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General hydrodynamics of ^{3}He-A in finite magnetic fields

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We have determined the hydrodynamic equations, including nonlinear terms, for superfluid ^{3}He-A in finite magnetic fields. The propagating and diffusive normal modes for a uniform texture have been studied, both for bulk and superleak geometries. In bulk, the longitudinal magnetization and the temperature produce two coupled modes; in a superleak, the longitudinal magnetization and the density produce two coupled modes. For both geometries, the coupled modes show crossing effects. (An appendix is devoted to considerations of an experiment designed to observe the coupled longitudinal modes, at "mode crossing.") The transverse spin waves have also been studied, but they do not show mode-crossing effects. For the bulk A_1 phase, the relationship of the bulk diffusive mode to the "magnetothermal effect," is discussed. For the A_1 phase in a superleak the relationship of the two diffusive longitudinal modes to the "magnetothermomechanical effects," and of the propagating longitudinal mode to the "magnetic fountain effect," are discussed. It is suggested that the longitudinal spin wave in the A phase, at any field, can be studied through its characteristic decay length, when the generator frequency lies below the gap frequency.

I. INTRODUCTION

Despite a great deal of work on the theory of ^{3}He-A, there remains a gap in our knowledge of its hydrodynamics as one increases the field $\vec{H}$, and thus moves from the A phase at $H=0$ (where the gaps for up and down spin pairing are equal) to the A_1 phase (where one of these gaps, conventionally taken to be $\Delta_\downarrow$, goes to zero). As recently pointed out by Liu,[1] the early work of Pleiner and Graham[2] on the A_1 phase neglected an important contribution to the superfluid velocity, and this term has very significant consequences. In particular, Liu finds that, in a superleak, fourth sound and the longitudinal spin wave couple strongly, yielding two new modes. At finite wave vectors the first has a frequency whose square is the sum of the squares of the uncoupled fourth sound and longitudinal sound frequencies, while the second has (when dissipation is neglected) zero frequency. He also shows that, for times t short compared to the longitudinal spin relaxation time T_1, there can be "magnetothermomechanical" effects in which a nonuniform magnetization coexists with a nonuniform temperature or pressure. Similar effects occur in bulk, with second sound and the longitudinal spin wave coupling strongly. There the frequency shift is much more pronounced, but more difficult to observe, due to the expected high attenuation rate for second sound in ^{3}He-A. Liu also shows that, in bulk, there is a "magnetothermal" effect, in which a nonuniform magnetization can coexist with a nonuniform temperature for $t << T_1$.

These effects are all the more interesting when one considers that, in the A phase at $H=0$, the corresponding modes *both* have finite frequencies, and are *pure* modes of density and longitudinal magnetization. In the transition region between the A phase at $H=0$ and the A_1 phase, one would expect interesting effects to occur, which might be as prominent as the effects predicted for the A_1 phase. For this reason we have developed the hydrodynamic theory of the A phase in finite H, where $\Delta_\uparrow \neq \Delta_\downarrow$. (For convenience we shall refer to this modified A phase simply as the A_2 phase, and we shall refer to the A phase at $H=0$ simply as the A phase. This should *not* be taken to imply that a thermodynamic transition occurs when H is turned on.) Indeed, we find mode-crossing to occur, where we employ the weak-coupling Ginzburg-Landau model to provide some of the reactive coefficients which appear in the theory (Appendix A). After completing our work, we found that somewhat less general results were also found for the A_2 phase by Gongadze, Gurgenishvili, and Kharadze,[3] who employed, at the outset, weak-coupling Ginzburg-Landau (GL) theory and the generalized Legget equations. Note that the hydrodynamic theory can predict the attenuation rates for these modes, as well as their frequencies, whereas the method of Ref. 3 can only obtain the frequencies. However, because there are presently no experimental results on the mode frequencies in the A_2 phase, not to mention the attenuation rates, the latter have not been computed. On the other hand, we have considered diffusive modes in some detail. In the A_1 phase we elucidate the relationship between the bulk and superleak diffusive modes and the "magnetothermal" and "magnetothermomechnical" effects; we also discuss the relationship of the propagating superleak longitudinal mode to the "magnetic fountain effect."

In Appendix B we discuss, in some detail, questions associated with the design of an experiment to detect the predicted longitudinal mode-crossing effect.

The reader may find it useful to be reminded of the symmetries associated with the spin part of the order parameter $A_{\mu i}$ for the A, A_2, and A_1 phases. For the A phase ($H=0$), the spin part of $A_{\mu i}$ is given by a unit vector $\hat{d}$, which in the case of an unbounded system can point in any direction. The system is completely unaffected by spin-space rotations about $\hat{d}$. For the A_2 phase ($0 < H < H_{A2}$), the spin part of $A_{\mu i}$ develops another unit vector $\hat{e}$ which is perpendicular to $\hat{d}$, and $\hat{d}$ and $\hat{e}$ are forced, in an unbounded system, to lie perpendicular to $\vec{H}$. The system now is affected by spin-space rotations about $\hat{d}$, so the A_2 phase has less symmetry than the A phase. For the A_1 phase ($H \geq H_{A2}$), the directions of $\hat{d}$ and $\hat{e}$ are no longer well defined (other than that they are perpendicular to $\vec{H}$); they can be changed by either an appropriate phase change or by an appropriate rotation of the orbital part of $A_{\mu i}$ about the unit vector $\hat{l}$. As a consequence, rotations of $\hat{f} \equiv \hat{d} \times \hat{e}$ about $\hat{d}$ and about $\hat{e}$ are energetically equivalent, and rotations about $\hat{f}$ are indistinguishable from appropriate phase changes or from appropriate rotations of the orbital part of $A_{\mu i}$ about $\hat{l}$. Thus, on going from the A_2 phase to the A_1 phase, one goes to a state of higher symmetry. See Ref. 3(a).

II. NOTATION

Before discussing the hydrodynamics of the A_2 phase, it is important to establish a notation. Unfortunately, previous authors have not been consistent with one another.[1,3–6] We will describe the order parameter $A_{\mu i}$ in the form

$$A_{\mu i} = \Delta e^{i\phi}(\hat{m} + i\hat{n})_i v_\mu \ , \tag{1}$$

$$v_\mu = a\hat{d} + ib\hat{e}, \quad |\vec{v}|^2 = 1 \ , \tag{2}$$

$$\begin{aligned} a &= (2\Delta)^{-1}(\Delta_\uparrow e^{i\phi_f} + \Delta_\downarrow e^{-i\phi_f}) \ , \\ b &= (2\Delta)^{-1}(\Delta_\uparrow e^{i\phi_f} - \Delta_\downarrow e^{-i\phi_f}) \ , \\ \Delta^2 &= \tfrac{1}{2}(\Delta_\uparrow^2 + \Delta_\downarrow^2) \ . \end{aligned} \tag{3}$$

(Latin subscripts denote a vector in real space, Greek subscripts denote a vector in spin space.) Here, the orbital part of the order parameter, $(\hat{m} + i\hat{n})$ (with $\hat{l} \equiv \hat{m} \times \hat{n}$) is conventional, as are the quantities $\Delta_\uparrow, \Delta_\downarrow, \Delta, \hat{d}$. However, $\phi_f, \hat{e}, \hat{f} \equiv \hat{d} \times \hat{e}$ are not. The quantities ϕ and ϕ_f are redundant, but useful: rotating $\hat{m}$ and $\hat{n}$ clockwise about $\hat{l}$ by an amount ϕ will eliminate ϕ from Eq. (1); and rotating $\hat{d}$ and $\hat{e}$ clockwise about $\hat{f}$ by an amount ϕ_f will eliminate ϕ_f from (3) and (1). We employ $\hat{d}$, $\hat{e}$, and $\hat{f}$ because they form a mnemonic for the spin part of the order parameter, and because in the A phase $\vec{v}$ reduces to $\hat{d}$, which is the conventional notation.[7] Note that $\phi, \phi_f, \hat{l}, \hat{f}$ are all odd under time-reversal T. However, $\hat{m}, \hat{n}, \hat{d}, \hat{e}$ have no definite signature under T. Note that the set $(\hat{m}, \hat{n}, \hat{d}, \hat{e})$ is equivalent to $(-\hat{m}, -\hat{n}, -\hat{d}, -\hat{e})$.

One of the problems confronting a theory of the A_2 phase is that it must employ the correct superfluid velocity $\vec{v}^s$ both for the A phase, where (with $\beta = \hbar/2m$)

$$\vec{v}^s = \beta m_i \vec{\nabla} n_i \ , \tag{4}$$

and for the A_1 phase, where[1]

$$\vec{v}^s = \beta(m_i \vec{\nabla} n_i + d_\alpha \vec{\nabla} e_\alpha) \ .$$

As a guide to the intermediate region, one might consider the quantity

$$\beta |A_{\mu i}|^{-2} \operatorname{Im}(A_{\mu i}^* \vec{\nabla} A_{\mu i}) = \beta(m_i \vec{\nabla} n_i + p d_\alpha \vec{\nabla} e_\alpha) \ ,$$

where (with $\phi_f = 0$)

$$p = 2ab = (\Delta_\uparrow^2 - \Delta_\downarrow^2)/(\Delta_\uparrow^2 + \Delta_\downarrow^2) \ . \tag{5}$$

This form takes the proper values for $p=0$ (A phase) and for $p=1$ (A_1 phase). Unfortunately, the parameter p depends upon temperature T, pressure P, etc., so that the value of $\partial_i v_j^s - \partial_j v_i^s$, needed in the hydrodynamic derivation, would become rather complicated. Since $\beta m_i \vec{\nabla} n_i$ and $\beta d_\alpha \vec{\nabla} e_\alpha$ appear independently in the Ginzburg-Landau (GL) expansion of the A_2-phase free energy (Appendix A), we must employ two "velocity" variables, although the choice is not unique. For convenience, we will work with the definitions

$$\vec{v}^s \equiv \beta m_i \vec{\nabla} n_i, \quad \vec{v}^{sp} \equiv \beta d_\alpha \vec{\nabla} e_\alpha \ , \tag{6}$$

and require that only the form $\vec{v}^s + \vec{v}^{sp}$ appear in the theory of the A_1 phase. With these definitions, we have[8]

$$\begin{aligned} \partial_i v_j^s - \partial_j v_i^s &= \beta \hat{l} \cdot (\partial_i \hat{l} \times \partial_j \hat{l}) \ , \\ \partial_i v_j^{sp} - \partial_j v_i^{sp} &= \beta \hat{f} \cdot (\partial_i \hat{f} \times \partial_j \hat{f}) \ . \end{aligned} \tag{7}$$

Note that $\vec{v}^s$ is a true Galilean velocity, whereas $\vec{v}^{sp}$ is not.[4] Also, note that $\vec{v}^{sp}$ constitutes a generalization of the quantity $\vec{v}^{sp}$ employed in Ref. 7. Physically, $(\vec{v}^s \pm \vec{v}^{sp})$ corresponds to the superfluid velocity of the "up" and "down" spin pairs.

The hydrodynamic analysis begins with the differential of the energy density ϵ:

$$\begin{aligned} d\epsilon = {} & [T\, dS + \mu\, d\rho + \vec{v}^n \cdot d\vec{g} + \vec{\lambda}^s \cdot d\vec{v}^s \\ & + \beta\psi_i\, dl_i + \beta\phi_{ij}\, d(\partial_j l_i)] \\ & + [\vec{\lambda}^{sp} \cdot d\vec{v}^{sp} + \beta\omega_f\, d\phi_f + \beta\pi_\alpha\, df_\alpha \\ & + \beta\pi_{\alpha i}\, d(\partial_i f_\alpha) + (h_\alpha - H_\alpha)\, dm_\alpha] \ . \end{aligned} \tag{8}$$

This equation will be useful for later reference. The first bracket contains purely real space (or orbital) vectors, and follows the notation of Refs. 9 and 10. Here S is the entropy density, ρ is the mass density, μ is the chemical potential, $\vec{v}^n$ is the normal fluid velocity, $\vec{g}$ is the momentum density, $\vec{\lambda}^s=\vec{g}-\rho\vec{v}^n$ by a Galilean transformation on ϵ and $\vec{g}$, $\beta\psi_i \equiv \partial\epsilon/\partial l_i$, and $\beta\phi_{ij}\equiv\partial\epsilon/\partial(\partial_j l_i)$. The second bracket contains quantities defined in terms of spin space vectors. Here $\vec{\lambda}^{sp}\equiv\partial\epsilon/\partial\vec{v}^{sp}$ is the density conjugate to $\vec{v}^{sp}$, $d\phi_f=-\hat{f}\cdot d\vec{\theta}_s$ (where $d\vec{\theta}_s$ is a local infinitesimal rotation in spin space), $\beta\omega_f\equiv\partial\epsilon/\partial\phi_f$, $\beta\pi_\alpha\equiv\partial\epsilon/\partial f_\alpha$, $\beta\pi_{\alpha i}\equiv\partial\epsilon/\partial(\partial_i f_\alpha)$, and $h_\alpha-H_\alpha \equiv\partial\epsilon/\partial m_\alpha$, where m_α is the magnetization. H_α is the (static) external magnetic field, and ϵ includes the interaction energy density, $-m_\alpha H_\alpha$, of the system with H_α. (Although we employ $\hat{f}$ rather than $\hat{w}$ of Ref. 2, we use the Ref. 2 definition of π_α, up to a factor of β; $\pi_{\alpha i}$ is chosen to be easily associated with π_α, unlike the symbol employed in Ref. 1; $\vec{\lambda}^{sp}$ and ω_f are new terms; h_α is the conventional symbol for the internal field.[11,12]) Note that the nuclear dipole-dipole interaction contributes to ϕ_i, ω_f, and π_α; in the cases of ω_f and π_α, their forms will be given explicitly in the GL regime, where they are relevant to the transverse and longitudinal motions (with respect to $\hat{f}$) of the spin vectors.

