Correlations in noisy Landau-Zener transitions

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We analyze the influence of colored classical Gaussian noise on Landau-Zener transitions during a two-level crossing in a time-dependent regular external field. Transition probabilities and coherence factors become random due to the noise. We calculate their two-time correlation functions, which describe the response of this two-level system to a weak external pulse signal. The spectrum and intensity of the magnetic response are derived. Although the noise enters the equation of motion for the Bloch vector in a multiplicative way, nonperturbative analytic results are obtained by a resummation of diagrams in the limit of a short-noise correlation time. Our results also cover regimes where fluctuations are of the same order of magnitude as averages.

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I. INTRODUCTION

The Landau-Zener (LZ) theory[1,2] (see, also Ref. 3) plays an important role in many different physical problems ranging from chemistry, biology, and the theory of collisions to the tunneling of Bose-condensates, dynamics of glasses and spin reversal in magnetic wires and nanomagnets, as well as quantum computing. The LZ theory treats a rather simple and general situation of two-level crossing. Near the point of (avoided) level crossing, adiabaticity is violated and transitions occur. The LZ Hamiltonian can be represented by a 2×2 matrix acting in a two-dimensional space of vectors spanned by the basis vectors $|\uparrow\rangle$ and $|\downarrow\rangle$,

$$H_{\rm LZ} = \begin{pmatrix} E_\uparrow & \Delta \\ \Delta^* & E_\downarrow \end{pmatrix}, \tag{1}$$

where $E_\uparrow = -E_\downarrow = -(\hbar/2)\nu t$ and the transition matrix element Δ does not depend on time. Without loss of generality one can assume Δ to be a positive constant. Landau and Zener have found the evolution matrix

$$U_{\rm LZ} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \tag{2}$$

which connects initial amplitudes in the diabatic basis[4] with their final values. The entries

$$a = e^{-\pi\gamma^2}, \quad b = -\frac{\sqrt{2\pi}}{\gamma\Gamma(i\gamma^2)} e^{-\pi\gamma^2/2 - i\pi/4} \tag{3}$$

depend only on the dimensionless parameter

$$\gamma = \frac{\Delta}{\hbar\sqrt{\nu}}, \tag{4}$$

which we call the LZ parameter.

In many applications the LZ theory must be modified to take into account thermal noise or noise of a different nature. In a pioneering work, Kayanuma[5] has solved such a problem on the basis of three simplifying assumptions: (i) the regular transition matrix element is equal to zero (i.e., $\Delta=0$); the noise is completely responsible for transitions, (ii) the correlation function of noise has a simple exponential form, and (iii) the noise is fast. The first two limitations were lifted in the work by Pokrovsky and Sinitsyn.[6] Their analysis of time scales for different processes distinguishes the correlation time of noise $\tau_{\rm n}$, the accumulation time $\tau_{\rm acc} = (\nu\tau_{\rm n})^{-1}$ during which the noise effectively produces transitions (i.e., for $t = \tau_{\rm acc}$, the bandwidth $\hbar/\tau_{\rm n}$ of the noise coincides with the level spacing $\hbar\nu t$), and the LZ time $\tau_{\rm LZ} = \Delta/(\hbar\nu)$ during which the standard LZ transition (without noise) proceeds. The noise is called fast, if $\tau_{\rm n} \ll \tau_{\rm acc}$ and $\tau_{\rm LZ} \ll \tau_{\rm acc}$. Such a separation of time scales, which is also the basic assumption for the present work, allows for two simplifications. The inequality $\tau_{\rm n} \ll \tau_{\rm acc}$ allows for a nonperturbative resummation of terms to infinite order in the noise amplitude. The inequality $\tau_{\rm LZ} \ll \tau_{\rm acc}$ allows to neglect the regular matrix element Δ on time scales $|t| \gtrsim \tau_{\rm LZ}$ where noise is effective, and to neglect the noise in the interval $|t| \lesssim \tau_{\rm acc}$ where regular LZ transitions occur. Since these intervals overlap, one can solve first the reduced problem with the noise-producing transitions and then match this solution with the LZ solution. In this way the authors of Ref. 6 have found the average value of the density matrix (see also Ref. 7 for a related numerical study).

The two-level problem is as usual equivalent to the problem of spin $\frac{1}{2}$ rotating in the time-dependent field containing regular and stochastic parts. In their second work[8] the same authors represented the density matrix in terms of the Bloch vector $\mathbf{g}$,

$$\hat{\rho}(t) = \tfrac{1}{2}[\hat{1} + \mathbf{g}(t)\cdot\hat{\boldsymbol{\sigma}}], \tag{5}$$

where $\hat{\boldsymbol{\sigma}}$ is the triplet of Pauli matrices for spin $\frac{1}{2}$. It is a well-known fact that the square of the Bloch vector is an integral of motion. In Ref. 8 this fact was used to calculate the square fluctuation

$$\langle[\delta\mathbf{g}(t)]^2\rangle = \mathbf{g}^2 - \langle\mathbf{g}(t)\rangle^2.$$

However, the fluctuations of separate components, $\langle[\delta g_i(t)]^2\rangle$, $(i=x,y,z)$ were not calculated, and neither were two-time correlators $\langle g_i(t)g_j(t')\rangle$ found.

Such fluctuations are observable, for example, in a system of magnetic molecules subject to the same magnetic field (with identical regular and random contributions). The measurement of the z component of the magnetic moment averaged over the system yields $m_z(t)=(\hbar/2)\mathrm{Tr}\,\hat{\sigma}_z\cdot\hat{\rho}(t)=(\hbar/2)g_z(t)$, which depends on the particular noise history of the measurement. The repetition of such a measurement for many independent noise histories then yields an average $\langle m_z(t)\rangle$ with fluctuations $\langle[\delta m_z]^2\rangle=(\hbar^2/4)\langle[\delta g_z(t)]\rangle$. Similarly, the twofold time dependence of the correlators $\langle g_i(t)g_j(t')\rangle$ determines spectral properties of the magnetization. Namely, this situation is realized in magnetic systems subject to an external regular and random magnetic field. In both cases we assume that the characteristic wavelength of the electromagnetic field is much larger than the linear size of the system.

In two-level systems the random field can be realized by an environment, for example, by thermal phonons or by the magnetic field of nuclei. In such a situation different two-level systems feel different random fields. Therefore, in the ensemble consisting of a large number N of the two-level systems the fluctuation will be suppressed proportional to $1/\sqrt{N}$. However, the two-time correlation function describes the linear response of such a system to a small perturbation $\delta\mathbf{h}(t)$ (identical for all members) according to a standard linear-response equation:

$$\delta\langle s_i\rangle = \int_{-\infty}^{t} dt'\langle s_i(t)s_j(t')\rangle\delta h_j(t'). \tag{6}$$

Such a perturbation can be realized as a pulse of electric magnetic or acoustic field. In this work we determine all these correlations under the same assumption of fast noise.

The outline of the paper is as follows. In Sec. II we formulate the model for LZ transitions in the presence of the noise we focus on. In Sec. III we calculate in a general form the time evolution of averages and autocorrelations of the Bloch vector in the absence of the regular transition matrix element Δ. These general findings are evaluated to obtain the averages and fluctuations of transition probabilities first for $\Delta=0$ in Sec. IV, and then also for $\Delta>0$ in Sec. V. Implications for the spectral width and intensity of fluctuations for transitions in a gas of colliding atoms or molecules are discussed in Sec. VI. We close with concluding remarks in Sec. VII. Some calculational details are collected in the Appendix.

II. FUNDAMENTAL

We consider the dynamics of a quantum-spin $\hat{\mathbf{S}}$ in the presence of a time-dependent effective magnetic field $\mathbf{B}(t)$. Absorbing the Landé factor and the Bohr magneton into this field, the Hamiltonian reads

$$\hat{H}(t) = -\mathbf{B}(t)\cdot\hat{\mathbf{S}}. \tag{7}$$

The effective field $\mathbf{B}(t)=\mathbf{b}(t)+\boldsymbol{\eta}(t)$ is composed of a regular part $\mathbf{b}$ and a noisy part $\boldsymbol{\eta}$ with zero average, $\langle\boldsymbol{\eta}(t)\rangle=\mathbf{0}$. In the LZ theory, an avoided level crossing is described by the regular field which may be chosen as

$$\mathbf{b}(t) = b_z(t)\mathbf{e}_z + b_x\mathbf{e}_x, \tag{8}$$

with a z component approximately linear in time,

$$b_z(t) \approx \nu t, \quad \nu \equiv \dot{b}_z(0), \tag{9}$$

and an approximately time-independent perpendicular component $b_x=-(2/\hbar)\Delta$. The noise is considered as Gaussian distributed and uncorrelated in the longitudinal and transverse directions with variances $(i,j=x,y,z)$

$$\langle\eta_i(t)\eta_j(t')\rangle = \delta_{ij}f_i(t-t'). \tag{10}$$

The correlation functions f_i naturally are even functions. A further planar symmetry $f_x=f_y\equiv f_\perp$ is assumed.

