OPTIMAL ALLOCATION OF INVENTORY AND DEMAND FOR MANAGING SUPPLY CHAIN REVENUES

A Dissertation

by

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ABSTRACT

This dissertation focuses on three distinct yet related problems that are motivated by practices of electronics manufacturers, who satisfy stochastic demand from multiple markets and multisource parts from several suppliers. The first problem investigates joint replenishment and allocation decisions for a supplier who satisfies stochastic demand from a primary market and a spot market. We formulate the problem as a multi-period stochastic dynamic program and show that the optimal policy is characterized by two quantities: the critical produce-up-to level and the critical retain-up-to level. We establish bounds for these two quantities, discuss their economic interpretation, and use them to construct a new and effective heuristic policy. We identify two practical benchmark policies and establish thresholds on the unit revenue earned from the spot market such that one of the two benchmark policies is optimal. Using a computational study, we quantify the benefits of the optimal policy relative to the benchmark policies and examine the effects of demand correlation.

The second problem investigates an important extension where a supplier faces stochastic demand from Class 1 along with price-sensitive stochastic demand from Class 2. We investigate the supplier's joint replenishment, allocation and pricing problem by formulating it as a multi-period, two-stage stochastic dynamic program. We show that a dynamic pricing policy is optimal at stage 2, and the stage 1 optimal policy is characterized by two quantities: the critical produce-up-to level and the critical amount of inventory to be protected from Class 1. In contrast to the optimal policy, myopic policies are less costly to evaluate, and hence, are more practical. We establish two sufficient conditions under which a myopic joint inventory and pricing policy is optimal. Using a computational study, we show that the benefits of dynamic pricing to Class 2 are higher than the benefits of discretionary sales to Class 1.

While the first two problems consider a supplier's decision under stochastic demand from multiple markets, the third problem considers decisions of a buyer who satisfies stochastic demand by multi-sourcing parts with percentage supply allocations (PSAs). We define PSA as a pre-negotiated percentage of a multi-sourced part's total demand that the buyer should allocate to a supplier. During a recent industry collaboration, we observed that in such settings the buyer's demand allocation decisions are challenging due to operational changes needed for (temporarily) switching suppliers, and lead to the bullwhip effect. Demand allocation policies that can meet PSAs and the resulting bullwhip effect have not been investigated in the literature before. We contribute to the existing literature by introducing and analyzing the concept of *bullwhip effect under multi-sourcing*. In addition, we propose and investigate three demand allocation policies: (i) random allocation policy (RAP), which benchmarks the current practice, (ii) time-based (CCP-T) and (iii) quantity-based cyclic consumption (CCP-Q) policies. We show that while RAP and CCP-T always lead to bullwhip effect, the bullwhip ratio under CCP-Q can be less than 1. We demonstrate that CCP-T and CCP-Q can reduce the supplier's bullwhip effect without increasing the buyer's expected long-run average number of supplier switches compared to RAP.

DEDICATION

To my family

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TABLE OF CONTENTS

LIST OF FIGURES

FIGURE Page

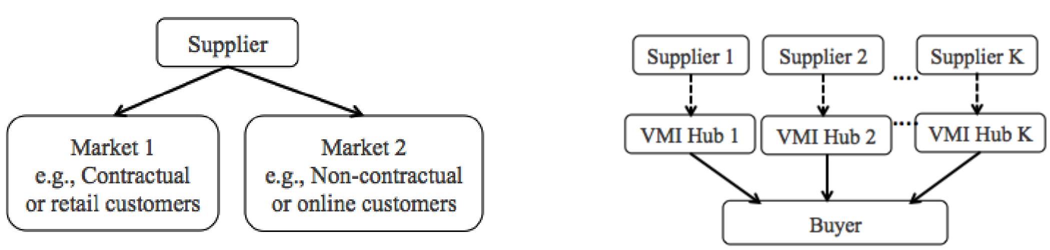
LIST OF TABLES

CHAPTER I

INTRODUCTION AND SIGNIFICANCE

This dissertation is motivated by practices in the electronics industry. In particular, on the outbound side, manufacturers satisfy stochastic demand from multiple markets. Similarly, on the inbound side, manufacturers multisource parts from several suppliers. Accordingly, the objective of this dissertation is to develop quantitative models to investigate optimal inventory and demand allocation decisions in the presence of multiple markets or suppliers as illustrated in Figure 1. In particular, this dissertation focuses on three distinct yet related problems. The first problem investigates joint replenishment and liquidation decisions for a supplier who satisfies demand from a contractual and a spot market. The second problem extends this to investigate a supplier's joint replenishment, allocation and pricing decisions under two markets. The third problem investigates a buyer's demand allocation decisions under multi-sourcing and their impact on the bullwhip effect.

Customer segmentation is the division of a supplier's market into different groups of customers such that customers in each group share similar characteristics and preferences. Suppliers may differentiate their customers based on price, fulfillment priority, contractual agreements, shortage costs and time of occurrence of demand. In particular, recently, several online sales platforms have been developed for a variety of products ranging from consumer



(a) Supplier with multiple markets (b) Buyer with multiple suppliers

Figure 1: Graphical representation of different problem settings

electronics to industrial equipment and jewelry. Various large scale suppliers frequently use such platforms, along with discount stores and foreign distributors to sell excess inventory. For example, computer hard drives are produced and sold by several suppliers (e.g., Samsung, Seagate, and Western Digital) whose prominent customers include powerful manufacturers (e.g., Apple, Dell, and HP). Each supplier may sign a long-term (typically 6 to 12 months) contract with one or more customers (manufacturers) and may also sell their products in online markets. It is estimated that, online trading accounts for upto one third of all procurement in the electronic component industry benefiting more than 80% of the suppliers, original equipment manufacturers (OEMs) and contract manufactures [2]. In fact, as noted in the recent literature, suppliers with traditional contractual or retail customers attain significant benefits from online markets [32]. More specifically, this provides the supplier opportunities to save on holding costs, better utilize the production capacity, and practice dynamic pricing. These opportunities translate into operational flexibility while improving profits and the supplier's ability to manage risk due to demand uncertainty.

Recognizing the potential benefits and increased relevance of such practices, the first problem in this dissertation considers a capacitated supplier facing stochastic demand from two markets: (i) a primary market with high priority contractual customers, and (ii) a secondary spot market with non-contractual customers. To maximize the expected net profit, the supplier has to make two decisions in each period: (i) how much to produce, and (ii) if there are excess units left after satisfying the primary market demand, how many of these to sell in the spot market (i.e., how much inventory to retain for the next period). We build on and contribute to the literature that focuses on production and allocation decisions under stochastic demand (e.g., [4, 31, 30, 26]). Duran et al. [31, 30] consider a similar setting as ours and assume that customers from one [30] or both [31] markets are willing to wait. As rightly noted by Duran et al. [31], customers' willingness to wait or not changes the form of the model, and results in [31] do not follow from the analysis in [30]. We extend these models by considering that unmet demand from both markets is lost.

We model this problem as a finite horizon, multi-period stochastic dynamic program. We show that the optimal policy is characterized by two quantities: the critical produce-up-to level and the critical retain-up-to level. We establish bounds for the two critical quantities, discuss their economic interpretation, and use them to construct a new and effective heuristic policy. We also identify two practical benchmark policies and establish thresholds on the unit revenue earned from the spot market such that one of the two benchmark policies is optimal. We provide closed form expressions to determine these thresholds for the infinite horizon problem under specific conditions on the available production capacity. In general, it is difficult, if not impossible, to theoretically determine these thresholds in closed form for the finite horizon problem. Hence, we report results of a computational study to gain insights regarding the behavior of the optimal policy with respect to the spot market revenue. Our computational results also quantify the benefits of the optimal policy relative to the benchmark policies and examine the effects of demand correlation.

The second problem in this dissertation investigates an important extension of the first problem to further improve supplier's operational flexibility and profits by dynamic pricing. Specifically, we consider a supplier facing stochastic demand from two customer classes: (i) Class 1 customers are charged an exogenously determined fixed price in each period, and (ii) customers in Class 2 with stochastic price-sensitive demand can be charged a different price in each period. To maximize the expected profit, the supplier has to make three decisions in each period: (i) how much to produce, (ii) how many units to protect from Class 1 to be able to satisfy a potentially higher profit demand from Class 2, and (iii) what price to charge to Class 2 customers? Thus, we extend the literature that focuses on production and allocation decisions under stochastic demand (e.g., [4, 31, 30, 26]) by considering the additional pricing decision. In addition, we build and contribute to the literature that focuses on simultaneous determination of optimal inventory and pricing decisions in a periodic review setting (e.g., [13, 24, 33, 46, 51, 59, 81]). We classify this literature as single demand Class models [24, 33, 51, 59], and multiple demand Class models [13, 46, 81]. The multiple demand Class models in [13, 46, 81] consider a single period problem without inventory allocation decisions.

For modeling purposes, we divide each period into two stages: Stage 1 starts at the beginning of the period and ends (and stage 2 starts) after the supplier observes and satisfies Class 1 demand. Stage 2 ends at the end of the period. We formulate the problem as a finite horizon, multi-period, two-stage stochastic dynamic program. We show that the optimal price charged to Class 2 customers is a function of the left-over inventory after satisfying Class 1 demand, i.e., a dynamic pricing policy is optimal. Furthermore, the stage 1 optimal policy is characterized by two quantities: the critical produce-up-to level and the critical amount of inventory to be protected from Class 1. That is, a discretionary sales policy is optimal for Class 1. Since there are three decisions for each period, and the optimal pricing policy is state-dependent, computing the optimal policy is not straightforward. This may make it less attractive from implementation perspective. In contrast, myopic policies are less costly to evaluate, and hence, are readily accepted by practitioners. A myopic policy makes decisions in each period by isolating it from the future periods. We establish two sufficient conditions under which a myopic joint inventory and pricing policy is optimal. For example, these conditions are satisfied when all cost and demand parameters are stationary and any units remaining at the end of the planning horizon are salvaged at a value equal to the unit production cost. Thus, we generalize the pure inventory models, which investigate the optimality of myopic replenishment policies in presence of a single demand Class [5, 14, 43, 47, 53, 58, 67, 68, 77]. We conduct a computational study and quantify the benefits of discretionary sales for Class 1 and dynamic pricing for Class 2.

Similar to the way suppliers benefit from selling to multiple markets, buyers can also benefit from multi-sourcing and supplier diversification. In fact multi-sourcing with percentage supply allocations (PSAs) is common across all industries including electronics, health care, supermarkets and retail supply chains [9, 48, 76]. We define PSA as a pre-negotiated percentage of the multi-sourced part's total demand that the buyer should allocate to a supplier in order to get discounts (commonly called as market share discounts [8, 57]) or avoid penalties, depending on the specific agreement. In the electronics industry a contract with PSAs is usually implemented along with vendor managed inventory (VMI) programs in the form of commitments, which are binding on the buyer and suppliers.

Accordingly, the third problem in this dissertation considers a buyer (e.g., electronics manufacturer), who faces stochastic demand for its end-product, which is an assembly of several parts. We consider a specific part that is multi-sourced via a VMI program with PSAs. An important decision that the buyer makes under multi-sourcing is how to allocate demand (or orders) among various suppliers. We address the buyer's demand allocation decision for the multi-sourced part with the objective to meet the PSAs for each supplier. The challenge is that demand is stochastic, and at any given time, inventory from only one supplier can be used and switching from one supplier to another is expensive [48]. During a recent industry collaboration with a major computer manufacturer, we observed that the demand allocation policy used in current practice creates two main challenges: (i) it may not be effective in meeting PSAs, (ii) it can significantly increase the variability of the demand observed by the suppliers, leading to the bullwhip effect [48]. Bullwhip effect is the amplification of demand variability as customer orders travel upstream of the supply chain and leads to supply chain inefficiencies [10, 12, 52].

In multi-sourcing settings, buyer's demand allocation policies that can meet PSAs and the resulting bullwhip effect are not addressed in the existing literature. To fill these gaps, we propose and investigate three practical policies: (i) random allocation policy (RAP), which benchmarks the current practice, (ii) time-based cyclic consumption policy (CCP-T) and (iii) quantity-based cyclic consumption policy (CCP-Q). We compare their performance based on (i) long-run fraction of total demand allocated to each supplier, (ii) buyer's expected long-run average number of supplier switches, and (iii) supplier's bullwhip effect under multi-sourcing. We contribute to the current literature by demonstrating the existence of bullwhip effect caused due to demand allocation policies under multi-sourcing. We term it as the *bullwhip effect under multi-sourcing* and emphasize its absence in single-sourcing systems. We show that while RAP and CCP-T always lead to bullwhip effect, the bullwhip ratio under CCP-Q can be less than 1. Our results offer new insights that substantiate the exclusionary, and hence, anti-competitive effects of a contract with PSAs under RAP. We demonstrate, analytically where possible and numerically if not, that CCP-T and CCP-Q can reduce the supplier's bullwhip effect without increasing the buyer's expected long-run average number of supplier switches compared to RAP. Furthermore, when negotiating a contract with the buyer, suppliers will find our results valuable to carefully select the service levels that they commit to provide based on the agreed upon PSA.

The remainder of the dissertation is organized as follows. In Chapter II, we introduce the first problem, discuss the relevant literature and present our results and findings. In Chapter III, we introduce the second problem, discuss the relevant literature and present our results and findings. In Chapter IV, we introduce the third problem, discuss the relevant literature and present our results and findings. We make concluding remarks and summarize the contributions of this dissertation in Chapter V.

CHAPTER II

JOINT REPLENISHMENT AND LIQUIDATION DECISIONS UNDER CONTRACTUAL AND SPOT MARKETS

II.1 Introduction

Recently, several web-based spot markets have been developed for a variety of products ranging from consumer electronics to industrial equipment and jewelry. Various large scale suppliers frequently use such platforms, along with discount stores and foreign distributors, for inventory liquidation. A prominent spot-market example includes Broker Forum (www.brokerforum.com), a Business-to-Business (B2B) spot market enabling online transactions for the sale of consumer electronics components.

For example, computer hard drives are produced and sold by several suppliers (e.g., Samsung, Seagate, and Western Digital) whose prominent customers include powerful manufacturers (e.g., Dell, HP, and Toshiba). Each supplier may sign a long-term (typically 6 to 12 months) contract with one or more customers (manufacturers) and also may trade in online spot markets like Broker Forum. In fact, as noted in the recent literature, suppliers with contractual customers attain significant benefits from a spot market [32]. Such a market provides the supplier opportunities to (i) liquidate excess inventory to save on holding costs; and (ii) utilize excess production capacity to earn higher profits, if available. These opportunities may translate into operational flexibility while improving profits and the supplier's ability to manage risk due to demand uncertainty.

In this chapter, we consider such a supplier (e.g., Samsung) that satisfies demands from contractual and spot markets during a finite planning horizon. On the supply side, contractual customers (e.g., make-to-order manufacturers like Dell) guarantee a market for the supplier's products. On the demand side, high volume buyers are guaranteed high priority deliveries at an agreed upon price under the contractual agreement, long before the actual demands are realized. Hence, we consider the case where demand from contractual customers is stochastic. There is additional stochastic demand from customers in a secondary spot market. Here, we use the term spot market broadly to include all customers who do not have a commitment from the supplier. Unsatisfied demand from the contractual customers incurs a lost sales penalty. Based on the contract, supplier's unit revenue from contractual customers and the lost sales penalty are fixed and constant for the entire planning horizon. In contrast, unit revenue from the spot market may vary from one period to another and there is no penalty for unsatisfied demand. To maximize the expected net profit, the supplier has to make two decisions in each period: (i) how much to produce, and (ii) if there are excess units left after satisfying the primary market demand, how many of these to liquidate in the spot market (i.e., how much inventory to retain for the next period).

Clearly, the supplier may manage such an inventory system by employing simple policies. For example, if the unit revenue from the spot market is high enough, the supplier may prefer to liquidate the entire left-over inventory, i.e., the excess stock after satisfying the demand from the contractual customers . We term this approach the *retain-none* policy. However, if the unit revenue from the spot market is too low, the supplier may choose to completely ignore it. The supplier may then choose to follow a *retain-all* policy and carry the entire left-over inventory to the next period. We refer to these two policies as *benchmark* policies. The benchmark policies are simple and easy to implement, and, in our experience, they are encountered in current practice. Clearly, the real question is not whether to liquidate or retain the entire left-over inventory, but how much to produce and to retain?

We model the problem as a finite horizon, multi-period stochastic dynamic program. We show that the optimal policy is a combination of a modified base-stock and a retain-up-to policy. In other words, the optimal policy is characterized by two quantities: the critical produce-up-to level and the critical retain-up-to level. We term the optimal inventory to be carried to the next period as the *optimal retain-up-to level.* Thus, even if selling in the spot market is profitable (i.e., unit revenue from the spot market is larger than the unit production cost), it may not be optimal to liquidate the entire left-over inventory. Instead, if one reserves an inventory equal to the optimal retain-up-to level, in anticipation of future high-priority or high-revenue demand, higher profits can be earned. In addition, the profit can further be increased by considering the replenishment (i.e., production) and liquidation decisions jointly. We establish bounds for the two critical levels of the optimal policy parameters for the general problem and two of its special cases. Furthermore, we discuss the economic interpretation of these bounds, and use them to construct a heuristic policy.

We study the impact of the unit revenue earned from the spot market on the optimal policy. We show that there are (lower and upper) thresholds on the unit revenue earned from the spot market such that one of the two benchmark policies is optimal. We provide closed form expressions to determine these thresholds for the infinite horizon problem under specific conditions on the available production capacity. In general, it is difficult, if not impossible, to theoretically determine these thresholds in closed form for the finite horizon problem. Hence, in this case, we rely on a detailed computational study to gain insights regarding the behavior of the optimal policy with respect to the spot market revenue. We investigate the performance of our heuristic policy and show that the optimality gap is less than 2% for more than 98% of the problem instances. Based on our numerical results, we also quantify the economic benefits of the optimal policy as opposed to the benchmark policies and examine the effects of demand correlation. We show that economic benefits of the optimal policy over each of the benchmark policies are significant, more than 5%, for some problem instances. Finally, we discuss interesting insights of the problem from a managerial perspective. For example, based on our computational study, we infer that if specific conditions on model parameters are satisfied, then one of the benchmark policies (retain-none and retain-all) is optimal. This leads to a better understanding of the benchmark policies, which are easier to implement.

In summary, along with analytically addressing the operational questions regarding (i) how much to produce and (ii) how much to retain, our numerical results also offer answers to the following key practical questions:

(iii) Under what conditions are the benchmark policies optimal? The results here will indicate when the simpler–yet, practical–benchmark policies that either completely ignore or focus on the spot market sales are optimal, and justify their practical value.

(iv) What is the value of the optimal policy relative to the benchmark policies? These results will quantify benefits of the optimal policy and justify its implementation in comparison to the benchmark policies.

(v) What is the effect of correlation between the primary and spot market demands in each period on the benefits of the optimal policy? The primary and spot market demand may be correlated, and hence, it is important to find how correlation affects the value of the optimal policy.

Furthermore, we analytically investigate two relevant and key questions:

(vi) What is the structure of the optimal policy if demand from the primary market is backlogged, rather than lost?

(vii) Is it always preferable to completely satisfy the primary market demand? This is important to determine and compare the value of the contractual customers in the primary market and the spot market.

The remainder of this chapter is organized as follows: In Section II.2, we provide a brief summary of the related literature and the contribution of this study. We present the notation and stochastic dynamic program formulation of the problem in Section II.3. In Section II.4, we characterize the structure of the optimal policy for a finite planning horizon and establish bounds associated with the optimal policy parameters. In Section II.4, we also study the impact of the unit revenue earned from the spot market on the optimal policy. Section II.5 examines the infinite horizon problem with stationary model parameters and demand distributions. Section II.6 presents insightful results of our computational study. Section II.7 studies the case where the primary market demand is backlogged. Section II.8 studies the case where all cost and revenue parameters are non-stationary, i.e., the unit revenue and lost sales penalty from the primary market are also non-stationary. This is followed by concluding remarks in Section II.9.

II.2 Related Literature

First and foremost, this research is related to the capacitated inventory models which investigate production/inventory decisions only under stochastic demand from a single customer class. The modeling approach followed by these papers is similar to the modeling approach that we use. Notable works include [6, 34, 35]. We generalize the existing work in this area by considering multiple markets, i.e., contractual and spot markets, with stochastic demand while modeling capacitated replenishment and liquidation decisions, simultaneously.

This research is also related to two streams of literature on stochastic production and inventory models under periodic review: The first stream places emphasis on the disposal of excess stock, which, in a sense, is similar to selling in the spot market in our model. The second stream focuses on production and/or inventory rationing decisions.

Majority of the first stream focuses on the determination of optimal disposal, or equivalently, optimal retention quantity, and, unlike our model, production ordering decisions are not considered explicitly. Hence, we refer to this stream of literature as *pure disposal* models [15, 41, 63, 66]. Based on whether excess inventory can be disposed of once or several times over a multi-period planning horizon, the pure disposal models can further be classified as single-disposal [15] and multi-disposal [41, 63, 66] models, and all of them assume that one can dispose of or order as many units as desired. In contrast, we consider a multi-disposal model with stochastic demand in the spot market and also investigate the production decisions under limited capacity. Fukuda also [39] considers a multi-period model such that the decision to be made at the beginning of each period is one of the following: order, dispose, or do nothing. However, unlike our model, one can dispose of or order as many units as desired under Fukuda's model, too.

The retain-up-to level in our model specifies how much demand to satisfy from the spot market, i.e., how to ration inventory to the spot market. Thus, the second stream of related literature investigates: (i) how much to produce, and/or (ii) how much demand to satisfy from each customer class [4, 22, 26, 31, 30, 36, 38, 60, 65, 69, 75, 78, 82]. We refer to these decisions as production and rationing decisions, respectively. We divide this stream of literature in three categories: (i) papers that study only production decisions [69, 78, 82], (ii) papers that study only rationing decisions (see Kleijn and Dekker [50] for a review), and (iii) papers that study production and rationing decisions jointly. The papers that study production and rationing decisions jointly can be further classified as single [65] or multiple demand class models [4, 22, 26, 30, 31, 36, 38, 60, 75]. The multiple demand class models can be further divided as single procurement [60], multi-procurement models with unlimited production capacity [22, 36, 38, 75] and multi-procurement models with limited production capacity [4, 31, 30, 26].

We also consider a multi-procurement model with limited production capacity and discuss such models, i.e., [4, 31, 30, 26], in detail. Araman and Özer [4] consider selling to a long-term channel along with selling/buying in a spot market with unlimited demand/supply. They consider that the production capacity for the entire planning horizon is limited and spot market decisions are made before the long-term channel demand is observed. In contrast, we consider only selling in the spot market with stochastic demand and limited production capacity in each period. Furthermore, in our model, spot market decisions are made after the long-term channel demand is satisfied. This change in the sequence of decisions and the stochastic (rather than unlimited) demand in the spot market change the model significantly, and results from [4] cannot be directly applied.

Duran et al. [30] consider two customer classes with independent demands and the classes are differentiated by their willingness to pay a premium price in order to have higher priority on resources (inventory and capacity). They assume that unless rejected, customers from both classes are willing to wait and show that policies of the (S, R, B) form are optimal, where S is the order up-to level, R is the reserve up-to amount to be protected from selling to current customers, and B is the backlog up-to amount. Duran et al. [31] extend this result to models where the high priority customers are willing to pay a premium price but are not willing to wait. As rightly noted by Duran et al. [31], customers' willingness to wait or not changes the form of the model, and results in [31] do not follow from the analysis in [30]. Thus, a natural question that follows is: What is the structure of the optimal policy when customers from neither class are willing to wait? We address this question in this chapter. Furthermore, differentiating customers as contractual (primary market) or non-contractual (spot market) allows us to extend the analysis in [31, 30] to settings where customers from the spot market may be willing to pay more than those from the primary market. We also allow demands from the two customer classes to be correlated. Both [31, 30] allow some units to be protected from the high priority customers and focus on the ability to differentiate between different classes. Although in our main model we do not allow discretionary sales to the primary market, we study it as an extension to our model. We also focus on the ability to increase the expected profit by making the production and liquidation decisions jointly.

Vericourt et al. [26] develop a multiple demand class queuing-based model to decide when to produce and whether the arriving demand should be satisfied or back-ordered. They study a infinite horizon problem with stationary cost and demand parameters. In contrast, we study a periodic review system with lost sales and non-stationary cost and demand parameters.

Finally, we note a more recent line of research studying the buyer's problem of procuring inventory from multiple sources, e.g., contractual supplier and spot market [40, 55]. We complement to this line of research by studying a supplier's problem of selling in multiple markets.

In summary, relative to the existing literature, we model stochastic demands in both contractual and spot markets explicitly under limited production capacity. We show that the optimal policy is a combination of a modified base-stock and a retain-up-to policy. Furthermore, we establish bounds on the critical policy parameters. Our practical contribution is that we identify alternate benchmark policies and show that there are (lower and upper) thresholds on the unit revenue earned from the spot market such that one of the alternate benchmark policies is optimal. Based on our computational study, we quantify the benefits of using the optimal policy over the benchmark policies and show that the potential savings are significant.

II.3 Model Formulation

We consider a supplier that sells a single product in two markets as illustrated in Figure 2. The selling horizon consists of T periods and each period has a finite production capacity of C_t units for $t = 1, \ldots, T$. At the beginning of period t, the supplier decides the number of units to be produced, q_t, based on the current on-hand inventory, I_t. We denote the produce-up-to level by $y_t = q_t + I_t$. Production cost is proportional to the quantity produced and c_t denotes the unit production cost in period t. We assume that the production is instantaneous. After production is completed, the stochastic demand from the primary market, X_{1t}, is realized and one of the following cases arises: If demand from the primary market exceeds the produce-up-to level, y_t, then excess demand is lost and a unit penalty of b is incurred in period t. On the other hand, if demand from the primary market in period t is less than y_t, then the remaining on-hand inventory can be liquidated by selling in the spot market. We denote the stochastic demand from the spot market in period t by X_{2t}. Hence, the second decision that the supplier makes in each period is how much to sell in the spot market, or equivalently, how much inventory to carry to the next period.

We term the maximum amount of inventory to be carried to the next period as the retain-up-to level and denote it by z_t in period t. That is, after sales to the primary market, any inventory in excess of the retain-up-to level, z_t, is available for sales in the spot market. There is no penalty for unsatisfied demand from the spot market. Thus, in our model, inventory is replenished immediately when needed, protected when scarce, and liquidated

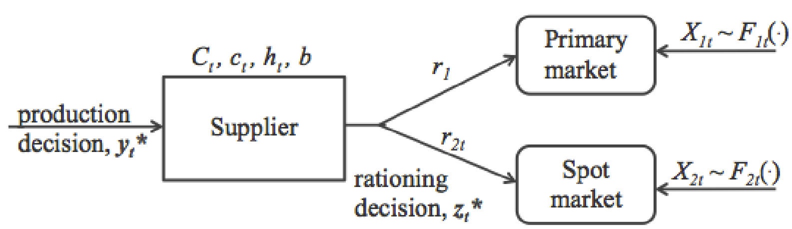


Figure 2: Graphical representation of the problem setting with a supplier, a primary market and a spot market

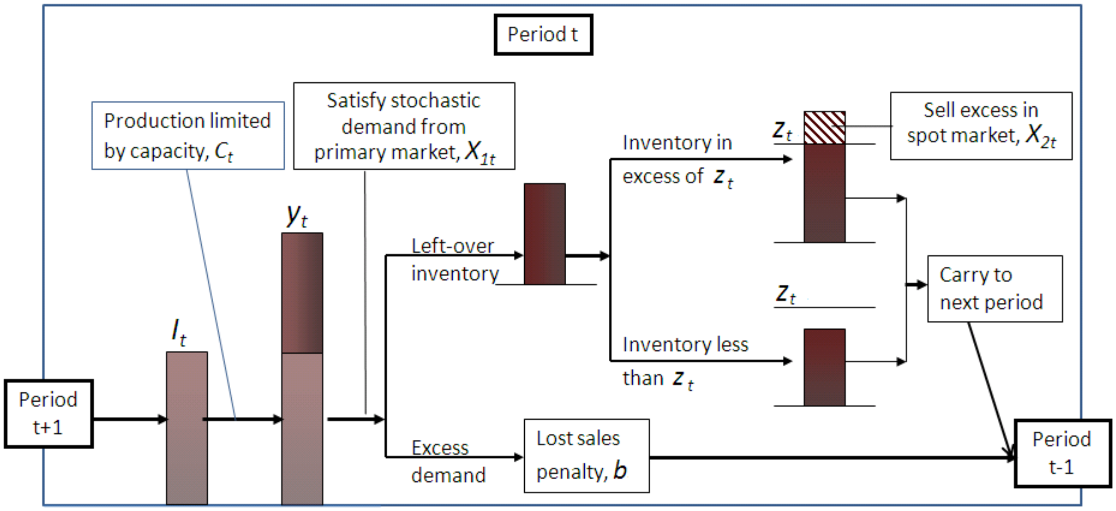


Figure 3: Graphical representation of the sequence of events for period t.

when it is in excess.

We consider a holding cost of h_t in period t for each unit carried to the next period. The unit revenue from the primary market is denoted by r_1. Note that both the unit revenue (r_1) and the unit lost sales penalty (b) associated with the primary market are constant throughout the planning horizon. These parameter values are determined based on the contractual agreement between the supplier and the primary market customers. On the other hand, the unit revenue from the spot market in period t, denoted by r_{2t}, may change from period to period. Clearly, $r_1 > c_t$ for $t = 1, \ldots, T$, in order for the problem to make economic sense. However, we do not restrict the value of r_{2t}. If $r_{2t} \geq c_t$, then selling in the spot market is profitable. On the other hand, if $r_{2t} < c_t$, then selling in the spot market is equivalent to salvaging. Furthermore, r_{2t} may be greater than or less than r_1. We consider a one-period discount factor of $0 \leq \beta \leq 1$. Figure 3 gives a graphical representation of the sequence of events that take place in a period and Table 1 summarizes the notation that we use in our model.

We index each period in terms of the number of periods remaining until the end of the horizon and formulate the problem as a stochastic dynamic program. Since, we define z_t as

T	total number of periods
X_{1t}	demand random variable for the primary market in period t
X_{2t}	demand random variable for the spot market in period t
$F_{it}(\cdot)$	cumulative distribution function of X_{it}, $i = 1, 2$
$f_{it}(\cdot)$	probability density function of X_{it}, $i = 1, 2$
$F_{it}^{-1}(\cdot)$	inverse of the cumulative distribution function of X_{it}, $i = 1, 2$
r_1	unit revenue from the primary market in periods $t = 1, \ldots, T$
r_{2t}	unit revenue from the spot market in period t
c_t	unit cost of production in period t
b	unit lost sales penalty for unsatisfied demand in primary market in periods $t = 1, \ldots, T$
h_t	unit holding cost in period t
C_t	production capacity in period t
I_t	on-hand inventory at the beginning of period t
q_t	number of units produced in period t
y_t	produce-up-to level in period t
z_t	retain-up-to level in period t
β	one period discount factor

Table 1: Notation for the joint replenishment and liquidation problem

the maximum, and not the actual, amount of inventory to be carried to the next period, the optimal retain-up-to level in period t, denoted by z_t^*, is independent of the realized value of X_{1t}. For mathematical simplicity, we formulate the problem such that z_t^* is determined before observing X_{1t}. $V_t(I)$ denotes the maximum expected discounted profit when there are t periods to-go until the end of the planning horizon and the starting inventory is I. For each period $t = 1, \ldots, T$, with starting inventory I, the expected profit earned in period t when the produce-up-to level is y and retain-up-to level is z units is given as

$$\begin{aligned} P_{0t}(I, y, z) = E\big[&- c_t(y - I) + [r_1 y - b(X_{1t} - y)]1(X_{1t} > y) \\ &+ r_1 X_{1t} 1(X_{1t} \leq y) - h_t(y - X_{1t})1(y - z \leq X_{1t} \leq y) \\ &+ [r_{2t}(y - X_{1t} - z) - h_t z]1(X_{1t} \leq y - z, X_{1t} + X_{2t} > y - z) \\ &+ [r_{2t} X_{2t} - h_t(y - X_{1t} - X_{2t})]1(X_{1t} + X_{2t} \leq y - z)\big]. \end{aligned} \quad (1)$$

The first term in the above equation represents the production cost. The second term is the total revenue earned and the lost sales penalty incurred when the primary market demand exceeds the produce-up-to level, y. The third term is the revenue earned from the primary market when y exceeds the demand from the primary market. The fourth term represents the holding cost incurred if demand from the primary market is greater than $y - z$ but less than y. In this case, the inventory remaining after satisfying the primary market demand is less than the retain-up-to level, and hence, all of it is carried to the next period. The fifth term represents the revenue earned from the spot market and the holding cost incurred if demand from primary market is less than $y - z$ and the sum of demands from both markets is greater than $y - z$. In this case, z units are carried to the next period, and part of the spot market demand is satisfied. The final term gives the same when the sum of demands from both markets is less than $y - z$. In this case, demands from both markets are fully satisfied.

For conciseness, we define the following functions:

$$G_{it}(u) = \int_u^\infty (x-u)dF_{it}(x), \tag{2}$$

$$H_t(u) = \int_u^\infty (x-u)dF_{st}(x) \text{ , where } F_{st}(x) \text{ is cdf and } f_{st}(x) \text{ is the pdf of } X_{1t} + X_{2t}, \tag{3}$$

$$\begin{aligned} P_t(y,z) = {} & (r_1 + h_t)E[X_{1t}] + (r_{2t} + h_t)E[X_{2t}] - (c_t + h_t)y - (r_1 + b + h_t)G_{1t}(y) \\ & + (r_{2t} + h_t)(G_{1t}(y-z) - H_t(y-z)). \end{aligned} \tag{4}$$

Using equation (1), $P_{0t}(y, z)$ can be rewritten as follows:

$$\begin{aligned} P_{0t}(y,z) = E\big[& -c_t(y-I) + [(r_1+b)y - bX_{1t}]1(X_{1t} > y) \\ & + r_1 X_{1t}(1 - 1(X_{1t} > y)) - h_t(y - X_{1t})(1(X_{1t} > y - z) - 1(X_{1t} > y)) \\ & + [r_{2t}(y - X_{1t} - z) - h_t z](1(X_{1t} + X_{2t} > y - z) - 1(X_{1t} > y - z)) \\ & + [r_{2t}X_{2t} - h_t(y - X_{1t} - X_{2t})](1 - 1(X_{1t} + X_{2t} > y - z))\big] \end{aligned}$$

$$
\begin{aligned}
P_{0t}(y,z) = E\big[& - c_t(y-I) - (r_1+b+h_t)(X_{1t}-y)1(X_{1t}>y) + r_1 X_{1t} \\
& + (r_{2t}+h_t)(X_{1t}-y+z)1(X_{1t}>y-z) \\
& - (r_{2t}+h_t)(X_{1t}+X_{2t}-y+z)1(X_{1t}+X_{2t}>y-z) \\
& + r_{2t}X_{2t} - h_t(y-X_{1t}-X_{2t})\big] \\
= {} & c_t I + (r_1+h_t)E[X_{1t}] + (r_{2t}+h_t)E[X_{2t}] - (c_t+h_t)y \\
& + E\big[- (r_1+b+h_t)(X_{1t}-y)1(X_{1t}>y) \\
& + (r_{2t}+h_t)\big((X_{1t}-y+z)1(X_{1t}>y-z) \\
& - (X_{1t}+X_{2t}-y+z)1(X_{1t}+X_{2t}>y-z)\big)\big]. \qquad (5)
\end{aligned}
$$

Using definitions (2), (3) and (4) on the right-hand side of (5), we have $P_{0t}(y,z) = c_t I + P_t(y,z)$. The maximum expected profit when there are t periods-to-go and the starting inventory is I can be written as

$$
V_t(I) = c_t I + \max\{J_t(y,z) : I \le y \le C_t + I,\ z \ge 0\}, \quad \text{where} \qquad (6)
$$

$$
\begin{aligned}
J_t(y,z) = P_t(y,z) + \beta\big[& E[V_{t-1}(0)1(X_{1t}>y)] \\
& + E[V_{t-1}(z)1(X_{1t} \le y-z, X_{1t}+X_{2t}>y-z)] \\
& + E[V_{t-1}(y-X_{1t}-X_{2t})1(X_{1t}+X_{2t} \le y-z)] \\
& + E[V_{t-1}(y-X_{1t})1(y-z < X_{1t} \le y)]\big], \qquad (7)
\end{aligned}
$$

for $t = 1, \ldots, T$. The first term in equation (7) represents the expected profit in period t. The remaining terms represent the discounted expected profit-to-go for the next $t-1$ periods conditioned on the demand realization in period t. More specifically, the second term represents the discounted expected profit-to-go if the primary market demand is greater than y, and hence, the on-hand inventory available at the beginning of next period is zero. The third term represents the discounted expected profit-to-go if demand from the primary market is less than $y - z$ and the sum of demands from both markets is greater than $y - z$. In this case, z units are carried to the next period. The fourth term gives the discounted expected profit-to-go if the sum of demands from both markets is less than $y-z$, and hence, demands from both markets are fully satisfied and the remaining units are carried to the next period. The last term gives the discounted expected profit-to-go if demand from the primary market is greater than $y-z$ but less than y. In this case, the inventory remaining after satisfying the primary market demand is less than the retain-up-to level, and hence, all of it is carried to the next period. There is no salvaging at the end of the planning horizon, and hence, $V_0(I) = 0$ for $I \geq 0$. The objective is to compute $V_T(I)$ and to determine the optimal y_t^* and z_t^* that achieve this maximum expected profit for each period t, $t = 1, \ldots, T$.

We observe that r_{2t} is the immediate revenue that can be obtained by selling a left-over unit in the spot market in period t. On the other hand, the opportunity cost of carrying a unit to the next period is $\beta(r_1 + b) - h_t$. If the cost and revenue parameters are such that $\beta(r_1 + b) \leq r_{2t} + h_t$ for $t = 1, \ldots, T$, then carrying inventory to the next period is not profitable and the liquidation decisions are trivial. In order to avoid such cases, we assume that $\beta(r_1 + b) > r_{2t} + h_t$ for $t = 1, \ldots, T$. Furthermore, for $0 < \beta < 1$ and $h_t \geq 0$ this implies that $(r_1 + b) > r_{2t}$, i.e., losing sales from the primary market is more expensive than losing sales from the spot market. This provides economic motivation to our problem setting where primary market customers are given higher priority. In the next section, we characterize the structure of the optimal policy.

II.4 Characterization of the Structure of the Optimal Policy

In this section, we first present a theorem which completely characterizes the structure of the optimal policy. Next, in Section II.4.1, we establish upper bounds on the critical policy parameters for the general problem studied in Section II.3 and two of its special cases, and discuss their value. In Section II.4.2, we study how the optimal policy parameters change with respect to the unit revenue earned from the spot market. We show that there exist lower and upper thresholds on the unit revenue earned from the spot market such that one of the two benchmark policies is optimal.

Theorem 1 presents how to determine the optimal policy parameters (y_t^*, z_t^*) by studying the functions $V_t(I)$ and $J_t(y, z)$, given by equations (6) and (7), respectively. In particular, it shows that the optimal policy parameters are characterized by two quantities: S_t and R_t. S_t denotes the critical produce-up-to level and R_t is the critical retain-up-to level in period t. We term it as the (S_t, R_t) policy.

Theorem 1. *$J_t(y,z)$ and $V_t(I)$ satisfy the following properties for $t = 1,\ldots,T$:*

(a) $J_t(y,z)$ has a finite maximizer denoted by (y_t^*, z_t^*) such that

$$z_t^* = \begin{cases} 0 & \text{if } \left.\dfrac{dV_{t-1}(z)}{dz}\right|_{z=0} \le \dfrac{r_{2t}+h_t}{\beta}, \\ \\ R_t & \text{if } \left.\dfrac{dV_{t-1}(z)}{dz}\right|_{z=0} > \dfrac{r_{2t}+h_t}{\beta}, \end{cases} \tag{8}$$

where R_t satisfies $\left.\dfrac{dV_{t-1}(z)}{dz}\right|_{z=R_t} = \dfrac{r_{2t}+h_t}{\beta}$ and

$$y_t^* = \begin{cases} I & \text{if } S_t < I, \\ S_t & \text{if } I \le S_t \le C_t + I, \\ C_t + I & \text{if } S_t > C_t + I, \end{cases} \tag{9}$$

where S_t is determined by $J_t(S_t, z_t^*) = \max\{J_t(y, z_t^*) : y \in \Re^+\}$.

(b) $V_t(I)$ is a concave function of I.

(c) $\dfrac{dV_t(I)}{dI} \le r_1 + b$ and $\lim_{I\to\infty} \dfrac{dV_t(I)}{dI} < 0$.

Proof. Before we proceed with a proof, using Leibniz's rule of differentiation of an integral, we present the first order derivatives of $J_t(y,z)$ with respect to y and z as follows:

$$\begin{aligned} \frac{\partial J_t(y,z)}{\partial y} &= -c_t - h_t + (r_1 + b + h_t)[1 - F_{1t}(y)] + (r_{2t} + h_t)[F_{1t}(y-z) - F_{st}(y-z)] \\ &\quad + \beta E\left[\frac{dV_{t-1}(y - X_{1t} - X_{2t})}{dy} 1(X_{1t} + X_{2t} \le y - z)\right] \\ &\quad + \beta E\left[\frac{dV_{t-1}(y - X_{1t})}{dy} 1(y - z \le X_{1t} \le y)\right], \end{aligned} \tag{10}$$

$$\frac{\partial J_t(y,z)}{\partial z} = \left[\beta \frac{dV_{t-1}(z)}{dz} - (r_{2t} + h_t)\right][F_{1t}(y-z) - F_{st}(y-z)]. \tag{11}$$

The proof follows by induction. We will first show that the properties (a)-(c) are true for the one-period problem. Since $V_0(\cdot) = 0$, we have that $dV_0(z)/dz = 0$. Substituting this in equation (11) for $t = 1$, it can be easily shown that $\partial J_1(y,z)/\partial z \leq 0$, and hence, $J_1(y,z)$ is decreasing in z. Therefore, $z_1^* = 0$, which satisfies the first part of equation (8). Next, we note that the assumption $\beta(r_1 + b) \geq r_{2t} + h_t$ and $0 < \beta < 1$ implies that $r_1 + b > r_{2t}$ for $t = 1, \ldots, T$. Then, taking the derivative of equation (10) with respect to y for $t = 1$ and evaluating the result at $z = 0$, we obtain:

$$\frac{\partial^2 J_1(y,0)}{\partial y^2} = -(r_1 + b - r_2)f_{11}(y) - (r_{21} + h_1)f_{s1}(y) < 0.$$

This shows that $J_1(y,0)$ is concave in y. Defining S_1 such that

$$\left.\frac{\partial J_1(y,0)}{\partial y}\right|_{y=S_1} = 0,$$

we have the optimal produce-up-to level in period 1 expressed as

$$y_1^* = \begin{cases} I & \text{if } S_1 < I, \\ S_1 & \text{if } I \leq S_1 \leq C_1 + I, \\ C_1 + I & \text{if } S_1 > C_1 + I. \end{cases}$$

Consequently, it follows from equation (6) that $V_1(I) = c_1 I + J_1(y_1^*(I), 0)$. The first order derivative of $V_1(I)$ is

$$\frac{dV_1(I)}{dI} = \begin{cases} r_1 + b - (r_1 + b - r_{21})F_{11}(I) - (r_{21} + h_1)F_{s1}(I) & \text{if } S_1 < I, \\ c_1 & \text{if } I \le S_1 \le C_1 + I, \\ r_1 + b - (r_1 + b - r_{21})F_{11}(C_1 + I) & \\ \quad -(r_{21} + h_1)F_{s1}(C_1 + I) & \text{if } S_1 > C_1 + I. \end{cases} \tag{12}$$

From equation (12), we observe that $dV_1(I)/dI \le r_1 + b$. Moreover, taking the derivative of equation (12) with respect to I, it can be easily shown that $d^2V_1(I)/dI^2 \le 0$. Thus, $V_1(I)$ is concave in I. For $S_1 < I$, evaluating the limit of equation (12) as I goes to infinity, we obtain:

$$\lim_{I \to \infty} \frac{dV_1(I)}{dI} = r_1 + b - (r_1 + b - r_{21}) - (r_{21} + h_1) < 0.$$

As a result, properties (a)-(c) are true for the one-period problem.

Suppose that properties (a)-(c) are true for periods $t-1, t-2, \ldots, 2$. We will show that they are true for period t. By the induction hypothesis based on part (b), $dV_{t-1}(z)/dz$ is a decreasing function of z. Suppose that $dV_{t-1}(z)/dz|_{z=0} \le (r_{2t}+h_t)/\beta$. Then, $dV_{t-1}(z)/dz \le (r_{2t} + h_t)/\beta$ for $z > 0$. Using this inequality and equation (11), we observe that $J_t(y, z)$ is a decreasing function of z for $y \in [I, C_t + I]$. Therefore, $z_t^* = 0$. On the other hand, if $dV_{t-1}(z)/dz|_{z=0} > (r_{2t} + h_t)/\beta$, then by the induction hypothesis based on part (c) ($\lim_{I\to\infty} dV_{t-1}(I)/dI < 0$) and the intermediate value theorem there exists a $R_t \in \Re^+$ such that

$$\left.\frac{dV_{t-1}(z)}{dz}\right|_{z=R_t} = \frac{r_{2t} + h_t}{\beta}. \tag{13}$$

Using equations (11) and (13), we observe that $J_t(y, z)$ is increasing in z for $z < R_t$, and decreasing in z for $z > R_t$. Therefore, $J_t(y, z)$ attains its maximum at $z = R_t$ for $y \in [I, C_t + I]$.

From the above discussion, we see that for $y \in [I, C_t + I]$, if

$$\left.\frac{dV_{t-1}(z)}{dz}\right|_{z=0} \leq \frac{r_{2t} + h_t}{\beta}, \tag{14}$$

then $z_t^* = 0$; otherwise, $z_t^* = R_t$.

Next, we show that $J_t(y, z_t^*)$ is concave in y. Evaluating expression (10) at $z = 0$ and taking the derivative of the result with respect to y, we obtain:

$$\begin{aligned}
\frac{\partial J_t(y,0)}{\partial y} &= -c_t - h_t + (r_1 + b + h_t)[1 - F_{1t}(y)] + (r_{2t} + h_t)[F_{1t}(y) - F_{st}(y)] \\
&\quad + \beta E\left[\frac{dV_{t-1}(y - X_{1t} - X_{2t})}{dy} 1(X_{1t} + X_{2t} \leq y)\right] \qquad \text{and} \\
\frac{\partial^2 J_t(y,0)}{\partial y^2} &= -(r_1 + b - r_{2t}) f_{1t}(y) + \left[\beta \left.\frac{dV_{t-1}(y - x_{1t} - x_{2t})}{dy}\right|_{x_{1t}+x_{2t}=y} - (r_{2t} + h_t)\right] f_{st}(y) \\
&\quad + \beta E\left[\frac{d^2 V_{t-1}(y - X_{1t} - X_{2t})}{dy^2} 1(X_{1t} + X_{2t} \leq y)\right] < 0.
\end{aligned} \tag{15}$$

The negativity of (15) follows from the induction hypothesis that $V_{t-1}(\cdot)$ is concave and inequality (14). On the other hand, if $z_t^* = R_t$, then R_t satisfies equality (13) by definition. Evaluating expression (10) at $z = R_t$ and taking the derivative of the resulting expression with respect to y, we obtain:

$$\begin{aligned}
\frac{\partial J_t(y,R_t)}{\partial y} &= -c_t - h_t + (r_1 + b + h_t)[1 - F_{1t}(y)] + (r_{2t} + h_t)[F_{1t}(y - R_t) - F_{st}(y - R_t)] \\
&\quad + \beta E\left[\frac{dV_{t-1}(y - X_{1t} - X_{2t})}{dy} 1(X_{1t} + X_{2t} \leq y - R_t)\right] \\
&\quad + \beta E\left[\frac{dV_{t-1}(y - X_{1t})}{dy} 1(y - R_t \leq X_{1t} \leq y)\right] \qquad \text{and} \\
\frac{\partial^2 J_t(y,R_t)}{\partial y^2} &= \left[\beta \left.\frac{dV_{t-1}(y - x_{1t})}{dy}\right|_{x_{1t}=y} - (r_1 + b + h_t)\right] f_{1t}(y) \\
&\quad - \left[\beta \left.\frac{dV_{t-1}(y - x_{1t})}{dy}\right|_{x_{1t}=y-R_t} - (r_{2t} + h_t)\right] [f_{1t}(y - R_t) - f_{st}(y - R_t)] \\
&\quad + \beta E\left[\frac{d^2 V_{t-1}(y - X_{1t} - X_{2t})}{dy^2} 1(X_{1t} + X_{2t} \leq y - R_t)\right] \\
&\quad + \beta E\left[\frac{dV_{t-1}^2(y - X_{1t})}{dy^2} 1(y - R_t \leq X_{1t} \leq y)\right] < 0.
\end{aligned} \tag{16}$$

In equation (16), the negativity of the first term follows from the fact that $dV_{t-1}(I)/dI \leq r_1 + b$. From equation (13) it can be seen that the second term is equal to zero. The last two terms are negative based on the induction hypothesis that $V_{t-1}(\cdot)$ is concave. Thus, from inequalities in (15) and (16), it follows that $J_t(y, z_t^*)$ is concave in y. Defining $S_t \in \Re$ such that

$$\left.\frac{\partial J_t(y, z_t^*)}{\partial y}\right|_{y=S_t} = 0, \tag{17}$$

we have the optimal produce-up-to level in period t, y_t^*, expressed as in equation (9). As a result, property (a) holds. Then, it follows from equation (6) that $V_t(I) = cI + J_t(y_t^*(I), z_t^*)$. The first-order derivative of $V_t(I)$ with respect to I evaluated at $y_t^*(I)$ from equation (9) is given by:

$$\frac{dV_t(I)}{dI} = c_t + \frac{dJ_t(y_t^*(I), z_t^*)}{dI}, \tag{18}$$

$$\frac{dV_t(I)}{dI} = c_t + \begin{cases} \left.\frac{\partial J_t(y,z_t^*)}{\partial y}\right|_{y=I} & \text{if } S_t < I, \\ 0 & \text{if } I \leq S_t \leq C_t + I, \\ \left.\frac{\partial J_t(y,z_t^*)}{\partial y}\right|_{y=C_t+I} & \text{if } S_t > C_t + I. \end{cases} \tag{19}$$

Taking the first-order derivative of equation (19) with respect to I and using the fact that $J_t(y, z_t^*)$ is concave in y in the resulting expression, it is easy to verify that $d^2V_t(I)/dI^2 \leq 0$. Hence, $V_t(I)$ is concave in I.

