

Nonlinear magnetic ringing of spin-ordered solid ^3He

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Exact and complete solution is presented on the nonlinear longitudinal magnetic ringing of spin-ordered bcc ^3He ensuing on the complete switchoff of a dc magnetic field \vec{H}_1 . The theory is based on the set of dynamic equations proposed by Osheroff *et al.*, which neglects relaxation effects. The complete motion is found to be characterized by three principal frequencies, of which only two enter into the ringing behavior of the longitudinal magnetization. An alternative way to prepare the initial state for observing the subsequent nonlinear ringing, viz., by applying an ac magnetic pulse at the zero-field resonance frequency instead of turning off a dc field \vec{H}_1 , is also analyzed, and found to be less than ideal, even though the subsequent zero-field ringing behavior is still exactly soluble.

I. INTRODUCTION

Recently, Osheroff, Cross, and Fisher¹ have shown that the low-field antiferromagnetic resonance spectrum of spin-ordered bcc ^3He exhibits large shifts from the Larmor frequency which can be well fitted by the formula

$$\omega^2 = \frac{1}{2} \{ \omega_L^2 + \Omega_0^2 \pm [(\omega_L^2 - \Omega_0^2)^2 + 4\omega_L^2 \Omega_0^2 \cos^2 \theta]^{1/2} \}, \quad (1)$$

where $\omega_L \equiv \gamma H$ is the Larmor frequency, and the \pm sign indicates that there are two resonance branches at any given applied field H . Furthermore, they have shown that this spectrum can be understood if one postulates that the spin-ordered state of bcc ^3He is an antiferromagnetic state with ferromagnetic planes arranged in an up-up-down-down sequence along a $[1,0,0]$ direction, which they denote with a unit vector \hat{l} . The direction of the sublattice magnetization at zero field is denoted with a unit vector \hat{d} . The dipole-dipole interaction of the ^3He spins then contributes a term $\frac{1}{2} \lambda (\hat{l} \cdot \hat{d})^2$ to the total energy density, and the spin dynamics of such an antiferromagnetic system is shown to obey the following two coupled equations:

$$\partial_t \hat{d} = \hat{d} \times (\gamma \vec{H} - \gamma^2 \chi_0^{-1} \vec{S}), \quad (2)$$

$$\partial_t \vec{S} = \gamma \vec{S} \times \vec{H} - \lambda (\hat{d} \cdot \hat{l}) (\hat{d} \times \hat{l}). \quad (3)$$

In these equations γ is the gyromagnetic ratio, χ_0 is the *transverse* susceptibility, \vec{S} is the spin density, and $\partial_t \hat{d}$ denotes the time derivative of \hat{d} . If Eqs. (2) and (3) are first *linearized* around the equilibrium solution for $\hat{l} \cdot \vec{H} = \cos \theta$ (where $\hat{H} \equiv \vec{H}/H$), viz., $\gamma \vec{S} = \chi_0 \vec{H}$, $\hat{d} \cdot \hat{l} = \hat{d} \cdot \vec{S} = 0$, and $\hat{l} \cdot \vec{S} = \cos \theta$, the resulting equations then describe the undamped resonance modes

of frequencies given by Eq. (1) with $\Omega_0^2 = \gamma^2 \lambda / \chi_0$.

Thus the cw NMR experiment of Osheroff *et al.*, which probed only the small oscillations of the spin-ordered solid ^3He , has, strictly speaking, verified the linearized version of Eqs. (2) and (3) only. In order to verify the full equations, it is necessary to perform a *nonlinear* experiment of some sort, such as the longitudinal magnetic ringing experiment performed on superfluid ^3He by Webb, Sager, and Wheatley.² In such an experiment, the spin system is first allowed to reach an equilibrium in a static magnetic field \vec{H}_1 . This field \vec{H}_1 is completely turned off at time $t=0$, in a time very short compared with all relevant intrinsic times, and the time-dependent longitudinal magnetization at $t > 0$ is then monitored. Ringing behavior has been observed in both $^3\text{He-A}$ and $^3\text{He-B}$, which has been theoretically explained.³ As a matter of fact, the ringing behavior in $^3\text{He-A}$ is described by a set of dynamic equations, which differ from Eqs. (2) and (3) only in the sign of the λ term, except that the initial condition is also quite different [i.e., $(\hat{d} \parallel \hat{l}) \perp (\vec{S} \parallel \vec{H})$ for $^3\text{He-A}$].³ All of these differences are the consequences of a single sign change in the dipole energy $E_D = \frac{1}{2} \lambda (\hat{d} \cdot \hat{l})^2$, which, nevertheless, leads to important differences in its topological implications. This is best seen by viewing the various orientations of \hat{d} as points on a unit sphere, with the north pole of the sphere representing the direction of \hat{l} . Then for $^3\text{He-A}$, for which $\lambda < 0$, the north and south poles form two isolated point minima, while the equator represents a line of degenerate maxima. Thus if the system is to change from one dipole-energy minimum to the other, it would have to cross the dipole-energy ridge on the equator, which can occur only if the total energy exceeds the critical value $\lambda/2$. On the other hand, for spin-ordered bcc ^3He , the roles of dipole-energy minima and maxima are interchanged, so that the system now has a degenerate

erate set of dipole-energy minima, and only two isolated point maxima. Thus a general motion of \hat{d} no longer has to pass through the dipole-energy maxima (except for special initial conditions), and it is now possible for \hat{d} to move along the dipole-energy valley without experiencing any dipole torque. All of these topological features should surface in a nonlinear magnetic resonance experiment, which makes such an experiment interesting.

