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Continuous-variable entanglement in a correlated spontaneous emission laser

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We discuss the generation and evolution of entangled light in a correlated spontaneous emission laser. The master equation for the two-mode field in a cavity is derived and solved analytically. The time-dependent characteristic function in the Wigner representation for the two-mode field is obtained. It shows that the two-mode field in the cavity evolves in a two-mode Gaussian state. The entanglement degree of the two-mode field in the cavity increases initially, then decreases, and finally vanishes as the field evolves from an initial vacuum. The period of the entanglement is extended as the intensity of the driving field is increased. It is found that the entanglement still exists even when the two-mode squeezing disappears. During the entanglement period, the intensity of the field is amplified. The entanglement for the initial field being a two-mode squeezed vacuum and the entanglement of the output field are also discussed.

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I. INTRODUCTION

In recent years, a number of possible practical applications of entangled states have been proposed. These include the quantum computation, the quantum communication, and the quantum cryptography [1]. Recently, it has been recognized that Gaussian continuous variable entangled states are important part of the quantum information theory [2]. With successful experiments on quantum teleportation with two-mode squeezed states [3] as well as the experimental realization of the entanglement in atomic ensembles [4], a lot of theoretical and experimental studies have been devoted to generating the continuous variable Gaussian entangled states and quantifying the entanglement of these states. For example, Simon [5] and Duan *et al.* [6] proposed, respectively, two criteria to determine the separability for two-mode Gaussian states. Fiurasek and Cerf [7] proposed a scheme to measure the entanglement of Gaussian states without homodyning, Josse *et al.* [8] presented an experiment to generate the continuous variable entanglement, and Li *et al.* [9] proposed a scheme to generate the continuous variable excitonic state in microcavity. On the other hand, any attempt to exploit entangled states in quantum information processing has to face the obvious difficulty that the coherence and the intensity of pure entangled states are unavoidably corrupted by the interaction between the system and its environment. How to prepare the entangled light in a large number of photons is also an important issue. In this paper we address this question by discussing in details a correlated spontaneous emission laser as an entanglement amplifier.

In the past 2 decades, research on the quantum amplifier has been carried out extensively. Scully and Zubairy [10] presented a theory of two-photon phase-sensitive amplification by a three-level atomic system in a cascade configuration, where the atomic coherence is induced by the initial atomic coherent superposition of the upper and the bottom levels. They showed that the quadratures of the field are amplified with equal gain and added noise in one of the quadratures goes to zero at the expense of increased noise in the other quadrature under certain conditions. Lu and Zhu [11] extended this model to the case of two modes. They found that the two-mode squeezing can be produced, and they discussed the quantum correlation of the two-mode field. Ansari *et al.* [12] considered a three-level atomic ensemble in which the atomic coherence is induced by a strong resonant external driving field for the single-mode and two-mode cases, and they found that the system behaves as an ideal parameter amplifier under some conditions. Huang, Zhu, and Zubairy [13] investigated the amplified cat state in a quantum amplifier and found that the nonclassicality of the cat state can be preserved during its amplification. By means of the phase-sensitive amplifier, Ahmad, Qamar, and Zubairy [14] proposed a scheme to measure quantum states and reconstruct multimode entangled states in a cavity. In a recent paper, Xiong, Scully, and Zubairy [15] have proposed a scheme for an entanglement amplifier based on a two-mode correlated spontaneous emission laser. They showed, using the entanglement measure of Ref. [6], that the two modes of the amplified laser field are in an entangled state even in the presence of cavity losses. This analysis is, however, based on the Einstein–Podolsky–Rosen (EPR) uncertainty which is only a sufficient nonseparability criterion for a two-mode state [6]. Recently Eisenberg *et al.* [16] proposed a scheme to generate multiple entangled photons.

In this paper we extend the analysis of Ref. [15]. We first obtain the time-dependent characteristic function in the Wigner representation for the two-mode field in a cavity and show that the field evolves in a two-mode Gaussian state. This Gaussian solution is helpful in studying the entanglement with the sufficient and necessary separability criteria for two-mode Gaussian states of Refs. [5,6]. We show that, as the field in the cavity evolves from an initial vacuum, the entanglement between the two modes increases initially, then decreases and eventually disappears, which is in agreement with the results of Ref. [15]. The entanglement period is extended with the increase of the intensity of the driving field, and the entanglement periods determined by the two separability criteria established respectively by Simon [5]

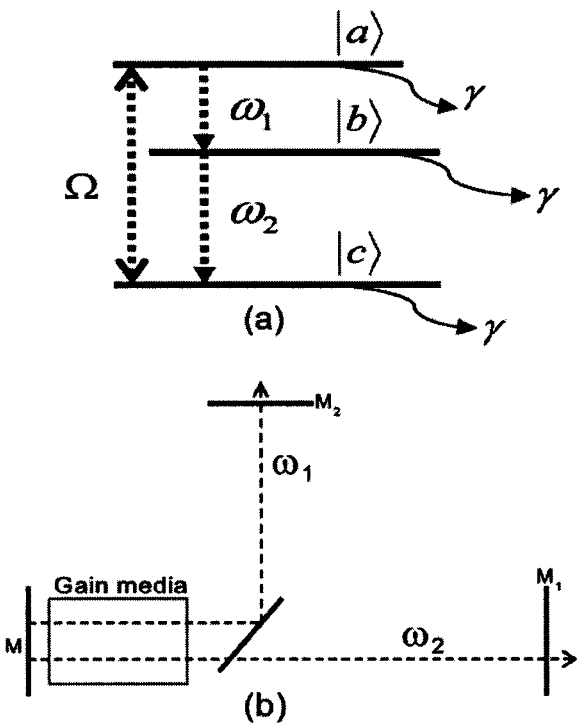

FIG. 1. (a) Three-level atom in a cascade configuration. The dipole-allowed transitions $|a\rangle \leftrightarrow |b\rangle$ and $|b\rangle \leftrightarrow |c\rangle$ with frequency ω_1 and ω_2; A strong external field induces the dipole-forbidden transition $|a\rangle \leftrightarrow |c\rangle$. γ is the spontaneous emission rate of the three levels into other levels. (b) Gain media (atomic ensemble) in a doubly resonant cavity with perfect reflecting mirror M and two high-reflection mirrors M_1 and M_2.

and Duan *et al.* [6] are found to be identical at the present problem. It is found that the entanglement still exists even when the two-mode squeezing disappears. During the entanglement period, the intensity of the entangled light is amplified. The entanglement for the initial field being a two-mode squeezed vacuum and the entanglement of the output field are also discussed.