To prove property (c), we first note from part (a) that $dV_{t-1}(z)/dz|_{z=z_t^*} \leq (r_{2t} + h_t)/\beta$. Using this inequality and the fact that $V_{t-1}(\cdot)$ is concave, we have

$$\left.\frac{dV_{t-1}(z)}{dz}\right|_{z\geq z_t^*} < \frac{r_{2t} + h_t}{\beta}. \tag{20}$$

For $S_t < I$, evaluating expression (10) at $y = I$ and substituting this on right-hand side of equation (19), we have

$$\frac{dV_t(I)}{dI} = -h_t + (r_1 + b + h_t)[1 - F_{1t}(I)] + (r_{2t} + h_t)[F_{1t}(I - z_t^*) - F_{st}(I - z_t^*)]$$
$$+ \beta E\left[\frac{dV_{t-1}(I - X_{1t} - X_{2t})}{dI} 1(X_{1t} + X_{2t} \leq I - z_t^*)\right]$$
$$+ \beta E\left[\frac{dV_{t-1}(I - X_{1t})}{dI} 1(I - z_t^* \leq X_{1t} \leq I)\right]. \quad (21)$$

Using inequality (20) and the fact that $dV_{t-1}(I)/dI \leq r_1 + b$ on the right-hand side of equation (21), after some algebraic manipulations, for $S_t < I$ we have

$$\frac{dV_t(I)}{dI} \leq r_1 + b - [(1-\beta)(r_1 + b) + h_t]F_{1t}(I) - [\beta(r_1 + b) - r_{2t} - h_t]F_{1t}(I - z_t^*) \leq r_1 + b. \quad (22)$$

The last in inequality in (22) follows from the assumption that $\beta(r_1+b) > r_{2t}+h_t$. Following a similar argument, we can show that for $S_t > C_t + I$, $dV_t(I)/dI < r_1 + b$. For $I \leq S_t \leq C_t + I$, $dV_t(I)/dI = c_t < r_1 + b$. Next, we recall that equation (21) gives the expression for $dV_{t-1}(I)/dI$ for $S_t < I$. Hence, taking limit of the right-hand side of equation (21) and using the induction hypothesis that $\lim_{I\to\infty} dV_{t-1}(I)/dI < 0$ in the resulting expression, we obtain:

$$\lim_{I\to\infty} \frac{dV_t(I)}{dI} = -h_t + \lim_{I\to\infty} \beta E\left[\frac{dV_{t-1}(I - X_{1t} - X_{2t})}{dI} 1(X_{1t} + X_{2t} \leq I - z_t^*)\right] < 0.$$

As a result, properties (a)-(c) are true for period t. □

Thus, Theorem 1 shows that the optimal policy is a combination of a modified base-stock and a retain-up-to policy. We use the term *modified* base-stock policy because the optimal produce-up-to level depends on the starting on-hand inventory and the production capacity. The retain-up-to policy indicates that after sales to the primary market, it is optimal to sell only the units in excess of the optimal retain-up-to level in the spot market. From part (a) of Theorem 1, we observe that the optimal retain-up-to level depends on the unit revenue from the spot market, the unit holding cost, the one-period discount factor and the derivative of the profit-to-go function with respect to the retain-up-to level. For period t, R_t denotes the retain-up-to level at which the discounted marginal value of retaining one more unit of inventory for the next $t-1$ periods is equal to the potential revenue and savings in the holding cost that can be realized by selling a unit in the spot market in period t. We show that if

$$\left.\frac{dV_{t-1}(z)}{dz}\right|_{z=0} \leq \frac{r_{2t}+h_t}{\beta}, \tag{23}$$

then $J_t(y,z)$ is a decreasing function of z for $y \geq 0$. Note that throughout the text, we use decreasing (increasing) to mean non-increasing (non-decreasing) for brevity. From equation (6), it follows that $z_t^* = 0$. On the other hand, if the inequality in (23) is violated, then $J_t(y,z)$ is unimodal in z for $y \geq 0$, and $z_t^* = R_t$. Hence, as the unit revenue from the spot market increases, it is optimal to liquidate more inventory. We would like to note that z_t^* may be higher than the optimal produce-up-to level y_t^* for some problem instances. This means that the optimal policy is equivalent to the retain-all policy (i.e., the left-over inventory after satisfying the primary market demand is carried to the next period), and hence, the spot market should be ignored.

Part (b) of Theorem 1 shows that the maximum expected profit-to-go is a concave function of the initial inventory. Further, part (c) shows that for every unit of on-hand inventory available at the beginning of a period, the increase in the optimal expected profit is bounded by the sum of the unit revenue from the primary market and unit lost sales penalty, i.e., r_1+b. Part (c) also shows that if the on-hand inventory available at the beginning of a period goes to infinity, then the increase in the optimal expected profit is negative.

II.4.1 Bounds for the optimal policy parameters

In this section, we develop upper bounds for R_t and S_t for the general problem studied in Section II.3 and two of its special cases. Furthermore, we discuss the economic interpretation of these bounds, which can be useful from a managerial perspective.

Proposition 1. *(a) The critical retain-up-to level R_t is bounded above as follows for $t = 2, \ldots, T$:*

If $c_{t-1} \le (r_{2t} + h_t)/\beta$, then $R_t \le F_{1,t-1}^{-1}\left(\dfrac{\beta(r_1+b) - r_{2t} - h_t}{\beta(r_1 + b - r_{2,t-1})}\right) - C_{t-1} + z_{t-1}^*$.

If $c_{t-1} > (r_{2t} + h_t)/\beta$, then $R_t \le F_{1,t-1}^{-1}\left(\dfrac{\beta(r_1+b) - r_{2t} - h_t}{\beta(r_1 + b - r_{2,t-1})}\right) + z_{t-1}^*$.

(b) The critical produce-up-to level S_t is bounded above as follows for $t = 1, \ldots, T$:

$$S_t \le F_{1t}^{-1}\left(\frac{r_1 + b - c_t}{r_1 + b - r_{2t}}\right) + z_t^*.$$

This upper bound is finite if $c_t > r_{2t}$, and infinite if $c_t \le r_{2t}$.

Proof. Since $V_t(\cdot)$ is concave, $dV_t(I)/dI$ is a decreasing function of I. Therefore, from equation (19), we observe that

$$\frac{dV_t(I)}{dI} \begin{cases} < c_t & \text{if } S_t < I, \\ = c_t & \text{if } I \le S_t \le C_t + I, \\ > c_t & \text{if } S_t > C_t + I, \end{cases} \tag{24}$$

for $t = 1, \ldots, T$. In addition, we know from Theorem 1 that R_t satisfies equation (13).

(a) If $c_{t-1} \le (r_{2t} + h_t)/\beta$, then applying equation (24) for period $t-1$, we observe that for all $I \ge S_{t-1} - C_{t-1}$, $dV_{t-1}(I)/dI \le c_{t-1} \le (r_{2t} + h_t)/\beta$. Since R_t should satisfy equation (13), we have $R_t < S_{t-1} - C_{t-1}$. Next, we evaluate $dV_{t-1}(I)/dI$ for $I = R_t$ when $R_t < S_{t-1} - C_{t-1}$. Using the last part of equation (19), we have

$$\left.\frac{dV_{t-1}(I)}{dI}\right|_{I=R_t} = c_{t-1} + \left.\frac{\partial J_{t-1}(y, z_{t-1}^*)}{\partial y}\right|_{y=C_{t-1}+R_t}. \tag{25}$$

Using equations (10) and (13), we can write equation (25) as

$$\begin{aligned}
&\frac{r_{2t}+h_t}{\beta} \\
&= -h_{t-1} + (r_1+b+h_{t-1})[1-F_{1,t-1}(C_{t-1}+R_t)] \\
&\quad + (r_{2,t-1}+h_{t-1})[F_{1,t-1}(C_{t-1}+R_t-z^*_{t-1}) - F_{s,t-1}(C_{t-1}+R_t-z^*_{t-1})] \\
&\quad + \beta E\left[\frac{dV_{t-2}(C_{t-1}+z-X_{1,t-1}-X_{2,t-1})}{dz}\bigg|_{z=R_t} 1(X_{1,t-1}+X_{2,t-1} \le C_{t-1}+R_t-z^*_{t-1})\right] \\
&\quad + \beta E\left[\frac{dV_{t-2}(C_{t-1}+z-X_{1,t-1})}{dz}\bigg|_{z=R_t} 1(C_{t-1}+R_t-z^*_{t-1} \le X_{1,t-1} \le C_{t-1}+R_t)\right], \quad (26)
\end{aligned}$$

for $t = 2, \ldots, T$. Using inequality (20) and part (c) of Theorem 1 on the right-hand side of equation (26), we have

$$\begin{aligned}
\frac{r_{2t}+h_t}{\beta} &\le -h_{t-1} + (r_1+b+h_{t-1})[1-F_{1,t-1}(C_{t-1}+R_t)] \\
&\quad + (r_{2,t-1}+h_{t-1})[F_{1,t-1}(C_{t-1}+R_t-z^*_{t-1}) - F_{s,t-1}(C_{t-1}+R_t-z^*_{t-1})] \\
&\quad + \beta\left(\frac{r_{2,t-1}+h_{t-1}}{\beta}\right) E\big[1(X_{1,t-1}+X_{2,t-1} \le C_{t-1}+R_t-z^*_{t-1})\big] \\
&\quad + \beta(r_1+b) E\big[1(C_{t-1}+R_t-z^*_{t-1} \le X_{1,t-1} \le C_{t-1}+R_t)\big] \\
&= r_1+b-(r_1+b+h_{t-1})F_{1,t-1}(C_{t-1}+R_t) \\
&\quad + (r_{2,t-1}+h_{t-1})F_{1,t-1}(C_{t-1}+R_t-z^*_{t-1}) \\
&\quad + \beta(r_1+b)[F_{1,t-1}(C_{t-1}+R_t) - F_{1,t-1}(C_{t-1}+R_t-z^*_{t-1})]. \qquad (27)
\end{aligned}$$

Using the fact that $\beta(r_1+b) \le r_1+b \le r_1+b+h_{t-1}$ on the right-hand side of equation (27), we have

$$\begin{aligned}
\frac{r_{2t}+h_t}{\beta} &\le r_1+b-(r_1+b+h_{t-1})F_{1,t-1}(C_{t-1}+R_t) + (r_{2,t-1}+h_{t-1})F_{1,t-1}(C_{t-1}+R_t-z^*_{t-1}) \\
&\quad + (r_1+b+h_{t-1})[F_{1,t-1}(C_{t-1}+R_t) - F_{1,t-1}(C_{t-1}+R_t-z^*_{t-1})].
\end{aligned}$$

After some algebra, the above inequality can be written as

$$R_t \leq F_{1,t-1}^{-1}\left(\frac{\beta(r_1+b)-r_{2t}-h_t}{\beta(r_1+b-r_{2,t-1})}\right) - C_{t-1} + z_{t-1}^*.$$

If $c_{t-1} > (r_{2t}+h_t)/\beta$, then applying equation (24) for period $t-1$, we observe that for all $I \leq S_{t-1}$, $dV_{t-1}(I)/dI \geq c_{t-1} > (r_{2t}+h_t)/\beta$. Since R_t should satisfy equation (13), we have $R_t > S_{t-1}$. Next, we evaluate $dV_{t-1}(I)/dI$ for $I = R_t$ when $R_t > S_{t-1}$. Using the first part of equation (19), we have

$$\left.\frac{dV_{t-1}(I)}{dI}\right|_{I=R_t} = c_{t-1} + \left.\frac{\partial J_{t-1}(y, z_{t-1}^*)}{\partial y}\right|_{y=R_t}. \tag{28}$$

Using equations (10) and (13), we can write equation (28) as

$$\begin{aligned}
\frac{r_{2t}+h_t}{\beta} = &-h_{t-1} + (r_1+b+h_{t-1})[1-F_{1,t-1}(R_t)] \\
&+ (r_{2,t-1}+h_{t-1})[F_{1,t-1}(R_t - z_{t-1}^*) - F_{s,t-1}(R_t - z_{t-1}^*)] \\
&+ \beta E\left[\left.\frac{dV_{t-2}(z - X_{1,t-1} - X_{2,t-1})}{dz}\right|_{z=R_t} 1(X_{1,t-1}+X_{2,t-1} \leq R_t - z_{t-1}^*)\right] \\
&+ \beta E\left[\left.\frac{dV_{t-2}(z - X_{1,t-1})}{dz}\right|_{z=R_t} 1(R_t - z_{t-1}^* \leq X_{1,t-1} \leq R_t)\right],
\end{aligned} \tag{29}$$

for $t = 2, \ldots, T$. Then, following a similar argument as for the $c_{t-1} \leq (r_{2t}+h_t)/\beta$, it can be shown that

$$R_t \leq F_{1,t-1}^{-1}\left(\frac{\beta(r_1+b)-r_{2t}-h_t}{\beta(r_1+b-r_{2,t-1})}\right) + z_{t-1}^*.$$

(b) Recall that $S_t \in \Re$ satisfies equation (17) by definition. Then, simultaneous use of equations (10) and (17) implies

$$\begin{aligned}
\left.\frac{\partial J_t(y, z_t^*)}{\partial y}\right|_{y=S_t} = 0 = &-c_t - h_t + (r_1 + b + h_t)[1 - F_{1t}(S_t)] \\
&+ (r_{2t} + h_t)[F_{1t}(S_t - z_t^*) - F_{st}(S_t - z_t^*)] \\
&+ \beta E\left[\left.\frac{dV_{t-1}(y - X_{1t} - X_{2t})}{dy}\right|_{y=S_t} 1(X_{1t} + X_{2t} \le S_t - z_t^*)\right] \\
&+ \beta E\left[\left.\frac{dV_{t-1}(y - X_{1t})}{dy}\right|_{y=S_t} 1(S_t - z_t^* \le X_{1t} \le S_t)\right], \qquad (30)
\end{aligned}$$

for $t = 1, \ldots, T$. Using equation (20) and part (c) of Theorem 1 on right-hand side of equation (30), we have

$$\begin{aligned}
0 &\le -c_t - h_t + (r_1 + b + h_t)[1 - F_{1t}(S_t)] + (r_{2t} + h_t)[F_{1t}(S_t - z_t^*) - F_{st}(S_t - z_t^*)] \\
&\quad + \beta\left(\frac{r_{2t} + h_t}{\beta}\right) E\big[1(X_{1t} + X_{2t} \le S_t - z_t^*)\big] + \beta(r_1 + b) E\big[1(S_t - z_t^* \le X_{1t} \le S_t)\big] \\
&= r_1 + b - c_t - (r_1 + b + h_t)F_{1t}(S_t) + (r_{2t} + h_t)F_{1t}(S_t - z_t^*) \\
&\quad + \beta(r_1 + b)\big[F_{1t}(S_t) - F_{1t}(S_t - z_t^*)\big]. \qquad (31)
\end{aligned}$$

Using the fact that $\beta(r_1 + b) \le r_1 + b + h_t$ on the right-hand side of equation (31), we obtain

$$\begin{aligned}
0 \le\; & r_1 + b - c_t - (r_1 + b + h_t)F_{1t}(S_t) + (r_{2t} + h_t)F_{1t}(S_t - z_t^*) \\
& + (r_1 + b + h_t)\big[F_{1t}(S_t) - F_{1t}(S_t - z_t^*)\big].
\end{aligned}$$

After some algebra, the above inequality can be written as

$$S_t \le F_{1t}^{-1}\left(\frac{r_1 + b - c_t}{r_1 + b - r_{2t}}\right) + z_t^*.$$

Note that $(r_1 + b - c_t)/(r_1 + b - r_{2t}) < 1$ if $p > r_{2t}$, and hence, this upper bound on S_t is finite. On the other hand, if $p \le r_{2t}$, this upper bound on S_t is infinite. □

Let us define K_{ut} as the cost of selling a unit in the spot market in period t, which will be needed in the primary market in period $t-1$ and K_{ot} as the cost of retaining a unit in period t, which will not be needed in the primary market in period $t-1$, $t = 1, \ldots, T$. If we assume that there is ample demand in the spot market, then it is easy to verify that $K_{ut} = \beta(r_1 + b) - r_{2t} - h_t$ and $K_{ot} = r_{2t} + h_t - \beta r_{2,t-1}$. We term $K_{ut}/(K_{ut} + K_{ot})$ as the critical ratio. Suppose $c_{t-1} \leq (r_{2t} + h_t)/\beta$. Then, Proposition 1 shows that R_t is bounded above as

$$R_t \leq F^{-1}_{1,t-1}\left(\frac{K_{ut}}{K_{ut} + K_{ot}}\right) - C_{t-1} + z^*_{t-1} \quad \text{for } t = 2, \ldots, T.$$

The inequality $c_{t-1} \leq (r_{2t} + h_t)/\beta$ implies that it is more profitable to produce for the primary market in period $t-1$ rather than carrying a unit from period t. Hence, if R_t units are carried to period $t-1$ and we produce up to C_{t-1}, then the probability of satisfying all demand from the primary market and retaining z^*_{t-1} units in period $t-1$ is at most equal to the critical ratio. Thus, R_t and its upper bound are positive only if

$$C_{t-1} < F^{-1}_{1,t-1}\left(\frac{K_{ut}}{K_{ut} + K_{ot}}\right) + z^*_{t-1}.$$

It is important to note that the upper bound on R_t, and hence, z^*_t depends on the primary market demand distribution and the available production capacity in the next period. However, the upper bound is independent of the spot market demand distribution.

On the other hand, if $c_{t-1} > (r_{2t} + h_t)/\beta$, then it is profitable to satisfy the primary market demand in period $t-1$ by carrying units from period t. Thus, independent of C_{t-1},

$$R_t \leq F^{-1}_{1,t-1}\left(\frac{K_{ut}}{K_{ut} + K_{ot}}\right) + z^*_{t-1} \quad \text{for } t = 1, \ldots, T.$$

That is, if R_t units are carried to period $t-1$ and we do not produce in period $t-1$, then the probability of satisfying all demand from the primary market and retaining z^*_{t-1} units in period $t-1$ is at most equal to the critical ratio. Thus, in this case, R_t has a strictly positive upper bound. We also note that the upper bound for R_t is tighter when $c_{t-1} \leq r_{2t} + h_t$.

Next, define K'_{ut} as the cost of producing one unit less and K'_{ot} as the cost of producing one unit more than the primary market demand in period t. Again, if we assume that there is ample demand in the spot market, then $K'_{ut} = r_1 + b - c_t$, $K'_{ot} = c_t - r_{2t}$, and the critical ratio is equal to $K'_{ut}/(K'_{ut} + K'_{ot})$. It follows that, in terms of K'_{ut} and K'_{ot}, S_t is bounded above as

$$S_t \leq F_{1t}^{-1}\left(\frac{K'_{ut}}{K'_{ut} + K'_{ot}}\right) + z_t^*.$$

If we produce up to S_t in period t, then the probability of satisfying all demand from the primary market and retaining z_{t-1}^* units in period t is at most equal to the critical ratio.

The above economic interpretation of bounds is interesting from a managerial perspective. Furthermore, these bounds allow us to narrow the search region while computing the optimal policy parameters. One drawback of these bounds is their additive nature, i.e., bounds in period t depend on the optimal retain-up-to decision in period $t-1$, z_{t-1}^*. Consequently, when z_{t-1}^* is large, bounds in periods $t, \ldots, T$ can be loose. It is difficult, if not impossible, to establish bounds on S_t and R_t based purely on problem parameters.

We complement Proposition 1 by establishing bounds on S_t and R_t based purely on problem parameters for two special cases: First, in Proposition 2, we provide lower bounds on S_t and R_t for the case with ample demand in the spot market such that, if desired, all the remaining inventory after satisfying the primary market demand can be sold in the spot market. Second, in Proposition 3, we provide upper bounds on S_t and R_t for the case with ample production capacity and stationary cost parameters.

Proposition 2. *When there is ample demand in the spot market (i.e. $X_{2t} = \infty$, for $t = 1, \ldots, T$)*

(a) The critical retain-up-to level R_t is bounded as follows for $t = 2, \ldots, T$:

If $c_{t-1} \leq (r_{2t} + h_t)/\beta$, then

$$F_{1,t-1}^{-1}\left(\frac{\beta(r_1 + b) - r_{2t} - h_t}{\beta(r_1 + b - r_{2,t-1})}\right) - C_{t-1} \leq R_t \leq F_{1,t-1}^{-1}\left(\frac{\beta(r_1 + b) - r_{2t} - h_t}{\beta(r_1 + b - r_{2,t-1})}\right) - C_{t-1} + z_{t-1}^*.$$

If $c_{t-1} > (r_{2t}+h_t)/\beta$, *then*

$$F^{-1}_{1,t-1}\left(\frac{\beta(r_1+b)-r_{2t}-h_t}{\beta(r_1+b-r_{2,t-1})}\right) \le R_t \le F^{-1}_{1,t-1}\left(\frac{\beta(r_1+b)-r_{2t}-h_t}{\beta(r_1+b-r_{2,t-1})}\right) + z^*_{t-1}.$$

(b) The critical produce-up-to level S_t *is bounded as follows for* $t = 2, \ldots, T$:

If $r_{2t} \ge c_t$, $y^*_t = C_t + I$.

If $r_{2t} < c_t$, *then* $F^{-1}_{1t}\left(\frac{r_1+b-c_t}{r_1+b-r_{2t}}\right) \le S_t \le F^{-1}_{1t}\left(\frac{r_1+b-c_t}{r_1+b-r_{2t}}\right) + z^*_t$.

Proof. Since, this is a special case of the problem considered in Section II.3, Theorem 1 and Proposition 1 still hold. Therefore, the upper bounds on R_t and S_t as given in Proposition 2 follow directly from Proposition 1.

Below, we provide derivations for the lower bounds on R_t and S_t. We note that ample demand in the spot market implies

$$E[1(X_{1t}+X_{2t} \le x)] = F_{st}(x) = 0 \tag{32}$$

for $x \ge 0$ and for $t = 1, \ldots, T$. In addition, based on the definition of R_t from Theorem 1, we note that $dV_{t-1}(z)/dz|_{z=R_t} = (r_{2t}+h_t)/\beta$. Using this inequality and the fact that $V_{t-1}(\cdot)$ is concave, we have

$$\left.\frac{dV_{t-1}(z)}{dz}\right|_{z\le R_t} \ge \frac{r_{2t}+h_t}{\beta}. \tag{33}$$

(a) If $c_{t-1} \le (r_{2t}+h_t)/\beta$, using equation (32) on the right-hand side of equation (26), we obtain

$$\begin{aligned}\frac{r_{2t}+h_t}{\beta} &= -h_{t-1} + (r_1+b+h_{t-1})[1-F_{1,t-1}(C_{t-1}+R_t)]\\ &\quad + (r_{2,t-1}+h_{t-1})F_{1,t-1}(C_{t-1}+R_t-z^*_{t-1})\\ &\quad + \beta E\left[\left.\frac{dV_{t-2}(C_{t-1}+z-X_{1,t-1})}{dz}\right|_{z=R_t} 1(C_{t-1}+R_t-z^*_{t-1} \le X_{1,t-1} \le C_{t-1}+R_t)\right],\end{aligned} \tag{34}$$

for $t = 2, \ldots, T$. Using part (a) of Theorem 1 for period $t-1$, we note that $z^*_{t-1} = 0$ or

$z^*_{t-1} = R_{t-1}$. If $z^*_{t-1} = 0$, then from equation (34), we have

$$\begin{aligned}
\frac{r_{2t}+h_t}{\beta} &= -h_{t-1} + (r_1+b+h_{t-1})[1-F_{1,t-1}(C_{t-1}+R_t)] + (r_{2,t-1}+h_{t-1})F_{1,t-1}(C_{t-1}+R_t) \\
&= r_1+b-(r_1+b-r_{2,t-1})F_{1,t-1}(C_{t-1}+R_t).
\end{aligned}$$

After some algebra, the above equation can be written as

$$R_t = F^{-1}_{1,t-1}\left(\frac{\beta(r_1+b)-r_{2t}-h_t}{\beta(r_1+b-r_{2,t-1})}\right) - C_{t-1}. \tag{35}$$

On the other hand, if $z^*_{t-1} = R_{t-1}$, then using inequality (33) on right-hand side of equation (34), we have

$$\begin{aligned}
\frac{r_{2t}+h_t}{\beta} \geq & -h_{t-1} + (r_1+b+h_{t-1})[1-F_{1,t-1}(C_{t-1}+R_t)] \\
& + (r_{2,t-1}+h_{t-1})F_{1,t-1}(C_{t-1}+R_t-R_{t-1}) \\
& + \beta\left(\frac{r_{2,t-1}+h_{t-1}}{\beta}\right)E[1(C_{t-1}+R_t-R_{t-1} \leq X_{1,t-1} \leq C_{t-1}+R_t)] \\
\frac{r_{2t}+h_t}{\beta} = & \; r_1+b-(r_1+b+h_{t-1})F_{1,t-1}(C_{t-1}+R_t)] + (r_{2,t-1}+h_{t-1})F_{1,t-1}(C_{t-1}+R_t).
\end{aligned}$$

After some algebra, the above inequality can be written as

$$R_t \geq F^{-1}_{1,t-1}\left(\frac{\beta(r_1+b)-r_{2t}-h_t}{\beta(r_1+b-r_{2,t-1})}\right) - C_{t-1}. \tag{36}$$

The lower bound for R_t follows from equations (35) and (36).

If $c_{t-1} > (r_{2t}+h_t)/\beta$, using equation (32) on the right-hand side of equation (29), we have

$$\begin{aligned}
\frac{r_{2t}+h_t}{\beta} = & -h_{t-1} + (r_1+b+h_{t-1})[1-F_{1,t-1}(R_t)] + (r_{2,t-1}+h_{t-1})F_{1,t-1}(R_t - z^*_{t-1}) \\
& + \beta E\left[\left.\frac{dV_{t-2}(z-X_{1,t-1})}{dz}\right|_{z=R_t} 1(R_t - z^*_{t-1} \leq X_{1,t-1} \leq R_t)\right],
\end{aligned} \tag{37}$$

for $t = 2, \ldots, T$. Then, following a similar argument as for the $c_{t-1} \le (r_{2t} + h_t)/\beta$, it can be shown that

$$R_t \ge F_{1,t-1}^{-1}\left(\frac{\beta(r_1 + b) - r_{2t} - h_t}{\beta(r_1 + b - r_{2,t-1})}\right). \tag{38}$$

(b) If $r_{2t} \ge c_t$, using equation (32) on the right-hand side of equation (10) for $z = z_t^*$, we get

$$\begin{aligned}\frac{\partial J_t(y, z_t^*)}{\partial y} &= -c_t - h_t + (r_1 + b + h_t)[1 - F_{1t}(y)] + (r_{2t} + h_t)F_{1t}(y - z_t^*) \\ &\quad + \beta E\left[\frac{dV_{t-1}(y - X_{1t})}{dy} 1(y - z_t^* \le X_{1t} \le y)\right].\end{aligned} \tag{39}$$

If $z_t^* = 0$, then from equation (39), we have

$$\begin{aligned}\frac{\partial J_t(y, 0)}{\partial y} &= -c_t - h_t + (r_1 + b + h_t)[1 - F_{1t}(y)] + (r_{2t} + h_t)F_{1t}(y) \\ &= -c_t - h_t + (r_1 + b + h_t)[1 - F_{1t}(y)] - (r_{2t} + h_t)[1 - F_{1t}(y)] + r_{2t} + h_t \\ &= r_{2t} - c_t + (r_1 + b - r_{2t})[1 - F_{1t}(y)] > 0.\end{aligned} \tag{40}$$

On the other hand, if $z_t^* = R_t$ then from equation (39), we have

$$\begin{aligned}\frac{\partial J_t(y, R_t)}{\partial y} &= -c_t - h_t + (r_1 + b + h_t)[1 - F_{1t}(y)] + (r_{2t} + h_t)F_{1t}(y - R_t) \\ &\quad + \beta E\left[\frac{dV_{t-1}(y - X_{1t})}{dy} 1(y - R_t \le X_{1t} \le y)\right].\end{aligned} \tag{41}$$

Using inequality (33) on right-hand side of equation (41), we observe that

$$\begin{aligned}\frac{\partial J_t(y, R_t)}{\partial y} &\ge -c_t - h_t + (r_1 + b + h_t)[1 - F_{1t}(y)] + (r_{2t} + h_t)F_{1t}(y - R_t) \\ &\quad + \beta\left(\frac{r_{2t} + h_t}{\beta}\right)E[1(y - R_t \le X_{1t} \le y)] \\ &= -c_t - h_t + (r_1 + b + h_t)[1 - F_{1t}(y)] + (r_2 + h)F_{1t}(y) > 0.\end{aligned} \tag{42}$$

Thus, from equations (40) and (42) it follows that $J_t(y, z_t^*)$ is increasing in y, and hence,

$y_t^* = C_t + I$.

On the other hand, if $r_{2t} < c_t$ using equation (32) on the right-hand side of equation (30), we obtain

$$\begin{aligned} 0 = &-c_t - h_t + (r_1 + b + h_t)[1 - F_{1t}(S_t)] + (r_{2t} + h_t)F_{1t}(S_t - z_t^*) \\ &+ \beta E\left[\frac{dV_{t-1}(y - X_{1t})}{dy}\bigg|_{y=S_t} 1(S_t - z_t^* \leq X_{1t} \leq S_t)\right], \end{aligned} \tag{43}$$

for $t = 1, \ldots, T$. If $z_t^* = 0$, then from equation (43), it follows that

$$0 = -c_t - h_t + (r_1 + b + h_t)[1 - F_{1t}(S_t)] + (r_{2t} + h_t)F_{1t}(S_t).$$

After some algebra, the above equation can be written as

$$S_t = F_{1t}^{-1}\left(\frac{r_1 + b - c_t}{r_1 + b - r_{2t}}\right). \tag{44}$$

If $z_t^* = R_t$, then from equation (43), we have

$$\begin{aligned} 0 = &-c_t - h_t + (r_1 + b + h_t)[1 - F_{1t}(S_t)] + (r_{2t} + h_t)F_{1t}(S_t - R_t) \\ &+ \beta E\left[\frac{dV_{t-1}(y - X_{1t})}{dy}\bigg|_{y=S_t} 1(S_t - R_t \leq X_{1t} \leq S_t)\right]. \end{aligned} \tag{45}$$

Using inequality (33) on right-hand side of equation (45), it follows that

$$\begin{aligned} 0 \geq &-c_t - h_t + (r_1 + b + h_t)[1 - F_{1t}(S_t)] + (r_{2t} + h_t)F_{1t}(S_t - R_t) \\ &+ \beta\left(\frac{r_{2t} + h_t}{\beta}\right)E[1(S_t - R_t \leq X_{1t} \leq S_t)], \\ 0 \geq &\; r_1 + b - p - (r_1 + b + h_t)F_{1t}(S_t) + (r_{2t} + h_t)F_{1t}(S_t). \end{aligned}$$

After some algebra, the above inequality can be written as

$$S_t \geq F_{1t}^{-1}\left(\frac{r_1 + b - c_t}{r_1 + b - r_{2t}}\right). \tag{46}$$

The lower bound for S_t follows from equations (44) and (46). □

We recall that for ample demand in the spot market, $K_{ut} = \beta(r_1 + b) - r_{2t} - h_t$ and $K_{ot} = r_{2t} + h_t - \beta r_{2,t-1}$. Suppose $c_{t-1} \leq (r_{2t} + h_t)/\beta$. Then, we have

$$F_{1,t-1}^{-1}\left(\frac{K_{ut}}{K_{ut} + K_{ot}}\right) - C_{t-1} \leq R_t \leq F_{1,t-1}^{-1}\left(\frac{K_{ut}}{K_{ut} + K_{ot}}\right) - C_{t-1} + z_{t-1}^*.$$

That is, the bounds for R_t are such that, if R_t units are carried to period $t-1$ and we produce up to C_{t-1}, then (a) the probability of satisfying all demand from the primary market in period $t-1$ is at least equal to the critical ratio, and (b) the probability of satisfying all demand from the primary market and retaining z_{t-1}^* units in period $t-1$ is at most equal to the critical ratio. On the other hand, if $c_{t-1} > (r_{2t} + h_t)/\beta$, R_t units are carried to period $t-1$ and we do not produce in period $t-1$, then (a) the probability of satisfying all demand from the primary market in period $t-1$ is at least equal to the critical ratio, and (b) the probability of satisfying all demand from the primary market and retaining z_{t-1}^* units in period $t-1$ is a at most equal to the critical ratio. As before, if $c_{t-1} \leq (r_{2t} + h_t)/\beta$, the bounds are independent of C_{t-1}.

Part (b) of Proposition 2 indicates that if $r_{2t} \geq c_t$, then selling in the spot market is profitable, and, hence, it is optimal to produce up to the capacity in each period. However, if $r_{2t} < c_t$, then it may not be optimal to produce up to the capacity. In this case, we again have $K'_{ut} = r_1 + b - c_t$ and $K'_{ot} = c_t - r_{2t}$. Then, in terms of K'_{ut} and K'_{ot}, S_t is bounded such that, if we produce up to S_t in period t, the (a) the probability of satisfying all demand from the primary market in period t is at least equal to the critical ratio, and (b) the probability of satisfying all demand from the primary market and retaining z_{t-1}^* units in period t is at most equal to the critical ratio, which is now equal to $K'_{ut}/(K'_{ut} + K'_{ot})$.

Next, Proposition 3 provides bounds for R_t and S_t based purely on the problem parameters for the case with ample production capacity and stationary cost parameters.

Proposition 3. *When the production capacity in each period is ample and the cost parameters are stationary (i.e., $C_t = \infty$, $r_{2t} = r_2$, $c_t = c$ and $h_t = h$ for $t = 1, \ldots, T$),* *(a) If* $c \leq (r_2 + h)/\beta$*, then* $z_t^* = 0$.

If $c > (r_2 + h)/\beta$*, then the critical retain-up-to level* R_t *is bounded above as follows* $t = 2, \ldots, T$*:*

$$R_t \leq F_{1,t-1}^{-1}\left(\frac{r_1 + b - (r_2 + h)/\beta}{r_1 + b + h - \beta c}\right).$$

This upper bound is finite if $(r_2 + h)/\beta < c \leq (r_2 + h(1 + \beta))/\beta^2$ *and infinite if* $c > (r_2 + h(1 + \beta))/\beta^2$.

(b) The critical produce-up-to level S_t *is bounded as follows for* $t = 1, \ldots, T$*,*

$$S_t \leq F_{1,t}^{-1}\left(\frac{r_1 + b - c}{r_1 + b + \min\{-r_2, h - \beta c\}}\right).$$

This upper bound is finite if $p > r_2$ *and infinite if* $c \leq r_2$.

Proof. Suppose that ample production capacity is available in each period and cost parameters are stationary (i.e., $C_t = \infty$, $r_{2t} = r_2$, $c_t = c$ and $h_t = h$ for $t = 1, \ldots, T$). Then, substituting $C_{t-1} = \infty$ in equation (24) for $t = 1, \ldots, T$, we have

$$\frac{dV_t(I)}{dI} \begin{cases} = c & \text{if } I \leq S_t, \\ < c & \text{if } I > S_t. \end{cases} \tag{47}$$

(a) If $c \leq (r_2 + h)/\beta$, then from the first part of (47), we have $dV_t(I)/dI|_{I=0} = c \leq (r_2 + h)/\beta$. Then, from part (a) of Theorem 1, it follows that $z_t^* = 0$. Next, suppose that $c > (r_2 + h)/\beta$. Since this is a special case of the problem considered in Section II.3, equation (29) holds with $r_{2t} = r_2$, $c_t = c$ and $h_t = h$ for $t = 1, \ldots, T$. Using inequalities (20) and (47) on the right-hand side of (29) and after some algebra, we have

$$\begin{aligned} \frac{r_2 + h}{\beta} &\leq r_1 + b - (r_1 + b + h - \beta c)F_{1,t-1}(R_t) - (\beta c - r - h)F_{1,t-1}(R_t - z_{t-1}^*) \\ &\leq r_1 + b - (r_1 + b + h - \beta c)F_{1,t-1}(R_t). \end{aligned}$$

Rearranging the above inequality, we have

$$R_t \leq F_{1,t-1}^{-1}\left(\frac{r_1 + b - (r_2 + h)/\beta}{r_1 + b + h - \beta c}\right).$$

Note that $(r_1 + b - (r_2 + h)/\beta)/(r_1 + b + h - \beta c) \leq 1$ if $p \leq (r_2 + h(1 + \beta))/\beta^2$, and hence, this upper bound on R_t is finite. If $c > (r_2 + h(1 + \beta))/\beta^2$, then this upper bound on R_t is infinite.

(b) If $c \leq (r_2 + h)/\beta$, then from part (a) $z_t^* = 0$. Substituting $z_t^* = 0$ in part (b) of Proposition 1, we have

$$S_t \leq F_{1,t}^{-1}\left(\frac{r_1 + b - c}{r_1 + b - r_2}\right). \tag{48}$$

Next, suppose that $c > (r_2 + h)/\beta$. Then, using the same argument as in part (a) above, we know that equation (30) holds with $r_{2t} = r_2$, $c_t = c$ and $h_t = h$ for $t = 1, \ldots, T$. Using inequalities (20) and (47) on the right-hand side of (29), and following a similar argument as used above for establishing the upper bound for R_t, it can be shown that

$$S_t \leq F_{1,t}^{-1}\left(\frac{r_1 + b - c}{r_1 + b + h - \beta c}\right). \tag{49}$$

We note that when $c \leq (r_2+h)/\beta$, we have $-r_2 \leq h-\beta c$, and hence, $\min\{-r_2, h-\beta c\} = -r_2$. On the other hand, if $c > (r_2 + h)/\beta$, then $\min\{-r_2, h - \beta c\} = h - \beta c$. Then, combining the inequalities in (48) and (49), the upper bound on S_t can be expressed as in part (b) of Proposition 3. Furthermore, similar to part (b) of Proposition 1 we see that if $c \leq r_{2t}$ then the upper bound on S_t as presented in part (b) of Proposition 3 is infinite. $\square$

In Figure 4, we summarize the collective results of Propositions 1, 2 and 3 to show the regions where we can find finite upper bounds for S_t and R_t. To this end, we let S_{tu} and R_{tu} denote the upper bounds on S_t and R_t, respectively, given by Proposition 1. Let R_{tl}^a denote the lower bound on R_t for the case with ample demand as given by Proposition 2, and R_{tu}^c denote the upper bound on R_t for the case with ample capacity and stationary cost parameters as given by Proposition 3. We use the results from Propositions 1, 2 and 3 to

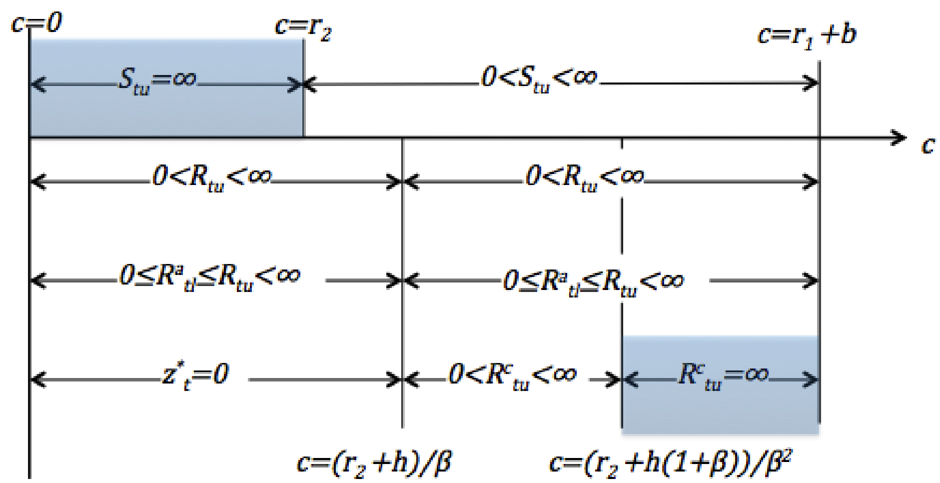


Figure 4: Graphical representation of conditions when bounds on S_t and R_t are finite

develop a heuristic policy in Section II.6.2 and demonstrate its effectiveness.

II.4.2 Impact of the spot market revenue on the optimal policy

From Section II.1, recall that the two practical benchmark policies, retain-none and retain-all, may be employed to manage the inventory system under consideration. These policies are actually special cases of the optimal policy. Under the retain-none policy, the entire left-over inventory is liquidated in the spot market after the demand from the primary market is satisfied. Hence, the retain-none policy is obtained by setting $z_t = 0$, for $t = 1, \ldots, T$ and using a modified base-stock policy for the production decision. Let $V_{t,none}(I)$ be the expected profit under the retain-none policy. Then, substituting $z = 0$ in equations (6) and (7) we obtain the following dynamic program

$$V_{t,none}(I) = c_t I + J_{t,none}(y_{t,none}, 0), \quad \text{where} \tag{50}$$

$$\begin{aligned} J_{t,none}(y, 0) = &(r_1 + h_t)E[X_{1t}] - (r_{2t} + h_t)(E[X_{2t}] - H_t(y)) - (c_t + h_t)y \\ &- (r_1 + b - r_{2t})G_{1t}(y) + \beta\big[E[V_{t-1,none}(0)1(X_{1t} + X_{2t} > y)] \\ &+ E[V_{t-1,none}(y - X_{1t} - X_{2t})1(X_{1t} + X_{2t} \le y)]\big], \end{aligned}$$

$$y_{t,none} = \begin{cases} I & \text{if } S_{t,none} < I, \\ S_{t,none} & \text{if } I \le S_{t,none} \le C_t + I, \\ C_t + I & \text{if } S_{t,none} > C_t + I, \end{cases} \tag{51}$$

and the base-stock or produce-up-to level, $S_{t,none}$, is determined by solving the first order condition for $J_{t,none}(y, 0)$.

On the other hand, under the retain-all policy, the entire left-over inventory is carried to the next period and the spot market is ignored, i.e., $z_t = y_t$ for $t = 1, \ldots, T$. Hence, to compute the optimal production decision under the retain-all policy we substitute $z = y$ in equations (6) and (7), which gives the following dynamic program:

$$V_{t,all}(I) = c_t I + \max\{J_{t,all}(y, y) : I \le y \le C + I\}, \quad \text{where} \tag{52}$$

$$\begin{aligned} J_{t,all}(y, y) = &(r_1 + h_t)E[X_{1t}] - (c_t + h_t)y - (r_1 + b + h_t)G_{1t}(y) \\ &+ \beta\big[E[V_{t-1,all}(0)1(X_{1t} > y)] + E[V_{t-1,all}(y - X_{1t})1(X_{1t} \le y)]\big], \end{aligned} \tag{53}$$

for $t = 1, \ldots, T$. We observe that (52) and (53) represents a multi-period inventory problem with limited capacity and non-stationary parameters, and hence, a modified base-stock policy is optimal for the production decision under the retain-all policy [6]. We observe that these alternate benchmark policies are easy to compute. Thus, one immediate question of interest is, "under what conditions are the benchmark policies optimal?" We address this question by studying the impact of the unit revenue from the spot market on the optimal policy.

Proposition 4. *For $t = 1, 2, \ldots, T$*

(a) There exist thresholds r^l_{2t} and r^u_{2t}, $r^l_{2t} \le r^u_{2t}$, given by $r^l_{2t} = \max\{r_{2t} \ge 0 : z^*_t \ge y^*_t\}$, and $r^u_{2t} = \min\{r_{2t} \ge 0 : z^*_t = 0\}$.

(i) If $r_{2t} \le r^l_{2t}$, then a retain-all policy is optimal in period t.

(ii) If $r_{2t} \ge r^u_{2t}$, then a retain-none policy is optimal in period t.

(b) r^l_{2t} decreases in C_t.

(c) r^u_{2t} *decreases in* C_{t-1}.

Proof. We first show that the following results are true: (1) z^*_t is a decreasing function of r_{2t}, and (2) y^*_t is an increasing function of r_{2t}. Then, we use these results to provide a proof for Proposition 4.

From Theorem 1, we know that

$$\left.\frac{dV_{t-1}(z)}{dz}\right|_{z=R_t} = \frac{r_{2t}+h_t}{\beta}.$$

We note that $dV_{t-1}(z)/dz$ is independent of r_{2t}. In addition, from concavity of $V(\cdot)$, it follows that $dV_{t-1}(z)/dz$ is a decreasing function of z. Therefore, R_t is a decreasing function of r_{2t}. Consequently, from equation (8) it follows that z^*_t is a decreasing function of r_{2t}.

We first show that S_t is an increasing function of r_{2t}. Evaluating expression (10) at $z = z^*_t$, we have

$$\begin{aligned}\frac{\partial J_t(y,z^*_t)}{\partial y} &= -c_t - h_t + (r_1+b+h_t)[1-F_{1t}(y)] + (r_{2t}+h_t)[F_{1t}(y-z^*_t) - F_{st}(y-z^*_t)] \\ &\quad + \beta E\left[\frac{dV_{t-1}(y-X_{1t}-X_{2t})}{dy}1(X_{1t}+X_{2t}\le y-z^*_t)\right] \\ &\quad + \beta E\left[\frac{dV_{t-1}(y-X_{1t})}{dy}1(y-z^*_t\le X_{1t}\le y)\right]. \end{aligned} \tag{54}$$

Let us denote the expression on the right-hand side of equation (54) with $G(y, z^*_t, r_{2t})$. Then, from equation (30), we have $G(S_t, z^*_t, r_{2t}) = 0$. By implicit differentiation, we have

$$\frac{\partial G}{\partial S_t}\partial S_t + \frac{\partial G}{\partial z^*_t}\partial z^*_t + \frac{\partial G}{\partial r_{2t}}\partial r_{2t} = 0. \tag{55}$$

From equation (55), we write

$$\frac{\partial S_t}{\partial r_{2t}} = \frac{-\frac{\partial G}{\partial z^*_t}\frac{\partial z^*_t}{\partial r_{2t}} - \frac{\partial G}{\partial r_{2t}}}{\frac{\partial G}{\partial S_t}} \tag{56}$$

We can express each term in (56) as follows:

$$\frac{\partial G}{\partial z_t^*} = \left.\frac{\partial^2 J_t(y, z_t^*)}{\partial y \partial z_t^*}\right|_{y=S_t}$$
$$= \left[\beta \left.\frac{dV_{t-1}(z)}{dz}\right|_{z=z_t^*} - (r_{2t} + h_t)\right][f_{1t}(S_t - z_t^*) - f_{st}(S_t - z_t^*)]. \quad (57)$$

$$\frac{\partial z_t^*}{\partial r_{2t}} \begin{cases} = 0 & \text{if } z_t^* = 0, \\ < 0 & \text{if } z_t^* = R_t. \end{cases} \quad (58)$$

$$\frac{\partial G}{\partial r_{2t}} = \left.\frac{\partial^2 J_t(y, z_t^*)}{\partial y \partial r_{2t}}\right|_{y=S_t} = F_{1t}(S_t - z_t^*) - F_{st}(S_t - z_t^*) > 0 \quad (59)$$

$$\frac{\partial G}{\partial S_t} = \left.\frac{\partial^2 J_t(y, z_t^*)}{\partial y^2}\right|_{y=S_t} < 0. \quad (60)$$

The negativity in expression (60) follows from the fact that $J_t(y, z_t^*)$ is concave in y. Observe that if $z_t^* = 0$, then $\partial z_t^* / \partial r_{2t} = 0$. On the other hand, if $z_t^* = R_t$, then $\partial G / \partial z_t^* = 0$. From equation (57), it follows that for $z_t^* \geq 0$ we can write equation (56), as

$$\frac{\partial S_t}{\partial r_{2t}} = \frac{-\frac{\partial G}{\partial r_{2t}}}{\frac{\partial G}{\partial S_t}} > 0, \quad (61)$$

where the last inequality follows from expressions (59) and (60). Thus, S_t is an increasing function of r_{2t}. Consequently, from equation (9), we observe that y_t^* is an increasing function of r_{2t}.

(a) Part (i) follows from the expression of r_{2t}^l and the fact that z_t^* is decreasing and y_t^* is increasing in r_{2t}. Part (ii) follows directly from the expression of r_{2t}^u and the fact that z_t^* is a decreasing function of r_{2t}.

(b) From equation (9), we observe that y_t^* is increasing in C_t. We also note that z_t^* is independent of C_t. Therefore, r_{2t}^l is decreasing in C_t.

(c) Similarly, from part (a) of Proposition 1, we observe that R_t is decreasing in C_{t-1}. Therefore, if follows that z_t^*, and hence, r_{2t}^u is decreasing in C_{t-1}. □

It is intuitive and easy to see that as r_{2t} increases it is optimal to retain less and produce more. We use this fact to provide insightful results. Part (a) of Proposition 4 shows that for all values of r_{2t} less than r_{2t}^l, it is optimal to ignore the spot market in period t. Thus, r_{2t}^l represents the maximum value of r_{2t} for which it is optimal to ignore the spot market in period t. On the other hand, for all r_{2t} greater than r_{2t}^u, it is optimal to liquidate the entire left-over inventory in the spot market in period t. Thus, r_{2t}^u represents the minimum value of r_{2t} for which it is optimal to liquidate the entire left-over inventory in the spot market in period t. Parts (b) and (c) show the impact of the production capacities in the current period and in the next period on r_{2t}^l and r_{2t}^u. Note that for $r_{2t}^l < r_{2t} < r_{2t}^u$ the optimal policy does not simplify to any of the benchmark policies.

Computing the thresholds r_{2t}^l and r_{2t}^u for given model parameters will considerably simplify the implementation of the optimal policy. In general, it is difficult, if not impossible, determine these thresholds in closed form. However, under certain conditions, these thresholds can be determined easily, as we discuss next.

First, we conjecture that $\beta c_{t-1} - h_t \leq r_{2t}^u \leq \beta(r_1 + b) - h_t$. For an intuitive explanation, observe that if $r_2 < \beta c_{t-1} - h_t$, then it is profitable to satisfy the primary market demand in period $t-1$ by carrying units from period t. Therefore, $z_t^* > 0$. On the other hand, if $r_{2t} > \beta(r_1 + b) - h_t$, then it is more profitable to sell in the spot market in period t than to sell in the primary market in period $t-1$. Hence, $z_t^* = 0$. Thus, we have $\beta c_{t-1} - h_t \leq r_{2t}^u \leq \beta(r_1 + b) - h_t$. Next, we compute r_{2t}^u under specific conditions on C_{t-1}. Suppose $S_{t-1} \leq C_{t-1}$ and $r_{2t} \geq \beta c_{t-1} - h_t$. It is easy to verify that

$$\left.\frac{dV_{t-1}(I)}{dI}\right|_{I=0} = c_{t-1} \leq \frac{r_{2t} + h_t}{\beta}.$$

Then, from Theorem 1, it follows that $z_t^* = 0$. Therefore, if $S_{t-1} \leq C_{t-1}$, i.e., there is enough production capacity available in the next period, then $r_{2t}^u = \beta c_{t-1} - h_t$.

In the next section, we provide closed form results to determine these thresholds for the infinite horizon problem. Furthermore, for the finite horizon problem, we examine the behavior of these thresholds computationally in Section II.6.

II.5 Infinite Planning Horizon

In this section, we show that for the infinite-horizon problem, a stationary $(S,\, R)$ policy is optimal. Moreover, for the infinite horizon case, we provide analytical formulae to determine the thresholds on the unit revenue earned from the spot market such that the optimal policy reduces to one of the benchmark policies.

Let us consider an infinite planning horizon with stationary model parameters and demand distributions such that $\beta(r_1 + b) > r_2 + h$. Let $V(I)$ be the maximum expected total discounted profit with starting inventory I and a discount factor $0 < \beta < 1$ over an infinite planning horizon. Then $V(I)$ satisfies the following optimality equation:

$$V(I) = pI + \max\{J(y,z) : I \le y \le C + I,\ z \ge 0\},\ \text{where} \tag{62}$$

$$\begin{aligned} J(y,z) = P(y,z) + \beta\big[&E[V(0)1(X_1 > y)] + \beta E[V(z)1(X_1 \le y - z, X_1 + X_2 > y - z)] \\ &+ E[V(y - X_1 - X_2)1(X_1 + X_2 \le y - z)] \\ &+ E[V(y - X_1)1(y - z \le X_1 \le y)]\big] \end{aligned} \tag{63}$$

and $P(y,z)$ is as defined in equation (4) without the subscript t. The objective is to compute $V(I)$ and to determine y^* and z^* that achieve this maximum expected discounted profit over an infinite planning horizon. Proposition 5 presents how to determine the optimal policy parameters y^* and z^* and shows that a stationary $(S,\, R)$ policy is optimal.

Proposition 5. *$J(y,z)$ and $V(I)$ satisfy the following properties:*

(a) $J(y,z)$ has a finite maximizer denoted by (y^*, z^*) such that

$$z^* = \begin{cases} 0 & \text{if } \left.\frac{dV(z)}{dz}\right|_{z=0} \le \frac{r_2+h}{\beta}, \\ \\ R & \text{if } \left.\frac{dV(z)}{dz}\right|_{z=0} > \frac{r_2+h}{\beta}, \end{cases}$$

where R satisfies $\left.\frac{dV(z)}{dz}\right|_{z=R} = \frac{r_2+h}{\beta}$ and

$$y^* = \begin{cases} I & \text{if } S < I \\ S & \text{if } I \le S \le C+I, \\ C+I & \text{if } S > C+I, \end{cases}$$

where S is determined by $J(S,z^*) = \max\{J(y,z^*) : y \in \Re^+\}$.

(b) $V(I)$ is a concave function of I.

(c) $\frac{dV(I)}{dI} \le r_1 + b$.