In this paper, we will discuss the nonlinear ringing behavior of spin-ordered solid ^3He , for two ways of preparing the initial state, and assuming that Eqs. (2) and (3) are the correct set of dynamic equations. This means that, in particular, relaxation effects will be neglected in this discussion. We shall show that within this approximation, the nonlinear ringing behavior becomes an exactly solvable problem, and all relevant measurable quantities can be given in terms of known functions and their quadratures. This should allow detailed comparison with experimental observations, for the purpose of either confirming the validity of Eqs. (2) and (3), or uncovering new physics about this system.

Three more sections follow this Introduction: Sec. II presents the exact solution of Eqs. (2) and (3) in a "complete-turn-off" situation, as defined in the second paragraph of this Introduction. This corresponds to one way of preparing the initial state. In Sec. III, we note a possible difficulty with this way of preparing the initial state, and then analyze a possible alternative. It is found that this alternative approach is less than ideal, even though the subsequent ringing behavior is still an exactly solvable problem. Section IV contains conclusion and discussion.

II. EXACT SOLUTION IN A COMPLETE-TURN-OFF SITUATION

We begin by introducing normalized quantities

$$\tau = \Omega_0 t, \quad \bar{\mathbf{h}} = \gamma \bar{\mathbf{H}} / \Omega_0, \quad \bar{\mathbf{s}} = \Omega_0 \bar{\mathbf{S}} / \lambda, \quad (4)$$

which converts Eqs. (2) and (3) to

$$\partial_\tau \hat{d} = \hat{d} \times (\bar{\mathbf{h}} - \bar{\mathbf{s}}), \quad (5)$$

$$\partial_\tau \bar{\mathbf{s}} = \bar{\mathbf{s}} \times \bar{\mathbf{h}} - (\hat{d} \cdot \hat{l})(\hat{d} \times \hat{l}). \quad (6)$$

In a complete-turn-off experiment, the system is allowed to first reach equilibrium in an external magnetic field $\bar{\mathbf{H}}_1$. This field is then completely switched off at $t=0$, during a time interval short compared with all the characteristic ringing times of the spins. Thus Eqs. (5) and (6) must be solved for $t > 0$, with $\bar{\mathbf{h}}$ set equal to zero, and for the initial condition

$$\begin{aligned} \bar{\mathbf{s}}(0) &= \bar{\mathbf{h}}_1, \quad \hat{d}(0) \perp \hat{l} \text{ and } \bar{\mathbf{s}}(0), \\ \bar{\mathbf{s}}(0) \cdot \hat{l} &= s(0) \cos \theta. \end{aligned} \quad (7)$$

In the following, we shall denote $\bar{\mathbf{s}}(0)$ and $\bar{\mathbf{d}}(0)$ simply as $\bar{\mathbf{s}}_0$ and $\bar{\mathbf{d}}_0$, and solve the problem in terms of these two quantities and θ . We choose a coordinate system in which

$$\hat{l} = (0, 0, 1), \quad \bar{\mathbf{s}}_0 = s_0(-\sin\theta, 0, \cos\theta), \quad \hat{d}_0 = (0, 1, 0). \quad (8)$$

It is straightforward to establish the following four invariants from Eqs. (5) and (6) with $\bar{\mathbf{h}}=0$

$$\hat{d}^2 = 1, \quad \bar{\mathbf{s}} \cdot \hat{d} = 0, \quad \bar{\mathbf{s}} \cdot \hat{l} = s_0 \cos\theta, \quad \bar{\mathbf{s}}^2 + (\hat{d} \cdot \hat{l})^2 = s_0^2, \quad (9)$$

where the last equation corresponds to conservation of energy. Equations (5) and (6) may thus be written as

$$\begin{aligned} \partial_\tau s_x &= -d_y d_z, \quad \partial_\tau s_y = d_x d_z, \quad \partial_\tau d_x = -d_y s_0 \cos\theta + s_y d_z, \\ \partial_\tau d_y &= d_x s_0 \cos\theta - s_x d_z. \end{aligned} \quad (10)$$

From Eq. (9) we also have

$$d_z^2 + s_z^2 = s_0^2 \sin^2\theta, \quad (11)$$

$$\bar{\mathbf{d}}_1 \cdot \bar{\mathbf{s}}_1 + d_z s_0 \cos\theta = 0. \quad (12)$$

We define

$$d_x = d_1 \cos\psi, \quad d_y = d_1 \sin\psi, \quad s_x = s_1 \cos\phi, \quad s_y = s_1 \sin\phi, \quad (13)$$

which converts Eq. (12) to

$$d_1 s_1 \cos(\psi - \phi) = -d_z s_0 \cos\theta. \quad (14)$$

After defining $d_z = \sin\eta$, we derive from Eq. (10)

$$\partial_\tau s_1 = -\sin\eta \cos\eta \sin(\psi - \phi), \quad (15)$$

$$\partial_\tau \eta = s_1 \sin(\psi - \phi). \quad (16)$$

Eliminating $(\psi - \phi)$ and s_1 among Eqs. (11), (14), and (16) gives

$$(\partial_\tau \eta)^2 + s_0^2 \cos^2\theta \tan^2\eta = s_0^2 \sin^2\theta - \sin^2\eta, \quad (16')$$

which can be integrated

$$\tau = - \int_0^\eta \frac{d\eta'}{(s_0^2 \sin^2\theta - s_0^2 \cos^2\theta \tan^2\eta' - \sin^2\eta')^{1/2}}, \quad (17)$$

where the minus sign is chosen based on the initial motion of \hat{d} . Introducing the two roots of the

denominator in $X \equiv \sin \eta'$

$$X_{\pm}^2 = \frac{1}{2} \{ (1 + s_0^2) \pm [(1 - s_0^2)^2 + 4s_0^2 \cos^2 \theta]^{1/2} \} , \quad (18)$$

we can evaluate the integral in Eq. (17) in terms of a Jacobian elliptical integral

$$\sin \eta (=d_z) = -X_- \operatorname{sn}(X_+ \tau | m) , \quad m \equiv (X_- | X_+)^2 , \quad (19)$$

which also gives the time dependence of $d_{\perp}^2 = 1 - d_z^2$, $s_{\perp}^2 = s_0^2 \sin^2 \theta - d_z^2$, and $s^2 = s_0^2 - d_z^2$.

