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Quantum theory of a micromaser operating on the atomic scattering from a resonant standing wave

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We study the amplification of a resonant standing-wave light field due to the interaction with a beam of monovelocity two-level atoms moving in the Raman-Nath regime and in the Bragg regime. The atomic density is low so that, at most, one atom is inside the cavity at a time. This system is very similar to the well-known micromaser but it is operating in the optical region of the field frequencies. Therefore, the situation corresponds to a microlaser. Unlike the micromaser system, the momentum transfer between the atoms and photons in the microlaser essentially effects the center-of-mass motion of the atoms and the evolution of the field.

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I. INTRODUCTION

Experimental realization of micromasers [1–3] has set the stage to comprehend the fundamental concepts, such as complimentarity and the nonlocal nature of the quantum world. In recent years, the invention of superconducting microcavities of high quality factors has made it possible to extend the subject to the cavity field of optical wavelength. This work has given birth to the field of microlasers [4,5].

In microlasers the de Broglie wavelength of atoms is comparable to the optical wavelength of the cavity field. Moreover, during an atom-field interaction, the momentum transfer from the atom to the field is inversely proportional to the wavelength of the field. Hence, in contrast to the micromasers, there occurs a much larger momentum transfer from the atom to the field. As a result the recoil effects may become significantly large and the inclusion of atomic scattering from the cavity field becomes important. This naturally has effects on the photon statistics of the cavity field [6,7].

In this paper we study the amplification of the optical field of a microcavity via resonant atoms. The momentum transfer between the atom and the field essentially effects the center-of-mass motion. Therefore, for our study, we consider two regimes of atomic deflection: (i) the Raman-Nath regime and (ii) the Bragg regime. In the Raman-Nath regime the recoil energy is much smaller than the interaction energy resulting in momentum transfer in a number of directions [8]. On the other hand, in the Bragg regime the recoil energy very much exceeds the interaction energy. Under a resonant condition the energy and momentum conservation restricts to only two directions [9–11].

We show that in the Raman-Nath regime the probability amplitudes are having periodic dependence on interaction time and for these values of interaction time the field inside the cavity becomes transparent to the atom. The atom leaves the cavity without contributing to photon statistics. In the Bragg regime we have a multiphoton process for the growth of the optical field in the microcavity. In the Bragg regime, the recoil frequency is much larger as compared with the Rabi frequency and there is an appreciable transfer of energy from the cavity field to the incoming atom in the form of an integral multiple of field quanta. As a result of energy conservation, there exists two values for the transferred field momenta corresponding to initial momentum of the atom. The atom that is initially coming in an excited state leaves the cavity after contributing photons to the cavity. The number of photons contributed to the cavity are proportional to the initial momentum of the atom.

The paper is organized as follows: In Sec. II we describe scattering of an atom in an optical regime by a quantized standing-wave field and lay foundations for our analytical calculations. In Sec. III we construct a master equation for the field in a microlaser operating on the atomic scattering. We examine the photon statistics of the field in Sec. IV. We devote Sec. V to study the decay of off-diagonal matrix elements and the linewidth of the microlaser field. We conclude our paper with discussion of our results in Sec. VI.

II. SCATTERING OF AN ATOM BY A QUANTUM RESONANT STANDING-WAVE FIELD

We consider a single two-level atom of mass M passing with the velocity v_z along the z axis through a single standing-wave mode of the electromagnetic field in a cavity. The cavity is aligned along the x axis. We neglect damping of the cavity mode and the spontaneous emission of the atom into other field modes. In order to study the interaction of an atom with the single-mode field we consider the dipole approximation and rotating-wave approximations. The initial momentum of the atomic center-of-mass motion along the z direction, Mv_z, is assumed to be large enough; therefore, we treat it classically. However, we quantize the motion of the atom along the standing-wave mode, together with the internal degrees of freedom of the atom, and the cavity field. The field is uniform in the y direction, and we do not consider motion of the atom in this direction.