Proof. The proof follows from the value iteration algorithm and induction. Let $V^k(I)$ and $J^k(y,z)$ be the values of $V(I)$ and $J(y,z)$ at the k-th iteration of the value-iteration algorithm, respectively. Let us set $V^0(I) = 0$ for $I \ge 0$ and define

$$V^k(I) = cI + \max\{J^k(y,z) : I \le y \le C+I, \ , z \ge 0\}, \tag{64}$$

$$\begin{aligned} J^k(y,z) = P(y,z) + \beta\big[&E[V^{k-1}(0)1(X_1 > y)] + E[V^{k-1}(z)1(X_1 \le y-z, X_1+X_2 > y-z)] \\ &+ E[V^{k-1}(y - X_1 - X_2)1(X_1 + X_2 \le y-z)] \\ &+ E[V^{k-1}(y-X_1)1(y-z \le X_1 \le y)]\big]. \end{aligned} \tag{65}$$

Evaluating expressions (64) and (65) for $k = 1$, it is easy to observe that $V^1(I) = V_1(I)$

and $J^1(y,z) = J_1(y,z)$ as given by equations (6) and (7) for $t = 1$ and stationary model parameters. In Theorem 1, we show that properties (a)-(c) are true for $V_1(I)$ and $J_1(y,z)$. Hence, properties (a)-(c) are true for $V^1(I)$ and $J^1(y,z)$.

Suppose that properties (a)-(c) are true at iterations $k-1, k-2, \ldots, 2$. We will show that they are true at iteration k. For iteration k, it is also straight forward to observe that equations (64) and (65) are the same as equations (6) and (7) with $t = k$. That is, $V^k(I) = V_k(I)$ and $J^k(y,z) = J_k(y,z)$. Thus, from Theorem 1 it follows that properties (a)-(c) are true at iteration k.

Then, from the convergence results of the value-iteration algorithm (see e.g., [61]), we have $\lim_{k\to\infty} V^k(I) = V(I)$, $\lim_{k\to\infty} J^k(y,z) = J(y,z)$ and the optimal policy is stationary. Thus, $V(I)$ and $J(y,z)$ satisfy properties (a)-(c). $\square$

Proposition 6 partially presents how to compute the stationary thresholds, r_2^l and r_2^u, on the unit revenue earned from the spot market such that the optimal policy reduces to one of the benchmark policies. Furthermore, it provides bounds on the critical produce-up-to level and conditions under which these bounds are tight.

Proposition 6. *For the infinite horizon case,*

(a) $r_2^l = \beta c - h$. *Therefore, if* $r_2 \leq \beta c - h$, *then* $z^* = \infty > y^*$, *and hence, the retain-all policy is optimal. Furthermore, the critical produce-up-to level is bounded below as follows:*

$$S \geq F_1^{-1}\left(\frac{r_1 + b - c}{r_1 + b + h - \beta c}\right). \tag{66}$$

The bound is tight when $C \geq F_1^{-1}\left(\frac{r_1 + b - c}{r_1 + b + h - \beta c}\right)$.

(b) Define $\bar{C}_1$ *as*

$$\bar{C}_1 = F_1^{-1}\left(\frac{\beta(r_1 + b) - r_2 - h}{\beta(r_1 + b - r_2)}\right). \tag{67}$$

(i) If $C \geq \bar{C}_1$, *then* $r_2^u = \beta c - h$. *Therefore, if* $C \geq \bar{C}_1$ *and* $r_2 \geq \beta c - h$, *then* $z^* = 0$, *and hence, the retain-none policy is optimal. Furthermore, the critical*

produce-up-to level satisfies the following inequality

$$(r_1 + b - r_2)F_1(S) + (r_2 + h - \beta c)F_s(S) \geq r_1 + b - c.$$

The above inequality is tight when $C \geq \max\{\bar{C}_1, \bar{C}_2\}$*, where* $\bar{C}_2$ *is such that*

$$0 = r_1 + b - c - (r_1 + b - r_2)F_1(\bar{C}_2) - (r_2 + h - \beta c)F_s(\bar{C}_2). \tag{68}$$

(ii) If $C < \bar{C}_1$*, then* $r_2^u \geq \beta c - h$*. Therefore, if* $C < \bar{C}_1$ *and* $r_2 \geq \beta c - h$*, then* $0 < z^* = R$*. Furthermore, the critical produce-up-to level is bounded below as in equation (66).*

Proof. Recall that $\lim_{t\to\infty} J_t(y, z) = J(y, z)$. Then, substituting $r_{2t} = r_2$, $c_t = c$, $h_t = h$, $C_t = C$, $F_{1t}(\cdot) = F_1(\cdot)$ and $F_{2t}(\cdot) = F_2(\cdot)$ for $t = 1, 2, \ldots$, on the right-hand side of equations (10) and (11), we have

$$\begin{aligned}
\frac{\partial J(y,z)}{\partial y} &= -c - h + (r_1 + b + h)[1 - F_1(y)] + (r_2 + h)[F_1(y - z) - F_s(y - z)] \\
&\quad + \beta E\left[\frac{dV(y - X_1 - X_2)}{dy} 1(X_1 + X_2 \leq y - z)\right] \\
&\quad + \beta E\left[\frac{dV(y - X_1)}{dy} 1(y - z \leq X_1 \leq y)\right], \qquad (69) \\
\frac{\partial J(y,z)}{\partial z} &= \left[\beta \frac{dV(z)}{dz} - (r_2 + h)\right][F_1(y - z) - F_s(y - z)]. \qquad (70)
\end{aligned}$$

Next, we note that the critical produce-up-to level, S, is the same for each period. Hence, even if $I > S$ at the beginning of the planning horizon, this initial inventory will deplete in time. Therefore, after a transient phase, we will have $I \leq S$ for each period. Then, using part (a) of Proposition 5, the optimal produce-up-to level can be expressed as

$$y^* = \begin{cases} S & \text{if } S \leq C + I, \\ C + I & \text{if } S > C + I. \end{cases} \tag{71}$$

Furthermore, using the fact that $\lim_{t\to\infty} V_t(I) = V(I)$ and equation (24), we have

$$\frac{dV(I)}{dI}\begin{cases} = c & \text{if } S \le C + I, \\ \ge c & \text{if } S > C + I. \end{cases} \tag{72}$$

(a) Suppose that $r_2 < \beta c - h$. Then, from equation (72), we note that $dV(I)dI \ge c > (r_2+h)/\beta$. Substituting this in equation (70), it can be easily shown that $J(y,z)$ is increasing for $z < y$, and constant for $z \ge y$. Therefore, $z^* = y$. Thus, $r_2^l = \beta c - h$. Next, using the definition of S (i.e., $\partial J(y,z^*)/\partial y|_{y=S} = 0$), $z^* = y$ and equation (69), we have

$$\left.\frac{\partial J(y,y)}{\partial y}\right|_{y=S} = 0 = -c - h + (r_1 + b + h)[1 - F_1(S)] + \beta E\left[\left.\frac{dV(y - X_1)}{dy}\right|_{y=S} 1(0 \le X_1 \le S)\right].$$

Using equation (72) on the right-hand side of the above expression and simplifying the resulting expression, we have

$$S \ge F_1^{-1}\left(\frac{r_1 + b - c}{r_1 + b - \beta c + h}\right). \tag{73}$$

Furthermore, the inequality in (73) is tight when the inequality in (72) is tight. That is, when there is *sufficient* capacity so that $S \le C$, then $S = F_1^{-1}((r_1+b-c)/(r_1+b-\beta c+h))$. In other words, the inequality in (73) is tight when $C \ge F_1^{-1}\big((r_1 + b - c)(r_1 + b - \beta c + h)\big)$.

(b) Next, suppose that $r_2 \ge \beta c - h$. From the proof for Theorem 1, we recall that $V^k(I) = V_k(I)$. Let z^{k*} be the optimal retain-up-to level at iteration k of the value iteration algorithm and R^k be such that $dV^{k-1}(z)/dz|_{z=R^k} = (r_2 + h)/\beta$. Then, $R^k = R_k$, and the upper bound for R_k as given by Proposition 1 also holds for R^k. More specifically, we have

$$R^k = R_k \le F_1^{-1}\left(\frac{\beta(r_1 + b) - r_2 - h}{\beta(r_1 + b - r_2)}\right) - C + z^*_{k-1} = \bar{C}_1 - C + z^*_{k-1}. \tag{74}$$

(b)(i) Define $\bar{C}_1$ as in equation (67). If $C \ge \bar{C}_1$, we have $R^k = R_k \le z^*_{k-1}$. From Theorem 1 and Proposition 5, we know that $z^{1*} = z^*_1 = 0$. Hence, for $k = 2$, $R^2 = R_2 \le 0$.

This implies that $z^{2*} = z_2^* = 0$. Continuing in this manner, we see that $z^{k*} = 0$ for all k. Therefore, $z^* = \lim_{k\to\infty} z^{k*} = 0$. Thus, $r_2^u = \beta c - h$. Next, using the definition of S (i.e., $\partial J(y, z^*)/\partial y|_{y=S} = 0$), $z^* = 0$, and equation (69), we have

$$\left.\frac{\partial J(y,0)}{\partial y}\right|_{y=S} = 0 = -c - h + (r_1 + b + h)[1 - F_1(S)] + (r_2 + h)[F_1(S) - F_s(S)] + \beta E\left[\left.\frac{dV(y - X_1 - X_2)}{dy}\right|_{y=S} 1(X_1 + X_2 \le S)\right].$$

Using equation (72) on the right-hand side of the above expression and simplifying the resulting expression, it follows that S is such that

$$0 \ge r_1 + b - c - (r_1 + b - r_2)F_1(S) - (r_2 + h - \beta c)F_s(S). \quad (75)$$

Again, from equation (72), we see that the inequality in (75) is tight when $S \le \bar{C}_2$, i.e., and $C \ge \max\{\bar{C}_1, \bar{C}_2\}$, where $\bar{C}_2$ is such that

$$0 = r_1 + b - c - (r_1 + b - r_2)F_1(\bar{C}_2) - (r_2 + h - \beta c)F_s(\bar{C}_2). \quad (76)$$

(b)(ii) Next, suppose that $C < \bar{C}_1$. From proof of part (c) of Proposition 4, we have z_k^* is decreasing in C and from part (b)(i) above, we know that if $C \ge \bar{C}_1$ then $z_k^* = 0$. Hence, if $C < \bar{C}_1$ then $z_k^* \ge 0$. Then, using $z^{k*} = z_k^*$ for all k, we have $z^* = \lim_{k\to\infty} z^{k*} \ge 0$. Hence, $r_2^u \ge \beta c - h$. Again, using the definition of S (i.e., $\partial J(y, z^*)/\partial y|_{y=S} = 0$), $z^* = R$ and following a similar argument as in part (a) above, it can be shown that S is bounded below as in equation (66).

□

Part (a) of Proposition 6 shows that for the infinite horizon problem $r_2^l = \beta c - h$, and hence, if $r_2 \le \beta c - h$ then the retain-all policy is optimal. Part (b) of Proposition 6 show that if there is enough production capacity available, as given by expression (67), then $r_2^l = r_2^u = \beta c - h$. In other words, if there is enough production capacity available, then the optimal policy reduces to one of the benchmark policies depending on whether $r_2 \leq \beta c - h$ or $r_2 \geq \beta c - h$. Thus, the quantity $\beta c - h$ represents an important threshold. As we discuss in the next section, these results prove to be insightful for the finite horizon model as well. On the other hand, if enough production capacity is not available, then from parts (a) and (b)(ii) of Proposition 6 we have $r_2^l = \beta c - h \leq r_2^u$. Hence, when r_2 is such that $r_2^l < r_2 < r_2^u$ the optimal policy does not reduce to either of the benchmark policies. Furthermore, Proposition 6 provides an lower bound on S, and shows that the bound is tight when production capacity is above a certain threshold.

II.6 Computational Results

In this section, we present the results of our computational study, which has a four objective. First, we develop a heuristic policy based on the bounds in Section II.4.1 and examine its performance. Second, we numerically examine the thresholds on the unit revenue from the spot market such that the optimal policy reduces to one of the benchmark policies. Third, we quantify the benefits of the optimal policy over the benchmark policies. Finally, we study the effect of demand correlation on the benefits of the optimal policy.

II.6.1 Experimental setup

In the interest of a more focused discussion of the key insights of our model, the computational results reported in this section assume that all model parameters are stationary, i.e., $r_{2t} = r_2$, $c_t = c$, $h_t = h$, and $C_t = C$ for $t = 1, \ldots, T$. We set $T = 9$, $\beta = 1$, and use the truncated Normal distribution for the primary and spot market demands. We analyze two experimental settings: In Setting 1, demand parameters $E[X_{1t}] \sim \text{Unif}[100, 150]$ and $E[X_{2t}] \sim \text{Unif}[40, 80]$ for $t = 1, \ldots, 9$. In setting 2, $E[X_{1t}] \sim \text{Unif}[40, 80]$ and $E[X_{2t}] \sim \text{Unif}[100, 150]$ for $t = 1, \ldots, 9$. Thus, under Setting 1 (Setting 2) the primary market size is larger (smaller) than the spot market size. We consider two different values for the standard deviation to examine the effect of demand uncertainty ($\sigma_{it} = \alpha E[X_{it}]$, $\alpha = 0.1$ or 0.3). Also, to study the effect of demand correlation we consider three values for correlation coefficient, $\rho = -0.9, 0, 0.9$. Based on parameter values presented in Table 2, we construct an experi-

Setting	α	c_t	h_t	r_1	r_{2t}	b	C_t	ρ
1,2	0.1	50	1.25	75	30	1.1(r_1-c_t)	150	-0.9
	0.3		2.5	150	$\vdots$		300	0.0
			5.0	300	80			0.9

Table 2: Experimental setup for the joint replenishment and liquidation problem

mental design corresponding to 1296 parameter settings. For each of these 1296 parameter settings, we generate five different sets of values for demand parameters $E[X_{1t}]$ and $E[X_{2t}]$ (see Tables 3 and 4). Hence, we consider a total of 6480 problem instances.

As observed in Section II.4 and Section II.5, $\beta c_{t-1} - h_t$ represents an important threshold for r_{2t}. Hence, in our numerical study, although c_t is fixed, we consider three and six different values for h_t and r_{2t}, respectively. Thus, we consider both cases: $r_{2t} \leq \beta c_{t-1} - h_t$ ($r_{2t} = 30, 40$) and $r_{2t} > \beta c_{t-1} - h_t$ ($r_{2t} = 50, \ldots, 80$). In order to investigate the impact of production capacity constraints, we consider two different values for the available production capacity ($C_t = 150, 300$). We note that for each of the 1296 parameter settings the inequality $r_1 + b > r_{2t} + h_t$ is satisfied, as assumed for our theoretical results.

Set	t	1	2	3	4	5	6	7	8	9
1	$E[X_{1t}]$	111	141	129	126	131	113	111	101	130
	$E[X_{2t}]$	70	77	56	79	45	42	56	63	55
2	$E[X_{1t}]$	132	113	142	144	144	104	146	104	121
	$E[X_{2t}]$	40	52	69	74	61	72	68	53	48
3	$E[X_{1t}]$	149	114	107	103	117	123	127	113	146
	$E[X_{2t}]$	43	69	69	62	58	60	40	50	77
4	$E[X_{1t}]$	103	141	126	138	137	128	137	101	107
	$E[X_{2t}]$	54	73	46	43	77	51	74	53	43
5	$E[X_{1t}]$	113	145	132	133	141	132	136	120	102
	$E[X_{2t}]$	56	60	74	49	52	67	72	60	76

Table 3: Demand parameters for Setting 1

Set	t	1	2	3	4	5	6	7	8	9
1	$E[X_{1t}]$	80	68	73	63	49	51	55	68	68
	$E[X_{2t}]$	114	114	119	120	145	101	132	129	131
2	$E[X_{1t}]$	70	47	57	71	58	56	58	55	71
	$E[X_{2t}]$	131	139	140	116	148	104	115	126	117
3	$E[X_{1t}]$	51	43	68	68	80	80	75	55	41
	$E[X_{2t}]$	108	113	128	106	115	145	114	105	116
4	$E[X_{1t}]$	48	65	61	47	41	40	66	44	78
	$E[X_{2t}]$	150	135	144	113	105	124	125	105	116
5	$E[X_{1t}]$	59	50	69	72	69	45	79	64	65
	$E[X_{2t}]$	122	144	105	134	122	118	147	105	126

Table 4: Demand parameters for Setting 2

II.6.2 Heuristic policy

In this section, we propose a heuristic policy to compute S_t and R_t values. This heuristic policy is based on Propositions 1 and 2, and a myopic newsvendor solution. We denote the produce-up-to and retain-up-to levels computed by the heuristic policy by S_{th} and R_{th}, respectively. Below, we present the heuristic, discuss each step of the heuristic and demonstrate its effectiveness. The heuristic is as follows:

1. Compute $\bar{S}_1$ as the critical produce-up-to level when $r_{21} = c_1$, i.e.,

$$\left.\frac{\partial J_1(y, 0|r_{21} = c_1)}{\partial y}\right|_{y=\bar{S}_1} = (r_1 + b - c_1)(1 - F_{11}(\bar{S}_1)) - (c_1 + h_1)F_{s1}(\bar{S}_1) = 0. \quad (77)$$

2. For $t = 1$, set $R_{1h} = z_1^* = 0$ and compute S_{1h} as follows:

$$\text{If } c_1 \geq r_{21}, \text{ then } \quad S_{1h} = \min\left\{F_{11}^{-1}\left(\frac{r_1 + b - c_1}{r_1 + b - r_{21}}\right), \bar{S}_1\right\}, \quad (78)$$

$$\text{else } \quad S_{1h} = \max\left\{F_{11}^{-1}\left(\frac{r_1 + b - c_1}{r_1 + b + h_1}\right) + F_{21}^{-1}\left(\frac{r_{21} - c_1}{r_{21} + h_1}\right), \bar{S}_1\right\}. \quad (79)$$

3. For $t > 1$ define $\bar{C}_{1,t-1} = F_{1,t-1}^{-1}\left(\frac{\beta(r_1 + b) - r_{2t} - h_t}{\beta(r_1 + b - r_{2,t-1})}\right)$, and set $z_t = R_{th}$ and compute R_{th} as follows:

(a) If $c_{t-1} \leq (r_{2t} + h_t)/\beta$, then set $R_{th} = \max\{\bar{C}_{t-1} - C_{t-1}, 0\}$.

(b) If $c_{t-1} > (r_{2t} + h_t)/\beta$, then set $R_{th} = \bar{C}_{t-1}$.

Furthermore, compute S_{th} as follows: $S_{th} = F_{1t}^{-1}\left(\frac{r_1 + b - c_t}{r_1 + b + h_t}\right) + F_{2t}^{-1}\left(\frac{r_{2t} - c_t}{r_{2t} + h_t}\right)$.

Step 1 of the heuristic computes $\bar{S}_1$ as the critical produce-up-to level for $t = 1$ by setting $r_{21} = c_1$. For $t = 1$, in Step 2 of the heuristic, we set $R_{1h} = z_1^* = 0$. Furthermore, we compute S_{1h} using the upper bound on S_1 as given by Proposition 1. We note that if $c_1 > r_{21}$ and $c_1 - r_{21} \approx 0$, or if $c_1 \leq r_{21}$, then the upper bound on S_1 as given by Proposition 1 goes to infinity, and hence, will be loose (see Figure 4). To address this, we note that S_1 is an increasing function of r_{21} (see proof of Proposition 4). Then, if $c_1 \geq r_{21}$, we bound S_{1h} from above by $\bar{S}_1$. On the other hand, when $c_1 < r_{21}$, we set S_{1h} as the maximum of $\bar{S}_1$ and the sum of single-period newsvendor solutions computed individually for the primary and the spot markets. For $t > 1$, Step 3 of the heuristic sets R_{th} to its upper bound as given by Proposition 1 when $z_{t-1}^* = 0$. Furthermore, the heuristic policy simplifies the computation of z_t by setting $z_t = R_{th}$. Based on extensive numerical studies, we observe that the upper bound on S_1 as given by Proposition 1 when $z_t^* = 0$ may not perform well for some parameter settings. Hence, for $t > 1$, we compute S_{1h} myopically as sum of single period newsvendor solutions computed individually for the primary and the spot markets.

Tables 5 and 6 presents performance of the heuristic policy, measured in terms of the percent optimality gap defined as

$$e = 100\left(\frac{\text{Expected profit under optimal policy} - \text{Expected profit under heuristic policy}}{\text{Expected profit under optimal policy}}\right)\%.$$

We list our observations as below:

O1 The heuristic policy is equivalent to the optimal policy, i.e., $e = 0$, for 10.2% and 23.9% instances for settings 1 and 2. We note that R_{th} is equal to the lower bound on R_t for the case with ample demand in the spot market. Hence, as $E[X_{2t}]$ increases in comparison to $E[X_{1t}]$, the percentage of instances with $e = 0$ increases.

$e = \%$ opti. gap	Setting 1, $T = 9$	
	% instances	Common features, if any
$e = 0$	10.2%	
$0 < e \leq 1$	86.6%	
$1 < e \leq 2$	2.7%	$r_2 > 50$, $\alpha > 0.1$
$2 < e \leq 3$	0.4%	$r_2 > 60$, $\alpha > 0.1$, $r_1 < 150$, $C < 300$, $h < 5$
$3 < e \leq 5$	0.1%	$r_2 > 70$, $\alpha > 0.1$, $r_1 < 150$, $C < 300$, $h < 2.5$

Table 5: Performance of the heuristic policy for Setting 1

O2 We observe that $e \leq 2\%$ for 99.5% and 96.1% of problem instances for settings 1 and 2, respectively. The maximum optimality gap over all instances is less than 10%. This indicates that the heuristic performs well compared to the optimal policy.

O3 Next, we investigate the problem parameters which may lead to poor performance of the heuristic policy. To this end, we determine if all the problem instances with optimality gap belonging to a particular range of e have any common features. Tables 5 and 6 enlist these common problem features, if any, for each range of e. We observe that e is higher when the difference between r_1 and r_2 is less (e.g., $r_2 > 50$ and $r_1 < 150$), demand variability, i.e., α, is high, and the production capacity is tight. To understand this, we recall that the heuristic sets R_{th} to its upper bound as given by Proposition 1 when $z^*_{t-1} = 0$. When the capacity is tight, z^*_{t-1} may not be zero, and hence, $R_t > R_{th}$. In addition, if the difference between r_1 and r_2 is less, then Step 3 of the heuristic shows

$e = \%$ opti. gap	Setting 2, $T = 9$	
	% instances	Common features, if any
$e = 0$	23.9%	
$0 < e \leq 1$	67.3%	
$1 < e \leq 2$	4.9%	$r_2 > 40$, $r_1 < 300$
$2 < e \leq 3$	2.2%	$r_2 > 40$, $\alpha > 0.1$, $r_1 < 150$
$3 < e \leq 5$	1.6%	$r_2 > 50$, $\alpha > 0.1$, $r_1 < 150$, $C < 300$, $\rho > -0.9$
$5 < e \leq 10$	0.1%	$r_2 > 50$, $\alpha > 0.1$, $r_1 < 150$, $C < 300$, $\rho > -0.9$, $h < 2.5$

Table 6: Performance of the heuristic policy for Setting 2

t	1	2	3	4	5	6	7	8	9
$E[X_{1t}]$	111	141	129	126	131	113	111	101	130
$E[X_{2t}]$	70	77	56	79	45	42	56	63	55

Table 7: Demand parameters for Setting 1 and value set 1

that R_{th} is reduced further. As a result, a smaller retain-up-to level is chosen, which may result in lost sales from the primary market in the next period leading to a low profit. On the other hand, when the capacity is high, the effect of a smaller retain-up-level can be addressed by producing more in the next period.

II.6.3 Behavior of the optimal policy with respect to the spot market revenue

From Section II.4.2, we know that there exist thresholds r_{2t}^l and r_{2t}^u such that (a) if $r_{2t} \leq r_{2t}^l$, then $z_t^* \geq y_t^*$ and retain-all policy is optimal, and (b) if $r_{2t} \geq r_{2t}^u$, then $z_t^* = 0$ and retain-none policy is optimal. Further, when $r_{2t}^l \leq r_{2t} \leq r_{2t}^u$ the optimal policy does not reduce to any of the benchmark policies. Thus, we say that the optimal policy has one of the following three forms: retain-all, retain-none and (S_t, R_t).

In Table 8, we illustrate this result for demand parameter values presented in Table 7. Note that, in Table 8, bold typeface indicates that enough production capacity is available for that period, i.e., $S_t \leq C_t$. We summarize our main observations as follows:

t, r_{2t}	30	40	50	60	70	80	90	100
1	**none**	**none**	**none**	none	none	none	none	none
2	(S_t,R_t)	(S_t,R_t)	none	none	none	none	none	none
3	all	all	(S_t,R_t)	(S_t,R_t)	(S_t,R_t)	(S_t,R_t)	(S_t,R_t)	none
4	all	all	(S_t,R_t)	(S_t,R_t)	(S_t,R_t)	(S_t,R_t)	none	none
5	all	all	(S_t,R_t)	none	none	none	none	none
6	**all**	**all**	**(S_t,R_t)**	(S_t,R_t)	(S_t,R_t)	(S_t,R_t)	none	none
7	**all**	**all**	**none**	none	none	none	none	none
8	**all**	**all**	**none**	none	none	none	none	none
9	all	all	none	none	none	none	none	none

Table 8: Form of the optimal policy: Setting 1, $\alpha = 0.1$, $h_t = 1.25$, $r_1 = 75$, $C_t = 150$.

O1 If there is only one period-to-go then the retain-none policy is optimal. This is intuitive because in the last period there is a single opportunity to earn revenue, and all inventory should be sold to make more profits.

O2 Under both settings 1 and 2, for $t \geq 3$ and $r_{2t} = 30, 40$ (i.e., $r_{2t} < \beta c_{t-1} - h_t$), the retain-all policy is optimal.

O3 Under Setting 1, for $t \geq 2$, $r_{2t} = 50, \ldots, 80$ (i.e., $r_{2t} \geq \beta c_{t-1} - h_t$) and $S_{t-1} \leq C_{t-1}$, the retain-none policy is optimal. Under Setting 2, we observe that $S_t < C_t$ for $t = 1, \ldots, 9$ for each experiment and for $r_{2t} = 50, \ldots, 80$, retain-none policy is optimal. Hence, for all 6480 problem instances, we verify that if $S_{t-1} \leq C_{t-1}$ then $r^u_{2t} = \beta c_{t-1} - h_t$.

O4 On the other hand, for $t \geq 2$, if $r_{2t} = 50, \ldots, 80$ (i.e., $r_{2t} \geq \beta c_{t-1} - h_t$) and $S_{t-1} > C_{t-1}$ then the magnitudes of the two critical quantities, S_t and R_t, and hence, the form of the optimal policy depends on several model parameters.

O5 In particular, as r_{2t} increases S_t increases while R_t decreases. Therefore, as seen from Table 8, the retain-none policy is optimal for higher values of r_{2t}.

O6 In contrast, both S_t and R_t increase as r_1 or the demand uncertainty, represented by α, increases. Similarly, both S_t and R_t decrease as h_t and C_{t-1} increase. These results are intuitive, for example, if unit revenue from the primary market is higher, profits can be increased by producing more and selling less in the spot market with a lower unit revenue. Therefore, an increase in r_1 or α widens the (S_t, R_t) region while an increase in h_t or C_{t-1} shrinks the (S_t, R_t) region.

O7 In addition, from an extended numerical study we observe that under stationary demand settings z^*_t decreases as the number of periods-to-go decreases. This is intuitive since the opportunities for sale decrease as we approach the end of the planning horizon, and hence, it is optimal to carry less inventory. Therefore, if r_{2t} is constant for the entire planning horizon, then it may be optimal to ignore the spot market for a few periods at the beginning of the planning horizon but liquidate more and more units as we move closer to the end of the planning horizon.

O8 In general, even under non-stationary demand settings z_t^* decreases as the number of periods-to-go decrease. For example, in Table 8 for $r_{2t} = 30, 40$, the retain-all policy is optimal for $t \geq 3$, the $(S_t,\, R_t)$ policy is optimal for $t = 2$, and the retain-none policy is optimal for $t = 1$. However, under non-stationary demand settings, z_t^* may increase due to change in demand parameters (mean and standard deviation) in the next period. The combined effect of t and demand parameters in the next period may lead to an increase in z_t^*. For example, in Table 8 for $r_{2t} = 60$, even though the retain-none policy is optimal for $t = 7$ (i.e., $z_7^* = 0$), the $(S_t,\, R_t)$ policy is optimal for $t = 6$ (i.e., $z_6^* = R_6 > 0$). This is because the mean demand increases from 113 for $t = 6$ to 131 for $t = 5$.

In summary, we note that when there are two or more periods-to-go, the two most important factors that determine z_t^*, and hence, the form of the optimal policy are: (1) relative values of r_{2t} and $\beta c_{t-1} - h_t$ and (2) relative values of S_{t-1} and C_{t-1}. Together, these factors define the boundary on which the optimal policy changes from one form to another. For example, when $S_{t-1} \leq C_{t-1}$ the optimal policy reduces to either one of the benchmark policies. In other words, when there is enough production capacity available in the next period $r_{2t}^l = r_{2t}^u$. On the other hand, when the production capacity available in the next period is not enough to meet the critical produce-up-to level (in the next period), we have $r_{2t}^l < r_{2t}^u$. Furthermore, we numerically verify the conjecture stated in SectionII.4.2 that $\beta c_{t-1} - h_t \leq r_{2t}^u \leq \beta(r_1 + b) - h_t$. The above observations are true for all of the 6480 problem instances.

II.6.4 Benefits of the optimal policy over the benchmark policies

We quantify the benefits of the optimal policy by computing the percentage increase in the maximum expected profit that can be achieved by employing the optimal policy rather than one of the benchmark policies, as follows

$$100\left(\frac{\text{Expected profit under optimal policy} - \text{Expected profit under benchmark policy}}{\text{Expected profit under optimal policy}}\right)\%.$$

We find that when there is only one period-to-go, the average (over all 6480 problem instances) profit improvement under the optimal policy compared to the retain-all policy is 15.9%, where the minimum improvement is 0.6% and maximum is 70.4%. That is when there is only one period-to-go ignoring the spot market under the retain-all policy results in a significant loss. However, in this case, we know that the retain-none policy is optimal. Furthermore, under Setting 1 for $C_t = 300$ and under Setting 2 for $C_t = 150, 300$, the production capacity constraint is never tight. Therefore, the optimal policy again reduces to one of the benchmark policies. However, under Setting 1 for $C_t = 150$ the capacity constraint is tight and the optimal policy may not reduce to either of the benchmark policies. Figure 5 shows that when the capacity constraint is tight ($C_t = 150$), the demand uncertainty is high

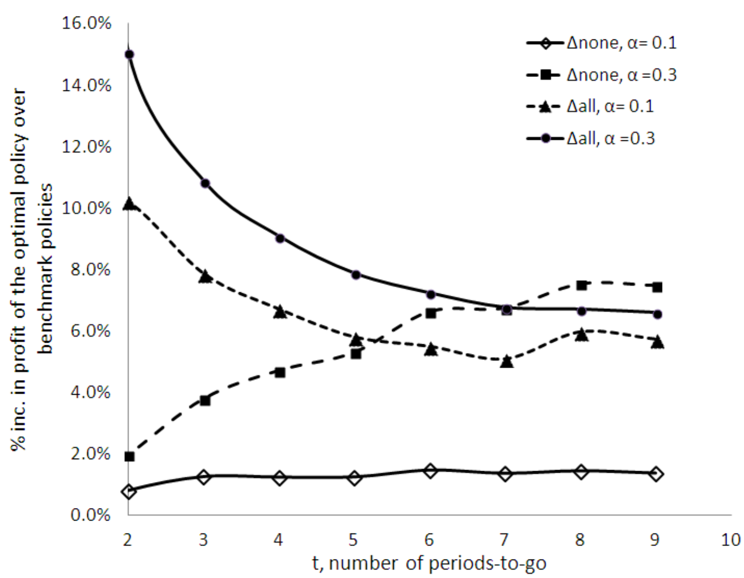


Figure 5: Average % improvement of the optimal policy over the benchmark policies: Setting 1, $C_t = 150$

($\alpha = 0.3$) and there are more than six periods-to-go, the profit improvement of the optimal policy over both the benchmark policies is significant, more than 5%.

We summarize our additional observations as follows:

O1 For $t \geq 2$, the optimal policy results in an average profit improvement of 7.1% over the retain-all policy, where the minimum benefit is 0.2% and the maximum benefit is 60.7%. On the other hand, for $t \geq 2$ the average profit improvement over the retain-none policy is 2.33%, where the minimum benefit is 0% and the maximum benefit is 25.3%.

O2 Profit improvement under the optimal policy is higher when the demand uncertainty is higher, i.e., $\alpha = 0.3$.

O3 Figure 5 illustrates that profit improvement under the optimal policy compared to the retain-all policy is high when the number of periods-to-go is less (8-12%) and decreases convexly with the increase in the number of periods-to-go.

O4 In contrast, profit improvement under the optimal policy compared to the retain-none policy is low when the number of periods-to-go is less (1-3%). When the demand uncertainty is low ($\alpha = 0.1$) and $t \geq 2$, the optimal policy leads to an average improvement of 1%. However, when the demand uncertainty is high ($\alpha = 0.3$), the benefits of the optimal policy increase concavely, more than 5%, with the increase in the number of periods-to-go.

II.6.5 Effect of demand correlation

Finally, we study the effect of correlation between the primary and spot market demands in each period t, denoted by ρ, on the benefits of the optimal policy. Our main observations as follow:

O1 For each of the 6480 problem instances, we find that the optimal retain-up-to level, z_t^* decreases as ρ increases. To understand this intuitively, we note that the distribution of the sum of the primary and spot market demands is important in computing the optimal policy parameters. As ρ increases, the variance of the the sum of the primary and spot

	Retain-none policy				Retain-all policy			
	$C = 150$		$C = 300$		$C = 150$		$C = 300$	
ρ, α	0.1	0.3	0.1	0.3	0.1	0.3	0.1	0.3
-0.9	1.3	5.9	0.8	2.2	4.1	8.2	8.9	11.2
0	1.2	4.0	0.8	2.0	4.1	7.2	8.0	10.2
0.9	1.2	4.4	0.8	1.8	4.0	6.3	7.8	9.3

Table 9: Effect of demand correlation on average % improvement of the optimal policy over the benchmark policies: Setting 1, $t \geq 2$

market demands increases. Therefore, as ρ increases the risk due to demand uncertainty increases, and hence, any opportunity to sell in the spot market should be well utilized. This in turn reduces the maximum optimal inventory to be carried to the next period, i.e., z_t^*.

O2 Table 9 shows that the average profit improvement of the optimal policy over the benchmark policies decreases as ρ increases. Furthermore, the effect of demand correlation is higher when demand uncertainty is higher, i.e., $\alpha = 0.3$.

Based on our observations so far, we conjecture that the optimal retain-up-to level in period t, z_t^*, is proportional to the difference between the mean primary market demand

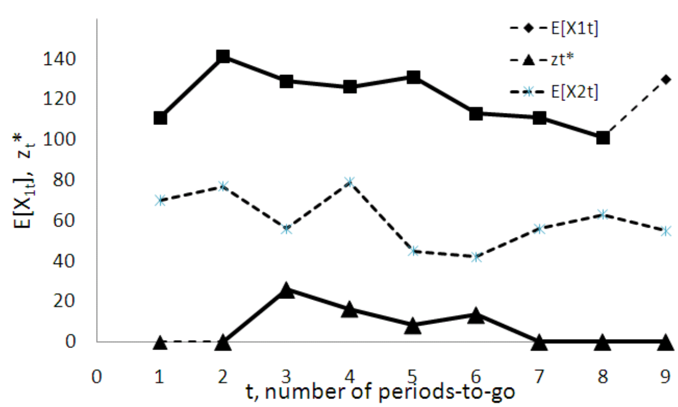


Figure 6: $E[X_{1t}]$, $E[X_{2t}]$ and z_t^* with respect to t: Setting 1, $\alpha = 0.1$, $h_t = 1.25$, $r_1 = 75$, $C_t = 150$

and the available production capacity in the next period, i.e., $E[X_{1,t-1}] - C_{t-1}$. We verify this conjecture numerically under constant production capacity. Figure 6 shows variation of $E[X_{1t}]$, $E[X_{2t}]$ and z_t^* with respect to the number of period-to-go for demand parameter values presented in Table 7. We observe that for periods $t = 2, \ldots, 9$, highlighted in bold, z_t^* follows a similar trend as $E[X_{1t}]$ but with a lag of one period. In other words, z_t^* is proportional to $E[X_{1,t-1}]$. This is intuitive because at the end of each period we retain in anticipation of future high-priority (primary market) demand. Therefore, if the primary market demand in the next period is high (low), it is optimal to retain more (less) at the end of the current period. (Though Figure 6 shows results for a particular problem instance, they are qualitatively true for other instances as well.) We also note that z_t^* does not seem to be affected by change in $E[X_{2t}]$. This is consistent with our observation that one of the most important factors that determines z_t^* is the relative value of C_{t-1} and S_{t-1}, which further depends on $E[X_{1,t-1}]$ (See Section II.6.3). In fact, this observation leads to an interesting question for future work: What happens if demand in different markets are correlated in time? In particular, what is the effect of correlation between the spot market demand in a given period and the primary market demand in the next period on the optimal policy parameters and benefits of the optimal policy?

Next, we consider the case where the primary market demand is backlogged.

II.7 Backlog Model

In this section, we consider that any unmet demand from the primary market in periods $t = 2, \ldots, T$ is backlogged and incurs a unit backlog penalty of $\bar{b}$. Furthermore, we consider that any unmet demand from the primary market in the last period, i.e., $t = 1$ is lost and incurs a unit penalty of b. We observe that the total cost of backlogging a unit from the primary market in period t is $\bar{b} + \beta c_{t-1}$. If the cost parameters are such that $\bar{b} + \beta c_{t-1} < r_{2t}$ for $t = 2, \ldots, T$, then it is more profitable to sell the current on-hand inventory in the spot market and backlog the primary market demand, and the retain-up-to decisions are trivial. In order to avoid such trivial cases, we assume that $\bar{b} + \beta c_{t-1} > r_{2t}$ for $t = 2, \ldots, T$, and thus, ensure that giving priority to the primary market customers is profitable.

Let $\bar{V}_t(I)$ be the maximum expected discounted profit when there are t periods to-go until the end of the planning horizon and the starting inventory is I. For each period $t = 2, \ldots, T$, with starting inventory I, the expected profit earned in period t when the produce-up-to level is y and retain-up-to level is z units is given as

$$\begin{aligned}\bar{P}_{0t}(I, y, z) = E\big[&- c_t(y - I) + r_1 X_{1t} - \bar{b}(X_{1t} - y)1(X_{1t} > y) - h_t(y - X_{1t})1(y - z \le X_{1t} \le y) \\ &+ [r_{2t}(y - X_{1t} - z) - h_t z]1(X_{1t} \le y - z, X_{1t} + X_{2t} > y - z) \\ &+ [r_{2t} X_{2t} - h_t(y - X_{1t} - X_{2t})]1(X_{1t} + X_{2t} \le y - z)\big], \qquad (80)\end{aligned}$$

and $\bar{P}_{01}(I, y, z) = P_{01}(I, y, z)$. For $t = 2, \ldots, T$, define $\bar{P}_t(y, z)$ as follows

$$\begin{aligned}\bar{P}_t(y, z) = (r_1 + h_t)E[X_{1t}] &+ (r_{2t} + h_t)E[X_{2t}] - (c_t + h_t)y - (\bar{b} + h_t)G_{1t}(y) \\ &+ (r_{2t} + h_t)(G_{1t}(y - z) - H_t(y - z)), \qquad (81)\end{aligned}$$

where $G_{1t}(\cdot)$ and $H_t(\cdot)$ are as defined previously by equations (2) and (3). Then, for $t = 2, \ldots, T$ it can be shown that $\bar{P}_{0t}(I, y, z) = c_t I + \bar{P}_t(y, z)$. The maximum expected profit when there are t periods-to-go and the starting inventory is I can be written as

$$\bar{V}_t(I) = c_t I + \max\{\bar{J}_t(y, z) : I \le y \le C_t + I,\ z \ge 0\}, \quad \text{where} \qquad (82)$$

$$\begin{aligned}\bar{J}_t(y, z) = \bar{P}_t(y, z) + \beta\big[&E[\bar{V}_{t-1}(y - X_{1t})1(X_{1t} > y - z)] \\ &+ E[\bar{V}_{t-1}(z)1(X_{1t} \le y - z, X_{1t} + X_{2t} > y - z)] \\ &+ E[\bar{V}_{t-1}(y - X_{1t} - X_{2t})1(X_{1t} + X_{2t} \le y - z)], \qquad (83)\end{aligned}$$

for $t = 2, \ldots, T$, and $\bar{V}_1(I) = V_1(I)$ as defined by equations (6) and (7) for $t = 1$. Similar to the lost sales model, we have $\bar{V}_0(I) = 0$ for $I \ge 0$. The objective is to compute $\bar{V}_T(I)$ and to determine the optimal $\bar{y}_t^*$ and $\bar{z}_t^*$ that achieve this maximum expected profit for each period t, $t = 1, \ldots, T$.

Theorem 2 shows that when the primary market demand is backlogged, similar to the lost sales case, the optimal policy for the backlog case is also characterized by the produce-up-to and the retain-up-to levels.

Theorem 2. *$\bar{J}_t(y,z)$ and $\bar{V}_t(I)$ satisfy the following properties for $t = 1,\ldots,T$:*

(a) $\bar{J}_t(y,z)$ has a finite maximizer denoted by $(\bar{y}_t^*, \bar{z}_t^*)$ such that

$$\bar{z}_t^* = \begin{cases} 0 & \text{if } \left.\dfrac{d\bar{V}_{t-1}(z)}{dz}\right|_{z=0} \leq \dfrac{r_{2t}+h_t}{\beta}, \\ \\ \bar{R}_t & \text{if } \left.\dfrac{d\bar{V}_{t-1}(z)}{dz}\right|_{z=0} > \dfrac{r_{2t}+h_t}{\beta}, \end{cases} \tag{84}$$

where $\bar{R}_t$ satisfies $\left.\dfrac{d\bar{V}_{t-1}(z)}{dz}\right|_{z=\bar{R}_t} = \dfrac{r_{2t}+h_t}{\beta}$ and

$$\bar{y}_t^* = \begin{cases} I & \text{if } \bar{S}_t < I, \\ \bar{S}_t & \text{if } I \leq \bar{S}_t \leq C_t + I, \\ C_t + I & \text{if } \bar{S}_t > C_t + I, \end{cases} \tag{85}$$

where $\bar{S}_t$ is determined by $\bar{J}_t(\bar{S}_t, z_t^*) = \max\{\bar{J}_t(y, \bar{z}_t^*) : y \in \Re^+\}$.

(b) $\bar{V}_t(I)$ is a concave function of I.

(c) $\lim_{I\to\infty} \dfrac{d\bar{V}_t(I)}{dI} < 0$.

Proof. We present the first order derivatives of $\bar{J}_t(y,z)$ with respect to y and z for $t = 2,\ldots,T$, as follows:

$$\begin{aligned} \frac{\partial \bar{J}_t(y,z)}{\partial y} &= -c_t - h_t + (\bar{b} + h_t)[1 - F_{1t}(y)] + (r_{2t} + h_t)[F_{1t}(y-z) - F_{st}(y-z)] \\ &\quad + \beta E\left[\frac{d\bar{V}_{t-1}(y - X_{1t} - X_{2t})}{dy} 1(X_{1t} + X_{2t} \leq y - z)\right] \\ &\quad + \beta E\left[\frac{d\bar{V}_{t-1}(y - X_{1t})}{dy} 1(X_{1t} > y - z)\right], \end{aligned} \tag{86}$$

$$\frac{\partial \bar{J}_t(y,z)}{\partial z} = \left[\beta \frac{d\bar{V}_{t-1}(z)}{dz} - (r_{2t} + h_t)\right][F_{1t}(y-z) - F_{st}(y-z)]. \tag{87}$$

The proof follows by induction. Since $\bar{V}_1(I) = V(I)$, from Theorem 1 we know that properties (a)-(c) are true for the one-period problem. Suppose that properties (a)-(c) are true for periods $t-1, t-2, \ldots, 2$. From equations (87) and (24), we observe that $\partial \bar{J}_t(y,z)/\partial z$ has the same structure as $\partial J_t(y,z)/\partial z$. Then, following the exact same steps as in the proof of Theorem 1, we can show that the optimal retain-up-to level in period t, $\bar{z}_t^*$ is as expressed in equation (84). In addition, for $y \in [I, C_t + I]$, if

$$\left.\frac{d\bar{V}_{t-1}(z)}{dz}\right|_{z=0} \leq \frac{r_{2t}+h_t}{\beta}, \tag{88}$$

then $\bar{z}_t^* = 0$; otherwise, $\bar{z}_t^* = \bar{R}_t$. Furthermore, using properties (a) and (b) for period t, we have

$$\left.\frac{d\bar{V}_{t-1}(z)}{dz}\right|_{z=0} \geq c_{t-1}. \tag{89}$$

Next, we show that $\bar{J}_t(y, \bar{z}_t^*)$ is concave in y. Evaluating expression (86) at $z = 0$ and taking the derivative of the result with respect to y, we obtain:

$$\begin{aligned}\frac{\partial \bar{J}_t(y,0)}{\partial y} &= -c_t - h_t + (\bar{b}+h_t)[1-F_{1t}(y)] + (r_{2t}+h_t)[F_{1t}(y) - F_{st}(y)] \\ &\quad + \beta E\left[\frac{d\bar{V}_{t-1}(y - X_{1t} - X_{2t})}{dy} 1(X_{1t}+X_{2t} \leq y)\right] \\ &\quad + \beta E\left[\frac{d\bar{V}_{t-1}(y - X_{1t})}{dy} 1(X_{1t} > y)\right],\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \bar{J}_t(y,0)}{\partial y^2} &= \left[\beta \left.\frac{d\bar{V}_{t-1}(y - x_{1t} - x_{2t})}{dy}\right|_{x_{1t}+x_{2t}=y} - (r_{2t}+h_t)\right] f_{st}(y) \\ &\quad - \left[\bar{b} - r_{2t} + \beta \left.\frac{d\bar{V}_{t-1}(y - x_{1t})}{dy}\right|_{x_{1t}=y}\right] f_{1t}(y) \\ &\quad + \beta E\left[\frac{d^2\bar{V}_{t-1}(y - X_{1t})}{dy^2} 1(X_{1t} > y - z)\right] \\ &\quad + \beta E\left[\frac{d^2\bar{V}_{t-1}(y - X_{1t} - X_{2t})}{dy^2} 1(X_{1t}+X_{2t} \leq y - z)\right] < 0.\end{aligned} \tag{90}$$

The negativity of (90) follows from the induction hypothesis that $V_{t-1}(\cdot)$ is concave and inequalities (88) and (89). On the other hand, if $\bar{z}_t^* = \bar{R}_t$, then $\bar{R}_t$ satisfies

$$\left.\frac{d\bar{V}_{t-1}(z)}{dz}\right|_{z=\bar{R}_t} = \frac{r_{2t}+h_t}{\beta} \tag{91}$$

by definition. Evaluating expression (86) at $z = \bar{R}_t$ and taking the derivative of the resulting expression with respect to y, we obtain:

$$\begin{aligned}\frac{\partial \bar{J}_t(y,\bar{R}_t)}{\partial y} &= -c_t - h_t + (\bar{b}+h_t)[1-F_{1t}(y)] + (r_{2t}+h_t)[F_{1t}(y-\bar{R}_t) - F_{st}(y-\bar{R}_t)] \\ &\quad + \beta E\left[\frac{d\bar{V}_{t-1}(y-X_{1t}-X_{2t})}{dy} 1(X_{1t}+X_{2t} \le y-\bar{R}_t)\right] \\ &\quad + \beta E\left[\frac{d\bar{V}_{t-1}(y-X_{1t})}{dy} 1(X_{1t} > y-\bar{R}_t)\right], \qquad \text{and}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \bar{J}_t(y,\bar{R}_t)}{\partial y^2} &= \left[r_{2t}+h_t - \beta \left.\frac{d\bar{V}_{t-1}(y-x_{1t})}{dy}\right|_{x_{1t}=y-\bar{R}_t}\right][f_{1t}(y-\bar{R}_t) - f_{st}(y-\bar{R}_t)] \\ &\quad - (\bar{b}+h_t)f_{1t}(y) + \beta E\left[\frac{d^2\bar{V}_{t-1}(y-X_{1t}-X_{2t})}{dy^2} 1(X_{1t}+X_{2t} \le y-\bar{R}_t)\right] \\ &\quad + \beta E\left[\frac{d^2\bar{V}_{t-1}(y-X_{1t})}{dy^2} 1(X_{1t} > y-\bar{R}_t)\right] < 0. \end{aligned} \tag{92}$$

From equation (91), it can be seen that the second term on the right-hand side of (92) is equal to zero, and the last two terms are negative based on the induction hypothesis that $V_{t-1}(\cdot)$ is concave. Thus, from inequalities in (90) and (92), it follows that $\bar{J}_t(y, \bar{z}_t^*)$ is concave in y. Defining $\bar{S}_t \in \Re$ such that

$$\left.\frac{\partial \bar{J}_t(y,\bar{z}_t^*)}{\partial y}\right|_{y=\bar{S}_t} = 0, \tag{93}$$

we have the optimal produce-up-to level in period t, y_t^*, expressed as in equation (85). As a result, property (a) holds. Then, following a similar argument as in the proof of parts (b) and (c) of Theorem 1, we can show that properties (b) and (c) are true for period t. □

Next, we consider the case where all cost and revenue parameters are non-stationary, i.e., the unit revenue and lost sales penalty from the primary market are also non-stationary.

II.8 Non-Stationary Revenue and Cost Parameters for the Primary Market

We now consider a generalization of our model where all cost and revenue parameters are non-stationary. Let r_{1t} and b_t denote the unit revenue and unit lost sales penalty for unsatisfied demand from the primary market in period t, respectively. Under this setting, there may be instances where it is optimal to ration inventory to both markets. That is, it may be profitable not to satisfy part or all of the demand from the primary market in order to retain some inventory to satisfy future high-revenue demand. As a result, in this setting, we have two decision variables that determine how much inventory to retain. In particular, z_{1t} denotes the inventory to be saved from sales to both primary and spot markets and z_{2t} denotes the *additional* inventory to be saved from the spot market in period t. Thus, z_{1t} and $z_{1t} + z_{2t}$, respectively, represent the minimum and the maximum amount of inventory to be retained in period t. As before, we assume that $r_{1t} + b_t > r_{2t} + h_t$ for $t = 1, \ldots, T$. For each period $t = 1, \ldots, T$, with starting inventory I, the expected profit earned in period t is given as

$$\begin{aligned}
&P_{0t}(y, z_1, z_2) \\
&= E[-c_t(y - I) + [r_{1t}(y - z_1) - h_t z_1 - b_t(X_{1t} - y + z_1)]1(X_{1t} > y - z_1) \\
&\quad + r_{1t}X_{1t}1(X_{1t} \leq y - z_1) - h_t(y - X_{1t})1(y - z_1 - z_2 \leq X_{1t} \leq y - z_1) \\
&\quad + [r_{2t}(y - z_1 - X_{1t} - z_2) - h_t(z_1 + z_2)]1(X_{1t} \leq y - z_1 - z_2, X_{1t} + X_{2t} > y - z_1 - z_2) \\
&\quad + [r_{2t}X_{2t} - h_t(y - X_{1t} - X_{2t})]1(X_{1t} \leq y - z_1 - z_2, X_{1t} + X_{2t} \leq y - z_1 - z_2)].
\end{aligned}$$

Let us define

$$\begin{aligned}
P_t(y, z_1, z_2) &= (r_{1t} + h_t)E[X_{1t}] + (r_{2t} + h_t)E[X_{2t}] - (c_t + h_t)y - (r_{1t} + b_t + h_t)G_{1t}(y - z_1) \\
&\quad + (r_{2t} + h_t)G_{1t}(y - z_1 - z_2) - (r_{2t} + h_t)H_t(y - z_1 - z_2), \qquad (94)
\end{aligned}$$

where $G_{it}(u)$ and $H_t(u)$ are as defined in equations (2) and (3). After some algebra, we can write the expected profit earned in period t as

$$P_{0t}(y, z_1, z_2) = cI + P_t(y, z_1, z_2).$$

The maximum expected profit when there are t periods-to-go and the starting inventory is I can be written as

$$\begin{aligned}
V_t(I) &= c_t I + \max\{J_t(y, z_1, z_2) : I \le y \le C_t + I,\ z_1, z_2 \ge 0\}, \quad \text{where} \qquad (95)\\
J_t(y, z_1, z_2) &= P_t(y, z_1, z_2) + \beta\big[E[V_{t-1}(z_1)1(X_{1t} > y - z_1)]\\
&\quad + E[V_{t-1}(z_1 + z_2)1(X_{1t} \le y - z_1 - z_2, X_{1t} + X_{2t} > y - z_1 - z_2)]\\
&\quad + E[V_{t-1}(y - X_{1t} - X_{2t})1(X_{1t} \le y - z_1 - z_2, X_{1t} + X_{2t} \le y - z_1 - z_2)]\\
&\quad + E[V_{t-1}(y - X_{1t})1(y - z_1 - z_2 \le X_{1t} \le y - z_1)]\big].
\end{aligned}$$

As before, $V_0(I) = 0$ for $I \ge 0$. Proposition 7 shows that the optimal policy parameters are characterized by three quantities: S_t, R_{1t} and R_{2t}. As before, S_t denotes the critical produce-up-to level, R_{1t} denotes the critical level of inventory to be saved from sales in either market and R_{2t} is the total critical level of inventory to be saved from sales in the spot market. R_{2t} is equivalent to R_t considered in Section II.4 and hence, it is the critical retain-up-to level for period t.