Evaluating $(\partial_{\tau} s_x)^2 + (\partial_{\tau} s_y)^2$ and $(\partial_{\tau} d_x)^2 + (\partial_{\tau} d_y)^2$ from Eq. (10), substituting into them Eqs. (13) and (15), gives

$$(\partial_{\tau} \phi)^2 = s_0^2 \cos^2 \theta \sin^4 \eta / s_{\perp}^4 , \quad (\partial_{\tau} \psi)^2 = s_0^2 \cos^2 \theta / \cos^4 \eta . \quad (20)$$

Choosing the proper signs when taking the square roots of both sides, these equations lead to the following solutions for ϕ and ψ :

$$\phi(\tau) = \pi - s_0 \cos \theta \int_0^{\tau} \frac{X_-^2 \operatorname{sn}^2(X_+ \tau' | m)}{s_0^2 \sin^2 \theta - X_-^2 \operatorname{sn}^2(X_+ \tau' | m)} d\tau' , \quad (21)$$

$$\psi(\tau) = \frac{\pi}{2} + s_0 \cos \theta \int_0^{\tau} \frac{1}{1 - X_-^2 \operatorname{sn}^2(X_+ \tau' | m)} d\tau' . \quad (22)$$

From Eq. (19), we see that d_z , d_{\perp} , s_{\perp} , and s are all periodic functions of time with the same characteristic frequency

$$\omega_{\eta} = \pi X_+ / 2K(m) , \quad (23)$$

where $K(m)$ is the complete elliptic integral. On the other hand, trigonometric functions of ϕ and ψ , such as those entering Eq. (13), are frequency modulated according to Eqs. (21) and (22). These functions, therefore, are not strictly periodic in time, and their Fourier transforms will contain one principal frequency, which we denote as ω_{ϕ} for $\phi(\tau)$, and ω_{ψ} for $\psi(\tau)$, and many satellite frequencies at $\omega_{\phi} \pm 2n\omega_{\eta}$ and $\omega_{\psi} \pm 2n\omega_{\eta}$, respectively, where $n = 1, 2, 3, \dots$. From Eqs. (21) and (22) it is easy to see that

$$\omega_{\phi} = s_0 \cos \theta \left\langle \frac{X_-^2 \operatorname{sn}^2(X_+ \tau | m)}{s_0^2 \sin^2 \theta - X_-^2 \operatorname{sn}^2(X_+ \tau | m)} \right\rangle_{\tau} , \quad (24)$$

and

$$\omega_{\psi} = s_0 \cos \theta \left\langle \frac{1}{1 - X_-^2 \operatorname{sn}^2(X_+ \tau | m)} \right\rangle_{\tau} , \quad (25)$$

where $\langle \rangle_{\tau}$ denotes a time average over the period $T_{\eta} \equiv 2\pi/\omega_{\eta}$. We therefore see that the complete solution of our problem is characterized by three characteristic frequencies ω_{η} , ω_{ϕ} , and ω_{ψ} , which we have plotted in Figs. 1–3 as functions of s_0 , for vari-

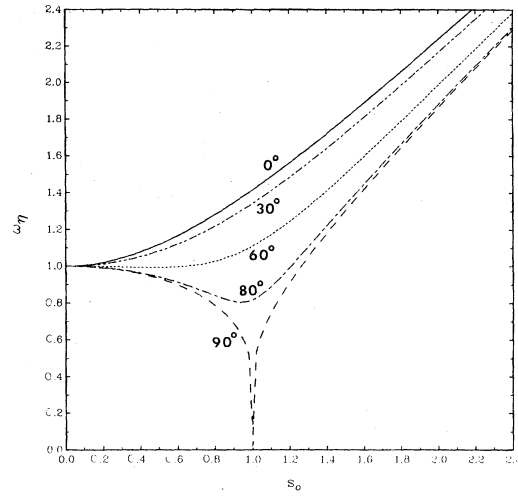


FIG. 1. Frequency ω_{η} of Eq. (23) is plotted as a function of $s_0 = \gamma H_1 / \Omega_0$, for several values of $\theta = \angle(\vec{H}_1, \hat{l})$. The frequencies $n\omega_{\eta}$, with $n = 0, \pm 1, \pm 2, \dots$ constitute the complete frequency spectrum of $\vec{d} \cdot \hat{l}$, $|\vec{d}_{\perp}| \equiv |\vec{d} - \hat{l}(\vec{d} \cdot \hat{l})|$, $|\vec{s}|$, and $|\vec{s}_{\perp}|$.

ous values of θ . In the longitudinal ringing experiments, the measured quantity is the component of magnetization along the direction of the magnetic field \vec{H}_1 which is turned off at $t = 0$. We shall denote this measured quantity as M_H . In our reduced units it is just s_H , where the relation is

$$M_H = \gamma \lambda s_H / \Omega_0 = \chi_0 \Omega_0 s_H / \gamma . \quad (26)$$

