Dissipation and coherent effects in narrow superconducting channels

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We apply the time-dependent Ginzburg-Landau (TDGL) equations to study small ac currents of frequency ω in superconducting channels narrow on the scale of London penetration depth. We show that TDGL equations have t-dependent and spatially uniform solutions that describe the order parameter with an oscillating part of the double frequency coexisting with an ac electric field. We evaluate the Ohmic losses (related neither to the flux flow nor to the phase slips) and show that the resistivity reduction on cooling through the critical temperature T_c should behave as $(T_c - T)^2 / \omega^2$. If the channel is cut out of an anisotropic material in a direction other than the principal axes, the transverse phase difference and the Josephson voltage between the channel sides are generated.

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I. INTRODUCTION

It is common knowledge that superconductors dissipate in the presence of the flux flow or, for large driving current densities, due to phase slips. It is also known that even a small ac current in zero applied field causes dissipation when none of the above sources of dissipation are present. For example, the resistive transition to the superconducting state recorded with small ac currents always has a finite width which for small enough currents and in zero field cannot always be relegated to the flux flow, sample inhomogeneities, or thermal fluctuations. A qualitative explanation of this dissipation employs the two-fluid model with the normal and superfluid densities n_n and n_s, constant in space and time.[1]

Ohmic losses in superconductors are absent for small dc currents. As was originally argued by Landau for superfluids, the flow of quasiparticles is stopped by the lattice (phonons) or by impurities and does not contribute to the current, whereas the creation of new excitations is prohibited by the gap in the quasiparticle spectrum. The situation is different for ac currents. During the ac period $2\pi/\omega$, the normal part of the Fermi liquid does not stop completely and, therefore, causes Ohmic losses. When ω significantly exceeds the phase relaxation rate τ_J^{-1}, but still is small relative to the normal carriers relaxation rate τ_n^{-1}, the Ohmic losses should approach their normal limit $\sim J^2/\sigma$.

These results were obtained within microscopic theory, see, e.g., Ref. 2. In this article we show that for low frequencies the time-dependent Ginzburg-Landau (TDGL) equation offers a general and simple method to approach the dissipation problem near the transition point without specific assumptions on the dissipation mechanism. We show that if superconducting wires (channels) are thin compared to the London depth λ and the ac current can be taken as uniform, the order parameter acquires a part oscillating in time with the frequency 2ω, where ω is the current frequency. The order parameter modulus stays constant in space since no vortices or phase slips are assumed to exist in zero applied

field and for sufficiently small currents. One can say that there is a periodic exchange between the superfluid condensate and the normal excitations, accompanied by an ac electric field E. In general, the phase shift between the field and the current depends on relative values of $\omega\tau_\Delta$ and $\omega\tau_J$ with τ_Δ and τ_J being the relaxation times for the order parameter and for the phase (i.e., for the current). As a consequence, the dissipation depends on these parameters too.

In anisotropic superconducting channels, the ac currents flowing in any but the principal crystal directions cause the electric field to have a component perpendicular to the current, i.e., across the channel. This is due to the anisotropy of the superconductor in use and due to anisotropy of the normal conductivity. We show that for $\omega\tau_J \ll 1$ the transverse field is caused by the transverse phase difference inherent to anisotropic superconductors. This offers a relatively simple probe of existence of this phase difference which has been recently predicted.[3]

II. ISOTROPIC CASE

To set notations, we start with the first GL equation

$$-\xi^2 \mathbf{\Pi}^2 \psi = \psi(1 - |\psi|^2/\psi_0^2). \tag{1}$$

Here, ξ is the coherence length and $\mathbf{\Pi} = \nabla + 2\pi i \mathbf{A}/\phi_0$ with $\mathbf{A}$ and ϕ_0 being the vector potential and the flux quantum. For the order parameter written as $\psi = f e^{i\chi}$, we have $\mathbf{\Pi}\psi = e^{i\chi}(\nabla f + i\mathbf{P}f)$, where $\mathbf{P}$ is proportional to the gauge-invariant vector potential

$$\mathbf{Q} = \phi_0 \nabla \chi/2\pi + \mathbf{A} = \phi_0 \mathbf{P}/2\pi. \tag{2}$$

Equation (1) contains $e^{i\chi}$ on both sides. After cancelling this factor and separating real and imaginary parts, one obtains for the real part

$$-\xi^2(\nabla^2 f - fP^2) = f(1 - f^2/f_0^2). \tag{3}$$

The imaginary part coincides with $\text{div}\mathbf{j} = \text{div}f^2\mathbf{P} = 0$.