Proposition 7. *$J_t(y, z_1, z_2)$ and $V_t(I)$ satisfy the following properties for $t = 1, \ldots, T$:*

(a) $J_t(y, z_1, z_2)$ has a finite maximizer denoted by $(y_t^*, z_{1t}^*, z_{2t}^*)$ such that

$$(z_{1t}^*, z_{2t}^*) = \begin{cases} (0, 0) & \text{if } \left.\frac{dV_{t-1}(z)}{dz}\right|_{z=0} \leq \frac{r_{2t} + h_t}{\beta}, \\ (0, R_{2t}) & \text{if } \frac{r_{2t} + h_t}{\beta} < \left.\frac{dV_{t-1}(z)}{dz}\right|_{z=0} \leq r_{1t} + b_t + h_t, \\ (R_{1t}, R_{2t} - R_{1t}) & \text{if } \left.\frac{dV_{t-1}(z)}{dz}\right|_{z=0} > r_{1t} + b_t + h_t, \end{cases} \tag{96}$$

where R_{2t} satisfies $\left.\frac{dV_{t-1}(z)}{dz}\right|_{z=R_{2t}} = \frac{r_{2t} + h_t}{\beta}$, R_{1t} satisfies $\left.\frac{dV_{t-1}(z)}{dz}\right|_{z=R_{1t}} = r_{1t} + b_t + h_t$ and

$$y_t^* = \begin{cases} I & \text{if } S_t < I, \\ S_t & \text{if } I \leq S_t \leq C_t + I, \\ C_t + I & \text{if } S_t > C_t + I, \end{cases} \tag{97}$$

where S_t is determined by $J_t(S_t, z_{1t}^*, z_{2t}^*) = \max\{J_t(y, z_{1t}^*, z_{2t}^*) : y \in \Re^+\}$.

(b) $V_t(I)$ is a concave function of I.

Proof. The first order derivatives of $J_t(y, z_1, z_2)$ with respect to y, z_1 and z_2 are given as follows:

$$\begin{aligned} \frac{\partial J_t(y, z_1, z_2)}{\partial y} &= -c_t - h_t + (r_{1t} + b_t + h_t)[1 - F_{1t}(y - z_1)] + (r_{2t} + h_t)[F_{1t}(y - z_1 - z_2) \\ &\quad - F_{st}(y - z_1 - z_2)] + \beta E\left[\frac{dV_{t-1}(y - X_{1t} - X_{2t})}{dy} 1(X_{1t} + X_{2t} \leq y - z_1 - z_2)\right] \\ &\quad + \beta E\left[\frac{dV_{t-1}(y - X_{1t})}{dy} 1(y - z_1 - z_2 \leq X_{1t} \leq y - z_1)\right], \end{aligned} \tag{98}$$

$$\frac{\partial J_t(y,z_1,z_2)}{\partial z_1} = \left[\beta\frac{dV_{t-1}(z_1)}{dz_1} - (r_{1t}+b_t+h_t)\right][1-F_{1t}(y-z_1)]$$
$$+\left[\beta\frac{dV_{t-1}(z_1+z_2)}{dz_1} - (r_{2t}+h_t)\right][F_{1t}(y-z_1-z_2)-F_{st}(y-z_1-z_2)], \quad (99)$$
$$\frac{\partial J_t(y,z_1,z_2)}{\partial z_2} = \left[\beta\frac{dV_{t-1}(z_1+z_2)}{dz_2} - (r_{2t}+h_t)\right][F_{1t}(y-z_1-z_2)-F_{st}(y-z_1-z_2)]. \quad (100)$$

The proof follows by induction. We will first show that the properties (a)-(c) are true for the one period problem. Since $V_0(\cdot)=0$, $dV_0(z)/dz=0$. Substituting this in equations (99) and (100) for $t=1$, it can be easily shown that $\partial J_1(y,z_1,z_2)/\partial z_1 \le 0$ and $\partial J_t(y,z_1,0)/\partial z_2 \le 0$. Hence, $J_1(y,z_1,z_2)$ is decreasing in both z_1 and z_2. Therefore, $z_{11}^*=0$ and $z_{21}^*=0$, which satisfies the first part of equation (96). In addition, taking the derivative of equation (98) with respect to y for $t=1$ and evaluating the result at $z_1=z_2=0$, we see that $J_1(y,0,0)$ is concave in y. Defining S_1 such that

$$\frac{\partial J_1(y,0,0)}{\partial y}\Big|_{y=S_1}=0,$$

we have the optimal produce-up-to level in period 1 expressed as in equation (97).

Consequently, if follows from equation (95) that $V_1(I)=c_1I+J_1(y_1^*(I),0,0)$. The first order derivative of $V_1(I)$ is

$$\frac{dV_1(I)}{dI} = \begin{cases} r_{11}+b_1-(r_{11}+b_1-r_{21})F_{11}(I)-(r_{21}+h_1)F_{s1}(I) & \text{if } S_1 < I, \\ c_1 & \text{if } I \le S_1 \le C_1+I, \\ r_{11}+b_1-(r_{11}+b_1-r_{21})F_{11}(C_1+I) & \text{if } S_1 > C_1+I. \\ \quad -(r_{21}+h_1)F_{s1}(C_1+I) & \end{cases} \quad (101)$$

From equation (101), we observe that $dV_1(I)/dI \le r_{11}+b_1$. Moreover, taking the derivative of equation (101) with respect to I, it can be easily shown that $d^2V_1(I)/dI^2 \le 0$. Thus, $V_1(I)$ is concave in I. As a result, properties (a)-(c) are true for the one-period problem.

Suppose that properties (a)-(b) are true for periods $t-1, t-2, \ldots, 2$. We will show that they are true for period t. By the induction hypothesis based on part (b), $dV_{t-1}(z)/dz$ is a decreasing function of z. Suppose that $dV_{t-1}(z)/dz|_{z=0} \leq (r_{2t}+h_t)/\beta$. Then,

$$\left.\frac{dV_{t-1}(z)}{dz}\right|_{z>0} \leq \frac{r_{2t}+h_t}{\beta}. \tag{102}$$

Using this inequality and equation (100), we observe that $J_t(y, z_1, z_2)$ is a decreasing function of z_2 for $z_1 \geq 0$ and $y \in [I, C_t+I]$. Therefore, $z_{2t}^* = 0$. Evaluating expression (99) at $z_2 = 0$, we have

$$\begin{aligned}\frac{\partial J_t(y, z_1, 0)}{\partial z_1} &= \left[\beta \frac{dV_{t-1}(z_1)}{dz_1} - (r_{1t}+b_t+h_t)\right][1 - F_{1t}(y-z_1)] \\ &+ \left[\beta \frac{dV_{t-1}(z_1)}{dz_1} - (r_{2t}+h_t)\right][F_{1t}(y-z_1) - F_{st}(y-z_1)].\end{aligned}$$

Using inequality (102) on right-hand side of the above equation, we have

$$\begin{aligned}\frac{\partial J_t(y, z_1, 0)}{\partial z_1} &= \left[\beta\left(\frac{r_{2t}+h_t}{\beta}\right) - (r_{1t}+b_t+h_t)\right][1 - F_{1t}(y-z_1)] \\ &+ \left[\beta\left(\frac{r_{2t}+h_t}{\beta}\right) - (r_{2t}+h_t)\right][F_{1t}(y-z_1) - F_{st}(y-z_1)] < 0.\end{aligned} \tag{103}$$

The negativity of (103) follows from the assumption that $\beta(r_{1t}+b_t) > r_{2t}+h_t$ and $0 \leq \beta \leq 1$. Therefore, $J_t(y, z_1, 0)$ is a decreasing function of z_1 for $y \in [I, C_t+I]$, and hence, $z_{1t}^* = 0$.

On the other hand, if $dV_{t-1}(z)/dz|_{z=0} > (r_{2t}+h_t)/\beta$, then for some $R_{2t} \in \Re^+$,

$$\left.\frac{dV_{t-1}(z)}{dz}\right|_{z=R_{2t}} = \frac{r_{2t}+h_t}{\beta}. \tag{104}$$

Using equations (100) and (104), we observe that $J_t(y, z_1, z_2)$ is increasing in z_2 for $z_2 < R_{2t} - z_1$, and decreasing in z_2 for $z_2 > R_{2t} - z_1$. Therefore, $J_t(y, z_1, z_2)$ attains its maximum at $z_{2t}^* = R_{2t} - z_1$ for $z_1 \geq 0$ and $y \in [I, C_t+I]$. Evaluating expression (99) at $z_2 = R_{2t} - z_1$, we have

$$\frac{\partial J_t(y, z_1, R_{2t} - z_1)}{\partial z_1}$$
$$= \left[\beta \frac{dV_{t-1}(z_1)}{dz_1} - (r_{1t} + b_t + h_t)\right][1 - F_{1t}(y - z_1)]$$
$$+ \left[\beta \frac{dV_{t-1}(z_1 + z_2)}{dz_1}\bigg|_{z_2 = R_{2t} - z_1} - (r_{2t} + h_t)\right][F_{1t}(y - R_{2t}) - F_{st}(y - R_{2t})]. \tag{105}$$

Next, suppose that $(r_{2t} + h_t)/\beta < dV_{t-1}(z)/dz|_{z=0} \leq (r_{1t} + b_t + h_t)/\beta$. Then,

$$\frac{dV_{t-1}(z)}{dz}\bigg|_{z>0} \leq \frac{r_{1t} + b_t + h_t}{\beta}. \tag{106}$$

Using inequality (106) and equation (104) on the right-hand side of expression (105), it is easy to see that $\partial J_t(y, z_1, R_{2t} - z_1)/\partial z_1 < 0$. Thus, $J_t(y, z_1, R_{2t} - z_1)$ is a decreasing function of z_1. Therefore, $z_{1t}^* = 0$ and $z_{2t}^* = R_{2t} - z_{1t}^* = R_{2t}$.

On the other hand, if $dV_{t-1}(z)/dz|_{z=0} > (r_{1t} + b_t + h_t)/\beta$, then for some $R_{1t} \in \Re^+$,

$$\frac{dV_{t-1}(z)}{dz}\bigg|_{z=R_{1t}} = r_{1t} + b_t + h_t. \tag{107}$$

Since $dV_{t-1}(z)/dz|_{z=R_{1t}} = r_{1t} + b_t + h_t > (r_{2t} + h_t)/\beta$ and $V_{t-1}(\cdot)$ is concave, $0 < R_{1t} < R_{2t}$. Then, using equations (104) and (107) on the right-hand side of expression (105), we observe that $J_t(y, z_1, R_{2t} - z_1)$ is increasing in z_1 for $z_1 < R_{1t}$ and decreasing in z_1 for $z_1 > R_{1t}$. Therefore, $J_t(y, z_1, R_{2t} - z_1)$ attains its maximum at $z_{1t}^* = R_{1t}$ for $y \in [I, C_t + I]$. Furthermore, $z_{2t}^* = R_{2t} - z_{1t}^* = R_{2t} - R_{1t}$. Thus, for $y \in [I, C_t + I]$, (z_{1t}^*, z_{2t}^*) can be expressed as in equation (96).

Next, we show that $J_t(y, z_{1t}^*, z_{2t}^*)$ is concave in y. From the above discussion, we note that for $y \in [I, C_t + I]$, if

$$\frac{dV_{t-1}(z)}{dz}\bigg|_{z=0} \leq \frac{r_{2t} + h_t}{\beta}, \tag{108}$$

then $z_{1t}^* = z_{2t}^* = 0$. Evaluating expression (98) at $z_1^* = z_2^* = 0$ and taking the derivative of the resulting expression with respect to y, we obtain:

$$\frac{\partial J_t(y,0,0)}{\partial y} = -c_t - h_t + (r_{1t} + b_t + h_t)[1 - F_{1t}(y)] + (r_{2t} + h_t)[F_{1t}(y) - F_{st}(y)] + \beta E\left[\frac{dV_{t-1}(y - X_{1t} - X_{2t})}{dy} 1(X_{1t} \le y, X_{1t} + X_{2t} \le y)\right],$$

$$\frac{\partial^2 J_t(y,0,0)}{\partial y^2} = -(r_{1t} + b_t - r_{2t}) f_{1t}(y) + \left[\beta \frac{dV_{t-1}(y - x_{1t} - x_{2t})}{dy}\bigg|_{x_{1t}+x_{2t}=y} - (r_{2t} + h_t)\right] f_{st}(y) + \beta E\left[\frac{d^2 V_{t-1}(y - X_{1t} - X_{2t})}{dy^2} 1(X_{1t} + X_{2t} \le y)\right] < 0. \tag{109}$$

The negativity of (109) follows from the induction hypothesis that $V_{t-1}(\cdot)$ is concave and inequality (108). On the other hand, if

$$\frac{r_{2t} + h_t}{\beta} < \frac{dV_{t-1}(z)}{dz}\bigg|_{z=0} \le \frac{r_{1t} + b_t + h_t}{\beta}, \tag{110}$$

then $z_{1t}^* = 0$ and $z_{2t}^* = R_{2t}$. Evaluating expression (98) at $z_1^* = 0$ and $z_2^* = R_{2t}$ and taking the derivative of the resulting expression with respect to y, we obtain:

$$\frac{\partial J_t(y,0,R_{2t})}{\partial y} = -c_t - h_t + (r_{1t} + b_t + h_t)[1 - F_{1t}(y)] + (r_{2t} + h_t)[F_{1t}(y - R_{2t}) - F_{st}(y - R_{2t})] + \beta E\left[\frac{dV_{t-1}(y - X_{1t} - X_{2t})}{dy} 1(X_{1t} + X_{2t} \le y - R_{2t})\right] + \beta E\left[\frac{dV_{t-1}(y - X_{1t})}{dy} 1(y - R_{2t} \le X_{1t} \le y)\right],$$

$$\frac{\partial^2 J_t(y,0,R_t)}{\partial y^2} = \left[\beta \frac{dV_{t-1}(y - x_{1t})}{dy}\bigg|_{x_{1t}=y} - (r_{1t} + b_t + h_t)\right] f_{1t}(y) - \left[\beta \frac{dV_{t-1}(y - x)}{dy}\bigg|_{x=R_{2t}} - (r_{2t} + h_t)\right][f_{1t}(y - R_{2t}) - f_{st}(y - R_{2t})] + \beta E\left[\frac{d^2 V_{t-1}(y - X_{1t} - X_{2t})}{dy^2} 1(X_{1t} + X_{2t} \le y - R_{2t})\right] + \beta E\left[\frac{dV_{t-1}^2(y - X_{1t})}{dy^2} 1(y - R_{2t} \le X_{1t} \le y)\right] < 0. \tag{111}$$

In equation (111), the negativity of the first term follows from equation (110). The negativity of the second term follows from equation (104). The last two terms are negative based on the induction hypotheses that $V_{t-1}(\cdot)$ is concave. Finally, if $dV_{t-1}(z)/dz|_{z=0} > (r_{1t}+b_t+h_t)/\beta$, then $z_{1t}^* = R_{1t}$ and $z_{2t}* = R_{2t} - R_{1t}$. Evaluating expression (98) at $z_1^* = R_{1t}$ and $z_2^* = R_{2t} - R_{1t}$ and taking the derivative of resulting expression with respect to y, we obtain:

$$\begin{aligned}
\frac{\partial J_t(y, R_{1t}, R_{2t}-R_{1t})}{\partial y} &= -c_t - h_t + (r_{1t}+b_t+h_t)[1-F_{1t}(y-R_{1t})] \\
&+ (r_{2t}+h_t)[F_{1t}(y-R_{2t}) - F_{st}(y-R_{2t})] \\
&+ \beta E\left[\frac{dV_{t-1}(y-X_{1t}-X_{2t})}{dy} 1(X_{1t}+X_{2t} \le y-R_{2t})\right] \\
&+ \beta E\left[\frac{dV_{t-1}(y-X_{1t})}{dy} 1(y-R_{2t} \le X_{1t} \le y-R_{1t})\right], \\
\frac{\partial^2 J_t(y, R_{1t}, R_{2t}-R_{1t})}{\partial y^2} &= \left[\beta \frac{dV_{t-1}(y-x_{1t})}{dy}\bigg|_{x_{1t}=y-R_{1t}} - (r_{1t}+b_t+h_t)\right] f_{1t}(y-R_{1t}) \\
&- \left[\beta \frac{dV_{t-1}(y-x)}{dy}\bigg|_{x=R_{2t}} - (r_{2t}+h_t)\right] [f_{1t}(y-R_{2t}) - f_{st}(y-R_{2t})] \\
&+ \beta E\left[\frac{d^2V_{t-1}(y-X_{1t}-X_{2t})}{dy^2} 1(X_{1t}+X_{2t} \le y-R_{2t})\right] \\
&+ \beta E\left[\frac{dV_{t-1}^2(y-X_{1t})}{dy^2} 1(y-R_{2t} \le X_{1t} \le y-R_{1t})\right] < 0. \qquad (112)
\end{aligned}$$

In equation (112), the negativity of first term follows from the equation (107). The negativity of second term follows from the equation (104). The last two terms are negative based on the induction hypothesis that $V_{t-1}(\cdot)$ is concave. Thus, from inequalities in (109), (111) and (112) it follows that $J_t(y, z_{1t}^*, z_{2t}^*)$ is concave in y. Defining $S_t \in \Re$, such that

$$\frac{\partial J_t(y, z_{1t}^*, z_{2t}^*)}{\partial y}\bigg|_{y=S_t} = 0,$$

we have the optimal produce-up-to level in period t, y_t^* expressed as in equation (97). As a result property (a) holds. Then, it follows from equation (95) that $V_t(I) = c_t I + J_t(y_t^*(I), z_{1t}^*, z_{2t}^*)$. The first-order derivative of $V_t(I)$ with respect to I evaluated at y_t^* from equation (97) is given by:

$$\frac{dV_t(I)}{dI} = c_t + \frac{dJ_t(y_t^*(I), z_{1t}^*, z_{2t}^*)}{dI}$$

$$\frac{dV_t(I)}{dI} = c_t + \begin{cases} \left.\dfrac{\partial J_t(y, z_{1t}^*, z_{2t}^*)}{\partial y}\right|_{y=I} & \text{if } S_t < I, \\ 0 & \text{if } I \leq S_t \leq C_t + I, \\ \left.\dfrac{\partial J_t(y, z_{1t}^*, z_{2t}^*)}{\partial y}\right|_{y=C_t+I} & \text{if } S_t > C_t + I. \end{cases} \tag{113}$$

Taking the first-order derivative of equation (113) with respect to I and using the fact that $J_t(y, z_{1t}^*, z_{2t}^*)$ is concave in y in the resulting expression, it is easy to verify that $d^2V_t(I)/dI^2 \leq 0$. Hence, $V_t(I)$ is concave in I. As a result, properties (a)-(b) are true for period t. □

In addition to the results in Theorem 1, Proposition 7 indicates that if

$$\left.\frac{dV_{t-1}(z)}{dz}\right|_{z=0} \leq r_{1t} + b_t + h_t, \tag{114}$$

then $J_t(y, z_1, R_{2t} - z_1)$ is a decreasing function of z_1 for $y \geq 0$, and hence, from equation (95), $z_{1t}^* = 0$. On the other hand, if inequality (114) is violated, then $J_t(y, z_1, R_{2t} - z_1)$ is unimodal in z_1 for $y \geq 0$ and $z_{1t}^* = R_{1t}$ whereas $z_{2t}^* = R_{2t} - R_{1t}$. We observe that if the unit revenue from the primary market is constant for the entire planning horizon, then there is no economic motivation to save inventory from sales in the primary market. Thus, for the problem setting considered in Section II.3 $z_{1t} = R_{1t} = 0$ for $t = 1, \ldots, T$. However, if the unit revenue and unit lost sales penalty for unsatisfied demand from the primary market are non-stationary then it may be optimal to save some units from sales in the primary market as well. This is especially true in periods with low values of r_{1t} and b_t compared to those in the very next period.

II.9 Summary of Contributions and Key Insights

This chapter considers a multi-period, integrated replenishment and liquidation problem for a capacitated supplier facing stochastic demand from two markets. This problem is particularly relevant to suppliers facing demand from a spot market along with its primary market (with higher priority contractual customers). In such settings, if one reserves the optimal amount of inventory for future high-priority or high-revenue demand, higher profits can be earned. If the unit revenue and the lost sales penalty for the primary market are stationary, we show that the optimal policy is characterized by two quantities: the critical produce-up-to and the critical retain-up-to levels.

We establish (lower and upper) thresholds on the unit revenue from the spot market such that a simple benchmark policy is optimal. However, if the unit revenue from the spot market is within these thresholds, then the optimal policy does not reduce to any of the benchmark policies and leads to significant profit improvement. Our numerical results indicate that the benefits of the optimal policy over both the benchmark policies are higher, more than 5%, when the production capacity is low, demand uncertainty is high and/or demands in the primary and spot market are negatively correlated.

Our computational study provides interesting managerial insights, which can be used to develop rules of thumb for better understanding of the alternative and more practical policies. For example, we observe that the two most important factors that determine the optimal retain-up-to level are: (i) comparison of the current value of the product in the spot market and savings in future production cost, and (ii) relative values of optimal produce-up-to level and production capacity available in the next period. We observe that when there is enough production capacity available in the next period the optimal policy reduces to either one of the benchmark policies. On the other hand, when the production capacity available is not enough to meet the produce-up-to level in the next period, the optimal policy may not reduce to either of the benchmark policies.

The theoretical and numerical results presented in this chapter suggest that even if selling in the spot market is profitable, it may be suboptimal to satisfy the spot market demand completely even if we have sufficient inventory. On the other hand, by assumption, demand from the primary market customers must be completely satisfied if there is enough inventory available. This result is based on the assumption that the unit revenue and the unit lost sales penalty for unsatisfied demand from the primary market customers are stationary. One immediate question of interest is what happens if this assumption is violated, i.e., if the unit revenue and unit lost sales penalty for unsatisfied demand from the primary market are non-stationary and customers in primary market are non-contractual, is it still preferable to completely satisfy the primary market demand in each period? The answer requires a proof of the structure of the optimal policy under non-stationary parameters. In this case, the optimal policy is characterized by three parameters, rather than just two, where the third parameter specifies conditions under which one may choose to retain inventory at the expense of losing primary market sales.

As we noted earlier, a future research direction relates to examining the effect of correlation between the spot market demand in a given period and the primary market demand in the next period. This extension explicitly considers that the demands can be correlated over time as well. Other important extensions include the cases where (i) the spot market price is unknown at the time of decision making and (ii) the demands are price-dependent.

CHAPTER III

JOINT REPLENISHMENT, ALLOCATION AND PRICING DECISIONS UNDER TWO MARKETS

III.1 Introduction

Revenue management (RM) focuses on demand-management decisions, and has emerged as one of the most effective approaches to improve revenue, and hence, profits [72]. A fundamental strategy used in RM is customer segmentation, which is the division of a supplier's market into different groups, such that customers in each group share similar characteristics and preferences. Suppliers may differentiate its customers based on price, fulfillment priority, contractual agreements, shortage costs and time of occurrence of demand. Depending on the basis of customer segmentation, existing models in the literature utilize either quantity-based RM or price-based RM as a tactical tool to improve profits. Quantity-based RM focuses on capacity/inventory allocation decisions to different customer segments and is used when suppliers have to commit to certain prices. Price-based RM focuses on pricing decisions and is used when prices can be changed over time. However, several real life business settings may require the supplier to use quantity-based RM for one demand class and price-based RM for another demand class.

For example, original equipment manufacturers (OEMs) (e.g., Dell, HP) may sell their products to high volume customers, which includes retailers like Best Buy, Walmart or directly to individual customers through web-based channels. The retailers may negotiate delivery at fixed prices and may place orders at the beginning of each month. In contrast, direct customers order throughout the month and the OEM can easily change prices on its website targeting these direct customers. In such settings, it is profitable for the supplier to use a quantity-based RM strategy for the retail customers, whose demand is satisfied at the beginning of each month at a fixed price. For example, the supplier can use discretionary sales for the retail customers and satisfy their demand only partially while saving inventory for potentially higher-revenue demand from individual customers. Similarly, the supplier can use a price-based RM strategy for the direct customers who can be charged a different price in each month.

In this chapter, we consider such a supplier (e.g., OEMs) who satisfies demands from two customer classes over a finite planning horizon. Class 1 customers consists of high-volume buyers who negotiate delivery at fixed price. In contrast, customers with price-sensitive demand in Class 2 may be charged a different price in each period. We consider the case where the supplier observes and satisfies the stochastic demand from Class 1 before observing the Class 2 demand in each period. Unsatisfied demand from Class 1 is lost and incurs a lost sales penalty. In contrast, there is no penalty for unsatisfied demand from Class 2. To maximize the expected profit, the supplier has to make three decisions in each period. Before observing the Class 1 demand, the supplier decides (i) how much to produce, and (ii) how much inventory to protect from Class 1 customers (i.e., how much inventory to allocate to Class 1 demand). After observing and satisfying the Class 1 demand, the supplier decides (iii) the price to be charged to Class 2 customers.

For modeling purposes, we divide each period into two stages as follows: Stage 1 starts at the beginning of the period and ends (and stage 2 starts) after the supplier observes and satisfies Class 1 demand. Stage 2 ends at the end of the period. We consider an additive demand function for the price-sensitive demand in Class 2. We formulate the problem as a finite horizon, multi-period, two-stage stochastic dynamic program. We show that the optimal price charged to Class 2 customers is a function of the left-over inventory after satisfying Class 1 demand. Furthermore, the stage 1 optimal policy is a combination of the produce-up-to policy with potential discretionary sales to Class 1. In other words, stage 1 optimal policy is characterized by two quantities: the critical produce-up-to level and the critical amount of inventory to be protected from Class 1. Specifically, we show that a discretionary sales policy for Class 1 is optimal if the supplier anticipates higher-revenue demand from Class 2.

In general, the supplier's optimal policy can be computed using the backward induction algorithm, which may take considerable time to solve (e.g., see [54, 68]). This may make the optimal policy less attractive from implementation perspective. In contrast, myopic policies are less costly to evaluate, and hence, are readily accepted by practitioners. A myopic policy makes decisions in each period by isolating it from the future periods. As a result, a myopic solution to a T-period problem can be obtained by solving T single period problems. Considering the practical relevance of our problem, we establish a set of sufficient conditions under which a myopic policy is optimal. The first sufficiency condition requires the on-hand inventory at the beginning of the planning horizon to be less than the myopic produce-up-to level. The second sufficiency condition is satisfied if the critical myopic produce-up-to level is non-decreasing. For example, this is true when all cost and demand parameters are stationary and any units remaining at the end of the planning horizon are salvaged at a value equal to the unit production cost.

We conduct a computational study to: (i) investigate the sensitivity of the optimal policy parameters to key model parameters, and (ii) quantify the benefits of discretionary sales for Class 1 and dynamic pricing for Class 2, and examine the effect of key model parameters on these benefits. Our results show that the benefits of dynamic pricing for Class 2 (average 11.7%) are significantly higher than the benefits of discretionary sales for Class 1 (average 0.9%). Benefits of discretionary sales for Class 1 increase with increase in Class 1 or Class 2 demand variances, and decrease in the slope of the Class 2 demand curve. Furthermore, benefits of dynamic pricing for Class 2: (i) increase with the increase in Class 1 or Class 2 demand variances, (ii) increase (decrease) with the increase in the slope of the Class 2 demand curve such that the resulting maximum price that can be charged to Class 2 is higher (lower) than the sum of the price charged to Class 1 and the unit loss sales penalty.

In summary, this research seeks answers to the following key practical questions:

(i) What is the structure of the supplier's joint optimal replenishment, allocation and pricing policy?

(ii) What myopic policy can the supplier use and under what conditions is it optimal?

(iii) What is the value of discretionary sales to Class 1?

(iv) What is the value of dynamic pricing to Class 2?

Implementing discretionary sales and/or dynamic pricing may be costly and may also result in loss of good will [72]. Our results will quantify the benefits of these strategies and justify their implementation in comparison to simpler policies.

The remainder of this chapter is organized as follows: In Section III.2, we provide a brief summary of the related literature and the contributions of our study. In Section III.3, we present two-stage stochastic dynamic program formulation of the problem and characterize the structure of the optimal policy. In Section III.4, we present an alternate Markov Decision Processes (MDP) formulation of the problem and the corresponding myopic policy. We establish the conditions under which this myopic policy is optimal, and discuss the restrictions that these conditions impose of the model parameters in Section III.5. In Section III.6, we present results of our computational study, followed by concluding remarks in Section III.7.

III.2 Literature Review

First and foremost, this research is related to pure inventory models which investigate optimality of myopic replenishment policies under stochastic demand from a single demand Class. The methodological approach followed by these papers is similar to ours. Notable works include [14, 43, 53, 67, 68, 77, 79]. We generalize the existing work in this area by investigating the optimality of joint myopic replenishment, allocation and pricing policy in presence of two demand classes.

This research builds on and generalizes two relevant streams of operations and marketing literature: the first stream investigates inventory replenishment and allocation decisions in the presence of multiple demand classes. The second stream focuses on joint replenishment and pricing decisions in the presence of one or more demand classes.

The decision regarding the amount of inventory to be protected from Class 1 specifies the maximum inventory that should be used to satisfy Class 1 demand. Thus, the first stream of related literature investigates: (i) how much to produce, and (ii) how much demand to satisfy from each customer Class. We refer to these decisions as inventory replenishment and allocation decisions, respectively. We divide this stream of literature into three categories: (1) papers that study only production decisions [69, 78, 82], (2) papers that study only allocation decisions (see Kleijn and Dekker [50] for a review), and (3) papers that study joint production and allocation decisions. The papers that study joint production and allocation decisions can be further Classified as single [65] or multiple demand Class models [4, 22, 26, 30, 31, 36, 38, 60, 75]. The multiple demand Class models can be further divided as single procurement [60], multi-procurement models with unlimited production capacity [22, 36, 38, 75] and multi-procurement models with limited production capacity [4, 26, 30, 31]. In this research, we assume that unlimited production capacity is available, and we extend the second stream of literature by investigating pricing decisions in addition to the replenishment and allocation decisions.

The second stream focuses on simultaneous determination of optimal inventory and pricing decisions in the face of price-sensitive stochastic demand in a periodic review setting. Based on how uncertainty of price-sensitive demand is modeled, we Classify this literature as: (i) models that consider additive demand function [24, 46, 59, 74, 80, 81] , (ii) models that consider multiplicative demand functions [13, 59, 70, 80], and (3) models that consider a general demand function that admits both additive and multiplicative demand functions [7, 17, 23, 33, 51]. We refer the reader to Chan et. al. [17] for a detailed review of this stream of literature. With the exception of [13, 46, 81], this stream of literature assumes that there is a single demand Class. We consider multiple demand classes and discuss such models in detail. Cachon and Kok [13] investigate a single-period problem with two demand classes: (i) there is stochastic demand from Class 1, which is charged a exogenously fixed price, and (ii) Class 2 demand is a deterministic function of the realized demand in Class 1 and the price charged to it, which is a decision variable. Karakul [46] considers a single-period problem with stochastic price-sensitive demand from a primary Class, and stochastic demand from a clearance Class with exogenously fixed discounted price. Zhang and Bell [81] consider a single-period problem with stochastic price-sensitive demand from two or more demand classes. They consider that demand from all classes is realized simultaneously, and is satisfied without priority. In contrast, we consider a multi-period model with two demand classes such that: (i) stochastic demand from Class 1 is satisfied at an exogenously fixed price before Class 2 demand is observed, and (ii) stochastic price-sensitive demand from Class 2 is independent of the Class 1 demand. Furthermore, we allow inventory rationing and discretionary sales to Class 1, and the decide the price to be charged to Class 2 after observing Class 1 demand.

In summary, we contribute to the existing literature by investigating joint replenishment, allocation and pricing decisions under stochastic demand from two customer classes. We investigate optimality of joint myopic replenishment, allocation and pricing policy in presence of two demand classes. Based on our computational study, we show that the benefits of discretionary sales for Class 1 and dynamic pricing for Class 2 are significant.

III.3 Model Formulation

We consider a supplier who faces stochastic demand from two customer classes, Class 1 and Class 2, as illustrated in Figure 7. The selling horizon consists of T periods. At the beginning of period t, the supplier observes the on-hand inventory I_t and makes two decisions: (i) the number of units to be produced, q_t, and (ii) the minimum number of units to be protected from Class 1 customers, z_t. That is, the supplier may choose to satisfy Class 1 demand only partially even if sufficient inventory is available. We denote the produce-up-to level by $y_t = q_t + I_t$. We assume that the production occurs instantaneously and c_t is the unit cost of production. After production is completed, the stochastic demand from Class 1 customers, D_{1t}, is realized. The supplier satisfies Class 1 demand while protecting atleast z_t units of inventory. If Class 1 demand exceeds $y_t - z_t$, then excess demand is lost and incurs a unit penalty of b. On the other hand, if Class 1 demand is less than $y_t - z_t$, then the entire Class 1 demand is satisfied and the left-over inventory is available for sales to the Class 2 customers.

While the supplier charges a fixed unit price of r_1 to Class 1 customers over the entire planning horizon, Class 2 customers may be charged a different price in each period. In particular, after satisfying Class 1 demand, the supplier decides the price to be charged to Class 2 customers, denoted by p_t in period t. Following this, stochastic demand from Class 2 customers is realized and satisfied at the determined price. More specifically, the price-sensitive demand from Class 2 customers can be written as

$$D_{2t}(p, \xi_t) = d_t(p) + \xi_t,$$

where $d_t(p)$ is a decreasing and deterministic function of the Class 2 price p in period t. Furthermore, ξ_t is a non-negative, continuous random variable defined in the range $(0, A_t)$ and $E[\xi_t] = \mu_t$. There is no penalty for unsatisfied demand from Class 2. Any inventory remaining after sales to Class 2 is carried to the next period at a unit holding cost of h_t.

To summarize, in each period t, the supplier makes three decisions. Before observing the Class 1 demand, the supplier decides y_t and z_t. After observing and satisfying the Class 1 demand, the supplier decides p_t. That is, inventory can be replenished at the start of the period and is first used to satisfy Class 1 demand. Both the price charged to Class 1 (r_1) and the unit lost sales penalty (b) for unsatisfied demand from Class 1 are constant for the entire planning horizon. Protecting z_t units of inventory from Class 1 customers allows the supplier to satisfy Class 1 demand only partially, and save inventory for potentially higher-profit customers from Class 2. Furthermore, since Class 2 customers may be charged a different price in each period, it allows the supplier to practice dynamic pricing. Any inventory remaining at the end of the planning horizon is salvaged at a unit value of c_0. We

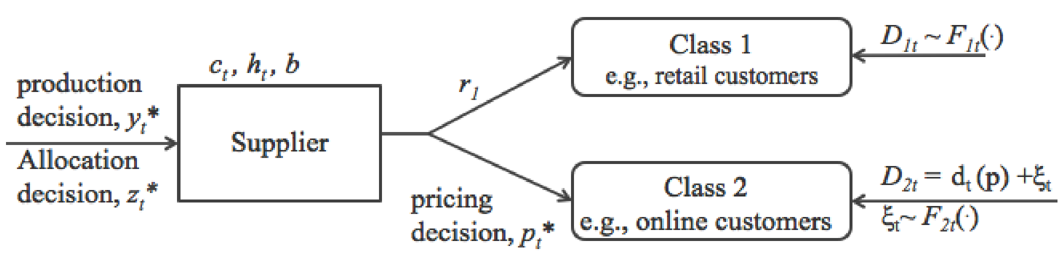


Figure 7: Graphical representation of the problem setting

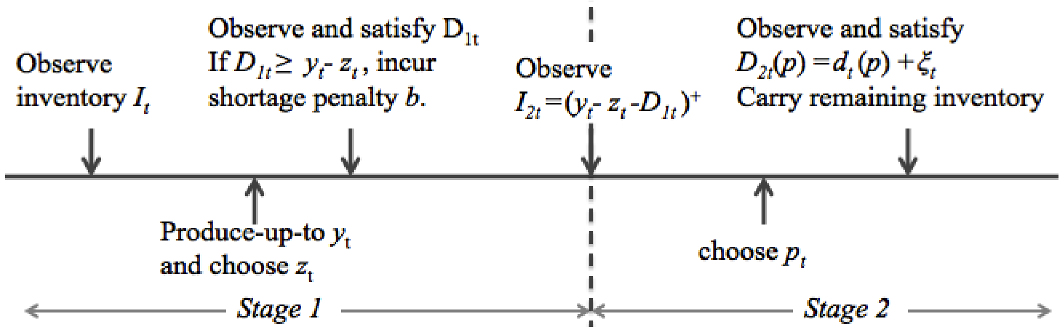


Figure 8: Graphical representation of the sequence of events for period t.

consider a one-period discount factor of $0 \leq \beta \leq 1$. Figure 8 gives a graphical representation of the sequence of events that take place in a period and Table 10 summarizes the notation that we use in our model.

We index each period in terms of the number of periods remaining until the end of the planning horizon. For modeling purposes, we divide each period into two stages as follows: Stage 1 starts at the beginning of the period and ends (and stage 2 starts) after the supplier observes and satisfies Class 1 demand. Stage 2 ends at the end of the period. We formulate the problem as a two-stage stochastic dynamic program. Since we define z_t as the minimum, and not the actual, amount of inventory to be protected from Class 1, the optimal value of z in period t, denoted by z_t^*, is independent of the realized value of D_{1t}. For mathematical simplicity, we formulate the problem such that z_t^* is determined before observing D_{1t}. $V_t(I)$ denotes the maximum expected discounted profit when there are t periods to go until the end of the planning horizon and the starting inventory is I. $V_{2t}(I_2)$ denotes the maximum expected discounted profit from stage 2 of period t until the end of the planning horizon when the left-over inventory after satisfying Class 1 demand is I_2. For each period $t = 1, \ldots, T$, the expected profit earned in stage 2 of period t when the left-over inventory is I_2 and Class 2 customers are charged a unit price of p is:

$$g_{2t}^0(p, I_2) = E\big[p \min\{D_{2t}(p, \xi_t), I_2\} - h_t(I_2 - D_{2t}(p, \xi_t))^+\big], \qquad (115)$$

T	total number of periods
D_{1t}	demand random variable for Class 1 in period t
ξ_t	random element of Class 2 demand in period t
$d_t(p)$	deterministic element of Class 2 demand when unit price in period t is p
$f_{1t}(\cdot)$, $F_{1t}(\cdot)$	probability density and cumulative distribution functions of D_{1t}
$f_{2t}(\cdot)$, $F_{2t}(\cdot)$	probability density and cumulative distribution functions of ξ_t
c_t	unit cost of production in period t
r_1	fixed unit price charged to Class 1 in periods $t = 1, ..., T$
b	unit lost sales penalty for unsatisfied demand from Class 1
h_t	unit holding cost in period t
I_t	on-hand inventory at the beginning of period t
I_{2t}	left-over inventory after satisfying Class 1 demand period t
q_t	number of units produced in period t
y_t	produce-up-to level in period t
z_t	minimum inventory to be protected from Class 1 in period t
p_t	unit price to be charged to Class 2 in period t
p_t^u	maximum price that can be charged to Class 2 customers in period t
β	one-period discount factor

Table 10: Notation

where $x^+ = \max\{x, 0\}$. For conciseness, we define the following functions:

$$G_{it}(u) = \int_u^\infty (x - u) dF_{it}(x), \tag{116}$$

$$g_{2t}(p, I_2) = (p + h_t)\big[d_t(p) + \mu_t - G_{2t}(I_2 - d_t(p))\big]. \tag{117}$$

Reorganizing the terms in (115) and after some algebraic manipulations, it can be shown that $g_{2t}^0(p, I_2) = -h_t I_2 + g_{2t}(p, I_2)$. Let p_t^u be the maximum price that can be charged to Class 2 customers in period t. The maximum expected discounted profit-to-go from stage 2 of period t when the left-over inventory after satisfying Class 1 demand is I_2 can be written as

$$V_{2t}(I_2) = -h_t I_2 + \max\{J_{2t}(p, I_2) : 0 \le p \le p_t^u\}, \quad \text{where} \tag{118}$$

$$\begin{aligned} J_{2t}(p, I_2) = {} & g_{2t}(p, I_2) + \beta E[V_{t-1}(I_2 - D_{2t}(p, \xi_t)) 1(D_{2t}(p, \xi_t) \le I_2)] \\ & + \beta E[V_{t-1}(0) 1(D_{2t}(p, \xi_t) > I_2)] \end{aligned} \tag{119}$$

for $t = 1, \dots, T$. The first term in equation (119) represents the expected profit in stage 2 of period t plus the holding cost for I_2 units. The remaining terms represent the expected discounted profit-to-go for the next $t-1$ periods conditioned on Class 2 demand realization in period t. Similarly, for each period $t = 1, \dots, T$, the expected profit earned in stage 1 of period t when the starting inventory is I, produce-up-to level is y, and z units are protected from Class 1 is given as

$$\begin{aligned} g_t^0(y, z, I) &= E\big[r_1 \min\{D_{1t}, y - z\} - c_t(y_t - I) - b(D_{1t} - y + z)^+\big] \\ &= cI + g_t(y, z), \\ \text{where } g_t(y, z) &= rE[D_{1t}] - (r_1 + b)G_{1t}(y - z) - c_t y. \end{aligned}$$

The maximum expected discounted profit when there are t periods-to-go and the starting inventory is I can be written as

$$V_t(I) = c_t I + \max\{J_t(y, z) : y \geq I, z \geq 0\}, \tag{120}$$

$$\text{where } J_t(y, z) = g_t(y, z) + E[V_{2t}(y - D_{1t})1(D_{1t} \leq y - z)] + E[V_{2t}(z)1(D_{1t} > y - z)] \tag{121}$$

for $t = 1, \dots, T$. The first term in equation (121) represents the expected profit in stage 1 of period t less the production cost for I units. The remaining terms represent the expected discounted profit-to-go from stage 2 of period t conditioned on Class 1 demand realization in period t. We assume that end-of-horizon salvage value, c_0, is such that $0 \leq c_0 \leq c_1$, and hence, $V_0(I) = c_0 I$. The objective is to compute $V_T(I)$ and to determine the optimal values of the decision variables denoted by y_t^*, z_t^* and p_t^* that achieve this maximum expected discounted profit.

We assume that $r_1 > c_t$ for the problem to make economic sense. Furthermore, for each period t, we make the following assumptions:

(A1) $d_t(p)$ is a decreasing and concave function of p for $0 \leq p \leq p_t^u$. That is, $d_t'(p) \leq 0$, $d_t''(p) \leq 0$.

(A2) ξ_t has increasing failure rate, i.e., $r(u) = f_{2t}(u)/(1 - F_{2t}(u))$ is increasing.

(A3) D_{1t} and ξ_t for $t = 1, \ldots, T$ are independent random variables.

Assumption A1 about $d_t(p)$ is mild and common in the economics and inventory management literatures (e.g., [24, 33, 59]). It encompasses power functions of the form $d_t(p) = \alpha - \theta p^\gamma$ ($\gamma > 1$, $\alpha, \theta > 0$), $\ln(\alpha - \theta p^\gamma)$ ($\gamma > 0$), and $a - \exp^{\gamma p}$ ($\gamma > 0$). Assumption A2 is also common in the literature (e.g., [24, 51, 59]). Examples of distributions with increasing failure rate include exponential, uniform, normal, truncated normal, lognormal, gamma with shape parameters greater than 1, beta with both shape and scale parameters greater than 1 (e.g., see Lariviere 2006 for more details). Assumption A3 means that demands in different classes and in consecutive periods are independent.

The maximum price that can be charged to Class 2 customers, p_t^u, is the smallest price such that $d_t(p) = 0$ (e.g., see [24]). We do not restrict the value of p_t^* to be greater than c_t, $t = 1, \ldots, T$. If $p_t^* \geq c_t$, then selling to Class 2 is profitable. On the other hand, if $p_t^* < c_t$, then selling to Class 2 is equivalent to salvaging. Furthermore, we consider that the inverse function of $d_t(\cdot)$, denoted by $d_t^{-1}(\cdot)$, is continuous and is defined as $d_t^{-1}(I) = p$ such that $d_t(p) = I$ for $0 \leq I \leq d_t(0)$. Furthermore, from assumption A1, we have $(d_t^{-1})'(d_t(p)) = 1/d_t'(p) \leq 0$ and $(d_t^{-1})''(d_t(p)) = -d_t''(p)/(d_t'(p))^3 \leq 0$. That is, $d_t^{-1}(\cdot)$ is decreasing and concave.

III.3.1 Characterization of the structure of the optimal policy

In this section, we first present an assumption that simplifies the analysis presented in rest of the chapter. Then, we present a theorem which completely characterizes the structure of the optimal policy. For a special case of our problem, we establish a lower bound on the optimal price and discuss the insights provided by this special case.

Similar to [80] and for the ease of exposition, we assume that for all $I_2 > 0$

$$\left.\frac{\partial J_{2t}(p, I_2)}{\partial p}\right|_{p=p_t^u} = \mu_t - G_{2t}(I_2) + d_t'(p_t^u)(p_t^u + h_t - \beta c_{t-1})F_{2t}(I_2) < 0. \qquad (122)$$

This assumption ensures that Class 2 is never charged the maximum price p_t^u and implies that μ_t is not too large. For example, when $d_t(p) = a_1 - a_2 p$ and ξ_t follows Uniform$(0, A_t)$

distribution, condition (122) implies that $\mu_t = 0.5A_t \leq 0.5(a_1 - a_2(\beta c_{t-1} - h_t))$. Results in Theorem 3 can be generalized to consider settings where condition (122) is not satisfied, albeit at the expense of a more tedious argument. In particular, when condition (122) is not satisfied for some left-over inventory $\bar{I}_2$ in period t, it can be shown that $J_{2t}(p, \bar{I}_2)$ is increasing for $0 < p \leq p_t^u$. Therefore, when the left-over inventory after satisfying Class 1 demand does not satisfy condition (122) the optimal price to be charged to Class 2 customers is p_t^u.

Before we characterize the structure of the optimal policy, we present a useful identity.

Lemma 1. *For any* $I_2 > 0$ *and* $0 < p < p_t^u$, $\big[d_t(p) + \mu_t - G_{2t}(I_2 - d_t(p))\big] > 0$ *for* $t = 1, \ldots, T$.

Proof. We recall that $\xi_t \sim (0, A_t)$, and hence, for $x \leq 0$ we have $dF_{2t}(x) = 0$. Then, using equation (116) for $0 < I_2 \leq d_t(p)$, we have

$$\begin{aligned} d_t(p) + \mu_t - G_{2t}(I_2 - d_t(p)) &= \int_0^{A_t} (d_t(p) + x) dF_{2t}(x) - \int_0^{A_t} (x - I_2 + d_t(p)) dF_{2t}(x) \\ &= I_2 > 0. \end{aligned}$$

On the other hand, if $I_2 \geq d_t(p)$, then

$$\begin{aligned} d_t(p) + \mu_t - G_{2t}(I_2 - d_t(p)) &= \int_0^{A_t} (d_t(p) + x) dF_{2t}(x) - \int_{I_2 - d_t(p)}^{A_t} (x - I_2 + d_t(p)) dF_{2t}(x) \\ &= \int_0^{I_2 - d_t(p))} (d_t(p) + x) dF_{2t}(x) + I_2\big[1 - F_{2t}\big(I_2 - d_t(p)\big)\big] > 0. \end{aligned}$$

□

Theorem 3 shows that the optimal price charged to Class 2 in period t is a function of the left-over inventory after satisfying Class 1 demand in period t. It also presents how to determine the optimal Class 2 price for a given I_2, denoted by $p_t^*(I_2)$, by studying the functions $V_{2t}(I_2)$ and $J_{2t}(p, I_2)$, given by equations (118) and (119). Theorem 3 also presents how to determine the stage 1 optimal policy parameters (y_t^*, z_t^*) by studying the functions $V_t(I)$ and $J_t(y, z)$ given by equations (120) and (121). In particular, it shows that the stage

1 optimal policy is characterized by two critical quantities: S_t is the critical produce-up-to level and R_t is the critical amount of inventory to be protected from Class 1 in period t.

Theorem 3. *$J_{2t}(p, I_2)$, $V_{2t}(I_2)$, $J_t(y, z)$ and $V_t(I)$ satisfy the following properties for $t = 1, \dots, T$:*

(a) For a given $I_2 \geq 0$, $J_{2t}(p, I_2)$ has a finite maximizer $p_t^*(I_2)$ such that $p_t^*(I_2) < p_t^u$ and

$$\left.\frac{\partial J_{2t}(p, I_2)}{\partial I_2}\right|_{p=p_t^*(I_2)} = 0.$$

(b) $p_t^*(I_2)$ is continuous and decreasing in I_2, and $\big(I_2 - d_t(p_t^*(I_2))\big)$ is increasing in I_2.

(c) $V_{2t}(I_2)$ is concave in I_2.

(d) $\dfrac{dV_{2t}(I_2)}{dI_2} \leq p_t^u$.

(e) $J_t(y, z)$ has a finite maximizer denoted by (y_t^*, z_t^*) such that

$$z_t^* = \begin{cases} 0 & \text{if } p^u \leq r_1 + b \\ R_t & \text{if } p^u > r_1 + b, \end{cases} \tag{123}$$

where $R_t > 0$ satisfies $\left.\dfrac{dV_{2t}(I_2)}{dI_2}\right|_{I_2=R_t} = r_1 + b$, and $y_t^* = \max\{S_t, I\}$ where S_t satisfies $\left.\dfrac{\partial J_t(y, z_t^*)}{\partial y}\right|_{y=S_t} = 0$.

(f) $V_t(I)$ is concave in I.

(g) $\dfrac{dV_t(I)}{dI} \leq c_t$.

