PHYSICAL REVIEW B VOLUME 56, NUMBER 6 1 AUGUST 1997-II

Long-wavelength anomalous diffusion mode in the two-dimensional XY dipole magnet

Ar. Abanov
Department of Physics, Texas A&M University, College Station, Texas 77843-4242

A. Kashuba
Department of Physics, Texas A&M University, College Station, Texas 77843-4242;
Laboratorium für Festkörper Physik der ETHZ, ETH-Hönggerberg, 8093, Zurich, Switzerland;
and Landau Institute for Theoretical Physics, Moscow, Russia

V. L. Pokrovsky
Department of Physics, Texas A&M University, College Station, Texas 77843-4242
and Landau Institute for Theoretical Physics, Moscow, Russia
(Received 24 July 1996; revised manuscript received 13 December 1996)

In a two-dimensional XY ferromagnet the dipole force induces a strong interaction between spin waves in the long-wavelength limit. The major effect of this interaction is the transformation of a propagating spin wave into a diffusion mode. We study the anomalous dynamics of such diffusion modes. We find that the Janssen–De Dominics functional, which governs this dynamics, approaches the non-Gaussian fixed point. A spin wave propagates by an anomalous anisotropic diffusion with the dispersion relation: $i\omega \sim k_y^{\Delta_y}$ and $i\omega \sim k_x^{\Delta_x}$, where $\Delta_y = 47/27$ and $\Delta_x = 47/36$. The low-frequency response to the external magnetic field is found. [S0163-1829(97)03329-8]

I. INTRODUCTION

The notion of the ground state and independent elementary excitations lies in the background of condensed-matter theory. The excitations, such as electrons, holes, phonons, excitons, spin waves, etc. normally have a propagating, wavelike nature. In a homogeneous medium they are characterized by their momentum or quasimomentum $\vec{\mathbf{p}}$ and their energy ω. Each kind of excitations has its specific dispersion relation or spectrum $\omega = \epsilon(\mathbf{p})$. A wide scope of physical problems can be solved assuming the excitation to be independent, or considering their interaction as a weak perturbation.[1] However, in recent years more and more problems have occurred to go beyond the simple picture of noninteracting or weakly interacting excitations. In his pioneering work Migdal[2] has indicated that electron-phonon interaction is not weak in a narrow range of energy, leading to a strong renormalization of the Fermi velocity and even to an instability of the Fermi surface.

Recently a growing number of physical systems revealed excitations which bare spectrum, obtained from linearized equations of motion, is strongly distorted by interaction with vacuum and thermal fluctuations. A few recent examples are: the so-called marginal Fermi liquid, in which electrons interacts via transverse magnetic-field fluctuations,[3] and $\nu = 1/2$ state of the fractional quantum Hall effect in which initial electrons transform into quite different fermions.[4]

On the other hand, many years ago, the mode-mode interaction has been recognized as a necessary element of the critical dynamics.[5–7] Particles, heat and spin diffusion must be considered as hydrodynamic modes in the long-wavelength limit, as well as propagating waves, such as sounds, spin waves, etc. Their interaction has been proven to be substantial not only in the critical region, but also for the hydrodynamics of liquid crystals[8,9] and for the charge-density-wave phason modes interacting with impurities.[10] The excitation spectrum of these systems is reconstructed by effects of strong interaction.

Here we present a solvable and experimentally feasible situation where the strong interaction between spin waves leads to the replacement of the propagating spin wave by a diffusion mode and to the appearance of a new soft mode in a certain range of momentum. This is the two-dimensional (2D) XY ferromagnet with dipolar interaction between spins.

The spin-diffusion mode appears naturally in the paramagnetic phase and in the vicinity of the Curie point.[11] We consider, on the other hand, a low-temperature ordered phase, where no diffusion is expected but rather a propagating and weakly dissipating spin-wave mode.

In a 2D XY ferromagnet at low temperatures the dipolar interaction is relevant in the long-wavelength limit, even despite the low density of spin waves. It was shown by one of the authors[12] that dipolar force induces an anomalous anisotropic scaling of spin-spin correlations in the ordered phase. In this article we find an analogous dynamical scaling.

This paper is organized as follows. In the next section we define the model and describe the spin-wave spectrum. In Sec. III we discuss the dynamics of the XY magnet with the dipole interaction and formulate the perturbation expansion for the model, using the Janssen-De Dominics[13] technique. Section IV is devoted to the solution of the Dyson equation. There we find the self-induced dissipation of the spin waves. In Sec. V the renormalization of the diffusion mode is considered and the anomalous anisotropic exponents are found. In Sec. VI the dynamical susceptibility is found. In conclusion we discuss prospects of the experimental observation of the anomalous modes. In Appendix A we use the Ward-

Takahashi identities to prove that the vertex corrections are small. Details of the Dyson equation solution can be found in Appendixes B and C. A brief report on main results of this article has been published earlier.[14]

II. THE HAMILTONIAN AND THE SPIN-WAVE SPECTRUM

Spin waves are fundamental excitations of the exchange magnet with spontaneously broken continuous symmetry. According to the so-called Adler principle,[15] the interaction between spin waves vanishes in the long-wavelength limit. At low temperatures the equilibrium density of spin waves is relatively small, and a long-wavelength nonequilibrium spin wave, excited by an external source, decays into other spin waves or scatters on an equilibrium spin wave slowly. Thus, the dynamical properties of the exchange magnet are determined by the well-defined spin-wave mode. It also means that the imaginary part of the poles of the dynamical response function becomes much smaller than their real part as the wavelength grows to infinity.

The 2D exchange magnet with the easy-plane anisotropy is described by the following classical Hamiltonian:

$$H_{ExA}[\mathbf{S}]=\int d^2x\left[\frac{J}{2}(\nabla\mathbf{S})^2+\frac{\lambda}{2}S_z^2\right]-g_G\mu_B\int d^2x\,\mathbf{SH}, \tag{1}$$

where J is the exchange coupling constant, λ is the strength of the easy-plane anisotropy $(\lambda>0)$,[16] g is the dipole interaction coupling constant, $\mathbf{H}$ is the external magnetic field, g_G is gyromagnetic ratio, μ_B is the Bohr magneton and the field $\mathbf{S}(\mathbf{x})$ represents the local spin of the magnet and can be normalized by a constraint

$$\mathbf{S}^2(\mathbf{x})=1. \tag{2}$$

In what follows we consider very large distances $L\sim 1\,\mu$m so the Ruderman-Kittel-Kasuga-Yosida type interactions being nonsingular on such scales are included in J.

The magnetic dipolar energy $H_{\rm dip}$ is represented by the sum:

$$H_{\rm dip}=\frac{g}{4\pi}\sum_{\mathbf{x}_i\neq\mathbf{x}_j}\frac{(\mathbf{S}_i\cdot\mathbf{S}_j)-3(\mathbf{S}_i\cdot\hat{\nu})(\mathbf{S}_j\cdot\hat{\nu})}{|\mathbf{x}_i-\mathbf{x}_j|^3}, \tag{3}$$

where $\hat{\nu}$ is a unit vector pointing from $\mathbf{x}_i$ to $\mathbf{x}_j$, $\mathbf{S}_i=\mathbf{S}(\mathbf{x}_i)$, and $g=2\pi(g_G\mu_B Sa^{-2})^2$ for the square lattice; for other lattices $g=2\pi(g_G\mu_B S\sigma)^2$, where σ is the inverse area of a plaquett of the lattice. The magnetic dipole energy can be separated into a short-range and a long-range part in the standard way.[17] The short-range part renormalizes the single-ion spin anisotropy and favors in-plane spin orientation. The long-range part of the magnetic dipole energy is conveniently expressed in terms of the Fourier transform $\mathbf{S_k}$ of the local magnetization field $\mathbf{S}(\mathbf{x})$,

$$H_{\rm dip}=\frac{g}{2}\sum_{\mathbf{k}}\frac{(\mathbf{S_k}k)(\mathbf{S}_{-\mathbf{k}}\mathbf{k})}{|\mathbf{k}|}. \tag{4}$$

Thus, the total magnetic Hamiltonian is

$$H=H_{ExA}+H_{\rm dip}. \tag{5}$$

The dipole forces are crucial in the long-wavelength limit. Under the scaling transformation the short-range exchange interaction scales like L^{d-2}, whereas the dipole force scales like L^{2d-3}, where $d=2,3$ is the spatial dimension of a magnet and L is the scale. The dipole energy is characterized by the dipole constant g and the exchange energy is characterized by the exchange constant J. Normally in ferromagnets ga^{d-1} is much smaller than J (a is the lattice constant). However, as the scaling transformation shows, beyond the characteristic scale $L_d\sim(J/g)$ in 2D and $L_d\sim\sqrt{J/g}$ in 3D the dipole interaction dominates the energy of a spin wave.

In the 2D XY ferromagnet, the dipole force stabilizes the long-range order.[18] According to the Landau-Peierls-Hohenberg-Mermin-Wagner theorem (see, e.g., Ref. 19) in the absence of a long-range interaction the 2D magnet with broken continuous symmetry exhibits the algebraic decay of the spin correlations instead of the long-range order, and the infinite susceptibility at zero magnetic field. Thus, the dipole force plays a special role in 2D XY ferromagnet. In addition, the dipolar force is crucial for the spin statics and dynamics in the ordered phase. In contrast to the 3D ferromagnet, where the spin waves are almost free, in 2D the interaction between spin waves induced by dipole force is dominant in the long-wavelength limit.

The Hamiltonian (5) has two different scales: the anisotropy scale $L_A=\sqrt{J/\lambda}$ and the dipole length $L_D=J/g$. We assume that the anisotropy λ is large compared to the dipolar energy $(L_A<L_D)$. We direct the y axis along the net magnetization of the magnet and the z axis perpendicular to the plane. The unit vector field $\mathbf{S}$ can be represented by two scalar fields $\phi(\mathbf{x},t)$ and $\pi(\mathbf{x},t)$

$$\mathbf{S}=(-\sqrt{1-\pi^2}\sin\phi;\sqrt{1-\pi^2}\cos\phi;\pi), \tag{6}$$

where both π and ϕ are small due to the fact that the dipole force stabilizes the long-range order.

With the precision to the fourth power of ϕ and π the Hamiltonian (5) is

$$H[\phi]=\frac{1}{2}\int_{\mathbf{k}}\int_{\omega}\Bigg((J\mathbf{k}^2+h)\phi_{\mathbf{k},\omega}\phi_{-\mathbf{k},-\omega}+\lambda\pi_{\mathbf{k},\omega}\pi_{-\mathbf{k},-\omega}+g\frac{(k_x(\phi_{\mathbf{k},\omega}-\phi^3_{\mathbf{k},\omega}/6)+k_y[\phi^2/2]_{\mathbf{k},\omega})(k_x(\phi_{-\mathbf{k},-\omega}-\phi^3_{-\mathbf{k},-\omega}/6)+k_y[\phi^2/2]_{-\mathbf{k},-\omega})}{|\mathbf{k}|}\Bigg). \tag{7}$$

Here $\phi_{\mathbf{k},\omega}$ and $\pi_{\mathbf{k},\omega}$ are the Fourier transforms of the fields $\phi(\mathbf{x},t)$ and $\pi(\mathbf{x},t)$ respectively. We expanded the in-plane magnetization components $\cos\phi$ and $\sin\phi$ up to the fourth power in small spin fluctuations ϕ. We take the uniform magnetic field $\mathbf{H}$ to be directed along the y axis and $h=g_G\mu_B SH$. The Fourier-transformed quantities are defined by

$$\phi(\mathbf{x},t)=\int_{\mathbf{k}}\int_{\omega}\phi_{\mathbf{k},\omega}e^{i(\mathbf{kx}-\omega t)}, \tag{8}$$

where an abbreviated notation

$$\int_{\mathbf{k}}\int_{\omega}\equiv\int\int\frac{d^2k}{(2\pi)^2}\frac{d\omega}{2\pi}$$

is used and $[\phi^2/2]_{\mathbf{k},\omega}$ denotes the Fourier transformation of $\phi^2(\mathbf{x},t)/2$.