The pressure is given by

$$P=TS-\epsilon+\mu\rho+\vec{v}^n\cdot\vec{g}+(h_\alpha-H_\alpha)m_\alpha \ . \tag{9}$$

The Gibbs-Duhem relation, esesntial to the nonlinear hydrodynamics, follows from (8) and (9):

$$\begin{aligned}\rho\, d\mu=[&-S\,dT+dP-\vec{g}\cdot d\vec{v}^n+\vec{\lambda}^s\cdot d\vec{v}^s\\ &+\beta\psi_i\, dl_i+\beta\phi_{ij}d(\partial_j l_i)]\\ &+[\vec{\lambda}^{sp}\cdot d\vec{v}^{sp}+\beta\omega_f\, d\phi_f+\beta\pi_\alpha\, df_\alpha\\ &\quad+\beta\pi_{\alpha i}\, d(\partial_i f_\alpha)-m_\alpha\, d(h_\alpha-H_\alpha)] \ .\end{aligned} \tag{10}$$

Note that ϕ_f, like ϕ, is not a globally well-defined quantity. However, $d\phi_f$ *is* well defined, and therefore we may employ it in Eqs. (8) and (10). In practice, we consider that $\phi_f=0$ in Eq. (3), and that spin-space rotations about $\hat{f}$ are our extra degree of freedom. For convenience, however, we employ $d\phi_f$ to denote such rotations.

III. DERIVATION OF HYDRODYNAMICS

The standard procedure of hydrodynamics involves writing down the equations of motion for the independent variables appearing in $d\epsilon$. Before doing this, we note that $\vec{v}^s$ and $\vec{v}^{sp}$ are nontrivial, since they may be generated either by rotations about $\hat{l}$ and $\hat{f}$ or by rotations of $\hat{l}$ and $\hat{f}$. If we let local changes in $\vec{v}^s$ be generated by the local infinitesimal orbital rotation $\delta\vec{\theta}_0$, and set $\delta\phi=-\hat{l}\cdot\delta\vec{\theta}_0$, then

$$\begin{aligned}\delta v_i^s&=\beta\epsilon_{jkl}[\delta\theta_{0k}m_l\nabla_i n_j+m_j\nabla_i(\delta\theta_{0k}n_l)]\\ &=-\beta l_j\nabla_i(\delta\theta_{0j})\\ &=-\beta\nabla_i(l_j\delta\theta_{0j})+\beta\delta\theta_{0j}\nabla_i l_j\\ &=\beta\nabla_i(\delta\phi)-\beta\delta\hat{l}\cdot(\hat{l}\times\nabla_i\hat{l}) \ .\end{aligned} \tag{11}$$

Hence,

$$\frac{\partial v_i^s}{\partial t}=\beta\nabla_i(\dot{\phi})-\beta(\hat{l}\times\nabla_i\hat{l})\cdot\dot{\hat{l}} \ . \tag{12}$$

Similarly, with $\delta\phi_f=-\hat{f}\cdot\delta\vec{\theta}_s$,

$$\frac{\partial v_i^{sp}}{\partial t}=\beta\nabla_i(\dot{\phi}_f)-\beta(\hat{f}\times\nabla_i\hat{f})\cdot\dot{\hat{f}} \ . \tag{13}$$

Thus it is unnecessary to obtain explicit equations of motion for $\vec{v}^s$ and $\vec{v}^{sp}$; they are contained in the motion of $(\phi,\hat{l})$ and $(\phi_f,\hat{f})$. The equations of motion we employ are, for the orbital variables,

$$\begin{aligned}&\dot{\rho}+\partial_i g_i=0,\quad \dot{g}_i+\partial_j\sigma_{ij}=0,\quad \dot{\epsilon}+\partial_i j_i^\epsilon=0 \ ,\\ &\beta\dot{\phi}+\mathcal{J}_\phi=0,\quad \beta\dot{l}_i+X_i=0 \ ,\\ &\dot{S}+\partial_i(Sv_i^n+q_i/T)=R/T\geq 0 \ ;\end{aligned} \tag{14}$$

and, for the spin space variables,

$$\begin{aligned}&\beta\dot{\phi}_f+\mathcal{J}_f=0,\quad \beta\dot{f}_\alpha+Z_\alpha=0 \ ,\\ &\dot{m}_\alpha+\partial_i j_{\alpha i}=-\gamma\epsilon_{\alpha\beta\gamma}[m_\beta(h_\gamma-H_\gamma)+\beta f_\beta\pi_\gamma\\ &\qquad\qquad+\beta(\partial_i f_\beta)\pi_{\gamma i}]+\gamma\beta\omega_f f_\alpha \ .\end{aligned} \tag{15}$$

Here γ is the gyromagnetic ratio of the ^{3}He nucleus (i.e., that of the unpaired neutron). Note that σ_{ij}, j_i^ϵ, $\mathcal{J}_\phi$, X_i, q_i, R, $\mathcal{J}_f$, Z_α, and $j_{\alpha i}$ are unknown, and must be determined. The terms on the right-hand side of the $\dot{m}_\alpha$ equation arise from the magnetic torque, $-\gamma\delta\epsilon/\delta\theta_{s\alpha}$.

The equilibrium conditions associated with the order parameter are determined by setting to zero the full variational derivatives with respect to ϕ, $\hat{l}$, ϕ_f, and $\hat{f}$. Using

$$\frac{\delta v_i^s}{\delta(\partial_j\phi)}=\beta\delta_{ij},\quad \frac{\delta v_i^s}{\delta l_j}=-\beta(\hat{l}\times\nabla_i\hat{l})_j \ , \tag{16}$$

which follow from (11), we find the $(\phi,\hat{l})$ equilibrium conditions to be

$$0=\vec{\nabla}\cdot\vec{\lambda}^s \ , \tag{17}$$

$$0=\hat{l}\times\vec{\Psi},\quad \Psi_i\equiv\psi_i-\partial_j\phi_{ij}-[\hat{l}\times(\vec{\lambda}^s\cdot\vec{\nabla})\hat{l}]_i \ , \tag{18}$$

or

$$0=\delta_{ij}^T\Psi_j,\quad \delta_{ij}^T\equiv\delta_{ij}-l_il_j \ . \tag{18$'$}$$

Similarly, the $(\phi_f,\hat{f})$ equilibrium conditions are found to be

$$0=\Omega_f\equiv\omega_f-\vec{\nabla}\cdot\vec{\lambda}^{sp}\ , \tag{19}$$

$$0=\hat{f}\times\vec{\Pi},\quad \Pi_\alpha\equiv\pi_\alpha-\partial_i\pi_{\alpha i}-[\hat{f}\times(\vec{\lambda}^{sp}\cdot\vec{\nabla})\hat{f}]_\alpha\ , \tag{20}$$

or

$$0=\delta^T_{\alpha\beta}\Pi_\beta,\quad \delta^T_{\alpha\beta}\equiv\delta_{\alpha\beta}-f_\alpha f_\beta\ . \tag{20'}$$

This identifies $\partial_i\lambda_i^s$, Ψ_i, Ω_f, and Π_α as thermodynamic forces which are zero in equilibrium.

By following the usual procedures for deriving the nonlinear hydrodynamics,[13] we obtain the dissipation function R in terms of a pure divergence term and a sum of products of (known) thermodynamic forces and (unknown) thermodynamic fluxes. It takes the form

$$\begin{aligned}R=\partial_i\{&q_i-j_i^\epsilon+(\epsilon-\vec{v}^n\cdot\vec{g})v_i^n+v_j^n\sigma_{ji}+(\mathcal{J}_\phi-\vec{v}^n\cdot\vec{v}^s)\lambda_i^s+[X_j-\beta(\vec{v}^n\cdot\vec{\nabla})l_j]\phi_{ji}\\&+(\mathcal{J}_f-\vec{v}^n\cdot\vec{v}^{sp})\lambda_i^{sp}+[Z_\alpha-\beta(\vec{v}^n\cdot\vec{\nabla})f_\alpha]\pi_{\alpha i}+J_{\alpha i}(h_\alpha-H_\alpha)\}\\&-(q_i/T)\partial_iT-\Sigma_{ij}\partial_jv_i^n-(\mathcal{J}_\phi-\mu-\vec{v}^n\cdot\vec{v}^s)\partial_i\lambda_i^s+[X_i-\beta(\vec{v}^n\cdot\vec{\nabla})l_i]\Psi_i+[\mathcal{J}_f-\gamma\beta\hat{f}_\alpha(h_\alpha-H_\alpha)-\vec{v}^n\cdot\vec{v}^{sp}]\Omega_f\\&+\{Z_\alpha-\beta(\vec{v}^n\cdot\vec{\nabla})f_\alpha-\gamma\beta[\hat{f}\times(\vec{h}-\vec{H})]_\alpha\}\Pi_\alpha-J_{\alpha i}\partial_i(h_\alpha-H_\alpha)\ ,\end{aligned}\tag{21}$$

where

$$\Sigma_{ij}\equiv\sigma_{ij}-P\delta_{ij}-g_iv_j^n-v_i^s\lambda_j^s-\beta(\partial_il_k)\phi_{kj}-v_i^{sp}\lambda_j^{sp}-\beta(\partial_if_\alpha)\Pi_{\alpha j}\ , \tag{22}$$

$$J_{\alpha i}\equiv j_{\alpha i}-m_\alpha v_i^n+\gamma\beta(\epsilon_{\alpha\beta\gamma}f_\beta\pi_{\gamma i}-f_\alpha\lambda_i^{sp})\ . \tag{23}$$

Using the principle that the fluxes (q_i/T, Σ_{ij}, $\mathcal{J}_\phi$, X_i, $\mathcal{J}_f$, Z_α, $J_{\alpha i}$) must be proportional to the thermodynamic forces [∂_iT, $\partial_jv_i^n$, $\vec{\nabla}\cdot\vec{\lambda}^s$, Ψ_i, Ω_f, Π_α, $\partial_i(h_\alpha-H_\alpha)$], and (because of the weak spin-orbit coupling) separately covariant under spin and space (orbital) rotations, we construct the fluxes. Keeping only coefficients which are nonzero in the uniform state, we find that the reactive parts of the fluxes are given by

$$q_i^R/T=A(\hat{l}\times\vec{\nabla}T)_i+(\alpha_x^{(1)}f_\alpha+\bar{\alpha}^{(1)}_{x\alpha\beta}m_\beta)(\hat{l}\times\vec{\nabla})_i(h_\alpha-H_\alpha)\ , \tag{24}$$

$$\mathcal{J}_\phi^R-\mu-\vec{v}^n\cdot\vec{v}^s=-\gamma'\hat{l}\cdot\vec{\nabla}\times\vec{v}^n+m_\alpha B_{\alpha\beta}\Pi_\beta\ , \tag{25}$$

$$X_i^R-\beta(\vec{v}^n\cdot\vec{\nabla})l_i=-\beta'(\hat{l}\times\vec{\Psi})_i-(\alpha_1\delta^T_{ij}l_k+\alpha_2\delta^T_{ik}l_j)\partial_jv_k^n\ , \tag{26}$$

$$\mathcal{J}_f^R-\gamma\beta\hat{f}_\alpha(h_\alpha-H_\alpha)-\vec{v}^n\cdot\vec{v}^{sp}=-\gamma''\hat{l}\cdot\vec{\nabla}\times\vec{v}^n+m_\alpha D_{\alpha\beta}\Pi_\beta\ , \tag{27}$$

$$Z_\alpha^R-\gamma\beta[\hat{f}\times(\vec{h}-\vec{H})]_\alpha-\beta(\vec{v}^n\cdot\vec{\nabla})f_\alpha=-\alpha^{(3)}_{x\alpha\beta}(\hat{f}\times\vec{\Pi})_\beta+m_\beta(B_{\beta\alpha}\vec{\nabla}\cdot\vec{\lambda}^s-D_{\beta\alpha}\Omega_f+E_{ij\beta\alpha}\partial_iv_j^n)\ , \tag{28}$$

$$J_{\alpha i}^R=(\alpha_x^{(1)}f_\alpha+\bar{\alpha}^{(1)}_{x\alpha\beta}m_\beta)(\hat{l}\times\vec{\nabla}T)_i-(\alpha^{(2)}_{ij\alpha\beta}f_\gamma+\bar{a}^{(2)}_{ij\alpha\beta}m_\gamma)\epsilon_{\beta\gamma\delta}\partial_j(h_\delta-H_\delta)\ , \tag{29}$$

$$\begin{aligned}\Sigma_{ij}^R=&-(\alpha_1\delta^T_{jk}l_i+\alpha_2\delta^T_{ik}l_j)\Psi_k-\gamma'\epsilon_{ijk}l_k\vec{\nabla}\cdot\vec{\lambda}^s\\&+(\gamma_{il}^{(1)}\epsilon_{jpq}+\gamma_{jq}^{(2)}\epsilon_{ipl}+\gamma_{jl}^{(3)}\epsilon_{ipq}+\gamma_{iq}^{(3)}\epsilon_{jpl})l_p\partial_qv_l^n+\gamma''\epsilon_{ijk}l_k\Omega_f+E_{ij\alpha\beta}m_\alpha\Pi_\beta\ .\end{aligned}\tag{30}$$