Proof. Before proceeding with a proof, we present the following first and second order derivatives of $J_{2t}(p, I_2)$ with respect to p as follows:

$$\begin{aligned} \frac{\partial J_{2t}(p, I_2)}{\partial p} &= d_t(p) + \mu_t - G_{2t}(I_2 - d_t(p)) + d_t'(p)(p + h_t)F_{2t}(I_2 - d_t(p)) \\ &\quad - d_t'(p)\beta E\left[\left.\frac{dV_{t-1}(I)}{dI}\right|_{I=I_2-d_t(p)-\xi_t} 1(\xi_t \leq I_2 - d_t(p))\right], \end{aligned} \tag{124}$$

$$\frac{\partial J_{2t}^2(p, I_2)}{\partial p^2} = [2d_t'(p) + (p + h_t)d_t''(p)]F_{2t}(I_2 - d_t(p))$$
$$- (d_t'(p))^2\left(p + h_t - \beta\frac{dV_{t-1}(I)}{dI}\bigg|_{I=0}\right)f_{2t}(I_2 - d_t(p))$$
$$+ (d_t'(p))^2\beta E\left[\frac{d^2V_{t-1}(I)}{dI^2}\bigg|_{I=I_2-d_t(p)-\xi_t} 1(\xi_t \leq I_2 - d_t(p))\right]$$
$$- d_t''(p)\beta E\left[\frac{dV_{t-1}(I)}{dI}\bigg|_{I=I_2-d_t(p)-\xi_t} 1(\xi_t \leq I_2 - d_t(p))\right]. \tag{125}$$

The proof follows by induction. We will first show that the properties (a)-(g) are true for the one-period problem. Since $V_0(I) = c_0 I$, we have $dV_0(I)/dI = c_0$ and $d^2V_0(I)/dI^2 = 0$. Substituting this in equation (124) for $t = 1$ and using Lemma 1 in the resulting expression, we observe that $\partial J_{2t}(p, I_2)/\partial p > 0$ for $p \leq \beta c_0 - h_1$. Therefore, the optimal price to be charged to Class 2 in period $t = 1$ is greater than $\beta c_0 - h_1$. Substituting $dV_0(I)/dI = c_0$ in equation (125) for $t = 1$, $p \geq \beta c_0 - h_1$, and using assumption A1 we have $\partial J_{21}^2(p, I_2)/\partial p^2 \leq 0$. Hence, $J_{21}^2(p, I_2)$ is a concave function of p for a given I_2. Define $H_t(p, I_2) = \partial J_{2t}(p, I_2)/\partial p$ for $t = 1, \ldots, T$, and $p_1^*(I_2)$ such that $H_1(p_1^*(I_2), I_2) = 0$. Then, from the first order condition we know that $p_1^*(I_2)$ maximizes $J_{21}(p, I_2)$. Furthermore, evaluating $H_1(p, I_2)$ at $p = p_1^u$ and using the inequality (122), it follows that $H_1(p_1^u, I_2) < 0$, and hence, $p_1^*(I_2) < p_1^u$ for all $I_2 > 0$. For simpler notation we write $p_t^*(I_2)$ simply as p_t^*, for $t = 1, \ldots, T$.

To prove property (b), for $t = 1, \ldots, T$ define $H_t^0(p, I_2)$ as follows:

$$H_t^0(p, I_2) = d_t(p) + \mu_t - G_{2t}(I_2 - d_t(p)) + d_t'(p)(p + h_t - \beta c_{t-1})F_{2t}(I_2 - d_t(p)). \tag{126}$$

Substituting $dV_0(I)/dI = c_0$ in equation (124) and evaluating equation (126) for $t = 1$, we observe that

$$H_1(p, I_2) = \frac{\partial J_{21}(p, I_2)}{\partial p} = H_1^0(p, I_2). \tag{127}$$

Using equation (116) and integration by parts, we rewrite $H_t^0(p, I_2)$ as follows:

$$\begin{aligned}
H_t^0(p, I_2) &= d_t(p) + \mu_t - \int_{I_2 - d_t(p)}^{\infty} (x - I_2 + d_t(p)) dF_{2t}(x) \\
&\quad + d_t'(p)(p + h_t - \beta c_{t-1}) F_{2t}(I_2 - d_t(p)) \\
&= I_2 + \int_0^{I_2 - d_t(p)} (x - I_2 + d_t(p)) dF_{2t}(x) + d_t'(p)(p + h_t - \beta c_{t-1}) F_{2t}(I_2 - d_t(p)) \\
&= I_2 - \int_0^{I_2 - d_t(p)} F_{2t}(x) dx + d_t'(p)(p + h_t - \beta c_{t-1}) F_{2t}(I_2 - d_t(p)) \\
&= \int_0^{I_2 - d_t(p)} \Big(1 - F_{2t}(x) + d_t'(p)(p + h_t - \beta c_{t-1}) f_{2t}(x)\Big) dx + d_t(p) \\
&= \int_{d_t(p)}^{I_2} \Big(1 - F_{2t}(x - d_t(p)) + d_t'(p)(p + h_t - \beta c_{t-1}) f_{2t}(x - d_t(p))\Big) dx + d_t(p) \\
&= \int_{d_t(p)}^{I_2} (1 - F_{2t}(x - d_t(p))) \left[1 + \frac{d_t'(p)(p + h_t - \beta c_{t-1}) f_{2t}(x - d_t(p))}{1 - F_{2t}(x - d_t(p))}\right] dx + d_t(p).
\end{aligned}$$

Using the fact that for $x < 0$, $f_{2t}(x) = F_{2t}(x) = 0$, we have

$$H_t^0(p, I_2) = \int_0^{I_2} (1 - F_{2t}(x - d_1(p))) \left[1 + \frac{d_t'(p)(p + h_t - \beta c_{t-1}) f_{2t}(x - d_t(p))}{1 - F_{2t}(x - d_t(p))}\right] dx.$$

For conciseness, define the the following functions for $t = 1, \ldots, T$

$$e_t(p, I_2) = \frac{-d_t'(p)(p + h_t - \beta c_{t-1}) f_{2t}(I_2 - d_t(p))}{1 - F_{2t}(I_2 - d_t(p))}, \tag{128}$$

$$\text{and} \quad Q_t(p, I_2) = [1 - F_{2t}(I_2 - d_t(p))](1 - e_t(p, I_2)). \tag{129}$$

Then, it follows that

$$H_t^0(p, I_2) = \int_0^{I_2} Q_t(p, x) dx. \tag{130}$$

Evaluating $H_1(p_1, I_2)$ at $p = p_1^*$ and using equations (127) and (130) for $t = 1$, we have $H_1(p_1^*, I_2) = \int_0^{I_2} Q_1(p_1^*, x) dx = 0$. Using implicit differentiation, we have

$$\frac{dp_1^*(I_2)}{dI_2} = \frac{-\partial H_1 / \partial I_2}{\partial H_1 / \partial p_1^*}. \tag{131}$$

We can express each term in (131) as follows:

$$\frac{\partial H_1}{\partial p_1^*} = \frac{\partial J_{21}^2(p, I_2)}{\partial p^2}\bigg|_{p=p_1^*} \le 0, \tag{132}$$
$$\frac{\partial H_1}{\partial I_2} = 1 - F_{21}(I_2 - d_1(p_1^*)) + d_1'(p_1^*)(p_1^* + h_1 - \beta c_s)f_{21}(I_2 - d_1(p_1^*)) = Q_1(p_1^*, I_2).$$

From assumptions A1 and A2, it follows that $e_t(p, I_2)$ is increasing in I_2 for $t = 1, \dots, T$. Then, for a fixed p, $Q_t(p, I_2)$ is decreasing in I_2 for $t = 1, \dots, T$. If $Q_1(p_1^*, I_2) > 0$, then for all $x \le I_2$, $Q_1(p_1^*, x) > 0$, and hence, $H_1(p_1^*, I_2) = \int_0^{I_2} Q_1(p_1^*, x)dx > 0$. This contradicts the definition of p_1^*, i.e., $H_1(p_1^*, I_2) = 0$. Therefore, we have $Q_1(p_1^*, I_2) \le 0$, and hence, $\partial H_1/\partial I_2 \le 0$. Using this and the inequality (132) on the right-hand side of (131), we have $dp_1^*(I_2)/dI_2 \le 0$, and hence, $p_1^*(I_2)$ is a decreasing function of I_2.

Next, evaluating $H_1(p, I_2) = \partial J_{2t}(p, I_2)/\partial p$ at $p = p_t^l = \beta c_0 - h_1$, we see that $H_1(p = \beta c_0 - h_1, I_2) = d_1(\beta c_0 - h_1) + \mu_t - G_{2t}(I_2 - d_1(\beta c_0 - h_1)) > 0$. Then, from concavity of $J_{2t}(p, I_2)$ it follows that $p_1^*(I_2) \ge \beta c_0 - h_1$ for $I_2 \ge 0$. Furthermore, we note that when $I_2 = 0$, $H_1(p, 0) = d_t(p) > 0$ for $p < p_1^u$ and $H_1(p^u, 0) = 0$. Hence, we have $p_1^*(0) = p_1^u$. Since $p_1^*(I_2)$ is a decreasing function of I_2, we have $p_t^*(I_2) \le p_1^u$. Thus, $p_1^*(I_2)$ always satisfies the constraint $p_1^l < p_1^*(I_2) \le p_1^u$ for $I_2 \ge 0$.

Next, lets us define $w_t(I_2) = I_2 - d_t(p_t^*(I_2))$, then

$$\frac{dw_t}{dI_2} = 1 - d_1'(p_1^*)\frac{dp_1^*(I_2)}{dI_2} = \frac{\partial H_1/\partial p_1^* + d_1'(p_1^*)\partial H_1/\partial I_2}{\partial H_1/\partial p_1^*}$$
$$= \frac{d_1'(p_1^*) + [d_1'(p_1^*) + d_1''(p_1^*)(p_1^* + h_1 - \beta c_0)]F_{21}(I_2 - d_1(p_1^*))}{\partial H_1/\partial p_1^*} > 0. \tag{133}$$

Using equation (118) for $t = 1$, we have $V_{21}(I_2) = -h_1 I_2 + J_{21}(p_1^*(I_2), I_2)$, and

$$\frac{dV_{21}(I_2)}{dI_2} = \beta c_0 - h_1 + (p_1^* + h_1 - \beta c_0)\big(1 - F_{21}(I_2 - d_1(p_1^*))\big) \tag{134}$$
$$\frac{d^2V_{21}(I_2)}{dI_2^2} = \frac{dp_1^*(I_2)}{dI_2}(1 - F_{21}(I_2 - d_1(p_1^*))$$
$$- (p_1^* + h_1 - \beta c_0)f_{21}(I_2 - d_1(p_1^*))\left(1 - d_1'(p_1^*)\frac{dp_1^*(I_2)}{dI_2}\right) \le 0.$$

Hence, $V_{21}(I_2)$ is concave in I_2. From equation (134) we have $dV_{21}(I_2)/dI_2|_{I_2=0} = p_1^*(0) = p_1^u$. Evaluating the limit of equation (134) as I_2 goes to infinity, we obtain

$$\lim_{I_2 \to \infty} \frac{dV_{21}(I_2)}{dI_2} = \beta c_0 - h_1.$$

Then, property (d) follows directly from concavity of $V_{21}(\cdot)$. Thus properties (a)-(d) are true for stage 2 for period $t = 1$.

Next, we consider the stage 1 problem for $t = 1$ and present the following first and second order derivatives, which will be used to prove properties (e)-(g):

$$\frac{\partial J_t(y,z)}{\partial z} = \left[\frac{dV_{2t}(z)}{dz} - (r_1 + b)\right](1 - F_{1t}(y - z)), \tag{135}$$

$$\frac{\partial J_t(y,z)}{\partial y} = (r_1 + b)(1 - F_{1t}(y - z)) - c_t + E\left[\frac{dV_{2t}(y - D_{1t})}{dy} 1(D_{1t} \le y - z)\right], \tag{136}$$

$$\frac{\partial^2 J_t(y,z)}{\partial y^2} = -\left[r_1 + b - \left.\frac{dV_{2t}(I_2)}{dI_2}\right|_{I_2=z}\right] f_{1t}(y - z) + E\left[\frac{d^2 V_{2t}(y - D_1)}{dy^2} 1(D_{1t} \le y - z)\right]. \tag{137}$$

Suppose that $p_1^u \le r_1 + b$, then from property (d), we have $dV_{21}(z)/dz \le p_1^u \le r_1 + b$ for $z > 0$. Using this inequality and equation (135) for $t = 1$, we observe that $J_1(y,z)$ is a decreasing function of z for $y \le I$. Therefore, $z_1^* = 0$. On the other hand, if $p_1^u > r_1 + b$, then using the fact that $c < r_1 + b$, property (d) for period $t = 1$ and the intermediate value theorem there exists a $R_1 \in \Re^+$ such that

$$\left.\frac{dV_{21}(I_2)}{dI_2}\right|_{I_2=R_1} = r_1 + b. \tag{138}$$

Evaluating expression (135) for $t = 1$ and using equation (138), we observe that $J_1(y,z)$ is increasing in z for $z < R_1$ and decreasing in z for $z > R_1$. Therefore, $J_1(y,z)$ attains its maximum at $z_1^* = R_1$ for $y \ge I$. From the above discussion, we see that

$$\left.\frac{dV_{21}(I_2)}{dI_2}\right|_{I_2=z_1^*} \le r_1 + b. \tag{139}$$

Evaluating expression (137) at $t = 1$ and $z = z_1^*$, and using the inequality in (139) and the concavity of $V_{21}(\cdot)$, we have $\partial^2 J_1(y, z_1^*)/\partial y^2 \leq 0$. Hence, $J_1(y, z_1^*)$ is concave in y. Defining S_1 such that $\partial J_1(y, z_1^*)/\partial y|_{y=S_1} = 0$, we have $y_1^*(I) = \max\{S_1, I\}$. From equation (120) it follows that $V_1(I) = cI + J_1(y_1^*(I), z_1^*)$, and

$$\frac{dV_1(I)}{dI} = c_1 + \left.\frac{\partial J_1(y, z_1^*)}{\partial y}\right|_{y=I} 1(I > S_1). \tag{140}$$

Then, taking the derivative of equation (140) with respect to I and using the concavity of $J_1(y, z_1^*)$ in y, it follows that $d^2V_1(I)/dI^2 \leq 0$. Hence, $V_1(I)$ is concave in I. Evaluating equation (140) at $I = 0$ we see that $dV_1(I)/dI|_{I=0} = c_1$, and using the concavity of $V_1(I)$ it is easy to see that property (g) is true for $t = 1$. As a result, properties (a)-(g) are true for $t = 1$.

Suppose that properties (a)-(g) are true for periods $t-1, t-2, \ldots, 2$. We will show that they are true for period t. Using assumption A1 and the properties (f) and (g) of the induction hypothesis on the right-hand side of (125), it is easy to see that $J_{2t}(p, I_2)$ is concave in p for $p_t^l \leq p \leq p_t^u$. Let $p_t^*(I_2)$ be the maximizer of $J_{2t}(p, I_2)$ for a fixed I_2. Then, from the first order condition we have (124) $H_t(p_t^*(I_2), I_2) = 0$, where $H_t(p, I_2) = \partial J_{2t}(p, I_2)/\partial p$. Furthermore, evaluating $H_t(p, I_2)$ at $p = p_1^u$ and using the inequality (122) and part (f) of the induction hypothesis, it follows that $H_t(p_t^u, I_2) < 0$, and hence, $p_t^*(I_2) < p_t^u$ for all $I_2 > 0$. Again, using implicit differentiation, we have

$$\frac{dp_t^*(I_2)}{dI_2} = \frac{-\partial H_t(p_t^*, I_2)/\partial I_2}{\partial H_t(p_t^*, I_2)/\partial p_t^*}. \tag{141}$$

We can express each term in (141) as follows:

$$\frac{\partial H_t(p_t^*, I_2)}{\partial p_t^*} = \left.\frac{\partial J_{2t}^2(p, I_2)}{\partial p^2}\right|_{p=p_t^*} \leq 0, \tag{142}$$

$$
\begin{aligned}
\frac{\partial H_t(p_t^*, I_2)}{\partial I_2} &= 1 - F_{2t}(I_2 - d_t(p_t^*)) + d_t'(p_t^*)(p_t^* + h_t - \beta c_{t-1}) f_{2t}(I_2 - d_t(p_t^*)) \\
&\quad - d_t'(p_t^*)\beta E\left[\frac{d^2 V_{t-1}(I)}{dI^2}\bigg|_{I = I_2 - d_t(p_t^*) - \xi_t} 1(\xi_t \le I_2 - d_t(p_t^*))\right].
\end{aligned} \tag{143}
$$

Thus, both the numerator and the denominator of the right-hand side of (141) exist, and hence, $p_t^*(I_2)$ is continuous in I_2^*. Taking the first order derivative of $w_t(I_2) = I_2 - d_t(p_t^*(I_2))$ with respect to I_2, we have

$$
\begin{aligned}
\frac{dw_t(I_2)}{dI_2} &= 1 - d_t'(p_t^*)\frac{dp_t^*(I_2)}{dI_2} = \frac{\partial H_t(p_t^*, I_2)/\partial p_t^* + d_t'(p_t^*)\partial H_t(p_t^*, I_2)/\partial I_2}{\partial H_t(p_t^*, I_2)/\partial p_t^*} \\
&= \frac{1}{\partial H_t/\partial p_t^*}\Bigg(d_t'(p_t^*) + [d_t'(p_t^*) + d_t''(p_t^*)(p_t^* + h_t)]F_{2t}(w_t(I_2)) \\
&\quad - d_t''(p_t^*)\beta E\left[\frac{dV_{t-1}(I)}{dI}\bigg|_{I = w_t(I_2) - \xi_t} 1(\xi_t \le w_t(I_2))\right]\Bigg).
\end{aligned}
$$

Using part (g) of the induction hypothesis that $dV_{t-1}(I)/dI \le c_{t-1}$ in the above equation, we have

$$
\begin{aligned}
\frac{dw_t(I_2)}{dI_2} &= 1 - d_t'(p_t^*)\frac{dp_t^*(I_2)}{dI_2} \\
&> \frac{d_t'(p_t^*) + [d_t'(p_t^*) + d_t''(p_t^*)(p_t^* + h_t - \beta c_{t-1})]F_{2t}(w_t(I_2))}{\partial H_t/\partial p_t^*} > 0.
\end{aligned} \tag{144}
$$

Thus, $w_t(I_2)$ is a increasing function of I_2.

From equation (141) and inequality (142), we note that to show that $p_t^*(I_2)$ is a decreasing function of I_2, we need to show that $\partial H_t(p_t^*, I_2)/\partial I_2 \le 0$. To this end, we rewrite $H_t(p_t, I_2)$. Using the induction hypothesis, we have $V_{t-1}(I) = c_{t-1}I + J_{t-1}(y_{t-1}^*(I), z_{t-1}^*)$, and hence,

$$
\frac{dV_{t-1}(I)}{dI} = c_{t-1} + \frac{\partial J_{t-1}(y, z_{t-1}^*)}{\partial y}\bigg|_{y=I} 1(I > S_{t-1}). \tag{145}
$$

Furthermore, from the definition of S_{t-1} and concavity of $V_{t-1}(\cdot)$, it follows that

$$
\frac{\partial J_{t-1}(y, z_{t-1}^*)}{\partial y}\bigg|_{y > S_{t-1}} \le 0. \tag{146}
$$

Then, using equation (124) and (145), $H_t(p, I_2) = \partial J_{2t}(p, I_2)/\partial p$ and $dH_t(p, I_2)/dI_2$ can be rewritten as

$$
\begin{aligned}
H_t(p, I_2) &= H_t^0(p, I_2) - K_t(p, I_2), && (147)\\
\frac{dH_t(p, I_2)}{dI_2} &= 1 - F_{2t}(I_2 - d_t(p)) + d_t'(p)(p + h_t - \beta c_{t-1}) f_{2t}(I_2 - d_t(p)) \\
&\quad - d_t'(p)\beta E\left[\frac{d^2 J_{t-1}(I, z_{t-1}^*)}{dI^2}\bigg|_{I=I_2-d_t(p)-\xi_t} 1(\xi_t \le (I_2 - d_t(p) - S_{t-1})^+)\right] \\
\frac{dH_t(p, I_2)}{dI_2} &= Q_t(p, I_2) - d_t'(p)\beta E\left[\frac{d^2 J_{t-1}(I, z_{t-1}^*)}{dI^2}\bigg|_{I=I_2-d_t(p)-\xi_t} 1(\xi_t \le (I_2 - d_t(p) - S_{t-1})^+)\right], \\
& && (148)
\end{aligned}
$$

where $H_t^0(p, I_2)$ is as defined in equation (126) and $K_t(p, I_2)$ is defined as follows:

$$
K_t(p, I_2) = d_t'(p)\beta E\left[\frac{dJ_{t-1}(I, z_{t-1}^*)}{dI}\bigg|_{I=I_2-d_t(p)-\xi_t} 1\big(\xi_t \le (I_2 - d_t(p) - S_{t-1})^+\big)\right] > 0. \quad (149)
$$

The non-negativity in (149) follows from the inequality in (146) and the assumption that $d_t'(p) < 0$.

Define $p_t^0(I_2)$ such that $H_t^0(p_t^0(I_2), I_2) = 0$. Then, following the exact same analysis as that for $t = 1$, it can be shown that $p_t^0(I_2)$ is a decreasing function of I_2. Next, since $p_t^*(I_2)$ is continuous, $w_t(I_2)$ is also continuous in I_2, and there exists a I_{2t}^s such that $w_t(I_{2t}^s) = I_{2t}^s - d_t(p_t^s) = S_{t-1}$, where $p_t^s = p_t^*(I_{2t}^s)$. Since, $w_t(I_2)$ is increasing in I_2, we have $w(I_2) = I_2 - d_t(p_t^*(I_2)) \le S_{t-1}$ for $0 \le I_2 \le I_{2t}^s$. Then, for $0 \le I_2 \le I_{2t}^s$ the last terms in equations (147) and (148) are equal to zero. Therefore, $H_t(p, I_2) = H_t^0(p, I_2)$, and hence, $p_t^*(I_2) = p_t^0(I_2)$ is a decreasing function of I_2. Consequently, for $0 \le I_2 \le I_{2t}^s$, we have $p_t^s \le p_t^*(I_2) \le p_t^u$.

Similarly, for $I_2 > I_{2t}^s$ we have $w(I_2) = I_2 - d_t(p_t^*(I_2)) \ge S_{t-1}$. Then, using $H_t(p_t^*(I_2), I_2) = 0$, equations (147) and the inequality in (149), we have, $H_t^0(p_t^*(I_2), I_2) > 0$. We know that $H_t^0(p, I_2)$ is a decreasing function of p and $H_t^0(p_t^0(I_2), I_2) = 0$. Therefore, $p_t^*(I_2) < p_t^0(I_2) < p_t^0(I_2 \le I_{2t}^s) = p_t^*(I_2 \le I_{2t}^s)$. Next, we make the following observations:

O1 Recall that for a fixed p, $Q_t(p, I_2)$ is decreasing in I_2. For a fixed p define $I_{2t}^Q(p)$ such that $Q_t(p, I_{2t}^Q(p)) = 0$. Then, from equation (130), we see that $H_t^0(p, I_2)$ is increasing

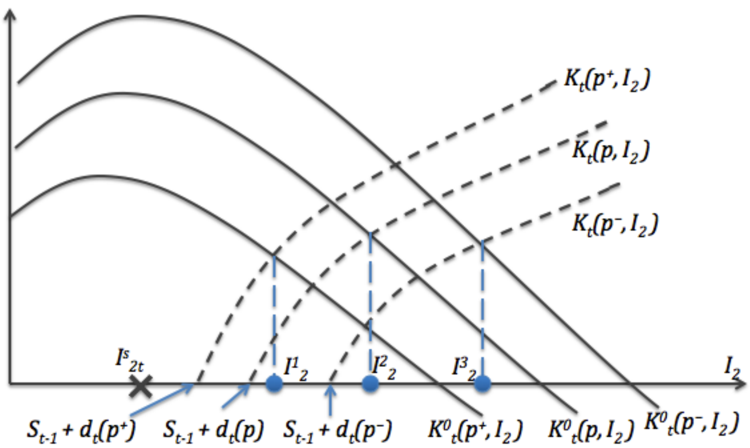


Figure 9: $H^0_t(p, I_2)$ and $K_t(p, I_2)$ w.r.t to I_2 for different values of p

for $I_2 \le I^Q_{2t}(p)$, and $H^0_t(p, I_2)$ is decreasing for $I_2 > I^Q_{2t}(p)$. Therefore, $I^Q_{2t}(p^s_t) < I^2_{2t}$.

O2 For a fixed I_2, $H^0_t(p, I_2)$ is decreasing in p.

O3 For a fixed p, $K_t(p, I_2)$ is increasing in I_2, and for a fixed I_2, $K_t(p, I_2)$ is increasing in p.

Using the above three observations, we can plot $H^0_t(p, I_2)$ and $K_t(p, I_2)$ for different values of I_2 and p as in Figure 9, where $p^- < p < p^+$. In Figure 9, I^1_2, I^2_2 and I^3_2 are such that $I^1_2 < I^2_2 < I^3_2$. We observe that $p^*_t(I^1_2) > p^*_t(I^2_2) > p^*_t(I^3_2)$, and hence, $p^*_t(I_2)$ is a decreasing function of I_2

Next, using equation (118), we have $V_{2t}(I_2) = -h_t I_2 + J_{2t}(p^*_t(I_2), I_2)$, and

$$\begin{aligned}
\frac{dV_{2t}(I_2)}{dI_2} &= -h_t + (p^*_t + h_t)\big(1 - F_{2t}(I_2 - d_t(p^*_t))\big) \\
&\quad + \beta E\left[\left.\frac{dV_{t-1}(I)}{dI}\right|_{I=I_2-d_t(p^*_t)-\xi_t} 1(\xi_t \le I_2 - d_t(p^*_t))\right] \qquad (150) \\
\frac{d^2V_{2t}(I_2)}{dI_2^2} &= \frac{dp^*_t(I_2)}{dI_2}(1 - F_{2t}(I_2 - d_t(p^*_t)) \\
&\quad - (p^*_t + h_t - \beta c_{t-1}) f_{2t}(I_2 - d_t(p^*_t))\left(1 - d'_t(p^*_t)\frac{dp^*_t(I_2)}{dI_2}\right) \\
&\quad + \beta E\left[\left.\frac{d^2V_{t-1}(I)}{dI^2}\right|_{I=I_2-d_t(p^*_t)-\xi_t} 1(\xi_t \le I_2 - d_t(p^*_t))\right] \le 0.
\end{aligned}$$

Hence, $V_{2t}(I_2)$ is concave in I_2. From equation (150) we have $dV_{2t}(I_2)/dI_2|_{I_2=0} = p^*_t(0) = p^u_t$, and property (d) follows directly from concavity of $V_{2t}(\cdot)$. Thus properties (a)-(d) are true for stage 2 for period $t = 1$.

Properties (e)-(g) for the stage 1 problem in period t follow by using the properties (a)-(d) for the stage 2 problem in period t and using the exact same analysis as for period $t = 1$. $\square$

Theorem 3 shows that the optimal policy is a combination of produce-up-to policy with potential discretionary sales to Class 1 and dynamic pricing to Class 2. For a given value of I_2, i.e., the left-over inventory after satisfying Class 1 demand, part (a) of Theorem 3 shows how to compute the optimal price to be charged to Class 2. Part (b) shows that $p_t^*(I_2)$ is an decreasing function of I_2 and $\big(I_2 - d_t(p_t^*(I_2))\big)$ is an increasing function of I_2. Part (c) shows that the maximum expected stage 2 profit-to-go is a concave function of I_2. Part (d) shows that for every unit of left-over inventory after sales to Class 1, the increase in the optimal expected profit at stage 2 is bounded by the maximum price that can be charged to Class 2.

We show that if $p^u \leq r_1 + b$, then $J_t(y, z)$ is a decreasing function of z for $y \geq 0$. Note that throughout the text, we use decreasing (increasing) to mean non-increasing (non-decreasing) for brevity. From equation (120) if $p^u \leq r_1 + b$, it follows that $z_t^* = 0$, and hence, it is not optimal to practice discretionary sales to Class 1. On the other hand, if $p^u > r_1 + b$ then $J_t(y, z)$ is a unimodal function of z for $y \geq 0$, and $z_t^* = R_t > 0$. That is, if $p^u > r_1 + b$, then it is optimal to practice discretionary sales by protecting $z_t^* = R_t > 0$ units from sales to Class 1. For period t, R_t denotes the inventory level at which the discounted marginal value of protecting one more unit of inventory from Class 1 is equal to sum of price charged to Class 1 and the unit lost sales penalty. Part (e) also shows that a produce-up-to policy is optimal in period t. Part (f) of Theorem 3 shows that the maximum expected profit-to-go in period t is a concave function of the initial inventory. Part (g) shows that for every unit of on-hand inventory available at the beginning of a period, the increase in the optimal expected profit is bounded by the unit production cost in period t.

Using the results presented in Theorem 3, the supplier's problem can be solved via a backward induction algorithm follows: First, compute and record $p_t^*(I_2)$, $V_{2t}(I_2)$ and

$dV_{2t}(I_2)/dI_2$ for each possible I_2 for period $t = 1$. Using these equations, compute and record R_t, S_t, $V_t(I)$ and $dV_t(I)/dI$ for each possible I for period $t = 1$. Then, repeat the above steps for each period $t = 2, \ldots, T$. Clearly, this requires the use of numerical methods, which may take considerable time to solve (e.g., see [54, 68]). This, in turn, may make the optimal policy less attractive from an implementation perspective. Considering the practical relevance of this problem, our goal is to determine conditions under which computations for the optimal policy parameters are simplified.

To this end, we first examine a special case of our problem and discuss relevant insights. In particular, we examine a special case such that the Class 2 demand is a deterministic function of its price, i.e., $\xi_t = 0$ and $D_{2t} = d_t(p)$ for $t = 1, \ldots, T$. Corollary 1 shows how $p_t^*(I_2)$ and z_t^* simplify for this special case.

Corollary 1. *When Class 2 demand is a deterministic function of its price, $p_t^*(I_2)$ and z_t^* simplify as follows for $t = 1, ..., T$:*
(a) $p_t^*(I_2) = \max\{d_t^{-1}(I_2), p_t^0\}$, where $p_t^0 > \beta c_{t-1} - h_t$ and satisfies

$$d_t(p_t^0) + (p_t^0 + h_t - \beta c_{t-1})d_t'(p_t^0) = 0. \tag{151}$$

(b) z_t^* is given by equation (123), and R_t satisfies the following equality:

$$r_1 + b = d_t^{-1}(R_t) + \frac{R_t}{d_t'(R_t)}. \tag{152}$$

Proof. Suppose that $\xi_t = 0$ and $D_{2t} = d_t(p)$ for $t = 1, \ldots, T$. Let $J_{2t}^d(p, I_2)$, $V_{2t}^d(I_2)$, $J_t^d(y, z)$ and $V_t^d(I)$ be reduced forms of $J_{2t}(p, I_2)$, $V_{2t}(I_2)$, $J_t(y, z)$ and $V_t(I)$ when $D_{2t} = d_t(p)$, respectively. Using this in (119), we obtain

$$J_{2t}^d(p, I_2) = \begin{cases} (p + h_t)I_2 + \beta V_{t-1}^d(0) & \text{if } I_2 \le d_t(p), \text{ i.e., } 0 \le p \le d_t^{-1}(I_2), \\ (p + h_t)d_t(p) + \beta V_{t-1}^d(I_2 - d_t(p)) & \text{if } I_2 > d_t(p), \text{ i.e., } d_t^{-1}(I_2) \le p \le p_t^u. \end{cases}$$

Taking the first order derivative of $J^d_{2t}(p, I_2)$ with respect to p, we have

$$\frac{dJ^d_{2t}(p, I_2)}{dp} = \begin{cases} I_2 & \text{if } 0 \le p < d_t^{-1}(I_2), \\ d_t(p) + \left(p + h_t - \beta \frac{dV^d_{t-1}(I)}{dI}\Big|_{I=I_2-d_t(p)}\right) d'_t(p) & \text{if } d_t^{-1}(I_2) < p \le p^u_t. \end{cases} \tag{153}$$

From the first part of equation (153), it is easy to see that for $0 \le p < d_t^{-1}(I_2)$, $J^d_{2t}(p, I_2)$ is increasing in p for $I_2 \ge 0$. Therefore, $p^*_t(I_2) \ge d_t^{-1}(I_2)$. Furthermore, from the second part of equation (153) and assumption (A1) for $d_t^{-1}(I_2) < p \le p^u_t$, we have

$$\begin{aligned} \frac{d^2 J^d_{2t}(p, I_2)}{dp^2} &= 2d'_t(p) + \left(p + h_t - \beta \frac{dV^d_{t-1}(I)}{dI}\Big|_{I=I_2-d_t(p)}\right) d''_t(p) \\ &\quad + (d'_t(p))^2 \beta \frac{d^2 V^d_{t-1}(I)}{dI^2}\Big|_{I=I_2-d_t(p)} \le 0. \end{aligned}$$

That is, $J^d_{2t}(p, I_2)$ is concave in p for $d_t^{-1}(I_2) < p \le p^u_t$. Evaluating the right-hand limit of $dJ^d_{2t}(p, I_2)/dp$ as p goes to $d_t^{-1}(I_2)$ and using part (g) of Proposition 3 in the resulting expression, we have

$$\begin{aligned} \lim_{p \downarrow d_t^{-1}(I_2)} \frac{dJ^d_{2t}(p, I_2)}{dp} &= d_t(p) + (p + h_t - \beta c_{t-1}) d'_t(p) \\ &= I_2 + (d_t^{-1}(I_2) + h_t - \beta c_{t-1}) d'_t(d_t^{-1}(I_2)). \end{aligned}$$

Define I^0_{2t} and p^0_t such that $d_t(p^0_t) = I^0_{2t}$ and

$$d_t(p^0_t) + (p^0_t + h_t - \beta c_{t-1}) d'_t(p^0_t) = 0.$$

If $I_2 \le I^0_{2t}$, i.e., $d_t^{-1}(I_2) \ge p^0_t$, then for $p \ge d_t^{-1}(I_2) \ge p^0_t$, $dJ^d_{2t}(p, I_2)/dp \le 0$ and $J^d_{2t}(p, I_2)$ is decreasing in p. Hence, $p^*_t(I_2) = d_t^{-1}(I_2)$. On the other hand, if $I_2 \ge I^0_{2t}$, i.e., $d_t^{-1}(I_2) \le p^0_t$, then for $d_t^{-1}(I_2) \le p \le p^0_t$, $dJ^d_{2t}(p, I_2)/dp \ge 0$ and for $p \ge p^0_t$, $dJ^d_{2t}(p, I_2)/dp \le 0$. Hence, $p^*_t(I_2) = p^0_t$. Therefore, $p^*_t(I_2) = \max\{d_t^{-1}(I_2), p^0_t\}$.

Next, from equation (118), we have $V_{2t}^d(I_2) = -h_t I_2 + J_{2t}^d(p_t^*(I_2), I_2)$, where

$$J_{2t}^d(p_t^*(I_2), I_2) = \begin{cases} (d_t^{-1}(I_2) + h_t)I_2 + \beta V_{t-1}^d(0) & \text{if } I_2 \le d_t(p_t^0), \\ (p_t^0 + h_t)d_t(p_t^0)) + \beta V_{t-1}^d(I_2 - d_t(p_t^0))) & \text{if } I_2 > d_t(p_t^0). \end{cases}$$

Then, taking the first order derivative of $V_{2t}^d(I_2)$, we have

$$\frac{dV_{2t}^d(I_2)}{dI_2} = \begin{cases} d_t^{-1}(I_2) + I_2(d_t^{-1})'(I_2) & \text{if } I_2 \le d_t(p_t^0), \\ \beta \left.\frac{dV_{t-1}^d(I)}{dI}\right|_{I=I_2-d_t(p_t^0))} - h_t & \text{if } I_2 > d_t(p_t^0). \end{cases} \tag{154}$$

Using part (g) of Theorem 3 for period $t-1$ in the second part of the equation (154), we observe that for all $I_2 > d_t(p_t^0)$, $dV_{2t}^d(I_2)/dI_2 \le \beta c_{t-1} - h_t \le r_1 + b$. Then, from the definition of R_t and the concavity of $V_{2t}^d(I_2)$, we have $R_t \le d_t(p_t^0)$. Then, using the first part of (154), R_t is such that

$$r_1 + b = d_t^{-1}(R_t) + R_t(d_t^{-1})'(R_t) = d_t^{-1}(R_t) + \frac{R_t}{d_t'(R_t)}.$$

□

Part (a) of Corollary 1 shows that when Class 2 demand is a deterministic function of its price, $p_t^*(I_2)$ is bounded below by p_t^0. Furthermore, p_t^0 is strictly greater than $\beta c_{t-1} - h_t$, and is independent of I_2. We observe that p_t^0, and hence, $p_t^*(I_2)$ depends only on I_2, $d_t(\cdot)$, h_t, c_{t-1} and β, and is independent of any other demand and cost parameters in the future periods. Similarly, part (b) of Corollary 1 shows that R_t, and hence, z_t^* depends only on $d_t(\cdot)$, r_1 and b. In other words, Corollary 1 implies that when Class 2 demand is a deterministic function of its price, $p_t^*(I_2)$ and z_t^* can be determined myopically.

A myopic policy ignores future consequences of the current decisions. As a result, a T-period problem can be solved by solving T single period problems. Computing myopic policies are much simpler than computing the optimal policy, which makes the myopic policies more attractive in practice. Therefore, in the next section, we investigate myopic policies for this problem and present a methodology to determine the myopic policy parameters when the Class 2 demand is stochastic. Furthermore, in Section III.5, we establish sufficient conditions under which a myopic policy is optimal for a T-period problem.

III.4 Myopic Inventory and Pricing Policies

A myopic policy makes decisions in each period t by isolating it from the future periods. As a result, a myopic solution to a T-period problem can be obtained by solving T single period problems. Hence, myopic policies are among the simplest policies. As rightly noted by Lovejoy [53], simple policies are less costly to evaluate and implement, and hence, are readily accepted by practitioners. In this section, we investigate myopic policies for our problem. More specifically, we present a methodology to determine the myopic policy parameters, denoted by S_t^m, R_t^m and $p_t^m(I_2)$ for $I_2 > 0$ and periods $t = 1, \ldots, T$.

Let I_t be the on-hand inventory available at the beginning of period t and I_{2t} be the left-over inventory after sales to Class 1 customers in period t. Given the values of y_t, z_t and p_t, the discounted profit for T periods can be expressed as

$$\Pi = \sum_{t=1}^{T} \beta^{T-t} \Big[r_1 \min\{D_{1t}, y_t - z_t\} - c_t(y_t - I_t) - b(D_{1t} - y_t + z_t)^+ \\ + p_t \min\{D_{2t}(p_t, \xi_t), I_{2t}\} - h_t(I_{2t} - D_{2t}(p_t, \xi_t))^+ \Big] + \beta^T c_0 I_0.$$

The expected discounted profit for T periods is given by $E[\Pi]$, and the T-period problem, denoted by $\mathcal{P}_T$, can be expressed as

$$\begin{aligned} \mathcal{P}_T : \qquad & \text{maximize} \quad E[\Pi] \\ & \text{subject to} \quad I_t \le y_t, \quad t = 1, \ldots, T, \\ & \qquad\qquad\quad z_t \ge 0, \quad t = 1, \ldots, T, \\ & \qquad\qquad\quad 0 \le p_t \le p_t^u, \quad t = 1, \ldots, T. \end{aligned}$$

Since excess demand from both the classes is lost, the inventory balance equations are

$$I_{2t} = \max\{y_t - D_{1t}, z_t\} = y_t - D_{1t} + (z_t - y_t + D_{1t})^+, \tag{155}$$

$$I_{t-1} = \big(I_{2t} - D_{2t}(p_t, \xi_t)\big)^+ = \big(y_t - D_{1t} + (z_t - y_t + D_{1t})^+ - D_{2t}(p_t, \xi_t)\big)^+. \tag{156}$$

We note that $\max\{u, v\} = u + (v - u)^+$ and $\min\{u, v\} = v - (v - u)^+$ for any $u, v \in \Re$. Using the inventory balance equations (155) and (156) to substitute for I_t and I_{2t}, we can simplify Π as follows:

$$\begin{aligned}
\Pi = & \sum_{t=1}^{T} \beta^{T-t} r_1\big(y_t - z_t - (y_t - z_t - D_{1t})^+\big) - c_T(y_T - I_T) \\
& - \sum_{t=1}^{T-1} \beta^{T-t} c_t \Big(y_t - (y_{t+1} - D_{1,t+1} + (z_{t+1} - y_{t+1} + D_{1,t+1})^+ - D_{2,t+1})^+ \Big) \\
& + \sum_{t=1}^{T} \beta^{T-t} \Big[- b(D_{1t} - y_t + z_t)^+ + p_t \Big(I_{2t} - \big(I_{2t} - D_{2t}\big)^+ \Big) - h_t\big(I_{2t} - D_{2t}\big)^+ \Big] \\
& + \beta^T c_0 (y_1 - D_{11} + (z_1 - y_1 + D_{11})^+ - D_{21})^+ \\
= & \sum_{t=1}^{T} \beta^{T-t} \Big[r_1\big(y_t - z_t - (y_t - z_t - D_{1t})^+\big) - c_t y_t - b(D_{1t} - y_t + z_t)^+ + p_t I_{2t} \\
& - (p_t + h_t)\big(I_{2t} - D_{2t}\big)^+ \Big] + \sum_{t=1}^{T} \beta^{T-t+1} c_{t-1} \big(y_t - D_{1t} + (z_t - y_t + D_{1t})^+ - D_{2t}\big)^+ \\
& + c_T I_T \\
= & \sum_{t=1}^{T} \beta^{T-t} \Big[r_1\big(y_t - z_t - (y_t - z_t - D_{1t})^+\big) - c_t y_t - b(D_{1t} - y_t + z_t)^+ \\
& + p_t(y_t - D_{1t} + (z_t - y_t + D_{1t})^+) - (p_t + h_t)\big(y_t - D_{1t} + (z_t - y_t + D_{1t})^+ - D_{2t}\big)^+ \Big] \\
& + \sum_{t=1}^{T} \beta^{T-t+1} c_{t-1} \big(y_t - D_{1t} + (z_t - y_t + D_{1t})^+ - D_{2t}\big)^+ + c_T I_T.
\end{aligned}$$

After some algebraic manipulations, we can rewrite Π as

$$\Pi = c_T I_T + \sum_{t=1}^{T} \beta^{T-t} \Gamma_t(y_t, z_t, p_t, D_{1t}, D_{2t}), \text{ where} \tag{157}$$

$$\begin{aligned}\Gamma_t(y_t, z_t, p_t, D_{1t}, D_{2t}) = {} & r_1\big(y_t - z_t - (y_t - z_t - D_{1t})^+\big) - c_t y_t - b(D_{1t} - y_t + z_t)^+ \\ & + p_t\big(y_t - D_{1t} + (z_t - y_t + D_{1t})^+\big) \\ & - (p_t + h_t - \beta c_{t-1})\big(y_t - D_{1t} + (z_t - y_t + D_{1t})^+ - D_{2t}\big)^+. \end{aligned} \tag{158}$$

We note that I_T is the on-hand inventory at the beginning of the planning horizon and its value is known. Then, using expression (157), $E[\Pi]$ can be written as

$$E[\Pi] = c_T I_T + \sum_{t=1}^{T} \beta^{T-t} E[\Gamma_t(y_t, z_t, p_t, D_{1t}, D_{2t})]. \tag{159}$$

We note that the expected value of $\Gamma_t(y_t, z_t, p_t, D_{1t}, D_{2t})$ depends on the distributions of D_{1t}, D_{2t} as well as the distributions of y_t, z_t and p_t (see equation (16) and the last paragraph on pg. 25 in [14] and pg. 65 in [42]). Proposition 8 shows that the expected value of $\Gamma_t(y_t, z_t, p_t, D_{1t}, D_{2t})$ depends only on the distributions of y_t, z_t and p_t for $t = 1, \ldots, T$.

Proposition 8. *For $t = 1, \ldots, T$, $E[\Gamma_t(y_t, z_t, p_t, D_{1t}, D_{2t})] = E[\Phi_t(y_t, z_t, p_t)]$, where*

$$\begin{aligned}\Phi_t(y_t, z_t, p_t) = {} & r_1 E[D_{1t}] - (r_1 + b) G_{1t}(y_t - z_t) - c_t y_t + \int_0^{y_t - z_t} \Omega_t(p_t, y_t - x) dF_{1t}(x) \\ & + (1 - F_{1t}(y_t - z_t)) \Omega_t(p_t, z_t), \end{aligned} \tag{160}$$

$$\Omega_t(p, I_2) = (\beta c_{t-1} - h_t) I_2 + (p + h_t - \beta c_{t-1}) \bigg(d_t(p) + \mu_t - G_{2t}(I_2 - d_t(p)) \bigg). \tag{161}$$

Proof. We observe that y_t and z_t depend only on the information available at the start of period t (i.e., $I_T, y_T, z_T, D_{1T}, p_T, D_{2T}, \ldots, I_{t+1}, y_{t+1}, z_{t+1}, D_{1,t+1}, p_{t+1}, D_{2,t+1}$) and are independent of D_{1t} and D_{2t}. Similarly, p_t depends only on the information available in period t before the Class 2 demand is realized (i.e., $I_T, y_T, z_T, D_{1T}, p_T, D_{2T}, \ldots, I_t, y_t, z_t, D_{1t}$) and is independent of Class 2 demand D_{2t}. Furthermore, $D_{11}, D_{21}, \ldots, D_{1T}, D_{2T}$ are independent.

We can write

$$E[\Gamma_t(y_t, p_t, z_t, D_{1t}, D_{2t})] = E\Big[E[\Gamma_t(y_t, p_t, z_t, D_{1t}, D_{2t})|y_t, z_t, p_t]\Big], \tag{162}$$

and define $\Phi_t(y_t, p_t, z_t) = E[\Gamma_t(y_t, z_t, p_t, D_{1t}, D_{2t})|y_t, z_t, p_t]$. From equation (158), we have

$$\begin{aligned}\Phi_t(y_t, p_t, z_t) = r_1\Big(y_t - z_t - \int_0^{y_t - z_t}(y_t - z_t - x)dF_{1t}(x)\Big) - b\int_{y_t - z_t}^{\infty}(x - y_t + z_t)dF_{1t}(x)\\ - c_t y_t + E\Big[E\big[p_t I_{2t} - (p_t + h_t - \beta c_{t-1})(I_{2t} - D_{2t})^+\big]\big|I_{2t} = y_t - D_{1t}\\ + (z_t - y_t + D_{1t})^+\Big].\end{aligned}$$

We define $\Omega_t(p, I_2)$ as follows:

$$\Omega_t(p, I_2) = E\big[pI_2 - (p + h_t - \beta c_{t-1})(I_2 - D_{2t})^+\big]. \tag{163}$$

Using the above definition of $\Omega_t(p, I_2)$ and the fact that $D_{2t}(p) = d_t(p) + \xi_t$, $\Phi_t(y_t, p_t, z_t)$ can be rewritten as follows:

$$\begin{aligned}\Phi_t(y_t, p_t, z_t) &= r_1\Big(\int_0^{\infty} x dF_{1t}(x) + \int_{y_t - z_t}^{\infty}(y_t - z_t - x)dF_{1t}(x)\Big) - b\int_{y_t - z_t}^{\infty}(x - y_t + z_t)dF_{1t}(x)\\ &\quad - c_t y_t + E\big[\Omega_t(p_t, I_{2t})\big|I_{2t} = y_t - D_{1t} + (z_t - y_t + D_{1t})^+\big]\\ &= r_1 E[D_{1t}] - (r_1 + b)G_{1t}(y_t - z_t) - c_t y_t + \int_0^{y_t - z_t}\Omega_t(p_t, y_t - x)dF_{1t}(x)\\ &\quad + (1 - F_{1t}(y_t - z_t))\Omega_t(p_t, z_t),\end{aligned}$$

where we used the fact that when $D_{1t} > y_t - z_t$, $I_{2t} = y_t - D_{1t} + (z_t - y_t + D_{1t})^+ = z_t$. Furthermore, using equation (163) and $D_{2t}(p) = d_t(p) + \xi_t$, $\Omega_t(p_t, I_2)$ can be simplified as

$$\begin{aligned}\Omega_t(p, I_2) &= pI_2 - (p + h_t - \beta c_{t-1})\int_0^{I_2 - d_t(p)}(I_2 - d_t(p) - x)dF_{2t}(x)\\ &= pI_2 - (p + h_t - \beta c_{t-1})\Big(I_2 - d_t(p) - \mu_t + \int_{I_2 - d_t(p)}^{\infty}(x - I_2 + d_t(p))dF_{2t}(x)\Big),\end{aligned}$$

$$\Omega_t(p, I_2) = (\beta c_{t-1} - h_t)I_2 + (p + h_t - \beta c_{t-1})\bigg(d_t(p) + \mu_t - G_{2t}(I_2 - d_t(p))\bigg).$$

Then, the result follows directly by using the definition of $\Phi_t(y_t, p_t, z_t)$ on the right-hand side of (162). □

The result in Proposition 8 simplifies further analysis. Using equation (159) and Proposition 8, $E[\Pi]$ can now be rewritten as

$$E[\Pi] = c_T I_T + \sum_{t=1}^{T} \beta^{T-t} E[\Phi_t(y_t, z_t, p_t)]. \tag{164}$$

Therefore, the problem $\mathcal{P}_T$ can be considered as a maximization problem with separable terms for each period t subject to the constraints $I_t \leq y_t$, $z_t \geq 0$ and $0 \leq p_t \leq p_t^u$. Each separable term $\beta^{T-t}E[\Phi_t(y_t, z_t, p_t)]$, $t = 1, \ldots, T$, of this problem is equal to the present value of expected profit for period t. Let S_t^m, z_t^m and p_t^m denote the values of the decisions variables that maximize $\beta^{T-t}E[\Phi_t(y_t, z_t, p_t)]$ for each period $t = 1, \ldots, T$. Proposition 9 presents how to determine S_t^m, z_t^m and p_t^m for a given $I_2 \geq 0$ by studying the functions $\Omega_t(p, I_2)$, and $\Phi_t(y, z, p_t^m)$ for $t = 1, \ldots, T$. Similar to Section III.3.1, we assume that condition (122) is true.

Proposition 9. *$\Omega_t(p, I_2)$ and $\Phi_t(y, z, p_t^m(I_2))$ satisfy the following properties for t, $t = 1, \ldots, T$:*

(a) For a given $I_2 \geq 0$, $\Omega_t(p, I_2)$ has a finite maximizer $p_t^m(I_2)$ such that $\beta c_{t-1} - h_t < p_t^m(I_2) \leq p_t^u$ and

$$\begin{aligned} 0 = {} & d_t(p_t^m(I_2)) + \mu_t - G_{2t}(I_2 - d_t(p_t^m(I_2))) \\ & + d_t'(p_t^m(I_2))\big(p_t^m(I_2) + h_t - \beta c_{t-1}\big)F_{2t}(I_2 - d_t(p_t^m(I_2))). \end{aligned} \tag{165}$$

(b) $p_t^m(I_2)$ is decreasing in I_2, and $\delta_t(I_2) = I_2 - d_t(p_t^m(I_2)) \geq 0$ is increasing in I_2.

(c) $\Omega_t(p_t^m(I_2), I_2)$ is concave in I_2.