The paper is organized as follows. In Sec. II, we derive the master equation for the two-mode field in the cavity. In Sec. III, the master equation of the field is solved exactly. In Sec. IV, we discuss in detail the conditions for the entanglement between the two modes in the cavity. Also the entanglement of the output field is evaluated in the parametric approximation in this section. In Sec. V we present a summary of our main results.

II. MODEL AND THE MASTER EQUATION

We consider a three-level atomic ensemble in a cascade configuration interacting resonantly with a two-mode field in a cavity [see Figs. 1(a) and 1(b)]. The atoms, initially in the lower level $|c\rangle$, are injected into the cavity at a rate r_a. The upper level $|a\rangle$ and the lower level $|c\rangle$ have the same parity and the intermediate level $|b\rangle$ as the opposite parity. A strong external field treated classically with Rabi frequency $\Omega e^{-i\phi}$ induces the dipole-forbidden transition $|a\rangle \leftrightarrow |c\rangle$ (through two photon transition). The interaction Hamiltonian is

$$V_{\text{int}} = \left(g_1 a_1 |a\rangle\langle b| + g_2 a_2 |b\rangle\langle c| - \tfrac{1}{2}\Omega e^{-i\phi}|a\rangle\langle c|\right) + \text{H.c.}, \tag{1}$$

where a_1 and a_2 are the annihilation operators for the first mode with frequency ω_1 and the second mode ω_2, respectively, g_1 and g_2 are the coupling constants associated with the transitions $|a\rangle \leftrightarrow |b\rangle$ and $|b\rangle \leftrightarrow |c\rangle$. In the interaction picture, the density operator for the two-mode field is governed by

$$\frac{d}{dt}\rho_f = -i\,\text{Tr}_{atom}[V_{\text{int}}, \rho_{af}], \tag{2}$$

where ρ_{af} is the atom-field density operator. With Eq. (1) we have

$$\frac{d}{dt}\rho_f = -ig_1[a_1^\dagger, \rho_{ab}] - ig_2[a_2^\dagger, \rho_{bc}] + \text{H.c.}, \tag{3}$$

where $\rho_{ab} = \langle a|\rho_{af}|b\rangle$ and $\rho_{bc} = \langle b|\rho_{af}|c\rangle$. Following the method in Ref. [17], ρ_{ab} and ρ_{bc} can be evaluated to the first order in the coupling constant g_i $(i=1,2)$ as

$$\frac{d}{dt}\rho_{ab} = -\gamma\rho_{ab} + \frac{i\Omega e^{-i\phi}}{2}\rho_{cb} - i(g_1 a_1 \rho_{bb} - g_1 \rho_{aa} a_1 - g_2 \rho_{ac} a_2^\dagger),$$

$$\frac{d}{dt}\rho_{bc} = -\gamma\rho_{bc} - \frac{i\Omega e^{-i\phi}}{2}\rho_{ba} - i(g_2 a_2 \rho_{cc} - g_2 \rho_{bb} a_2 + g_1 a_1^\dagger \rho_{ac}). \tag{4}$$

Here γ is the spontaneous emission rate of the three levels of the atoms into other levels. The zeroth-order equations of motion for ρ_{ii} $(i=a,b,c)$ and ρ_{ac} are

$$\frac{d}{dt}\rho_{aa} = -\gamma\rho_{aa} + \frac{i\Omega}{2}(e^{-i\phi}\rho_{ca} - e^{i\phi}\rho_{ac}),$$

$$\frac{d}{dt}\rho_{cc} = -\gamma\rho_{cc} + \frac{i\Omega}{2}(e^{i\phi}\rho_{ac} - e^{-i\phi}\rho_{ca}) + r_a\rho_f,$$

$$\frac{d}{dt}\rho_{ac} = -\gamma\rho_{ac} - \frac{i\Omega}{2}e^{i\phi}(\rho_{aa} - \rho_{cc}),$$

and $\rho_{bb}=0$. On substituting the steady-state solutions of the above equations in Eq. (4), and integrating from $-\infty$ to t, we obtain the expressions of ρ_{ab} and ρ_{bc}. It follows on combining Eq. (3) and taking into account the dissipations of each mode in the vacuum environment that we obtain the motion equation of ρ_f as

$$\begin{aligned}\frac{d}{dt}\rho_f = &-B_1(\rho_f a_1 a_1^\dagger - a_1^\dagger a_1 \rho_f) - \kappa_1(a_1^\dagger a_1 \rho_f - a_1 \rho_f a_1^\dagger)\\ &-(B_2+\kappa_2)(a_2^\dagger a_2 \rho_f - a_2 \rho_f a_2^\dagger) + C_1(a_1^\dagger a_2^\dagger \rho_f - a_1^\dagger \rho_f a_2^\dagger)\\ &+ C_2(\rho_f a_1^\dagger a_2^\dagger - a_1^\dagger \rho_f a_2^\dagger) + \text{H.c.},\end{aligned} \tag{5}$$

where we have

$$B_1 = \frac{3r_a g_1'^2 \Omega'^2}{(1+\Omega'^2)(4+\Omega'^2)}, \quad B_2 = \frac{r_a g_2'^2}{(1+\Omega'^2)},$$

$$C_1 = \frac{ie^{-i\phi} r_a g_1' g_2' \Omega' (\Omega'^2 - 2)}{(1+\Omega'^2)(4+\Omega'^2)} = ie^{-i\phi} C_{11},$$

$$C_2 = -\frac{ie^{-i\phi} r_a g_1' g_2' \Omega'}{(1+\Omega'^2)} = -ie^{-i\phi} C_{22},$$

with $\Omega' = \Omega/\gamma$, $g_i' = g_i/\gamma$, and κ_i $(i=1,2)$ is the damping constant of each mode. Here B_1 is the gain term for the first mode, B_2 describes the absorption of the second mode, and C_1, C_2 represent the coupling between the two modes.