Hence, the Hamiltonian of the atom-field system is

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$$\hat{H}\equiv\frac{\hat{p}^2}{2M}+\frac{1}{2}\hbar\omega_0\hat{\sigma}_z+\hbar\omega\hat{a}^\dagger\hat{a}+\hbar g\cos(k\hat{x})(\hat{\sigma}_+\hat{a}+\hat{\sigma}_-\hat{a}^\dagger). \tag{1}$$

Here, the operators $\hat{x}$ and $\hat{p}$ describe the position and momentum of the atom along the cavity mode, and $\hat{\sigma}_z$ and $\hat{\sigma}_\pm$ are the Pauli spin operators. Moreover, $\hat{a}$ and $\hat{a}^\dagger$ denote the annihilation and creation operators of the cavity mode, ω_0 is the transition frequency of the atom, and ω is the frequency of the cavity mode. We describe the vacuum Rabi frequency as g, and the wave number of the cavity mode as k. For simplicity, we consider the exact resonance between the field and the atom, that is $\omega=\omega_0$.

In the interaction picture, the atom-field system is described by the Hamiltonian

$$\hat{H}_{\rm int}\equiv\frac{\hat{p}^2}{2M}+\hbar g\cos(k\hat{x})(\hat{\sigma}_+\hat{a}+\hat{\sigma}_-\hat{a}^\dagger). \tag{2}$$

The corresponding time evolution of the combined system is governed by the unitary operator,

$$\hat{U}(t)=\exp(-i\hat{H}_{\rm int}t/\hbar). \tag{3}$$

Starting from an initial atomic momentum eigenstate $|p_0\rangle$, the energy operator $\hat{H}_{\rm int}$ expressed in Eq. (2) mixes in only the components $|p_0+l\hbar k\rangle$, where l is any integer. Since the atom is initially prepared in the excited internal state $|a\rangle$, and the field is initially in the number state $|n\rangle$, the wave function of the combined system after interaction time t is

$$\hat{U}(t)|p_0,a,n\rangle=\exp\left(-i\frac{p_0^2t}{2M\hbar}\right)\sum_{l=-\infty}^{\infty}|p_l\rangle[C_l^{(a,n)}(t)|a,n\rangle+C_l^{(b,n+1)}(t)|b,n+1\rangle], \tag{4}$$

where we express the momentum of the atoms after interaction as $p_l=p_0+l\hbar k$.

The probability amplitudes $C_l^{a,n}$ and $C_l^{b,n+1}$ are governed by the Schrödinger equations

$$i\frac{d}{dt}C_l^{(a,n)}(t)=(\Delta_0 l+\omega_{\rm r}l^2)C_l^{(a,n)}(t)+\frac{1}{2}g\sqrt{n+1}[C_{l+1}^{(b,n+1)}(t)+C_{l-1}^{(b,n+1)}(t)], \tag{5}$$

$$i\frac{d}{dt}C_l^{(b,n+1)}(t)=(\Delta_0 l+\omega_{\rm r}l^2)C_l^{(b,n+1)}(t)+\frac{1}{2}g\sqrt{n+1}[C_{l+1}^{(a,n)}(t)+C_{l-1}^{(a,n)}(t)], \tag{6}$$

with the initial conditions defined as

$$C_l^{(a,n)}(0)=\delta_{l,0},\qquad C_l^{(b,n+1)}(0)=0. \tag{7}$$

Here we have the initial Doppler shift $\Delta_0\equiv p_0k/M$ and the recoil frequency, associated with a photon recoil, $\omega_r=\hbar k^2/2M$. Note that the normalization condition

$$\sum_{l=-\infty}^{\infty}[|C_l^{(a,n)}(t)|^2+|C_l^{(b,n+1)}(t)|^2]=1 \tag{8}$$

of the wave function is automatically satisfied.

A. Raman-Nath regime

A simple analytical solution of Eqs. (5) and (6) can be achieved when the recoil energy is smaller than the interaction energy, that is,

$$\omega_r\ll g\sqrt{n+1}. \tag{9}$$

It is the Raman-Nath regime [10]. For this regime, we [12] find the analytical solution as

$$C_{2l}^{(a,n)}(t)=\exp[-i(\Delta_0t+\pi)l]J_{2l}\left(\frac{2g\sqrt{n+1}}{\Delta_0}\sin\frac{\Delta_0t}{2}\right), \tag{10}$$

$$C_{2l+1}^{(b,n+1)}(t)=\exp[-i(\Delta_0t+\pi)(l+1/2)]J_{2l+1}\left(\frac{2g\sqrt{n+1}}{\Delta_0}\sin\frac{\Delta_0t}{2}\right), \tag{11}$$

where J_m is the mth-order Bessel function and all the other probability amplitudes are zero.