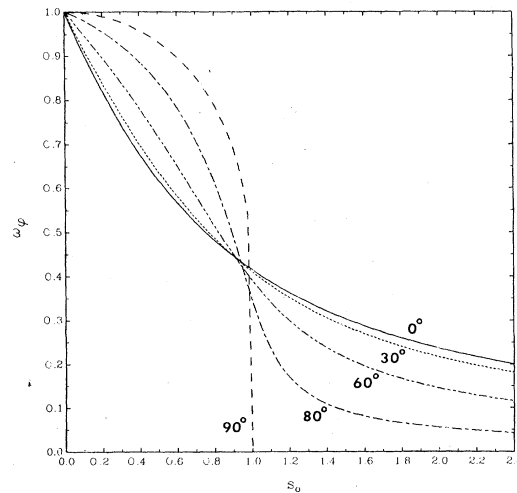


FIG. 2. Frequency ω_{ψ} of Eq. (24) is plotted in the same way as in Fig. 1. The frequencies $\pm \omega_{\psi} + 2n\omega_{\eta}$, with $n = 0, \pm 1, \pm 2, \dots$ constitute the complete frequency spectrum of any component of the magnetization that is not parallel to \hat{l} , including the component measured in a longitudinal ringing experiment if $\theta \neq 0^\circ$.

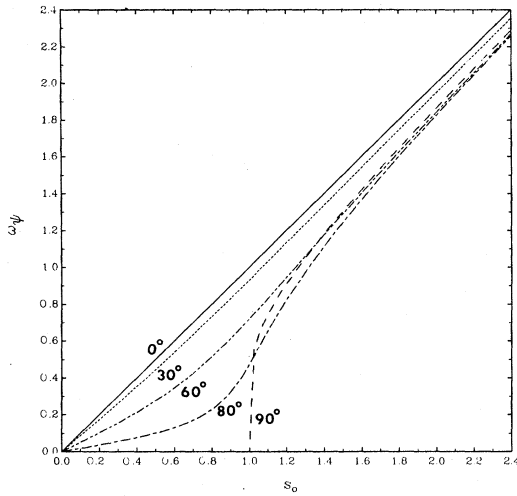


FIG. 3. Frequency ω_ψ of Eq. (25) is plotted in the same way as in Fig. 1. The frequencies $\pm\omega_\psi + 2n\omega_\eta$, with $n = 0, \pm 1, \pm 2, \dots$ constitute the complete frequency spectrum of any component of \hat{d} that is not parallel to \hat{l} .

In the coordinate system chosen,

$$s_H = s_z \cos\theta - s_x \sin\theta = s_0 \cos^2\theta - s_\perp \cos\phi \sin\theta \quad (27)$$

The first term is a constant, while the second term is a product of two time-dependent quantities, s_\perp and $\cos\phi$. The Fourier transform of s_H , therefore, is a convolution of those of s_\perp and $\cos\phi$. Thus the Fourier spectrum of s_H should also contain one prin-

cipal frequency ω_ϕ and many satellite peaks at $\omega_\phi \pm 2n\omega_\eta$. The frequency ω_ψ is therefore not observed in such an arrangement. In Figs. 4–6, we have plotted s_H as a function of τ for various values of θ and s_0 . These plots clearly show the two-frequency behavior of s_H , except in the small s_0 limit, where $\omega_\phi \sim \omega_\eta$, and a beatlike behavior appears. These curves are presumably what one will observe in an actual longitudinal ringing experiment, if Eqs. (5) and (6) are valid dynamic equations of the system. In extracting the frequency ω_η directly from such curves, one is reminded that, because of the dependence of s_H on the product of s_\perp and $-\cos\phi$, all maxima of s_\perp appear as minima in s_H whenever $-\cos\phi$ is negative. This is clearly illustrated by the case of $\theta = 75^\circ$, $s_0 = 0.6$. For this case, $s_0 \cos^2\theta = 0.04$ is nearly zero, and we see in Fig. 6 that the peaks above this value and the troughs below this value together form a regular time sequence.

Before we close this section, we will discuss the limiting behavior of our solution as θ approaches 0° and 90° . In the limit $\theta \rightarrow 0^\circ$, we obtain

$$X_+ \rightarrow (1 + s_0^2)^{1/2}, \quad X_- \sim s_0 \sin\theta / (1 + s_0^2)^{1/2} \rightarrow 0,$$

$$d_z = \sin\eta \sim -X_- \sin(X_+ \tau) \rightarrow 0, \quad \omega_\eta \rightarrow X_+,$$

$$\phi \rightarrow \pi - \frac{s_0}{X_+} \int_0^{X_+ \tau} \frac{\sin^2 \chi'}{1 + s_0^2 - \sin^2 \chi'} d\chi', \quad (28)$$

$$\psi \rightarrow \frac{\pi}{2} + s_0 \tau, \quad \omega_\psi \rightarrow s_0, \quad \omega_\phi \rightarrow X_+ - s_0,$$