III. GENERAL SOLUTION FOR THE MASTER EQUATION

The master equation (5) can be solved analytically by using the characteristic function. The characteristic function of a two-mode field in the Wigner representation is defined as [17,18]

$$\chi(\xi_1,\xi_2,t) = \mathrm{Tr}[\rho(t)\exp(\xi_1 a_1^\dagger - \xi_1^* a_1)\exp(\xi_2 a_2^\dagger - \xi_2^* a_2)]. \tag{6}$$

With standard operator corresponding we find that the characteristic function of the two-mode field under consideration obeys

$$\chi(\xi_1,\xi_2,t) = \exp\left\{(B_1-\kappa_1)\xi_1\frac{\partial}{\partial\xi_1} - (B_2+\kappa_2)\xi_2\frac{\partial}{\partial\xi_2} - C_1\xi_2^*\frac{\partial}{\partial\xi_1} + C_2\xi_1^*\frac{\partial}{\partial\xi_2} - \frac{(B_1+\kappa_1)}{2}\xi_1^*\xi_1 - \frac{(B_2+\kappa_2)}{2}\xi_2^*\xi_2 + \frac{(C_1^*+C_2^*)}{2}\xi_1\xi_2 + \mathrm{c.c}\right\}\chi(\xi_1,\xi_2,0). \tag{7}$$

Defining a set of operators which form a closed Lie algebra and using the operators ordering theorems [18] we can solve Eq. (7) exactly. We assume that the two-mode field is initially in a symmetry two-mode Gaussian state whose characteristic function is $\chi(\xi_1,\xi_2,0)=\exp\left(-\frac{1}{2}\xi V_0 \xi^\dagger\right)$ [19] with $\xi=(\xi_1^*,\xi_1,\xi_2^*,\xi_2)$, where

$$V_0 = \begin{pmatrix} n_1 & m_1 & m_s & m_c \\ m_1^* & n_1 & m_c^* & m_s^* \\ m_s^* & m_c & n_2 & m_2 \\ m_c^* & m_s & m_2^* & n_2 \end{pmatrix}, \tag{8}$$

is covariance matrix, $\langle a_i^\dagger a_i\rangle = n_i - 1/2$, $\langle a_i^2\rangle = -m_i$ $(i=1,2)$, $\langle a_1 a_2\rangle = -m_c$, and $\langle a_1 a_2^\dagger\rangle = m_s$. The time-dependent solution of Eq. (7) is obtained as $\chi(\xi_1,\xi_2,t)=\exp\left(-\frac{1}{2}\xi V_t \xi^\dagger\right)$ with

$$V_t = \begin{pmatrix} h_1 & h_{11}^* & h_{12}^* & h_3^* \\ h_{11} & h_1 & h_3 & h_{12} \\ h_{12} & h_3^* & h_2 & h_{22}^* \\ h_3 & h_{12}^* & h_{22} & h_2 \end{pmatrix}. \tag{9}$$

The elements of matrix V_t are given in the Appendix. As a result, the state of the two-mode field at time t also evolves in a two-mode Gaussian state. Here we limit our discussion to the case of the initial two-mode field being in a two-mode squeezed vacuum $\Psi(0)=S(r)|0,0\rangle$ with a two-mode squeezing operator $S(r)=\exp[-r(a_1^\dagger a_2^\dagger e^{-i\varphi} - a_1 a_2 e^{i\varphi})]$ [17,18]. For such an initial state, the nonzero elements of matrix V_0 are $n_1=n_2=\cosh(2r)/2$, $m_c=\sinh(2r)e^{-i\varphi}/2$, and

$$V_t = \begin{pmatrix} h_1 & 0 & 0 & h_3^* \\ 0 & h_1 & h_3 & 0 \\ 0 & h_3^* & h_2 & 0 \\ h_3 & 0 & 0 & h_2 \end{pmatrix}. \tag{10}$$

IV. ENTANGLEMENT ANALYSIS OF THE SYSTEM

A. Entanglement of the two modes in the cavity

We now discuss the entanglement of the two mode in the cavity based on the time-dependent solution of the master equation (3). The entanglement property of the two-mode Gaussian state with covariance matrix (10) can be analyzed by the use of two sufficient and necessary separability criteria for the two-mode Gaussian states established by Simon [5] and Duan *et al.* [6].

Simon's separability criterion is equivalent to judging whether or not the quantum state is P representable [19]. A two-mode Gaussian state is P representable, and hence, separable, if and only if we have $V-\frac{1}{2}I \geqslant 0$, where I is a 4×4 unit matrix and V is the covariance matrix for the Wigner characteristic function. The eigenvalues of $V_t-\frac{1}{2}I$ for the two-mode field with the covariance matrix (10) are twofold degenerate, and they are

$$\mu_{1,2}(t) = \frac{(h_1+h_2)\pm\sqrt{(h_1-h_2)^2+4|h_3|^2}}{2} - \frac{1}{2}. \tag{11}$$

It is easy to verify that the eigenvalue $\mu_1(t)$ is always nonnegative at any condition, and has no effect on establishing the nonseparability of the state. The eigenvalue $\mu_2(t)$ may be negative and determines the nonseparablity of the state of the two-mode field.