(d) $\Phi_t(y, z, p_t^m(I_2))$ *has a finite maximizer denoted by* (S_t^m, z_t^m) *such that*

$$z_t^m = \begin{cases} 0 & \text{if } p^u \leq r_1 + b, \\ R_t^m & \text{if } p^u > r_1 + b, \end{cases} \tag{166}$$

where R_t^m *satisfies* $\left.\frac{d\Omega_t(p_t^m(I_2), I_2)}{dI_2}\right|_{I_2=R_t^m} = r_1 + b$, *and*
S_t^m *satisfies* $\left.\frac{d\Phi_t(y, z_t^m, p_t^m(I_2))}{dI_2}\right|_{y=S_t^m} = 0.$

Proof. $\Omega_t(p, I_2)$ and $\Phi_t(y, z, p_t^m)$ are the one-period counterparts of $J_{2t}(p, I_2)$ and $J_t(y, z)$ with cost and demand parameters corresponding to period t, respectively. The proof follows by working out the same steps as in the proof of Theorem 3 for $t = 1$. In particular, taking the first and second order derivatives with respect to p of $\Omega_t(p, I_2)$ defined in equation (161), we have

$$\begin{aligned} \frac{\partial \Omega_t(p, I_2)}{\partial p} &= d_t(p) + \mu_t - G_{2t}(I_2 - d_t(p)) \\ &\quad + (p + h_t - \beta c_{t-1})\Big(d_t'(p) - d_t'(p)\big[1 - F_{2t}(I_2 - d_t(p))\big]\Big) \\ &= d_t(p) + \mu_t - G_{2t}(I_2 - d_t(p)) + d_t'(p)(p + h_t - \beta c_{t-1})F_{2t}(I_2 - d_t(p)), \qquad (167) \\ \frac{\partial^2 \Omega_t(p, I_2)}{\partial p^2} &= d_t'(p) - d_t'(p)[1 - F_{2t}(I_2 - d_t(p))] + d_t''(p)(p + h_t - \beta c_{t-1})F_{2t}(I_2 - d_t(p)) \\ &\quad + d_t'(p)F_{2t}(I_2 - d_t(p)) - (d_t'(p))^2(p + h_t - \beta c_{t-1})f_{2t}(I_2 - d_t(p)) \\ &= \big(2d_t'(p) + (p + h_t - \beta c_{t-1})d_t''(p)\big)F_{2t}(I_2 - d_t(p)) \\ &\quad - (d_t'(p))^2(p + h_t - \beta c_{t-1})f_{2t}(I_2 - d_t(p)). \qquad (168) \end{aligned}$$

(a) From equation (167), we observe that $\partial\Omega_t(p, I_2)/\partial p > 0$ for $p \leq \beta c_{t-1} - h_t$. Therefore, $p_t^m(I_2) \geq \beta c_{t-1} - h_t$. Furthermore, using assumption A1, we observe that $\partial^2\Omega_t(p, I_2)/\partial p^2 < 0$ for $p > \beta c_{t-1} - h_t$. Hence, $\Omega_t(p, I_2)$ is a concave function of p for fixed $I_2 \geq 0$ when $p > \beta c_{t-1} - h_t$. Furthermore, using the assumption that condition (122) is true, it can be shown that $\partial\Omega_t(p, I_2)/\partial p|_{p=p_t^u} < 0$. Therefore, there is a unique maximum of $\Omega_t(p, I_2)$ in interval $[\beta c_{t-1} - h_t, p_t^u]$ with respect to p. Define $p_t^m(I_2)$ such that

$$\frac{\partial \Omega_t(p, I_2)}{\partial p}|_{p=p_t^m(I_2)} = 0. \tag{169}$$

Then, $p_t^m(I_2)$ is the unique maximizer of $\Omega_t(p, I_2)$. Furthermore, evaluating equation (167) at $p = d_t^{-1}(I_2)$, i.e., $d(p) = I_2$, and simplifying the resulting expression we have

$$\left.\frac{\partial \Omega_t(p, I_2)}{\partial p}\right|_{p=d_t^{-1}(I_2)} = d_t(p) = I_2 \geq 0.$$

Then, from equation (169) and the concavity of $\Omega_t(p, I_2)$ in p, it follows that $p_t^m(I_2) \geq d_t^{-1}(I_2)$. Let us define $\delta_t(I_2) = I_2 - d_t(p_t^m(I_2))$, then using assumption A1 that $d_t(p)$ is a decreasing function of p it follows that $\delta_t(I_2) \geq 0$ for $I_2 \geq 0$.

(b) From equations (126), (130) and (167), we observe that $\partial \Omega_t(p, I_2)/\partial p = H_t^0(p, I_2) = \int_0^{I_2} Q_t(p, x)dx$, where $Q_t(p, I_2)$ is given by equation (129). Using this on the left-hand side of (169), we have $\partial \Omega_t(p, I_2)/\partial p|_{p=p_t^m(I_2)} = \int_0^{I_2} Q_t(p_t^m(I_2), x)dx = 0$. Using implicit differentiation, we have

$$\frac{dp_t^m(I_2)}{dI_2} = \frac{-\partial^2 \Omega_t(p, I_2)/\partial p \partial I_2|_{p=p_t^m(I_2)}}{\partial^2 \Omega_t(p, I_2)/\partial p^2|_{p=p_t^m(I_2)}}. \tag{170}$$

From the concavity of $\Omega_t(p, I_2)$ in p, we have

$$\left.\frac{\partial^2 \Omega_t(p, I_2)}{\partial p^2}\right|_{p=p_t^m(I_2)} \leq 0. \tag{171}$$

Furthermore, using equation (167), taking the first order derivative of $\partial \Omega_t(p, I_2)/\partial p$ with respect to I_2 and evaluating it at $p = p_t^m(I_2)$, we have

$$\begin{aligned}\left.\frac{\partial^2 \Omega_t(p, I_2)}{\partial p \partial I_2}\right|_{p=p_t^m(I_2)} &= 1 - F_{2t}\big[I_2 - d_t(p_t^m(I_2))\big] \\ &\quad + d_t'(p_t^m(I_2))\big[p_t^m(I_2) + h_t - \beta c_{t-1}\big] f_{2t}(I_2 - d_t(p_t^m(I_2))) \\ &= Q_t(p_t^m(I_2), I_2).\end{aligned}$$

From assumptions A1 and A2, it follows that $e_t(p, I_2)$ is increasing in I_2 for $t = 1, \ldots, T$. Then, for a fixed p, $Q_t(p, I_2)$ is decreasing in I_2 for $t = 1, \ldots, T$. If $Q_t(p_t^m(I_2), I_2) > 0$, then for all $x \le I_2$, $Q_t(p_t^m(I_2), x) > 0$, and hence, $\int_0^{I_2} Q_t(p_t^m(I_2), x)dx > 0$. This contradicts the definition of $p_t^m(I_2)$, i.e., $\int_0^{I_2} Q_t(p_t^m(I_2), x)dx = 0$. Therefore, we have $Q_t(p_t^m(I_2), I_2) \le 0$, and hence,

$$\left.\frac{\partial^2 \Omega_t(p, I_2)}{\partial p \partial I_2}\right|_{p=p_t^m(I_2)} = Q_t(p_t^m(I_2), I_2) \le 0. \tag{172}$$

Using the inequalities in (171) and (172) on the right-hand side of (170), we have $dp_t^m(I_2)/dI_2 \le 0$, and hence, $p_t^m(I_2)$ is a decreasing function of I_2. From part (a) above, we recall that $\delta_t(I_2) = I_2 - d_t(p_t^m(I_2)) \ge 0$, then

$$\begin{aligned}\frac{d\delta_t(I_2)}{dI_2} &= 1 - d_t'(p_t^m(I_2))\frac{dp_t^m(I_2)}{dI_2} \\ &= \frac{\partial^2\Omega_t(p, I_2)/\partial p^2|_{p=p_t^m(I_2)} + d_t'(p_t^m(I_2))\partial^2\Omega_t(p, I_2)/\partial p \partial I_2|_{p=p_t^m(I_2)}}{\partial^2\Omega_t(p, I_2)/\partial p^2|_{p=p_t^m(I_2)}} \\ &= \frac{d_t'(p_t^m(I_2)) + \big[d_t'(p_t^m(I_2)) + d_t''(p_t^m(I_2))(p_t^m(I_2) + h_t - \beta c_{t-1})\big]F_{2t}\big(I_2 - d_t(p_t^m(I_2))\big)}{\partial^2\Omega_t(p, I_2)/\partial p^2|_{p=p_t^m(I_2)}} \\ &> 0.\end{aligned}$$

Thus, $\delta_t(I_2) = I_2 - d_t(p_t^m(I_2))$ is increasing in I_2.

(c) Next, evaluating equation (161) at $p = p_t^m(I_2)$ and taking the first and the second order derivatives of the resulting expression with respect to I_2, we have

$$\begin{aligned}\frac{\partial\Omega_t(p_t^m(I_2), I_2)}{\partial I_2} &= \beta c_{t-1} - h_t + (p_t^m(I_2) + h_t - \beta c_{t-1})\big[1 - F_{2t}\big(I_2 - d_t(p_t^m(I_2))\big)\big], \\ \frac{\partial^2\Omega_t(p_t^m(I_2), I_2)}{\partial I_2^2} &= \frac{dp_t^m(I_2)}{dI_2}\big[1 - F_{2t}\big(I_2 - d_t(p_t^m(I_2))\big)\big] \\ &\quad - (p_t^m(I_2) + h_t - \beta c_{t-1})f_{2t}\big(I_2 - d_t(p_t^m(I_2))\big)\frac{d\delta_t(I_2)}{dI_2} \le 0.\end{aligned}$$

Thus, $\Omega_t(p_t^m(I_2), I_2)$ is a concave function of I_2. Furthermore, $d\Omega_t(p_t^m(I_2), I_2)/dI_2|_{I_2=0} = p_t^m(0) = p_t^u$. Then, from the concavity of $\Omega_t(p_t^m(I_2), I_2)$, we have $d\Omega_t(p_t^m(I_2), I_2)/dI_2 \le p_t^u$.

(d) Evaluating equation (160) at $p = p_t^m(I_2)$ and taking the first order derivative of the resulting expression with respect to z, we have

$$\frac{d\Phi_t(y,z,p_t^m(I_2))}{dz} = -\left(r_1 + b - \left.\frac{d\Omega_t(p_t^m(I_2),I_2)}{dI_2}\right|_{I_2=z_t}\right)(1 - F_{1t}(y-z)).$$

Then, using the fact that $d\Omega_t(p_t^m(I_2),I_2)/dI_2 \le p_t^u$ and following the exact same analysis as in the proof of part (e) of Theorem 3, we have z_t^m as expressed in equation (166). Furthermore, we can show that $d\Omega_t(p_t^m(I_2),I_2)/dI_2|_{I_2=z_t^m} \le r_1 + b$.

Evaluating equation (160) at $p = p_t^m(I_2)$, $z = z_t^m$ and taking the first order derivative of the resulting expression with respect to y, we have

$$\begin{aligned}
\frac{d\Phi_t(y,z_t^m,p_t^m(I_2))}{dy} &= -c_t + (r_1+b)(1 - F_{1t}(y - z_t^m)) \\
&\quad + \int_0^{y-z_t^m} \left.\frac{d\Omega_t(p_t^m(I_2),I_2)}{dI_2}\right|_{I_2=y-x} dF_{1t}(x) \\
\frac{d^2\Phi_t(y,z_t^m,p_t^m(I_2))}{dy^2} &= -\left(r_1 + b - \left.\frac{d\Omega_t(p_t^m(I_2),I_2)}{dI_2}\right|_{I_2=z_t^m}\right) f_{1t}(y) \\
&\quad + \int_0^{y-z_t^m} \left.\frac{d^2\Omega_t(p_t^m(I_2),I_2)}{dI_2^2}\right|_{I_2=y-x} dF_{1t}(x) \le 0.
\end{aligned}$$

Then, using $d\Omega_t(p_t^m(I_2),I_2)/dI_2|_{I_2=z_t^m} \le r_1 + b$ we see that $\Phi_t(y, z_t^m, p_t^m(I_2))$ is concave in y. Define S_t^m such that $\Phi_t(y, z_t^m, p_t^m(I_2))|_{y=S_t^m} = 0$, then S_t^m maximizes $\Phi_t(y, z_t^m, p_t^m(I_2))$. □

We note that $\big(y_t = S_t^m,\ z_t = z_t^m,\ p_t = p_t^m(I_2)\big)$ is the supplier's policy in period t obtained by isolating it from the future periods, and hence, is myopic. Part (a) of Proposition 9 shows that the myopic price charged to Class 2 in period t is a function of the left-over inventory after satisfying Class 1 demand in period t. It also shows that the myopic price is bounded from below by $\beta c_{t-1} - h_t$.

We note that any left-over inventory after satisfying Class 1 demand can either be sold to Class 2 at price p, or carried to period $t-1$ at a unit holding cost of h_t. In the latter case, a unit production cost of c_{t-1} is saved in period $t-1$, and hence, the present value of the carry option is equal to $\beta c_{t-1} - h_t$. It is now intuitive that the first option of selling inventory to Class 2 is profitable only when $p > \beta c_{t-1} - h_t$. This generalizes the results observed in Chapter II for stationary demand and cost parameters.

Part (d) of Proposition 9 shows that when $p_t^u \leq r_1 + b$, $z_t^m = z_t^* = 0$. That is, when $p_t^u \leq r_1 + b$, the myopic solution for the inventory to be protected from Class 1 is optimal. Part (d) also shows that the stage 1 myopic policy is characterized by two critical quantities: S_t^m is the myopic critical produce-up-to level and R_t^m is the myopic critical amount of inventory to be protected from Class 1 in period t.

We recall that Corollary 1 shows that for the special case where Class 2 demand is a deterministic function of its price, $p_t^*(I_2)$ and z_t^* can be determined myopically. In fact, using Proposition 9, we can show that for this special case $p_t^m(I_2) = p_t^*(I_2) = \max\{d_t^{-1}(I_2), p_t^0\}$ and $z_t^m = z_t^*$ for $t = 1, ..., T$. Next, we provide a simple example to illustrate the computations of the myopic policy.

Example: Suppose that Class 2 demand is a deterministic function of its price, $d_t(p) = a_{1t} - a_{2t}p$ and $D_{1t} \sim \text{Unif}[0, A_{1t}]$. Then, $d_t^{-1}(I_2) = (a_{1t} - I_2)^+/a_{1t}$, and from equation (151), we have

$$p_t^0 = \frac{a_{1t} + a_{2t}(\beta c_{t-1} - h_t)}{2a_{2t}} > 0 \quad and \quad d_t(p_t^0) = \frac{a_{1t} - a_{2t}(\beta c_{t-1} - h_t)}{2}.$$

The myopic price $p_t^m(I_2)$ is optimal and can be computed easily using Corollary 1 as

$$p_t^m(I_2) = p_t^*(I_2) = \max\left\{\frac{a_{1t} - I_2}{a_{2t}}, \frac{a_{1t} + a_{2t}(\beta c_{t-1} - h_t)}{2a_{2t}}\right\}.$$

If $a_{1t}/a_{2t} \leq r_1 + b$, then $z_t^m = z_t^* = 0$, else $z_t^m = z_t^* = R_t^m$, where R_t^m satisfies equation (152). It is easy to show that $R_t^m = 0.5(a_{1t} - a_{2t}(r_1 + b))$. From Proposition 9, S_t^m is such that

$$0 = -c_t + (r_1 + b)(1 - F_{1t}(S_t^m - z_t^m)) + \int_0^{S_t^m - z_t^m} \frac{d\Omega_t(p_t^m(I_2), I_2)}{dI_2}\bigg|_{I_2 = S_t^m - x} dF_{1t}(x),$$

$$\text{where} \quad \frac{d\Omega_t(I_2)}{dI_2} = \begin{cases} (a_{1t} - 2I_2)/a_{2t} & \text{if } I_2 \le d_t(p_t^0), \\ \beta c_{t-1} - h_t & \text{if } I_2 > d_t(p_t^0). \end{cases}$$

When $a_{1t}/a_{2t} \le r_1 + b$, $z_t^m = z_t^* = 0$ and some algebraic manipulations, we obtain:

$$S_t^m = \frac{2a_{2t}A_{1t}(r_1 + b - c_t) + (a_{1t} - a_{2t}(\beta c_{t-1} - h_t)^2}{2a_{2t}(r_1 + b - \beta c_{t-1} + h_t)}.$$

The above example illustrates that closed form solutions can be derived for the myopic policy parameters when the Class 2 demand is a deterministic function of its price, and Class 1 demand is a uniform random variable. Next, Proposition 10 establishes a lower bound on the myopic price, that is tighter than $\beta c_{t-1} - h_t$. Furthermore, it compares the myopic and the optimal prices.

Proposition 10. *(a) The myopic price is bounded as* $\beta c_{t-1} - h_t < p_t^{m0} \le p_t^m(I_2) \le p_t^u$*, where* p_t^{m0} *is such that*

$$d(p_t^{m0}) + \mu_t + d_t'(p_t^{m0})(p_t^{m0} + h_t - \beta c_{t-1}) = 0.$$

(b) $p_t^*(I_2) \le p_t^m(I_2)$*, and the inequality is tight when* $I_2 - d_t(p_t^m(I_2)) \le S_t$*.*

Proof. (a) Using equation (167) and taking the limit of $\partial\Omega_t(p, I_2)/\partial I_2$ as I_2 goes to infinity we have

$$\lim_{I_2 \to \infty} \frac{\partial\Omega_t(p, I_2)}{\partial p} = d_t(p) + \mu_t + d_t'(p)(p + h_t - \beta c_{t-1}).$$

Define $p_t^{m0} = \lim_{I_2 \to \infty} p_t^m(I_2)$, then by definition p_t^{m0} is such that

$$\lim_{I_2 \to \infty} \frac{\partial\Omega_t(p, I_2)}{\partial p}\bigg|_{p = p_t^{m0}} = d_t(p_t^{m0}) + \mu_t + d_t'(p_t^{m0})(p_t^{m0} + h_t - \beta c_{t-1}) = 0.$$

Since, $p_t^m(I_2)$ is a decreasing function of I_2, we have $p_t^m(I_2) \ge p_t^{m0}$. Furthermore, it is easy to see that $\lim_{I_2 \to \infty} \partial\Omega_t(p, I_2)/\partial p|_{p = \beta c_{t-1} - h_t} > 0$. Then, from the concavity of $\Omega_t(p, I_2)$ and the definition of p_t^{m0}, it follows that $\beta c_{t-1} - h_t < p_t^{m0} \le p_t^m(I_2)$.

(b) Evaluating equation (124) at $p = p_t^m(I_2)$ and using equation (169) in the resulting expression, we have

$$\left.\frac{\partial J_{2t}(p, I_2)}{\partial p}\right|_{p=p_t^m(I_2)}$$
$$= d_t'(p_t^m(I_2))\beta E\left[c_{t-1} - \left.\frac{dV_{t-1}(I)}{dI}\right|_{I=I_2-d_t(p_t^m(I_2))-\xi_t} 1\big(\xi_t \le I_2 - d_t(p_t^m(I_2))\big)\right] \quad (173)$$
$$\le 0.$$

The above inequality follows directly from assumption A1 and part (g) of Proposition 3. Then, from the definition of $p_t^*(I_2)$ and concavity of $V_{t-1}(I)$, we have $p_t^m(I_2) \le p_t^*(I_2)$. Furthermore, if $I_2 - d_t(p_t^m(I_2)) \le S_t$, then using equation (145) on the right-hand side of (173), we have $\partial J_{2t}(p, I_2)/\partial p|_{p=p_t^m(I_2)} = 0$. From the definition uniqueness of $p_t^*(I_2)$, it follows that $p_t^m(I_2) = p_t^*(I_2)$. □

Part (a) of Proposition 10 shows that the myopic price $p_t^m(I_2)$ is bounded below by p_t^{m0}, where p_t^{m0} is the optimal myopic price when I_2 goes to infinity. Furthermore, p_t^0 is strictly greater than $\beta c_{t-1} - h_t$ and independent of I_2. Part (b) of Proposition 10 shows the myopic price is an upper bound on the optimal price. This can be explained as follows. For the optimal policy, any inventory remaining at the end of period t, denoted by I_{t-1} has a value equal to $V_{t-1}(I_{t-1})$, which is a concave function of I_{t-1}. As a result, S_t, R_t and $p_t^*(I_2)$ depend on the future periods. Furthermore, a careful observation reveals that the myopic policy considers that any inventory remaining at the end of a period has a salvage value equal to $c_{t-1} \ge dV_{t-1}(I_{t-1})/dI_{t-1}$. Hence, we have $p_t^*(I_2) \le p_t^m(I_2)$. Part (b) of Proposition 10 also provides a condition under which the upper bound is tight, and hence, the myopic policy is optimal. In general, finding whether $I_2 - d_t(p_t^m(I_2)) \le S_t$ requires computing the myopic policy parameters. Nevertheless, it is easy to verify that this condition is satisfied when $I_2 < S_t$, which is true when demand and cost parameters are stationary and the on-hand inventory available at the beginning of the horizon is zero.

III.5 Optimality of the Myopic Policy

As discussed before, myopic policies are easy to compute, and hence, attractive from the implementation perspective (also see [68]). In this section, we establish sufficient conditions under which the joint myopic inventory and pricing policy of Proposition 9 is optimal for a finite horizon problem with T periods. Furthermore, we discuss the restrictions that these conditions impose on the demand and cost parameters.

Theorem 4. *The myopic policy with $y_t = \max\{I_t, S_t^m\}$, $z_t = z_t^m$ and $p_t = p_t^m(I_2)$ is optimal for a T-period planning horizon if $I_T \leq S_T^m$, and*

$$S_t^m - d_t(p_t^m(S_t^m)) \leq S_{t-1}^m \quad \textit{for } t = 1, \ldots, T. \tag{174}$$

Proof. From part (d) of Proposition 9, we know that $\Phi_t(S_t^m, z_t^m, p_t^m(I_2)) \geq \Phi_t(y_t, z_t, p_t)$ for all y_t, z_t, p_t and $t = 1, \ldots, T$. Therefore, $\Phi_t(S_t^m, z_t^m, p_t^m(I_2)) \geq E[\Phi_t(y_t, z_t, p_t)]$ for $t = 1, \ldots, T$, and hence,

$$\sum_{t=1}^{T} \beta^{t-1} \Phi_t(S_t^m, z_t^m, p_t^m(I_2)) \geq \sum_{t=1}^{T} \beta^{t-1} E[\Phi_t(y_t, z_t, p_t)]. \tag{175}$$

Before we proceed with the proof, we prove the following result, which will be useful later: If (a) $I_t \leq S_t^m$, (b) the myopic policy is used in period t, and (c) the condition (174) is true then $I_{t-1} \leq S_{t-1}^m$. Suppose that the above conditions (a)-(c) are true for period t. Then, $y_t = S_t^m$. Using the fact that $D_{1t} \geq 0$ and the inventory balance equations, we have $I_{2t} = \max\{S_t^m - D_{1t}, z_t^m\} \leq S_t^m$. Using this inequality and the inventory balance equatio (156) for period $t-1$, we have

$$I_{t-1} \leq \left(S_t^m - d_t(p_t^m(S_t^m) - \xi_t\right)^+.$$

From part (b) or Proposition 9, we have $\delta_t(I_2 = S_t^m) = S_t^m - d_t(p_t^m(S_t^m) > 0$. Then, using the fact that $\xi_t \geq 0$ and the inequality in (174), we have

$$I_{t-1} \leq \left(S_t^m - d_t(p_t^m(S_t^m) - \xi_t\right)^+ \leq S_t^m - d_t(p_t^m(S_t^m) \leq S_{t-1}^m.$$

Suppose that $I_T \leq S_T^m$, then $y_T = S_T^m$, $z_T = z_T^m$ and $p_T = p_T^m(I_2)$ are feasible with respect to all three constraints of the problem $\mathcal{P}_T$, namely, (i) $I_t \leq y_t$, (ii) $z_t \geq 0$, and (iii) $0 \leq p_t \leq p_t^u$. Furthermore, if the condition (174) is true for period T, then from the result above it follows that $I_{T-1} \leq S_{T-1}^m$. Using the same argument as for period T, it follows that $y_{T-1} = S_{T-1}^m$, $z_{T-1} = z_{T-1}^m$ and $p_{T-1} = p_{T-1}^m(I_2)$ are feasible for period $T-1$. Repeating this argument for periods $t = T-2, \ldots, 1$, we see that if the condition (174) is true then $y_t = S_t^m$, $z_t = z_t^m$ and $p_t = p_t^m(I_2)$ is feasible for each period $t = 1, 2, \ldots, T$. Then, using equation (175), it follows that $y_t = S_t^m$, $z_t = z_t^m$ and $p_t = p_t^m(I_2)$ is optimal for each period $t = 1, 2, \ldots, T$. □

The proof for Theorem 4 consists of two parts: feasibility and the optimality of the myopic policy. The conditions presented in Theorem 4 are sufficient to show that the myopic policy is feasible in each period. The first sufficiency condition is $I_T \leq S_T^m$, i.e., the on-hand inventory at the beginning of the planning horizon is less than the myopic produce-up-to level. It ensures that $y_T = S_T^m$ is feasible with respect to the constraint $y_T \geq I_T$. The second sufficiency condition given by the inequality (174) ensures that $I_t \leq S_t^m$, and hence, $y_t = S_t^m$ is feasible in periods $t = 1, \ldots, T-1$. As shown in Proposition 9, $z_t = z_t^m$ and $p_t = p_t^m(I_2)$ are always feasible in each period. The optimality of the myopic policy then follows from Proposition 9.

The second sufficiency condition given by the inequality (174) generalizes the condition for the optimality of a myopic inventory policy provided in [43, 79] for the case with non-stationary costs and demand distributions. Verification of this condition, in general, requires computing S_t^m, z_t^m and $p_t^m(I_2)$ for each period $t = 1, \ldots, T$. Nevertheless, Corollary 2 provides easy to verify conditions on the demand and cost parameters such that the second sufficiency condition is satisfied.

Corollary 2. *The myopic policy with $y_t = \max\{I_t, S_t^m\}$, $z_t = z_t^m$ and $p_t = p_t^m(I_2)$ is optimal for a T-period planning horizon if (1) $I_T \leq S_T^m$, and*

(2a) all the demand and cost parameters are stationary,

(2b) any units at the end of the planning horizon can be salvaged at a value equal to the unit production cost.

Proof. Suppose that demand and cost parameters are stationary and any units at the end of the planning horizon can be salvaged at a value equal to the unit production cost, i.e., $c_t = 0$, $t = 0, 1, \ldots, T$. Then, from Proposition 9, it follows that $p_t^m(I_2) = p^m(I_2)$, $z_t^m = z^m$ and $S_t^m = S^m$ for $t = 1, \ldots, T$. Since $d_t(\cdot) = d(\cdot) \geq 0$, condition (174) is satisfied. Then, using Theorem 4 and the inequality $I_T < S_T^m = S^m$, it follows that the myopic policy is optimal. □

When all demand and cost parameters are stationary, and any units remaining at the end of the planning horizon can be salvaged at a value equal to the unit production cost, we show that a stationary myopic policy is optimal. Since $d_t(\cdot) \geq 0$, it is easy to see that condition (174) is satisfied, and hence, the myopic policy is optimal. For example, the pure inventory model in [43, 53, 77], and the joint inventory and pricing model in [70] assume that conditions (2a) and (2b) are true. Furthermore, by letting T go to infinity, we see that the myopic policy is optimal for the infinite horizon problem with stationary demand and cost parameters.

Next, we briefly discuss the reason why the myopic policy is not optimal when $I_t > S_t^m$, $t = 1, \ldots, T$. When $I_t > S_t^m$, $y_t = \max\{I_t, S_t^m\} = I_t$ and it is possible that $I_{2t} = I_t - D_{1t} > S_{t-1} + d_t(p_t^m(I_{2t}))$. In this case, from part (b) of Proposition 10, we have $p_t^*(I_2) < p_t^m(I_2)$, and hence, the myopic pricing policy is not optimal.

III.6 Computational Study

In this section, we present the results of our computational study, which has two goals. Our first goal is to investigate the sensitivity of the optimal policy parameters to key model parameters and develop managerial insights. We note that the optimal policy is a combination of two strategies: discretionary sales for Class 1 and dynamic pricing for Class 2. Our second goal is to quantify the benefits of these strategies, and examine the effect of key model parameters on these benefits. The key model parameters that we consider include the variances of Class 1 and Class 2 demands, and the slope of the Class 2 demand curve.

III.6.1 Experimental setup

The computational results reported in this section assume that (i) all demand and cost parameters are stationary, (ii) any units remaining at the end of the planning horizon are salvaged at the unit production cost, and (iii) there is no on-hand inventory available at the beginning of the planning horizon. As a result, a stationary myopic policy is optimal, and it is used to address the goals of our computational study.

We set $\beta = 1$, and use the gamma distribution for the Class 1 demand, which is commonly encountered in practice, especially, for electronics products with short life-cycles (e.g., see [11, 25, 48]). We investigate two different settings: $E[D_1] \sim \text{Unif}(1000, 2000)$ for Setting 1 and $E[D_1] \sim \text{Unif}(2000, 3000)$ for Setting 2. We consider three different values for the standard deviation of Class 1 demand such that $\sigma_1 = \alpha E[D_1]$, α =0.25, 0.5 or 1.0, and examine the cases with low, medium and high Class 1 demand variability. For Class 2, we consider a linear demand model such that $d(p) = a_1 - a_2 p$, and $p^u = a_1/a_2$. Furthermore, we consider the case where $\xi \sim \text{Unif}(0, A)$. To investigate the effect of the slope of demand curve and Class 2 demand variance, we consider three different values of a_2 and A. Based on the parameter values listed in Table 11, we develop a factorial design corresponding to 486 parameter settings. For each parameter setting, we generate five different sets of values for demand parameter $E[D_1]$, and hence, study a total of 2430 problem instances.

Setting	α	c	h	r_1	b	a_1	a_2	A
1	0.25	400	10	500	1.1(r_1-c)	3000	1	250
2	0.5	400	20	1000			2	500
	1.0	400	40	1500			4	1000

Table 11: Experimental setup

As observed in Sections III.3.1 and III.4, relative values of p^u and $r_1 + b$ are important when considering the need for discretionary sales for Class 1. Hence, in our computational study, we consider both cases: $p^u \leq r_1 + b$ and $p^u > r_1 + b$. Since we consider $\alpha \leq 1$, shape parameter of the gamma distribution is greater than 1 as required to satisfy assumption A2 presented in Section III.3. We note that for each of the 486 parameter settings $A < a_1 - a_2(c - h)$, and hence, condition (122) is satisfied.

III.6.2 Sensitivity of optimal policy parameters

Our first goal is to investigate the sensitivity of the optimal policy parameters to key model parameters and develop managerial insights. Based on our computational results, we make the following observations:

O1 An increase in $E[D_1]$ or α (i.e., variance of Class 1 demand) increases S^m, and it has no effect on z^m or $p^m(z^m)$ and, in general, on $p^m(I_2)$ for $I_2 \geq 0$.

O2 An increase in A leads to an increase in both mean and the variance of Class 2 demand, and we observe that S^m, z^m and $p^m(z^m)$ increase as can be seen from Table 12 for a representative set of parameters. Figure 10 shows how the optimal price $p^m(I_2)$ changes with respect to the left-over inventory I_2 for different values of A. It shows that as A increases, $p^m(I_2)$ becomes flatter in I_2. This results in an increase in both $p^m(I_2)$ and p^{m0}, which is the optimal myopic price when I_2 goes to infinity and a lower bound on $p^m(I_2)$ (see Proposition 10 for more details).

O3 As a_2 increases, any increase in price leads to a larger decrease in Class 2 demand, and accordingly, we observe that both z^m and $p^m(z^m)$ decrease. This in turn leads to a

A	S^m	z^m	$p^m(z^m)$
250	5038.6	998.5	1082.5
500	5188.0	1112.3	1110.6
1000	5496.2	1354.5	1168.0

Table 12: Effect of A on the optimal policy parameters: Setting 1, $E[D_1] = 2626$, $r_1 = 500$, $\alpha = 0.5$, $h = 40$, $a_2 = 2$

a_2	S^m	z^m	$p^m(z^m)$
1	5469.7	1493.4	1926.8
2	5188.0	1112.3	1110.6
4	4618.3	350.6	693.7

Table 13: Effect of a_2 on the optimal policy parameters: Setting 1, $E[D_1] = 2626$, $r_1 = 500$, $\alpha = 0.5$, $h = 40$, $A = 500$

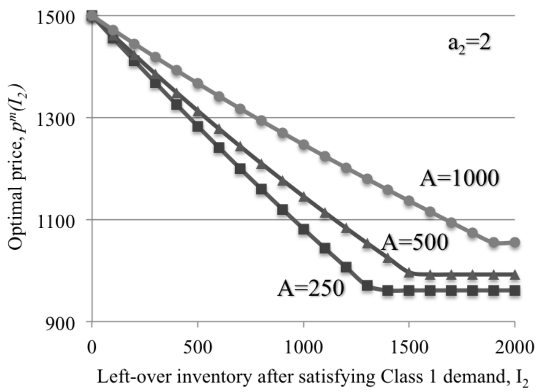


Figure 10: Optimal price with respect to I_2 and A: Setting 2, $E[D_1] = 2626$, $r_1 = 500$, $\alpha = 0.5$, $h = 40$, $a_2 = 2$

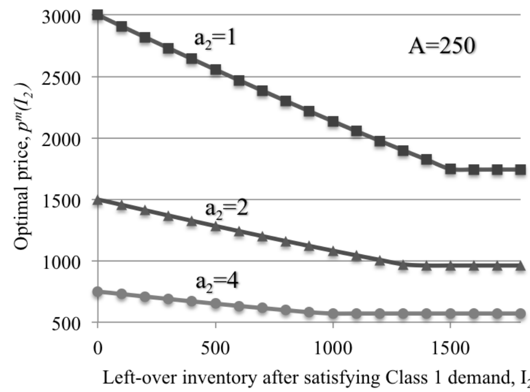


Figure 11: Optimal price with respect to I_2 and a_2: Setting 2, $E[D_1] = 2626$, $r_1 = 500$, $\alpha = 0.5$, $h = 40$, $A = 250$

decrease in S^m as can be seen from Table 13 for a representative set of parameters. Figure 11 shows how the optimal price $p^m(I_2)$ changes with respect to the left-over inventory I_2 for different values of a_2. We observe that as a_2 increases, $p^m(I_2)$ becomes flatter in I_2. This provides additional insights about the results in Proposition 10, which shows that $\beta c - h \leq p^{m0} \leq p^m(I_2) \leq p^u$. In particular, it shows that although both p^{m0} and $p^u = a_1/a_2$ decrease with respect to a_2, rate of decrease in p^u is higher than that in p^{m0}. This decreases the difference between the lower and the upper bounds on the optimal price that is charged to Class 2.

O4 As r_1 increases, S^m increases and z^m decreases, simultaneously. This is intuitive because if Class 1 price increases, profits can be increased by selling more to Class 1, which can be achieved by producing more and protecting less units from Class 1.

III.6.3 Benefits of discretionary sales and dynamic pricing

A prevalent practice in real life and the corresponding assumption in the inventory management literature are to fully satisfy demand if sufficient stock is available. This is primarily motivated by the goal to provide better service levels and prevent loss of good will. That is, in such models $z = 0$ and there are no discretionary sales. However, as we show in this chapter, discretionary sales is optimal when the seller anticipates a higher-revenue demand from a different customer Class. Similarly, while several recent models examine joint inventory and pricing decisions, a significant amount of literature assumes that the supplier (seller) is a price-taker and focus only on the inventory decisions (e.g. see [31, 49]). As demonstrated by our results and similar joint inventory and pricing models in the literature, dynamic pricing results in higher profits. However, implementing discretionary sales and dynamic pricing requires customer segmentation, is costly and may result in loss of good will. Hence, one of the goals of this numerical study is to quantify the benefits of these strategies.

To isolate the benefits of these two strategies, we consider two benchmark policies. Policy P1 considers there are no discretionary sales to Class 1, i.e., $z = 0$, and dynamic pricing is used for Class 2, i.e., $p(I_2) = p^m(I_2)$. Policy P2 considers discretionary sales to Class 1, i.e., $z = z^m$, but Class 2 is charged a pre-specified price. We note that at least z^m units are protected from Class 1 for sales to Class 2, and hence, $I_2 \geq z^m$. Furthermore, since $p^m(I_2)$ is a decreasing function of I_2, we have $p^m(I_2) \leq p^m(z^m)$. Hence, $p^m(z^m)$ is the highest price that can be charged to Class 2 demand. In order to provide a conservative estimate of the benefits of dynamic pricing for Class 2, we consider that the pre-specified price for policy P2 is $p^m(z^m)$. For each problem instance, we compute the per period expected profit loss under the benchmark policy as follows:

$$e_i = 100\left(\frac{V_1(0) - \text{Expected profit per period for policy P}i\text{ with zero starting inventory}}{V_1(0)}\right)\%,$$

where $V_1(0) = E[\phi(y_t^m, z_t^m, p_t^m(0))]$ is the maximum expected profit for a one-period problem with zero starting inventory. We note that the service level offered to Class 1 under the

Policy	Avg. (%)	Max (%)
P1	0.9	9.1
P2	11.7	44.2

Table 14: Average and maximum percentage expected profit loss for the benchmark policies

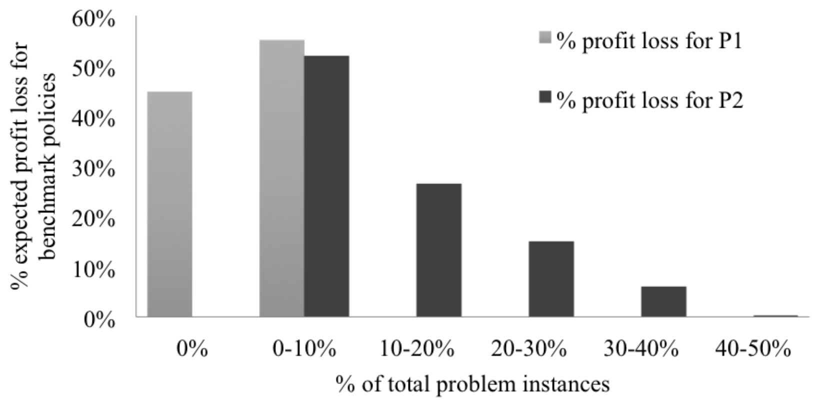


Figure 12: Distribution of percentage expected profit loss for the benchmark policies

optimal policy is equal to $SL_1 = F_1^{-1}(S^m - z^m)$. We compute SL_1 for each problem instance and use it to explain our results below.

The computational results that are summarized in Table 14 suggest that while both discretionary sales for Class 1 and dynamic pricing for Class 2 are valuable, the benefits of dynamic pricing for Class 2 are significantly higher. However, further examination of these results is essential. In particular, Figure 12 shows that there is no profit loss under policy P1 for 47.4% of the experiments. This percentage accounts for 1152 problem instances out of which (i) for 1080 instances $p^u \leq r_1 + b$, and hence, $z^m = 0$ and policy P1 is optimal, (ii) for 72 problem instances $p^u > r_1 + b$ and $SL_1 > 0.999$. For the remaining 1278 problem instances $p^u > r_1 + b$ and the average value of SL_1 is equal to 0.95, with a minimum of 0.84.

The profit loss under policy P2 is greater than zero for all problem instances. Consequently, ignoring discretionary sales under P1 results in a average expected profit loss of 0.9%, but ignoring dynamic pricing under policy P2 results in a significant average expected profit loss of 11.7%. However, these results should be used with caution, since the maximum expected profit loss under policy P1 is 9.1%, which is significantly high. A careful investigation of our numerical results suggests that the expected profit loss under policy P1 can be more than 5% for cases with higher Class 1 market size, low values of $r_1 + b$ compared to p^u, and high values of α and h.

α	Policy P1†		Policy P2		SL_1
	Avg. (%)	Max. (%)	Avg. (%)	Max. (%)	
0.25	0.2	1.7	11.3	40.8	0.982
0.5	0.8	5.2	11.6	41.4	0.971
1.0	1.7	9.1	12.3	44.2	0.964

† indicates results when $p^u > r_1 + b$

Table 15: Percentage expected profit loss for the benchmark policies with respect to α

Next, we examine the impact of key model parameters on the profit loss under policies P1 and P2. For policy P1, we only present the results for the problem instances with $p^u > r_1+b$. Otherwise, policy P1 is optimal (i.e., $z^m = 0$). We list our observations below:

1. Table 15 shows that the average and the maximum expected profit loss under both policies P1 and P2 increase with respect to α. To explain this, we note that as α increases, SL_1 decreases. This leads to a higher profit loss under policy P1. Furthermore, an increase in α leads to an increase in the variance of Class 1 demand, which in turn increases the variance of the left-over inventory after satisfying Class 1 demand. Using an extended computational study, we find that $p^m(I_2)$ is a relatively flat function of I_2 for smaller values of α compared to higher values of α. As a result, the profit loss under policy P2 with fixed price increases with α.
2. Table 16 shows that the average expected profit loss under policies P1 and P2 decreases

A	Policy P1†		Policy P2		SL_1
	Avg. (%)	Max. (%)	Avg. (%)	Max. (%)	
250	1.0	9.1	14.4	44.2	0.970
500	0.9	8.7	12.2	37.0	0.972
1000	0.8	7.9	8.6	25.4	0.975

† indicates results when $p^u > r_1 + b$

Table 16: Percentage expected profit loss for the benchmark policies with respect to A

a_2	Policy P1†		Policy P2	
	Avg. (%)	Max. (%)	Avg. (%)	Max. (%)
1	0.8	9.1	12.5	40.5
2	1.7	6.8	15.1	44.2
4	0.4	1.3	7.6	25.4

† indicates results where $p^u > r_1 + b$

Table 17: Percentage profit loss for the benchmark policies with respect to a_2

as A, and hence, the mean and variance of Class 2 demand increase. Again, this seemingly non-intuitive result, especially, with respect to policy P1 can be explained by studying SL_1. From Table 12, we note that although z^m increases as A increases, so does S^m and SL_1. Hence, the profit loss under policy P1 decreases. We note that an increase in A results in two changes with opposing effects on the expected profit: (i) It increases the mean of the stochastic and price-independent component of Class 2, and hence, the total expected Class 2 demand. This in turn increases the expected profit for both the optimal and P2 policies. (ii) It decreases the difference between the lower and the upper bounds on the optimal price that is charged to Class 2 (see item O2 in section III.6.2). This decreases the percentage expected profit loss due to the fixed price strategy under policy P2. Results in Table 16 show that the combined effect of these factors due to an increase in A leads to a decrease in expected profit loss under policy P2. This indicates that the effect due to increase in demand is dominated by the effect due to a comparatively flatter optimal price curve.

3. Table 17 shows that the average expected profit loss under policies P1 and P2 increase when a_2 increases from 1 to 2, but decreases as a_2 further increases to 4. We further examine this result by conducting an extended study with several values of a_2. Figure 13 shows the behavior of expected profit loss with respect to a_2. It shows that as a_2 increases, the profit loss under policy P1 decreases. As a_2 increases, z^m decreases (see item O3 in section III.6.2). Figure 13 also shows that the profit loss under policy P2

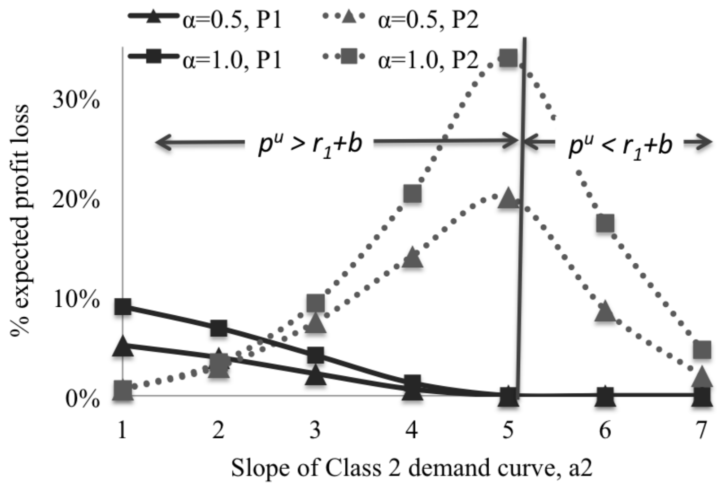


Figure 13: Effect of a_2 on the expected profit loss under the benchmark policies: Setting 2, $E[D_1] = 2626$, $r_1 = 500$, $\alpha = 1.0$, $h = 40$, $A = 250$

increases (decreases) with respect to a_2 when a_2 is such that $p^u = a_1/a_2 > r_1 + b$ ($p^u = a_1/a_2 \leq r_1 + b$). To explain this, we note that an increase in a_2 results in two opposing effects on the expected profit: (i) It reduces the deterministic component of demand, and hence, the total Class 2 demand. This in turn reduces the expected profit for both the optimal and P2 policies. Furthermore, when $p^u = a_1/a_2 \leq r_1 + b$, $p^m(z^m) = p^m(0) = p^u \geq p^m(I_2)$. As a result, there is higher decrease in Class 2 demand and the expected profit under policy P1, compared to that under the optimal policy. (ii) It decreases the difference between the lower and the upper bounds on the optimal price that is charged to Class 2 (see item O3 in section III.6.2). As a result, the percentage profit loss due to the fixed price strategy under policy P2 decreases. Figure 13 indicates that when $p^u = a_1/a_2 > r_1 + b$, the effect due to decrease in demand is lower than the effect due to flattening of the price curve, and hence, the profit loss under policy P2 decreases, and vice-a-versa.

III.7 Summary of Contributions and Key Insights

In this chapter, we consider a multi-period, joint replenishment, allocation and pricing problem for a supplier facing stochastic demand from two customer classes. This problem is particularly relevant for electronics manufacturers who sell their products to retailers at a fixed price, and to individual buyers via web-based channels where prices can be changed easily. We model the price-sensitive demand in Class 2 using an additive demand function. We show that the optimal policy is a combination of produce-up-to policy with potential discretionary sales to Class 1 and dynamic pricing to Class 2. To facilitate the implementation of the optimal policy, we establish two sufficiency conditions under which a myopic policy is optimal. The first sufficiency condition requires the on-hand inventory at the beginning of the planning horizon to be less than the myopic produce-up-to level. The second sufficiency condition is satisfied if the critical myopic produce-up-to level is increasing. For example, this is true when all cost and demand parameters are stationary and any units remaining at the end of the planning horizon are salvaged at a value equal to the unit production cost. Our results show that the dynamic pricing for Class 2 leads to significant benefits with an average of 11.7%. On the other hand, benefits of discretionary sales for Class 1 can be more than 5% in the following cases: (i) Class 1 market size is higher, (ii) sum of unit price charged to Class 1 and the lost sales penalty is less than the maximum price charged to Class 2, and (iii) variance of Class 2 demand is high.

CHAPTER IV

DEMAND ALLOCATION DECISIONS UNDER MULTI-SOURCING AND THEIR IMPACT ON THE BULLWHIP EFFECT

IV.1 Introduction

In this chapter, we consider a real life problem faced by a major electronics manufacturer (buyer), who faces stochastic demand for its end-product, which is an assembly of several parts. We consider a specific part that is multi-sourced via a vendor managed inventory (VMI) program with percentage supply allocations (PSAs) as illustrated in Figure 14. We define PSA as a pre-negotiated percentage of the multi-sourced part's total demand that the buyer should allocate to a supplier in order to get discounts (commonly called as market share discounts [8, 57]) or avoid penalties, depending on the specific agreement. We address the buyer's demand allocation decision for the multi-sourced part with the objective to meet the PSAs for each supplier.

Multi-sourcing with PSAs is common across many industries including electronics, health care, supermarkets and retail supply chains [9, 76]. For example, in the health care and retail industries, a contract with PSAs is implemented in the form of a market share discount, which

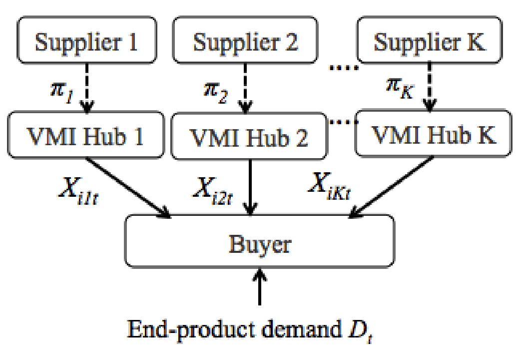


Figure 14: Supply chain setting

is a price-break given by a supplier based on the percentage of the buyer's total demand that is allocated to that particular supplier [8]. In the electronics industry, a contract with PSAs is usually implemented along with VMI programs in the form of commitments, which are binding on the buyer and the suppliers. As part of the VMI program, the buyer regularly shares forecasts with its suppliers, who manage replenishments to VMI hubs located close to the buyer's facility. Furthermore, the buyer (e.g., a computer manufacturer) may commit to allocate a certain percentage (e.g., 50%) of its total demand to a supplier. In return, the supplier provides the buyer a price guarantee and timely delivery of items, even when overall market demand for that item temporarily exceeds the overall market supply [73]. For example, the supplier provides a constant service level by continuously holding a minimum level of inventory at its VMI hub. At the end of the contract term, if the actual percentage of total demand allocated to a supplier is less than the committed PSA, then the buyer is directly liable for this deficit. In particular, the buyer either pays a penalty or simply purchases the amount equivalent to the percentage deficit, even if there is no immediate demand for it. Similarly, the supplier pays penalty for any shortages that occur when the actual inventory level falls below the specified minimum.

An important decision that the buyer makes under multi-sourcing is how to allocate demand (or orders) among various suppliers. A contract with PSAs poses unique challenges to the demand allocation decisions due to operational constraints, especially, in the electronics industry. In the absence of any operational constraints, the demand allocation decisions are straightforward. For example, if a hospital uses a contract with PSAs for surgical instruments, then a practical and effective policy is to split every among suppliers proportional to the PSAs. In contrast, consider a computer manufacturer who multi-sources motherboards from suppliers M1 and M2, and hard-drives from suppliers H1 and H2. Suppose that motherboards from M1 (M2) are compatible only with hard drives from H1 (H2), and either of the compatible pair can be used in the end-product. At any given time, inventory from only one supplier can be used and when the buyer moves from one supplier to another, we say that the *buyer switches suppliers*. Suppose that motherboards from M1 are currently being used to assemble the end-product and the manufacturer wants to switch to motherboards from M2. This implies switching to hard drives from H2, and entails operational changes on the factory floor, e.g., replacing inventories of M1 and H1 from the assembly line with those of M2 and H2. During a recent industry collaboration with a major computer manufacturer, we observed that when all components of a computer are considered, the operational changes necessary to (temporarily) switch suppliers for even a single part are significant, affect productivity and are expensive [48]. As a result, developing a practical demand allocation policy that meets the PSAs with minimum number of supplier switches is extremely important.

In addition, during our industry collaboration, we observed that the demand allocation policy used in current practice creates two main challenges: (i) it is not effective in meeting PSAs, (ii) it can significantly increase the variability of the demand observed by the suppliers, leading to the bullwhip effect. Bullwhip effect is the amplification of demand variability as customer orders travel upstream of the supply chain and leads to supply chain inefficiencies [10, 12, 52]. Furthermore, we note that a contract with PSAs in the form of market share discounts has attracted antitrust scrutiny because they may have exclusionary effects when offered by a major supplier with significant market power. For example, Intel's use of such discounts for a central processing unit led to antitrust proceedings by AMD and Federal Trade Commission, resulting in monetary fines and prohibition of future use of such discounts by Intel [1, 44]. This has motivated several economic models, which conclude that like most pricing practices, market share discounts can be used for both pro-competitive as well as anti-competitive purposes [56, 57, 71, 76].

As noted before, the objective of this chapter is to address the operational challenges under multi-sourcing with PSAs by developing effective demand allocation policies that are practical in the context of electronics industry. To this end, we first benchmark the demand allocation policy observed in the electronics industry using random allocation policy (RAP). In addition, we propose two alternate policies: time-based cyclic consumption (CCP-T) and quantity-based cyclic consumption policies (CCP-Q). These policies prescribe guidelines to systematically make the supplier switching decisions. We evaluate and compare the performances of these policies based on (i) long-run fraction of total demand allocated to each supplier, (ii) buyer's expected long-run average number of supplier switches, and (iii) supplier's bullwhip effect under multi-sourcing.

Demand allocation policies that can meet PSAs have not been studied in the literature before. In contrast, the bullwhip effect is investigated by a vast and growing body of academic and practice oriented literature. Empirical evidence demonstrates that the bullwhip effect is widespread across all industries and leads to supply chain inefficiencies [10, 12, 52]. Existing models in the literature study the bullwhip effect in single-sourcing supply chains with one supplier and one or more buyers [18, 19, 20, 21, 28, 27, 29, 37, 52]. However, there is no literature on bullwhip effect under multi-sourcing. We contribute to the current literature by demonstrating the existence of bullwhip effect caused due to demand allocation policies in multi-sourcing systems. We term it as the *bullwhip effect under multi-sourcing* and emphasize its absence in single-sourcing systems. We quantify the bullwhip effect under multi-sourcing and study its impact on the supplier's replenishment and inventory holding costs. While we do not focus on the pro-competitive or anti-competitive effects of a contract with PSAs, our results offer new insights showing that these effects may be strengthened or weakened depending on the demand allocation policy used by the buyer.

In particular, we provide analytical expressions that quantify the performances of RAP, CCP-T and CCP-Q, and show that they can meet PSAs in the long run. We demonstrate that RAP *always* leads to the bullwhip effect under multi-sourcing and offer new insights that substantiate the exclusionary, and hence, anti-competitive effects of a contract with PSAs. We also show that while CCP-T also leads to the bullwhip effect, it is less than that under RAP. On one hand, we show that CCP-Q may not always lead to the bullwhip effect. However, when the coefficient of variation of the buyer's demand is high, the bullwhip effect under CCP-Q can be higher than that under RAP. When the total number of suppliers increase while keeping the PSA for a particular supplier fixed, the bullwhip effect observed by that supplier: (i) remains the same under RAP and CCP-T, (ii) decreases under CCP-Q. In contrast, when PSAs are distributed equally among all suppliers, the bullwhip effect under each policy increases as the total number of suppliers increases. Considering the importance of the buyer's expected long-run number of supplier switches, we provide a method to compute policy parameters for CCP-T and CCP-Q to achieve a target value of the number of supplier switches. Furthermore, when negotiating a contract with the buyer, suppliers will find our results valuable to carefully select the service levels that they commit to provide based on the agreed upon PSA.