Note that, in the case $\Delta_0\neq0$, which implies that $p_0\neq0$, the probability amplitudes $C_l^{(a,n)}(t)$ and $C_l^{(b,n+1)}(t)$ are periodic functions of t with the periodicity

$$T=2\pi/\Delta_0=\lambda/v_0, \tag{12}$$

where $\lambda=2\pi/k$ is the wavelength of the field and $v_0=p_0/M$ is the velocity of the atomic motion along the cavity.

If the interaction time t is an integer multiple of the period, given in Eq. (12), we have $C_0^{(a,n)}(t)=1$ while all the other probability amplitudes are zero. For these values of the time of interaction the atom completes full cycles and leaves the cavity in an excited state without contributing to the cavity field. Hence, in these cases, the cavity field, the atomic internal state, and the atomic center-of-mass motion remain unchanged. This implies that in this case the quantum standing-wave field, acting as a medium, becomes perfectly transparent with respect to the incident atomic de Broglie wave. The atom leaves the cavity after a displacement v_0t which is an integer multiple of the wavelength of the field.

B. Bragg regime

In the Bragg regime the recoil energy is larger than the interaction energy [10],

$$\omega_r\gg g\sqrt{n+1}. \tag{13}$$

In this case, conservation of the kinetic energy becomes important leading to resonance between the two solutions of the equation,

$$\Delta_0 l+\omega_r l^2=0. \tag{14}$$

The solution $l=0$ corresponds to the incoming atomic beam, and the second solution to Eq. (14),

$$l=-l_0\equiv-\frac{\Delta_0}{\omega_r}=-\frac{2p_0}{\hbar k}, \tag{15}$$

gives the scattered component, conserving energy and momentum. The relation expressed in Eq. (15) corresponds to the Bragg condition in x-ray scattering from crystals.

When the condition expressed in Eq. (15) is fulfilled for a positive integer l_0, an adiabatic approximation allows us to eliminate the amplitudes corresponding to the nonresonant atomic beams. Following the procedures of Ref. [10], we find the analytical solution

$$C_0^{(a,n)}(t)=\exp[-i(n+1)\nu t]\cos[(n+1)^{l_0/2}\kappa t] \tag{16}$$

for the incoming atomic beam. Here we have introduced the notations ν and κ, which we define as

$$\nu=\begin{cases}-g^2/8\omega_r, & l_0=1\\ g^2/2\omega_r(l_0^2-1), & l_0>1\end{cases} \tag{17}$$

and

$$\kappa=\frac{g^{l_0}}{2^{l_0}\omega_r^{l_0-1}[(l_0-1)!]^2}. \tag{18}$$

For the scattered beam, we find the probability amplitude

$$C_{-l_0}^{(a,n)}(t)=-i\exp[-i(n+1)\nu t]\sin[(n+1)^{l_0/2}\kappa t] \tag{19}$$

if l_0 is even, or

$$C_{-l_0}^{(b,n+1)}(t)=-i\exp[-i(n+1)\nu t]\sin[(n+1)^{l_0/2}\kappa t] \tag{20}$$

if l_0 is odd, whereas all the other probability amplitudes are negligible.

The above equations show that the flipping between the two resonant momentum eigenstates $|p_0\rangle$ and $|p_{-l_0}\rangle$ occurs at the frequency $(n+1)^{l_0/2}\kappa$. In addition to the main flipping, there is a modulation at the frequency $(n+1)\nu$. Since $\kappa/g \propto(g/\omega_r)^{l_0-1}$ and $\nu/g\propto g/\omega_r$, the flipping frequency is larger, of the same order, or smaller than the modulation frequency if l_0 is equal to one, two, or larger than two, respectively.

III. MICROLASER OPERATING ON ATOMIC SCATTERING

In this section we study a micromaser operating on the scattering of the atoms. We take the atoms initially excited and prepared in the transverse-momentum eigenstate $|p_0\rangle$. They pass through the cavity according to a Poissonian process with an average rate r. The atomic flux is assumed to be so low that only one atom is in the cavity at a time. We consider the atomic decay as negligible. The time of interaction of each atom with the cavity field is much shorter than the cavity damping time so that the relaxation of the cavity field can be ignored while an atom is inside the cavity. For simplicity, we suppose that the injected atoms have the same longitudinal velocity and, therefore, interact with the cavity field for the same interval of time τ.