The separability criterion of Duan *et al.* is equivalent in determining the entanglement where the necessary and sufficient condition for having entanglement for the Gaussian states is that the parameter Δ is negative. The parameter Δ is defined as [6]

$$\Delta = \left\langle\left(aX_1 + \frac{1}{|a|}X_2\right)^2\right\rangle + \left\langle\left(aP_1 - \frac{1}{|a|}P_2\right)^2\right\rangle - \left(a^2 + \frac{1}{a^2}\right), \tag{12}$$

where a is a nonzero real number, and the position and momentum operators (X_i,P_i) for each mode are $X_i=(a_i+a_i^\dagger)/\sqrt{2}$ and $P_i=-i(a_i-a_i^\dagger)/\sqrt{2}$, respectively. To get the expression of Δ, we need rewrite the characteristic function $\chi(\xi_1,\xi_2,t)$ with the covariance matrix (10) in the standard form shown in Ref. [6]. First we perform a local rotation of $\Psi/2$ of the phase angle to h_3 in phase space, which does not change the degree of the entanglement, and results in $\chi(\xi_1,\xi_2,t)=\exp[-h_1|\xi_1|^2 - h_2|\xi_2|^2 - |h_3|(\xi_1\xi_2+\xi_1^*\xi_2^*)]$. Now we rewrite this characteristic function in the form of

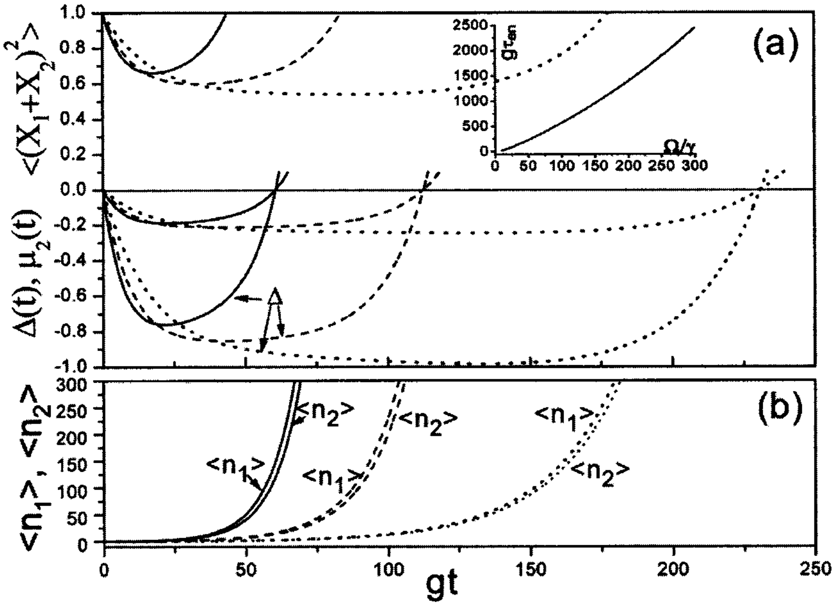

FIG. 2. (a) The time evolution of the variance $\langle (X_1+X_2)^2\rangle$ and negativities $\mu_2(t)$ and $\Delta(t)$, and (b) the mean photon numbers $\langle n_1\rangle$ and $\langle n_2\rangle$ of the two-mode field as the field evolves from an initial vacuum for $\phi=-\pi/2$, $\gamma=r_a=g_1=g_2$, $\kappa_1=\kappa_2=0.001\gamma$, $\Omega/\gamma=20$ (solid), $\Omega/\gamma=30$ (dashed), and $\Omega/\gamma=50$ (dotted). In the inset, we plot the time period of the entanglement as a function of Ω/γ.

$\chi(\xi_1,\xi_2,t)=\chi(\lambda,t)=\exp(-\frac{1}{2}\lambda M\lambda^T)$ with the parameter $\lambda=(\lambda_1^I,\lambda_1^R,\lambda_2^I,\lambda_2^R)$ where λ_i^R and λ_i^I are the real and imaginary parts of ξ_i $(i=1,2)$. Here M is a 4×4 matrix with the nonzero elements $M_{11}=M_{22}=2h_1$, $M_{33}=M_{44}=2h_2$, $M_{13}=M_{31}=-2|h_3|$ and $M_{24}=M_{42}=2|h_3|$. From Ref. [6], we can find $a^2=\sqrt{(M_{11}-1)/(M_{33}-1)}=\sqrt{(2h_2-1)/(2h_1-1)}$. Equation (12) reduces to

$$\Delta(t)=(2h_1-1)a^2+(2h_2-1)/a^2-4|h_3|. \tag{13}$$

In addition, the mean photon number of each mode can be obtained as $n_1=h_1-1/2$ and $n_2=h_2-1/2$. The variances

$$\langle (X_1+X_2)^2\rangle=\langle (P_1-P_2)^2\rangle=(h_1+h_2)-2|h_3|\cos\Psi \tag{14}$$

being smaller than one means having the two-mode squeezing, which also is a sufficient non-separability criterion for a two-mode field in our present system [6].

For the initial cavity field being in vacuum (i.e., $r=0$), from the expression of h_3, the phase angle Ψ is equal to $\phi+\pi/2$. Because Eqs. (11) and (13) are independent of the phase Ψ, which implies that the separability of the entanglement between the two modes is independent of the phase angle ϕ of the driving filed for the initial vacuum input. However, for the two-mode squeezing, from Eq. (14), it depends on ϕ. Obviously to get the maximum squeezing, we should choose the phase angle $\Psi=0$ and $\phi=-\pi/2$.