$$s_\perp \rightarrow 0, \quad s_H = s_z \rightarrow s_0.$$

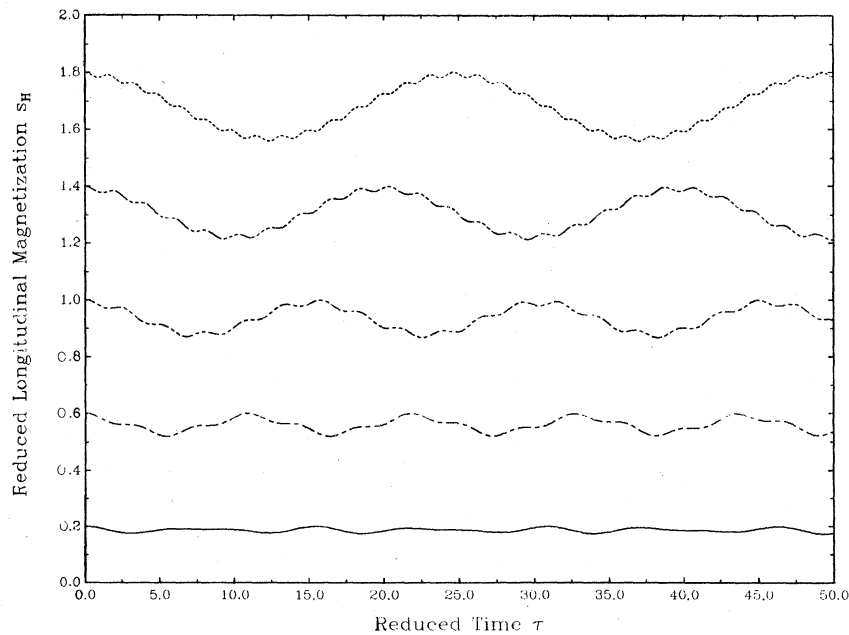
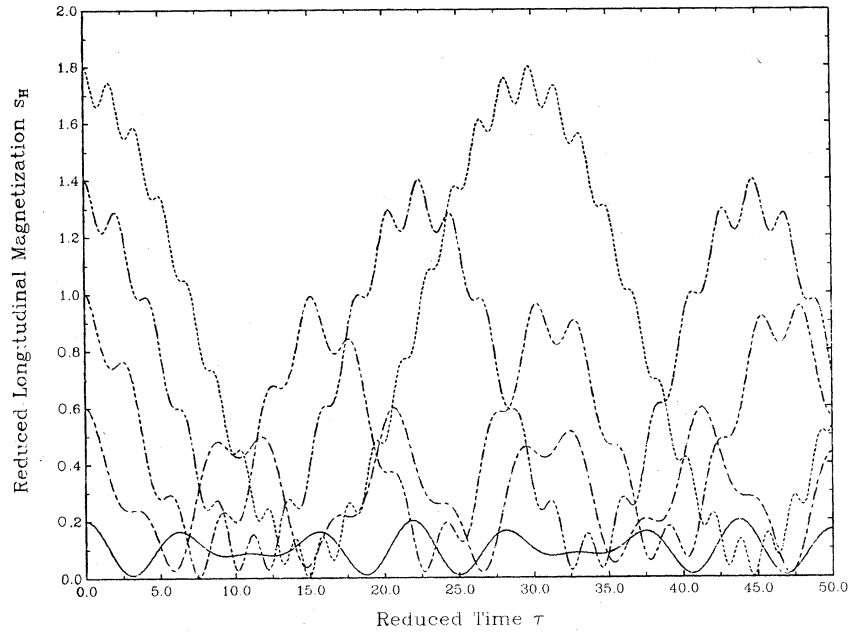
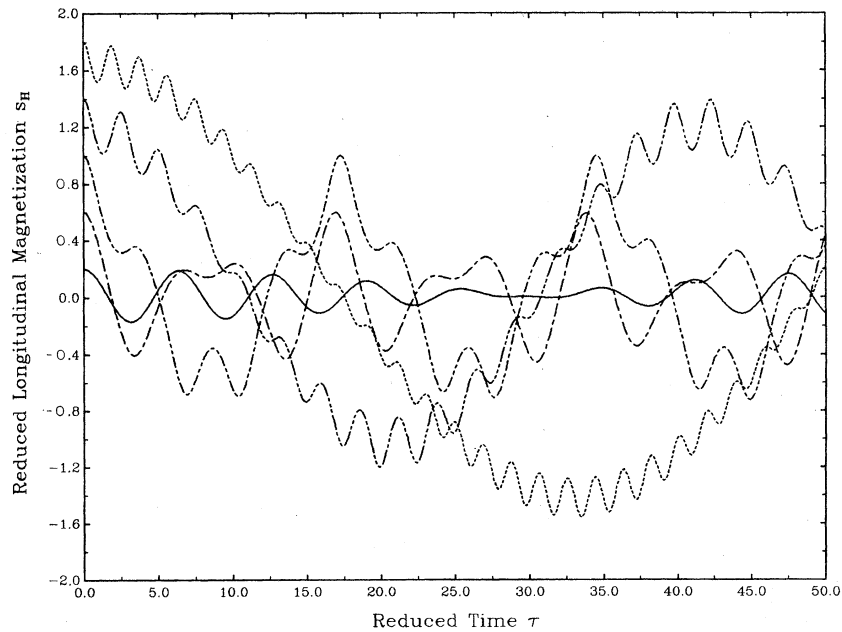


FIG. 4. Plotted is the reduced longitudinal magnetization s_H [c.f. Eqs. (26) and (27)] vs the reduced time $\tau \equiv \Omega_0 t$, for $\theta = 15^\circ$ and five values of $s_0 = 0.2, 0.6, 1.0, 1.4,$ and 1.8 . (s_0 is the initial value of s_H .)

FIG. 5. Same as in Fig. 4 except that $\theta = 45^\circ$.

Thus (a) no ringing behavior in s_H will be observed in this limit; (b) the spin-vector \hat{d} will simply precess in the plane perpendicular to \hat{l} (or \vec{H}_1) at the constant frequency s_0 ; (c) the variable $\phi(\tau)$ and the two frequencies ω_η and ω_ϕ become physically meaningless quantities in this limit due to the tendency of d_z and s_\perp to vanish.