Under the action of the Hamiltonian (7), the dynamics of the density matrix is equivalent to the classical equation of motion (Bloch equation)

$$\dot{\mathbf{g}}(t) = -\mathbf{B}(t)\times\mathbf{g}(t). \tag{11}$$

For the analysis of the Bloch equation it is convenient to introduce

$$g_\pm \equiv \frac{1}{\sqrt{2}}(g_x \pm i g_y), \tag{12}$$

as well as $\eta_\pm$, $b_\pm$, and $B_\pm$, by analogous definitions. In these terms the Bloch equation can be rewritten as

$$\dot{g}_\pm = \mp iB_z g_\pm \pm iB_\pm g_z, \tag{13a}$$

$$\dot{g}_z = i[B_- g_+ - B_+ g_-]. \tag{13b}$$

From these equations the longitudinal field component $B_z(t)$ can be eliminated by going over to the interaction representation with respect to the Hamiltonian

$$\hat{H}_0(t) = -B_z(t)\hat{S}_z. \tag{14}$$

Using the time-evolution operator

$$\hat{U}_0(t) = \exp\left\{\frac{1}{i\hbar}\int_{t_0}^{t} dt'\hat{H}_0(t')\right\}, \tag{15}$$

we rewrite the density matrix (5) as

$$\hat{\rho}(t) = \tfrac{1}{2}[\hat{1} + \hat{U}_0(t)\tilde{\mathbf{g}}(t)\cdot\hat{\sigma}\hat{U}_0^\dagger(t)] \tag{16}$$

in terms of the transformed Bloch vector

$$\tilde{g}_{\pm}(t) = g_{\pm}(t)\exp\{\pm i\int_{t_0}^{t} dt' B_z(t')\}. \tag{17}$$

The transformation does not affect the longitudinal component, $\tilde{g}_z(t)=g_z(t)$. Using analogous transformations for $\tilde{\eta}$, $\tilde{\mathbf{b}}$, and $\tilde{\mathbf{B}}$, the Bloch equation simplifies to

$$\dot{\tilde{g}}_{\pm} = \pm i\tilde{B}_{\pm}\tilde{g}_z, \tag{18a}$$

$$\dot{\tilde{g}}_z = i[\tilde{B}_{-}\tilde{g}_{+} - \tilde{B}_{+}\tilde{g}_{-}]. \tag{18b}$$

In the limit of short noise correlation time τ_{n}, the correlations for the transformed noise read

$$\langle\tilde{\eta}_{+}(t)\tilde{\eta}_{+}(t')\rangle = \langle\tilde{\eta}_{-}(t)\tilde{\eta}_{-}(t')\rangle = 0, \tag{19a}$$

$$\langle\tilde{\eta}_{+}(t)\tilde{\eta}_{-}(t')\rangle = \tilde{f}_{\perp}(t,t'), \tag{19b}$$

with the correlator

$$\tilde{f}_{\perp}(t,t') \equiv f_{\perp}(t-t')\exp\left\{i\int_{t'}^{t} dt'' B_z(t'')\right\} \approx f_{\perp}(t-t')e^{i\nu(t^2-t'^2)/2}. \tag{19c}$$

Since $f_{\perp}(t-t')$ decays for $|t-t'| \gg \tau_{\mathrm{n}}$ one may neglect the contribution of $\eta_z(t'')$ to the time integral provided,

$$\tau_n^2\langle\eta_z^2\rangle \ll 1. \tag{20}$$

This condition is assumed in the following. It can be considered as a further limitation for the noise correlation time or as a limitation for the noise amplitude. However, it is not crucial and is introduced only for simplification. If the longitudinal noise is independent of the transverse one, its contribution to the averaged exponent in Eq. (19a)–(19c) can be reduced to the Debye-Waller factor $W(t,t')$ $=\exp\left[-\frac{1}{2}\int_{t'}^{t} dt_1\int_{t'}^{t} dt_2\langle\eta(t_1)\eta(t_2)\rangle\right]$. The theory becomes less transparent and more cumbersome, but in essence remains the same, except for the range of very strong longitudinal noise.

Although the longitudinal noise is not effective in $\tilde{f}_{\perp}$, it cannot be neglected in general. From Eq. (17) one recognizes that η_z leads to a diffusive precession of the transverse components of $\mathbf{g}$. Therefore, the expectation values $\langle g_{\pm}\rangle$ decay exponentially on the dephasing time

$$\tau_{\phi} = \frac{2}{F_{z0}}, \tag{21}$$

with $F_{z0} \equiv \int_{-\infty}^{\infty} dt\langle\eta_z(t)\eta_z(0)\rangle$. For short noise correlations, $\tau_{\phi} \sim 1/[\tau_{\mathrm{n}}\langle\eta_z^2\rangle]$. Thus, longitudinal dephasing can be neglected during the accumulation period $|t| \lesssim \tau_{\mathrm{acc}}=1/\nu\tau_{\mathrm{n}}$ of transverse noise for $\langle\eta_z^2\rangle \ll \nu$. On the other hand, on sufficiently large time scales $|t| \gg \tau_{\phi}$ dephasing always sets in. If the dynamics of the system are followed over long times, the averages of g_x and g_y vanish because of the random precession around the z axis. Nevertheless, in this case the transverse amplitude

$$g_{\perp} \equiv (g_x^2+g_y^2)^{1/2} = (2g_{+}g_{-})^{1/2} = (2\tilde{g}_{+}\tilde{g}_{-})^{1/2}, \tag{22}$$

which is identical to the absolute value of the off-diagonal element of the density matrix, can have a finite expectation value.

III. NOISY DYNAMICS

In this section, we temporarily neglect regular transitions (we set $b_x=0$) and focus on transitions solely due to noise. To solve the equations of motion for the propagation from an initial time t_0 to $t>t_0$, we integrate the equations of motion (18) to

$$\tilde{g}_{\pm}(t) = \tilde{g}_{\pm}(t_0) \pm i\int_{t_0}^{t} dt'\,\tilde{\eta}_{\pm}(t')\tilde{g}_z(t'), \tag{23a}$$

$$\tilde{g}_z(t) = \tilde{g}_z(t_0) - \int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2 w(t_1,t_2)\tilde{g}_z(t_2) - \int_{t_0}^{t} dt'[i\tilde{\eta}_{+}(t')\tilde{g}_{-}(t_0) + \text{c.c.}], \tag{23b}$$

with the vertex pair function

$$w(t_1,t_2) \equiv \tilde{\eta}_{+}(t_1)\tilde{\eta}_{-}(t_2) + \text{c.c.} \tag{24}$$

Wherever "c.c." occurs, it stands for the complex conjugation of the preceding term.

Equation (23b) may be fed back iteratively into itself and into Eq. (23a), to express $\tilde{\mathbf{g}}(t)$ entirely in terms of $\tilde{\mathbf{g}}(t_0)$. Each term arising from this iteration can be visualized by a diagram. Defining the integral series

$$W_{t_0}^{t}(t') \equiv \delta(t_0-t') - \int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2 w(t_1,t_2)\delta(t_2-t') + \int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\int_{t_0}^{t_2} dt_3 \times\int_{t_0}^{t_3} dt_4 w(t_1,t_2)w(t_3,t_4)\delta(t_4-t') - \ldots \tag{25}$$

one obtains

$$\tilde{g}_z(t) = \int_{-\infty}^{\infty} dt' W_{t_0}^{t}(t')\left\{\tilde{g}_z(t_0) - \int_{t_0}^{t'} dt''[i\tilde{\eta}_{+}(t'')\tilde{g}_{-}(t_0) + \text{c.c.}]\right\}, \tag{26a}$$

$$\tilde{g}_{\pm}(t) = \tilde{g}_{\pm}(t_0) \pm i\int_{t_0}^{t} dt'\,\tilde{\eta}_{\pm}(t')\int_{-\infty}^{\infty} dt'' W_{t_0}^{t'}(t'')\left\{\tilde{g}_z(t_0) - \int_{t_0}^{t''} dt'''[i\tilde{\eta}_{+}(t''')\tilde{g}_{-}(t_0) + \text{c.c.}]\right\}. \tag{26b}$$