Here, A, $\alpha_x^{(1)}$, $\bar{\alpha}^{(1)}_{x\alpha\beta}$, γ', $B_{\alpha\beta}$, β', α_1, α_2, γ'', $D_{\alpha\beta}$, $\alpha^{(3)}_{x\alpha\beta}$, $\alpha^{(2)}_{ij\alpha\beta}$, $\bar{\alpha}^{(2)}_{ij\alpha\beta}$, $E_{ij\alpha\beta}$, $\gamma_{ij}^{(1)}$, $\gamma_{ij}^{(2)}$, $\gamma_{ij}^{(3)}$ are all undetermined, with characteristic structures (where appropriate) of the form

$$\bar{\alpha}^{(1)}_{x\alpha\beta}=\bar{\alpha}_d^{(1)}d_\alpha d_\beta+\bar{\alpha}_e^{(1)}e_\alpha e_\beta+\bar{\alpha}_f^{(1)}f_\alpha f_\beta\ ,$$

$$\alpha_{ij}^{(2)}=\alpha_\parallel^{(2)}l_il_j+\alpha_\perp^{(2)}\delta^T_{ij}\ .$$

(No $d_\alpha e_\beta$ term appears because it is odd under time reversal T, and no $d_\alpha f_\beta$ and $e_\alpha f_\beta$ terms appear because they have no definite signature under T.) Note, that, since $\hat{f}$ should be irrelevant to the behavior of the other variables as $|\vec{H}|\to 0$ (A phase), we expect that $\alpha_x^{(1)}$, $\alpha^{(2)}_{ij\alpha\beta}\to 0$ as $|\vec{H}|\to 0$. Such a statement cannot be made of $\alpha^{(3)}_{x\alpha\beta}$, because $\hat{f}$ can remain relevant to its own behavior. Also, because $\hat{f}$ is not truly a spontaneously broken symmetry, it is likely that $\alpha_x^{(1)}$ and $\alpha^{(2)}_{ij\alpha\beta}$ vary as $\vec{m}\cdot\hat{f}$, for small $\vec{m}$. Note that $B_{\alpha\beta}f_\beta=D_{\alpha\beta}f_\beta=E_{\alpha\beta ij}f_\beta=0$; thus, in bulk equilibrium, where $\vec{m}\parallel\hat{f}$, the $B_{\alpha\beta}$, $D_{\alpha\beta}$, and $E_{\alpha\beta ij}$ terms are zero.

Some of the remaining terms are determined (or partially determined) by the requirement that Σ_{ij}^R must enable us automatically to satisfy angular momentum conservation. Let

$$L_i=(\vec{r}\times\vec{g})_i+\gamma^{-1}(\vec{m})_i \tag{31}$$

denote the angular momentum density, which must satisfy

$$\frac{\partial L_i}{\partial t}+\partial_jJ_j^{L_i}=(\vec{m}\times\vec{H})_i\ . \tag{32}$$

This problem is nearly identical to the one considered in Ref. 9. In the present case, we must be able to write

$$\sigma_{ij}=\sigma_{ij}^{(s)}+\partial_kf_{ijk}+\epsilon_{ijk}\left[\left(\frac{\partial\epsilon}{\partial\vec{\theta}_s}\right)+(\vec{m}\times\vec{H})\right]_k\ , \tag{33}$$

where $\sigma_{ij}^{(s)} \equiv \frac{1}{2}(\sigma_{ij}+\sigma_{ji})$ and $f_{ijk}=-f_{jik}$, in which case

$$J_j^{L_i} = \epsilon_{ikl}(r_k\sigma_{lj}+f_{klj})+\gamma^{-1}j_{ij} \quad . \tag{34}$$

From $d\epsilon = \vec{m}\times\vec{H}\cdot d\vec{\theta}$ under a simultaneous rotation $d\vec{\theta}$ of both spin and space coordinates we obtain the identity[14,15]

$$\begin{aligned}0=\epsilon_{ijk}[&v_i^n g_j+\lambda_i^s+\beta\psi_i l_j\\&+\beta(\phi_{il}\partial_l l_j+\phi_{li}\partial_j l_l)\\&+{}_i^{sp}v_j^{sp}+\beta\omega_f d_i e_j+\beta\pi_i f_j\\&+\beta(\pi_{il}\partial_l f_j+\pi_{li}\partial_j f_l)+h_i m_j] \quad ,\end{aligned} \tag{35}$$

which must be used to enforce Eq. (32). As a result, one obtains the restrictions that

$$\gamma' = \tfrac{1}{2}\beta = \alpha_2-\alpha_1, \quad \gamma''=0 \quad , \tag{36}$$

$$\gamma_\parallel^{(1)}=\gamma_\parallel^{(2)}=\gamma_\parallel^{(3)}, \quad \gamma_\perp^{(1)}=\gamma_\perp^{(2)} \quad . \tag{37}$$

Because of the restrictions in (37), explicit consideration of the coefficient of $\partial_q v_l^n$ in (30) shows that the only forms that ultimately appear are

$$\gamma_\parallel^{(1)}, \quad \tfrac{1}{2}(\gamma_\perp^{(1)}+\gamma_\perp^{(3)}) \quad . \tag{38}$$

In addition, we find that

$$f_{ijk}^R = \tfrac{1}{2}\beta[\epsilon_{ijl}l_l\lambda_k^s-(l_i\phi_{jk}-l_j\phi_{ik})] \quad . \tag{39}$$

The dissipative parts of the fluxes are obtained from considerations similar to those for the reactive parts. The dissipative parts have a behavior under time reversal which is opposite to that of the reactive parts. Furthermore, they must satisfy the Onsager principle, and they must yield $R\geq 0$ (positive entropy production). We find, when angular momentum considerations are included, that

$$\begin{aligned}q_i^D/T=-(\kappa_{ij}/T)\partial_j T-(\alpha_{ij}^{(1)}f_\alpha+\bar{\alpha}_{ij\alpha\beta}^{(1)}m_\beta)\\ \times\partial_j(h_\alpha-H_\alpha) \quad ,\end{aligned} \tag{40}$$

$$\mathcal{G}_\phi^D = -\xi\vec{\nabla}\cdot\vec{\lambda}^s-\xi_{ij}\partial_j v_i^n+\xi'\Omega_f \quad , \tag{41}$$

$$X_i^D = \eta\delta_{ij}^T\Psi_j = \xi_{kji}\partial_k v_j^n \quad , \tag{42}$$

$$\mathcal{G}_f^D = \xi_{ij}''\Omega_f-\xi_{ij}'\partial_j v_i^n-\xi'\vec{\nabla}\cdot\vec{\lambda}^s \quad , \tag{43}$$

$$Z_\alpha^D = \nu_{\alpha\beta}\Pi_\beta, \quad \nu_{\alpha\beta}=\nu_d d_\alpha d_\beta+\nu_e e_\alpha e_\beta \quad , \tag{44}$$

$$\begin{aligned}J_{\alpha j}^D = j_{\alpha i}^D = -\mu_{\alpha i\beta j}\partial_j(h_\beta-H_\beta)\\ -(\alpha_{ij}^{(1)}f_\alpha+\bar{\alpha}_{ij\alpha\beta}^{(1)}m_\beta)\partial_j T \quad ,\end{aligned} \tag{45}$$

$$\begin{aligned}\Sigma_{ij}^D=\sigma_{ij}^D=-\nu_{ijkl}(\partial_k v_l^n+\partial_l v_k^n)\\ -\xi_{ij}\vec{\nabla}\cdot\vec{\lambda}^s+\xi_{ijk}\Psi_k+\xi_{ij}'\Omega_f \quad ,\end{aligned} \tag{46}$$

where κ_{ij}, ξ_{ij}, ξ_{ij}', and $\alpha_{ij}^{(1)}$ have the form

$$\kappa_{ij}=\kappa_\parallel l_i l_j+\kappa_\perp\delta_{ij}^T \quad ,$$

$$\xi_{kji}=\xi(l_j\epsilon_{kip}+l_k\epsilon_{jip})l_p \quad ,$$

$\bar{\alpha}_{ij\alpha\beta}^{(1)}$ and $\mu_{\alpha i\beta j}$ have the form[11]

$$\mu_{\alpha i\beta j}=\delta_{ij}^T(\mu_2 d_\alpha d_\beta+\mu_{1e}e_\alpha e_\beta+\mu_{1f}f_\alpha f_\beta)+l_i l_j(\mu_4 d_\alpha d_\beta+\mu_{3e}e_\alpha e_\beta+\mu_{3f}f_\alpha f_\beta) \quad ,$$

and[16]

$$\nu_{ijkl}=\nu_1 l_i l_j l_k l_l+\nu_2\delta_{ij}^T\delta_{kl}^T+\tfrac{1}{2}\nu_3(\delta_{ij}^T l_k l_l+\delta_{kl}^T l_i l_j)+\tfrac{1}{2}\nu_4(\delta_{ik}^T\delta_{jl}^T+\delta_{il}^T\delta_{jk}^T)+\tfrac{1}{2}\nu_5(\delta_{ik}^T l_j l_l+\delta_{jl}^T l_i l_k+\delta_{il}^T l_j l_k+\delta_{jk}^T l_i l_l) \quad .$$

Note that $\mu_{1e}=\mu_{1f}$ and $\mu_{3e}=\mu_{3f}$ in the A phase.[11] Since $\sigma_{ij}^D=\sigma_{ji}^D$, we have that $f_{ijk}^D=0$. Note that, because $R\geq 0$, the coefficents appearing in (40)–(46) are subject to a number of constraints, which we do not write down explicitly. It should also be observed that we have omitted a term $-\alpha_x^{(2)}[\hat{f}\times(\hat{l}\times\vec{\nabla})]_i\times(\vec{h}-\vec{H})_\alpha$ from the $j_{\alpha i}^D$ of Ref. 1, which does not give $\bar{R}\geq 0$ unless $\alpha_x^{(2)}=0$. We expect that $\alpha_{ij}^{(1)}f_\alpha$ and $\alpha_{ij\alpha\beta}^{(1)}m_\beta$ are of the same order of magnitude, both going to zero as $|\vec{H}|\to 0$. Also, ξ' and $\xi_{ij}'\to 0$ as $|\vec{H}|\to 0$.