The interaction between the spin waves is described by the nonquadratic part of the Hamiltonian (7):

$$H_{\text{int}}=\int_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}\int_{\omega_1,\omega_2,\omega_3}f(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\phi_{\mathbf{k}_1,\omega_1}\phi_{\mathbf{k}_2,\omega_2}\phi_{\mathbf{k}_3,\omega_3}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\delta(\omega_1+\omega_2+\omega_3)$$
$$+\int_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4}\int_{\omega_1,\omega_2,\omega_3,\omega_4}g(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)\phi_{\mathbf{k}_1,\omega_1}\phi_{\mathbf{k}_2,\omega_2}\phi_{\mathbf{k}_3,\omega_3}\phi_{\mathbf{k}_4,\omega_4}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\delta(\omega_1+\omega_2+\omega_3+\omega_4), \tag{9}$$

where the three-leg bare vertex $f(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$ is

$$f(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)=\frac{g}{3}\sum_{i=1}^{3}\frac{k_{ix}k_{iy}}{|\mathbf{k}_i|} \tag{10}$$

and the four-leg vertex is defined as

$$g(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)=\frac{g}{24}\sum_{i>j=1}^{4}\frac{(k_{iy}+k_{jy})^2-(k_{ix}+k_{jx})^2}{|\mathbf{k}_i+\mathbf{k}_j|}. \tag{11}$$

The vertices (10) and (11) decrease as the momentum $\mathbf{k}$ goes to zero. Besides that, these vertices are singular at $|\mathbf{k}|\to 0$. Nevertheless, the interaction between spin waves asymptotically vanishes in the long-wavelength limit for a 3D magnet.[15] No renormalization of the bare correlator

$$K(\mathbf{k})=\langle\delta S_x(\mathbf{k})\delta S_x(-\mathbf{k})\rangle=\langle\phi_{\mathbf{k}}\phi_{-\mathbf{k}}\rangle=\frac{T}{J\mathbf{k}^2+g(k_x^2/|\mathbf{k}|)} \tag{12}$$

appears in this limit (we set $h=0$).

In 2D the situation changes drastically: the interaction grows with the wavelength, resulting in strong renormalization of critical exponents. To show the difference we calculate the upper marginal dimension of the Hamiltonian (7). Let us consider an arbitrary diagram from the perturbation expansion of some correlator. In order to add an internal line to such a diagram, we need to add three bare correlators K given by Eq. (12), two vertices f and one integration over $dk_x d^{(D-1)}k_y$ (this particular method of regularization is chosen in order to preserve the rotation symmetry which will be considered in the Appendix A in detail). From Eq. (12) we see that for small momenta, $k_x\sim k_y^{3/2}$ and $K\sim k_y^{-2}$. Equation (10) gives $f\sim k_x\sim k_y^{3/2}$. Hence, requiring the one-line insertion be dimensionless, we obtain $3\cdot(-2)+2\cdot 3/2+3/2+(D-1)=0$ or $D=5/2$. It means that the theory is renormalizable in $D\leqslant 5/2$. It turns out that in the static 2D case the critical exponents can be found exactly.

Following the work,[12] we rescale the field $\phi^2\to\phi^2/\sqrt{Jg}$ and rewrite the Hamiltonian (7) in a slightly different form:

$$H=\int_{\mathbf{k}}\Bigg(ak_y^2\phi_{\mathbf{k}}\phi_{-\mathbf{k}}+a^{-1}\frac{k_x^2}{|k_y|}\phi_{\mathbf{k}}\phi_{-\mathbf{k}}+2w\frac{k_xk_y}{|k_y|}\phi_{\mathbf{k}}\left[\frac{\phi^2}{2}\right]_{-\mathbf{k}}$$
$$+w^2a|k_y|\left[\frac{\phi^2}{2}\right]_{\mathbf{k}}\left[\frac{\phi^2}{2}\right]_{-\mathbf{k}}\Bigg), \tag{13}$$

where $a=\sqrt{J/g}$, $w^2=\sqrt{g/J^3}$ and we have taken into account that for small $\mathbf{k}$ one can substitute $|\mathbf{k}|\to k_y$ [see Eq. (12)]. Requiring that the Hamiltonian (13) does not change under scale transformation $k_y\to lk_y$, $k_x\to l^{\Delta_x^0}k_x$, $\phi_{\mathbf{k}}\to l^{\Delta_\phi^0}\phi_{\mathbf{k}}$, $a\to l^{\Delta_a^0}a$, $w\to l^{\Delta^0}w$, we find the bare exponents $\Delta_x^0=3/2$, $\Delta_a^0=0$, $\Delta_\phi^0=-9/4$, $\Delta_w^0=1/4$ (where we have taken into account that $[\phi^2/2]$ has one more integration). Now, according to the standard procedure,[20] we introduce the renormalized field and charges: $\phi^2=Z_\phi\phi_R^2$, $a=Z_a a_R$, $w=Z_w\mu^{1/4}\widetilde{w}_R$, where μ is the scale at which the dimensionless $\widetilde{w}_R$ is observed. The second term in the Hamiltonian (13) is nonanalytic at small momenta. It means that there can be no corrections to this term in the regular perturbation expansion.

The symmetry group consists of simultaneous rotations of the coordinate system and spins. In Appendix A it is shown that, due to this symmetry, there is no correction to the third term either. Hence, we can establish the relations between the renormalization coefficients:

$$Z_a^{-1}Z_\phi=1, \tag{14}$$

$$Z_\phi^{3/2}Z_w\mu^{1/4}=1. \tag{15}$$

In the renormalization-group (RG) scheme the dimensionless charge $\widetilde{w}$ reaches a finite value at fixed point and determines all the critical exponent, similar to the standard procedure with the dimensionless four-leg vertex in the theory φ.[4] The reader can check this fact and find the fixed value of $\widetilde{w}$ in Sec. V. Here we skip the calculation to avoid repetition. Since $\widetilde{w}_R=\text{const}$ in the fixed point, one finds $Z_w\sim(\Lambda/\mu)^{1/4}$, where Λ is the scale at which the coupling constant equals w. Introducing the critical exponents $Z_\phi\sim\Lambda^{2\Delta_\phi}$, $Z_a\sim\Lambda^{\Delta_a}$, we find from Eqs. (14) and (15) that $\Delta_\phi=-1/12$ and $\Delta_a=-1/6$. Demanding that the first and the second terms in the Hamiltonian (13) have the same dimensions, we find $\Delta_x = 4/3$. Finally, from the Callan-Symanzik equation,[20] the long-range limit of the two-point correlation function can be found:

$$K(\mathbf{x}) = (x^2 + |y|^{8/3})^{-1/4} f(x/|y|^{4/3}),$$

where $f(x)$ is an arbitrary finite function. The dynamical properties of the magnet, however, are still to be found.

In the conclusion of this section we draw the readers attention to a type of critical behavior appearing in the considered model. It displays the scaling behavior everywhere below the transition point. It happened earlier, for example, in the XY model with short-range interaction. The scaling is caused by a fluctuating Goldstone mode. This situation is characterized in the renormalization-group theory as a line of fixed points. However, unlike the classical XY model, the XY model with dipolar interaction produces fixed critical exponents, independent of the point of line. In this respect our theory reminds us more of the theory of critical points. The reason for this intermediate behavior is that the critical fluctuations on large distances are governed by the singular dipolar interaction, and this singularity is not renormalized by the thermal motion.

III. THE DYNAMICS OF XY MAGNET

In this section we introduce equations of motion for the magnetization, describe the path-integral formulation of the problem and establish an approximate form of Dyson equation which governs the nonlinear dynamics in the low-frequency range.

At $T=0$ the classical magnet with the Hamiltonian (5), obeying the constraint (2), follows the Landau-Lifshitz equations (see, e.g., Ref. 21):

$$S\hbar \frac{\partial \mathbf{S}(\mathbf{x},t)}{\partial t} = \mathbf{S}(\mathbf{x},t) \times \frac{\delta H}{\delta \mathbf{S}(\mathbf{x},t)}, \tag{16}$$

where S is the absolute value of a spin localized on a magnetic ion. In terms of the canonically conjugated fields π and ϕ, Eq. (16) can be rewritten in the Hamiltonian form:

$$\hbar S \partial_t \pi(\mathbf{x},t) = \frac{\delta H}{\delta \phi(\mathbf{x},t)}, \tag{17}$$

$$-\hbar S \partial_t \phi(\mathbf{x},t) = \frac{\delta H}{\delta \pi(\mathbf{x},t)} \approx \lambda \pi(\mathbf{x},t). \tag{18}$$

In the harmonic approximation equations (17,18) imply the dispersion relation for the spin-wave mode[16] in a XY magnet:

$$\epsilon^2(\mathbf{k}) = \lambda \left(J\mathbf{k}^2 + g\frac{k_x^2}{|\mathbf{k}|} \right) = c^2 \left(\mathbf{k}^2 + p_0 \frac{k_x^2}{|\mathbf{k}|} \right), \tag{19}$$

where $c = \sqrt{\lambda J}$ is the spin-wave velocity and $p_0 = g/J$. The out-of-plane anisotropy λ affects the dynamics in the long-wavelength limit $k \ll p_\lambda = \sqrt{\lambda/J}$ considered in this article. Thus, p_λ is the upper cutoff momentum in our theory. A spin wave with $k \gg p_0$ has the phononlike isotropic spectrum $\epsilon = ck$. The range $p_0 \ll k \ll p_\lambda$ will be called acoustic shell ($\mathcal{A}$ shell). At lower momenta $p \ll p_0$ the spin-wave spectrum is dominated by the dipolar interaction: $\epsilon(\mathbf{k}) \approx c\sqrt{p_0 k} \sin\theta$, where θ is the angle between the direction of the spontaneous magnetization and the wave vector. This range of momenta will be called dipolar shell ($\mathcal{D}$ shell). The effect of the presence of the dipolar force in the 2D XY magnet is not limited to the change of free spin-wave spectrum. As we have mentioned earlier, it leads to strong spin-wave interaction and to a crucial transformation of the spin propagation.

Without dipolar forces the dynamics of the 2D XY ferromagnet is well described by noninteracting spin waves. The high level of fluctuations leads to a strong temperature dependence of the dynamic spin correlators, which have algebraic character, just as static ones.[22,23] The renormalized two-point spin-correlation function features the pole with the temperature-dependent power exponent. The dipole force suppresses such strong spin fluctuations, but not entirely.

In addition, the dipolar interaction induces decay processes. As a result finite spin-wave lifetime $\Gamma(\mathbf{k})$ or the width of the level $b(\mathbf{k}) = \Gamma^{-1}(\mathbf{k})$ appears. In 3D at low temperature and at small momentum $|\mathbf{k}|$, the width $b(\mathbf{k})$ is much smaller than $\omega(\mathbf{k})$. In 2D, however, the interaction is essential and must be considered seriously.

To take into account the dissipation induced by thermal fluctuations at a temperature T, we introduce a phenomenological dissipation functional:[24]

$$R[\phi] = \int dt d^2x d^2x' R(\mathbf{x}-\mathbf{x}') \dot{\phi}(\mathbf{x}) \dot{\phi}(\mathbf{x}'). \tag{20}$$

Eliminating π from Eqs. (17,18) and adding a proper dissipation term, one obtains the following equation for $\phi(\mathbf{x},t)$:

$$-\frac{1}{\lambda}\partial_t^2 \phi(\mathbf{x},t) = \left[\frac{1}{\lambda} \epsilon^2(\mathbf{k}) \phi(\omega,\mathbf{k}) \right]_{\mathbf{x},t} + \frac{\delta H_{\text{int}}}{\delta \phi(\mathbf{x},t)} + \frac{\delta R}{\delta \dot{\phi}(\mathbf{x},t)} + \eta(\mathbf{x},t) - h(\mathbf{x},t), \tag{21}$$

where $h(\mathbf{x},t)$ is the external magnetic field, and the interaction part of the Hamiltonian is determined by Eq. (9). We have introduced the random noise $\eta(\mathbf{x},t)$ in Eq. (21). The noise, in effect, generates dissipation. We note, that on scales under consideration the anisotropy destroys spin conservation in the x-y plane first, and on even larger distances the dipole-dipole interaction destroys the conservation of the z projection of the spin (see also Ref. 24). As usual, the random noise is assumed to obey the Gaussian statistics. Its correlation function is determined by the fluctuation-dissipation theorem:[25]

$$\langle \eta_{\mathbf{k}} \eta_{-\mathbf{k}} \rangle = 2TR(\mathbf{k}). \tag{22}$$

Here $R(\mathbf{k})$ is the Fourier transform of the function $R(\mathbf{x}-\mathbf{x}')$. The dissipation in the exchange ferromagnet vanishes in the long-wavelength limit:[15] $R(\mathbf{k}) = b\mathbf{k}^2$. In 2D XY dipole magnet the dissipation does not vanish in the long-wavelength limit.