The gauge-invariant form of TDGL involves the scalar potential φ:[4–6]

$$\tau_\Delta\left(\frac{\partial}{\partial t} - i\frac{2\pi c}{\phi_0}\varphi\right)\psi = \psi\left(1 - \frac{f^2}{f_0^2}\right) + \xi^2\mathbf{\Pi}^2\psi, \qquad (4)$$

where τ_Δ is the order parameter relaxation time. Separating real and imaginary parts we have

$$\tau_\Delta\frac{\partial f}{\partial t} = f\left(1 - \frac{f^2}{f_0^2}\right) + \xi^2(\nabla^2 f - fP^2), \qquad (5)$$

$$-\tau_\Delta cf\Phi = \nabla f \cdot \mathbf{Q} + \nabla \cdot (f\mathbf{Q}), \qquad (6)$$

where $\Phi = \varphi - (\phi_0/2\pi c)\partial_t\chi$.

III. SPATIALLY UNIFORM SOLUTIONS OF TDGL

We are interested in coordinate independent solutions $f(t)$ and $Q_i(t)$. The system (5) and (6) then takes the form

$$\frac{\tau_\Delta}{2}\frac{\partial u}{\partial t} = u(1 - u) - \xi^2 P^2 u, \quad u = \frac{f^2}{f_0^2}, \qquad (7)$$

$$\mathbf{\Phi} = 0. \qquad (8)$$

This should be complemented with equations for the current. A uniform current $\mathbf{J}$ consists, in general, of normal and superconducting parts

$$J = \sigma E - \frac{2e^2}{Mc}f^2 Q, \qquad (9)$$

where E is the electric field directed along the channel and σ is the conductivity for the quasiparticles flow. We aim to describe the system response to ac currents; then σ is, in general, ω dependent. If, however, the frequencies are bound by inequality $\omega\tau_n \ll 1$ with τ_n being the scattering time for the normal excitations, one can consider σ as a real ω-independent quantity.

The electric field is expressed in terms of gauge invariant potentials

$$\mathbf{E} = -\nabla\mathbf{\Phi} - \frac{\partial\mathbf{Q}}{c\partial t} = -\frac{\partial\mathbf{Q}}{c\partial t} \qquad (10)$$

so that the total current is

$$J = -\frac{1}{c}\left(\sigma\frac{\partial Q}{\partial t} + \frac{2e^2}{M}f^2 Q\right). \qquad (11)$$

At a given current, Eqs. (7) and (11) form a complete system for two functions $u(t)$ and $Q(t)$. It is convenient to introduce dimensionless vector potential

$$q = Q\frac{2\pi\xi}{\phi_0} \qquad (12)$$

and to measure the current density in units of the depairing value[7] $j = J/J_{GL}$ with

$$J_{GL} = \frac{e^2\phi_0 f_0^2}{\pi\xi cM} = \frac{c\phi_0}{4\pi^2\xi\lambda^2}, \quad \lambda^2 = \frac{Mc^2}{4\pi e^2 f_0^2}. \qquad (13)$$

Also, we use the so-called "current relaxation time"

$$\tau_J = \frac{M\sigma}{2e^2 f_0^2} \sim \frac{n_n}{n_s}\tau_n. \qquad (14)$$

In our view, a better term for this quantity is the "phase relaxation time" which we use in what follows, but we keep the standard notation τ_J. When $T \to T_c$, $\tau_J \propto 1/(T_c - T)$ and so does τ_Δ.[5] Then, the system of equations to solve takes the form

$$\tau_\Delta\dot{u}/2 = u - u^2 - q^2 u, \qquad (15)$$

$$-\tau_J\dot{q} - uq = j, \qquad (16)$$

where overdots stand for d/dt.