We plot the time evolution of the variance $\langle (X_1+X_2)^2\rangle$, and negativities $\mu_2(t)$ and $\Delta(t)$ in Fig. 2(a) for the vacuum input. The entanglement and the two-mode squeezing of the two modes in the cavity increase initially, then decrease and finally vanish after a period that depends on Ω/γ. We find that there exists entanglement even when the two-mode squeezing disappears. The entanglement periods (from $t=0$ to the time of the entanglement vanishing) τ_{en} are identical for the Simon's and Duan's criteria determined by $\mu(\tau_{en})=0$ and $\Delta(\tau_{en})=0$. Increasing Ω/γ leads to longer entanglement period τ_{en} [see the inset in Fig. 2(a)]. The mean photon numbers $\langle n_1\rangle$ and $\langle n_2\rangle$ of the two modes are plotted in Fig. 2(b). We can see that, during the entanglement period, the mean photon numbers increase. For $\Omega/\gamma=50$, we can have maximal entanglement degree with total mean photon number above 200. As a result, in order to have entanglement with high intensity, high Rabi frequency (strong driving field) is needed. To maintain the mean photon number of the field growing, according to the expressions of h_1 and h_2, the condition $B_1-B_2+\varpi>\kappa_1+\kappa_2$ should be held. As $\kappa_1+\kappa_2$ is the loss of the cavity, we can set $B_1-B_2+\varpi=\kappa_1+\kappa_2$ as the threshold condition. That is to say the system is operating above the threshold. For example, for $\kappa_1=\kappa_2=0.001\gamma$, $r_a=g_1=g_2=\gamma$, at the threshold (or above), the strength of the driving field is $\Omega/\gamma\leqslant 1000$. When Ω/γ above this value, the system is operating below threshold and the field could not increase continuously. As time goes to infinity, the system reach a quasisteady state, where the entanglement is determined by Eq. (11) with $h_1^s=[C_{11}C_{22}(B_1+\alpha_2)-B_1\alpha_2(\alpha_1+\alpha_2)]/D+1/2$, $h_2^s=-C_{11}^2\kappa_1/D+1/2$ and $h_3^s=ie^{i\phi}\kappa_1C_{11}\alpha_3/D$ with $D=(\alpha_1+\alpha_2)(\alpha_1\alpha_2-C_{11}C_{22})$. For $\Omega/\gamma\gg 1,(\Omega/\gamma)^{-2}$ is much smaller than $(\Omega/\gamma)^{-1}$, and therefore B_1 and B_2 are much smaller than C_{11} and C_{22} and we have approximately $C_{11}=C_{22}=c\approx r_a g_1' g_2'/\Omega'$. Further more, if B_1 and B_2 are much smaller than κ_1 and κ_2, B_1 and B_2 in Eq. (5) can be neglected. As a result Eq. (5) is reduced to the master equation for a nondegenerate parametric amplifier under the vacuum dissipation (parametric approximation). Under this approximation the expressions for h_i $(i=1,2,3)$ are simplified to

$$h_1=h_2=\frac{c[c\cosh(2ct)+\kappa\sinh(2ct)]e^{-2\kappa t}-\kappa^2}{2(c^2-\kappa^2)},$$

$$h_3=ie^{i\phi}\frac{c[c\sinh(2ct)+\kappa\cosh(2ct)]e^{-2\kappa t}-c\kappa}{2(c^2-\kappa^2)}, \tag{15}$$

where we have assumed $\kappa_1=\kappa_2=\kappa$. The total mean photon number of the field becomes $\langle n\rangle=\langle n_1\rangle+\langle n_2\rangle=c[(c$

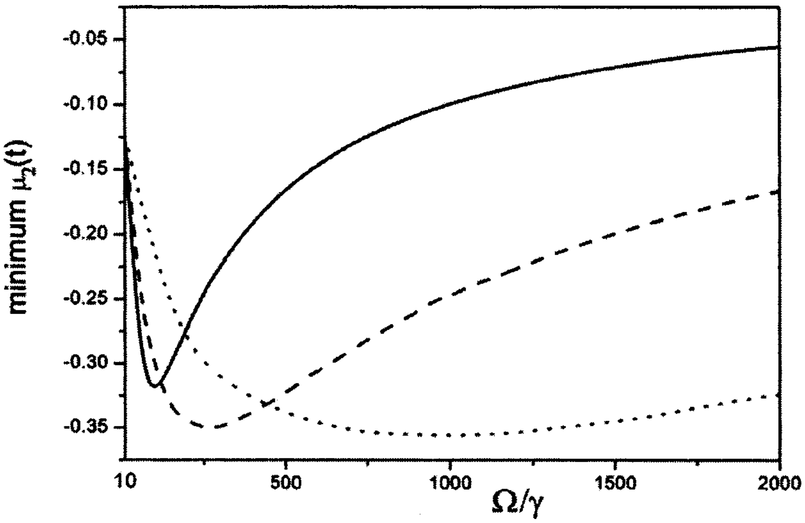

FIG. 3. The minimum $\mu_2(t)$ vs Ω/γ for $r_a=\gamma$, $\kappa_1=\kappa_2=0.001\gamma$, $g_1=g_2=0.5\gamma$ (solid), $g_1=g_2=\gamma$ (dashed), $g_1=g_2=2.0\gamma$ (dotted).

$+\kappa)e^{2(c-\kappa)t}+(c-\kappa)e^{-2(c+\kappa)t}]/2(c^2-\kappa^2)+c^2/(\kappa^2-c^2)$. When the system operates above the threshold, i.e., $c>\kappa$, we have the total photon number increased exponentially. The negativity is $\mu_2=c[e^{-2(c+\kappa)t}-1]/2(c+\kappa)$. So the minimum $\mu_2(t)$ is equal to $-c/2(c+\kappa)\geqslant -0.5$, which increases (or decreases) on the increase of Ω/γ (or g_i). Meanwhile it can also be shown $\Delta_{\min}=4u_{2\min}$, i.e., the two criteria are the same with a difference of scale.

In Fig. 3 we plot the minimum $\mu_2(t)$ (which corresponds to the maximal entanglement degree) as a function of Ω/γ. We find that there exists a critical value of Ω/γ such that in the left region of it, the maximal entanglement degree increases with the increase of Ω/γ, which is in the opposition to that in the right region. As the coupling constant g_i $(i=1,2)$ increases, the entanglement is significantly enhanced for the large values of Ω/γ.