We emphasize that the finite lifetime Γ is determined self-consistently by the processes of the decay and scattering of spin waves. We neglect the spin-wave-electron and spin-wave-sound interactions. The first interaction is not weak, but the Fermi velocity is much higher than the spin-wave velocity and the spin-electron interaction is not effective for long wavelength. Even the sound velocity can be much larger than the spin-wave velocity, since the latter is proportional to small $\sqrt{\lambda}$.

Our aim then is to calculate the linear-response function $G(\mathbf{x},t)$ to a weak external field $h(\mathbf{x},t)$ [see Eq. (21)] averaged over the thermal fluctuations $\eta(\mathbf{x},t)$. We apply the Janssen–De Dominicis functional method[13] to reformulate stochastic equation (21) in terms of the path integral. The probability distribution for the noise $\eta(\mathbf{x},t)$ is

$$W[\eta]\sim\exp\left[-\frac{1}{4T}\int d^2x d^2x'\int dt\,\eta(\mathbf{x},t)\times R^{-1}(\mathbf{x}-\mathbf{x}')\,\eta(\mathbf{x}',t)\right]. \tag{23}$$

Following the standard dynamic field theory,[20,9] upon averaging over the noise distribution and introducing auxiliary response field $\hat{\phi}(\mathbf{x},t)$, one can reduce the solution of the stochastic differential equation (21) to the calculation of the dynamical partition function:

$$Z[j,\hat{j}]=\int \mathcal{D}[\phi]\mathcal{D}[i\hat{\phi}]\times\exp\left(\mathcal{J}[\phi,\hat{\phi}]+\int d^2x\int dt[\hat{j}\hat{\phi}+j\phi]\right) \tag{24}$$

and its derivatives over the currents j and $\hat{j}$. Here $\mathcal{J}(\phi,\hat{\phi})$ is the Janssen–De Dominics functional (JDF):

$$\mathcal{J}[\phi,\hat{\phi}]=\int d^2x\int d^2x'\int dt\,\hat{\phi}(\mathbf{x},t)TR(\mathbf{x}-\mathbf{x}')\,\hat{\phi}(\mathbf{x}',t)-\hat{\phi}(\mathbf{x}',t)\left[\frac{1}{\lambda}\partial_t^2\phi(\mathbf{x},t)\,\delta(\mathbf{x}'-\mathbf{x})+\frac{\delta H}{\delta\phi(\mathbf{x},t)}\delta(\mathbf{x}'-\mathbf{x})+R(\mathbf{x}'-\mathbf{x})\partial_t\phi(\mathbf{x},t)\right]. \tag{25}$$

By differentiation of the JDF over j and $\hat{j}$ one can obtain any correlation function.

From the quadratic part of the JDF one finds the bare propagator:

$$\hat{\mathbf{G}}_0=\begin{pmatrix} 0 & G_0^*(\omega,\mathbf{k}) \\ G_0(\omega,\mathbf{k}) & D_0(\omega,\mathbf{k})\end{pmatrix}, \tag{26}$$

where we define the bare dynamical response and spin-spin correlation function as follows:

$$G_0(\omega,\mathbf{k})=\frac{\lambda}{\omega^2-\epsilon^2(\mathbf{k})-i\omega\lambda R(\mathbf{k})}, \tag{27}$$

$$D_0(\omega,\mathbf{k})=\frac{2T\lambda R(\mathbf{k})}{[\omega^2-\epsilon^2(\mathbf{k})]^2+\omega^2\lambda^2R^2(\mathbf{k})}\ . \tag{28}$$

They obey the standard fluctuation-dissipation relation: $D_0=2T/\omega\,\mathrm{Im}G_0$. The same relation is correct for the total dynamic correlation $D(\omega,\mathbf{k})$ and the total linear-response function $G(\omega,\mathbf{k})$:

$$D=\frac{2T}{\omega}\mathrm{Im}G.$$

The anharmonic (interaction) part of $\mathcal{J}$ is

$$\mathcal{J}_{\mathrm{int}}=\sum_{\omega,\mathbf{k}}\hat{\phi}_{\omega,\mathbf{k}}\frac{\delta H_{\mathrm{int}}}{\delta\phi_{\omega,\mathbf{k}}}=\sum_{\omega,\mathbf{k}}\hat{\phi}_{\omega,\mathbf{k}}\left(3\sum_{\omega_2,\omega_3}\sum_{\mathbf{k}_2,\mathbf{k}_3}f(\mathbf{k},\mathbf{k}_2,\mathbf{k}_3)\,\phi_{\mathbf{k}_2,\omega_2}\phi_{\mathbf{k}_3,\omega_3}\delta(\mathbf{k}+\mathbf{k}_2+\mathbf{k}_3)\,\delta(\omega+\omega_2+\omega_3)+4\sum_{\omega_2,\omega_3,\omega_4}\sum_{\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4}g(\mathbf{k},\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)\,\phi_{\mathbf{k}_2,\omega_2}\phi_{\mathbf{k}_3,\omega_3}\phi_{\mathbf{k}_4,\omega_4}\delta(\mathbf{k}+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\,\delta(\omega+\omega_2+\omega_3+\omega_4)\right). \tag{29}$$

We define in a common way the self-energy operator $\hat{\mathbf{\Sigma}}(\omega,\mathbf{k})$ by the relation

$$\hat{\mathbf{G}}^{-1}(\omega,\mathbf{p})=\hat{\mathbf{G}}_0^{-1}(\omega,\mathbf{p})-\hat{\mathbf{\Sigma}}(\omega,\mathbf{p}), \tag{30}$$

where

$$\hat{\mathbf{G}}=\begin{pmatrix} 0 & G^*(\omega,\mathbf{k}) \\ G(\omega,\mathbf{k}) & D(\omega,\mathbf{k})\end{pmatrix}, \tag{31}$$

and $G(\omega,\mathbf{k})$ and $D(\omega,\mathbf{k})$ are the complete response function and correlator, respectively.

The self-energy $\hat{\mathbf{\Sigma}}(\omega,\mathbf{k})$ satisfies the Dyson equation:

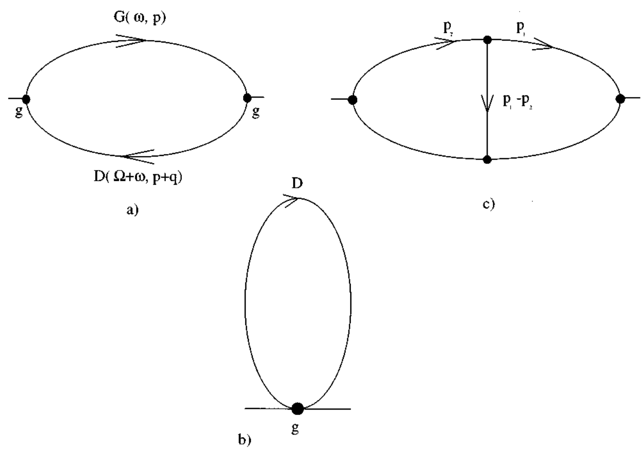

FIG. 1. (a,b) The main contribution to the self-energy. The functions G and D are given by Eq. (31). The three-leg vertices in (a) and the four-leg vertex in (b) are from Eq. (29). (c) Two-loop correction to the self-energy. Momenta of internal lines are indicated.

$$\hat{\mathbf{\Sigma}}(\omega,\mathbf{k})=g\int_{\Omega}\int_{\mathbf{p}}\hat{\mathbf{G}}(\Omega,\mathbf{p})\hat{\Lambda}_0(\mathbf{p},\mathbf{k};\omega,\Omega)\hat{\Lambda}(\mathbf{p},\mathbf{k};\omega,\Omega)\times\hat{\mathbf{G}}(\omega-\Omega,\mathbf{k}-\mathbf{p}), \tag{32}$$

where $\hat{\Lambda}(\mathbf{p},\mathbf{k};\omega,\Omega)$ is the full vertex and $\hat{\Lambda}_0(\mathbf{p},\mathbf{k};\omega,\Omega)$ is the bare vertex which can be readily found from the interaction part of action (29).

Further we consider a limit $R(\mathbf{k})\to+0$. According to the fluctuation-dissipation theorem (FDT), the matrix $\hat{\mathbf{\Sigma}}(\omega,\mathbf{p})$ must have a form:

$$\hat{\mathbf{\Sigma}}(\omega,\mathbf{p})=\begin{pmatrix}\frac{2T}{\lambda\omega}\mathrm{Im}\Sigma(\omega,\mathbf{p}) & \frac{1}{\lambda}\Sigma^*(\omega,\mathbf{p})\\ \frac{1}{\lambda}\Sigma(\omega,\mathbf{p}) & 0\end{pmatrix}, \tag{33}$$

where the self-energy function $\Sigma(\omega,\mathbf{p})$ is associated with the complete response function $G(\omega,\mathbf{p})$ by the same relationship:

$$G^{-1}(\omega,\mathbf{p})=G_0^{-1}(\omega,\mathbf{p})-\frac{1}{\lambda}\Sigma(\omega,\mathbf{p}). \tag{34}$$

In Appendix A we show that the full vertex Λ can be approximated with a good accuracy by its bare value $f(\omega,\mathbf{p})$ in the low-frequency range. In this respect our theory is similar to the Migdal theory of the interacting electron-phonon system.[2] In the Migdal theory the simplification is due to a narrow scale of the energy shell in which the interaction proceeds. In our theory we assume that the frequency of spin fluctuations is small instead. Under this condition the two-loop corrections are small, and the diagram in Fig. 1(b) contributes to a negligible change of the spectrum (19).[20] Such neglect of the two-loop diagrams (vertex correction) was a major assumption in the so-called mode-coupling methods.[11] This approximation serves well in the theory of the 3D critical dynamics with the dipole force being included. Later we prove this assumption for 2D. Thus, the Dyson equation for our problem is as follows:

$$\Sigma(\Omega,\mathbf{q})=18\lambda^3T\int_{\mathbf{p}}\int_{\omega}f^2(\mathbf{p},\mathbf{q})D(\omega,\mathbf{p})G(\omega+\Omega,\mathbf{q}-\mathbf{p}). \tag{35}$$

IV. SOLUTION OF DYSON EQUATION: SOFT MODES

In this section we solve Dyson equation (35) for the special conditions formulated below. They include low-temperature, low-frequency (wave vector) and weak dipolar interaction. We notify the real and the imaginary part of the self-energy term as $\Sigma=a^2(\omega,\mathbf{p})-i\omega b(\omega,\mathbf{p})$. Thus, the Green function (30) reads

$$G^{-1}(\omega,\mathbf{p})=\omega^2-\epsilon^2(\mathbf{p})-a^2(\omega,\mathbf{p})+i\omega b(\omega,\mathbf{p}), \tag{36}$$

while the spin-spin correlation function is

$$D(\omega,\mathbf{p})=\frac{b(\omega,\mathbf{p})}{[\omega^2-\epsilon^2(\mathbf{p})-a^2(\omega,\mathbf{p})]^2+\omega^2b^2(\omega,\mathbf{p})} \tag{37}$$

(we have slightly changed the definitions of G and D and referred the factor $\sqrt{2T\lambda^3}$ to the vertex).

We employ the reduced temperature $t=T/4\pi J$ and the ratio $g/\sqrt{J\lambda}=p_0/\sqrt{\lambda/J}$ as small parameters. The latter means that the $\mathcal{A}$ shell is much larger than the $\mathcal{D}$ shell. We also use the notation $L=\ln(\sqrt{J\lambda}/g)$.