In the diagrammatic representation, every factor W corresponds to an arbitrary number of vertex pairs w. Correspond-

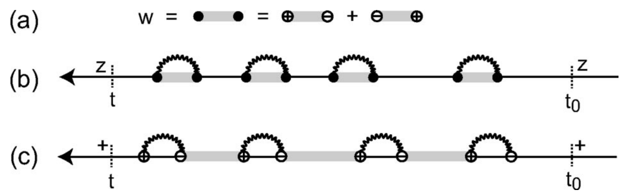

FIG. 1. Diagrammatic representation of w and contributions to $G_{\alpha,\alpha}(t,t_0)$. The time axis is chosen to point to the left-hand side. An encircled + or − represents a noise vertex $\tilde{\eta}_\pm$. (a) A pair of vertices connected by a gray bond represents $-w(t_1,t_2)$. A pair of bold circles contains the sum over both possible polarities. (b) To leading order in small $\tau_{\rm n}$, only diagrams with noncrossing vertex contractions contribute. $G_{z,z}(t,t_0)$ consists only of noise contraction (wiggly lines) within vertex pairs. (c) $G_{+,+}(t,t_0)$ consists of a negative excess vertex as the first vertex (closest to t_0) and a positive excess vertex as the last vertex. Contractions can be performed only from pair to pair.

ing to the definition (24), two "polarities" have to be considered per pair [cf. Fig. 1(a)]. During the time evolution (26) of $\tilde{\mathbf{g}}$, besides such paired vertices, "excess" vertices can appear as first or last vertices during the evolution period.

According to the Wick theorem for Gaussian fields, the noise averaging corresponds to all possible contractions between noise vertices. The correlations (19a)–(19c) imply that contractions can be performed only between vertices with opposite charge (in a pictorial language, we associate a "charge" ±1 with every vertex $\tilde{\eta}_\pm$). Therefore, only neutral diagrams (where the number of vertices $\tilde{\eta}_+$ equals the number of vertices $\tilde{\eta}_-$) do not vanish.

A. Dynamics of averages

Before addressing correlations, we analyze which diagrams contribute to the average $\langle\tilde{\mathbf{g}}(t)\rangle$. Because of the neutrality constraint, this average reduces to

$$\langle\tilde{g}_z(t)\rangle = \int_{-\infty}^{\infty} dt' \langle W_{t_0}^{t}(t')\rangle\langle\tilde{g}_z(t_0)\rangle, \tag{27a}$$

$$\langle\tilde{g}_\pm(t)\rangle = \Bigg\{1 - \int_{t_0}^{t} dt' \int_{-\infty}^{\infty} dt'' \int_{t_0}^{t''} dt''' \times\langle\tilde{\eta}_\pm(t') W_{t_0}^{t'}(t'')\tilde{\eta}_\mp(t''')\rangle\Bigg\}\langle\tilde{g}_\pm(t_0)\rangle. \tag{27b}$$

In factorizing the expectation values we assume that noise before and after t_0 is statistically independent.

To leading order in $\tau_{\rm n}\ll\tau_{\rm acc}$, only contractions between neighboring vertices (in time order) contribute. In $\langle\tilde{g}_z(t)\rangle$, all contractions are performed within the pair functions $w(t,t')$ [cf. Fig. 1(b)], which (for $t-t_0\gg\tau_{\rm n}$) give rise to factors

$$\int_{t_0}^{t} dt' \langle w(t,t')\rangle = F_\perp(\nu t),$$

with the Fourier transform

$$F_\perp(\Omega) \equiv \int_{-\infty}^{\infty} d\tau \cos(\Omega\tau) f_\perp(\tau). \tag{28}$$

Returning to Eq. (27a), we find that the series can be easily summed up to

$$\langle\tilde{g}_z(t)\rangle = G_{z,z}(t,t_0)\langle\tilde{g}_z(t_0)\rangle \tag{29a}$$

with

$$G_{z,z}(t,t_0) \equiv \int_{-\infty}^{\infty} dt' \langle W_{t_0}^{t}(t')\rangle = \exp\left\{-\int_{t_0}^{t} dt' F_\perp(\nu t')\right\}. \tag{29b}$$

This result was obtained in the Ref. 6 via the differential equation

$$\frac{d\langle\tilde{g}_z(t)\rangle}{dt} = -F_\perp(\nu t)\langle\tilde{g}_z(t)\rangle, \tag{30}$$

which immediately follows from the integral formula (29a) and (29b) and confirms the statistical independence of the events before and after some moment of time t on a time scale much larger than $\tau_{\rm n}$.

Likewise, to leading order in $\tau_{\rm n}\ll\tau_{\rm acc}$, $\langle\tilde{g}_+(t)\rangle$ consists only of contractions linking neighboring pairs due to the presence of excess vertices [cf. Fig. 1(c)]. Such a contraction leads to a factor

$$\int_{t_0}^{t} dt' \langle\tilde{\eta}_\pm(t)\tilde{\eta}_\mp(t')\rangle = F_\perp^\pm(\nu t)$$

with

$$F_\perp^\pm(\Omega) \equiv \int_0^{\infty} d\tau\, e^{\pm i\Omega\tau} f_\perp(\tau).$$

Apparently, $F_\perp = F_\perp^+ + F_\perp^-$ and $F_\perp^+ = F_\perp^{-\,*}$ because of the even parity of the noise correlator $f_\perp(t)$. Hence, resummation of diagrams gives

$$\langle\tilde{g}_+(t)\rangle = G_{+,+}(t,t_0)\langle\tilde{g}_+(t_0)\rangle \tag{31}$$

with

$$\begin{aligned} G_{+,+}(t,t_0) &\equiv 1 - \int_{t_0}^{t} dt' \int_{-\infty}^{\infty} dt'' \int_{t_0}^{t''} dt''' \\ &\quad\times\langle\tilde{\eta}_\pm(t') W_{t_0}^{t'}(t'')\tilde{\eta}_\mp(t''')\rangle \\ &= \exp\left\{-\int_{t_0}^{t} dt' F_\perp^\pm(\nu t')\right\}. \end{aligned} \tag{32}$$

In summary, the dynamics of averages can be brought into the compact form ($\alpha=z,\pm$)

$$\langle\tilde{g}_\alpha(t)\rangle = G_{\alpha,\alpha}(t,t_0)\langle\tilde{g}_\alpha(t_0)\rangle, \tag{33a}$$

$$G_{\alpha,\alpha}(t,t_0) \equiv e^{-[\vartheta_\alpha(t)-\vartheta_\alpha(t_0)]} \tag{33b}$$

by means of noise integrals

$$\vartheta_{\pm}(t) \equiv \int_{-\infty}^{t} dt' F_{\perp}^{\pm}(\nu t'), \tag{34a}$$

$$\vartheta_{z}(t) \equiv \int_{-\infty}^{t} dt' F_{\perp}(\nu t'). \tag{34b}$$

Note that the subscripts of ϑ relate to the components of the Bloch vector rather than to the noise components.

B. Dynamics of pair correlations

Using the diagrammatic approach established above, we now proceed to calculate pair correlations of the Bloch vector for $b_x=0$.

1. Reduction to equal-time correlators

We start by realizing that

$$\langle \tilde{g}_{\alpha}(t)\tilde{g}_{\beta}(\overline{t})\rangle = e^{-[\vartheta_{\alpha}(t)-\vartheta_{\alpha}(\overline{t})]}\langle \tilde{g}_{\alpha}(\overline{t})\tilde{g}_{\beta}(\overline{t})\rangle, \tag{35}$$

since noise is uncorrelated before and after $\overline{t}$, assuming (without loss of generality) that $t>\overline{t}$. Thus we need to calculate only equal-time correlators

$$\tilde{C}_{\alpha\beta}(t) \equiv \langle \tilde{g}_{\alpha}(t)\tilde{g}_{\beta}(t)\rangle.$$

The correlations of principal interest are

$$\langle g_z(t)g_z(t)\rangle = \langle \tilde{g}_z(t)\tilde{g}_z(t)\rangle, \tag{36a}$$

$$\langle g_+(t)g_-(t)\rangle = \langle \tilde{g}_+(t)\tilde{g}_-(t)\rangle. \tag{36b}$$

They are invariant under the transformation to the interaction representation.