In Fig. 4(a) we plot the entanglement evolution for $r\neq 0$ (i.e., the entangled light input of the initial cavity field being the two-mode squeezed vacuum). It shows that the entanglement period decreases on the increase of the degree of the initial squeezing of the cavity field. Besides the amplification of the intensity, the system also can enhance the entanglement degree for a relative small initial squeezing r. When r exceeds a certain value $(r=0.4)$, although there is no entanglement amplification, we still have the intensity of the entanglement amplified to a certain degree. For example, the mean photon number of the two-mode entangled light is about 4.0 [when $u_2(t)=-0.2$] for the initial squeezed light input with $r=0.6$ which has mean photon number about 0.8 [and $u_2(0)\approx -0.35$]. For the given squeezing parameter r, the enhancement of the entanglement depends on the strength of the diving field. The optimal entanglement increases with the increase of the driving field, see Fig. 4(b). Also we find the entanglement is dependent on the relative phase between the initial squeezed intracavity field and the external driving field, which is plotted in Fig. 4(c). From it the entanglement at $\varphi-\theta=\pi/2$ is larger than that at other values of the relative phase.

B. Entanglement of the output field

Now we consider the entanglement of the output field. As the field outside the cavity is continuum of frequencies, we need to calculate the entanglement between different components of frequencies. Because, for the system operates above the threshold, the steady entanglement cannot be achieved (except for the parametric approximation), the analysis of the entanglement of the output field involves definition of the spectrum of an unstable field, which will make the calculation complicated. So here we only consider the entanglement of the output field under the parametric approximation discussed above. Under this approximation the master equation Eq. (5) can be derived from the effective Hamiltonian (as a parametric amplification Hamiltonian) $H_{\text{int}}^{\text{eff}}=-c(a_1a_2e^{i\phi}+a_1^\dagger a_2^\dagger e^{-i\phi})$. Using the input-output theorem [20], we can easily calculate out the quantity Σ^ω which is defined as the summation of the variances for quadrature variables X_i and P_i

$$\Sigma^\omega=\langle (X_1+X_2)^2\rangle_\omega+\langle (P_1-P_2)^2\rangle_\omega \tag{16}$$

in the frequency domain associated with two frequency components $\omega_1+\omega$ and $\omega_2-\omega$ with a frequency shift ω. According to Ref. [6], if $\Sigma^\omega<2$ we can say sufficiently that the two components are entangled. Here we will do not give the derivation and only present the main results which has been derived in Ref. [21]. For the output field with frequencies $\omega_1+\omega$ and $\omega_2-\omega$, Σ_{out}^ω can be calculated as

$$\Sigma_{out}^\omega=\frac{8[2\tilde{c}^2+\tilde{c}(1+\tilde{c}^2+\tilde{\omega}^2)\sin\phi]}{|(1-i\tilde{\omega})^2-\tilde{c}^2|^2}+2, \tag{17}$$

with the total mean photon number at the frequency shift ω

$$\langle n\rangle_\omega=\frac{8\tilde{c}^2}{|(1-i\tilde{\omega})^2-\tilde{c}^2|^2}, \tag{18}$$

where $\tilde{c}=c/\kappa$ and $\tilde{\omega}=\omega/\kappa$. Here we can see, from Eq. (17), that the maximum entanglement of the output field is obtained at $\phi=-\pi/2$. At the central frequency $(\omega=0)$, we have $\Sigma_{out}^{\omega=0}=2-8\tilde{c}/(1+\tilde{c})^2$. On the threshold $(\tilde{c}=1)$, we can have the perfect entangled light at the central frequency $(\Sigma_{out}^{\omega=0}=0)$. On the other hand, from Eq. (15), we have $\Sigma_{\text{int}ra}=2/(1+\tilde{c})$ for the two-mode intracavity field. So we have $\Sigma_{\text{int}ra}-\Sigma_{out}^{\omega=0}=2\tilde{c}(\tilde{c}-3)/(\tilde{c}+1)^2$, which tells us the entanglement degree of the output field at the central frequency is