The main contribution to the self-energy is given by the one-loop diagrams shown in Figs. 1(a) and 1(b). Our theory is valid only if the temperature is small:

$$t\ln(\sqrt{J\lambda}/g)=tL\ll 1. \tag{38}$$

The functions $b(\omega,\mathbf{p})$ and $a(\omega,\mathbf{p})$ are even in both arguments.[1] The imaginary part of the self-energy is odd in ω: $\mathrm{Im}\Sigma(\Omega,\mathbf{q})=-\Omega b(\Omega,\mathbf{q})$. Hence, the equation for the dissipation function reads

$$b(\Omega,\mathbf{q})=9\lambda^3T\int_{\mathbf{p}}\int_{\omega}f^2(\mathbf{p},\mathbf{q})D(\omega,\mathbf{p})D(\omega+\Omega,\mathbf{q}-\mathbf{p}). \tag{39}$$

The integrand in Eq. (39) is positive. Thus, the main contribution to $b(\Omega,\mathbf{q})$ comes from the region where poles of the two D functions coincide. The function $D(\omega,\mathbf{p})$ has poles at $\omega\approx\pm\epsilon(\mathbf{p})$ in the $\mathcal{A}$ shell. Following the terminology of the field theory, we call the surface $\omega^2=\epsilon^2(\mathbf{p})$ the mass shell. The self-energy in the $\mathcal{A}$ shell is small as it is shown in Appendix B and we neglect it. Because the dissipation is small, the D function can be represented as a sum of δ functions:

$$D(\omega,\mathbf{p})\approx\sum_{\pm}\frac{\pi}{2\epsilon^2(\mathbf{p})}\delta(\Delta\omega_{\pm}), \tag{40}$$

where $\Delta\omega_{\pm}=\omega\pm\epsilon(\mathbf{p})$ (Ref. 26) measures the deviation from the mass shell. After integrating ω out from Eq. (39) with the D functions from Eq. (40), we recover the Fermi golden rule for the probability of the spin-wave decay and scattering processes.

Looking for the long-wavelength quasiexcitations, we need the self-energy at very small momenta $q\ll p_0$, which we denote as Σ_0. We anticipate the quasiexcitations to be soft: $\Omega\ll cq$. Here we restrict the quasiexcitation wave vector $\mathbf{q}$ to be directed almost along the magnetization:

$|q_x| \ll q$ (arbitrarily directed $\mathbf{q}$ are considered in Appendix C). The essential contribution to the integral in Eq. (35) comes from the internal momentum p being in the $\mathcal{A}$ shell and the internal frequency $\omega = \epsilon(\mathbf{p})$. Performing the integration over ω with the D function from Eq. (40), we find

$$\Sigma_0 = \frac{c^2 p_0^2 t}{4\pi} \int \frac{c^4 p^3 dp}{\epsilon^4(\mathbf{p})} \frac{\Omega \sin^2(2\psi) d\psi}{\Omega - cq\cos\psi + i b_1}, \tag{41}$$

where b_1 is the dissipation function $b(\omega,\mathbf{p})$ of a spin wave inside the $\mathcal{A}$ shell. Details of the derivation of Eq. (41) are given in Appendix B. We neglect the real part of the self-energy in the $\mathcal{A}$ shell as it is justified in the same appendix. Note, that $\mathrm{Re}\Sigma_0$ vanishes in the static limit $\Omega = 0$.

If $cq \gg b_1$, we make the integral over ψ in Eq. (41) to find

$$\Sigma_0(\chi) = c^2 p_0^2 t L \cos^2\chi \exp(-2i\chi), \tag{42}$$

where χ, defined by the equation $\cos\chi = \Omega/cq$, measures the deviation from the mass shell. More generally, we introduce a notation r for the ratio Ω/cq ($r = \cos\chi$ if $r \leq 1$). Then

$$\Sigma_0(r) = c^2 p_0^2 L t r^2 (2r^2 - 1 - 2r\sqrt{r^2-1}). \tag{43}$$

Note that $\Sigma_0(r) \approx -c^2 p_0^2 t/4$ when $r \to \infty$ and $|\Sigma_0(r)| < c^2 p_0^2 t/4$ at any r. The self-energy $\Sigma_0(r)$ is real for $r > 1$.

If q is so small that $cq \ll b_1$, Eq. (41) implies the q-independent dissipation constant:

$$b_0 = c^2 p_0^2 t L \int \frac{d\psi}{4\pi} \frac{\sin^2(2\psi)}{b_1(\psi)}. \tag{44}$$

In this calculation we have used the fact that the dissipation of a spin wave in the $\mathcal{A}$ shell b_1 depends only on the angle ψ between the direction of magnetization and the spin-wave wave vector $\mathbf{p}$ which we prove below.

Now we need to calculate $b_1(\psi)$. An unusual feature of our theory is that the dissipation process in the $\mathcal{A}$ shell is mediated by an off-mass-shell virtual spin wave. Indeed, the dispersion relation (19) does not allow for decay or merging processes. Alternatively, as we will show, the dissipation of a spin wave in the $\mathcal{A}$ shell, propagating along the direction specified with the angle ψ ($\sin\psi = q_x/q$), is mediated by an internal virtual spin wave in Eq. (9), with a momentum of $p \ll p_0$ and a frequency of $\omega < cp$, propagating along the direction very close to the y axis $\varphi^2 \ll 1$ (where $\sin\varphi = p_x/p$), to provide a finite attenuation of this state. The integration over ω with one of the D functions in Eq. (39), taken in the form (40), leads to the following equation:

$$b_1 = -9c^4 t \frac{f^2(\mathbf{0},\mathbf{q})}{8\pi J^2 q^2} \int d^2\mathbf{p} D[\epsilon(\mathbf{p}+\mathbf{q}) - \epsilon(\mathbf{q}), \mathbf{p}]. \tag{45}$$

Since $\omega = \epsilon(\mathbf{p}+\mathbf{q}) - \epsilon(\mathbf{q})$, we conclude that $\omega = cp\cos\Phi$, where $\Phi = \psi - \varphi \approx \psi$ is the angle between the vectors $\mathbf{q}$ and $\mathbf{p}$. According to fluctuation-dissipation theorem: $D(\epsilon_{\mathbf{p}+\mathbf{q}} - \epsilon_{\mathbf{q}}, \mathbf{p}) = 1/\epsilon_{\mathbf{p}+\mathbf{q}} - \epsilon_{\mathbf{q}} \mathrm{Im} G(\epsilon_{\mathbf{p}+\mathbf{q}} - \epsilon_{\mathbf{q}}, \mathbf{p})$. Invoking the definition of the angle χ for virtual spin wave, we find that $\chi = \Phi \approx \psi$. Substituting $f^2(\mathbf{0},\mathbf{q}) = (1/9) q^2 \sin^2(2\psi)$, $\epsilon^2(\mathbf{p}) = c^2p^2 + c^2 p_0 p \sin^2\varphi \approx c^2 p^2 + c^2 p_0 p \varphi^2$ and taking into account that Σ_0 from Eq. (42) depends only on $\chi = \psi$, we write

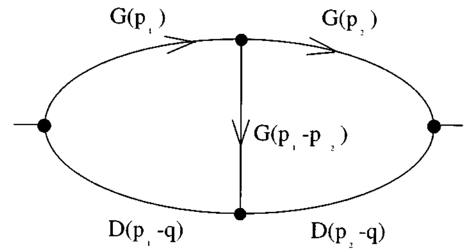

FIG. 2. The most ''dangerous'' two-loop diagram.

$$b_1(\psi) = \frac{c^2 p_0^2 t}{2\pi} \mathrm{Im} \int \frac{\sin^2\psi \cos\psi \, dp \, d\varphi}{p^2 \sin^2\psi + p_0 p \varphi^2 + \Sigma_0(\psi)/c^2}. \tag{46}$$

Note that the most dangerous region of integration is the region of small p, such that $\varphi^2 \sim p/p_0 \ll 1$ in Eq. (46). In other words, the dissipation of a short-wavelength spin wave, propagating in the direction ψ, is determined by the scattering on the long-wavelength virtual spin wave, with the momentum along the $\hat{y}$ direction, which lies on a specific distance off the mass shell: $\omega/cp = \cos\psi$. The integration over p in Eq. (46) is confined towards the crossover region: $p \sim p_c = p_0\sqrt{tL}$.

To find the anisotropic dissipation of a spin-wave mode in the $\mathcal{A}$ shell, we plug $\Sigma_0(\psi)$ from Eq. (42) into Eq. (46). After a change of variables $(p,\varphi) \to (\rho,\vartheta)$, given by formulas $p = p_0 \rho^2 \cos\vartheta$ and $\varphi = \rho \sin\psi \sin\vartheta \cos^{-1/2}\vartheta$ ($-\infty < \rho < \infty$ and $0 < \vartheta < \pi/2$), the integration becomes trivial and gives

$$b_1(\psi) = \beta_1 t^{3/4} c p_0 \frac{\sin^{3/2}(2\psi)\sin(\psi/2)}{L^{1/4}\cos\psi}, \tag{47}$$

where the direction of the spin wave is limited to the fundamental quadrant: $0 < \psi < \pi/2$, and $\beta_1 = \Gamma^2(1/4)/4\sqrt{2\pi} \approx 1.31$.

Let us return to the range of very low momenta $p \ll b_1/c$. Plugging Eq. (47) into Eq. (44), one finds

$$b_0 = \beta_0 c p_0 t^{1/4} L^{5/4},$$

where $\beta_0 \approx 1.24$. The condition $cp_{DM} \sim b_1$ defines the crossover wave vector: $p_{DM} \sim \beta_1 p_0 t^{3/4}/L^{1/4}$, between the self-energies (42) and (44). The dissipation functions (42, 44, and 47) represent the self-consistent solution of the Dyson equation (35, 39).

Finally, we verify that the two-loop correction [see Fig. 1(c)] is negligible. There exist several diagrams with different arrangements of G and D functions. On each of the two short loops on the diagram Fig. 1(c) there exists at least one D function but there may be two of them. We consider only the most ''dangerous'' diagram with each short loop having exactly one D function (see Fig. 2). Note that the main contribution to the diagram Fig. 2 comes from regions of internal momenta, $\mathbf{p}_1$ and $\mathbf{p}_2$ are restricted to the $\mathcal{A}$ shell. Inside the $\mathcal{A}$ shell the Green and the D functions have strong singularities on the mass shell. As it was done in Appendix B, we integrate in both short loops the internal frequencies ω_1 and ω_2 and find that only the nonstatic term is nonzero:

$$\Sigma_0 \sim t^2 \int \frac{d^2p_1 d^2p_2}{\epsilon^3(\mathbf{p_1})\epsilon^3(\mathbf{p_2})} \frac{\Omega f^2(\mathbf{p_1},\mathbf{p_2}) f(\mathbf{p_1},\mathbf{p_1}) f(\mathbf{p_2},\mathbf{p_2})}{(\Omega - cq\cos\phi_1 - ib_1)(\Omega - cq\cos\phi_2 - ib_2)} \frac{\epsilon^{-2}(\mathbf{p_1}-\mathbf{p_2})}{[\epsilon(\mathbf{p}_1)-\epsilon(\mathbf{p}_2)-\epsilon(\mathbf{p_1}-\mathbf{p_2})]}, \tag{48}$$

where ϕ_1 and ϕ_2 is the direction of the momenta $\mathbf{p_1}$ and $\mathbf{p_2}$. We assumed that $\phi_1 \approx \phi_2$. Since the three spin-wave processes are not allowed by the conservation laws, the Green function $G(\omega_1-\omega_2,\mathbf{p_1}-\mathbf{p_2})$ is off the mass shell (the corresponding last denominator in Eq. (48) reads $(p_1+p_2)(\phi_1-\phi_2)^2+p_0\cos^2(\phi_1)>0$ if $p_1,p_2 \gg p_0$). Now we can count the momenta powers in Eq. (48) to verify that the integration is convergent towards the dipolar momentum p_0, and, thus has no logarithm. A simple counting of temperatures shows that Eq. (48) $\sim t^2/b_1^2 \sim t^{1/2}$. Hence the two-loop dissipation function is $b_0' = b_0(1+t^{1/4}/L)$. Similar consideration shows that the function $b_1'(\psi,q)-b_1(\psi)$, which represents the two-loop corrections for b_1, is small in $t^{1/4}$, and is also small in the ratio p_0/q.