We describe the evolution of the equal-time correlators

$$\tilde{C}_{\alpha\beta}(t) = \sum_{\gamma\delta} G_{\alpha\beta,\gamma\delta}(t,t_0)\tilde{C}_{\gamma\delta}(t_0)$$

in terms of propagators $G_{\alpha\beta,\gamma\delta}(t,t_0)$. In the diagrammatic representation, these propagators consist only of ladder-like diagrams: The times axes t and $\overline{t}$ represent the legs of the ladder. Noise contractions between the sides are the rungs. A contribution of any diagram with intersecting or overlapping lines is smaller than that of the main sequence by a factor of the order $\tau_{\rm n}/\tau_{\rm acc}$. Since only "neutral" diagrams survive averaging, many propagators vanish. The surviving propagators satisfy "charge conservation"

$$\chi(\alpha) + \chi(\beta) = \chi(\gamma) + \chi(\delta),$$

with the charge function

$$\chi(\alpha) \equiv \begin{cases} 1 & \text{for} \quad \alpha = +, \\ 0 & \text{for} \quad \alpha = z, \\ -1 & \text{for} \quad \alpha = -, \end{cases}$$

which now refers to propagator indices, not to noise indices. At the initial time t_0, the propagator charge is opposite to the charge of the first excess vertex. At the final time, the propagator charge coincides with the charge of the last excess vertex.

Among the nonvanishing propagators, several propagators are mutually dependent. In particular, there is a trivial symmetry $G_{\alpha\beta,\gamma\delta}=G_{\beta\alpha,\delta\gamma}$ corresponding to the exchange of the legs of the ladder. Additional relations follow from the time-reversal invariance or the complex conjugation, such as $G_{z-,z-}=G^{*}_{z+,z+}$.

We therefore find the time evolution of the equal-time correlators captured by the relations

$$\tilde{C}_{zz}(t) = G_{zz,zz}(t,t_0)\tilde{C}_{zz}(t_0) + 2G_{zz,+-}(t,t_0)\tilde{C}_{+-}(t_0), \tag{37a}$$

$$\tilde{C}_{+-}(t) = G_{+-,zz}(t,t_0)\tilde{C}_{zz}(t_0) + [G_{+-,+-}(t,t_0) + G_{+-,-+}(t,t_0)]\tilde{C}_{+-}(t_0), \tag{37b}$$

$$\tilde{C}_{++}(t) = G_{++,++}(t,t_0)\tilde{C}_{++}(t_0), \tag{37c}$$

$$\tilde{C}_{z+}(t) = [G_{z+,z+}(t,t_0) + G_{z+,+z}(t,t_0)]\tilde{C}_{z+}(t_0). \tag{37d}$$

It is important to realize that the dynamics of the "neutral" correlations $\tilde{C}_{z,z}$ and $\tilde{C}_{+,-}$ is decoupled from the "non-neutral" $\tilde{C}_{z,\pm}$ and $\tilde{C}_{\pm,\pm}$. This implies that the latter correlations will not be generated by noise if they are absent initially. However, such correlations can be generated in general by the transverse components of the magnetic field.

For the convenience of the reader we anticipate the results of our subsequent explicit calculation of the following propagators:

$$G_{zz,zz}(t,t_0) = \tfrac{1}{3}(1 + 2e^{-3\theta_z}), \tag{38a}$$

$$G_{+-,zz}(t,t_0) = \tfrac{1}{3}(1 - e^{-3\theta_z}), \tag{38b}$$

$$G_{zz,+-}(t,t_0) = G_{+-,zz}(t,t_0), \tag{38c}$$

$$G_{+-,+-}(t,t_0) = \tfrac{1}{6}(2 + 3e^{-\theta_z} + e^{-3\theta_z}), \tag{38d}$$

$$G_{+-,-+}(t,t_0) = \tfrac{1}{6}(2 - 3e^{-\theta_z} + e^{-3\theta_z}), \tag{38e}$$

introducing the abbreviation

$$\theta_{\alpha} \equiv \vartheta_{\alpha}(t) - \vartheta_{\alpha}(t_0). \tag{39}$$

These explicit calculations are based on integral equations which we now derive and solve for the propagators $G_{zz,zz}$ and $G_{zz,+-}$ as two representative examples. The diagrammatic derivation of the other propagators is deferred to the Appendix. We refrain from presenting the expressions for $G_{++,++}$, $G_{z+,z+}$, and $G_{z+,+z}$ since they are not relevant for our purposes.

2. Propagator $G_{zz,zz}$

We start with the propagator for the longitudinal correlator,

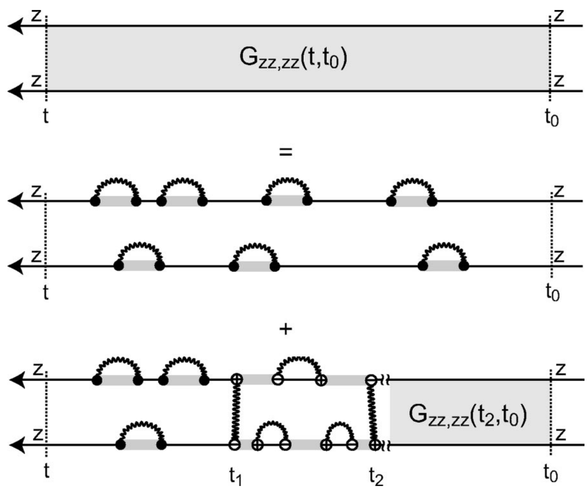

FIG. 2. Diagrams contributing to $G_{zz,zz}(t,t_0)$. First, there is the "ladder" without rungs. The diagrams with rungs can be represented as propagators $G_{z,z}(t,t_1)$ on both legs of the ladder, following the ring contraction between the last rungs as times t_1 and t_2. Prior to t_2 additional leg propagators and rings may occur. In the Bethe-Salpeter equation, these additional contractions sum up to $G_{zz,zz}(t_2,t_0)$.

$$G_{zz,zz}(t,t_0) = \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} d\overline{t'} \langle W_{t_0}^t(t') W_{t_0}^t(\overline{t'})\rangle .$$

This expression follows directly from the contribution to the autocorrelation of Eq. (26a) stemming from the initial value $\tilde{g}_z(t_0)$.

To leading order in $\tau_{\mathrm{n}}/\tau_{\mathrm{acc}}$, the contributing diagrams have the ladder structure mentioned previously (cf. Fig. 2). On leg segments between two neighboring rungs of the ladder, an arbitrary number of noncrossing vertex contractions can be performed. Since there are no final excess vertices close to time t, all contractions between the last rung (at time t_1) and the final time t are intrapair contractions. Between the last rung and the next to last rung, all contractions are interpair contractions, forming a ring structure limited by time t_1 and t_2. Then, the next contractions on the legs prior to t_2 can be intrapair contractions again.

Since all possible diagrams prior to t_2 have the same structure as the diagrams prior to t, the propagator satisfies a Bethe-Salpeter integral equation:

$$\begin{aligned} G_{zz,zz}(t,t_0) = G_{z,z}^2(t,t_0) + 2\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 G_{z,z}^2(t,t_1) \\ \times f_\perp(\nu t_1) G_{+,+}(t_1,t_2) G_{-,-}(t_1,t_2) f_\perp(\nu t_2) \\ \times G_{zz,zz}(t_2,t_0). \end{aligned} \tag{40}$$

The first term represents the diagram without rungs. In the second term, the last pair of rungs at times t_1 and t_2 is separated from possible additional pairs prior to t_2. On the ring between t_1 and t_2, the intrapair noise contractions are "oriented." The explicit factor 2 arises from the two possible orientations.

To simplify the integral equation, it is convenient to substitute the time variables by variables $y \equiv \vartheta_z(t) - \vartheta_z(t_0)$ and $y_i \equiv \vartheta_z(t_i) - \vartheta_z(t_0)$. This leads to

$$\begin{aligned} G_{zz,zz}(y,0) = e^{-2y} + 2\int_0^y dy_1 \int_0^{y_1} dy_2 \\ \times e^{-2(y-y_1)} e^{-(y_1-y_2)} G_{zz,zz}(y_2,0). \end{aligned} \tag{41}$$

This integral equation can now be transformed into a differential equation in two steps:

$$\frac{d}{dy}[e^{2y}G_{zz,zz}(y,0)] = 2\int_0^y dy_2 e^{y+y_2} \times G_{zz,zz}(y_2,0), \tag{42a}$$

$$\frac{d}{dy}\left\{ e^{-y}\frac{d}{dy}[e^{2y}G_{zz,zz}(y,0)]\right\} = 2e^{y}G_{zz,zz}(y,0). \tag{42b}$$

Performing the derivatives, the last equation simplifies to

$$\frac{d^2}{dy^2}G_{zz,zz}(y,0) + 3\frac{d}{dy}G_{zz,zz}(y,0) = 0.$$

This differential equation must be solved with the initial conditions

$$G_{zz,zz}(0,0) = 1, \tag{43a}$$

$$\frac{d}{dy}G_{zz,zz}(y,0)\Big|_{y=0} + 2G_{zz,zz}(0,0) = 0. \tag{43b}$$

The first one expresses the trivial fact that there is no time evolution over a vanishing time interval and follows from Eq. (40). The second one stems directly from Eq. (42a). Both determine a surprisingly simple form of the final result (38a) for $G_{zz,zz}(t,t_0)$.