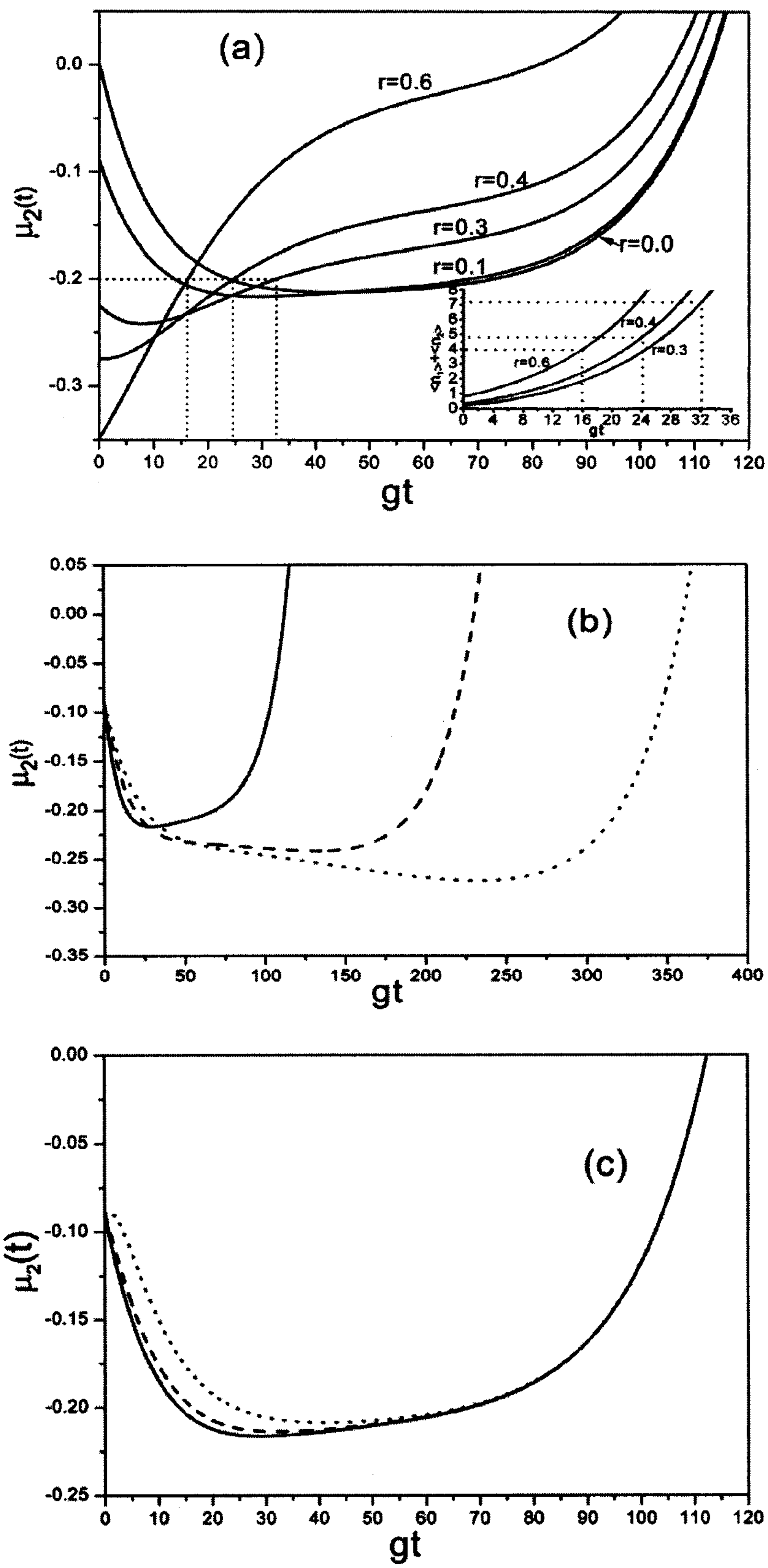

FIG. 4. (a) The entanglement evolution for the initial two-mode squeezed vacuum for $\Omega/\gamma=30$, $\gamma=r_a=g_1=g_2$, $\kappa_1=\kappa_2=0.001\gamma$, $\varphi-\theta=\pi/2$ and the inset plots time development of the total mean photon number. (b) The dependence of the entanglement amplification on the strength of the driving field for $\Omega/\gamma=30$ (solid), $\Omega/\gamma=50$ (dashed), $\Omega/\gamma=70$ (dotted). (c) The dependence of the entanglement on the relative phase angle $\varphi-\theta$ between the driving field and the initial two-mode squeezed vacuum with $r=0.1$, $\varphi-\theta=0(\pi)$ (dotted), $\varphi-\theta=\pi/4$ (dashed), $\varphi-\theta=\pi/2$ (solid). The other parameters are the same as in (a).

stronger than that of the cavity field for $\tilde{c}<3$ under the parametric approximation.

In Figs. 5(a) and 5(b), we plot the dependence of the summation of quantum fluctuations Σ_{out}^{ω} and the total photon number $\langle n\rangle_\omega$ on the frequency $\tilde{\omega}$ for the system operating under, on, or above the threshold. From them we can see that the entanglement degree and the photon number get their maximum at the central frequency and decreases with the increasing of the frequency $\tilde{\omega}$. At the central frequency and near the threshold, the entanglement state of intense field

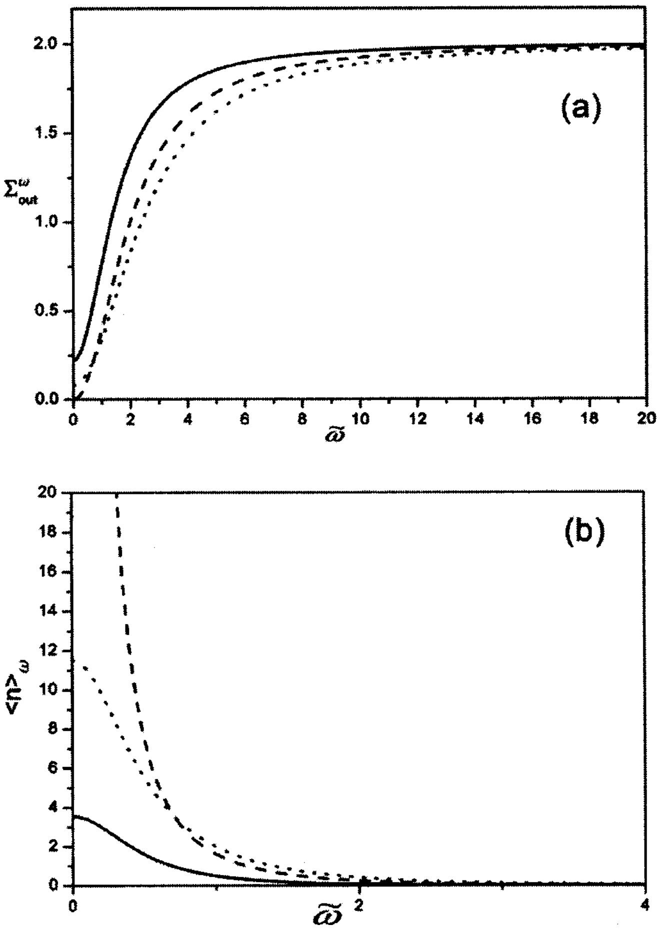

FIG. 5. The dependences of the quantum fluctuations of quadrature variables (a) and the total mean photon number (b) on the dimensionless frequency with $\tilde{c}=0.5$ (solid), and $\tilde{c}=1.0$ (dashed), and $\tilde{c}=1.5$ (dotted).

(which is large number of photons) can be achieved, which can also be seen from Eq. (18).