It is useful to rewrite (15) with (23) in mind. We then find that

$$\begin{aligned}\dot{m}_\alpha+\partial_i(m_\alpha v_i^n+J_{\alpha i})=-\gamma[\vec{m}\times(\vec{h}-\vec{H})]_\alpha\\ +\gamma\beta(\hat{f}\Omega_f-\hat{f}\times\vec{\Pi})_\alpha \quad ,\end{aligned} \tag{47}$$

thus making it clear that $\dot{m}_\alpha=0$ in equilibrium [where $\vec{v}^n=0$, $J_{\alpha i}=0$, $\vec{m}\times(\vec{h}-\vec{H})=0$, $\Omega_f=0$, and $\hat{f}\times\vec{\Pi}=0$]. Observe that, in an ordinary material, a finite T_1 is needed to also make $\vec{m}\cdot(\vec{h}-\vec{H})=0$ in equilibrium. Here, because of the $\dot{\phi}_f$ equation, we have $\hat{f}\cdot(\vec{h}-\vec{H})=0$ in equilibrium. Since $\hat{f}\parallel\vec{m}$ in equilibrium, we also have $\vec{m}\cdot(\vec{h}-\vec{H})=0$ in equilibrium. Hence $\vec{h}=\vec{H}$ in equilibrium here, *without* spin relaxation. Physically, this is caused by the internal Josephson effect,[7] which disappears for the A_1 phase.

Before leaving this section, we remind the reader that our results must go over to the results of Ref. 9 for the A phase, and to the results of Ref. 1 for the A_1 phase. This has a number of consequences for the parameters appearing in the theory. Of particular importance are $\vec{\lambda}^s$ and $\vec{\lambda}^{sp}$, which take the form (when $\hat{l}\times\vec{\nabla}\rho$ and $\hat{l}\times\vec{\nabla}S$ terms are neglected[17])

$$\vec{\lambda}^s=\vec{\rho}^s\cdot(\vec{v}^s-\vec{v}^n)+\beta\vec{C}\cdot\vec{\nabla}\times\hat{l}+\vec{\rho}^{sp}\cdot\vec{v}^{sp} \quad , \tag{48}$$

$$\vec{\lambda}^{sp}=\vec{\tau}^s\cdot(\vec{v}^s-\vec{v}^n)+\beta\vec{C}^{sp}\cdot\vec{\nabla}\times\hat{l}+\vec{\tau}^{sp}\cdot\vec{v}^{sp} \quad . \tag{49}$$

[Note that $\vec{\nabla}\times\hat{f}$ terms do not appear in (48) and (49) because $\hat{f}$ is a spin vector.] To establish the dependence on $\vec{v}^s-\vec{v}^n$, it is useful to work in the

$\vec{v}^n=\vec{0}$ frame. Here $\epsilon_0=\epsilon-\vec{v}^n\cdot\vec{g}+\frac{1}{2}\rho(\vec{v}^n)^2$ gives

$$d\epsilon_0=\{T\,dS+[\mu+\tfrac{1}{2}(\vec{v}^n)^2]\,d\rho+\vec{\lambda}^s\cdot d(\vec{v}^s-\vec{v}^n)+\beta\psi_i\,dl_i+\beta\phi_{ij}\,d(\partial_j l_i)\}$$
$$+[\vec{\lambda}^{sp}\cdot d\vec{v}^{sp}+\beta\omega_f\,d\phi_f+\beta\pi_\alpha\,df_\alpha+\beta\pi_{\alpha i}\,d(\partial_i f_\alpha)+(h_\alpha-H_\alpha)\,dm_\alpha]\ , \tag{50}$$

thus making it clear that $\vec{\lambda}^s$ and $\vec{\lambda}^{sp}$ depend upon $\vec{v}^s-\vec{v}^n$. In addition, the following Maxwell relations hold:

$$\frac{\partial\lambda_i^s}{\partial v_j^{sp}}=\frac{\partial\lambda_j^{sp}}{\partial(v_i^s-v_i^n)}\to\rho_{ij}^{sp}=\tau_{ji}^s\ ; \tag{51}$$

$$\frac{\partial\lambda_i^s}{\partial(\partial_j l_k)}=\beta\frac{\partial\phi_{kj}}{\partial(v_i^s-v_i^n)}\to\beta\phi_{kj}$$
$$\ni\beta C_{il}^{sp}\epsilon_{ljk}(v_i^s-v_i^n)\ ; \tag{52}$$

$$\frac{\partial\lambda_i^{sp}}{\partial(\partial_j l_k)}=\beta\frac{\partial\phi_{kj}}{\partial v_i^{sp}}\to\beta\phi_{kj}$$
$$\ni\beta C_{il}^{sp}\epsilon_{ljk}v_i^{sp}\ . \tag{53}$$

[In (52) and (53), the symbol "$\ni$" denotes "contains the term."] When we consider the A_1 phase, where only $\vec{v}^s+\vec{v}^{sp}$ can appear in the theory (and where rotations about $\hat{f}$ can cost no energy, so that $\omega_f=0$), we have $\vec{\lambda}^s=\vec{\lambda}^{sp}$ (which implies that $\vec{\rho}^s=\vec{\tau}^s$, $\vec{C}=\vec{C}^{sp}$, $\vec{\rho}^{sp}=\vec{\tau}^{sp}$) and $\vec{\rho}^s=\vec{\rho}^{sp}$, $\vec{\tau}^s=\vec{\tau}^{sp}$. On the other hand, when we consider the A phase, rotations about $\hat{d}$ cost no energy, so that $\pi_\alpha e_\alpha=0$ and $\pi_{\alpha i}e_\alpha=0$.

IV. NORMAL MODES OF THE UNIFORM SYSTEM

In a uniform system, the equilibrium $\hat{f}$ will align either along or against the external field $\vec{H}$ and the equilibrium magnetization $\vec{m}_0$. Thus we define[1]

$$\vec{m}_0\cdot\hat{f}_0\equiv m_0M_s\ , \tag{54}$$

where $m_0=|\vec{m}_0|$ and $M_s=\pm1$. It will be necessary only to consider the behavior of $(\rho,S,\vec{g},\phi,\phi_f,\hat{f},\vec{m})$, since the motion of $\hat{l}$ is known to be very slow.[18] Furthermore, its behavior has already been discussed quite adequately for the A phase,[10,19,20] and the A_2 phase presents no qualitatively different situations.[5]

We will begin by discussing the coupled motions of $\delta\hat{f}\cdot\hat{d}$, $\delta\hat{f}\cdot\hat{e}$, $\delta\vec{m}\cdot\hat{d}$, $\delta\vec{m}\cdot\hat{e}$, which yield the transverse spin waves of the A_2 phase. Note that spin hydrodynamics for the A phase has been discussed in Refs. 11 and 21, for the A_1 phase in Ref. 2, and for the A_2 phase in Ref. 5. However, the work of Pleiner and Graham,[2] and of Pleiner,[5] is built around the fact that $\hat{f}$ is not a truly hydrodynamic variable, and therefore does not have any gapless modes. These works carefully study the transverse spin modes in the infinite wavelength limit, and include dissipation, but they do not attempt systematic studies of the wave-vector dependence of the eigenfrequencies. Our work, on the other hand, neglects the interaction energy that makes $\hat{f}$ not a truly hydrodyamic variable, and concentrates upon the wave-vector dependence of the eigenfrequencies. In addition, we have neglected dissipation. Our results are more general than those of Ref. 3. Although the interaction that causes the transverse spin modes to develop gaps does dominate the long-wavelength behavior of the modes, our study is illuminating, for it shows how one continuously goes from the spin waves of the A phase to those of the A_1 phase.

The discussion of transverse spin waves will then be followed by a discussion of the coupled motions of $\rho,S,\vec{g},\phi,\phi_f,\delta\vec{m}\cdot\hat{f}$.

A. Transverse spin waves

Neglecting dissipation and the unknown term in $\alpha_{x\alpha\beta}^{(3)}$ in Eq. (28) for Z_i^R, and linearizing about $\vec{m}\parallel\hat{f}$, Eq. (15) yields

$$\frac{\partial\hat{f}}{\partial t}=\gamma\hat{f}\times(\vec{H}-\vec{h})\ , \tag{55}$$

where

$$\vec{h}=\vec{\chi}^{-1}\cdot\vec{m}$$
$$=\chi_d^{-1}(\vec{m}\cdot\hat{d})\hat{d}+\chi_e^{-1}(\vec{m}\cdot\hat{e})\hat{e}+\chi^{-1}(\vec{m}\cdot\hat{f})\hat{f}\ . \tag{56}$$

Since $\chi_d^{-1}\approx\chi_e^{-1}\approx\chi^{-1}$ when (small) susceptibility anisotropy effects are neglected, and $\vec{H}=\chi^{-1}\vec{m}_0$, Eq. (56) becomes, with $\vec{m}=\vec{m}_0+\delta\vec{m}$, and $\delta\vec{m}\cdot\vec{m}_0=0$,

$$\vec{h}\approx\chi^{-1}\vec{m}=\vec{H}+\chi^{-1}\delta\vec{m}\ . \tag{57}$$

With $\delta\hat{f}=(\delta\hat{f}\cdot\hat{d})\hat{d}+(\delta\hat{f}\cdot\hat{e})\hat{e}$ and $\delta\vec{m}(\delta\vec{m}\cdot\hat{d})\hat{d}+(\delta\vec{m}\cdot\hat{e})\hat{e}$, and assuming an $e^{-i\omega t}$ dependence on time, (55) and (57) yield

$$-i\omega(\delta\hat{f}\cdot\hat{d})=\gamma\chi^{-1}(\delta\vec{m}\cdot\hat{e})\ ,$$
$$-i\omega(\delta\hat{f}\cdot\hat{e})=-\gamma\chi^{-1}(\delta\vec{m}\cdot\hat{d})\ . \tag{58}$$

When one neglects the unknown $\vec{\nabla}T$ terms in Eq. (29) for $J_{\alpha i}^R$, the linearized version of the transverse component of Eq. (47) is given by

$$\delta_{\alpha\beta}^T\left(\frac{\partial\vec{m}}{\partial t}\right)_\beta=[\gamma\vec{m}_0\times(-\delta\vec{h})]_\alpha-(\gamma\beta\hat{f}\times\vec{\Pi})_\alpha$$
$$+\tilde{\alpha}_{ij\alpha\beta}^{(2)}m_0M_sf_\gamma\epsilon_{\beta\gamma\delta}\partial_i\partial_j(\delta h_\delta)\ ,$$
$$\tilde{\alpha}_{ij\alpha\beta}^{(2)}\equiv\bar{\alpha}_{ij\alpha\beta}^{(2)}+(m_0M_s)^{-1}\alpha_{ij\alpha\beta}^{(2)}\ , \tag{59}$$

where $\delta\vec{h} = \chi^{-1}\delta\vec{m}$, and $\Pi_\alpha \approx \pi_\alpha - \partial_i \pi_{\alpha i}$. For our purposes, π_α is due to the nuclear dipole interaction ϵ_D, given by[6,22]

$$\epsilon_D = -\tfrac{3}{5} g_D [a^2(\hat{d}\cdot\hat{l})^2 + b^2(\hat{e}\cdot\hat{l})^2] \quad . \tag{60}$$

Since $a^2 > b^2$, $(\hat{d}\cdot\hat{l})^2 = 1$ in equilibrium. Evaluating π_α slightly away from equilibrium, one obtains

$$\beta\pi_\alpha = \frac{\partial \epsilon_D}{\partial f_\alpha} = \tfrac{6}{5} g_D a^2 d_\alpha (\delta\hat{f}\cdot\hat{d}) \quad . \tag{61}$$

Furthermore, $\pi_{\alpha i}$ must have the form

$$\beta\pi_{\alpha i} = M^d_{ij} d_\alpha (\hat{d}\cdot\partial_j \hat{f}) + M^e_{ij} e_\alpha (\hat{e}\cdot\partial_j \hat{f}) \quad , \tag{62}$$

where $M^d_{ij} = M^d_\parallel l_i l_j + M^d_\perp \delta^T_{ij}$, and similarly for M^e_{ij}. Utilizing (61) and (62), and assuming an $\exp(i\vec{q}\cdot\vec{r} - i\omega t)$ dependence on space and on time, (59) becomes

$$\begin{aligned} -i\omega(\delta\vec{m}\cdot\hat{d}) &= \gamma\chi^{-1} m_0 M_s (1 + \gamma^{-1}\tilde{\alpha}^{(2)}_{ijd} q_i q_j)(\delta\vec{m}\cdot\hat{e}) \\ &\quad + \gamma M^e_{ij} q_i q_j (\delta\hat{f}\cdot\hat{e}) \quad , \\ -i\omega(\delta\vec{m}\cdot\hat{e}) &= -\gamma\chi^{-1} m_0 M_s (1 + \gamma^{-1}\tilde{\alpha}^{(2)}_{ije} q_i q_j)(\delta\vec{m}\cdot\hat{d}) \\ &\quad - \gamma M^d_{ij} q_i q_j (\delta\hat{f}\cdot\hat{d}) - \gamma(\tfrac{6}{5} g_D a^2)(\delta\hat{f}\cdot\hat{d}) \quad . \end{aligned} \tag{63}$$