Considering an initial state with $\tilde{\mathbf{g}}(t_0) = \tilde{g}_z(t_0)\mathbf{e}_z$, the conservation of $\mathbf{g}^2$ implies the normalization

$$G_{zz,zz} + 2G_{+-,zz} = 1. \tag{44}$$

This implies the result (38b) for $G_{+-,zz}(t,t_0)$, which is derived also diagrammatically in the Appendix.

3. Propagator $G_{zz,+-}$

As an example for the diagrammatic calculation of the other propagators, we consider here

$$\begin{aligned} G_{zz,+-}(t,t_0) = \int_{-\infty}^{\infty} dt' \int_{t_0}^{t'} dt'' \int_{-\infty}^{\infty} d\overline{t'} \int_{t_0}^{\overline{t'}} d\overline{t''} \\ \times \langle W_{t_0}^t(t') W_{t_0}^t(\overline{t'}) \tilde{\eta}_-(t'') \tilde{\eta}_+(\overline{t''})\rangle. \end{aligned} \tag{45}$$

This expression follows directly from the contribution to the autocorrelation of Eq. (26a) stemming from the initial value $\tilde{g}_\pm(t_0)$. Its diagrams (cf. Fig. 3) have no excess vertices close to the final time t but opposite excess vertices close to the initial time t_0. Therefore, close to t_0 only interpair contractions are possible, which are connected through the first rung at the time t_1. All possible contractions between t_1 and t have

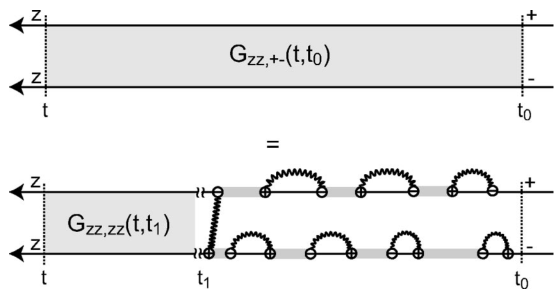

FIG. 3. The propagator $G_{zz,+-}$ consists of diagrams in which the first vertices are excess vertices. They can be contracted with a certain number of pairs until the first rung occurs at time t_1. The contraction of all later pairs yields $G_{zz,zz}(t,t_1)$.

the same structure as the diagrams of $G_{zz,zz}$. Therefore, the diagrams can be summed implicitly to

$$G_{zz,+-}(t,t_0) = \int_{t_0}^{t} dt_1 G_{zz,zz}(t,t_1) f(\nu t_1) \times G_{+,+}(t_1,t_0) G_{-,-}(t_1,t_0). \tag{46}$$

After a transition to variables y the integral can be performed explicitly and yields $G_{zz,+-}=G_{+-,zz}$.

The normalization condition

$$G_{zz,+-}(t,t_0) + G_{+-,+-}(t,t_0) + G_{+-,-+}(t,t_0) = 1,$$

which expresses the conservation of $\mathbf{g}^2$ for an initial transverse state, determines already the sum of the propagators $G_{+-,+-}(t,t_0)$ and $G_{+-,-+}(t,t_0)$, which enter Eqs. (37) only in this combination. Nevertheless, both propagators $G_{+-,+-}(t,t_0)$ and $G_{+-,-+}(t,t_0)$ are calculated separately in the Appendix.

IV. TRANSITIONS PRODUCED BY NOISE ONLY

On the basis of the propagators derived in the previous section we now evaluate transition probabilities and their fluctuations, still focusing on the subcase $b_x=0$. We consider an arbitrary initial state at $t_0=-\infty$ with arbitrary Bloch vector $\mathbf{g}(-\infty)$. We are interested in the final time $t=\infty$, where the average is given by[8]

$$\langle g_z(\infty)\rangle = e^{-\Theta} g_z(-\infty), \tag{47a}$$

$$\langle \tilde{g}_{\pm}(\infty)\rangle = e^{-\Theta/2} \tilde{g}_{\pm}(-\infty). \tag{47b}$$

according to Eqs. (33). Since the noise correlators are even functions of time differences, only the real parts of ϑ_α contribute to the difference $\vartheta_\alpha(\infty)-\vartheta_\alpha(-\infty)$. We have introduced

$$\Theta \equiv \vartheta_z(\infty) = \frac{\pi}{\nu}\langle \eta_x^2 + \eta_y^2\rangle$$

for abbreviation. It is important to realize that the decay of the averaged Bloch vector depends only on the *instantaneous* noise correlation and that the final value can be finite only for colored noise.

Although the final slow amplitudes $\tilde{g}_{\pm}(\infty)$ will be finite if the initial amplitudes $\tilde{g}_{\pm}(-\infty)$ were finite, the original amplitudes vanish,

$$\langle g_{\pm}(\infty)\rangle = 0, \tag{48}$$

due to the diffusive random precession leading to a decay on the dephasing time τ_ϕ given in Eq. (21). However, the transverse amplitude $g_\perp$ decays like $\tilde{g}_{\pm}$,

$$\langle g_\perp(\infty)\rangle = e^{-\Theta/2} g_\perp(-\infty). \tag{49}$$

For the variances we obtain—suppressing the obvious time arguments $(\infty,-\infty)$ of the propagators—

$$\begin{aligned}\langle g_z^2(\infty)\rangle &= G_{zz,zz} g_z^2(-\infty) + G_{zz,+-} g_\perp^2(-\infty)\\ &= \tfrac{1}{3}\{\mathbf{g}^2 + e^{-3\Theta}[2g_z^2(-\infty) - g_\perp^2(-\infty)]\},\end{aligned} \tag{50a}$$

$$\begin{aligned}\langle g_\perp^2(\infty)\rangle &= 2G_{+-,zz} g_z^2(-\infty) + [G_{+-,+-} + G_{+-,-+}] g_\perp^2(-\infty)\\ &= \tfrac{1}{3}\{2\mathbf{g}^2 - e^{-3\Theta}[2g_z^2(-\infty) - g_\perp^2(-\infty)]\},\end{aligned} \tag{50b}$$

and for the fluctuations

$$\begin{aligned}\langle[\delta g_z(\infty)]^2\rangle &\equiv \langle g_z^2(\infty)\rangle - \langle g_z(\infty)\rangle^2\\ &= \tfrac{1}{3}\{\mathbf{g}^2 + [2e^{-3\Theta} - 3e^{-2\Theta}] g_z^2(-\infty)\\ &\quad - e^{-3\Theta} g_\perp^2(-\infty)\},\end{aligned} \tag{51a}$$

$$\begin{aligned}\langle[\delta g_\perp(\infty)]^2\rangle &\equiv \langle g_\perp^2(\infty)\rangle - \langle g_\perp(\infty)\rangle^2\\ &= \tfrac{1}{3}\{2\mathbf{g}^2 - 2e^{-3\Theta} g_z^2(-\infty)\\ &\quad - [e^{-3\Theta} - 3e^{-\Theta}] g_\perp^2(-\infty)\}.\end{aligned} \tag{51b}$$

For systems prepared initially in a state with a well-defined $\mathbf{g}$, the correlations $C_{z\pm}$ and $C_{\pm\pm}$ do not necessarily vanish. They are not explicitly considered here since they do not couple to C_{zz} and C_{+-} according to Eq. (37). Furthermore, on long-time scales, phase diffusion due to η_z annihilates these non-neutral correlations anyway.