Having explicit expressions for the self-energy we can analyze the dispersion relation $\omega^2=\epsilon^2(\mathbf{p})+\Sigma(\omega,\mathbf{p})$ in the range of small ω and p. New results are expected for the region $p<p_c=p_0\sqrt{tL}$ in which Σ_0 becomes comparable with $\epsilon^2(\mathbf{p})$. In a range of momentum $p_{DM} \ll p \ll p_c$ and angles $\psi \ll \sqrt{p_0 tL/p}$, we find a new propagating soft mode with the dispersion:

$$\omega = cp(p^2+p_0 p\psi^2)^{1/2}/p_0\sqrt{tL}. \tag{49}$$

The dissipation of the soft mode grows to the boundary of the region and becomes of the order of its energy at $\psi \sim \sqrt{p_0 tL/p}$ or $p \sim p_{DM}$. There is no soft mode beyond the indicated range. The spin-wave mode persists at $p>p_0 tL$. In a range $p \ll p_{DM}$ and small angles a new diffusion mode occurs with the dispersion:

$$\omega = -i\epsilon^2(\mathbf{p})t^{-1/4}L^{5/4}/\beta_0 c p_0. \tag{50}$$

The angular range of the diffusion mode increases with decreasing p and captures the entire circle at $p<p_0 tL$.

At the end of this section we would like to remind the reader that Eq. (44) was obtained for the quasiexcitation directed along the y axis. It can be easily checked that in case of arbitrarily directed $\mathbf{q}$ one must write $\sin^2(2\psi+2\phi)$ instead of $\sin(2\psi)$ in Eq. (41), where $\sin\phi = q_x/q$. However, in this case the integral becomes singular, so one must treat this equation more carefully. We will come back to this question in Appendix C.

V. RENORMALIZATION OF THE DIFFUSION MODE

In this section we concern ourself with the renormalization of the diffusion mode. As we established in the previous section, at wave vectors $p<p_{DM}$ the diffusive dynamics term dominates ($\lambda \to 0$ limit) in the harmonic part of JDF [Eqs. (27) and (28)]. The interaction between ''diffusons,'' given by the anharmonic part of the JDF (32), effectively ''renormalizes'' the diffuson dispersion at very small wave vectors $p<p_a \ll p_{DM}$. We shall determine the anomalous diffusion onset wave vector p_a in the end of this section.

To simplify further calculations, we introduce a scale transformation of the fields ϕ, $\hat{\phi} \to (Jg/T^2)^{-1/4}\phi$, $(Jg/T^2)^{-1/4}\hat{\phi}$. In these notations the JDF (28,32) is

$$\mathcal{J}[\phi,\hat{\phi}] = \sum_{\omega,\mathbf{k}} \hat{\phi}_{-\omega,-\mathbf{k}} \Bigg[a\frac{T}{\Gamma_0}\hat{\phi}_{\omega,\mathbf{k}} - \left(ak_y^2 + \frac{k_x^2}{a|k_y|} \right)\phi_{\omega,\mathbf{k}} - a\frac{i\omega}{\Gamma_0}\phi_{\omega,\mathbf{k}} - w\frac{k_x k_y}{|k_y|}\left[\frac{\phi^2}{2}\right]_{\omega,\mathbf{k}} - w\sum_{\Omega,\mathbf{p}} \phi_{\omega-\Omega,\mathbf{k}-\mathbf{p}}\frac{p_x p_y}{|p_y|}\phi_{\Omega,\mathbf{p}}$$
$$- w^2 a \sum_{\Omega,\mathbf{p}} \phi_{\omega-\Omega,\mathbf{k}-\mathbf{p}}|p_y|\left[\frac{\phi^2}{2}\right]_{\Omega,\mathbf{p}} \Bigg], \tag{51}$$

where $\Gamma_0 = J\Gamma$, $a=\sqrt{J/g}$ and

$$w = (T^2 g/J^3)^{1/4}. \tag{52}$$

Note, that it is possible to get rid of the spatial anisotropy charge a by rescaling the x coordinate only.

We have already seen from the statics consideration, (see Sec. II) that the mean-field scaling dimensions are: $\Delta_a^0=0$, $\Delta_{k_x}^0=3/2$, $\Delta_\phi^0=1/4$, and $\Delta_w^0=1/4$ (we remind the reader that the dimension of k_y is accepted to be 1). Similar simple calculations give the dynamic mean-field exponents: $\Delta_{1/\Gamma}^0=0$, $\Delta_\omega^0=2$, and $\Delta_{\hat{\phi}}^0=9/4$.

According to the standard renormalization-group procedure, we introduce renormalization constants: $\phi = Z_\phi^{1/2}\phi_R$, $\hat{\phi} = Z_{\hat{\phi}}^{1/2}\hat{\phi}_R$, $a=Z_a a_R$, $1/\Gamma_0 = Z_{1/\Gamma} 1/\Gamma_R$ and $\tilde{w} = Z_w \tilde{w}_R$, where

$$\tilde{w} = l^{1/4} w. \tag{53}$$

First we note that the fields ϕ and $\hat{\phi}$ have the same renormalization coefficients. It immediately follows from the fluctuation-dissipation theorem. Indeed, according to this theorem

$$D(\mathbf{x},t;\mathbf{y},t)=\langle\phi(\mathbf{x},t)\phi(\mathbf{y},t)\rangle=2T\mathrm{Im}\int_0^\infty G(\mathbf{x},t;\mathbf{y},0)dt=2T\mathrm{Im}\int_0^\infty\langle\hat{\phi}(\mathbf{x},t)\phi(\mathbf{y},0)\rangle dt.$$

In the standard formulation of the renormalization theory only the charges (a,Γ) and the fields $(\phi,\hat{\phi})$ acquire the renormalization coefficients, not coordinate, time or temperature. Thus, we conclude $Z_\phi=Z_{\hat{\phi}}$.

Let us divide the Janssen–De Dominics functional into two parts:

$$J=J_R+\Delta J.$$

The first one is the ''renormalized'' functional $\mathcal{J}_R$

$$\mathcal{J}_R[\phi_R,\hat{\phi}_R]=\sum_{\omega,\mathbf{k}}\hat{\phi}_{R-\omega,-\mathbf{k}}\left[a_R\frac{T}{\Gamma_R}\hat{\phi}_{R\omega,\mathbf{k}}-\left(a_Rk_y^2+\frac{k_x^2}{a_R|k_y|}\right)\phi_{R\omega,\mathbf{k}}-a_R\frac{i\omega}{\Gamma_R}\phi_{R\omega,\mathbf{k}}-\tilde{w}_Rl^{1/4}\frac{k_xk_y}{|k_y|}\left[\frac{\phi_R^2}{2}\right]_{\omega,\mathbf{k}}\right.$$
$$\left.-\tilde{w}_Rl^{1/4}\sum_{\Omega,\mathbf{p}}\phi_{R\omega-\Omega,\mathbf{k}-\mathbf{p}}\frac{p_xp_y}{|p_y|}\phi_{R\Omega,\mathbf{p}}-\tilde{w}_R^2a_Rl^{1/2}\sum_{\Omega,\mathbf{p}}\phi_{R\omega-\Omega,\mathbf{k}-\mathbf{p}}|p_y|\left[\frac{\phi_R^2}{2}\right]_{\Omega,\mathbf{p}}\right]. \tag{54}$$

The second one $\Delta\mathcal{J}$ contains the counterterms:

$$\Delta\mathcal{J}=\sum_{\omega,\mathbf{k}}\hat{\phi}_{R-\omega,-\mathbf{k}}\left[a_R\frac{T}{\Gamma_R}(Z_\phi Z_aZ_{1/\Gamma}-1)\hat{\phi}_{R\omega,\mathbf{k}}-\left(a_R(Z_\phi Z_a-1)k_y^2+(Z_\phi Z_a^{-1}-1)\frac{k_x^2}{a_R|k_y|}\right)\phi_{R\omega,\mathbf{k}}\right.$$
$$-a_R\frac{i\omega}{\Gamma_R}(Z_\phi Z_aZ_{1/\Gamma}-1)\phi_{R\omega,\mathbf{k}}-(Z_wZ_\phi^{3/2}-1)\tilde{w}_Rl^{1/4}\frac{k_xk_y}{|k_y|}\left[\frac{\phi_R^2}{2}\right]_{\omega,\mathbf{k}}-(Z_wZ_\phi^{3/2}-1)\tilde{w}_Rl^{1/4}$$
$$\left.\times\sum_{\Omega,\mathbf{p}}\phi_{R\omega-\Omega,\mathbf{k}-\mathbf{p}}\frac{p_xp_y}{|p_y|}\phi_{R\Omega,\mathbf{p}}-(Z_w^2Z_aZ_\phi^2-1)\tilde{w}_R^2a_Rl^{1/2}\sum_{\Omega,\mathbf{p}}\phi_{R\omega-\Omega,\mathbf{k}-\mathbf{p}}|p_y|\left[\frac{\phi_R^2}{2}\right]_{\Omega,\mathbf{p}}\right]. \tag{55}$$

A simple power counting shows that all corrections are only logarithmically divergent in the dimension of 5/2 (see Sec. II). Hence, in what follows the ε regularization scheme with $\varepsilon=1/2$ is assumed.

Evaluating the diagram shown in Fig. 1(a) up to second order in k_y (using the bare G_0 and D_0 functions), one finds the one-loop correction to the second term of the JDF:

$$-\frac{18}{128\pi}a_R\tilde{w}_R^2l^{1/2}\int\frac{dk_y}{k_y^{3/2}}-(Z_\phi Z_a-1)a_R.$$

In order to cancel the divergency, we set

$$Z_\phi Z_a=1-\frac{18}{128\pi}\tilde{w}_R^2l^{1/2}\int\frac{dk_y}{k_y^{3/2}}. \tag{56}$$

The same procedure for the fourth term gives

$$Z_aZ_\phi Z_{1/\Gamma}=1-\frac{1}{32\pi}\tilde{w}_R^2l^{1/2}\int\frac{dk_y}{k_y^{3/2}}. \tag{57}$$

In Appendix A we show that both the three-leg and four-leg vertices do not have one-loop corrections. Hence

$$Z_wZ_\phi^{3/2}=1. \tag{58}$$

One can easily see that the one-loop correction to the term $\hat{\phi}k_x^2/|k_y|\phi$ in Eq. (58) vanishes as well (see Refs. 12,20). It means that

$$Z_a^{-1}Z_\phi=1. \tag{59}$$

Equations (56–59) have the following solution:

$$Z_a=Z_\phi=1-\frac{9}{128\pi}\tilde{w}_R^2l^{1/2}\int\frac{dk_y}{k_y^{3/2}}, \tag{60}$$

$$Z_{1/\Gamma}=1+\frac{7}{64\pi}\tilde{w}_R^2l^{1/2}\int\frac{dk_y}{k_y^{3/2}}, \tag{61}$$

$$Z_w=\left(1+\frac{27}{256\pi}\tilde{w}_R^2l^{1/2}\int\frac{dk_y}{k_y^{3/2}}\right). \tag{62}$$

Next we introduce the Gell-Mann-Low β function $\beta=\mu(\partial w_R/\partial\mu)|_{w,\Lambda}$ and the Callan-Simanzik anomalous dimensions $\gamma_a=(\mu/a_R)(\partial a_R/\partial\mu)$, $\gamma_\Gamma=\mu\Gamma_R(\partial 1/\Gamma_R/\partial\mu)$ and $\eta_\phi=(\mu/Z_\phi)(\partial Z_\phi/\partial\mu)$. We denote by μ the scale at which the coupling constant is equal to $\tilde{w}_R$ and denote by Λ the scale at which the coupling constant is equal to $\tilde{w}$. From Eq. (62) and the definition of Z_w, one has

$$w_R=w\left(\frac{\Lambda}{\mu}\right)^{1/4}\left(1-\frac{27}{128\pi}\tilde{w}^2\Lambda^{1/2}\int_\mu^\Lambda\frac{dk_y}{k_y^{3/2}}\right). \tag{63}$$

And finally, we find the β function, which coincides with that found in the statics case[12]

$$\beta(\tilde{w}_R)=-\frac{1}{4}\tilde{w}_R+\frac{27}{256\pi}\tilde{w}_R^3. \tag{64}$$

The fixed point of the renormalization-group flow is

$$\tilde{w}_R^{*2} = \frac{64\pi}{27}.$$

After performing this procedure for anomalous dimensions one gets

$$\gamma_a = -\frac{9}{128\pi}\tilde{w}_R^2, \tag{65}$$

$$\gamma_{1/\Gamma} = \frac{7}{64\pi}\tilde{w}_R^2, \tag{66}$$

$$\eta_\phi = \frac{9}{128\pi}\tilde{w}_R^2. \tag{67}$$

Using the result for $\tilde{w}_R^*$, it is straightforward to find $\gamma_a{}^* = -1/6$, $\gamma_{1/\Gamma}{}^* = 7/27$, and $\eta_\phi{}^* = 1/6$. The scaling dimensionality of the frequency Δ_ω can be found from comparison of the second and third terms in the expansion (51): $\Delta_\omega = 2 - \gamma_{1/\Gamma} = 47/27$.