Here $\tilde{\alpha}^{(2)}_{ijd} = \tilde{\alpha}^{(2)}_{ij\alpha\beta} \hat{d}_\alpha \hat{d}_\beta$, and similarly for $\tilde{\alpha}^{(2)}_{ije}$. The solutions of (58) and (63) are given by

$$\begin{aligned} \omega^2_\pm &= \tfrac{1}{2}(\omega_L^2 + \omega_F^2 + \omega_D^2 + \omega_d^2 + \omega_e^2) \\ &\quad \pm \tfrac{1}{2}[(\omega_L^2 + \omega_F^2 + \omega_D^2 + \omega_d^2 - \omega_e^2)^2 \\ &\quad + 2\omega_e^2(\omega_L^2 + \omega_F^2)]^{1/2} \quad , \end{aligned} \tag{64}$$

where

$$\begin{aligned} \omega_L^2 &\equiv (\gamma\chi^{-1} m_0 M_s)^2 = (\gamma H)^2 \quad , \\ \omega_F^2 &= \omega_L^2 \gamma^{-1} (\tilde{\alpha}^{(2)}_{ijd} + \tilde{\alpha}^{(2)}_{ije}) q_i q_j \quad , \\ \omega_D^2 &\equiv (\gamma^2/\chi) \tfrac{6}{5} g_D a^2 \equiv \omega_A^2 a^2 \quad , \\ \omega_d^2 &\equiv (\gamma^2/\chi) M^d_{ij} q_i q_j \quad , \\ \omega_e^2 &\equiv (\gamma^2/\chi) M^e_{ij} q_i q_j \quad . \end{aligned}$$

When the terms specific to ^{3}He-A are dropped, Eq. (64) yields the spin waves for a ferromagnet, where ω_F^2 may be thought of as due to the q dependence of the susceptibility. Note that, if the $\alpha^{(3)}_{x\alpha\beta}$ terms were kept in Eq. (55), the analog of Eq. (64) would become quite complex. These terms do not matter for $q = 0$. Nevertheless, in principle, they do affect the spin-wave dispersion.

When $H = 0$, $\omega_L = 0$, $a^2 = 1$, and $\omega_e^2 = 0$, so

$$\omega^2_\pm = \tfrac{1}{2}(\omega_A^2 + \omega_d^2) \pm \tfrac{1}{2}(\omega_A^2 + \omega_d^2) \quad .$$

ω_+^2 corresponds to one of the usual transverse spin waves[11]; $\omega_-^2 (=0)$ corresponds to the diffusive magnetization mode discussed in Ref. 11, and to another mode involving rotation about $\hat{d}$, which is unphysical for the A phase, and thus irrelevant to Ref. 11.

It is instructive to study the effects of a small field on $\omega^2_\pm$. With $(\omega_D^2 + \omega_d^2) >> (\omega_L^2 + \omega_F^2 + \omega_e^2)$, (64) yields

$$\omega_+^2 \approx \omega_D^2 + \omega_d^2 + \omega_L^2 + \omega_F^2, \quad \omega_-^2 \approx \omega_e^2 \quad .$$

Thus ω_-^2, which is largely associated with $\delta\vec{m}\cdot\hat{d}$ and $\delta\hat{f}\cdot\hat{e}$, corresponds to a gapless spin wave of low velocity v_e, when one is near the A phase. Since $M^e_{ij} \propto H^2$ for small H [see Eqs. (65) and (66) below] $v_e \propto H$.

On the other hand, more generally one has, for $(\omega_d^2 + \omega_e^2) << (\omega_D^2 + \omega_L^2 + \omega_F^2)$,

$$\begin{aligned} \omega_+^2 &\approx (\omega_D^2 + \omega_L^2 + \omega_F^2) + (\omega_d^2 + \omega_e^2) \\ &\quad - \omega_e^2[\omega_D^2/(\omega_D^2 + \omega_L^2)] \quad , \\ \omega_-^2 &\approx \omega_e^2[\omega_D^2/(\omega_D^2 + \omega_L^2)] \quad . \end{aligned}$$

These results are in agreement with those of Ref. 3, when ω_F is set to zero and notational differences are accounted for. {Reference 3 employs β, which equals $(1-p^2)^{1/2}$ in the present notation, so that $a^2 = \frac{1}{2}[1 + (1-p^2)^{1/2}] = \frac{1}{2}(1+\beta)$.} In the Appendix, we present the weak-coupling GL expressions for various quantities appearing in the theory. From (9.54) of Ref. 7. near T_c, M^d_{ij} and M^e_{ij} may be written (with $\beta = \hbar/2m$)

$$\begin{aligned} M^d_{ij} &= \tfrac{1}{2}\beta^2 \rho^s_{ij} [1 + (1-p^2)^{1/2}] \quad , \\ M^e_{ij} &= \tfrac{1}{2}\beta^2 \rho^s_{ij} [1 - (1-p^2)^{1/2}] \quad , \\ \rho^s_{ij} &= (12\rho/5)(1 + F_1/3)^{-1}(1 - T/T_c) \\ &\quad \times (\delta_{ij} - \tfrac{1}{2} l_i l_j)(\Delta C/1.42 C_n) \quad , \end{aligned} \tag{65}$$

where F_1 is a Landau parameter. It is also useful to note that, for T near T_c we may rewrite p as

$$\begin{aligned} p &= H/H_{A_2}(T) \quad , \\ H_{A_2}(T) &\equiv \frac{(1 - T/T_c)(2k_B T/\gamma\hbar)}{\eta(1-\delta)/(1+\delta)} \quad , \end{aligned} \tag{66}$$

where we have employed the same notation for η and δ as in Refs. 6 and 23. Approximate values at the melting pressure P_m are $\delta \approx 0.25$ and $\eta \approx 5 \times 10^{-2}$.[23] Because of the linear dependence of p upon H, it would be convenient to fix T and monitor ω_+^2 and ω_-^2 as functions of H. This may not be practical. Clearly, H_{A_2} is the field which destroys the A_2 phase.

Finally, we evaluate (64) neglecting ω_F and in the limit of large $|\vec{q}|$, so $\omega_d^2, \omega_e^2 >> \omega_L^2, \omega_D^2$. In that case we find

$$\omega_+^2 \approx \omega_d^2, \quad \omega_-^2 \approx \omega_e^2 \quad .$$

The eigenvectors in these cases are also simple; ω_+ involves only $\delta\vec{m}\cdot\hat{e}$ and $\delta\hat{f}\cdot\hat{d}$ and ω_- involves only $\delta\vec{m}\cdot\hat{d}$ and $\delta\hat{f}\cdot\hat{e}$. For $\omega_F \neq 0$, the corresponding modes are more complex.

If Z_α^D were included in f_α, the modes would develop a q-independent relaxation time; and if the interaction which orients f_α with m_α were included, both modes would develop a gap at $q=0$.[2,5]

B. Longitudinal modes

We first consider the case where the normal fluid is "clamped," so $\vec{v}^n = 0$. The linearized entropy equation, without dissipation, then tells us that $S=\text{const}$. Mass conservation [Eq. (14)] gives, with $\vec{g}\approx\vec{\lambda}^s$ and Eq. (48) for $\vec{\lambda}^s$,

$$-\omega\delta\rho + q_i(\rho_{ij}^s v_j^s + \rho_{ij}^{sp} v_j^{sp}) = 0 \ , \tag{67}$$

where $\delta\rho$ is the deviation of the mass density from its equilibrium value. [We assume an $\exp(i\vec{q}\cdot\vec{r} - i\omega t)$ dependence on space and time.] The linearized equation of motion for $\delta\phi$, Eqs. (14) and (25), with $\delta\mu \approx (\partial\mu/\partial\rho)\delta\rho$, gives

$$-i\beta\omega\delta\phi + \left(\frac{\partial\mu}{\partial\rho}\right)\delta\rho = 0 \ . \tag{68}$$

The linearized equation of motion for $\delta\phi_f$, Eqs. (14) and (27), with $\vec{h}-\vec{H}\approx\chi^{-1}\delta\vec{m}$, gives

$$-i\beta\omega\delta\phi_f + \gamma\beta\chi^{-1}(\delta\vec{m}\cdot\hat{f}) = 0 \ . \tag{69}$$

Finally, the $\hat{f}\cdot(\partial\vec{m}/\partial t)$ equation, Eqs. (47) and (29) (neglecting the unknown coupling to $\hat{l}\times\vec{\nabla}T$), gives

$$-i\omega(\delta\vec{m}\cdot\hat{f}) = \gamma\beta(\omega_f - iq_i\lambda_i^{sp}) \ . \tag{70}$$

We obtain ω_f from the dipole interaction, Eq. (60), evaluated slightly off equilibrium:

$$\beta\omega_f = \frac{\partial\epsilon_0}{\partial\phi_f} = \tfrac{6}{5}g_D(1-p^2)^{1/2}\delta\phi_f \ . \tag{71}$$

Using λ_i^{sp} from Eq. (49), Eq. (70) becomes

$$-i\omega(\delta\vec{m}\cdot\hat{f}) = \gamma\tfrac{6}{5}g_D(1-p^2)\delta\phi_f - i\gamma\beta q_i(\tau_{ij}^s v_j^s + \tau_{ij}^{sp} v_j^{sp}) \ . \tag{72}$$

Use of

$$v_j^s \approx \beta(iq_j)\delta\phi, \quad v_j^{sp}\approx\beta(iq_j)\delta\phi_f \ , \tag{73}$$

in (67)–(69) and (72), enables us to solve these equations in terms of $\delta\phi$, $\delta\phi_f$, $\delta\rho$, and $\delta\vec{m}\cdot\hat{f}$. We obtain a quadratic equation for ω^2

$$\omega^4 - \omega^2(\tilde{\omega}_D^2 + \omega_{f0}^2 + \omega_{40}^2) + [(\tilde{\omega}_D^2+\omega_{f0}^2)\omega_{40}^2 - \omega_{f1}^2\omega_{41}^2] = 0 \ , \tag{74}$$

where

$$\tilde{\omega}_D^2 \equiv (\gamma^2/\chi)\tfrac{6}{5}g_D(1-p^2)^{1/2} \equiv \omega_A^2(1-p^2)^{1/2} \ ,$$

$$\omega_{f0}^2 \equiv \beta^2(\gamma^2/\chi)q_iq_j\tau_{ij}^{sp} \ ,$$

$$\omega_{40}^2 \equiv (\partial\mu/\partial\rho)q_iq_j\rho_{ij}^s \ ,$$

$$\omega_{f1}^2 \equiv \beta^2(\gamma^2/\chi)q_iq_j\tau_{ij}^s \ ,$$

$$\omega_{41}^2 \equiv (\partial\mu/\partial\rho)q_iq_j\rho_{ij}^{sp} \ .$$

In the A phase, $p=0$ and $\rho_{ij}^{sp}=\tau_{ij}^s=0$, and the solutions are

$$\omega_+^2 = \omega_A^2 + \omega_{f0}^2, \quad \omega_-^2 = \omega_{40}^2 \ ;$$

in the A_1 phase $p=1$, so $\tilde{\omega}_D^2 = 0$, and $\rho_{ij}^{sp}=\rho_{ij}^s=\tau_{ij}=\tau_{ij}^{sp}$, so the solutions are

$$\omega_+^2 = \omega_{f0}^2 + \omega_{40}^2, \quad \omega_-^2 = 0 \ .$$

These results are in agreement with Refs. 3 and 11 for the A phase, and with Refs. 1 and 3 for the A_1 phase. If we consider the weak-coupling GL regime, where $\tau_{ij}^s = \rho_{ij}^{sp} = p\rho_{ij}^s = p\tau_{ij}^{sp}$, we have

$$\omega_\pm^2 = \tfrac{1}{2}[\omega_A^2(1-p^2)^{1/2} + \omega_{f0}^2 + \omega_{40}^2] \pm \tfrac{1}{2}\{[\omega_A^2(1-p^2)^{1/2}+\omega_{f0}^2-\omega_{40}^2]^2 + 4p^2\omega_{40}^2\omega_{f0}^2\}^{1/2} \ ; \tag{75}$$

here, all the p dependence is explicit. This equation, which is in agreement with that of Ref. 3 for the A_2 phase, enables us to track the eigenfrequencies as one continuously increases p, moving across the A_2 phase from A to A_1. If the "unperturbed" modes are considered to have

$$\omega_{0+}^2 = \omega_A^2(1-p^2)^{1/2} + \omega_{f0}^2, \quad \omega_{0-}^2 = \omega_{40}^2 \ ,$$

then "crossing" occurs when $\omega_{0+}^2 = \omega_{0-}^2$. Since, near T_c,[2,24]

$$\omega_{f0}^2 = \omega_{40}^2[(1+\tfrac{1}{4}Z_0)/(1+F_0)] << \omega_{40} \ ,$$

crossing is primarily determined by $\omega_{40}^2 \approx \omega_A^2(1-p^2)^{1/2}$, and the fractional shift of the modes at crossing is given by $\pm(p/2)(\omega_{f0}/\omega_{40})^{1/2}$. Note that a similar analysis for the transverse modes [cf., Eq. (64)] yields no "crossing."