To discuss the statistics of transitions between the two levels, we consider the system to be located initially in the state $|\uparrow\rangle$, i.e., $\mathbf{g}(-\infty)=\mathbf{e}_z$. A measurement which detects at time t whether the system is in the state $|\uparrow\rangle$ or $|\downarrow\rangle$ yields the probabilities

$$\Pi_{\uparrow\uparrow/\uparrow\downarrow}(t) = \mathrm{Tr}\ \tfrac{1}{2}(\hat{1} \pm \hat{\sigma}_z)\hat{\rho}(t) = \tfrac{1}{2}[1 \pm g_z(t)]. \tag{52}$$

The trace accounts for the average over quantum fluctuations for a *given* realization of noise. The additional averaging over the noise leads to the final expectation values,

$$\langle \Pi_{\uparrow\uparrow}(\infty)\rangle = \tfrac{1}{2}[1 + e^{-\Theta}], \langle \Pi_{\uparrow\downarrow}(\infty)\rangle = \tfrac{1}{2}[1 - e^{-\Theta}]. \tag{53a}$$

Since the probabilities $\Pi_{\uparrow\uparrow/\uparrow\downarrow}$ depend on the noise, they are fluctuating quantities themselves. Their fluctuations are

$$\langle[\delta\Pi_{\uparrow\downarrow}(\infty)]^2\rangle = \tfrac{1}{12}(1 - 3e^{-2\Theta} + 2e^{-3\Theta}). \tag{53b}$$

Remarkably, the fluctuations of probabilities in general have the same order of magnitude as average values. They are weak only if the noise is weak, i.e., $\langle[\delta\Pi_{\uparrow\downarrow}(\infty)]^2\rangle \approx \Theta^2/4$ for $\Theta \ll 1$. In the opposite limiting case $\Theta \gg 1$ of strong noise the two levels become equipopulated, $\langle\Pi_{\uparrow\uparrow}(\infty)\rangle = \langle\Pi_{\uparrow\downarrow}(\infty)\rangle = \frac{1}{2}$, and the fluctuation is equal to $\langle[\delta\Pi_{\uparrow\downarrow}(\infty)]^2\rangle = \frac{1}{12}$.

V. LZ SYSTEM WITH NOISE

In this section we assume that not only the noise, but also the regular part of the Hamiltonian, has a finite nondiagonal matrix element Δ corresponding to a finite transverse component of the regular magnetic field. As it was shown in Ref. 6, if the noise is fast, there exists a well-defined time separation for the transition due to the noise and due to the regular part of the Hamiltonian. Consider matching at times $\pm t_\times$ with $t_{\mathrm{LZ}} \ll t_\times \ll \tau_{\mathrm{acc}}$. The time evolution may be decomposed into three intervals: (i) From $t=-\infty$ to $t=-t_\times$ one may neglect Δ. The transition is only under the influence of noise. Since $t_\times \ll \tau_{\mathrm{acc}}$, this time evolution is approximately the same as from $t=-\infty$ to $t=0$ in the absence of Δ. (ii) From $t=-t_\times$ to $t=t_\times$ the LZ transition occurs. Since $t_\times \gg \tau_{\mathrm{LZ}}$, time evolution is approximately the same as from $t=-\infty$ to $t=\infty$ in the absence of noise. (iii) From $t=t_\times$ to $t=\infty$ the situation is again analogous to the regime (i).

We represent the time-evolution operator as

$$\hat{U}(t,t_0) = \hat{U}_0(t)\hat{\tilde{U}}(t,t_0)\hat{U}_0^\dagger(t_0), \tag{54}$$

and approximate

$$\hat{\tilde{U}}(\infty,-\infty) = \hat{\tilde{U}}_{\mathrm{iii}}\hat{\tilde{U}}_{\mathrm{ii}}\hat{\tilde{U}}_{\mathrm{i}}, \tag{55a}$$

$$\hat{\tilde{U}}_{\mathrm{i}} = \hat{\tilde{U}}(0,-\infty) \quad \text{for } \Delta = 0, \tag{55b}$$

$$\hat{\tilde{U}}_{\mathrm{ii}} = \hat{\tilde{U}}(\infty,-\infty) \quad \text{for } \eta = 0, \tag{55c}$$

$$\hat{\tilde{U}}_{\mathrm{iii}} = \hat{\tilde{U}}(\infty,0) \quad \text{for } \Delta = 0 \tag{55d}$$

for the time evolution operator acting on the slow vector.

We assume that at $t=-\infty$ the density matrix is diagonal (complete decoherence), i.e.,

$$\mathbf{g}(-\infty) = g_z(-\infty)\mathbf{e}_z.$$

Then,

$$\langle g_z(-\infty)g_z(-\infty)\rangle = g_z^2(-\infty)$$

is the only nonvanishing correlator at $t=-\infty$. Since $\tau_{\mathrm{n}} \ll t_x$, the noise in intervals (i) and (iii) is statistically independent and the averaging can be performed separately for both intervals.

A. Interval (i)

Neglecting Δ in the first time interval, we find at $t=-t_x$ the averages

$$\langle \tilde{g}_z(-t_\times)\rangle = e^{-\Theta/2}g_z(-\infty), \tag{56a}$$

$$\langle \tilde{g}_\pm(-t_\times)\rangle = 0, \tag{56b}$$

and correlations

$$\tilde{C}_{zz}(-t_\times) = G_{zz,zz}(0,-\infty)g_z^2(-\infty) = \tfrac{1}{3}(1+2e^{-3\Theta/2})g_z^2(-\infty), \tag{57a}$$

$$\tilde{C}_{+-}(-t_\times) = G_{+-,zz}(0,-\infty)g_z^2(-\infty) = \tfrac{1}{3}(1-e^{-3\Theta/2})g_z^2(-\infty). \tag{57b}$$

According to Eqs. (37) all other independent pair correlations vanish at $t=-t_\times$.

B. Interval (ii)

Time evolution at intermediate times from $-t_\times$ to $t_\times$ is given by

$$\hat{U}_{\mathrm{ii}} = \hat{U}_{\mathrm{LZ}}.$$

Hence the density matrix evolves according to

$$\hat{\rho}(t_\times) = \hat{U}_{\mathrm{LZ}}\hat{\rho}(-t_\times)\hat{U}_{\mathrm{LZ}}^\dagger$$

which implies the transformation

$$\tilde{\mathbf{g}}(t_\times)\cdot\hat{\sigma} = \hat{U}_{\mathrm{LZ}}\tilde{\mathbf{g}}(-t_\times)\cdot\hat{\sigma}\hat{U}_{\mathrm{LZ}}^\dagger$$

for the transformed Bloch vector $\tilde{\mathbf{g}}$. The explicit transformation of its components reads

$$\begin{pmatrix} \tilde{g}_+(t_\times) \\ \tilde{g}_z(t_\times) \\ \tilde{g}_-(t_\times) \end{pmatrix} = \mathbf{U}^{\mathrm{LZ}} \cdot \begin{pmatrix} \tilde{g}_+(-t_\times) \\ \tilde{g}_z(-t_\times) \\ \tilde{g}_-(-t_\times) \end{pmatrix} \tag{58a}$$

with the rotation matrix

$$\mathbf{U}^{\mathrm{LZ}} = \begin{pmatrix} a^{*2} & -\sqrt{2}a^*b^* & -b^{*2} \\ \sqrt{2}a^*b & |a|^2-|b|^2 & \sqrt{2}ab^* \\ -b^2 & -\sqrt{2}ab & a^2 \end{pmatrix}. \tag{58b}$$

From these relations we obtain

$$\langle \tilde{g}_z(t_\times)\rangle = U_{zz}^{\mathrm{LZ}}\langle \tilde{g}_z(-t_\times)\rangle = (|a|^2-|b|^2)e^{-\Theta/2}g_z(-\infty), \tag{59a}$$

$$\langle \tilde{g}_+(t_\times)\rangle = U_{+z}^{\mathrm{LZ}}\langle \tilde{g}_z(-t_\times)\rangle = -\sqrt{2}a^*b^*e^{-\Theta/2}g_z(-\infty) \tag{59b}$$

for the averages. Analogously, correlations are transformed according to

$$\tilde{C}_{\alpha\beta}(t_\times) = \sum_{\gamma\delta} U_{\alpha\gamma}^{\mathrm{LZ}}U_{\beta\delta}^{\mathrm{LZ}}\tilde{C}_{\gamma\delta}(-t_\times). \tag{60}$$

Since we are ultimately interested only in the correlations $\tilde{C}_{zz}(\infty)$ and $\tilde{C}_{+-}(\infty)$, and since the time evolution in interval (iii) again follows Eqs. (37), we need to evaluate at time $t=t_\times$ only the two correlations

$$\begin{aligned}\tilde{C}_{zz}(t_\times) &= (U_{zz}^{\mathrm{LZ}})^2\tilde{C}_{zz}(-t_\times) + 2U_{z+}^{\mathrm{LZ}}U_{z-}^{\mathrm{LZ}}\tilde{C}_{+-}(-t_\times) \\ &= \tfrac{1}{3}[1+2(1-6|a|^2|b|^2)e^{-3\Theta/2}]g_z^2(-\infty),\end{aligned} \tag{61a}$$

$$\begin{aligned}\tilde{C}_{+-}(t_\times) &= U_{+z}^{\mathrm{LZ}}U_{-z}^{\mathrm{LZ}}\tilde{C}_{zz}(-t_\times) + [U_{++}^{\mathrm{LZ}}U_{--}^{\mathrm{LZ}} + U_{+-}^{\mathrm{LZ}}U_{-+}^{\mathrm{LZ}}]\tilde{C}_{+-}(-t_\times) \\ &= \tfrac{1}{3}[1-(1-6|a|^2|b|^2)e^{-3\Theta/2}]g_z^2(-\infty).\end{aligned} \tag{61b}$$

As a check of these results, we verify the normalization relation $\tilde{C}_{zz}(t_\times)+2\tilde{C}_{+-}(t_\times)=g_z^2(-\infty)$.

