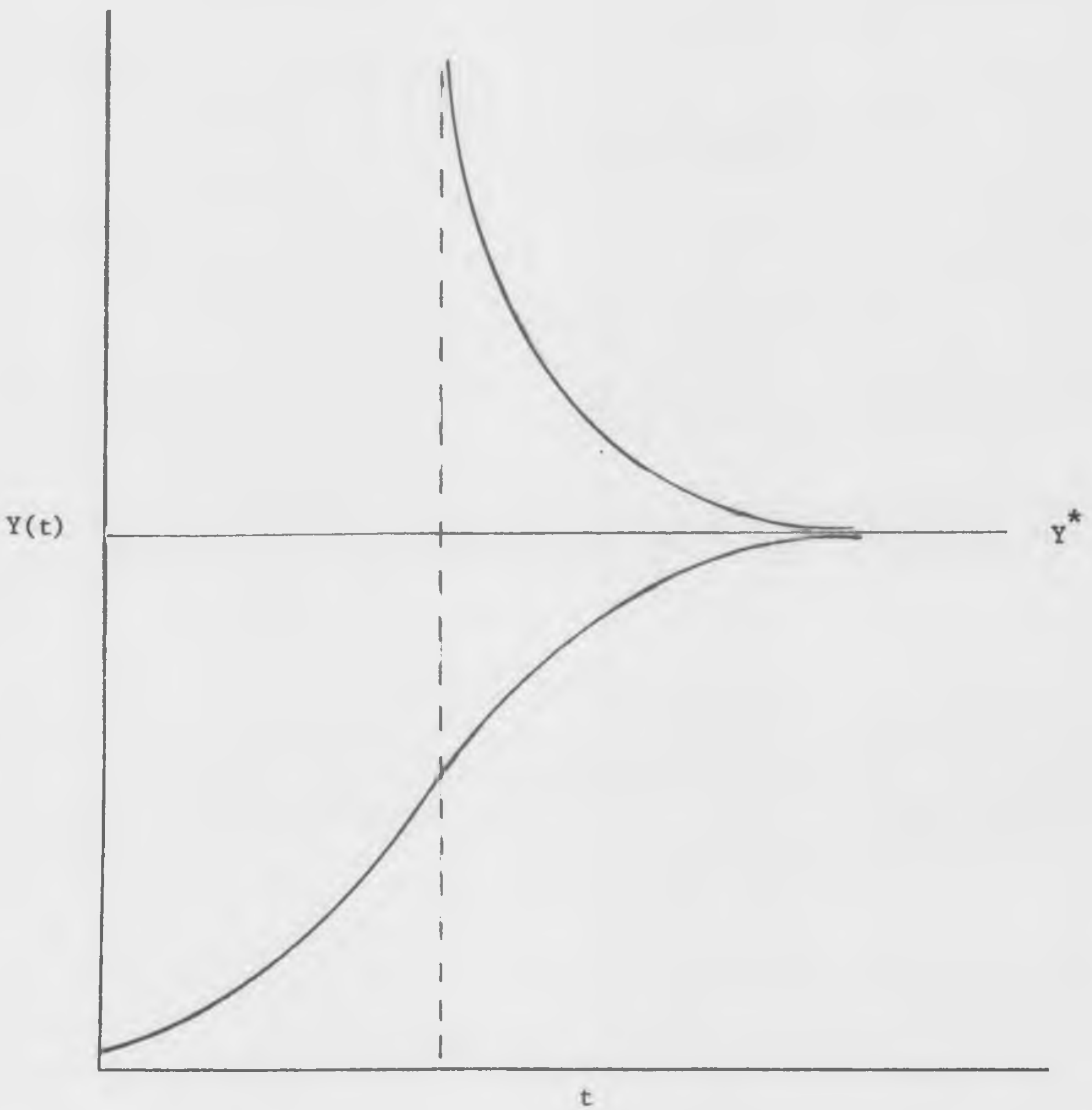

Figure 2. Logistic growth and decline curves illustrated (adopted from Lotka (1956:68).