Consider next the limit $\theta \rightarrow 90^\circ$. We must distinguish between two cases: (i) case I, $0 \leq s_0 \leq 1$;

FIG. 6. Same as in Fig. 4 except that $\theta = 75^\circ$.

we find

$$\begin{aligned} X_- &\rightarrow s_0, \quad X_+ = 1, \quad d_z = \sin\eta \rightarrow -s_0 \operatorname{sn}(\tau|s_0^2), \\ \omega_\eta &\rightarrow \pi/2K(s_0^2), \\ \phi &\rightarrow [1 - [\frac{1}{2}(4\tau/T_\eta + 1)]]\pi, \quad \omega_\phi \rightarrow \omega_\eta, \\ \sin\phi &\rightarrow 0, \quad \cos\phi \rightarrow -\operatorname{sgn}[\operatorname{cn}(\tau|s_0^2)], \\ \psi &\rightarrow \pi/2 \text{ for all } \tau, \quad \omega_\psi \rightarrow 0, \end{aligned} \quad (29)$$

and

$$\begin{aligned} d_x &\rightarrow 0, \quad d_y \rightarrow (1 - d_z^2)^{1/2}, \\ s_y &\rightarrow 0, \quad s_z \rightarrow 0, \quad s_x = -s_H \rightarrow -s_0 \operatorname{cn}(\tau|s_0^2). \end{aligned}$$

In the above, “sgn” means “the sign of,” and the notation $[x]$ in the ϕ equation means “the integer part of x .” To obtain this limiting behavior for ϕ , it is necessary to notice that the denominator in the integrand of Eq. (21) can give nonintegrable divergences, if $\theta = 90^\circ$ is directly substituted in, while the front factor of the integral, $s_0 \cos\theta$, becomes zero, so that a delicate limiting process is required in order to obtain the result given.

We conclude that in this case \hat{d} performs an orientational oscillation in the yz plane about \hat{y} , while \vec{s} is confined to have an x component only, which oscillates between $-s_0$ and $+s_0$. The whole motion is now characterized by a single frequency ω_η . (ii) Case II, $s_0 \geq 1$: In this case we find

$$\begin{aligned} X_- &= 1, \quad X_+ = s_0, \quad d_z = \sin\eta \rightarrow -\operatorname{sn}(s_0\tau|s_0^{-2}), \\ \omega_\eta &\rightarrow \pi s_0/2K(s_0^{-2}), \\ \phi &\rightarrow \pi \text{ for all } \tau, \quad \omega_\phi \rightarrow 0, \\ \psi &\rightarrow \{\frac{1}{2} + [\frac{1}{2}(4\tau/T_\eta + 1)]\}\pi, \quad \omega_\psi \rightarrow \omega_\eta, \\ \cos\psi &\rightarrow 0, \quad \sin\psi \rightarrow \operatorname{sgn}[\operatorname{cn}(s_0\tau|s_0^{-2})], \end{aligned} \quad (30)$$

and

$$\begin{aligned} d_x &\rightarrow 0, \quad d_y \rightarrow \operatorname{cn}(s_0\tau|s_0^{-2}), \\ s_y &\rightarrow 0, \quad s_z \rightarrow 0, \quad s_x = -s_H = -(s_0^2 - d_z^2)^{1/2}, \end{aligned}$$

where the symbols “sgn” in $\sin\psi$, and “[x]” in the ψ equation, have the same meanings as in the previous case, and the delicate limiting process is now required for obtaining the correct limiting behavior for ψ .

We conclude that in this case \hat{d} rotates in the plane

perpendicular to $-\hat{x}$ (i.e., \vec{H}_1) right handedly at a constant angular frequency ω_η , while \vec{s} remains in the $-\hat{x}$ direction, with its amplitude making *two* identical oscillations between s_0 and $(s_0^2 - 1)^{1/2}$ in each period T_η . The observed frequency in s_H for this case should therefore be identified as $2\omega_\eta$. Finally, we note that the limit $\theta \rightarrow 90^\circ$, including both cases of $0 \leq s_0 \leq 1$ and $s_0 \geq 1$, is exactly identical to the corresponding nonlinear longitudinal magnetic ringing behavior of superfluid $^3\text{He-A}$, as is studied by Maki and Tsuneto.⁴ This similarity ceases to hold, however, as soon as θ deviates from 90° , as a result of the topological differences pointed out in the Introduction.

III. ANALYSIS OF AN ALTERNATIVE WAY TO PREPARE THE INITIAL STATE

In this section, we discuss the possibility of whether there is an alternative way to prepare the initial state for the subsequent observation of nonlinear magnetic ringing in spin-ordered solid ^3He . This possibility is worthy of consideration because the characteristic field of the system, $H_c \equiv \Omega_0/\gamma$, ranges between 254 G at 0 K, to 162 G at the Néel temperature T_N . Thus in order to perform the complete-turn-off experiment, and observe the full nonlinear ringing spectrum, it is necessary to rapidly switch off a few hundred gauss of an applied magnetic field, which could pose a severe heating problem to the cryostat. One is therefore interested in probing the possibility of preparing the initial state, not by switching off a large dc field, but rather by applying a *weak* ac magnetic pulse at the zero-field resonance frequency $\Omega_0/2\pi$ for a sufficient duration, in order to drive the system to a finite deviation from the initial equilibrium state at zero field. In this way, it is hoped that a large \vec{s}_0 can be generated at an arbitrary direction, which would then “ring” after the pulse is over. Actually, as we shall show below, this way of preparing the initial state is less than ideal, because the initial state so obtained cannot be precisely predicted (except by numerical simulation), even though the subsequent ringing behavior is still an exactly solvable problem.