V. CONCLUSION

In conclusion, the generation and evolution of the entanglement between the two modes in a correlated spontaneous emission laser is investigated. We derive and solve analytically the master equation for the two-mode field in the cavity and obtain the time-dependent characteristic function of the field in the Wigner representation. It shows that the two-mode field in the cavity evolves in a two-mode Gaussian state. The entanglement between the two modes increases initially, then decreases and vanishes eventually as the field evolves from an initial vacuum. The entanglement period is extended as the intensity of the driving field increases. For the present problem, we show that the entanglement periods determined by the separability criteria established, respectively, by Simon and Duan *et al.* are found to be identical under some conditions. The maximal entanglement degree is enhanced with the increase of Ω/γ for Ω/γ being smaller than a certain critical value, and then decreases when the value of Ω/γ exceeds this critical value. The entanglement between the two modes still exists even when the squeezing disappears. The system can also amplify the entangled fields as the initial two modes being entangled (a two-mode squeezed vacuum) with relative small amount of squeezing. During the entanglement period, the intensity of the entangled light is amplified, thus leading to an entanglement amplifier. When the system behaves as a nondegenerate parametric amplifier, we still have, at the central frequency and near the threshold, relative large degree of the entanglement and the intensity of the output field.

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APPENDIX: EXPLICIT FORM OF THE MATRIX ELEMENTS IN Eq. (9)

$$h_1=\frac{n_1\Omega_1^4+4n_2C_{22}^2\Omega_1^2\sinh^2(\varpi t/2)-2iC_{22}\Omega_1^3\sinh(\varpi t/2)(m_c^*e^{-i\phi}-\mathrm{c.c.})}{\varpi^2\Omega_1^2}f-f_1,$$

$$h_2=\frac{n_1C_{11}^2\Omega_1^2\sinh^2(\varpi t/2)+n_2\varpi^2\Omega_2^2-2iC_{11}\Omega_1\Omega_2^2\sinh(\varpi t/2)(m_c^*e^{-i\phi}-\mathrm{c.c.})}{\varpi^2\Omega_1^2}f-f_2,$$

$$h_{11}=\frac{m_1^*\Omega_1^4-4e^{2i\phi}m_2C_{22}^2\Omega_1^2\sinh^2(\varpi t/2)+4ie^{i\phi}m_s^*C_{22}\Omega_1^3\sinh(\varpi t/2)}{\varpi^2\Omega^2}f,$$

$$h_{22}=\frac{-4e^{-2i\phi}m_1C_{11}^2\Omega_1^2\sinh^2(\varpi t/2)+4ie^{i\phi}m_sC_{11}\Omega_1\Omega_2^2\sinh(\varpi t/2)+m_2^*\Omega_2^4}{\varpi^2\Omega_1^2}f$$

$$h_{12}=\frac{-2ie^{-i\phi}m_1^*C_{11}\Omega_1^2\sinh(\varpi t/2)+2ie^{i\phi}m_2C_{22}\Omega_2^2\sinh(\varpi t/2)+m_s^*(2\Omega_2^2-\varpi^2)}{\varpi^2\Omega_1}f,$$

$$h_3 = \frac{2ie^{i\phi}n_1C_{11}\Omega_1^3\sinh(\varpi t/2) + 2ie^{i\phi}n_2C_{22}\Omega_1\Omega_2^2\sinh(\varpi t/2) + m_c^*\Omega_1^2\Omega_2^2 - m_c e^{2i\phi}\Omega_1^2(\Omega_2^2 - \varpi^2)}{\varpi^2\Omega_1^2} \times f - f_3^* = |h_3|e^{i\Psi},$$

$$f_i = \frac{1}{2\varpi^2}[A_{i1}(fe^{\varpi t} - 1) - A_{i2}(fe^{-\varpi t} - 1)] + \frac{A_{i3}}{2\varpi^2}(f-1) \quad (i = 1,2,3),$$

$$A_{11(2)} = \frac{[b_1\varpi \pm (C_{22}^2 - C_{11}C_{22})](\alpha_1 - \alpha_2 \pm \varpi) \pm 2(C_{22}^2 b_2 - C_{11}C_{22}b_1)}{\alpha_1 + \alpha_2 \pm \varpi},$$

$$A_{1(2)3} = \frac{2(C_{11}C_{22} - C_{22}^2)(\alpha_1 - \alpha_2) \pm 4C_{22(11)}(C_{11}b_1 - C_{22}b_2)}{(\alpha_1 + \alpha_2)},$$

$$A_{21(2)} = \frac{[b_2\varpi \pm (C_{11}^2 - C_{11}C_{22})](\alpha_2 - \alpha_1 \pm \varpi) \pm 2(C_{11}^2 b_1 - C_{11}C_{22}b_2)}{\alpha_1 + \alpha_2 \pm \varpi},$$

$$A_{31(2)} = ie^{-i\phi}\frac{\pm 2C_{11}C_{22}(C_{11} - C_{22}) \mp C_{11}b_1(\alpha_1 - \alpha_2 \pm \varpi) \pm b_2C_{22}(\alpha_1 - \alpha_2 \mp \varpi)}{\alpha_1 + \alpha_2 \pm \varpi},$$

$$A_{33} = ie^{-i\phi}\frac{(C_{11} - C_{22})(\alpha_1 - \alpha_2)^2 + 2(C_{11}b_1 - C_{22}b_2)(\alpha_1 - \alpha_2)}{\alpha_1 + \alpha_2},$$

$$f = e^{(\alpha_1+\alpha_2)t},\ \Omega_1 = \varpi\cosh(\varpi t/2) + (\alpha_1 - \alpha_2)\sinh(\varpi t/2),\ \Omega_2 = \sqrt{\Omega_1^2 + 4C_{11}C_{22}\sinh(\varpi t/2)},$$

$$\alpha_1 = B_1 - \kappa_1,\ \alpha_2 = b_2 = -B_2 - \kappa_2,\ \varpi = \sqrt{(\alpha_1 - \alpha_2)^2 + 4C_{11}C_{22}},\ b_1 = -B_1 - \kappa_1.$$

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