The long-range limit of the functions G and D is found from the Callan-Symanzik equation:[20]

$$G(t,\mathbf{x}) = \frac{1}{(x^2+|y|^{8/3})^{65/72}} f\left(\frac{x}{|y|^{4/3}}, \frac{t}{|y|^{47/27}}\right), \tag{68}$$

$$D(t,\mathbf{x}) = \frac{1}{(x^2+|y|^{8/3})^{1/4}} \tilde{f}\left(\frac{x}{|y|^{4/3}}, \frac{t}{|y|^{47/27}}\right), \tag{69}$$

where $f(x,y)$ and $\tilde{f}(x,y)$ are arbitrary functions.

In the static limit $(t=0)$ the exponents in the correlation function (68) are exact, as was previously found by one of the authors:[12]

$$D(\mathbf{r}) = \langle \phi(\mathbf{r})\phi(0)\rangle \sim (x^{-1/2}, y^{-2/3}). \tag{70}$$

For the Fourier components of the Green function Eq. (68), we find in the region of anomalous diffusion:

$$G(\omega,\mathbf{k}) = f_1\left(\frac{k_x}{k_y^{4/3}}, \frac{\omega}{k_y^{47/27}}\right). \tag{71}$$

The anomalous dispersion of the diffusion mode announced in the abstract follows from the last equation. We see that the static dipole contribution is not renormalized in dynamics, as it was suggested in Refs. 27,28. We also note that the exchange coupling acquires an anomalous dimension $\Delta_J = 1/3$, whereas the dynamic term ω/Γ acquires anomalous a dimension $+2/27$. Taking into account the anomalous dimension $\gamma_{1/\Gamma} = 7/27$, we conclude that the anomalous dimension of ω is $\gamma_\omega = 1/6$. The interaction between the diffusons in the scaling limit reduces the dissipation or, in other words, hardens the diffusion.

Now we can estimate the wave vector p_a, an upper boundary for anomalous diffusion. We assume that temperature is small Eq. (38). Initially, according to Eq. (52), the bare vertex $w_0 = \sqrt{T} g^{1/4}/J^{3/4}$, and is also small. Under the renormalization flow, the vertex w_R grows with the inverse wave vector as the power 1/4 Eq. (53). The RG flow starts at p_{DM} and approaches the fixed point at the root of the Gell-Mann-Low function $\beta(w_R) = 0$. Invoking Eq. (64), we find the fixed-point solution $w_R \sim 1$. Thus, the wave vector p_a is defined as the wave vector at which $w_R \sim 1$:

$$w_0(p_{DM}/p_a)^{1/4} \sim 1. \tag{72}$$

We see that

$$p_a \sim T^2 T^{3/4}, \tag{73}$$

that is very small. Even if t is not small, $p_a \sim p_{DM} w_0^4 \sim p_0(ga/J)$. It is much smaller than p_0.

VI. SUSCEPTIBILITIES

In this section we find the susceptibility to the magnetic field directed along the average magnetization $\langle \mathbf{S} \rangle$ (y axis), the so-called longitudinal susceptibility. We consider the magnetic field in the form $H = H_0 + \delta H(\mathbf{x},t)$ where H_0 is independent of $\mathbf{x}$ and t. When an additional magnetic field δH is imposed, a new vertex $\delta h_{\omega,\mathbf{k}}[\phi\hat{\phi}]_{-\omega,-\mathbf{k}}$ emerges in the JDF (25) (we denote $h = g_G \mu_B S H; \delta h = g_G \mu_B S \delta H$). It leads to a correction $\delta D(\omega,\mathbf{k})$ to the correlation function $D(\omega,\mathbf{k})$:

$$\delta D(\mathbf{x}_1=\mathbf{x}_2, t_1=t_2) = \int_\Omega \int_{\mathbf{k}} \int_\omega \int_{\mathbf{q}} h(\omega,\mathbf{q}) D_0(\Omega,\mathbf{k})[G_0(\Omega + \omega, \mathbf{k}+\mathbf{q}) + G_0^*(\Omega-\omega,\mathbf{k}-\mathbf{q})],$$

where $D_0(\omega,\mathbf{k})$ and $G_0(\omega,\mathbf{k})$ are taken from Eqs. (27) and (28).

By definition, the susceptibility χ is

$$\chi(\omega,\mathbf{k},h) = \frac{\delta}{\delta H(\omega,\mathbf{k})}\langle S_y\rangle = -\frac{1}{2}\frac{\delta}{\delta H(\omega,\mathbf{k})}\langle \phi(\mathbf{x},t)\phi(\mathbf{x},t)\rangle = -\frac{g_G\mu_B S}{2}\frac{\delta}{\delta h(\omega,\mathbf{k})} D(\mathbf{x}_1=\mathbf{x}_2, t_1=t_2). \tag{74}$$

Hence,

$$\chi(\omega,\mathbf{q}) = -g_G\mu_B S \int_\Omega \int_{\mathbf{k}} D_0(\Omega,\mathbf{k}) G_0(\Omega+\omega,\mathbf{k}+\mathbf{q}). \tag{75}$$

In the most interesting case, when $\mathbf{q}=0$, all integrals can be evaluated and the final answer is

$$\chi(\omega,h) = g_G\mu_B S \frac{\Gamma^2(3/4)}{4\pi\sqrt{\pi}} T \left(\frac{J^3}{g^2}\right)^{1/4} \frac{2\Gamma}{\omega}\left[h^{1/4} - \left(h - \frac{i\omega}{2\Gamma}\right)^{1/4}\right]. \tag{76}$$

In the limiting case $\omega = 0$, the susceptibility reads

$$\chi=\mathrm{const}h^{-3/4}.$$

This result has been found earlier.[18]

VII. CONCLUSION

In conclusion we discuss how new modes can be observed in experiment. The new modes appear on a macroscopic scale of the length of the order of magnitude $1\,\mu$m. Even rather weak in-plane anisotropy can suppress or disguise the new modes. Therefore, we can expect that the new dynamics will be observed in films with very weak in-plane anisotropy. The best known candidates for this role are films grown on hexagonal substrates. The hexagonal anisotropy is naturally weaker than the tetragonal one, because they are proportional to a higher degree of the relativistic parameter. Besides, the hexagonal anisotropy totally vanishes at large distances in a range of temperature from $(4/9)T_{\mathrm{BKT}}$ until T_{BKT}, where T_{BKT} is the temperature of Berezinskii-Kosterlitz-Thouless transition.[22,29] The simplest idea is to use the (111) face of fcc crystals, such as Ag, Au, Cu. An iron film on the (111) face of Ar has been grown by Bader and co-workers.[30] Recently the Ru film has been grown on the hexagonal graphite substrate.[31] Thus, 2D ferromagnets with exact XY symmetry are available.

An important question is whether the interaction of spins with the conductivity electrons in metallic films leaves an opportunity to observe the anomalous dissipation and soft modes. It seems surprising, but no dissipation of spin waves by electrons occurs in the long-wave limit. Indeed, if a conductivity electron emits or absorbs a spin wave, the electron's spin projection changes sign. Thus, the electron transits from majority to minority band or vice versa. Since we are interested in very long spin waves, the electron momentum almost does not change in such a reaction. It means that the electron energy changes by the value of self-consistent exchange field energy, much larger than the energy of a low-wave-vector spin wave. Thus, this process is forbidden by the conservation laws.

However, not only the film, but the substrate is also metallic. The varying magnetization in the film creates the eddy currents in the substrate which undergo Ohmic losses. The dissipation due to this mechanism is easy to calculate and to compare with the spin-wave energy. The result is

$$\frac{\gamma(\mathbf{q})}{\omega(\mathbf{q})}\approx\frac{\pi^2\sigma\mu_B^2\sqrt{\lambda}}{c^2\hbar(qa)^2\sqrt{J}},$$

where $\gamma(\mathbf{q})$ is the width of the spin-wave energy level due to Ohmic losses, σ is the substrate conductivity, μ_B is the Bohr magneton and a is the lattice constant. For $q=p_0=10^{-4}$ cm, $a=3$ Å, $\rho=1/\sigma=1.56\ \mu\Omega$ cm, and $\lambda/J=1/20$, we find the ratio $\gamma/\omega\approx10^{-8}$. This estimate shows that the dissipation due to eddy currents is neglegibly small.

The next question is: what dynamic effects can be observed and at what conditions? As we have noted earlier, if the in-plane anisotropy is not especially small (less than 1 K in energy scale), the only opportunity is to use a sixfold substrate in the range of temperature, higher than $(4/9)T_{\mathrm{BKT}}$. It has been predicted[22,29] that sixfold anisotropy vanishes on large scale in this range of temperature. At such high temperatures the observation of propagating soft modes described in Sec. III seems to be improbable, since all they require $t\ll1$, i.e. $T\ll T_c$. However, the anomalous diffusion can be observed even at $T\sim T_c$, given a sufficiently large scale of length ($\geqslant 10\ \mu$m). The best way to observe it is to apply a short and inhomogeneous pulse of magnetic field, and follow when the secondary signal will arrive to fixed indicators. Such a picosecond-pulse technique has been recently used for investigation of the film dynamics.[32] We propose to use the same pulse technique to different films.

A quick estimation of the time $t_{x,y}$ needed for the secondary signal to reach the indicator at distances L_x and L_y along $\hat{x}$ and $\hat{y}$ directions shows: $t_{x,y}\sim(\hbar/Jp_0a)(L_{x,y}/a)^{\Delta^a_{x,y}}$, where a is the lattice constant and $\Delta^a_{x,y}$ are the anomalous-diffusion dimensions for axes $\hat{x}$ and $\hat{y}$, respectively. Thus, $t_x\sim\hbar/ga(L_x/a)^{47/36}\sim10^{3.5}$ s and $t_y\sim\hbar/ga(L_y/a)^{47/27}\sim1$ s, where we have assumed $L_x\sim L_y\sim1$ cm.

The retardation time for the secondary signal is much longer than the time for the primary signal propagation. The strong size and direction dependence of the propagation time can be used for detecting the anomalous diffusion.

ACKNOWLEDGMENTS

The work of A.K. was suported in part by the Swiss National Fond under Oststaaten-Soforthilfemassnahmen Grant No. N7GUPJ038620. This work was also partly supported by DOE Grant No. DE-FG03-96ER45598. Our thanks are due to M.V. Volpert for her help in preparation of the manuscript.