We are interested in calculating the system response to ac currents $J = J_0\cos\omega t$ with amplitudes $J_0 \ll J_{GL}$, i.e., for $j \ll 1$. In this situation the order parameter u is close to unity and $q \ll 1$. The system to solve can be simplified ($u \approx 1 - v, v \ll 1$):

$$\tau_\Delta\dot{v}/2 + v = q^2, \qquad (17)$$

$$\tau_J\dot{q} + q = -j_0\cos\omega t. \qquad (18)$$

The second equation here is *linear*, moreover, it is decoupled from the equation for v and is easily solved. The solution consists of a transient part depending on initial conditions (the general solution of the homogeneous equation) and the long time asymptotics of our interest (the particular solution of the full equation). The latter can be readily found by looking for q of the form $A\sin\omega t + B\cos\omega t$:

$$q(t \to \infty) = -\frac{j_0(\omega\tau_J\sin\omega t + \cos\omega t)}{1 + \omega^2\tau_J^2}. \qquad (19)$$

This can also be written in a more familiar form

$$q_\infty = -\frac{j_0}{\sqrt{1 + \omega^2\tau_J^2}}\sin(\omega t + \alpha), \quad \tan\alpha = \frac{1}{\omega\tau_J}. \qquad (20)$$

In the following we are interested only in the stationary long time asymptotics and omit the subscript ∞.

Substituting the obtained q in Eq. (17) we can find the long time asymptotics for v. To this end, we look for $v = v_0 + v_1\cos 2\omega t + v_2\sin 2\omega t$ and obtain

$$v_0 = \frac{j_0^2}{2(1 + \omega^2\tau_J^2)}, \qquad (21)$$

$$v_1 = \frac{j_0^2(1 - \omega^2\tau_J^2 - 2\omega^2\tau_J\tau_\Delta)}{2(1 + \omega^2\tau_J^2)^2(1 + \omega^2\tau_\Delta^2)}, \qquad (22)$$

$$v_2 = \frac{j_0^2\omega[2\tau_J + \tau_\Delta(1 - \omega^2\tau_J^2)]}{2(1 + \omega^2\tau_J^2)^2(1 + \omega^2\tau_\Delta^2)}. \qquad (23)$$

This yields

$$u = 1 - \frac{j_0^2}{2(1 + \omega^2\tau_J^2)} - \frac{j_0^2}{2}\sin(2\omega t + \beta), \qquad (24)$$

$$\tan \beta = \frac{v_1}{v_2}. \tag{25}$$

Thus, in the stationary state reached when $t \gg \max(\tau_\Delta, \tau_J)$, the order parameter has a part oscillating with frequency 2ω near the average value given in the first two terms of Eq. (24). Clearly, the frequency doubling is due to the order parameter independence on the current direction. The zero frequency limit of Eq. (24) coincides with the known GL result for the order parameter suppression by a dc current $u = 1 - j_0^2$.

Oscillations of the order parameter *per se* are difficult to measure. This is not the case for the electric field E and the dissipation density $W = JE$. The field E of Eq. (10) in the stationary long time state is

$$E = -\frac{\phi_0 \dot{q}}{2\pi \xi c} = \frac{\phi_0}{2\pi \xi c} \frac{j_0 \omega}{\sqrt{1 + \omega^2 \tau_J^2}} \cos(\omega t + \alpha). \tag{26}$$

The dissipation averaged over the oscillations period follows:

$$\bar{W} = \frac{\pi J_0^2 \lambda^2 \omega^2 \tau_J}{c^2 (1 + \omega^2 \tau_J^2)}. \tag{27}$$

It is worth noting that for small currents both the electric field and dissipation are not affected by the order parameter relaxation time τ_Δ. For $\omega \tau_J \ll 1$, the field $E \propto \omega$ and $\bar{W} \propto \omega^2$; they are ω independent for $\omega \tau_J \gg 1$.