We address two key research questions in this chapter. The first is which policy is preferred by the buyer and each supplier? Clearly, the buyer prefers the policy with the lowest expected long-run average number of supplier switches, and each supplier prefers the policy that results in the lowest bullwhip effect for that supplier. To address this, we rank the three policies based on each performance measure. We demonstrate, analytically where possible and numerically if not, that both CCP-T and CCP-Q can reduce the supplier's bullwhip effect without increasing the buyer's expected long-run average number of supplier switches compared to that under RAP. Furthermore, we establish a threshold value of PSA such that suppliers with higher (lower) PSA than this threshold observe the lowest bullwhip effect under CCP-Q (CCP-T), and hence, prefer it over other policies.

The second question that we investigate is: which policy is the best considering the total system consisting of the buyer and all the suppliers? To address this, we choose parameters of CCP-T and CCP-Q so that the resulting expected long-run average number of supplier switches under each policy is the same, and hence, all policies are equivalent for the buyer. We consider that each supplier replenishes its VMI hub using an produce-up-to policy and compare the demand allocation policies based on the system cost, i.e., the total replenishment and inventory holding costs across all suppliers. Our numerical results show that, compared to RAP, both CCP-T and CCP-Q result in significant savings in the total system cost with an average of more than 15%. These savings are higher for smaller values of coefficient of variation of the buyer's demand. Furthermore, CCP-Q performs mildly better than CCP-T with an average benefit of 0.44% and a maximum of 2.62%. We note that CCP-T is easier to implement than CCP-Q, and hence, depending on the magnitude of the actual savings

CCP-T or CCP-Q may be used.

The remainder of this chapter is organized as follows: In Section IV.2, we present the problem definition and introduce the performance measures used to evaluate the demand allocation policies. In Sections IV.3, IV.4 and IV.5, we present the models and analyses for RAP, CCP-T and CCP-Q demand allocation policies, respectively. We define RAP, CCP-T and CCP-Q demand allocation policies. We also derive the expressions of the performance measures and evluate the effects of model parameters on the performance measures for each policy. In Section IV.6, we compare and rank the three demand allocation policies based on the three performance measures listed above. Section IV.7 compares the three policies based on the total system cost. Section IV.8 summarizes our findings and the key insights.

IV.2 Problem Setting and Definition

In this section, we discuss the problem setting, introduce our notation and the performance measures used to evaluate the demand allocation policies.

We consider a buyer (e.g., electronics manufacturer) who faces stochastic demand for its end-product, which is an assembly of several parts. The demand for the buyer's end-product arrives periodically. We consider a specific part that is multi-sourced from K suppliers via a VMI program with PSAs. Let π_j be the PSA for supplier j such that $0 < \pi_j < 1$ for $j = 1, \ldots, K$ and $\sum_{j=1}^{K} \pi_j = 1$. Without loss of generality, we assume that $\pi_1 \leq \pi_2 \leq \ldots \leq \pi_K$. Under a VMI program, suppliers maintain inventory at a VMI hub close to the buyer's facility as illustrated in Figure 14. As a result, the buyer does not carry any inventory and pulls it from suppliers' VMI hubs after observing the demand in each period. We address the buyer's demand allocation decisions subject to an operational constraint. More specifically, the operational constraint dictates that the buyer can switch from one supplier to other at most once per period (e.g., a day). This is a common practice observed in the electronics industry and the aim is to lower the number of supplier switches. Accordingly, our model allows switching suppliers only at the beginning of a period.

At the beginning of each period t, the buyer observes the random demand, denoted by D_t. Then, based on the demand allocation policy, the buyer selects one supplier among its

K suppliers and allocates the entire demand for that period to the selected supplier. In other words, the demand allocation policy guides the buyer in selecting a supplier for each period. The buyer's objective is to meet the PSAs in the long-term for each supplier with the minimum number of supplier switches. We benchmark the demand allocation policy observed in the electronics industry using RAP, and demonstrate that it always leads to bullwhip effect under multi-sourcing. To mitigate this, we propose two alternate policies, CCP-T and CCP-Q, which are based on a novel concept of a consumption cycle [48]. We define the consumption cycle as below:

Definition 1. *A consumption cycle allocates demand to different suppliers in a sequential manner such that during every cycle, either the total time or the minimum quantity of demand allocated to each supplier is fixed. Any sequence of suppliers may be chosen based on operational ease.*

Depending on whether the supplier switching decisions are made by time or quantity, we define time-based (CCP-T) and quantity-based cyclic consumption (CCP-Q) policies. For modeling purposes we choose the $1 - 2 - \ldots K$ sequence, and emphasize that our results regarding performance of CCP-T and CCP-Q are independent of the chosen sequence.

IV.2.1 Underlying stochastic process

Next, we discuss the underlying demand process and introduce our notation. We consider that $\{D_t : t = 1, 2, \ldots\}$ is a sequence of independent and identically distributed (i.i.d.) non-negative random variables with cumulative distribution function $F(\cdot)$. Let us denote the mean and standard deviation of D_t by μ and σ, respectively. Let $CV[D_t] = \sigma/\mu$ be the coefficient of variation of D_t. Define Y_n as the total demand observed by the buyer up to period n, i.e.,

$$Y_n = \sum_{t=1}^{n} D_t. \tag{176}$$

Let i be the index for the demand allocation policy used by the buyer, $i = 1$ for RAP, $i = 2$ for CCP-T, and $i = 3$ for CCP-Q. For $i = 1, 2, 3$, and $t, n = 1, 2, \ldots$, define

K_{it} = supplier to which demand is allocated in period t under policy i

X_{ijt} = demand allocated to supplier j in period t under policy i

Y_{ijn} = the total demand allocated to supplier j up to period n under policy i

Then, for each policy i, $K_{it} \in \{1, 2, \ldots, K\}$, and

$$X_{ijt} = 1(K_{it} = j)D_t \tag{177}$$

$$Y_{ijn} = \sum_{t=1}^{n} X_{ijt}. \tag{178}$$

Next, we introduce the performance measures used to evaluate the demand allocation policies.

IV.2.2 Performance measures

During a recent industry collaboration, we observed that under the demand allocation policy used in current practice, inventory consumption from some suppliers may exceed their PSAs, while not meeting the same for other suppliers. This satisfies the buyer's percentage commitment for some suppliers, but violates it for other suppliers. Furthermore, since supplier switching affects productivity the buyer prefers a policy with less number of supplier switches. In addition, we observed that the demand allocation policy used in current practice leads to the bullwhip effect. Table 18 presents empirical evidence based on actual industry data. It highlights that the coefficient of variation (CV) of orders placed to supplier S3 was almost twice the CV of the buyer's demand. This increases costs and leads to supply chain inefficiencies. For example, due to increased demand variability, suppliers have to hold more inventory at the VMI hubs to achieve the committed service level. Furthermore, the bullwhip effect increases the supplier's replenishment and inventory holding costs [21]. Consequently, we investigate and compare the three demand allocation policies, based on the following three long-run performance measures:

	S1 orders	S2 orders	S3 orders	Sales
Average daily demand	1568	951	481	3000
Standard deviation of daily demand	5431	1513	1938	6153
Coefficient of Variation (CV)	3.46	1.59	4.03	2.05

Table 18: Variability of orders placed with suppliers v/s variability of sales.

1. Since the buyer's primary objective is to meet PSAs, the first performance measure that we consider is the expected long-run fraction of total demand allocated to supplier j under policy i, denoted by α_{ij}. Mathematically, we can write $\alpha_{ij} = \lim_{n\to\infty} \alpha_{ij}(n)$, where

$$\alpha_{ij}(n) = E\left[\frac{Y_{ijn}}{Y_n}\right]. \tag{179}$$

2. The second performance measure that we consider is the buyer's expected long-run average number of supplier switches under policy i, denoted by γ_i. As discussed before, the buyer tends to choose the policy which results in less number of supplier switches. Consequently, we discuss how to compute the parameters of a demand allocation policy given a specific target value of γ, denoted by γ_0.

3. Our third performance measure is the bullwhip effect observed by supplier j due to demand allocation policy i, denoted by β_{ij}. We term this as the *bullwhip effect under multi-sourcing.* Traditionally, for single-supplier single-buyer systems, the bullwhip effect is defined as the ratio of the variance of supplier's demand (i.e., buyer's orders) and the variance of buyer's demand. In such systems, the supplier's and the buyer's long-run average demands are equal. In contrast, in multi-supplier systems, the average demand observed by each supplier is less than the average demand observed by the buyer. Since CV measures variation of a random variable relative to its mean, for

multi-supplier systems, we define the bullwhip ratio as

$$\beta_{ij} = \frac{CV[\text{supplier } j\text{'s demand under policy } i]}{CV[\text{buyer's demand}]}. \tag{180}$$

If $\beta_{ij} \geq 1$ ($\beta_{ij} < 1$), then orders placed with supplier j are more (less) variable than the demand observed by the buyer, and we say that supplier j observes (does not observe) bullwhip effect under policy i.

It is worthwhile to note that there is no bullwhip effect in the single-sourcing version of our supply chain setting, where the traditional measures to reduce the bullwhip effect are already in place, e.g., VMI, zero lead time (between the VMI hub and buyer's facility) [18, 19, 20, 28, 27, 29]. Furthermore, other causes of bullwhip effect like updating of forecasts based on observed demand, supply shortages leading to inventory rationing, batching of orders, and price fluctuations are also absent in our setting [21, 52].

IV.3 Random Allocation Policy

The random allocation policy (RAP) benchmarks the demand allocation policy observed in the electronics industry. In practice, demand can be allocated to a random supplier in each period providing full flexibility to the buyer [45]. Accordingly, we model RAP by considering K_{1t} as a random variable such that $\mathrm{P}(K_{1t} = j) = r_j$, where $0 < r_j < 1$ and $\sum_{j=1}^{K} r_j = 1$ for $j = 1, \ldots, K$ and $t = 1, 2, \ldots$. As a result, RAP is characterized by $(r_1, \ldots, r_K)$ and works as follows: In each period t, the buyer observes the realized value of K_{1t} and assigns the entire demand of the period to that supplier.

We illustrate RAP using an example. Suppose there are two suppliers, i.e., $K = 2$ and the buyer uses a RAP with parameters $r_1 = r_2 = 0.5$. Suppose that for $t = 1$ and $k_{11} = 1$, the buyer allocates D_1 to supplier 1 and no demand is allocated to supplier 2. Next, suppose that for $t = 2$, we also have $k_{12} = 1$. Then, the buyer allocates D_2 to supplier 1, and so on. Figure 15 illustrates a realization of the buyer's cumulative demand process Y_n, and supplier j's cumulative demand process Y_{1jn} under RAP when $k_{1t} = j$ for $t = 1, 2, 8, 10$.

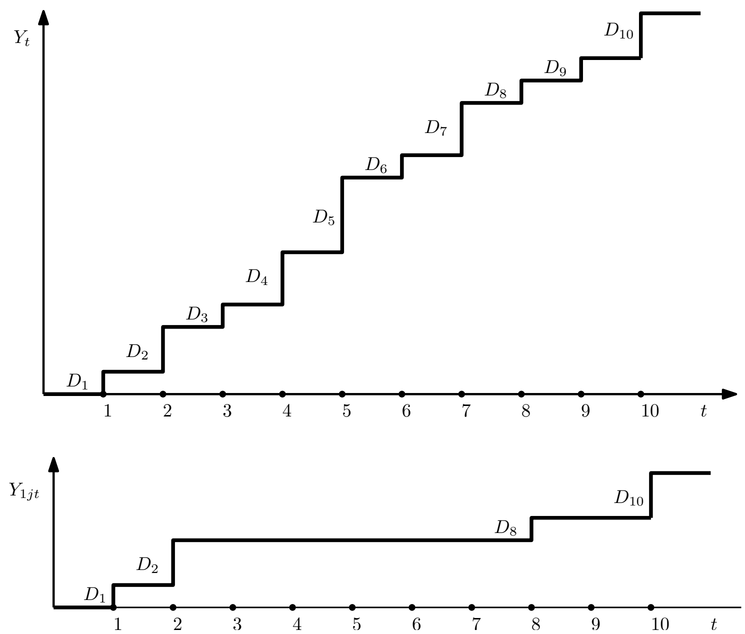


Figure 15: Cumulative demand process for the buyer and an arbitrary supplier j under RAP

We note that K_{1t} and D_t are independent random variables and K_{1t} are i.i.d. Using equation (177) for $i = 1$, we see that $X_{1jt} = 1(K_{1t} = j)D_t$, $t = 1, 2, \ldots$ are also i.i.d. Proposition 11 provides expressions for the performance of RAP with parameters $(r_1, \ldots, r_K)$.

Proposition 11. *For $j = 1, \ldots, K$, the following are true for RAP with parameters $(r_1, \ldots, r_K)$:*

(a) $\alpha_{1j} = r_j$,

(b) $\beta_{1j} = \sqrt{\dfrac{1}{r_j} + \dfrac{1 - r_j}{r_j CV^2[D]}} \geq \dfrac{1}{\sqrt{r_j}} > 1$,

(c) $\gamma_1 = 1 - \sum_{j=1}^{K} r_j^2$.

Proof. Before we proceed with the proof, we derive some useful identities. First, using linearity of expectation, we have

$$1 = E\left[\frac{\sum_{t=1}^{n} D_t}{Y_n}\right] = \sum_{t=1}^{n} E\left[\frac{D_t}{Y_n}\right].$$

Since $D_t,\ t = 1, 2, \ldots$ are i.i.d., $E[D_t/Y_n] = E[D_{t'}/Y_n]$ for $t \neq t'$, then using the above equality we have

$$E[D_t/Y_n] = 1/n \quad \text{for } t = 1, 2, \ldots, n. \tag{181}$$

Next, using the fact that K_{1t} and D_t are independent and $P(K_{1t} = j) = r_j$, we have

$$E[X_{1jt}] = E\big[1(K_{1t} = j)D_t\big] = r_j\mu \tag{182}$$

$$\begin{aligned} Var[X_{1jt}] = Var\big[1(K_{1t} = j)D_t\big] &= r_j E[D^2] - r_j^2\mu^2 \\ &= r_j\sigma^2 + r_j(1 - r_j)\mu^2 \\ &= r_j^2\mu^2\left(\frac{CV^2[D] + 1 - r_j}{r_j}\right). \end{aligned} \tag{183}$$

(a) Using equations (177), (178), (179) for $i = 1$, the fact that K_{1t} and D_t/Y_n are independent and the identity (181), we have

$$\alpha_{1j}(n) = E\left[\frac{\sum_{t=1}^{n} 1(K_{1t} = j)D_t}{Y_n}\right] = \sum_{t=1}^{n} E[1(K_{1t} = j)]E\left[\frac{D_t}{Y_n}\right] = \sum_{t=1}^{n} r_j\frac{1}{n} = r_j.$$

(b) Using the equations (182) and (183), we compute $CV[X_{1jt}]$ as

$$CV[X_{1jt}] = \frac{\sqrt{Var[X_{1jt}]}}{E[X_{1jt}]} = r_j\mu\sqrt{\frac{CV^2[D] + 1 - r_j}{r_j}}\,\frac{1}{r_j\mu} = \sqrt{\frac{CV^2[D] + 1 - r_j}{r_j}}. \tag{184}$$

X_{1jt} are i.i.d. random variables, and hence, the bullwhip effect under RAP can be defined as follows

$$\begin{aligned} \beta_{1j} = \frac{CV[X_{1jt}]}{CV[D_t]} &= \frac{1}{CV[D]}\sqrt{\frac{CV^2[D] + 1 - r_j}{r_j}} \\ &= \sqrt{\frac{1}{r_j} + \frac{1 - r_j}{r_j CV^2[D]}} \geq \frac{1}{\sqrt{r_j}} > 1. \end{aligned}$$

(c) If supplier j is chosen in period t, then the probability that the buyer switches to another supplier at the end of period t is equal to $Pr(K_{1,t+1} \neq j) = 1 - r_j$. Since K_{1t}, $t = 1, 2, \ldots$ are i.i.d., γ_1 can be computed by conditioning on K_{1t} as follows

$$\begin{aligned}\gamma_1 &= E\big[E[\text{Number of supplier switches in a period}|K_{1t} = j]\big] \\ &= \sum_{j=1}^{K}(1 - r_j)r_j = 1 - \sum_{j=1}^{K} r_j^2,\end{aligned}$$

where we use the fact that $\sum_{j=1}^{K} r_j = 1$. □

From part (a) of Proposition 11, it follows that RAP will meet PSA for supplier j, $j = 1, \ldots, K$, in the long-run if $r_j = \pi_j$. Part (b) of Proposition 11 shows that each supplier observes bullwhip effect when the buyer uses RAP. This is intuitive because under RAP, in addition to the uncertainty in the order size, the supplier also faces the uncertainty due to the buyer's random selection process. In addition, part (b) of Proposition 11 provides a lower bound on β_{1j}. Part (c) of Proposition 11 presents an expression for the buyer's long-run average number of supplier switches under RAP, γ_1. We observe that γ_1 depends on PSAs. Next, Corollary 3 examines the effect of model parameters on β_{1j}.

Corollary 3. *The following are true for RAP with parameters* $(r_1, \ldots, r_K)$*:*
(a) β_{1j} *is a convex decreasing function of* r_j*,*
(b) β_{1j} *is a convex decreasing function of* $CV[D]$*,*
(c) β_{1j} *is a super-modular function of* r_j *and* $CV[D]$*.*

Proof. (a) Taking the first and second order derivatives of β_{1j} with respect to r_j, we have

$$\begin{aligned}\frac{\partial \beta_{1j}}{\partial r_j} &= \frac{-1}{2CV[D]}\left(\frac{CV^2[D] + 1 - r_j}{r_j}\right)^{-1/2}\left(\frac{CV^2[D] + 1}{r_j^2}\right) \\ &= \frac{-(CV^2[D] + 1)}{2CV^2[D]r_j^2\beta_{1j}} \leq 0,\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 \beta_{1j}}{\partial r_j^2} &= \frac{-(CV^2[D]+1)}{2CV^2[D]}\left(\frac{-2}{r_j^3\beta_{1j}} - \frac{1}{r_j^2\beta_{1j}^2}\frac{\partial \beta_{1j}}{\partial r_j}\right)\\
&= \frac{-(CV^2[D]+1)}{2CV^2[D]r_j^3\beta_{1j}}\left(-2 + \frac{CV^2[D]+1}{CV^2[D]r_j\beta_{1j}^2}\right)\\
&= \frac{(CV^2[D]+1)}{2CV^2[D]r_j^3\beta_{1j}}\left(\frac{CV^2[D]+1+2r_j}{CV^2[D]+1-r_j}\right) \geq 0.
\end{aligned}
$$

Thus, β_{1j} is a convex decreasing function of r_j.

(b) Taking the first and second order derivatives of β_{1j} with respect to $CV[D]$, we have

$$
\begin{aligned}
\frac{\partial \beta_{1j}}{\partial CV[D]} &= \frac{-(1-r_j)}{\beta_{1j}r_jCV^3[D]} \leq 0\\
\frac{\partial^2 \beta_{1j}}{\partial (CV[D])^2} &= \frac{-(1-r_j)}{r_j}\left(\frac{-1}{\beta_{1j}^2CV^3[D]}\frac{\partial \beta_{1j}}{\partial CV[D]} - \frac{3}{\beta_{1j}CV^4[D]}\right)\\
&= \frac{-(1-r_j)}{\beta_{1j}r_j\beta_{1j}CV^4[D]}\left(\frac{(1-r_j)CV^2[D]}{CV^2[D]+1-r_j} - 3\right)\\
&= \frac{(1-r_j)\big[CV^2[D](2+r_j)+3(1-r_j)\big]}{r_j\beta_{1j}CV^4[D]\big(CV^2[D]+1-r_j\big)} \geq 0.
\end{aligned}
$$

Thus, β_{1j} is a convex decreasing function of $CV[D]$.

(c) Taking the first order derivative of $\partial\beta_{1j}/\partial r_j$ with respect to $CV[D]$, we have

$$
\begin{aligned}
\frac{\partial \beta_{1j}}{\partial r_j \partial CV[D]} &= \frac{-1}{2r_j}\left(\frac{1}{\beta_{1j}^2}\left(1+\frac{1}{CV^2[D]}\right)\frac{\partial \beta_{1j}}{\partial CV[D]} - \frac{2}{\beta_{1j}CV^3[D]}\right)\\
&= \frac{-1}{2r_j\beta_{1j}CV^2[D]}\left(\frac{(1-r_j)(CV^2[D]+1)}{\beta_{1j}^2r_jCV^3[D]} - \frac{2}{CV[D]}\right)\\
&= \frac{-1}{2r_j\beta_{1j}CV^3[D]}\left(\frac{(1-r_j)(CV^2[D]+1)}{CV^2[D]+1-r_j} - 2\right)\\
&= \frac{-1}{2r_j\beta_{1j}CV^3[D]}\left(\frac{1-r_j+CV^2[D](1+r_j)}{CV^2[D]+1-r_j} \geq 0.\right.
\end{aligned}
$$

Thus, β_{1j} is a super-modular function of r_j and $CV[D]$ [42]. □

Part (a) of Corollary 3 presents an important result that β_{1j} is a convex decreasing function of r_j. That is, as the probability of choosing supplier j (r_j), and hence, the percentage of demand allocated to supplier j increases supplier j's bullwhip effect decreases.

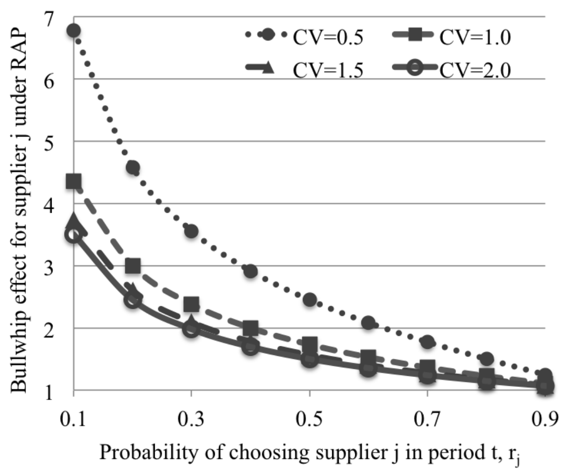


Figure 16: β_{1j} with respect to r_j for different values of $CV[D]$

Therefore, when $r_j = \pi_j$ for $j = 1, \ldots$ PSAs are met and a major supplier with high PSA observes lower bullwhip effect compared to a minor supplier with low PSA. An important and counter-intuitive result is that β_{1j} is a convex decreasing function of $CV[D]$. One may expect that as $CV[D]$ increases, the supplier's demand variance increases, and hence, β_{1j} also increases. However, we note that as $CV[D]$ increases both the buyer's and the supplier's demand variances increase. Furthermore, the result in part (b) of Corollary 3 shows that the magnitudes of the increases in the buyer's and the supplier's demand variances are such that β_{1j} decreases. Part (c) of Corollary 3 shows that β_{1j} is a super-modular function of r_j and $CV[D]$, i.e., the decrease in β_{1j} due to an increase in r_j is larger for smaller values of $CV[D]$. Figure 16 complements the analytical results of Corollary 3 and further expounds the effect of r_j and CV on β_{1j}. It illustrates the result that β_{1j} is a convex decreasing function of r_j and $CV[D]$, and β_{1j} can be significantly high for small values of r_j and/or $CV[D]$.

Corollary 4 examines the trade-off between the bullwhip effect observed by different suppliers and the buyer's long-run average number of supplier switches under dual-sourcing, i.e., $K = 2$. We consider that $r_j = \pi_j$ for $j = 1, 2$, and hence, PSAs are met. We recall that $\pi_1 \leq \pi_2$.

Corollary 4. *The following is true under dual-sourcing, i.e., $K = 2$, and RAP with parameters $r_j = \pi_j$ for $j = 1, 2$:*

(a) As π_1 increases, β_{11} decreases while β_{12} increases such that the rate of decrease in β_{11} is greater than the rate of increase in β_{12}. When $\pi_1 = \pi_2 = 0.5$, $\beta_{11} = \beta_{12} = \sqrt{2 + 1/CV^2[D]}$.

(b) γ_1 is a concave increasing function of π_1.

Proof. Suppose that $r_j = \pi_j$ for $j = 1, 2$. Using this in part (a) of Corollary 3 it follows that as π_1 increases, β_{11} decreases while β_{12} increases and when $\pi_1 = \pi_2 = 0.5$, $\beta_{11} = \beta_{12} = \sqrt{2 + 1/CV^2[D]}$. Next, using the chain rule of differentiation, we have

$$\frac{\partial \beta_{12}}{\partial \pi_1} = \frac{\partial \beta_{12}}{\partial \pi_2}\frac{\partial \pi_2}{\partial \pi_1} = -\frac{\partial \beta_{12}}{\partial \pi_2} > 0. \tag{185}$$

Using part (a) of Corollary 3, the fact that when $\pi_1 = \pi_2 = 0.5$, $\beta_{11} = \beta_{12}$ and equation (185), we have

$$\left.\frac{\partial \beta_{11}}{\partial \pi_1}\right|_{\pi_1 < 0.5} < \left.\frac{\partial \beta_{11}}{\partial \pi_1}\right|_{\pi_1 = 0.5} = \left.\frac{\partial \beta_{12}}{\partial \pi_2}\right|_{\pi_2 = 0.5} < \left.\frac{\partial \beta_{12}}{\partial \pi_2}\right|_{\pi_2 > 0.5} = -\left.\frac{\partial \beta_{12}}{\partial \pi_1}\right|_{\pi_1 < 0.5} < 0.$$

Therefore, the rate of decrease in β_{11} with respect to π_1, i.e., $-\partial\beta_{11}/\partial\pi_1$, is greater than the rate of increase in β_{12} with respect to π_1, i.e., $\partial\beta_{12}/\partial\pi_1$. Similarly, using part (c) of Proposition 11 we have

$$\gamma_1 = 1 - \pi_1^2 - (1 - \pi_1)^2 = 2\pi_1 - 2\pi_1^2.$$

Taking the first and second order derivatives of γ_1 with respect to π_1 and using the fact that $\pi_1 \leq 0.5 \leq \pi_2$, we have

$$\begin{aligned}\frac{d\gamma_1}{d\pi_1} &= 2 - 4\pi_1 \geq 2 - 4 \times 0.5 = 0,\\ \frac{d^2\gamma_1}{d\pi_1^2} &= -4 \leq 0.\end{aligned} \tag{186}$$

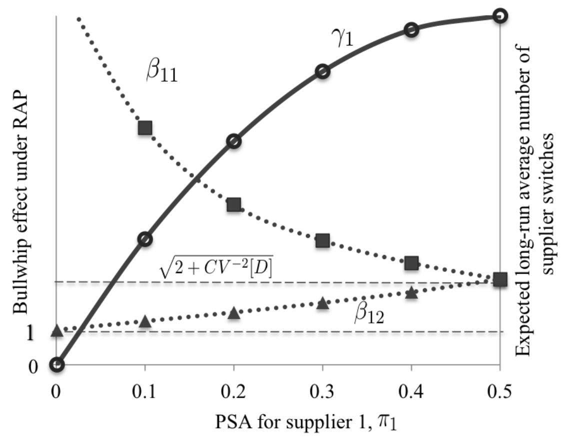


Figure 17: β_{11} , β_{12} and γ_1 with respect to r_1 for $K = 2$

Thus, γ_1 is a concave increasing function of π_1. □

Figure 17 illustrates the effect of π_1 on β_{11}, β_{12} and γ_1 when $K = 2$. It shows that as π_1 increases, β_{11} decreases and β_{12} increases such that the rate of decrease in β_{11} is higher than the rate of increase in β_{12}. From Corollary 4 and Figure 17, we gain the following key insights regarding RAP, and hence, the demand allocation policy used in practice:

O1 In a dual-sourcing system, the buyer prefers a skewed, rather than a comparable distribution of PSA among its two suppliers due to lower expected long-run average number of supplier switches. As a result, if the major supplier offers a lucrative discount for 100% of buyer's market share, the buyer may prefer single-sourcing from the major supplier rather than dual-sourcing.

O2 In a dual-sourcing system, the minor supplier with low PSA is at a significant disadvantage: not only is its market share (of buyer's demand) small, but it also observes high bullwhip effect, which increases his replenishment and inventory holding costs at the VMI hub [21]. As a result, it may not be profitable for the minor supplier with lower market/negotiating power to trade with a buyer who insists on low PSA.

Thus, for the demand allocation policy used in practice, our results offer new insights that substantiate the exclusionary effects of a contract with PSAs when offered by a major supplier with significant market power. Furthermore, when negotiating a contract with the buyer, suppliers will find our results valuable to carefully select the service levels that they commit to provide based on the agreed upon PSA.

In Sections IV.4 and IV.5, we propose alternate demand allocation policies which can reduce the supplier's bullwhip effect without increasing the buyer's expected long-run average number of supplier switches.

IV.4 Time-based Cyclic Consumption Policy

When implementing a consumption cycle to make demand allocations, the buyer may make supplier switching decisions based on time. More specifically, in every cycle, the buyer may allocate demand to each supplier for a specific number of periods before switching to the next supplier. We term this approach as the time-based cyclic consumption policy (CCP-T). CCP-T prescribes guidelines to construct a fixed schedule for switching suppliers, and each supplier knows the time periods during which demand will be allocated to its VMI hub. The CCP-T is characterized by K parameters $(L_1, \ldots, L_K)$, where L_j is the fixed number of periods during which demand is allocated to supplier j in each consumption cycle.

For modeling purposes, we choose the $1-2-\ldots K$ sequence for cycling through the supplies. Nevertheless, our results regarding performance of CCP-T and CCP-Q in Section IV.5 are independent of the chosen sequence. We model CCP-T as follows: Let L denote the fixed length of each consumption cycle, then we have

$$L = \sum_{j=1}^{K} L_j. \tag{187}$$

Furthermore, K_{2t} is known for each t and can be written as follows:

$$K_{2t} = \begin{cases} j & \text{if } (t \text{ mod } L) = 1 + \sum_{k=1}^{j-1} L_k, \ldots, \sum_{k=1}^{j} L_k \text{ and } 1 \le j < K \\ K & \text{if } (t \text{ mod } L) = 0, 1 + \sum_{k=1}^{K-1} L_k, \ldots, \sum_{k=1}^{K} L_k - 1. \end{cases} \tag{188}$$

Next, we illustrate CCP-T with an example. Suppose $K = 2$ and the buyer uses a CCP-

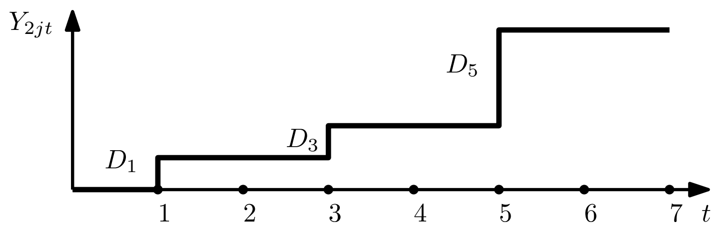


Figure 18: Cumulative demand process for an arbitrary supplier under CCP-T.

T with parameters ($L_1 = 1, L_2 = 2$). Then, $L = 3$ and demand in periods $t = 1, 4, 7, 10, \ldots$ is allocated to supplier 1, and demand in periods $t = 2, 3, 5, 6, 8, 9, \ldots$ is allocated to supplier 2. Figure 18 illustrates a realization of supplier 1's demand process for this example.

Using equation (177) for $i = 2$, we know that demand allocated to supplier j in period t is given by $X_{2jt} = 1(K_{2t} = j)D_t$. Since K_{2t} depends on t, X_{2jt} are not i.i.d. We emphasize that K_{2t} is deterministic and we know the periods when $X_{2jt} = 0$. However, all time-based consumption cycles are i.i.d., and hence, we can define the bullwhip ratio under CCP-T as

$$\beta_{2j} = \frac{CV[\text{demand observed by supplier } j \text{ in a consumption cycle}]}{CV[\text{demand observed by the buyer in a consumption cycle}]}. \tag{189}$$

Proposition 12 provides expressions for the performance measures of CCP-T with parameters $(L_1, \ldots, L_K)$.

Proposition 12. *For $j = 1, \ldots, K$, the following are true for a CCP-T with parameters $(L_1, \ldots, L_K)$:*

(a) $\alpha_{2j} = L_j/L$,

(b) $\beta_{2j} = \sqrt{L/L_j} > 1$,

(c) $\gamma_2 = K/L$.

Proof. (a) We first compute $\alpha_{2j}(n)$ and then take its limit as n goes to infinity to compute

α_{2j}. Using equations (177), (178), (179) and (181), we have

$$\alpha_{2j}(n) = E\left[\frac{\sum_{t=1}^{n} 1(K_{2t}=j)D_t}{Y_n}\right] = \sum_{t=1}^{n} 1(K_{2t}=j)\frac{1}{n}. \tag{190}$$

We can express n in terms of L as $n = n_1 L + n_2$, where $n_1 = \lfloor n/L \rfloor$ is the quotient and $n_2 = (n \bmod L)$ is the remainder of the division of n by L. Then, from equation (188), we have

$$\alpha_{2j}(n) = \begin{cases} (n_1 L_j)/n & \text{if } (n \bmod L) \le \sum_{m=1}^{j-1} L_m \\ \left(n_1 L_j + \left(n_2 - \sum_{m=1}^{j-1} L_m\right)\right)/n & \text{if } \sum_{m=1}^{j-1} L_m < (n \bmod L) < \sum_{m=1}^{j} L_m \\ (n_1+1)L_j/n & \text{if } (n \bmod L) \ge \sum_{m=1}^{j} L_m, \end{cases} \tag{191}$$

where $\sum_{m=1}^{0} L_m = 0$. Since $n_2 = (n \bmod L) < L$, when $\sum_{m=1}^{j-1} L_m < (n \bmod L) < \sum_{m=1}^{j} L_m$, we have $(n_2 - \sum_{m=1}^{j-1} L_m) < L_j$. As a result, the following inequality holds

$$\frac{n_1 L_j}{n} \le \alpha_{2j}(n) \le \frac{(n_1+1)L_j}{n}. \tag{192}$$

Furthermore, writing $n_1 = (n - (n \bmod L))/L$, we have

$$\lim_{n\to\infty} \frac{n_1 L_j}{n} = \lim_{n\to\infty} \frac{(n-(n \bmod L))L_j}{Ln} = \lim_{n\to\infty}\left(1 - \frac{(n \bmod L)}{n}\right)\frac{L_j}{L} = \frac{L_j}{L}, \tag{193}$$

$$\lim_{n\to\infty} \frac{(n_1+1)L_j}{n} = \lim_{n\to\infty}\left(1 + \frac{L-(n \bmod L)}{n}\right)\frac{L_j}{L} = \frac{L_j}{L}. \tag{194}$$

Then, using (192)-(194) and the sandwich theorem (see pg. 39 in [62]), it follows that $\lim_{n\to\infty} \alpha_{2j}(n) = L_j/L$.

(b) Let U_{jm} be the total demand observed by supplier j in the mth consumption cycle, then

U_{jm}, $m = 1, 2, \ldots$ are i.i.d., and

$$U_{jm} \sim \sum_{t=1}^{L} 1(K_{2t} = j) D_t.$$

Using equation (189) and the definition of U_{jm}, we have

$$\beta_{2j} = \frac{CV[U_{jm}]}{CV[Y_L]}. \tag{195}$$

From equation (188), in each consumption cycle the total number of periods that demand is allocated to supplier j is equal to L_j. Hence, we have

$$E[U_{jm}] = E\big[\sum_{t=1}^{L} 1(K_{2t} = j) D_t\big] = \sum_{t=1}^{L} E[1(K_{2t} = j) D_t] = L_j \mu, \tag{196}$$

$$Var[U_{jm}] = Var\big[\sum_{t=1}^{L} 1(K_{2t} = j) D_t\big] = \sum_{t=1}^{L} Var[1(K_{2t} = j) D_t] = L_j \sigma^2, \tag{197}$$

$$CV[U_{jm}] = \frac{\sqrt{L_j \sigma^2}}{L_j \mu} = CV[D]\sqrt{\frac{1}{L_j}}. \tag{198}$$

Furthermore, since D_t, $t = 1, 2, \ldots$ are i.i.d, $CV[Y_L] = CV[D]/\sqrt{L}$. Using this and equation (198) in (195), we have $\beta_{2j} = \sqrt{L/L_j}$. Since $L > L_j$, it follows that $\beta_{2j} > 1$.

(c) During each cycle, the buyer switches suppliers K times. Therefore, γ_2 can be written as

$$\gamma_2 = \frac{\text{Number of switching suppliers per cycle}}{\text{length of each cycle}} = \frac{K}{L}.$$

□

Part (a) of Proposition 12 shows that when the buyer uses CCP-T, PSAs will be met in the long-run, i.e., $\alpha_{2j} = \pi_j$, if

$$L_j = L\pi_j \tag{199}$$

for $j = 1, \ldots, K$. The operational constraint limiting the number of supplier switches to a maximum one per period will be satisfied if $L_j \geq 1$ for $j = 1, \ldots, K$. Furthermore, when L_j values are integers, suppliers are switched only at the beginning of a period. Recall that $\pi_1 < \pi_j$, for $j = 2, \ldots, K$. Hence, if condition (199) is true then the operational constraint will be satisfied if $L_1 \geq 1$ so that $L \geq 1/\pi_1$ and L_j, $j = 1, \ldots, K$ are all integers. Part (b) of Proposition 12 shows that CCP-T also leads to bullwhip effect. Furthermore, when $L_j = L\pi_j$, we have

$$\beta_{2j} = \frac{1}{\sqrt{\pi_j}} > 1.$$

One interesting and non-intuitive observation is that β_{2j} depends only on the ratio L/L_j and is independent of μ, σ and $CV[D]$. Part (c) of Proposition 12 provides an expression to compute γ_2 and shows that γ_2 increases in K and decreases in L. More importantly, part (c) of Proposition 12 demonstrate the inherent flexibility of CCP-T, which allows it to meet the buyer's target for the expected long-run average number of supplier switches, denoted by γ_0. In particular, given K and γ_o, part (c) of Proposition 12 shows that L can be determined as $L(\gamma_0) = K/\gamma_0$. Furthermore, the result in Corollary 5 follows directly from Proposition 12.

Corollary 5. *β_{2j} is a convex decreasing function of L/L_j.*

Next, in Corollary 6, we set $L_j = L\pi_j$ and examine the trade-off between the bullwhip effect observed by the suppliers and the buyer's expected long-run average number of supplier switches for different combination of π_1 and π_2 under dual sourcing, i.e., $K = 2$.

Corollary 6. *The following are true under dual-sourcing, i.e., $K = 2$, and CCP-T with parameters $L_j = L\pi_j$ for $j = 1, 2$:*

(a) $\beta_{21} \geq \beta_{22}$,

(b) As π_1 increases, β_{21} decreases while β_{22} increases such that the rate of decrease in β_{21} is greater than the rate of increase in β_{22}. When $\pi_1 = \pi_2 = 0.5$, $\beta_{21} = \beta_{22} = \sqrt{K}$.

(c) γ_2 is a linear increasing function of π_1, and a linear decreasing function of L.

Proof. Suppose that $L_j = L\pi_j$ for $j = 1, 2$. Parts (a) and (b) follow directly using Corollary 5 and following the same steps as in the proof of Corollary 4. Using part (c) of Proposition

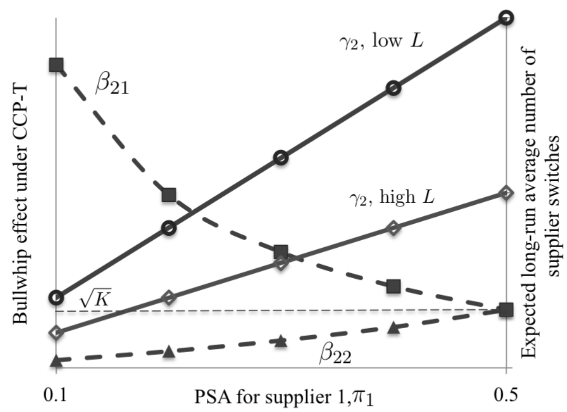


Figure 19: β_{21}, β_{22} and γ_2 with respect to π_1 for $K = 2$

11, we have $\gamma_2 = K/L = K\pi_j/L_j$. Thus, γ_2 is a linear increasing function of π_1, and a linear decreasing function of L. □

Figure 19 illustrates the effect of π_1 on β_{21}, β_{22} and γ_2 when $K = 2$. It shows that as π_1 increases, β_{21} decreases while β_{22} increases such that the rate of decrease in β_{21} is greater than the rate of increase in β_{22}. Furthermore, it shows that as L increases γ_2 decreases. Therefore, a wise choice of L and L_j will allow the buyer to meet PSAs and its target for the expected long-run average number of supplier switches, while reducing the bullwhip effect observed by the suppliers.

IV.5 Quantity-based Cyclic Consumption Policy

When implementing a consumption cycle, the buyer may make supplier switching decisions based on quantity. More specifically, the buyer may decide to allocate a pre-defined number of units to supplier j in each cycle, $j = 1, \ldots, K$, before switching to the next supplier. We propose the following quantity-based cyclic consumption policy (CCP-Q): (1) CCP-Q is characterized by K parameters, which are $(Q_1 \ldots, Q_K)$. Similar to CCP-T, we assume that the buyer cycles through all suppliers in the $1 - 2 - \ldots - K$ sequence. (2) In each cycle, the buyer allocates at least Q_j units to supplier j in consecutive periods and

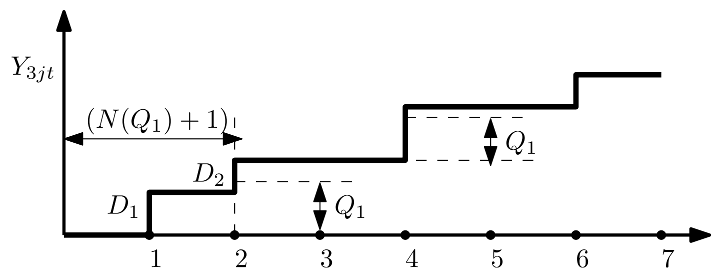


Figure 20: Cumulative demand process for an arbitrary supplier under CCP-Q

switches to supplier $j+1$ after the period in which the total number of units allocated to supplier j in that cycle reaches or exceeds Q_j units for the first time, $j = 1, \ldots, K$.

We model CCP-Q by introducing additional notation. We recall equation (176) and define

$$N(d) = \sup\{n : Y_n < d\}, \qquad \text{where } N(0) = 0. \tag{200}$$

That is, $N(d)$ is a renewal process that counts the maximum number of periods such that the buyer's total demand is up to d units. We denote the associated renewal function as $M(d) = E[N(d)]$. Hence, the total number of periods during which demand is allocated to supplier j in each cycle is equal to $N(Q_j) + 1$. Since $N(d) \geq 0$, it follows that any $Q_j \geq 0$ will satisfy the operational requirement and limit switching suppliers to a maximum of one per period.

We illustrate CCP-Q using an example. Suppose $K = 2$ and the buyer uses a CCP-Q with parameters $Q_1 = Q_2 = 50$. Consider the first consumption cycle, which starts in period $t = 1$. The buyer observes d_1, the realized value of D_1, allocates it to supplier 1. Suppose that $d_1 = 40 < Q_1$, then the buyer does not switch suppliers and D_2 is also allocated to supplier 1. Suppose that $d_2 = 30$, i.e., $d_1 + d_2 > Q_1$. The buyer has now exceeded Q_1 units quota for supplier 1 for the first time. Hence, the buyer switches to supplier 2, and allocates D_3 to supplier 2. Suppose $D_3 = 55 > Q_2$, i.e., the buyer has now allocated more than

Q_2 units to supplier 2. Therefore, the first consumption cycle ends. The same process is repeated during each consumption cycle. In this example, for the first cycle $N(Q_1) = 1$ and $N(Q_2) = 0$. As a result, demand was allocated to supplier 1 for 2 periods and to supplier 2 for 1 period. Figure 20 illustrates supplier 1's demand process for this example.

We note that that all quantity-based consumption cycles are i.i.d. Let L_q be the length of a quantity-based consumption cycle, then

$$L_q = \sum_{j=1}^{K}(N(Q_j) + 1). \tag{201}$$

In addition, the demand allocated to each supplier in a consumption cycle is also random. Let $\mathbf{Q} = [Q_1, Q_2, \ldots, Q_K]$ be a vector of size K. Proposition 13 provides performance measures of CCP-Q with parameters $\mathbf{Q}$.

Proposition 13. *For $j = 1, \ldots, K$, the following are true for CCP-Q with parameters $\mathbf{Q}$:*

$$(a) \quad \alpha_{3j}(\mathbf{Q}) = \frac{M(Q_j) + 1}{\sum_{i=1}^{K}(M(Q_i) + 1)},$$

$$(b) \quad \beta_{3j}(\mathbf{Q}) = \frac{\sum_{i=1}^{K}(M(Q_i) + 1)}{(M(Q_j) + 1)} \sqrt{\frac{(M(Q_j) + 1)(CV^2[D] - M(Q_j)) + H(Q_j)}{\sum_{i=1}^{K}\left((M(Q_i) + 1)(CV^2[D] - M(Q_i)) + H(Q_j)\right.}}, \text{ where}$$

$$H(Q_j) = 2\mu^{-1}\left(Q_j M(Q_j) - \int_0^{Q_j} M(q)dq\right), \tag{202}$$

$$(c) \quad \gamma_3(\mathbf{Q}) = \frac{K}{\sum_{j=1}^{K}(M(Q_j) + 1)}.$$

Proof. We first derive some useful results which will be used to prove parts (a) -(c). Let V_{jm}, $j = 1, \ldots, K$, be the total demand allocated to supplier j in the mth consumption cycle. Then, $V_m = \sum_{j=1}^{K} V_{jm}$ is the buyer's total demand in the mth consumption cycle. Let $K_3'(d)$ be the supplier to whom the buyer allocates demand when the buyer's cumulative demand is d. During the mth consumption cycle, $K_3'(d)$ is equal to j for a random demand-quantity equal to V_{jm}. Then, $\{K_3'(d), d \geq 0\}$ is a regenerative process with state space $\{1, 2, \ldots, K\}$. The regenerative points are the epochs at which a new consumption cycle starts. That is, $K_3'(d)$ consists of i.i.d consumption cycles. The tuples $(V_{1m}, \ldots, V_{Km}, V_m)$, $m \geq 1$, are i.i.d.

Since D_t has a finite mean $\mu < \infty$, $F(0) < 1$, we have $E[V_{jm}] < \infty$ and $E[V_m] < \infty$. Furthermore,

$$V_{jm} \sim \sum_{t=1}^{N(Q_j)+1} D_t,$$

i.e., V_{jm} follows the same distribution as that of $\sum_{t=1}^{N(Q_j)+1} D_t$. Since $N(Q_j)+1$ is a stopping time for $\{D_t, t = 1, 2, \ldots\}$ [64], we have

$$E[V_{jm}] = \mu(M(Q_j) + 1). \tag{203}$$

Furthermore, using the results from [16], we have

$$E[V_{jm}^2] = E[D^2](M(Q_j) + 1) + 2\mu\Big(Q_j M(Q_j) - \int_0^{Q_j} M(q)dq\Big).$$

Using the fact that $Var[V_{jm}] = E[V_{jm}^2] - E^2[V_{jm}]$, after some algebraic simplifications we obtain

$$\begin{aligned} Var[V_{jm}] &= (M(Q_j) + 1)(\sigma^2 - \mu^2 M(Q_j)) + 2\mu\Big(Q_j M(Q_j) - \int_0^{Q_j} M(q)dq\Big) \\ &= \mu^2\big((M(Q_j) + 1)(CV^2[D] - M(Q_j)) + H(Q_j)\big), \end{aligned} \tag{204}$$

where $H(Q_j)$ is as defined in (202). Furthermore, using the fact that $V_m = \sum_{j=1}^{K} V_{jm}$ and V_{jm}, $j = 1, \ldots, K$, are independent, we have

$$E[V_m] = \sum_{j=1}^{K} E[V_{jm}], \tag{205}$$

$$Var[V_m] = \sum_{j=1}^{K} Var[V_{jm}]. \tag{206}$$

(a) Using the renewal reward theorem [64],α_{3j} can be written as

$$\alpha_{3j}(Q) = \frac{E[V_{jm}]}{E[V_m]} = \frac{M(Q_j)+1}{\sum_{j=1}^{K}(M(Q_j)+1)}.$$

(b) Similar to CCP-T, we define the bullwhip effect under CCP-Q as follows

$$\beta_{3j}(Q) = \frac{CV[V_{jm}]}{CV[V_m]} = \frac{E[V_m]}{E[V_{jm}]}\sqrt{\frac{Var[V_{jm}]}{Var[V_m]}} = \frac{\sum_{j=1}^{K} E[V_{jm}]}{E[V_{jm}]}\sqrt{\frac{Var[V_{jm}]}{\sum_{j=1}^{K} Var[V_{jm}]}}, \tag{207}$$

where we substituted for $E[V_m]$ and $Var[V_m]$ using equations (205) and (206). The result follows by using equations (203) and (204) on the right-hand side of (207).

(c) During each cycle, the buyer switches suppliers K times. Then, using equation (201), γ_3 can be written as

$$\begin{aligned}\gamma_3(Q) &= \frac{\text{number of supplier switches per cycle}}{E[\text{length of consumption cycle}]}\\ &= \frac{K}{E[\sum_{j=1}^{K}(N(Q_j)+1)]} = \frac{K}{\sum_{j=1}^{K}(M(Q_j)+1)}.\end{aligned}$$

□

From part (a) of Proposition 13, we observe that the buyer can meet PSAs in the long-run (i.e., $\alpha_{3j} = \pi_j$) if Q_j, $j = 1, \ldots, K$, satisfy certain conditions. Corollary 7 derives these conditions. From part (b) of Proposition 13, we observe that β_{3j} depends on: (i) the parameters of the buyer's demand distribution: μ, σ and $M(\cdot)$, which depend on $F(\cdot)$, and (ii) the parameters of CCP-Q, $(Q_1, \ldots, Q_K)$, $j = 1, \ldots, K$. Part (c) of Proposition 13 provides an expression to compute γ_3.

Corollary 7. *If Q_j, $j = 1, \ldots, K$, are such that*

$$\pi_1(M(Q_j)+1) = \pi_j(M(Q_1)+1), \tag{208}$$

then the following are true for CCP-Q:

(a) $\alpha_{3j}(p, Q_1) = \pi_j$,

(b) $\beta_{3j}(p, \mathbf{Q})) = \dfrac{1}{\pi_j}\sqrt{\dfrac{(M(Q_j)+1)(CV^2[D]-M(Q_j))+H(Q_j)}{\sum_{i=1}^{K}\big((M(Q_i)+1)(CV^2[D]-M(Q_i))+H(Q_j)\big)}}$,

(c) $\gamma_3(p, Q_1) = \dfrac{K\pi_1}{M(Q_1)+1}$.

Proof. Suppose that Q_j, $j = 1, \ldots, K$, satisfy equation (208). Using part (a) of Proposition 13, equation (208) and the fact that $\sum_{j=1}^{K}\pi_j = 1$, we obtain

$$\alpha_{3j}(p, Q_1) = \frac{\pi_j(M(Q_1)+1)}{\pi_1(M(Q_1)+1)\sum_{j=1}^{K}\pi_j/\pi_1} = \pi_j.$$

Similarly, part (b) follows by using part (b) of Proposition 13 and equation (208). Using part (c) of Proposition 13, equation (208) and the fact that $\sum_{j=1}^{K}\pi_j = 1$, we obtain

$$\gamma_3(p, Q_1) = \frac{K}{(M(Q_1)+1)\sum_{j=1}^{K}\pi_j/\pi_1} = \frac{K\pi_1}{M(Q_1)+1}.$$

□

Part (a) of Corollary 7 shows that if condition (208) is satisfied, then the expected long-run fraction of total demand allocated to supplier j is equal to π_j, $j = 1, \ldots, K$. Thus, in order to meet PSAs, the buyer needs to choose Q_j for any one of the suppliers, say supplier 1, and compute Q_j for other suppliers by solving equation (208). Corollary 7 also provides expressions for β_{3j} and γ_3 when condition (208) is satisfied.