To analyze this question quantitatively, we must solve Eqs. (5) and (6) with $\vec{h}(\tau) = \vec{h}_{ac} \sin\tau$, and for the initial conditions $\vec{s}(0) = 0$, $\hat{d}(0) \perp \hat{l}$. Since we shall limit ourselves to the case when $h_{ac} \ll 1$, we can study the *initial* growth by linearizing Eqs. (5) and (6) with respect to the initial state. The solution can then be easily found:

$$\begin{aligned} \vec{s} &\approx -\frac{1}{2}(\hat{d}_0 \times \hat{l})[\vec{h}_{ac} \cdot (\hat{d}_0 \times \hat{l})](\tau \cos\tau - \sin\tau), \\ \hat{d} &= \hat{d}_0 - [\vec{h}_{ac} \times \hat{d}_0 - (\vec{h}_{ac} \times \hat{d}_0 \cdot \hat{l})\hat{l}](1 - \cos\tau) \\ &\quad - \frac{1}{2}(\vec{h}_{ac} \times \hat{d}_0 \cdot \hat{l})\hat{l}\tau \sin\tau. \end{aligned} \quad (31)$$

Thus after an *integer* number of oscillations in the applied ac field, we find \hat{d} remaining in the original direction \hat{d}_0 , but with \vec{s} grown linearly to

$$-(\hat{d}_0 \times \hat{l})[\vec{h}_{ac} \cdot (\hat{d}_0 \times \hat{l})] \tau .$$

This \vec{s} is, unfortunately, always in the direction $\hat{d}_0 \times \hat{l}$, and is therefore perpendicular to \hat{l} . In order to generate an \vec{s} in an arbitrary direction relative to \hat{l} , it is necessary to let the system evolve into the *non-linear* regime. Then the problem can no longer be analyzed exactly by us. However, numerical integration of Eqs. (5) and (6) indicates that it is indeed possible to generate in this way a general state characterized by $\vec{s} \cdot \hat{l} = s \cos \theta$ and $\hat{d} \cdot \hat{l} = \cos \alpha$ but with $\vec{s} \cdot \hat{d} = 0$. [It can be shown from Eqs. (5) and (6) that $\vec{s} \cdot \hat{d}$ is a constant of motion even when \vec{h} represents an arbitrary time-dependent field.] The only shortcomings of this approach are that (1) the values of s , θ , and α cannot be precisely predicted except through numerical simulation, and (2) it is not clear whether there are limitations on the ranges of values for s , θ , and α that can be reached this way.

If only an initial state with $\vec{s}_0 \cdot \hat{l} = s_0 \cos \theta$, $\hat{d}_0 \cdot \hat{l} = \cos \alpha$, and $\vec{s}_0 \cdot \hat{d}_0 = 0$ has been somehow created, the subsequent ringing behavior in zero applied field is still an exactly solvable problem. The solution is very similar to the complete-turn-off situation analyzed in the previous section, so we shall omit the

$$\phi = \pi - s_0 \cos \theta \int_0^\tau \frac{X_-^2 \operatorname{sn}^2[X_+(\tau' - \tau_0)|m]}{s_0^2 \sin^2 \theta + \cos^2 \alpha - X_-^2 [X_+(\tau' - \tau_0)|m]} d\tau' \quad (36)$$

and

$$\psi = \cos^{-1}(\operatorname{ctn} \theta \operatorname{ctn} \alpha) + s_0 \cos \theta \int_0^\tau \frac{1}{1 - X_-^2 \operatorname{sn}^2[X_+(\tau' - \tau_0)|m]} d\tau' , \quad (37)$$

which generalize Eqs. (21) and (22) to the case when $\hat{d}_0 \cdot \hat{l} = \cos \alpha \neq 0$.

As for the three principal frequencies ω_η , ω_ϕ , and ω_ψ , we find that Eqs. (23), (24), and (25) remain valid, if these equations are modified according to the prescription given in Eq. (34).

Finally, we remark that the initial state considered in this section is already the most general initial state that can be generated by an arbitrary time-dependent external magnetic field.

IV. CONCLUSION AND DISCUSSION

In this paper we have analyzed the nonlinear magnetic ringing behavior of spin-ordered bcc ^3He at zero external magnetic field, and for two ways of preparing the initial state.

In a complete-turn-off experiment, the initial state is prepared by first establishing an equilibrium state

details and give the results only.

Choosing still the coordinate system so that $\hat{l} = (0, 0, 1)$, $\vec{s}_0 = s_0(-\sin \theta, 0, \cos \theta)$, but

$$\hat{d}_0 = (\sin \alpha \cos \beta, \sin \alpha \sin \beta, \cos \alpha) ,$$

we demand $\vec{s}_0 \cdot \hat{d}_0 = 0$ by requiring that

$$\cos \beta = \operatorname{ctn} \theta \operatorname{ctn} \alpha . \quad (32)$$

We also define the variables η , ϕ , and ψ in the same way as in Sec. II. We then find

$$d_z = \sin \eta = -X_- \operatorname{sn}[X_+(\tau - \tau_0)|m] , \quad (33)$$

where m is still defined as $(X_-/X_+)^2$, but X_\pm are now given by Eq. (18) with the following changes:

$$s_0^2 \rightarrow s_{0,\text{eff}}^2 \equiv s_0^2 + \cos^2 \alpha , \quad (34)$$

$$\cos \theta \rightarrow \cos \theta_{\text{eff}} \equiv s_0 \cos \theta / s_{0,\text{eff}} .$$

Furthermore,

$$\begin{aligned} \tau_0 &= X_+^{-1} \int_0^{X_+^{-1} \cos \alpha} [(1-z^2)(1-mz^2)]^{-1/2} dz \\ &\equiv X_+^{-1} F[\sin^{-1}(X_-^{-1} \cos \alpha)|m] , \end{aligned} \quad (35)$$

where $F[X|m]$ is the elliptic integral of the first kind. We also obtain

in an external magnetic field \vec{H}_1 , and this field \vec{H}_1 is then rapidly switched off at $t=0$. In Sec. II of this paper, we have shown that the nonlinear ringing behavior ensuing on the turnoff of \vec{H}_1 is an exactly solvable problem, if Eqs. (2) and (3) are assumed to govern the spin dynamics of this system. These equations have neglected spin-relaxation effects.