Since τ_J diverges when $T \to T_c$, we obtain in this limit the dissipation in the normal state $\bar{W} = J_0^2 / 2\sigma$, as expected. Expanding $\bar{W}$ of Eq. (27) in the small parameter $1/\omega^2 \tau_J^2$ and keeping the first correction we obtain

$$\bar{W} \approx \frac{J_0^2}{2\sigma} \left(1 - \frac{4e^4 f_0^4}{M^2 \sigma^2 \omega^2} \right). \tag{28}$$

While looking at the T dependence of the dissipation near T_c, it should be noted that the quasiparticle conductivity $\sigma = n_n e^2 \tau_n / M$ decreases linearly in $(T_c - T)$ due to a decrease of the quasiparticles density n_n. This causes an initial increase of $\bar{W}$, which can be considered as manifestation of well-studied coherence effects in electromagnetic absorption. However, for frequencies of our interest $\omega \tau_n \ll 1$, the "bump" (the maximum) in the dissipation $\Delta \bar{W} \sim (J_0^2 / \sigma) \omega^2 \tau_n^2$ is situated at $T \approx T_c (1 - \omega^2 \tau_n^2)$, i.e., very close to T_c. Out of this narrow temperature domain the dissipation reduction on cooling through T_c should behave as $(T_c - T)^2 / \omega^2$.

IV. ANISOTROPIC CHANNEL

In isotropic superconductors in the presence of persistent currents, the gauge invariant gradient $\nabla \Theta = \nabla \chi + 2\pi A / \phi_0$ is directed along the current. Recently Kogan and Pokrovsky showed that in anisotropic superconductors the transverse phase difference may appear if the driving current does not point in any of the principal crystal directions.[3] In particular, this situation is realized in current carrying channels cut out of anisotropic crystals with a long side in any but a principal
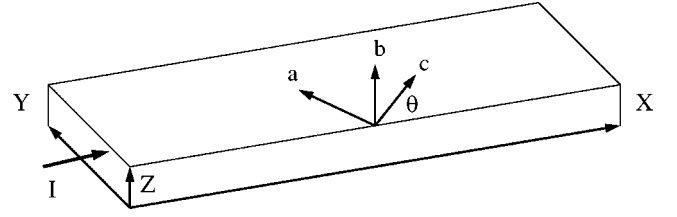

FIG. 1. The superconducting channel with a long dimension along x, the direction of an ac current. The crystal axis b is directed along z. The other crystal axes are in the xy plane with a misalignment angle θ between c and x.

direction and which are narrow on the scale λ. One of the possible ways to observe the transverse phase is to measure the voltage V generated by time-dependent phase difference according to the Josephson formula $V = (\hbar/2e)\partial \Delta \Theta / \partial t$. This can be achieved by driving an ac current through the said channel, a simpler possibility to observe the transverse phase than that suggested in Ref. 3.

In the static case, the supercurrent density is given by $J_i = 2e\hbar M_{ik}^{-1} |\Delta|^2 P_k$, where M_{ij} is the superconducting mass tensor; the summation is implied over repeated indices. It is convenient to introduce the normalized inverse mass tensor $\mu_{ik} = M_{ik}^{-1} M$ with $M = (M_a M_b M_c)^{1/3}$; then the eigenvalues are related by $\mu_a \mu_b \mu_c = 1$. In the uniaxial case, $\mu_a^2 \mu_c = 1$, the inverse masses can be expressed in terms of a single anisotropy parameter $\gamma^2 = \mu_a / \mu_c$: $\mu_a = \gamma^{2/3}$, $\mu_c = \gamma^{-4/3}$.

In the coordinates of Fig. 1, the components μ_{ik} are

$$\mu_{xx} = \gamma^{-4/3} (\gamma^2 \sin^2 \theta + \cos^2 \theta),$$

$$\mu_{yy} = \gamma^{-4/3} (\gamma^2 \cos^2 \theta + \sin^2 \theta), \tag{29}$$

$$\mu_{xy} = \gamma^{-4/3} (1 - \gamma^2) \sin \theta \cos \theta,$$

whereas $\mu_{zz} = \mu_b = \gamma^{2/3}$ and $\mu_{zx} = \mu_{zy} = 0$. To describe t-dependent situations, we again employ the TDGL model, the generalization of which to the anisotropic situation is straightforward: one has to replace the operator $\xi^2 \Pi^2$ in Eq. (1) with $\xi^2 \mu_{ik} \Pi_i \Pi_k$ (see, e.g., Ref. 8). Then, we employ the same procedure as in the isotropic case to make the model dimensionless. The scalar quantities ξ and λ now have meaning of averages $(\xi_a \xi_b \xi_c)^{1/3}$ and $(\lambda_a \lambda_b \lambda_c)^{1/3}$, respectively.