Because the "longitudinal resonance" develops a "fourth-sound" component, that mode should be observable with a fourth-sound apparatus; and because the fourth-sound mode develops a longitudinal-resonance component, that mode should be observable with a longitudinal-resonance apparatus. A detailed discussion of questions associated with the design of an experiment to detect this mode crossing is given in Appendix B.

One may also study the theoretical properties of the longitudinal modes for an unconfined geometry.

One finds a mode which is basically first sound, and two modes which are basically coupled longitudinal-resonance and second-sound modes. They may be obtained from the equations of motion for S, ϕ, ϕ_f, and $\delta\vec{m}\cdot\hat{f}$, subject to $\dot{\rho}\approx 0$ and $\vec{g}\approx 0$. We find that

$$\omega^4-\omega^2(\tilde{\omega}_D^2+\tilde{\omega}_{f0}^2+\omega_{20}^2)+[(\tilde{\omega}_D^2+\omega_{f0}^2)\omega_{20}^2-\tilde{\omega}_{f1}^2\omega_{21}^2]=0\ ,$$
$$\omega_{20}^2=\left[\frac{-S\,\partial\mu}{\partial S}\right]_P q_iq_j\rho_{ik}^s(\rho^n)_{kj}^{-1}\ ,$$
$$\omega_{21}^2=\left[\frac{-S\,\partial\mu}{\partial S}\right]_P q_iq_j\rho_{ik}^{sp}(\rho^n)_{kj}^{-1}\ , \qquad (76)$$
$$\tilde{\omega}_{f0}^2=\beta^2(\gamma^2/\chi)q_iq_j[\tau_{ij}^{sp}+\tau_{ik}^s(\rho^n)_{kl}^{-1}\rho_{lj}^s]\ ,$$
$$\tilde{\omega}_{f1}^2=\beta^2(\gamma^2/\chi)q_iq_j[\tau_{ij}^s+\tau_{ik}^s(\rho^n)_{kl}^{-1}\rho_{lj}^s]\ .$$

Since $\omega_{20}^2 << \omega_{40}^2$, the shift in the second-sound frequency on going from the A phase to the A_1 phase [from $(\omega_{20}^2+\omega_{f0}^2)^{1/2}$] is much more pronounced than the shift in the fourth-sound frequency, as noted by Liu.[1] However, because it is harder to generate second sound than fourth sound (due to the high attenuation rate associated with second sound[25]), it will be more difficult to study these modes. In the A_1 phase, where $\tilde{\omega}_D^2=0$, $\tilde{\omega}_{f0}^2=\tilde{\omega}_{f1}^2$, and $\omega_{20}^2=\omega_{21}^2$, we have

$$\omega_+^2=\tilde{\omega}_{f0}^2+\omega_{20}^2,\quad \omega_-^2=0\ .$$

The $\omega_-^2=0$ mode corresponds to a "magnetothermomechanical effect," and will be discussed in Sec. V, where we consider diffusive modes, static solutions, and related behavior.

V. DIFFUSIVE MODES, STATIC SOLUTIONS, AND RELATED BEHAVIOR

We now consider the static solutions to the hydrodynamic equations, restricting our considerations to the longitudinal variables. (Hence shearing motion of $\vec{v}^n$, and motion of $\hat{f}$, and $\hat{l}$ will not be discussed.) To appreciate the physical significance of these solutions, it is useful to consider the normal modes as a function of frequency. Although they have not been considered yet in this paper, the diffusive modes, for which $\omega=-iDq^2$, will be shown to very often dominate the low-frequency behavior of the system. To illustrate this point, we will first consider an ordinary liquid in the bulk and in the superleak geometries.

Ordinary liquid. In bulk, the normal modes of an ordinary liquid are ordinary sound and a thermal diffusion mode, with $D=[\kappa/T\rho(\partial s/\partial T)_P]$, where $s\equiv S/\rho$. This latter has a fluid velocity which is prc portional to q^2. At high frequencies, because the thermal diffusion mode decays in space rapidly as one moves away from the source, only the propagating (sound) mode is noticeable far from the source. However, at low frequencies, since the wavelength $\lambda\propto\omega^{-1}$ for a propagating mode, whereas the "wavelength" $\lambda\propto|q|^{-1}\propto\omega^{-1/2}$ for the diffusive mode, the sound wavelength can become much larger than the distance between observer and source, so the pressure appears nearly constant; nevertheless, the thermal diffusion mode can still extend so far that it simulates a linear-temperature profile. Indeed, the steady-state solutions correspond to a uniform pressure and a linear-temperature profile. In other words, far from the source, the high-frequency behavior is dominated by the propagating mode, whereas the low-frequency behavior is dominated by the diffusive mode. This behavior is a characteristic one.

Inside a superleak, where the (normal) fluid velocity is subject to a great deal of viscous drag, one can set up a steady-state linear-pressure profile, accompanied by a small but steady mass flow (i.e., the Joule-Thomson porous plug). This mode corresponds to the dc version of the normal mode for sound propagation along a long narrow tube. In addition to this mode and its dc behavior (which will not be considered when we discuss superfluids), there is also a thermal diffusion mode (as in the bulk) whose dc behavior gives a linear-temperature profile. We now turn to the behavior of an ordinary superfluid, ^{4}He II.

Ordinary superfluid. In bulk, the normal modes of an ordinary superfluid are first sound (primarily a pressure wave) and second sound (primarily a temperature wave). There are no longitudinal diffusive modes. These two normal modes give the steady-state behavior of the system: the pressure is uniform because pressure differences are evened out by dc first sound (which can carry momentum), and temperature differences are evened out by dc second sound (which can carry heat). Note that the decay of temperature, as one moves away from a dc heat source, is exponential, with a characteristic length on the order of a mean free path.[26]

In a superleak, the normal modes are fourth sound (primarily a pressure wave, due to compression and rarefaction of the superfluid) and a thermal diffusion mode. This latter mode occurs at constant μ (so $d\vec{v}^s/dt=0$), whereas the bulk thermal diffusion mode for an ordinary fluid occurs at constant P (so $d\vec{v}^n/dt=0$). As a consequence, the diffusion constant is now $D=[\kappa/T(\partial S/\partial T)_\mu]$. The fourth-sound mode is not excited in the dc limit (unless material is flowing through a superleak which connects two chambers of bulk superfluid, as in the case of the fountain effect).[27] However, the dc thermal diffusion mode can be excited, to obtain a linear-temperature and -pressure profile across the superleak. (This is what occurs in the thermomechanical effect.[27]) We now consider the more complex case of ^{3}He-A.

^{3}He-A. In bulk, the longitudinal normal modes of

^{3}He-A are first sound, second sound, and the longitudinal spin wave (LSW). There are no diffusive modes. Both first and second sound directly give the steady-state response to pressure and temperature disturbances, causing pressure and temperature differences to be evened out (just as for an ordinary superfluid, except that the superfluid density is now a tensor). The static longitudinal magnetic response of the system, however, is complicated by the presence of the nuclear dipole-dipole interaction [ω_f in Eq. (15)], which is nonzero for $(\hat{l}\cdot\hat{d})^2 \neq 1$. This occurs if $\phi_f \neq 0$, and is responsible for the gap ω_A in the longitudinal spin wave, and for the absence of any uniform, steady-state solution with $\vec{v}^{sp} \neq 0$.[28] Indeed the steady-state solution which follows from the LSW dispersion relation, $\omega^2=\omega_A^2+c_{f0}^2 q^2$ (where c_{f0} is the spin-wave velocity for $\omega_A=0$), gives a finite (in fact, imaginary) value for q, rather than giving the value $q=0$ which corresponds to a uniform solution.[29] In other words, the steady-state solutions to the hydrodynamic equations carry no spin currents. In the bulk, $\hat{f}\cdot(\vec{h}-\vec{H})=0$ is the steady-state solution, as a consequence of Eqs. (15) and (27).

In a superleak, the normal modes are fourth sound, the LSW, and a thermal diffusion mode. The LSW is unchanged from its corresponding bulk solution, and hence there is no uniform steady-state solution with $\vec{v}^{sp} \neq 0$. The fourth-sound mode is like that for an ordinary superfluid, except that the superfluid density is a tensor. The thermal diffusion mode is like that for an ordinary superfluid, except that $\kappa \rightarrow \kappa(\hat{q})=\kappa_{ij}\hat{q}_i\hat{q}_i$. The implicactions for steady-state behavior are that $\hat{f}\cdot(\vec{h}-\vec{H})$ is zero throughout the superleak; and that fourth-sound and the thermal diffusion modes manifest themselves as in the case of an ordinary superfluid, respectively, supporting the fountain effect and the thermomechanical effect.

^{3}He-A_2. In bulk, the longitudinal normal modes of ^{3}He-A_2 are first sound, and two propagating modes wherein the second-sound and LSW modes of the A phase are mixed [except at $q=0$, cf. Eq. (76)]. There are no diffusive modes. In the $q\rightarrow 0$ limit both first and second sound directly give the steady-state response to pressure and temperature disturbances just as for ^{3}He-A. Note that, since the spin current has a $\vec{\lambda}^{sp}$ term proportional to $\vec{v}^s$ [cf., Eqs. (23) and (49)], second sound carries a magnetization current proportional to $\vec{v}^s$, in addition to the small magnetization current $m_\alpha\vec{v}^n$ which in principle also occurs for ^{3}He-A. ($\vec{v}^{sp}\rightarrow\vec{0}$ for second sound as $q\rightarrow 0$, so the $\vec{v}^{sp}$ term in $\vec{\lambda}^{sp}$ does not contribute here.) The static longitudinal magnetic response of the system is also very similar to that for ^{3}He-A. Again, $\hat{f}\cdot(\vec{h}-\vec{H})=0$ is the bulk steady-state solution.

In a superleak, the normal modes are: two propagating modes wherein the fourth-sound and LSW modes of the A phase are mixed [except at $q=0$, cf. Eq. (74)]; and a thermal diffusion mode similar to that for $H=0$. The steady-state behavior is very similar to that for ^{3}He-A in a superleak: $\hat{f}\cdot(\vec{h}-\vec{H})$ is zero throughout the superleak, the $q=0$ fourth-sound mode can support the fountain effect, and the thermal diffusion mode can support the thermomechanical effect. We now consider ^{3}He-A_1.

^{3}He-A_1. In bulk, the propagating longitudinal normal modes are first sound, and a propagating mode wherein the second sound and LSW modes of the A phase are mixed.[1] There is also a diffusive mode involving temperature and magnetization. Since $\vec{v}^s$ and $\vec{v}^{sp}$ are no longer independent variables, appearing only in the combination $\vec{v}^s+\vec{v}^{sp}$ ($\vec{v}^{sp}$ may be nonzero, since $\omega_f=0$ in the A phase), there is one less degree of freedom and one less mode than for the A and A_2 phases. (Mode counting involves counting each propagating mode twice, and each diffusive mode once; thus the bulk A and A_2 phases have six modes, whereas the bulk A_1 phase has five modes.) As usual, first sound causes pressure differences to be evened out, and can carry momentum in the dc limit. The propagating mixed mode evens out a linear combination of temperature and magnetization differences, carrying both a heat and a magnetization current. The diffusive mode, which involves a constant value for the effective chemical potential $\mu_{\text{eff}}=\mu+\gamma\beta\hat{f}\cdot(\vec{h}-\vec{H})$, can set up a linear profile of temperature and magnetization ("magnetothermal effect"). (For diffusion, the equation for momentum conservation implies that $\delta P\approx 0$, and thus $\delta\mu_{\text{eff}}=0$ involves no δP term for the diffusive mode.) Of course, as Liu has pointed out,[1] one can set up such a linear profile only for times short compared to T_1. Note that the diffusion constant takes the value

$$D(\hat{q})=\left[\kappa(\hat{q})+\left(\frac{TS}{\gamma\beta\rho}\right)\alpha^{(1)}(\hat{q})\right]\Bigg/\left[T\rho\left(\frac{\partial s}{\partial T}\right)_P\right], \tag{77}$$

where $\kappa(\hat{q})\equiv\kappa_{ij}\hat{q}_i\hat{q}_j$ and $\alpha^{(1)}(q\hat{q})\equiv(\alpha_{ij}^{(1)}+\bar{\alpha}_{ij\alpha\beta}^{(1)}m_\beta f_\alpha)q_iq_j$. In obtaining Eq. (77), small corrections proportional to $\omega_{20}^2/\omega_{f0}^2$, and terms due to the m_α dependence of S and ρ, were neglected.