C. Interval (iii)

As already mentioned, the time evolution in interval (iii) is absolutely analogous to the evolution in interval (i). We therefore obtain

$$\langle \tilde{g}_z(\infty)\rangle = G_{z,z}(\infty,0)\langle \tilde{g}_z(t_\times)\rangle = (|a|^2 - |b|^2)e^{-\Theta}g_z(-\infty), \tag{62a}$$

$$\langle \tilde{g}_+(\infty)\rangle = G_{+,+}(\infty,0)\langle \tilde{g}_z(t_\times)\rangle = -\sqrt{2}a^*b^*e^{-[\vartheta_+(\infty)-\vartheta_+(0)]} \times e^{-\Theta/2}g_z(-\infty). \tag{62b}$$

These results were obtained already in Ref. 8. In terms of **g** they mean

$$\langle g_z(\infty)\rangle = (|a|^2 - |b|^2)e^{-\Theta}g_z(-\infty), \tag{63a}$$

$$\langle g_\perp(\infty)\rangle = \sqrt{2}|a||b|e^{-3\Theta/4}g_z(-\infty). \tag{63b}$$

We recall that the averages $\langle g_x(\infty)\rangle = \langle g_y(\infty)\rangle = 0$ vanish due to the presence of the longitudinal noise component. The correlations follow from Eqs. (37) with $t=\infty$ and $t_0 = t_\times$:

$$\tilde{C}_{zz}(\infty) = \tfrac{1}{3}[1 + 2(1 - 6|a|^2|b|^2)e^{-3\Theta}]g_z^2(-\infty), \tag{64a}$$

$$\tilde{C}_{+-}(\infty) = \tfrac{1}{3}[1 - (1 - 6|a|^2|b|^2)e^{-3\Theta}]g_z^2(-\infty). \tag{64b}$$

In the presence of regular transitions, the transition probabilities and their fluctuations—given in Eqs. (53a) and (53b) in the absence of Δ—finally become

$$\langle \Pi_{\uparrow\uparrow/\uparrow\downarrow}(\infty)\rangle = \tfrac{1}{2}[1 \pm (|a|^2 - |b|^2)e^{-\Theta}] \tag{65a}$$

and

$$\langle [\delta\Pi_{\uparrow\uparrow/\uparrow\downarrow}(\infty)]^2\rangle = \tfrac{1}{12}[1 - 3(1 - 4|a|^2|b|^2)e^{-2\Theta} + 2(1 - 6|a|^2|b|^2)e^{-3\Theta}]. \tag{65b}$$

To obtain the explicit dependence on the LZ parameter one may use $|a|^2 = e^{-2\pi\gamma^2}$ and the unitarity relation $|b|^2 = 1 - |a|^2$.

VI. RESPONSE SPECTRUM AND INTENSITY

Let a short pulse of an effective field directed along axis i act on the LZ system at some moment of time t. Its response at a later moment of $t' = t + \tau$ is determined by the correlation function, $K_{i,j}(t,\tau) = \langle s_i(t)s_j(t+\tau)\rangle$. It is reasonable to measure the spectral content of the response at a fixed time of initial pulse t. It is given by the Fourier transform

$$K_{i,j}(t;\omega) = \int_0^\infty K_{i,j}(t,\tau)e^{i\omega\tau}d\tau. \tag{66}$$

According to general relations (35) and (33), in the limit of $\tau\to\infty$ such a function vanishes or saturates to a finite value. For example, in the case of $i=j=z$ and $t \gg t_\times$ it is equal to $\tilde{C}_{zz}(t)\exp\left(-\int_t^\infty F_\perp(\nu\tau')d\tau'\right)$. Therefore, the spectral density of the response contains a $1/\omega$ component at a small frequency with an intensity that depends on the time of excitation. This is not surprising since the LZ process violates the homogeneity of time. At infinite time t this intensity reaches a finite limit $\tilde{C}_{zz}(\infty)$. It happens since the noise becomes ineffective at a sufficiently large time and does not randomize the quantum amplitudes any more. Therefore, any quantum amplitude or a component of the Bloch vector **g** changes after this time in an almost deterministic way. On the other hand, the spectral intensity remains finite even at $t\to\infty$ since the value of **g** reached at the moment, when the noise becomes ineffective, is random.

Another (incoherent) contribution to the spectral density arises from the variation of the correlation function at a finite time,

$$K_{zz}(\tau,t) - K_{zz}(\infty,t) = K_{zz}(\infty,t) \times\left[\exp\left(\int_{t+|\tau|}^\infty F_\perp(\nu\tau')d\tau'\right) - 1\right]. \tag{67}$$

To estimate the spectral width of this contribution we assume that either time t is large enough or the noise is weak. Then the exponent in Eq. (67) can be expanded to the linear term. Its spectral width is easily estimated as $\Delta\omega_1 = 1/\tau_{\rm acc}$. The spectral intensity of this line at $t\sim\tau_{\rm acc}$ can be estimated as $I_1 \propto \tilde{C}_{zz}(\infty)\langle\eta_x^2 + \eta_y^2\rangle/\nu$. It decreases rapidly with growing time t.

If the LZ processes are repeated randomly with the average time $\tau_{\rm coll}$ between the events (as it happens in a gas of colliding atoms or molecules), the $1/\omega$ line is smeared out to the width $\Delta\omega_0 \sim 1/\tau_{\rm coll} \ll \Delta\omega_1$, whereas its average intensity I_0 is proportional to $\tilde{C}_{zz}(\infty)$. The width of the incoherent line remains the same $\Delta\omega_1 = 1/\tau_{\rm acc}$, but its intensity is stabilized at a value $I_1(\tau_{\rm coll}) \propto I_1\varphi(\tau_{\rm coll}/\tau_{\rm acc})$ the smaller the larger is $\tau_{\rm coll}$ [here $\varphi(x)$ is a function describing the decay of the noise correlations, which depends on details].

Thus, the main contribution to the spectral width of the population or induced field fluctuations consists of two narrow lines. The first has the width determined by collisions and a permanent intensity per particle. The second has a permanent width, but its intensity is determined by collision time. In the rarefied gas, or more generally if $\tau_{\rm coll} \gg \tau_{\rm acc}$, the first line is narrower and stronger than the second one.

VII. CONCLUSIONS

We have calculated the correlation functions of the Bloch vector components in the LZ process subject to a fast Gaussian noise. These correlators determine the linear response of the LZ system to a weak, time-dependent probe signal. Two-time correlation functions are factorizable at the time scale much larger than the noise correlation time $\tau_{\rm n}$ due to the statistical independence of random processes. Thus the main problem is the calculation of simultaneous averages. The condition of the fast noise allows to use for these averages the ladder graphs only. The resulting Bethe-Salpeter equation can be reduced to differential equations which are exactly solvable. In general, fluctuations are strong and of the same order of magnitude as the average values.

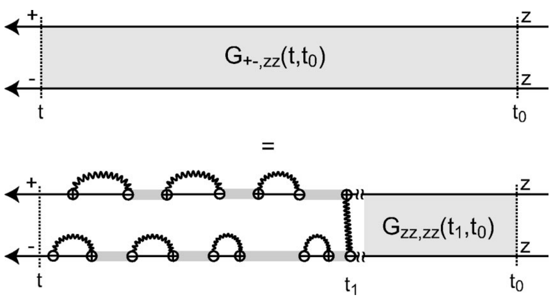

FIG. 4. The diagrams $G_{+-,zz,}$ are analogous to the diagrams of $G_{zz,+-}$ with the only difference being that now the excess vetices are the last vertices.