As a result, the system (15) and (16) is replaced with

$$\tau_\Delta \dot{u}/2 = u - u^2 - \mu_{ik} q_i q_k u, \tag{30}$$

$$\tau_J s_{ik} \dot{q}_k + u \mu_{ik} q_k = -j_0 \delta_{ix} \cos \omega t. \tag{31}$$

The tensor $s_{ik} = \sigma_{ik} / \sigma$ is the normalized conductivity with $\sigma = (\sigma_a \sigma_b \sigma_c)^{1/3}$. As with the mass tensor, we can introduce for the uniaxial case the conductivity anisotropy parameter $\gamma_\sigma^2 = \sigma_a / \sigma_c$ so that $s_a = s_b = \gamma_\sigma^{2/3}$ and $s_c = \gamma_\sigma^{-4/3}$. With these definitions, the components of s_{ik} are given by formulas (29) in which γ is replaced with γ_σ.

For small currents, we have for $v = 1 - u$:

$$-\tau_\Delta \dot{v}/2 = v - \mu_{ik} q_i q_k, \tag{32}$$

$$\tau_J s_{ik} \dot{q}_k + \mu_{ik} q_k = -j_0 \delta_{ix} \cos \omega t. \tag{33}$$

As in the isotropic case, the equation for q_i is decoupled from that for v. One can look for a particular solution of Eq. (33) in the form $q_i = A_i \sin \omega t + B_i \cos \omega t$ to obtain a linear system of equations for A_i, B_i.

Perhaps, the easiest is to deal with Eq. (33) in the crystal frame (a, b, c) where all material tensors are diagonal. In this frame, the equation to solve reads

$$\tau_J s_\alpha \dot{q}_\alpha + \mu_\alpha q_\alpha = -j_{0\alpha} \cos \omega t, \quad \alpha = a, c,$$

$$j_{0a} = -j_0 \sin \theta, \quad j_{0c} = j_0 \cos \theta. \tag{34}$$

The long time asymptotics is easily obtained:

$$q_\alpha = -\frac{j_{0\alpha} \sin(\omega t + \beta_\alpha)}{\sqrt{\omega^2 \tau_J^2 s_\alpha^2 + \mu_\alpha^2}}, \quad \tan \beta_\alpha = \frac{\mu_\alpha}{\omega \tau_J s_\alpha}. \tag{35}$$

The electric field components in the channel frame (x, y) and the dissipation read

$$E_x = \frac{\phi_0 j_0 \omega}{2\pi \xi c} \left[\frac{\sin^2 \theta \cos(\omega t + \beta_a)}{\sqrt{\omega^2 \tau_J^2 s_a^2 + \mu_a^2}} + \frac{\cos^2 \theta \cos(\omega t + \beta_c)}{\sqrt{\omega^2 \tau_J^2 s_c^2 + \mu_c^2}} \right],$$

$$E_y = \frac{\phi_0 j_0 \omega \sin 2\theta}{4\pi \xi c} \left[\frac{\cos(\omega t + \beta_c)}{\sqrt{\omega^2 \tau_J^2 s_c^2 + \mu_c^2}} - \frac{\cos(\omega t + \beta_a)}{\sqrt{\omega^2 \tau_J^2 s_a^2 + \mu_a^2}} \right],$$

$$\bar{W} = \frac{\pi J_0^2 \lambda^2 \omega^2 \tau_J}{c^2} \left[\frac{s_a \sin^2 \theta}{\omega^2 \tau_J^2 s_a^2 + \mu_a^2} + \frac{s_c \cos^2 \theta}{\omega^2 \tau_J^2 s_c^2 + \mu_c^2} \right]. \tag{36}$$

Clearly, these expressions have the correct isotropic limit. It is instructive to consider a few limiting situations.

(1) For a dc current ($\omega = 0$), Eq. (32) gives $v = \mu_{ik} q_i q_k$ whereas Eq. (33) yields $q_i = -j_0 \mu_{xi}^{-1}$. We then have

$$u = 1 - j_0^2 \mu_{xx}^{-1}, \tag{37}$$

so that the order parameter suppression by a dc current depends on the current direction. We will not write down a cumbersome expression for u in the general case, but the physics here is the same as for the isotropic case: the order parameter has a small part oscillating with frequency 2ω.