In a superleak, the normal modes are a propagating mode wherein the fourth-sound and LSW modes of the A phase are mixed (with $\delta S=0$, $\hat{f}\cdot\delta\vec{m}=-\gamma\beta\delta\rho$),[1] and two diffusive modes [with $\delta\mu+(\gamma\beta/\chi)\hat{f}\cdot\delta\vec{m}=0$] wherein pressure, temperature, and magnetization are strongly coupled. (Altogether, the A_1 phase in a superleak has four modes, whereas the A and A_2 phases in a superleak have five modes.) The propagating mode can carry both momentum and magnetization, and is responsible for the "magnetic fountain effect" which can occur in ^{3}He-A_1. The two diffusive modes can be generated in an arbitrary linear combination, and it is this extra degree of freedom which permits

$$0=\delta\mu_{\rm eff}=-(S/\rho)\delta T+(1/\rho)\delta P+\gamma\beta\hat{f}\cdot\delta(\vec{h}-\vec{H}) \tag{78}$$

to be satisfied by various ratios of δT, δP, and $\hat{f}\cdot\delta(\vec{h}-\vec{H})$. This equation implies that "magnetothermomechanical effects" occur in ^{3}He-A_1. One possibility, discussed by Liu,[1] is that one can prepare the superleak with $\delta T=0$, so that for times short compared with T_1, by application of an $\vec{H}$ with a linear profile one can set up a magnetization-induced pressure head: $\delta P=-\gamma\beta\rho\hat{f}\cdot(\vec{h}-\vec{H})$. (This effect suggests a method to experimentally determine the direction of $\hat{f}$.[30])

There are a number of other consequences to this equation, when considered in the context of a superleak which connects two chambers of bulk ^{3}He-A_1. When one of the chambers is heated, superfluid will flow in from the other chamber until $\mu_{\rm eff}$ equilibrates. If both chambers have vapor in them, the change in $\mu_{\rm eff}$ in the third term in Eq. (78) far outweighs the change in the second term, unless the superleak is to raise the height of a fluid volume in a capillary whose cross sectional area A obeys $A/V<g\chi/\gamma^2\beta^2\rho\approx(4000\ {\rm cm})^{-1}$, where V is the volume of the heated chamber and g is the acceleration of gravity. On the other hand, if both of the chambers are closed and are initially filled, then the increase in the second term of Eq. (78) outweighs the increase in the third term (in the ratio of the square of the fourth-sound velocity to the square of the LSW velocity, which is about 400). Thus a δT generates mostly δP in this case, with some $\hat{f}\cdot\delta\vec{m}$. It is clear from the above that, in any given situation, the nature of the magnetothermomechanical effect will depend upon details of the experimental design.

The diffusion constants can be found by solving the simultaneous equations

$$\left[-i\omega\left(\frac{\partial S}{\partial T}\right)_\mu+\frac{q^2\kappa(\hat{q})}{T}\right]\delta T+\left[i\omega\left(\frac{\partial S}{\partial P}\right)_T\left(\frac{\gamma\beta\rho}{\chi}\right)+\frac{q^2\alpha^{(1)}(\hat{q})}{\chi}\right](\hat{f}\cdot\delta\vec{m})=0\ , \tag{79}$$

$$\left\{-i\omega\left[1+\left(\frac{\gamma^2\beta^2\rho}{\chi}\right)\left(\frac{\partial\rho}{\partial P}\right)_T\right]+\frac{q^2\mu(\hat{q})}{\chi}\right\}(\hat{f}\cdot\delta\vec{m})+\left[-i\omega(-\gamma\beta)\left(\frac{\partial\rho}{\partial T}\right)_\mu+q^2\alpha^{(1)}(\hat{q})\right]\delta T=0\ . \tag{80}$$

[Here, $\mu(\hat{q})=\mu_{\alpha i\beta j}f_\alpha f_\beta\hat{q}_i\hat{q}_j$ is an effective transport coefficient, as are $\kappa(\hat{q})$ and $\alpha^{(1)}(\hat{q})$.] These equations are obtained from the $\dot{m}_\alpha$ and $\dot{S}$ equations under the restriction that $\delta\mu_{\rm eff}=0$, and they employ the $\dot{\rho}$ equation to eliminate the term $\Omega_f(=-\vec{\nabla}\cdot\vec{\lambda}^s$ in the A_1 phase) appearing in the $\dot{m}_\alpha$ equation. In addition, the dependence of S and ρ upon m_α has been neglected. Because $(\gamma^2\beta^2\rho/\chi)(\partial\rho/\partial P)_T<<1$, and because $(\partial S/\partial T)_\mu>>(\gamma^2\beta^2\rho/\chi)(\partial S/\partial P)_T\times(\partial\rho/\partial T)_\mu$, it is a consistent approximation to simplify Eqs. (79) and (80) by neglecting the $\partial S/\partial P$, $\partial\rho/\partial T$, and $\partial\rho/\partial P$ terms, thus obtaining

$$\left[-i\omega\left(\frac{\partial S}{\partial T}\right)+\frac{q^2\kappa(\hat{q})}{T}\right]\delta T+q^2\frac{\alpha^{(1)}(\hat{q})}{\chi}(\hat{f}\cdot\delta\vec{m})=0\ , \tag{81}$$

$$[-i\omega+q^2\mu(\hat{q})/\chi](\hat{f}\cdot\delta\vec{m})+q^2\alpha^{(1)}(\hat{q})\delta T=0\ . \tag{82}$$

The diffusion constants obtained in this way are

$$D_\pm=\tfrac{1}{2}(D_T+D_M)\pm\tfrac{1}{2}[(D_T-D_M)^2+4\bar{D}^2]^{1/2}\ , \tag{83}$$

where $D_T\equiv[\kappa(\hat{q})/T(\partial S/\partial T)_\mu]$, $D_M\equiv[\mu(\hat{q})/\chi]$, and $\bar{D}^2\equiv[\alpha^{(1)}(\hat{q})]^2/\chi(\partial S/\partial T)_\mu$. Note that $D_\pm>0$ for stability, so the condition $\bar{D}^2<D_TD_M$ must hold. These modes involve $(\delta P,\delta T,\hat{f}\cdot\delta\vec{m})$, where δP is obtained from the condition $\delta\mu_{\rm eff}=0$.

VI. SUMMARY AND DISCUSSION

We have derived the nonlinear equations of hydrodynamics for ^{3}He in the A_2 phase. Our results for the propagating normal modes are in agreement with the results of hydrodynamic theories developed for the A phase[9,10] and for the A_1 phase,[1] and with the results of a time-dependent GL theory (with no normal-fluid velocity) for the A_2 phase.[3] By studying the diffusive modes, it was possible to shed light upon the bulk "magnetothermal" and superleak "magnetothermomechanical" effects for the A_1 phase.

One aspect of the problem, which we find quite interesting, is due to the unusual dispersion relation exhibited by some of the spin-wave modes. In the A phase it is well known that for the LSW, $\omega^2(\vec{q})=\omega_A^2+c_{f0}^2(\hat{q})q^2$. For $\omega>\omega_A$, this is a propagating mode; for $\omega<\omega_A$, the mode does not propagate, but rather decays exponentially in space. To date, there have been no direct measurements on this mode, so that, although ω_A is well studied, c_{f0} is not. One reason direct measurements of this mode may be difficult is that the group velocity is low, because it possesses a great deal of dispersion, with $|\vec{v}_g(\vec{q})|=c_{f0}^2(\hat{q})q/\omega(\vec{q})$. A time-of-flight measurement will certainly involve a complicated waveform, whose interpretation may be nontrivial. An alternative approach would be to use a phase-sensitive detection scheme in a continuous-wave (CW) experiment, with the frequency $\omega > \omega_A$ chosen to give a convenient value of q. However, perhaps an easier approach would be to perform a CW experiment for $\omega < \omega_A$, and measure the decay in amplitude (e.g., of an NMR signal) as a function of position to obtain q, and thus deduce $c_{f0}^2 = (\omega_A^2 - \omega^2)/|\vec{q}|^2$.

In the A_2 phase, it should also be possible to apply this technique. Here the results would be more complicated. The larger the q value to be studied, the greater the coupling of the pure LSW mode to the pure second-sound mode [cf., Eq. (76)]. As a consequence, an NMR coil operated below the A_2 gap frequency will generate two mixed modes, only one of which is propagating. Their relative phases and amplitudes will be such that, at the coil, $(\Delta T)_{\rm LSW} + (\Delta T)_{\rm 2nd} = 0$ [where $(\Delta T)_{\rm LSW}$ is the temperature change at the coil due to generation of the primarily LSW mode; and similarly for $(\Delta T)_{\rm 2nd}$]. However, only $(\Delta T)_{\rm 2nd}$ will propagate away, so $(\Delta T)_{\rm LSW}$ will remain at the coil. As a consequence there will be a localized temperature change, and a propagating temperature change. A similar argument can be made for a heater, which will generate $(\hat{f}\cdot\delta\vec{m})_{\rm LSW} + (\hat{f}\cdot\delta\vec{m})_{\rm 2nd} = 0$. Thus, near a heater in the bulk A_2 phase, there will be a localized magnetization change (due to the primarily LSW mode); in addition, there will be a propagating magnetization change (due to the primarily second-sound mode). Similarly, in a superleak, an NMR coil can generate a $(\Delta\mu)_{\rm LSW}$ which is localized and a $(\Delta\mu)_{\rm 4th} = -(\Delta\mu)_{\rm LSW}$ which propagates; and a heater or transducer can generate an $(\hat{f}\cdot\delta\vec{m})_{\rm LSW}$ which is localized and an $(\hat{f}\cdot\delta\vec{m})_{\rm 4th} = -(\hat{f}\cdot\delta\vec{m})_{\rm LSW}$ which propagates. We repeat that to have significant mode coupling one must have reasonably large values of q.

In the A_1 phase, the LSW gap disappears, and there is only a propagating mixed mode, both in bulk and in superleak. Thus an NMR coil or a heater can generate only one mode.

It would be useful to summarize the new aspects of the present work, over and above the details implicit in the transverse and longitudinal mode dispersion relations, and the discussion in Sec. V on the diffusive modes of the A_1 phase.

Besides the longitudinal mode crossing which is unique to the A_2 phase, there are two other significant effects on the normal modes in finite fields. First, for $\vec{H} = 0$ (A phase), spatially nonuniform rotations of the spin part of the order parameter about $\hat{e}$ and $\hat{f}$ are energetically equivalent, and have a restoring torque which leads to degenerate spin waves, whereas a similar rotation about $\hat{d}$ has no restoring torque and leads to a diffusive behavior for the magnetization change $\delta\vec{m}$ along $\hat{d}$. However, for H small but finite (A_2 phase), a restoring torque exists, and a spin wave develops with velocity proportional to H. This mode cannot be obtained by an extrapolation of the $H = 0$ hydrodynamics, and hence Eq. (19) of Ref. 11 is not valid. Also, for $H \geq H_{A_2}$ (A_1 phase), spatially nonuniform rotations of the order parameter about $\hat{d}$ and $\hat{e}$ are equivalent, and have a restoring torque which leads to degenerate spin waves, whereas a similar rotation about $\hat{f}$ (which in the A_1 phase is equivalent to a rotation about $\hat{l}$), when added to an equal and opposite rotation about $\hat{l}$, has no restoring torque[1] and leads to a diffusive behavior for $\delta\vec{m}$ along $\hat{f}$. However, for $H < H_{A_2}$ (A_2 phase), a restoring torque develops due to the dipolar interaction, and a finite-frequency mode occurs. This mode is basically the same (both for bulk and superleak geometries) as the longitudinal NMR mode discussed in Refs. 6 and 33. In addition, for the A_1 phase, when rotations about $\hat{f}$ and $\hat{l}$ are in the same sense, there is a restoring torque leading to the mixed spin wave and second-sound (or fourth-sound) mode discussed by Liu.[1] This mode is only slightly affected as one moves into the A_2 phase, until mode-crossing effects become significant.