APPENDIX A: ONE-LOOP VERTICES CORRECTIONS

First we note that the JDF (51) has the following nontrivial symmetry:[12]

$$\phi(\mathbf{k},\omega)\to\phi(\mathbf{k},\omega)+\varepsilon\,\delta^2(\mathbf{k})\,\delta(\omega), \tag{A1}$$

$$\hat{\phi}(\mathbf{k},\omega)\to\hat{\phi}(\mathbf{k},\omega), \tag{A2}$$

$$k_x\to k_x-\varepsilon awk_y, \tag{A3}$$

$$k_y\to k_y. \tag{A4}$$

The partition function (24) must have the same symmetry $\delta Z[j,\hat{j}]=0$. By a standard procedure,[20] we find the implications of the symmetry (A1–A4), known as Ward-Takahashi identities to the so-called ‘‘generating functional for proper vertices’’ $\mathbf{\Gamma}[\varphi,\hat{\varphi}]$ (it is the Legendre transform of $\ln Z[j,\hat{j}]$ with respect to the fields φ and $\hat{\varphi}$):

$$\int_{\mathbf{k}}\int_{\omega}\Bigg(\delta(\omega)\,\delta(\mathbf{k})\,\frac{\delta\mathbf{\Gamma}}{\delta\varphi(-\mathbf{k},-\omega)}-aw\Bigg[k_y\varphi(-\mathbf{k},-\omega)\times\frac{\partial}{\partial k_x}\,\frac{\delta\mathbf{\Gamma}}{\delta\varphi(-\mathbf{k},-\omega)}+k_y\hat{\varphi}(-\mathbf{k},-\omega)\times\frac{\partial}{\partial k_x}\,\frac{\delta\mathbf{\Gamma}}{\delta\hat{\varphi}(-\mathbf{k},-\omega)}\Bigg]\Bigg)=0. \tag{A5}$$

Now, writing down the Taylor-like expansion of $\mathbf{\Gamma}[\varphi,\hat{\varphi}]$ over fields φ and $\hat{\varphi}$ and plugging it into Eq. (A5), we find

$$\mathbf{\Gamma}_{\varphi\varphi\hat{\varphi}}(0,\overline{k},-\overline{k})-awk_y\frac{\partial}{\partial k_x}\mathbf{\Gamma}_{\varphi\hat{\varphi}}(\overline{k},-\overline{k})=0, \tag{A6}$$

where $\overline{k}$ is used for $(\mathbf{k},\omega)$. It was proven[20] that there is no correction to the term $k_x^2/|k_y|$ due to analyticity. According to Eq. (A6), it means that corrections to the three-leg vertex vanish if we set one of the two frequencies, which the vertex depends upon, equal to zero. Hence, the corrections must depend on the product of the two ω's and are small in the framework of Sec. III. Considering loop-wise expansion of $\mathbf{\Gamma}$ in the same spirit, one finds

$$\mathbf{\Gamma}=\sum_{n=0}^{\infty} w_n\mathbf{\Gamma}^{(n)}\approx\mathbf{\Gamma}^{(0)}+w\mathbf{\Gamma}^{(1)}+\ \ldots .$$

Looking only at the divergent parts of the corrections $\mathbf{\Gamma}^{(0)}$ and $\mathbf{\Gamma}^{(1)}$, we see that, because $\mathbf{\Gamma}^{(0)}$ is just the bare action and has no divergencies at all, the one-loop corrections to the ω-independent three-leg vertex do not diverge. As a consequence of this fact we obtain Eq. (58).

APPENDIX B: CALCULATION OF SELF-ENERGY

Let us start with the Dyson Eq. (42) corresponding to a one-loop diagram on Fig. 1(a). We may integrate Eq. (42) over the contour in complex plane ω such that the poles of the G function are outside the contour:

$$\Sigma_0=\frac{p_0^2c^6t}{2\pi}\int f^2(\mathbf{p}-\mathbf{q}/2,\mathbf{p}+\mathbf{q}/2)\{[\Omega+\epsilon(\mathbf{p}-\mathbf{q}/2)]^2-\epsilon^2(\mathbf{p}+\mathbf{q}/2)+ib_1\}^{-1}+\{[\Omega-\epsilon(\mathbf{p}-\mathbf{q}/2)]^2-\epsilon^2(\mathbf{p}+\mathbf{q}/2)-ib_1]^{-1}\}\frac{d^2p}{\epsilon^2(\mathbf{p}-\mathbf{q}/2)}. \tag{B1}$$

It is convenient to change in the second G function $\mathbf{p}$ on $-\mathbf{p}$:

$$\Sigma_0=\frac{p_0^2c^6t}{2\pi}\int\frac{f^2(\mathbf{p}-\mathbf{q}/2,\mathbf{p}+\mathbf{q}/2)d^2p}{\Omega+\epsilon(\mathbf{p}-\mathbf{q}/2)-\epsilon(\mathbf{p}+\mathbf{q}/2)+ib_1}\left[\frac{\epsilon^{-2}(\mathbf{p}-\mathbf{q}/2)}{[\Omega+\epsilon(\mathbf{p}+\mathbf{q}/2)+\epsilon(\mathbf{p}-\mathbf{q}/2)]}+\frac{\epsilon^{-2}(\mathbf{p}+\mathbf{q}/2)}{[\Omega-\epsilon(\mathbf{p}+\mathbf{q}/2)-\epsilon(\mathbf{p}-\mathbf{q}/2)]}\right]. \tag{B2}$$

Keeping only the lowest order in small momentum $\mathbf{q}$ and frequency Ω and after a simple expansion in the brackets we find

$$\Sigma_0=\frac{p_0^2c^6t}{2\pi}\int\frac{f^2(\mathbf{p},\mathbf{p})d^2p}{\Omega+\epsilon(\mathbf{p}-\mathbf{q}/2)-\epsilon(\mathbf{p}+\mathbf{q}/2)+ib_1}\,\frac{-2\Omega+4[\epsilon(\mathbf{p}+\mathbf{q}/2)-\epsilon(\mathbf{p}-\mathbf{q}/2)]}{4\epsilon^4(\mathbf{p})}. \tag{B3}$$

At this point we separate the above integral into the static Ω-independent part and the rest into the "dynamical" self-energy. The static Ω-independent self-energy (it is always real) reads

$$\Sigma_{\mathrm{st}}=-\frac{p_0^2c^6t}{2\pi}\int\frac{f^2(\mathbf{p},\mathbf{p})}{\epsilon^4(\mathbf{p})}d^2p. \tag{B4}$$

Now let us take into account the static self-energy given by the diagram in Fig. 1(b):

$$\Sigma_b=\frac{p_0c^4t}{4\pi}\int\frac{p_y^2-p_x^2}{p^2\epsilon^2(\mathbf{p})}d^2p. \tag{B5}$$

Comparing Eq. (B4) [remember that $f(\mathbf{p},\mathbf{p})=p_x^2p_y^2/p^2$] and Eq. (B5), we conclude that these cancel each other in the $\mathcal{A}$ shell. Thus, we have verified explicitly that the $\Omega,\mathbf{q}$-independent part of the self-energy is strictly zero as guaranteed by the Ward identity due to the rotation symmetry of the system (see Appendix A). To get nonzero static self-energy we have to expand self-energies like Eqs. (B4,B5) in powers of the transferred momentum $\mathbf{q}$. This will be done in Sec. IV using the static field-theoretical technique, not dynamical as in this appendix. The result is that $\mathbf{q}$-dependent static self-energy only matters when the spin-wave momentum $\mathbf{q}$ is so small that anomalous diffusion sets up. On shorter wavelengths like $q<p_{DM}$ we may safely neglect the contribution of Eqs. (B4) and (B5).

Now we return to the Ω-dependent dynamical part of the self-energy:

$$\Sigma_0(\chi,q)=\frac{p_0^2c^6t}{2\pi}2\Omega\int\frac{\sin^2(2\phi)}{\Omega-q\cos\phi+ib_1}\,\frac{dp\,d\phi}{4c^4p}. \tag{B6}$$

The integral over p gives exactly the logarithmic factor L:

$$\Sigma_0(\chi,q)=\frac{p_0^2c^2t}{\pi}L\Omega\int\frac{\sin^2\phi\cos^2\phi}{\Omega-q\cos\phi+ib_1}d\phi. \tag{B7}$$

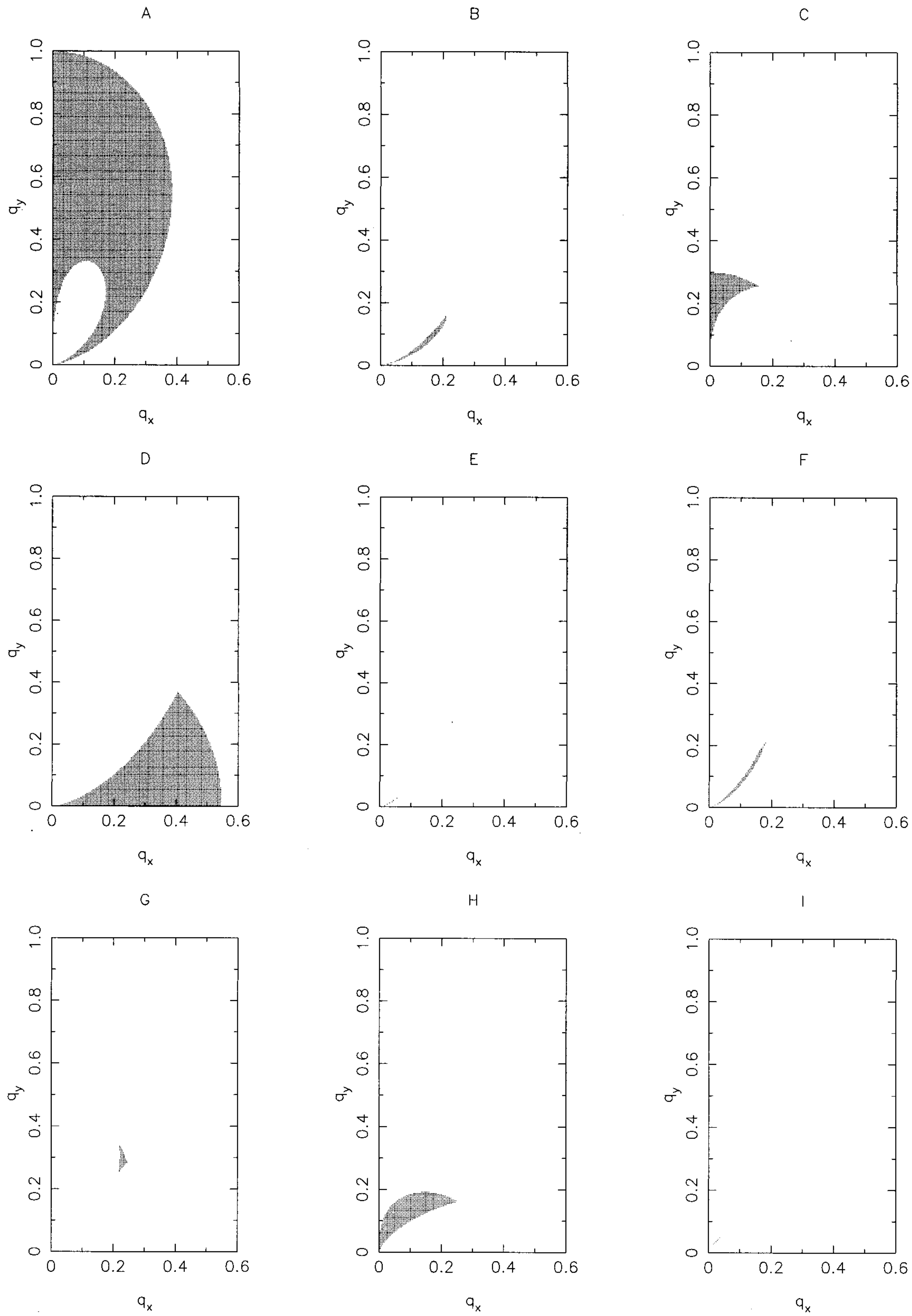

FIG. 3. Maps of the regions with different dispersion relations.

The imaginary part of Eq. (B7) could be easily found in the limit $\Omega, cq \gg b_1$. In this case we use the formula

$$\frac{1}{x-i0}=P\frac{1}{x}+i\pi\delta(x)$$

to find the $\mathfrak{I}$ part of Eq. (46). The $\mathfrak{R}$ part is also simple to calculate:

$$\mathfrak{R}\Sigma_0(\chi,q)=\frac{p_0^2c^2t}{\pi}L\frac{\Omega}{q}\times\int\frac{\sin^2\phi\cos\phi[(-\Omega+q\cos\phi)+\Omega]}{\Omega-q\cos\phi}d\phi, \tag{B8}$$

and then simplify

$$\Re\Sigma_0(\chi,q)=\frac{p_0^2c^2t}{\pi}L\frac{\Omega^2}{q}\int\frac{\sin^2\phi\cos\phi}{\Omega-q\cos\phi}d\phi. \quad \text{(B9)}$$

We repeat the same step

$$\Re\Sigma_0(\chi,q)=\frac{p_0^2c^2t}{\pi}L\frac{\Omega^2}{q^2}\int\frac{\sin^2\phi(-\Omega+q\cos\phi+\Omega)}{\Omega-q\cos\phi}d\phi \quad \text{(B10)}$$

with further simplification

$$\Re\Sigma_0(\chi,q)=\frac{p_0^2c^2t}{\pi}L\frac{\Omega^3}{q^2}\int\frac{\sin^2\phi}{\Omega-q\cos\phi}d\phi-\frac{p_0^2c^2t}{\pi}L\frac{\Omega^2}{q^2}\frac{2\pi}{2}. \quad \text{(B11)}$$

One could easily continue the same procedure to find

$$\Re\Sigma_0(\chi,q)=p_0^2c^2tL\frac{\Omega^2}{q^2}\left(2\frac{\Omega^2}{q^2}-1\right)=p_0^2c^2tL\cos^2\chi\cos(2\chi), \quad \text{(B12)}$$

which is exactly Eq. (46) of Sec. III.