Corollary 8. *When condition (208) is satisfied, the following are true for CCP-Q with parameters* $(Q_1, \ldots, Q_K)$*:*

(a) $$\frac{d\beta_{3j}}{d\mu} = \frac{\sum_{i=1}^{K}\big(Var[V_{im}]dVar[V_{jm}]/d\mu - Var[V_{jm}]dVar[V_{im}]/d\mu\big)}{2\pi_j^2\beta_{3j}\big(Var[V_{jm}]\big)^2},$$

(b) $$\frac{d\beta_{3j}}{dQ_1} = \frac{\sum_{i=1}^{K}\big(Var[V_{im}]dVar[V_{jm}]/dQ_1 - Var[V_{jm}]dVar[V_{im}]/dQ_1\big)}{2\pi_j^2\beta_{3j}\big(Var[V_{jm}]\big)^2},$$

$$(c) \quad \frac{d\beta_{3j}}{dCV[D]} = \frac{2\mu^2 CV[D](M(Q_1)+1)\big(\sum_{i=1}^{K} Var[V_{im}]\pi_j - Var[V_{jm}]\big)}{\pi_j^2 \beta_{3j}\big(Var[V_{jm}]\big)^2}, \quad where$$

$$\frac{dVar[V_{jm}]}{d\mu} = -2\mu M(Q_j)\big(M(Q_j)+1\big) + 2\mu H(Q_j), \tag{209}$$

$$\frac{dVar[V_{jm}]}{dQ_1} = \frac{\mu^2 \pi_j}{\pi_1} \frac{dM(Q_1)}{dQ_1}\left(CV^2[D] + \frac{2Q_j}{\mu} - \frac{2\pi_j}{\pi_1}(M(Q_1)+1) - 1\right). \tag{210}$$

Proof. When condition (208) is satisfied, using Corollary 7 and equation (207), we have

$$\beta_{3j}(Q) = \frac{1}{\pi_j}\sqrt{\frac{Var[V_{jm}]}{\sum_{j=1}^{K} Var[V_{jm}]}}.$$

Then, parts (a), (b) and equation (209) follow directly by taking the first order derivative of β_{3j} and $Var[V_{jm}]$ with respect to μ and Q_1, respectively. Differentiating both sides of (208) with respect to Q_1 and reorganizing the resulting expression, we have

$$\frac{dM(Q_j)}{dQ_1} = \frac{\pi_j}{\pi_j}\frac{dM(Q_1)}{dQ_1}. \tag{211}$$

Taking the derivative of $Var[V_{jm}]$ with respect to Q_1, and using equations (211) and (208) in the resulting expression, we have

$$\begin{aligned}\frac{dVar[V_{jm}]}{dQ_1} &= \mu^2 \frac{dM(Q_j)}{dQ_1}\left(CV^2[D] - M(Q_j) - M(Q_j) - 1 + \frac{2Q_j}{\mu}\right)\\ &= \frac{\mu^2 \pi_j}{\pi_1}\frac{dM(Q_1)}{dQ_1}\left(CV^2[D] + \frac{2Q_j}{\mu} - \frac{2\pi_j}{\pi_1}(M(Q_1)+1) - 1\right).\end{aligned}$$

Furthermore, taking the derivative of $Var[V_{jm}]$ with respect to $CV[D]$, we have

$$\frac{dVar[V_{jm}]}{dCV[D]} = \frac{2\mu^2 \pi_j (M(Q_1)+1)CV[D]}{\pi_1}. \tag{212}$$

Taking the derivative of β_{3j} using equation (212) in the resulting expression, we have

$$\begin{aligned}\frac{d\beta_{3j}}{dCV[D]} &= \frac{2\mu^2 CV[D](M(Q_1)+1)\big(\sum_{i=1}^{K} Var[V_{im}]\pi_j - Var[V_{jm}]p_i\big)}{\pi_j^2\beta_{3j}\big(Var[V_{jm}]\big)^2},\\ &= \frac{2\mu^2 CV[D](M(Q_1)+1)\big(\sum_{i=1}^{K} Var[V_{im}]\pi_j - Var[V_{jm}]\big)}{\pi_j^2\beta_{3j}\big(Var[V_{jm}]\big)^2}.\end{aligned}$$

□

Based on the results presented so far, we observe that in contrast to RAP and CCP-T, the expressions for the performance measures of CCP-Q are less amenable to intuition. Hence, to gain additional insights, we investigate performance of CCP-Q for two special cases. Corollary 9 investigates the case where $Q_j = q \geq 0$ for $j = 1, 2, \ldots, K$, and Corollary 10 investigates the case where the buyer's demand is exponentially distributed. Furthermore, in Section IV.5.1 we study the effect of model parameters on β_{3j} using an extensive computational study.

Corollary 9. *If $Q_j = q \geq 0$ for $j = 1, 2, \ldots, K$, then (a) $\alpha_{3j} = 1/K$, (b) $\beta_{3j} = \sqrt{K}$, and (c) $\gamma_3(q) = 1/(M(q)+1)$.*

Proof. Suppose that $Q_j = q$ for $j = 1, \ldots, K$. Then, the results follow directly by substituting $Q_j = q$ for $j = 1, \ldots, K$ in parts (a)-(c) of Proposition 13, and simplifying the resulting expressions. □

Corollary 9 verifies the intuitive notion that if $Q_j = q$ for all suppliers, then the expected fraction of demand allocated to each supplier is equal to $1/K$. That is, in the long-run, buyer's total demand is equally allocated among all K suppliers. An interesting and less obvious result is that if $Q_j = q$ for $j = 1, 2, \ldots, K$, then $\beta_{3j} = \sqrt{K}$ for $j = 1, 2, \ldots, K$ independent of the parameters of the buyer's demand distribution as well as the choice of q. As a result, the buyer can choose q to its benefit, without amplifying the bullwhip effect observed by its suppliers. For example, the buyer can choose a target value for the expected long-run average number of supplier switches and use part (c) of Corollary (9) to solve for

q.

Corollary 10. *If the buyer's demand is exponentially distributed, i.e.,* $F(d) = 1 - e^{-d/\mu}$ *and* Q_j*,* $j = 1, \ldots, K$*, are such that*

$$Q_j = \frac{(\pi_j - \pi_1)\mu + \pi_j Q_1}{\pi_1}, \tag{213}$$

then (a) $\alpha_{3j}(p, Q_1) = \pi_j$*, (b)* $\beta_{3j}(p, Q) = (\pi_j \sqrt{K})^{-1}$*, and (c)* $\gamma_3(p, Q_1) = K\pi_1\mu/(Q_1 + \mu)$*.*

Proof. Suppose that $F(d) = 1 - e^{-d/\mu}$, for $d \geq 0$, then we know that $N(d)$ is a Poisson process with mean d/μ. Therefore, $M(d) = Var[N(d)] = d/\mu$ for $d \geq 0$ and condition (208) simplifies as follows:

$$\pi_1\left(\frac{Q_j}{\mu} + 1\right) = \pi_j\left(\frac{Q_1}{\mu} + 1\right) \Rightarrow \quad Q_j \quad = \frac{(\pi_j - \pi_1)\mu + \pi_j Q_1}{\pi_1} = \frac{\pi_j(\mu + Q_1) - \pi_1\mu}{\pi_1}. \tag{214}$$

Then, part (a) follows directly from Corollary 7. To prove part (b), we note that when demand is exponentially distributed, $\mu = \sigma$. Substituting this and the fact that $M(Q_j) = Var[N(Q_j)] = Q_j/\mu$ in expression (204), we have

$$\begin{aligned} Var[V_{jm}] &= \left(\frac{Q_j}{\mu} + 1\right)\left(\mu^2 - \mu^2\frac{Q_j}{\mu}\right) + 2\mu\left(\frac{Q_j^2}{\mu} - \int_0^{Q_j} \frac{q}{\mu} dq\right) \\ &= \frac{Q_j + \mu}{\mu}(\mu^2 - \mu Q_j) + Q_j^2 = \mu^2. \end{aligned} \tag{215}$$

Using equations (208) and (215) on the right hand side of (207), we have

$$\beta_{3j}(p, Q) = \frac{(M(Q_1) + 1)\sum_{j=1}^K \pi_j/\pi_1}{(M(Q_1) + 1)\pi_j/\pi_1} \sqrt{\frac{\mu^2}{\sum_{j=1}^K \mu^2}} = \frac{1}{\pi_j\sqrt{K}}.$$

Part (c) follows by using $M(Q_j) = Q_j/\mu$ and part (c) of Corollary 7. □

For the special case where buyer's demand is exponentially distributed, Corollary 10 provides a simple expression (213) to compute Q_j for $j = 2, \ldots, K$ for a given value of Q_1 so that PSAs are met in the long-run. Interestingly, part (b) of Corollary 10 shows that the bullwhip ratio under CCP-Q can be less than 1, i.e., CCP-Q does not always lead to bullwhip effect. In fact, we have

$$\beta_{3j} \begin{cases} \leq 1 & \text{if } \pi_j \geq 1/\sqrt{K} \\ > 1 & \text{if } \pi_j < 1/\sqrt{K}. \end{cases} \tag{216}$$

Thus, when $\pi_j \geq 1/\sqrt{K}$, supplier j will prefer CCP-Q to RAP and CCP-T, which lead to bullwhip effect. We refer the reader to Cachon et al. [12] for practical examples of cases where the bullwhip ratio is less than 1. Part (b) of Corollary 10 also shows that β_{3j} depends only on π_j and K, and is independent of Q_1. Part (c) of Corollary 10 provides an expression for γ_3. It also shows that γ_3 is an increasing function of K, μ, π_1 and a decreasing function of Q_1. More importantly, part (c) of Corollary 10 demonstrates the inherent flexibility of CCP-Q, which allows it to meet the buyer's target for the expected long-run average number of supplier switches, denoted by γ_0. In particular, given γ_0, part (c) of Proposition 12 shows that Q_1 can be determined as

$$Q_1 = \frac{Kc\pi_1\mu}{\gamma_0} - \mu.$$

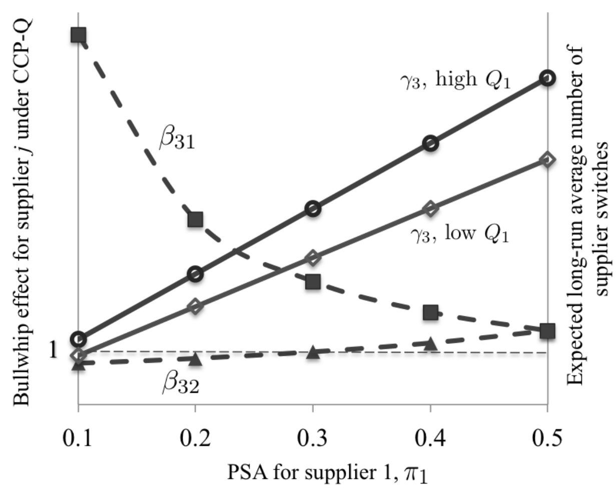


Figure 21: β_{31}, β_{32} and γ_3 with respect to $\pi_1 = 1 - \pi_2$ for $K = 2$

Figure 21 graphically illustrates the results in Corollary 10. It shows that for $K = 2$ as π_1 increases, β_{31} decreases while β_{32} increases such that the rate of decrease in β_{31} is greater than the rate of decrease in β_{32}. Furthermore, it shows that as Q_1 increases γ_3 decreases. Therefore, the buyer can manage its expected long-run average number of supplier switches by an appropriate choice of Q_1. That is, a wise choice of Q_1 will allow the buyer to meet PSAs and its target for the expected long-run average number of supplier switches, while reducing the bullwhip effect observed by the suppliers.

IV.5.1 Sensitivity of bullwhip ratio under CCP-Q to model parameters

In this section, we numerically investigate the sensitivity of the performance of CCP-Q to model parameters. For this purpose, we assume that condition (208) is satisfied, and hence, $\alpha_{3j} = \pi_j$ and β_{3j} as given by Corollary 7.

We assume that the buyer's demand follows gamma distribution, which is defined only for non-negative values and commonly encountered in practice (e.g., see [48]). For our numerical study, we design an experimental setup by considering the parameter values summarized in Table 19. We consider eight different values of μ and seven different values of $CV[D]$, which in turn imply a wide range of distribution shapes. As the number of suppliers increases, the buyer's administrative and contract management costs increase linearly. Hence, in practice buyers contract with up to five preferred suppliers (e.g., see [3]), allocating at least 10% of their demand to each supplier. As a result, we consider a maximum of five suppliers and only investigate those cases where $\pi_j \geq 0.1$ for $j = 1, \ldots, K$. In addition, we ignore the problem instances where $\pi_j = 1/K$ for $j = 1, \ldots, K$ since analytical results for this case are provided in Corollary 9. There are 48 combinations of PSAs where $\pi_j \geq 0.1$ and $\pi_j \neq 1/K$ for

μ	$CV[D]$	K	π_j/π_1	$w = Q_1/\mu$
50, 100, …, 6400	0.5, 0.75, …, 2	2, 3,4, 5	1, 2, 3, 4, 5	0.25, 0.5, …, 2

Table 19: Experimental setup to investigate the sensitivity of performance of CCP-Q to model parameters

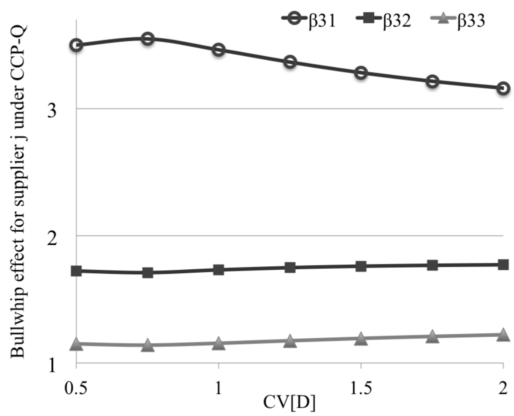


Figure 22: β_{3j} with respect to $CV[D]$: $\mu = 400$, $K = 3$, $\pi_1 = 1/6$, $\pi_2 = 1/3$, $\pi_3 = 1/2$, $w = 1$

$j = 1, \ldots, K$. We consider eight different values of Q_1, such that $w = Q_1/\mu = 0.25, 0.5, \ldots, 2$. Thus, we study a total of $8 \times 7 \times 48 \times 8 = 21,504$ problem instances.

We list our observations as follows:

1. β_{3j} is independent of μ.

2. When $K = 2$ the behavior of β_{3j}, $j = 1, 2$ with respect to π_1 is similar to the one observed in Figure 21.

3. As can be seen from Figure 22 for a representative set of parameter values, if $\pi_j < 1/K$, then β_{3j} is a concave function of $CV[D]$. Otherwise, β_{3j} is a convex function of $CV[D]$.

4. The behavior of β_{3j} with respect to w, and hence, Q_1 depends on π_j and $CV[D]$ as can be seen from Figure 23 for a representative setting. We note when $CV[D] = 1.0$, the gamma distribution with mean μ reduces to the exponential distribution with mean μ. Furthermore, from Corollary 10, we recall that when the buyer's demand follows exponential distribution $\beta_{3j} = (\pi_j\sqrt{K})^{-1}$ is independent of Q_1. When the buyer's demand follows the gamma distribution, we find that $\pi_j < 1/K$ and $CV[D] < 1.0$ ($CV[D] > 1.0$), then β_{3j} decreases convexly (increases concavely) and converges to $(\pi_j\sqrt{K})^{-1}$. In contrast, if $\pi_j \geq 1/K$ and $CV[V] < 1.0$ ($CV[D] > 1.0$), then β_{3j}

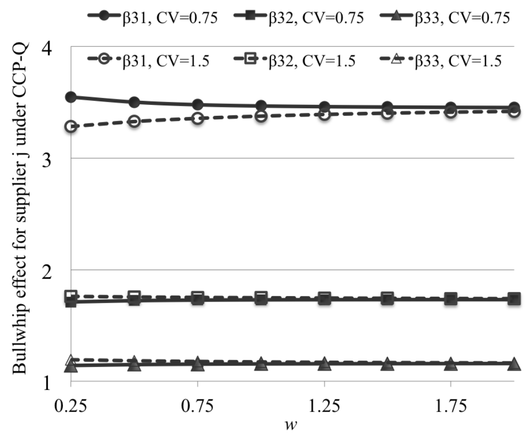


Figure 23: β_{3j} with respect to w: $\mu = 400$, $K = 3$, $\pi_1 = 1/6$, $\pi_2 = 1/3$, $\pi_3 = 1/2$

increases concavely (decreases convexly) and converges to $(\pi_j\sqrt{K})^{-1}$. In addition, we note that the magnitude of variation in β_{3j} with respect to w, and hence, Q_1 is so small that it may seem to be independent of Q_1, especially when $\pi_j \geq 1/K$.

5. In general, when π_j is fixed, β_{3j} decreases with respect to K, and the specific values depend on PSAs of other suppliers. Table 20 illustrates how β_{3j} changes when $\pi_1 = 0.2$ and K varies from 2 to 5. For example, as K increases from 2 to 3, β_{31} may decrease from 3.64 to 3.00 or 2.94, depending on PSAs of other suppliers when $K = 3$. This shows that as long as π_j stays the same, supplier j benefits if the buyer diversifies its supply base. That is, CCP-Q corroborates the pro-competitive effects of a contract

K	π_1	π_2	π_3	π_3	π_5	β_{31}
2	0.20	0.80				3.64
3	0.20	0.40	0.40			3.00
3	0.20	0.20	0.60			2.94
4	0.20	0.20	0.20	0.40		2.54
5	0.20	0.20	0.20	0.20	0.20	2.20

Table 20: β_{31} with respect to K: $\mu = 400$, $CV = 0.75$ and $w = 1$.

with PSAs. In contrast, Sections IV.3 and IV.4 show that the bullwhip effects under RAP and CCP-T are independent of K and PSAs of other suppliers.

IV.6 Comparison of Demand Allocation Policies

In this section, we compare the three demand allocation policies from the perspective of the buyer and the suppliers based on the three performance measures we defined in section IV.2.2. Clearly, the buyer prefers the policy with the lowest expected long-run average number of supplier switches, and each supplier prefers the policy that results in the lowest bullwhip effect for that supplier. To facilitate the comparison of the policies, we consider the following:

A1 Parameters for each demand allocation policy satisfy the respective conditions to meet the PSAs, i.e., $r_j = \pi_j$, $L_j = L\pi_j$ and Q_j satisfy condition (208) for $j = 1, \ldots, K$.

A2 Each policy limits the maximum number of supplier switches to one per period, i.e., $L \geq 1/\pi_1$. Alternatively, if $v = L\pi_1$, then $v \geq 1$.

A3 Lengths of the time-based and quantity-based consumption cycles are equal. From A1 and A2 and equation (201), we have

$$L = E[L_q] = E\Bigg[\sum_{j=1}^{K}(N(Q_j)+1)\Bigg] = \frac{(M(Q_1)+1)\sum_{j=1}^{K}\pi_j}{\pi_1} = \frac{M(Q_1)+1}{\pi_1}.$$

That is, $M(Q_1) = v - 1 \geq 0$.

From A1 it follows that $\alpha_{ij} = \pi_j$ for $i = 1, 2, 3$ and $j = 1, \ldots, K$. Hence, we compare the policies based on the bullwhip ratios and the expected long-run average number of supplier switches. Proposition 14 compares the performance of RAP and CCP-T. Propositions 15 and 16 compare the three demand allocation policies for special cases.

Proposition 14. *When conditions A1-A3 are met, RAP and CCP-T compare as follows for $j = 1, \ldots, K$: (a) $\beta_{1j} > \beta_{2j}$, and (b) if $L \geq L_{min}$, where L_{min} is such that*

$$L_{min} = \frac{K}{1 - \sum_{j=1}^{K}\pi_j^2}, \tag{217}$$

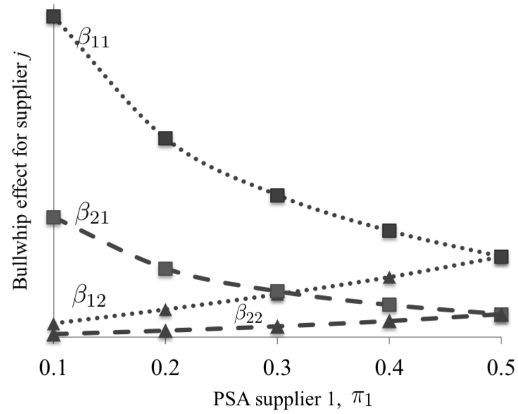


Figure 24: The bullwhip effect under RAP and CCP-T with respect to π_1 for $K = 2$

then $\gamma_1 \geq \gamma_2$*, else* $\gamma_1 < \gamma_2$.

Proof. (a) Using $r_j = \pi_j$ in part (b) of Proposition 11, we obtain $\beta_{1j} \geq 1/\sqrt{\pi_j}$. Similarly, substituting $L_j = L\pi_j$ in part (b) of Proposition 12, we obtain $\beta_{2j} = 1/\sqrt{\pi_j}$, and the result follows directly from this.

(b) Using $r_j = \pi_j$ in part (c) of Proposition 11, we have $\gamma_1 = 1 - \sum_{j=1}^{K} \pi_j^2$. From part (c) of Proposition 12, we have $\gamma_2 = K/L$. Hence, if $L > L_{min}$, where $L_{min} = K(1 - \sum_{j=1}^{K} \pi_j^2)^{-1}$, then $\gamma_1 \geq \gamma_2$, else $\gamma_1 < \gamma_2$. □

Part (a) of Proposition 14 shows that the bullwhip effect under CCP-T is always lower than that under RAP. Figure 24 illustrates this graphically, and provides a detailed comparison of RAP and CCP-T with respect to π_1 for $K = 2$. It shows that the bullwhip effect under CCP-T is significantly lower than the bullwhip effect under RAP for the minor supplier with $\pi_1 \leq 0.5$. Part (b) of Proposition 14 shows that when $L \geq L_{min}$, we have $\gamma_1 \geq \gamma_2$. Figure 25 shows that for $K = 2$, L_{min} is a convex decreasing function of π_1. Thus, Proposition 14 shows that for $L \geq L_{min}$, both the expected long-run average number of supplier switches and the bullwhip effect under CCP-T will be lower than those under RAP.

Proposition 15 extends this analysis for comparison of all three policies when PSAs are equally distributed among the suppliers.

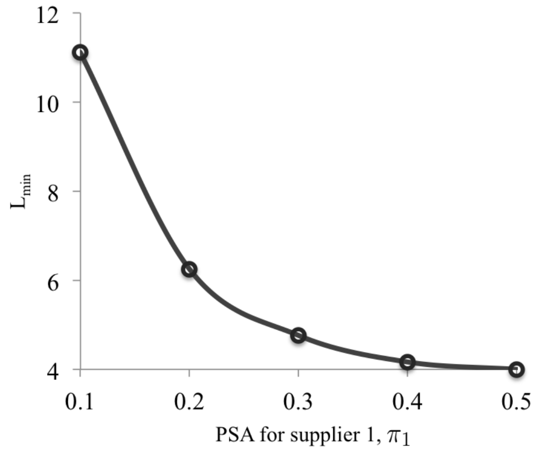


Figure 25: L_{min} with respect to π_1 for $K = 2$

Proposition 15. *When conditions A1-A3 are met and $\pi_j = 1/K$ for $j = 1, 2, \ldots, K$, then*

(a) $\beta_{1j} = \sqrt{K + \dfrac{K-1}{CV^2[D]}} > \beta_{2j} = \beta_{3j} = \sqrt{K} > 1.$

(b) If $L = E[L_q] \geq \bar{L}_{min}$ and $Q_1 \geq \bar{Q}_{1,min}$ then $\gamma_1 \geq \gamma_2 = \gamma_3$, where $\bar{L}_{min} = \dfrac{K^2}{K-1}$ and $\bar{Q}_{1,min} = \dfrac{1}{K-1}$. Else $\gamma_1 < \gamma_2 = \gamma_3$.

Proof. (a) Using Corollary 9, we have $\beta_{3j} = \sqrt{K} = 1/\sqrt{\pi_j}$ for $j = 1, \ldots, K$. From the proof for part (a) of Proposition 14, we know that $\beta_{1j} \geq \beta_{2j} = 1/\sqrt{\pi_j}$. It follows that $\beta_{1j} \geq \beta_{2j} = \beta_{3j}$.

(b) Evaluating γ_i as given by parts (c) of Proposition 11, Proposition 12, and Corollary 7 at $r_j = \pi_j = 1/K$, and $L = E[L_q] = (M(Q_1) + 1)/\pi_1$ for $j = 1, \ldots, K$, and simplifying the resulting expression we have: (i) $\gamma_1 = (K-1)/K$, (ii) $\gamma_2 = \gamma_3 = K/L$. Define $\bar{L}_{min} = K^2/(K-1)$ and $\bar{Q}_{1,min}$ such that $\bar{L}_{min} = (M(\bar{Q}_{1,min}) + 1)/\pi_1$, then $\bar{Q}_{1,min} = 1/(K-1)$. Furthermore, from (i) and (ii) above, we see that if $L \geq \bar{L}_{min}$ and $Q_1 \geq \bar{Q}_{1,min}$ then $\gamma_1 \geq \gamma_2 = \gamma_3$, else $\gamma_1 < \gamma_2 = \gamma_3$. □

Proposition 15 shows that when PSAs are equally distributed among all the suppliers, CCP-T and CCP-Q result in the same performance. Part (b) of Proposition 14 shows that

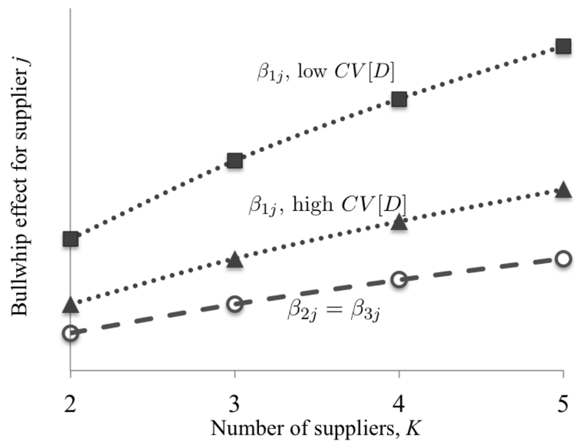


Figure 26: β_{ij} with respect to K when $\pi_j = 1/K$ for $j = 1, \ldots, K$

for $L \geq \bar{L}_{min}$ and $Q_1 \geq \bar{Q}_{1,min}$, both the expected long-run average number of supplier switches and the bullwhip effect under CCP-T as well as CCP-Q will be lower than those under RAP. Figure 26 graphically illustrates the increase in β_{1j} and $\beta_{2j} = \beta_{3j}$ with respect to K when PSAs are equally distributed among all the suppliers. As a result, the less is the number of suppliers that the buyer has, the higher is the PSA for each suppliers, and hence, the smaller is the bullwhip effect observed by each supplier. Next, we compare the three policies for the special case where the buyer's demand is exponentially distributed.

Proposition 16. *When conditions A1-A3 are met and the buyer's demand is exponentially distributed, then for $j = 1, \ldots, K$ we have*

(a) If $\pi_j < 1/K$, and (i) if $\pi_j K(2 - \pi_j) \leq 1$ then $\beta_{3j} \geq \beta_{1j} > \beta_{2j}$, (ii) if $\pi_j K(2 - \pi_j) > 1$ then $\beta_{1j} > \beta_{3j} > \beta_{2j}$. On the other hand, if $\pi_j \geq 1/K$, then $\beta_{1j} > \beta_{2j} > \beta_{3j}$.

(b) If $L = E[L_q] \geq L_{min}$ where L_{min} is as defined in equation (217) then $\gamma_1 > \gamma_2 = \gamma_3$, else $\gamma_1 < \gamma_2 = \gamma_3$. Furthermore, $E[L_q] \geq L_{min}$ is true when $Q_1 \geq Q_{1,min}$ and

$$Q_{1,min} = \frac{K\pi_1\mu}{1 - \sum_{j=1}^{K} \pi_j^2} - \mu. \tag{218}$$

Proof. Suppose that conditions A1-A3 are met and the buyer's demand is exponentially distributed. Then $CV = 1$ and from Proposition 11, for $r_j = \pi_j$ we have $\beta_{1j} = \sqrt{(2-\pi_j)/\pi_j}$. From Corollary 10, we have $\beta_{3j} = 1/(\pi_j\sqrt{K})$. If $\pi_j < 1/K$, then we have

$$\beta_{3j} = \frac{1}{\pi_j\sqrt{K}} \geq \frac{\sqrt{\pi_j}}{\pi_j} = \beta_{2j}. \tag{219}$$

In addition, if $\pi_j K(2-\pi_j) \leq 1$, then

$$\beta_{1j}^2 = \frac{2-\pi_j}{\pi_j} \leq \frac{1}{\pi_j^2 K} = \beta_{3j}^2,$$

and hence, $\beta_{1j} \leq \beta_{3j}$. From part (a) of Proposition 14, we know that $\beta_{1j} \geq \beta_{2j}$ for $j = 1, \ldots, K$. Thus, it follows that if $\pi_j < 1/K$ and $\pi_j K(2-\pi_j) \leq 1$ then $\beta_{3j} \geq \beta_{1j} > \beta_{2j}$. Similarly, if $\pi_j K(2-\pi_j) \leq 1$ then $\beta_{1j} > \beta_{3j}$. Combining this with equation (219), we see that if $\pi_j < 1/K$ and $\pi_j K(2-\pi_j) > 1$ then $\beta_{1j} > \beta_{3j} > \beta_{2j}$. On the other hand, if $\pi_j \geq 1/K$, then following the same steps as above we can show that $\beta_{1j} > \beta_{2j} > \beta_{3j}$. Part (b) follows from part (b) of Proposition 14 and using $L = E[L_q] = (M(Q_1)+1)/\pi_1$ from A3. □

Part (a) of Proposition 16 ranks RAP, CCP-T and CCP-Q based on the bullwhip ratio and specific conditions on π_j and K when the buyer's demand is exponentially distributed. The key insights of part (a) are: (i) If $\pi_j < 1/K$, then CCP-T results in the lowest bullwhip ratio, which is greater than 1 based on the results in Proposition 12. (ii) If $\pi_j \geq 1/K$, then CCP-Q results in the lowest bullwhip ratio, which may be greater or less than 1 as shown by Corollary 10. (iii) CCP-T never results in the highest bullwhip ratio, and (iv) RAP never results in the lowest bullwhip ratio. Hence, if $\pi_j < 1/K$, then supplier j prefers CCP-T. Otherwise, supplier j prefers CCP-Q among all the three demand allocation policies. Part (b) of Proposition 16 shows that for $L \geq L_{min}$ and $Q_1 \geq Q_{1,min}$, both the expected long-run average number of supplier switches and the bullwhip effect under CCP-T as well as CCP-Q will be lower than those under RAP.

IV.6.1 Numerical study

In this section, we compare the demand allocation policies by considering more realistic assumptions about the buyer's demand process. We consider that conditions A1-A3 are met, and hence, $\alpha_{ij} = \pi_j$ for $i = 1, 2, 3$ and $j = 1, \ldots, K$. Furthermore, we set parameters of CCP-T and CCP-Q so that $\gamma_1 = \gamma_2 = \gamma_3$. As a result, based on the expected long-run average number of supplier switches, all policies are equivalent to the buyer. Hence, we compare the three policies based on their bullwhip effect, i.e., from the supplier's perspective.

We consider the same experimental setup that is used in Section IV.5.1 and is summarized in Table 19 with the following changes: Since β_{ij}, $i = 1, 2, 3$ is independent of μ, we set $\mu = 400$. Furthermore, we consider that $Q_1 = Q_{1,min}$ and $L = L_{min}$ so that $\gamma_1 = \gamma_2 = \gamma_3$. Thus, we consider a total of $48 \times 7 = 336$ problem instances. We list our observations as follows:

O1 When $K = 2$, we have $\pi_1 \leq 0.5$ and $\pi_2 \geq 0.5$. Furthermore, as illustrated in Figure 27, when $K = 2$ and $\pi_1 < 0.5$, supplier 1 always prefers CCP-T, while supplier 2 always prefers CCP-Q. Furthermore, we observe that the bullwhip effect observed by supplier 1 is significantly lower under CCP-T than under CCP-Q. In contrast, the bullwhip effect

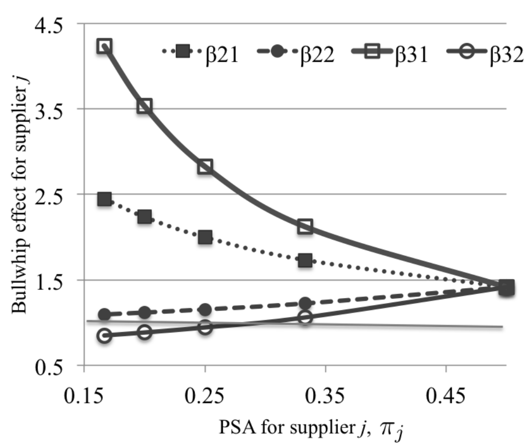


Figure 27: Comparison of the bullwhip effect for CCP-T and CCP-Q with respect to π_1 for $K = 2$, $CV[D] = 1$, $Q = \mu$

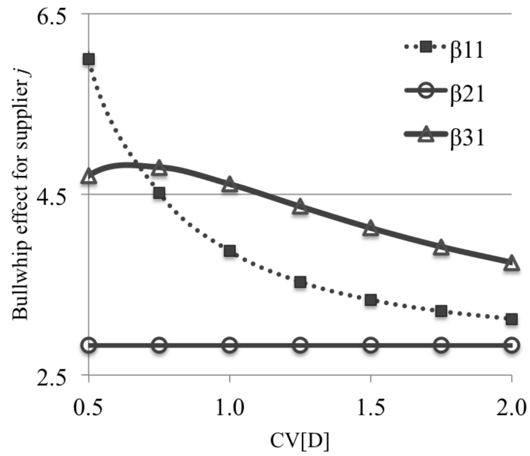


Figure 28: Comparison of β_{i1} with respect to $CV[D]$ when $\pi_1 < 1/K$ and $\pi_1 K(2-\pi_1) \leq 1$: $K = 3$, $\pi_1 = 0.125$, $\pi_2 = 0.25$, $\pi_3 = 0.625$

observed by supplier 2 is, relatively, only mildly higher under CCP-T than under CCP-Q. From Proposition 14, we already know that $\beta_{1j} > \beta_{2j}$ for $j = 1, \ldots, K$. Hence, for $K = 2$ and $\pi_1 < 0.5$, from a system perspective, CCP-T may perform better than CCP-Q. Similarly, when $K = 2$ and $\pi_1 = \pi_2 = 0.5$, perfromances of CCP-T and CCP-Q are exactly the same and better than RAP as shown in Proposition 15.

O2 For all problem instances, we observe that when $\pi_j < 1/K$ and $\pi_j K(2-\pi_j) \leq 1$, there exists a critical value u_1 such that if $CV[D] \leq u_1$ then $\beta_{1j} \geq \beta_{3j} > \beta_{2j}$, else if $CV[D] > u_1$ then $\beta_{3j} \geq \beta_{1j} > \beta_{2j}$. This is illustrated by Figure 28 for a representative setting where $K = 3$, $\pi_1 = 0.125$, $\pi_2 = 0.25$, $\pi_3 = 0.625$, and hence, $\pi_j < 1/K$ and $\pi_j K(2-\pi_j) \leq 1$ for $j = 1$. We observe that $u_1 \approx 0.70$. Furthermore, recalling the results from Section IV.5.1, it follows that β_{3j}, and hence, u_1 depends on K, and the PSAs of other suppliers.

O3 For all problem instances, we observe that when $\pi_j < 1/K$ and $\pi_j K(2-\pi_j) > 1$, there exists a critical value u_2 such that if $CV[D] \leq u_2$ then $\beta_{1j} \geq \beta_{3j} > \beta_{2j}$. Otherwise $\beta_{3j} \geq \beta_{1j} > \beta_{2j}$. This is illustrated by Figure 29 for a representative setting where

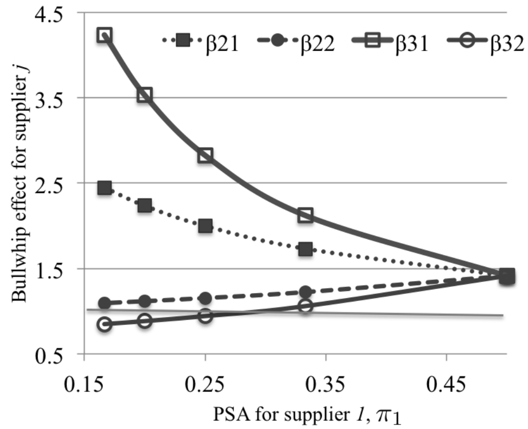


Figure 29: Comparison of β_{i2} with respect to $CV[D]$ when $\pi_2 < 1/K$ and $\pi_2 K(2-\pi_2) \leq 1$: $K = 3$, $\pi_1 = 0.125$, $\pi_2 = 0.25$, $\pi_3 = 0.625$

$K = 3$, $\pi_1 = 0.125$, $\pi_2 = 0.25$, $\pi_3 = 0.625$, and hence, $\pi_j < 1/K$ and $\pi_j K(2-\pi_j) > 1$ for $j = 2$. We observe that $u_2 \approx 1.350$. Again, u_2 depends on K, and the PSAs of other suppliers. This generalizes the results in item (ii) of part (a) of Proposition 16.

In fact, we can combine observations O2 and O3 as follows: If $\pi_j < 1/K$, there exists a critical value u such that if $CV[D] \leq u$, then $\beta_{1j} \geq \beta_{3j} > \beta_{2j}$. Otherwise $\beta_{3j} \geq \beta_{1j} > \beta_{2j}$.

O4 When $\pi_j \geq 1/K$, for each of the 384 problem instances, we observe that $\beta_{1j} \geq \beta_{2j} > \beta_{3j}$ as can be seen from Figure 30 for a representative set of parameters. This observation is consistent with part (a) of Proposition 16.

O5 Figures 28, 29 and 30 also show that the differences in the bullwhip ratios under different policies are relatively high for small values of $CV[D]$, compared to the large values of $CV[D]$. This is because (i) β_{1j} decreases with respect to $CV[D]$ as shown in Proposition 11, (ii) β_{2j} is independent of $CV[D]$ as shown in Proposition 11, and (iii) β_{3j} is either a concave or a convex function of $CV[D]$ as shown in Section IV.5.1.

In general, some suppliers will have PSAs less than $1/K$ and some others will have PSAs greater than $1/K$. As a result, there is no policy which is unanimously preferred by the buyer

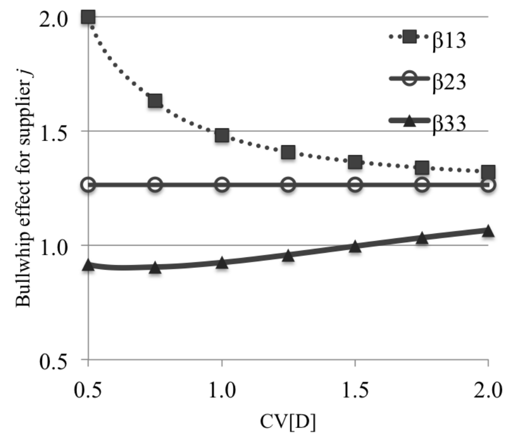


Figure 30: Comparison of β_{i3} with respect to $CV[D]$ when $\pi_3 > 1/K$: $K = 3$, $\pi_1 = 0.125$, $\pi_2 = 0.25$, $\pi_3 = 0.625$

and all the suppliers. The natural question then is: which policy is the best considering the total system consisting of the buyer and K suppliers? We address this question in the next section.

IV.7 System Cost and the Supplier's Perspective

So far, we have compared the demand allocation policies based on individual performance measures. In this section, we compare the demand allocation policies by considering the total system consisting of the buyer and K suppliers. Similar to Section IV.6.1, we consider that A1 is satisfied, and set the parameters of CCP-T and CCP-Q so that $\gamma_1 = \gamma_2 = \gamma_3$. That is, all policies meet the PSAs in the long-run and result in the same expected long-run average number of supplier switches. Consequently, all three demand allocation policies are equivalent to the buyer and we compare them by considering the system cost, i.e., the total replenishment and inventory holding costs incurred by K suppliers.

For this purpose, we consider that each supplier replenishes its VMI hub using an order-up-to policy. Furthermore, when the buyer uses RAP or CCP-T suppliers replenish the VMI hub inventory once every L periods. Similarly, when the buyer uses CCP-Q, suppliers replenish the VMI hub once every consumption cycle, for example, at the start of each consumption cycle. Let $\pi = [\pi_1, \pi_2, \ldots, \pi_K]$ be the vector of PSAs. Let $C_j(\pi_j)$ be supplier j's replenishment and inventory holding cost when its PSA is π_j and the bullwhip ratio is equal to 1, i.e., $\beta_{ij} = 1$. For example, $C_j(\pi_j)$ can be supplier j's replenishment and inventory holding cost when the buyer allocates demand to each supplier by splitting demand in each period proportional to the PSAs. Let $V_i(\beta_{ij}, \pi)$ be the total VMI hub replenishment and inventory holding cost incurred by all suppliers under policy i, $i = 1, 2, 3$, when PSAs are given by π. Then, using the results by Chen and Lee [21], we have

$$V_i(\beta_{ij}, \pi) = \sum_{j=1}^{K} C_j(\pi_j)\sqrt{\beta_{ij}}.$$

We assume that $C_j(\pi_j)$ is a linear function of π_j such that $C_j(\pi_j) = A_j\pi_j$ for $j = 1, \ldots, K$. Let $e_{ii'}$ be the percentage savings in replenishment and inventory holding costs across all suppliers due to policy i compared to policy i'. From the above discussion, Definition 2 follows immediately.

Definition 2. *If $C_j(\pi_j)$ is a linear function of π_j such that $C_j(\pi_j) = A_j\pi_j$ for $j = 1, \ldots, K$, then for $i, i' = 1, 2, 3$ we have*

$$e_{ii'} = \left(1 - \frac{\sum_{j=1}^{K} A_j\pi_j\sqrt{\beta_{ij}}}{\sum_{j=1}^{K} A_j\pi_j\sqrt{\beta_{i'j}}}\right) \times 100\%.$$

Clearly, if $e_{ii'} > 0$ then policy i results in lower costs compared to policy i'. To gain additional insights from Definition 2, we compute $e_{ii'}$ for each of the 384 problem instances of Section IV.6.1 by assuming that $A_j = A$ for $j = 1, \ldots, K$. We list our observations as follows:

O1 Our results show that both CCP-T and CCP-Q result in significant cost savings compared to RAP such that: (i) average value of e_{21} is 15.62% with a minimum of 1.87% and a maximum of 47.99%, and (ii) average value of e_{31} is 15.99% with a minimum of 1.85% and a maximum of 47.96%. Figure 31 shows the distribution of e_{21} and e_{31} over

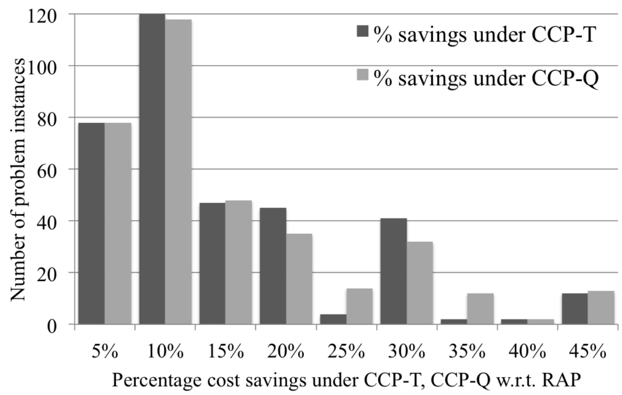


Figure 31: Distribution of cost savings under CCP-T and CCP-Q compared to RAP

384 problem instances.

O2 In addition, we find that on an average CCP-Q performs better than CCP-T, such that average value of e_{32} is 0.44% with a minimum of -0.37% and a maximum of 2.62%. Recall that CCP-T prescribes guidelines to construct a fixed schedule for switching suppliers, and is easier to implement than CCP-Q. Hence, depending on the magnitude of the actual savings CCP-T or CCP-Q may be used.

O3 Compared to RAP, the average cost savings under both CCP-T and CCP-Q decrease

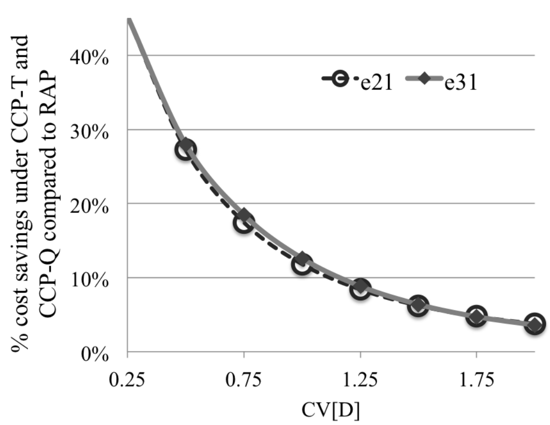


Figure 32: Average cost savings under CCP-T and CCP-Q compared to RAP with respect to $CV[D]$

convexly with respect to $CV[D]$. This is illustrated by Figure 32 for CCP-T, and follows from our observations in item O5 of Section IV.6.1.

In summary, compared to RAP, both CCP-T and CCP-Q result in significant savings in the total system cost with an average of more than 15% without increasing the buyer's expected long-run average number of supplier switches. These savings are higher for smaller values of $CV[D]$.

IV.8 Summary of Contributions and Key Insights

In this chapter, we propose and investigate three demand allocation policies for a buyer to fulfill percentage supply allocations in multi-sourcing systems. We benchmark the demand allocation policy observed in the electronics industry using RAP, and propose CCP-T and CCP-Q as alternate policies to improve performance. We evaluate and compare the performances of these policies based on (i) long-run fraction of total demand allocated to each supplier, (ii) buyers expected long-run average number of supplier switches, and (iii) suppliers bullwhip effect under multi-sourcing. We show that all three polices can meet PSAs in the long run. Our main contribution is that we introduce the concept of bullwhip effect under multi-sourcing, demonstrate its existence and quantify it. We show that while RAP and CCP-T always lead to bullwhip effect, the bullwhip ratio under CCP-Q can be less than 1. Counter to the intuition, the bullwhip effect for a supplier decreases (is independent of) with respect to the coefficient of variation of the buyer's demand under RAP (CCP-T). In contrast, under CCP-Q the bullwhip effect for a supplier is a concave (convex) function of the coefficient of variation of the buyer's demand if the product of PSA for that particular supplier and the total number of suppliers is less (greater) than unity.

We show that under dual sourcing systems and RAP, the buyer prefers a skewed, rather than a comparable distribution of PSAs among its two suppliers due to lower expected long-run average number of supplier switches. As a result, if the major supplier offers lucrative discount for 100% of buyer's market share, the buyer may prefer single-sourcing rather than dual-sourcing. In contrast, CCP-T and CCP-Q are flexible such that with a wise choice of parameters, the buyer can meet its target for the expected long-run average number of supplier switches, while reducing the bullwhip effect observed by the suppliers. Thus, the demand allocation policy used by the buyer influences the pro-competitive and anti-competitive effects of a contract with PSAs.

We rank the demand allocation policies based on each performance measure. We establish a threshold value of PSA such that suppliers with higher (lower) PSA than this threshold observe the lowest bullwhip effect under CCP-Q (CCP- T), and hence, prefer it over other policies. Furthermore, our numerical results show that, compared to RAP, both CCP-T and CCP-Q result in significant savings in the total system cost with an average of more than 15%. These savings are higher for smaller values of coefficient of variation of the buyer's demand. Furthermore, CCP-Q performs mildly better than CCP-T with an average benefit of 0.44% and a maximum of 2.62%. We note that CCP-T is easier to implement than CCP-Q, and hence, depending on the magnitude of the actual savings CCP-T or CCP-Q may be used. When negotiating a contract with the buyer, suppliers will find our results valuable to carefully select the service levels that they commit to provide based on the agreed upon PSAs.

CHAPTER V

CONCLUSIONS

This dissertation is motivated by practices in the electronics industry. In particular, on the out-bound side, manufacturers satisfy stochastic demand from multiple markets. On the in-bound side, manufacturers multi-source parts from several suppliers. Recognizing the potential benefits and increased relevance of such practices, this dissertation focuses on three distinct yet related problems. The first problem investigates joint replenishment and liquidation decisions for a supplier who satisfies demand from a contractual and a spot market. The second problem extends this to investigate a supplier's joint replenishment, allocation and pricing decisions under two markets. The third problem investigates a buyer's demand allocation decisions under multi-sourcing with percentage supply allocations and their impact on the bullwhip effect. The contributions of this dissertation can be summarized as follows:

1. We provide a stochastic dynamic programming (SDP) formulation for joint replenishment and allocation decisions of a supplier facing stochastic demand from both contractual and spot markets under production capacity limitations. In such settings, if one reserves the optimal amount of inventory for future high-priority or high-revenue demand, higher profits can be earned. If the unit revenue and the lost sales penalty for the primary market are stationary, we show that for both the lost sales and the backlog cases, the optimal policy is characterized by two quantities: the critical produce-up-to and the critical retain-up-to levels. We establish bounds for the two critical quantities, discuss their economic interpretation, and use them to construct a new and effective heuristic policy.

2. Our practical contribution is that we identify alternate benchmark policies and show that there are (lower and upper) thresholds on the unit revenue earned from the spot

market such that one of the alternate benchmark policies is optimal. Based on our computational study, we quantify the benefits of using the optimal policy over the benchmark policies and show that the potential savings are significant.

3. In addition, we study a generalization where all cost and revenue parameters are non-stationary, i.e., the unit revenue and lost sales penalty from the primary market are also non-stationary. We show that the optimal policy is characterized by three critical quantities rather than just two: the critical produce-up-to level, the critical level of inventory to be saved from sales in either market and the critical level of inventory to be saved from sales in the spot market.

4. We provide a stochastic dynamic programming formulation for joint replenishment, allocation and pricing decisions of a supplier facing stochastic demand from two customer classes. This problem is relevant for manufacturers who sell their products to high volume customers (e.g., retailers) as well as directly to individual customers through their web-based channel. We show that the optimal price charged to Class 2 customers is a function of the left-over inventory after satisfying Class 1 demand, i.e., a dynamic pricing policy is optimal. Furthermore, the stage 1 optimal policy is characterized by two quantities: the critical produce-up-to level and the critical amount of inventory to be protected from Class 1. That is, a discretionary sales policy is optimal for Class 1. We conduct a computational study and quantify the benefits of discretionary sales for Class 1 and dynamic pricing for Class 2.

5. We contribute to the existing literature by investigating the optimality of joint myopic replenishment, allocation and pricing policies for a supplier facing stochastic demand from two demand classes.

6. Although common across all industries, multi-sourcing under percentage supply allocations (PSAs) leads to unique challenges due to the operational changes needed for (temporarily) switching suppliers in the electronics industries. In such settings, buyer's demand allocation policies that can meet PSAs have not been studied in the

literature before. To fill this gap, we propose and investigate three demand allocation policies: (i) random allocation policy (RAP) benchmarks the current practice, (ii) time-based cyclic consumption (CCP-T) and (iii) quantity-based cyclic consumption policies (CCP-Q).

7. Furthermore, although a vast and growing body of academic and practice oriented literature investigates bullwhip effect in single-sourcing supply chains, there is no literature on bullwhip effect in multi-sourcing supply chains. We contribute to the current literature by demonstrating the existence of bullwhip effect caused due to demand allocation policies in multi-sourcing systems. We term it as the *bullwhip effect under multi-sourcing* and emphasize its absence in single-sourcing systems. We show that while RAP and CCP-T always lead to bullwhip effect, the bullwhip ratio under CCP-Q can be less than 1. We show that compared to RAP, both CCP-T and CCP-Q result in significant savings in total replenishment and inventory holding costs across all suppliers with an average of more than 15% without increasing the buyer's expected long-run average number of supplier switches.

8. Our results offer new insights showing that the pro-competitive and anti-competitive effects of a contract with PSAs may be strengthened or weakened depending on the demand allocation policy used by the buyer.

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