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APPENDIX A

The general Ginzburg-Landau free-energy density is given by

$$f = K_1|\partial_i A_{\mu i}|^2 + K_2(\partial_i A_{\mu j}^*)(\partial_i A_{\mu j}) + K_3(\partial_i A_{\mu j}^*)(\partial_j A_{\mu i}) \ , \tag{A1}$$

$$A_{\mu i} = \Delta(\hat{m} + i\hat{n})_i(a\hat{d} + ib\hat{e})_\mu \ . \tag{A2}$$

The K's are the same as those of Fetter,[31] with $A_{\mu i}$ normalized such that $|A_{\mu i}|^2 = 2\Delta^2$. With $p = 2ab$, $(1-p^2)^{1/2} = a^2 - b^2$, and $K_{13} = K_1 + K_3$, $K_{123} = K_2 + K_{13}$, $K_{1223} = 2K_2 + K_{13}$, we have [with $\vec{v}^s$ and $\vec{v}^{sp}$ defined in Eq. (6)]

$$\begin{aligned}(f/\Delta^2) = {} & K_{1223}\beta^{-2}(\vec{v}^s)^2 - K_{13}\beta^{-2}(\hat{l}\cdot\vec{v}^s)^2 + 2K_1\beta^{-1}\vec{v}^s\cdot\vec{\nabla}\times\hat{l} - 2K_{13}\beta^{-1}(\vec{v}^s\cdot\hat{l})(\hat{l}\cdot\vec{\nabla}\times\hat{l}) \\ & + K_2(\vec{\nabla}\cdot\hat{l})^2 + K_{123}[\hat{l}\times(\vec{\nabla}\times\hat{l})]^2 + K_2(\hat{l}\cdot\vec{\nabla}\times\hat{l})^2 + K_2\partial_i(l_j\partial_j l_i - l_i\partial_j l_j) + K_{1223}\beta^{-2}(\vec{v}^{sp})^2 - K_{13}\beta^{-2}(\hat{l}\cdot\vec{v}^{sp})^2 \\ & + 2p\beta^{-1}\vec{v}^{sp}\cdot[K_{1223}\beta^{-1}\vec{v}^s - K_{13}\beta^{-1}\hat{l}(\hat{l}\cdot\vec{v}^s) + K_1(\vec{\nabla}\times\hat{l}) - K_{13}\hat{l}(\hat{l}\cdot\vec{\nabla}\times\hat{l})] + \tfrac{1}{2}K_{1223}(\partial_i f)^2 - \tfrac{1}{2}K_{13}[(\hat{l}\cdot\vec{\nabla})f]^2 \\ & + \tfrac{1}{2}K_{1223}(1-p^2)^{1/2}[(\hat{d}\cdot\partial_i\hat{f})^2 - (\hat{e}\cdot\partial_i\hat{f})^2] - \tfrac{1}{2}K_{13}(1-p^2)^{1/2}\{[\hat{d}\cdot(\hat{l}\cdot\vec{\nabla})f]^2 - [\hat{e}\cdot(\hat{l}\cdot\vec{\nabla})\hat{f}]^2\} \\ & - (K_1 - K_3)p\beta^{-1}\hat{l}\cdot\vec{\nabla}\times\vec{v}^{sp} \ . \end{aligned} \tag{A3}$$

For the A phase ($p=0$), this reduces to Eqs. (5)–(7) of Hu *et al.*[32] Note that for the A_1 phase ($p=1$), only the form $\vec{v}^s+\vec{v}^{sp}$ appears, *except* for $\vec{\nabla}\times\vec{v}^s$ and $\vec{\nabla}\times\vec{v}^{sp}$ terms, which are textural in nature. Also, note that Eq. (A3) can be made to depend only on K_2 and K_{13} if surface terms are neglected, as required by Eq. (A1). In the weak-coupling limit, where $K_1=K_2=K_3=K$,

$$\vec{\lambda}^s=\frac{\partial f}{\partial \vec{v}^s}=K\Delta^2[8\beta^{-2}\vec{v}^s-4\beta^{-2}\hat{l}(\hat{l}\cdot\vec{v}^s)+2\beta^{-1}(\vec{\nabla}\times\hat{l}) -4\beta^{-1}\hat{l}(\hat{l}\cdot\vec{\nabla}\times\hat{l})+8p\beta^{-2}\vec{v}^{sp} -4p\beta^{-2}\hat{l}(\hat{l}\cdot\vec{v}^{sp})]\ , \tag{A4}$$

giving

$$\rho^s_{ij}=8K\Delta^2\beta^{-2}(\delta_{ij}-\tfrac{1}{2}l_il_j)\ ,$$
$$\rho^{sp}_{ij}=p\rho^s_{ij}\ , \tag{A5}$$
$$C_{ij}=2K\Delta^2(\delta_{ij}-2l_il_j)\ ;$$

and

$$\vec{\lambda}^{sp}=\frac{\partial f}{\partial\vec{v}^{sp}} =K\Delta^2[8\beta^{-2}\vec{v}^{sp}-4\beta^{-2}\hat{l}(\hat{l}\cdot\vec{v}^{sp}) +8p\beta^{-2}\vec{v}^s-4p\beta^{-2}\hat{l}(\hat{l}\cdot\vec{v}^s) +2p\beta^{-1}(\vec{\nabla}\times\hat{l})-4p\beta^{-1}\hat{l}(\hat{l}\cdot\vec{\nabla}\times\hat{l})]\ , \tag{A6}$$

giving

$$\tau^s_{ij}=\rho^{sp}_{ij}=p\rho^2_{ij}\ ,$$
$$\tau^{sp}_{ij}=\rho^s_{ij}\ , \tag{A7}$$
$$C^{sp}_{ij}=pC_{ij}\ ;$$

and

$$\beta\pi_{\alpha i}=\frac{\partial f}{\partial(\partial_i f_\alpha)}=K\Delta^2(4\partial_i\hat{f}_\alpha-2l_i(\hat{l}\cdot\vec{\nabla})\hat{f}_\alpha+4(1-p^2)^{1/2}[\hat{d}_\alpha(\hat{d}\cdot\partial_i\hat{f})-\hat{e}_\alpha(\hat{e}\cdot\partial_i\hat{f})] -2(1-p^2)^{1/2}\{\hat{d}_\alpha\hat{l}_i[\hat{d}\cdot(\hat{l}\cdot\vec{\nabla})\hat{f}]-\hat{e}_\alpha\hat{l}_i[\hat{e}\cdot(\hat{l}\cdot\vec{\nabla})\hat{f}]\})\ , \tag{A8}$$

giving, on comparison with (62) and (A5),

$$M^d_{ij}=\tfrac{1}{2}\beta^2\rho^s_{ij}[1+(1-p^2)^{1/2}]\ ,$$
$$M^e_{ij}=\tfrac{1}{2}\beta^2\rho^s_{ij}[1-(1-p^2)^{1/2}]\ . \tag{A9}$$

APPENDIX B: EXPERIMENTAL DESIGN CONSIDERATIONS

In this Appendix we (1) consider the geometry of an actual experimental cell that might be used to detect the longitudinal-mode crossing discussed in Sec. IV B; (2) translate from the theoretically convenient parameter p of Sec. IV B to the experimentally convenient parameter t' employed by Osheroff and Anderson[33]; and (3) indicate how one may efficiently obtain the design parameters needed for the cell to operate at convenient values of field, frequency, etc.

(1) An actual experimental cell might have the following design: two closely spaced parallel plates with $\vec{H}$ perpendicular to their normal $\hat{N}$, and an NMR coil whose axis is along $\vec{H}$. In this geometry, for a mode with $\vec{q}$ along $\vec{H}$, the coil is an efficient radiator and detector of the mixed fourth-sound and LSW modes. Furthermore, $\hat{l}$ is pinned along $\hat{N}$ by the boundary conditions and by its dipolar interaction with $\hat{d}$, which is perpendicular to $\vec{H}$ by the anisotropic susceptibility of the A_2 phase. Thus we have $\vec{q}\perp\hat{l}$, as considered in the estimates made later in this Appendix.

Note that it is preferable to perform resonance experiments, which measure a frequency (and, therefore, a phase velocity), rather than time-of-flight measurements, which measure a group velocity, $\vec{v}_g=\partial\omega/\partial\vec{q}$, and must be compared to a more complex theoretical form [i.e., Eq. (75) must be differentiated].

(2) In their paper on transverse and longitudinal NMR in the A_1 and A_2 phases, Osheroff and Anderson employ the reduced temperature[33]

$$t'\equiv(T_{c2}-T)/(T_{c1}-T_{c2})\ , \tag{B1}$$

where T_{c1} and T_{c2} are the field-dependent temperatures at which the A_1 and A_2 phases develop ($T_{c1}>T_{c2}$). For the Ginzburg-Landau regime, in which we are interested, Takagi has shown that[6]

$$T_{c1}/T_c=1+\eta h\ ,$$
$$T_{c2}/T_c=1-\eta h(1-\delta)/(1+\delta)\ , \tag{B2}$$

where $h=(\gamma\hbar/2k_BT_c)H$. Comparison of (B1), (B2), and (66) yields

$$t'=\tfrac{1}{2}(1-\delta)(p^{-1}-1)\approx 0.375(p^{-1}-1)\ , \tag{B3}$$

where we have employed the value $\delta\approx0.25$ appropriate to the melting pressure P_m.[23]

(3) We now consider questions pertaining to experimental design. This requires a knowledge of the

condition for mode crossing to occur. It is given by

$$\omega_{40}^2=\omega_A^2(1-p^2)^{1/2}+\omega_{f0}^2 \;,$$

$$q^2=\omega_A^2(1-p)^{1/2}/(c_{40}^2-c_{f0}^2) \;.$$

Since $c_{40}^2 \approx 400c_{f0}^2$ at P_m (where $c_{40}^2=\omega_{40}^2/q^2$, etc.), we may employ

$$q \approx (\omega_A/c_{40})(1-p^2)^{1/4} \;. \tag{B4}$$

Now, using the empirical result $(\omega_A/2\pi)=235 \times 10^3(1-T/T_c)^{1/2}$ sec^{-1} found by Webb *et al.*,[34] at $P=33$ bar, and scaling to $P_m=34.36$ bar using (8.23) of Ref. 23, we find the relation

$$\omega_A \approx 2\pi(240\times 10^3)(1-T/T_c)^{1/2}\ \text{sec}^{-1} \;. \tag{B5}$$

Next, with $c_4 \approx c_1(\rho_s/\rho)^{1/2}$, where $\rho_s=\rho_{sij}q_iq_j$ is evaluated for $\vec{q}\perp\hat{l}$, use of (65) gives the result that

$$c_4=c_1[\tfrac{12}{5}(1+F_1/3)^{-1}(\Delta C/1.42C_n)(1-T/T_c)]^{1/2} \;.$$

Using values for c_1, F_1, and $\Delta C/1.42C_n$ from Ref. 23, we find that

$$c_4 \approx 2.92\times 10^4(1-T/T_c)^{1/2}\ \text{cm/sec} \;, \tag{B6}$$

so that (B4) becomes

$$q \approx 51.6(1-p^2)^{1/4}\ \text{cm}^{-1} \;. \tag{B4$'$}$$

Thus we get, for $p=0$, $q_{\max} \approx 51.6$ cm^{-1}, corresponding to

$$\lambda_{\min}=2\pi/q_{\max} \approx 0.122\ \text{cm} \;. \tag{B7}$$

In other words, any cell designed to observe such crossing must be able to conveniently accommodate such a wavelength.

One approach to cell design is to choose a convenient wavelength λ, which then determines p, from (B4$'$):

$$p=\left[1-\left(\frac{\lambda_{\min}}{\lambda}\right)^4\right]^{1/2} \approx \left[1-\left(\frac{0.122}{\lambda}\right)^4\right]^{1/2} \;. \tag{B8}$$

For example, $\lambda=0.15$ cm gives $p=0.750$. From (B3) we then find that $t'=0.125$. Now let us choose a value for the longitudinal resonance frequency without mode-crossing effects. If we take $\omega_A(1-p^2)^{1/4}=2\pi\times 10^4$ sec^{-1}, then (B5) enables us to determine that $(1-T/T_c)=2.62\times 10^{-3}$. From (66) evaluated at P_m we then find that

$$H=pH_{A_2}(T)\approx p(1-T/T_c)(1.11\times 10^6\ \text{G}) \;, \tag{B9}$$

which then gives $H=2.16\times 10^3$ G in the present case.

The fractional shifts in the mode frequencies at crossing are given, near P_m, by [cf. Eq. (75)]

$$\pm(p/2)(\omega_{f0}/\omega_{40})^{1/2} \approx \pm(p/2)(0.23) \;. \tag{B10}$$

In the present example, we would thus expect to see modes at frequencies of $10^4(1\pm 0.10)$ Hz.

It should be noted that (B8) does not really give much flexibility in one's choice of values for λ. For example, $\lambda=0.20$ cm gives $p=0.928$, so $t'=0.029$, a rather small value; and $\lambda=0.25$ cm, gives $p=0.971$, so $t'=0.011$, an even smaller value. If t' gets too small, temperature stability becomes a concern.

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