Finally let us show that the self-energy in the $\mathcal{A}$ shell is negligible. We first assume that it does negligible: $a^2(\omega,\mathbf{p})\ll\epsilon^2(\mathbf{p})$. Then we consider the self-energy Eq. (B2), provided the external frequency and momentum lies on the mass shell inside the $\mathcal{A}$ shell: $\Omega=\epsilon(\mathbf{q})$. Let also $\sqrt{\lambda/J}\gg q<p_0$. One could easily verify that the main contribution comes if $q\ll p$. In this case we may use Eq. (B6):

$$\Sigma_0=\frac{p_0^2c^6t}{2\pi}2cq\int\frac{\sin^2(2\phi)}{c\sqrt{q^2+p_0q_x^2/q}-q\cos\phi+ib_1}\frac{dp\,d\phi}{4c^4p}. \quad \text{(B13)}$$

Integration over p gives $\ln(\sqrt{\lambda/J}/q)$, whereas the integration over the relative direction of internal and external spin waves gives factor $\sqrt{q^3/p_0q_x^2}$. Thus, the real self-energy

$$a^2(\Omega,\mathbf{q})\sim tLc^2p_0\sqrt{p_0q}.$$

At small t this result justifies our neglect of the self-energy in the $\mathcal{A}$ shell.

APPENDIX C: THE ANGULAR DEPENDENCE OF THE SELF-ENERGY OPERATOR

In this appendix we analyze the angular dependence of the self-energy part found in Sec. III. Repeating all arguments, one can find that Eq. (41) must be written in a slightly modified form:

$$\Sigma_0(\Omega,q,\phi)=\frac{c^2p_0^2t}{4\pi}\int\frac{c^4p^3dp}{\epsilon^4(\mathbf{p})}\frac{\Omega\sin^2(2\psi+2\phi)d\psi}{\Omega-cq\cos\psi+ib_1}. \quad \text{(C1)}$$

Now again, assuming $cq\gg b_1$, one finds

$$\Sigma_0(r,\phi)=c^2p_0^2tL\left[r^2(2r^2-1-2r\sqrt{r^2-1})\cos(4\phi)+\frac{1}{2}\frac{r}{\sqrt{r^2-1}}\sin^2(2\phi)\right], \quad \text{(C2)}$$

where $r=\Omega/cq$. In the case where $\phi=0$ and $r=\cos\chi$, formula (42) is recovered.

The main contribution to the integral in Eq. (45) comes from a region of very small angles ϕ. Hence, the correction to Σ_0 we have found above is not important in this calculations and Eq. (46) still holds.

By the next step we need to plug $b_1(\psi)$ from Eq. (46) to the nonzero-angle form of Eq. (44)

$$\Sigma(\Omega,q,\phi)=i\Omega c^2p_0^2tL\int_0^{2\pi}\frac{d\psi}{4\pi}\frac{\sin^2(2\psi+2\phi)}{ib_1(\psi)+\Omega-cq\cos\psi}. \quad \text{(C3)}$$

Evaluating this integral, one finds

$$\Sigma(|\Omega|\ll cq)=i\Omega c\tilde{p}\left[\cos^2(2\phi)-\alpha\left(\frac{q}{p_{DM}}\right)^{-3/5}\sin^2(2\phi)\right] \quad \text{(C4)}$$

$$\Sigma(|\Omega|\gg cq)=i\Omega c\tilde{p}\left[\cos^2(2\phi)-\alpha\left(\frac{|\Omega|}{cp_{DM}}\right)^{-3/5}\sin^2(2\phi)\right], \quad \text{(C5)}$$

where we denote $\tilde{p}=\beta_0t^{1/4}L^{5/4}p_0$, $p_{DM}=\beta_1t^{3/4}L^{-1/4}p_0$, and $\alpha=2^{-1/5}/\beta_0\beta_1\approx0.54$. In what follows we will also use the notation $p_c=p_0\sqrt{tL}$. While Eq. (C2) for Σ holds for $cq,\Omega\gg b_1(\psi)$, Eqs. (C4) and (C5) are valid in the opposite case $cq,\Omega\ll b_1(\psi)$.

Now we can analyze the dispersion relation $\omega^2(\mathbf{q})=\epsilon^2(\mathbf{q})+\Sigma(\omega,\mathbf{q})$ more accurately. In experiment usually $t\ll1$ so $p_0\gg\tilde{p}\gg p_c\gg p_{DM}$. Easy, but tedious calculations show that there can exist up to nine asymptotic regions in the momentum space with different dispersion relations:

$$\text{(a)}\quad \omega^2=c^2p_0q\sin^2\phi,$$

$$\text{if}\quad p_{DM}\frac{\sin^{3/2}(2\phi)\sin(\phi/2)}{\cos\phi}\ll q\ll p_0\sin^2\phi,$$

$$\text{(b)}\quad \omega^2=\frac{1}{2}c^2q^3\frac{p_0}{p_c^2}\phi^2,$$

$$\text{if}\quad \frac{p_c^2}{p_0}\phi^2\left(1+\frac{p_0p_{DM}}{p_c^2}\phi^{1/2}\right)\ll q\ll\frac{p_c^2p_0\phi^2}{p_0^2\phi^4+p_c^2},$$

$$\text{(c)}\quad \omega^2=\frac{1}{2}c^2q^3\frac{p_0}{p_c^2},$$

$$\text{if}\quad p_{DM}(\pi/2-\phi)^{1/2}\ll q\ll\frac{p_c^2}{p_0},$$

$$\text{(d)}\,\omega^2=\frac{c^2}{2}\frac{q^4}{p_c^2},$$

$$\text{if}\quad p_0\phi^2\ll q\ll p_c,$$

(e) $\omega=-icq^2\frac{p_0}{p_c^2}, \text{if} p_{DM}\phi^{5/2}\ll q\ll\frac{p_c^2}{p_0}\phi^2,$

(f) $\omega=-icq\frac{p_0}{\tilde{p}}\phi^2,$

if $p_{DM}\phi^{10/3}\ll q\ll p_{DM}\phi^{5/2}; \phi\ll\sqrt{\tilde{p}/p_0},$

(g) $\omega=-icq\frac{p_0}{\tilde{p}}\phi^2,$

if $p_{DM}\phi^{4/3}\ll q\ll p_{DM}\phi^{1/2}; \quad \sqrt{\tilde{p}/p_0}\ll\phi\ll 1,$

(h) $\omega=-icq\frac{p_0}{\tilde{p}},$

if $\frac{\tilde{p}p_{DM}}{p_0}(\pi/2-\phi)^{10/3}\ll q\ll\frac{\tilde{p}p_{DM}}{p_0}(\pi/2-\phi)^{1/2},$

(i) $\omega^{2/5}=-iq\frac{c^{2/5}p_0}{4\alpha\tilde{p}p_{DM}^{3/5}}\frac{1}{\cos^2\phi},$

if $p_{DM}\left(\frac{\tilde{p}}{p_0}\right)^{5/3}(\pi/2-\phi)^{10/3}\ll q\ll\frac{\tilde{p}p_{DM}}{p_0}\phi^{4/3}(\pi/2-\phi)^{10/3},$

where $0<\phi<\pi/2$. In Fig. 3 we show the case $t\sim0.3$, $L\sim1$, and $p_0=1$.

---

[1] A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinsky, *Methods of Quantum Field Theory in Statistical Physics* (Dover, New York, 1975).

[2] A. B. Migdal, Zh. Eksp. Teor. Fiz. **34**, 417 (1958).

[3] D. V. Khveshchenko and P. C. E. Stamp, Phys. Rev. Lett. **71**, 2118 (1993); B. L. Altshuler, L. B. Ioffe, and A. Millis, Phys. Rev. B **50**, 14 048 (1994).

[4] A. Stern and B. I. Halperin, Phys. Rev. B **52**, 5890 (1995).

[5] K. Kawasaki, Phys. Rev. **150**, 291 (1966).

[6] L. P. Kadanoff and J. Swift, Phys. Rev. **166**, 89 (1968).

[7] P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1977).

[8] R. Zeyher, Ferroelectrics **66**, 217 (1986).

[9] E. I. Kats and V. V. Lebedev, *Fluctuational Effects in the Dynamics of Liquid Crystals* (Springer-Verlag, Berlin, 1993).

[10] G. Grüner, Rev. Mod. Phys. **60**, 1129 (1988).

[11] E. Frey and F. Schwabl, Adv. Phys. **43**, 577 (1994).

[12] A. Kashuba, Phys. Rev. Lett. **73**, 2264 (1994).

[13] P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A **8**, 423 (1973); C. De Dominicis and L. Peliti, Phys. Rev. B **18**, 353 (1978); H. K. Janssen, Z. Phys. B **23**, 372 (1976).

[14] A. Kashuba, Ar. Abanov, and V. L. Pokrovsky, Phys. Rev. Lett. **77**, 2554 (1996).

[15] A. V. Chubukov and M. I. Kaganov, Sov. Phys. Usp. **30**, 1015 (1987).

[16] The classical exchange constant J and anisotropy λ used in this paper differs from that in quantum Heisenberg model: $J_{\mathrm{cl}}=J_{\mathrm{qua}}S^2$ and $\lambda_{\mathrm{cl}}=\lambda_{\mathrm{qua}}S^2$.

[17] A. Kashuba and V. L. Pokrovsky, Phys. Rev. B **48**, 10 335 (1993).

[18] S. V. Maleev, Zh. Eksp. Teor. Fiz. **71**, 2375 (1976); V. L. Pokrovsky and M. V. Feigelman, *ibid.* **72**, 557 (1977).

[19] N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966).

[20] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Oxford, New York, 1993).

[21] L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics: Electrodynamics of Continuous Media* (Pergamon, New York, 1960), Vol. 9.

[22] A. Ya. Blank, V. L. Pokrovsky, and G. V. Uimin, J. Low Temp. Phys. **14**, 459 (1974).

[23] V. L. Berezinsky, Zh. Eksp. Teor. Fiz. **59**, 907 (1970) [Sov. Phys. JETP **32**, 493 (1971)].

[24] An allowed by symmetry term $\sim R'\dot{\pi}\pi$ in the dissipative function is of higher order in time derivatives, since $\dot{\pi}\sim\ddot{\phi}$. The dissipation functional is also irrelevant.

[25] L. D. Landau and E. M. Lifshitz, *Course on Theoretical Physics: Statistical Physics* (Pergamon, New York, 1980), Vol. 5, Pt. 1.

[26] The a function is negligible in the $\mathcal{A}$ shell. If one assumes that $a^2(\omega,\mathbf{p})\ll\epsilon^2(\mathbf{p})$ inside the $\mathcal{A}$ shell, then from Eq. (35) one finds: $a^2(\epsilon(\mathbf{q}),\mathbf{q})\sim tLc^2p_0\sqrt{p_0q}$, justifying the assumption.

[27] A. Aharony, in *Phase Transition and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1976).

[28] R. A. Pelcovits and B. I. Halperin, Phys. Rev. B **19**, 4614 (1979).

[29] J. V. Jose, L. P. Kadanoff, S. Kirckpatrick, and D. R. Nelson, Phys. Rev. B **16**, 1217 (1977).

[30] Z. Q. Qiu, J. Pearson, and S. D. Bader, Phys. Rev. Lett. **67**, 1646 (1991).

[31] R. Pfandzelter, G. Steierl, and C. Rau, Phys. Rev. Lett. **74**, 3467 (1995).

[32] H. C. Siegmann, E. L. Garwin, C. Y. Prescott, J. Feldman, D. Mauri, D. Weller, R. Allenspach, and W. Weber, J. Magn. Magn. Mater. **151**, L8 (1955).