(2) $\omega \tau_J \ll 1$. This situation corresponds to temperatures not too close to the critical temperature because $\tau_J \to \infty$ when $T \to T_c$. We have

$$\bar{W} = \frac{J_0^2}{2\sigma} \omega^2 \tau_J^2 \left(\frac{s_a}{\mu_a^2} \sin^2 \theta + \frac{s_c}{\mu_c^2} \cos^2 \theta \right). \tag{38}$$

In the linear approximation in the small $\omega \tau_J$, the time averaged dissipation is absent; the energy during each period is

pumped from the condensate to the normal excitations and back in equal amounts. The electric fields are

$$E_x = \frac{\phi_0 \omega j_0}{2\pi \xi c} \mu_{xx}^{-1} \sin \omega t, \quad E_y = E_x \frac{\mu_{xy}^{-1}}{\mu_{xx}^{-1}}. \tag{39}$$

The conductivity tensor s_{ik} does not enter these expressions. One may say that these electric fields are due to the t dependence of the phase differences (the Josephson relation mentioned above). In particular, the very fact that the transverse field $E_y \neq 0$ is proof of the existence of the transverse phase. Hence, measuring the transverse and longitudinal voltages on a channel similar to the shown in Fig. 1, one can, in principle, verify Eq. (39) and therefore observe the transverse phase difference.

(3) $\omega \tau_J \gg 1$, the situation taking place in particular when $T \to T_c$. The dissipation of Eq. (36) reduces to a form similar to that of the isotropic case:

$$\bar{W} \approx \frac{J_0^2}{2} \sigma_{xx}^{-1} \left(1 - \eta(\theta) \frac{4e^4 f_0^4}{M^2 \sigma^2 \omega^2} \right) \tag{40}$$

with

$$\eta = (\gamma_\sigma \gamma^2)^{-4/3} \frac{\gamma^4 \sin^2 \theta + \gamma_\sigma^6 \cos^2 \theta}{\sin^2 \theta + \gamma_\sigma^2 \cos^2 \theta}. \tag{41}$$

At T_c, $\bar{W} = J_0^2 \sigma_{xx}^{-1}/2$ is the normal state dissipation. Thus, on cooling through T_c, the resistivity drop should behave as $(T_c - T)^2/\omega^2$ with an angular-dependent coefficient. It should be noted that γ_σ may exceed substantially the superconducting anisotropy γ causing a strong angular dependence of η.

We have also performed the linear stability analysis of our solutions of the TDGL equation to show that the homogeneous solution is stable unless the current reaches the magnitude of the order of J_{GL}. One can argue that vortices might be generated near the boundaries at smaller currents thus destroying the uniform time-dependent states. Without going into a detailed discussion of this restriction, we note that our solutions for small currents are certainly stable.

[1] M. Tinkham, *Introduction to Superconductivity* (McGraw-Hill, New York, 1996), Chap. 2.5.

[2] D. C. Matteis and J. Bardeen, Phys. Rev. **111**, 412 (1958).

[3] V. G. Kogan and V. L. Pokrovsky, Phys. Rev. Lett. **90**, 067004 (2003).

[4] A. Schmid, Phys. Kondens. Mater. **5**, 302 (1966).

[5] N. B. Kopnin, *Theory of Nonequilibrium Superconductivity* (Oxford University Press, New York, 2001).

[6] L. P. Gor'kov and N. B. Kopnin, Usp. Fiz. Nauk **116**, 413 (1975); Sov. Phys. Usp. **18**, 496 (1976).

[7] For convenience, we define the depairing current J_{GL} as having the order parameter suppression by small currents to be $f^2/f_0^2 = 1 - J^2/J_{GL}^2$. This definition differs by a constant factor from a standard definition of J_{GL} as a maximum value for which the GL equations still have stable solutions in the presence of large currents (Ref. 1).

[8] V. G. Kogan, J. R. Clem, J. M. Deang, and M. D. Gunzburger, Phys. Rev. B **65**, 094514 (2002).