The results are given by Eqs. (18), (19), (21), and (22), where the variables ϕ and ψ are defined in Eq. (13), and the coordinate system is such that the initial state is given by Eq. (8). The complete solution is found to be characterized by three principal frequencies as given in Eqs. (23)–(25), but only two of these frequencies (ω_η and ω_ϕ) can be observed, if the measured quantity is the longitudinal magnetization. Figures 1–3 summarize the dependence of all three frequencies on s_0 and θ , while Figs. 4–6 reveal the explicit time dependence of the longitudinal magnetization for various values of s_0 and θ as predicted by Eq. (2) and (3). The frequency ω_ψ is observable

only if the motion of \hat{d} can be directly monitored.

An alternative way to prepare the initial state is through the application of a weak ac magnetic pulse at the zero-field resonance frequency $\Omega_0/2\pi$ for a sufficient duration, and under zero applied dc magnetic field. In this way, no large dc magnetic field needs to be switched off rapidly at $t=0$, in order to avoid a possible heating problem. In the linear limit, we find the initial state so generated to satisfy $\vec{s}_0 \cdot \hat{l} = 0$ and

$$\hat{d}_0 \cdot \hat{l} = -\frac{1}{2} (\vec{h}_{ac} \times \hat{d}_0 \cdot \hat{l}) \tau \sin \tau ,$$

which, according to Eq. (34) is equivalent to a $\hat{d}_0 \cdot \hat{l} = 0$ situation, but with effective

$$s_{0,\text{eff}}^2 = s_0^2 + (\hat{d}_0 \cdot \hat{l})^2 ,$$

and

$$\cos \theta_{\text{eff}} = s_0 \cos \theta / s_{0,\text{eff}} = 0 .$$

However, by definition of the linear limit, the so-generated $s_{0,\text{eff}}$ must be much smaller than unity. Furthermore, because $\cos \theta_{\text{eff}} = 0$, the initial state so generated can only be used to test the $\theta = 90^\circ$ case discussed in Sec. II. Thus in order to generate an $s_{0,\text{eff}} \geq 1$ and/or $\theta_{\text{eff}} \neq 90^\circ$, for observing the full nonlinear ringing spectrum, it is necessary to apply the ac magnetic pulse for so long as to drive the system into the nonlinear regime. We can then no longer make exact predictions on the values for $s_{0,\text{eff}}$ and θ_{eff} except by numerical simulations, even though the subsequent ringing behavior after the ac pulse is switched off is still an exact analyzable problem, with the ringing frequencies appearing exactly the same as in the complete-turn off situation, if only $s_{0,\text{eff}}$ and θ_{eff} are used to replace s_0 and θ . The essential point we have learned in this part of the analysis is that the effects of the ac magnetic pulse at the resonance frequency $\Omega_0/2\pi$ cannot be understood by

treating Ω_0 as an effective Larmor frequency γH_{eff} , as in the analysis of the effects of a tipping pulse applied to a usual NMR system.⁵

In this paper, we have not studied the more general case of nonlinear magnetic ringing of spin-order solid ^3He in a finite applied dc magnetic field. This could be observed, for example, in a "partial-turn-off experiment," in which the system is first allowed to reach equilibrium in the total external field $\vec{H}_0 + \vec{H}_1$, and then at $t=0$ the field \vec{H}_1 alone is switched off, leaving the field \vec{H}_0 still acting on the system. We have not found any analytic method to study this more general nonlinear ringing problem, so that to study it, a purely numerical method would have to be employed. We only remark that even in this case, the field H_1 still has to be of the order of a few hundred Gauss, if the full nonlinear ringing spectrum is to be observed. Thus the difficulty we have discussed earlier cannot be removed by applying the additional field \vec{H}_0 . The alternative approach for preparing the initial state, viz., replacing the field \vec{H}_1 by an ac magnetic pulse at the resonance frequency, would now serve, in the limit of a very large H_0 , to tip the equilibrium magnetization to a new orientation. This approach then reduced to the standard approach for a *transverse* ringing experiment. If such an experiment turns out to be a more easily doable experiment, we will try to analyze this situation in a future publication. But in view of the likelihood that such an analysis will rely heavily on numerical method, we still recommend the zero-field longitudinal ringing as a most readily interpretable experiment.

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³See, for example, the review article by W. F. Brinkman and M. C. Cross, in *Progress in Low Temperature Physics*, edited by D. F. Brewer (North-Holland, Amsterdam 1978), Vol. VIIA, Chap. 2, Sec. 4.1.

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⁵This last statement is a direct answer to a question raised by W. P. Halperin at the Cornell Symposium on Liquid and Solid ^3He , where a preliminary report of this paper has been presented in a Poster Session. We thank Professor Halperin for pointing out the alternative possibility for preparing the initial state, which we analyzed in Sec. III of this paper.