Since the noise is ineffective for transitions at sufficiently large time $t \gg \tau_{\rm acc}$, the transition probabilities or the components of the Bloch vector remain deterministically coherent after this time. Therefore, the spectrum of the fluctuations contains a narrow $1/\omega$ line, whose intensity is determined by the distribution of these values at the time when the noise became ineffective. Besides this line, there exists another narrow line with a width of about $1/\tau_{\rm acc}$ and an intensity depending on the pulse time and going to zero when this time goes to infinity.

ACKNOWLEDGMENTS

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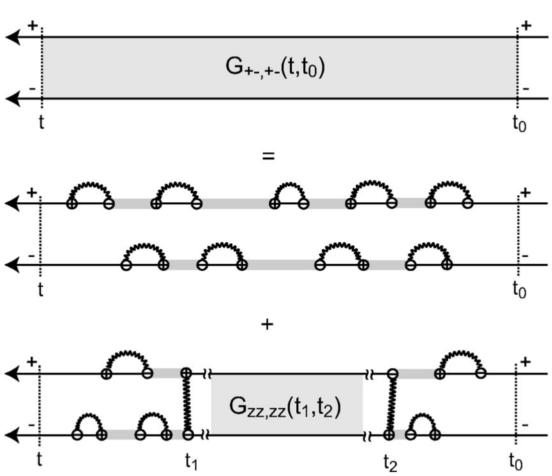

FIG. 5. $G_{+-,+-}$ consists of diagrams with excess vertices as first and last vertices on both time legs. There are disconnected diagrams (where the "ladder" has no rungs) and connected diagrams (where the ladder has rungs). In the second case, the excess vertices are are connected by rungs as t_1 and t_2, and between t_1 and t_2 additional pairs can be contracted as in $G_{zz,zz}(t_1,t_2)$.

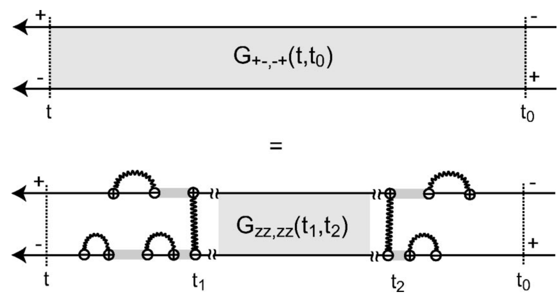

FIG. 6. The diagrams of $G_{+-,-+}$ as similar to these of $G_{+-,+-}$ with the difference being that the polarity of the first excess vertices is inverted. Therefore, no disconnected diagrams can be formed.

APPENDIX: DETAILS FOR CORRELATORS

Here we provide some details on the calculation of correlation propagators.

1. Propagator $G_{+-,zz}$

The calculation of the propagator $G_{+-,zz}$ can be performed largely in parallel to the calculation of $G_{zz,+-}$ in Sec. III B 3. The starting point is the equation

$$G_{+-,zz}(t,t_0) = \int_{t_0}^{t} dt' \int_{t_0}^{t} d\overline{t'} \int_{-\infty}^{\infty} d\overline{t'}' \int_{-\infty}^{\infty} dt'' \times \langle \tilde{\eta}_-(t')\tilde{\eta}_+(\overline{t'}) W_{t_0}^{t'}(t'') W_{t_0}^{\overline{t'}}(\overline{t'}')\rangle.$$

Using the Wick theorem, we perform the contractions,

$$G_{+-,zz}(t,t_0) = \int_{t_0}^{t} dt_1 G_{+,+}(t,t_1) G_{-,-}(t,t_1) \times f(\nu t_1) G_{zz,zz}(t_1,t_0),$$

where t_1 is the time of the last rung of the ladder (last means closest to t, see Fig. 4). After a substitution from t_1 to y_1 the integral is elementary with the result

$$G_{+-,zz}(t,t_0) = \tfrac{1}{3}(1 - e^{-3[\vartheta_z(t)-\vartheta_z(t_0)]}).$$

2. Propagator $G_{+-,+-}$

From the equations of motion (26) one extracts

$$\begin{aligned} G_{+-,+-}&(t,t_0) \\ &= 1 - \int_{t_0}^{t} dt' \int_{-\infty}^{\infty} dt'' \int_{t_0}^{t''} dt''' \langle \tilde{\eta}_-(t') W_{t_0}^{t'}(t'') \tilde{\eta}_+(t''')\rangle \\ &- \int_{t_0}^{t} d\overline{t'} \int_{-\infty}^{\infty} d\overline{t''} \int_{t_0}^{\overline{t''}} d\overline{t'''} \langle \tilde{\eta}_+(\overline{t'}) W_{t_0}^{\overline{t'}}(\overline{t''}) \tilde{\eta}_-(\overline{t'''})\rangle \\ &+ \int_{t_0}^{t} dt' \int_{-\infty}^{\infty} dt'' \int_{t_0}^{t''} dt''' \int_{t_0}^{t} d\overline{t'} \int_{-\infty}^{\infty} d\overline{t''} \int_{t_0}^{\overline{t''}} d\overline{t'''} \\ &\times \langle \tilde{\eta}_-(t') W_{t_0}^{t'}(t'') \tilde{\eta}_+(t''') \tilde{\eta}_+(\overline{t'}) W_{t_0}^{\overline{t'}}(\overline{t''}) \tilde{\eta}_-(\overline{t'''})\rangle. \end{aligned}$$

After averaging, the sum over the ladder diagrams can be captured by a Bethe-Salpeter equation:

$$G_{+-,+-}(t,t_0) = G_{+,+}(t,t_0)G_{-,-}(t,t_0) \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \times G_{+,+}(t,t_1)G_{-,-}(t,t_1)f(\nu t_1)G_{zz,zz}(t_1,t_2)f(\nu t_2) \times G_{+,+}(t_2,t_0)G_{-,-}(t_2,t_0),$$

where t_1 is the time of the last rung and t_2 the time of the first rung (see Fig. 5). All contractions in between sum up to $G_{zz,zz}(t_1,t_2)$. Since the latter propagator is explicitly known, the integrals can be performed directly, yielding Eq. (38d).

3. Propagator $G_{+-,-+}$

Similar to the previous propagator, one now has

$$G_{+-,-+}(t,t_0) = \int_{t_0}^{t} dt' \int_{-\infty}^{\infty} dt'' \int_{t_0}^{t''} dt''' \int_{t_0}^{t} d\overline{t'} \int_{-\infty}^{\infty} d\overline{t''} \int_{t_0}^{\overline{t''}} d\overline{t'''} \langle \times \tilde{\eta}_-(t') W_{t_0}^{t'}(t'') \tilde{\eta}_-(t''') \tilde{\eta}_+(\overline{t'}) W_{t_0}^{\overline{t'}}(\overline{t''}) \tilde{\eta}_+(\overline{t'''}) \rangle .$$

The configuration of the excess vertices necessitates at least two rungs at times t_1 and t_2 (see. Fig. 6). The contractions in between again sum up to $G_{zz,zz}(t_1,t_2)$, leading to

$$G_{+-,-+}(t,t_0) = \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 G_{+,+}(t,t_1)G_{-,-}(t,t_1)f(\nu t_1)G_{zz,zz} \times (t_1,t_2)f(\nu t_2)G_{+,+}(t_2,t_0)G_{-,-}(t_2,t_0),$$

and eventually to Eq. (38e).

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[4] We use the term "adiabatic" as usual, e.g., the state $|\uparrow\rangle$ at $t=-\infty$ is adiabatically connected to the state $\langle\downarrow|$ at $t=\infty$ (avoided level crossing). Analogously, $|\uparrow\rangle$ at $t=-\infty$ evolved "diabatically" into $|\uparrow\rangle$ at $t=\infty$ (crossing the second diabatic level). The evolution matrix (2) connects the values of the "slow" amplitudes $C'_{\uparrow',\downarrow}(t)$ which are defined by representing the state vector in a form $|t\rangle = C'_{\uparrow}(t)e^{-i\Phi_{\uparrow}(t)}|\uparrow\rangle + C'_{\downarrow}(t)e^{-i\Phi_{\downarrow}(t)}|\downarrow\rangle$. The dynamic phases $\Phi_{\uparrow,\downarrow}(t) = (1/\hbar)\int_0^t dt' E'_{\uparrow,\downarrow}(t')$ depend on the diabatic eigenstates of the Hamiltonian (1). Note that the diabatic energies are an odd function of time, whereas the adiabatic energies are an even function of time.

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