The time evolution of the density matrix [6,7,13], that is,

$$\hat{\varrho}=\sum_{n,n'=0}^{\infty}\varrho_{n,n'}|n\rangle\langle n'| \tag{21}$$

of the cavity field in the interaction picture is governed by the equation

$$\dot{\hat{\varrho}}=r\,\delta_\tau\hat{\varrho}+L\hat{\varrho}. \tag{22}$$

Here, $\delta_\tau\hat{\varrho}$ is the change in $\hat{\varrho}$ due to an atom interacting with the field for the time τ, and $L\hat{\varrho}$ is the Liouvillian operator which describes losses due to the coupling of the cavity mode to a thermal bath.

The expression of the gain operator $\delta_\tau\hat{\varrho}$ is

$$\delta_\tau\hat{\varrho}=\mathrm{Tr_A}\{\hat{U}(\tau)(|p_0,a\rangle\langle p_0,a|\otimes\hat{\varrho})\hat{U}^\dagger(\tau)\}-\hat{\varrho}. \tag{23}$$

When we use Eq. (4), the matrix elements of the gain operator are found to be

$$\langle n|\delta_\tau\hat{\varrho}|n'\rangle=-a_{n,n'}\varrho_{n,n'}+b_{n-1,n'-1}\varrho_{n-1,n'-1}, \tag{24}$$

where

$$a_{n,n'}=1-\sum_{l=-\infty}^{\infty}C_l^{(a,n)}(\tau)C_l^{*(a,n')}(\tau),$$

$$b_{n,n'}=\sum_{l=-\infty}^{\infty}C_l^{(b,n+1)}(\tau)C_l^{*(b,n'+1)}(\tau). \tag{25}$$

It is interesting to note that the normalization condition expressed in Eq. (8) yields $a_{n,n}=b_{n,n}$.

The loss operator $L\hat{\varrho}$ [6,7,13] is given by

$$L\hat{\varrho}=\frac{1}{2}\mathcal{C}(n_T+1)(2\hat{a}\hat{\varrho}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}\hat{\varrho}-\hat{\varrho}\hat{a}^\dagger\hat{a})+\frac{1}{2}\mathcal{C}n_T(2\hat{a}^\dagger\hat{\varrho}\hat{a}-\hat{a}\hat{a}^\dagger\hat{\varrho}-\hat{\varrho}\hat{a}\hat{a}^\dagger). \tag{26}$$

Here n_T is the number of photons in thermal equilibrium, and $\mathcal{C}$ is the cavity damping rate. The matrix elements of the loss operator read

$$\langle n|L\hat{\varrho}|n'\rangle = \mathcal{C}(n_T+1)\left[\sqrt{(n+1)(n'+1)}\varrho_{n+1,n'+1} - \frac{1}{2}(n+n')\varrho_{n,n'}\right] + \mathcal{C}n_T\left[\sqrt{nn'}\varrho_{n-1,n'-1} - \frac{1}{2}(n+n'+2)\varrho_{n,n'}\right]. \quad (27)$$

By substituting Eqs. (24) and (27) into Eq. (22), we obtain an expression for $\dot{\varrho}_{n,n'}$, such that

$$\dot{\varrho}_{n,n'} = -ra_{n,n'}\varrho_{n,n'} + rb_{n-1,n'-1}\varrho_{n-1,n'-1} + \mathcal{C}(n_T+1)\times\left[\sqrt{(n+1)(n'+1)}\varrho_{n+1,n'+1} - \frac{1}{2}(n+n')\varrho_{n,n'}\right] + \mathcal{C}n_T\left[\sqrt{nn'}\varrho_{n-1,n'-1} - \frac{1}{2}(n+n'+2)\varrho_{n,n'}\right]. \quad (28)$$

As it stands, the master equation (28) is the basic equation for the micromaser operating on the scattering of two-level atoms.

IV. PHOTON STATISTICS OF THE MICROLASER FIELD

We may obtain a rate equation for $\varrho_{n,n} \equiv P_n$ from Eq. (28) as

$$\dot{P}_n = -ra_nP_n + ra_{n-1}P_{n-1} + \mathcal{C}(n_T+1)[(n+1)P_{n+1} - nP_n] + \mathcal{C}n_T[nP_{n-1} - (n+1)P_n]. \quad (29)$$

Here P_n is the probability of having n photons in the cavity and

$$a_n = \sum_{l=-\infty}^{\infty} |C_l^{(b,n+1)}(\tau)|^2, \quad (30)$$

is the probability for the scattered atom to go from the excited state to the ground state in the presence of n photons in the cavity. Various terms on the right-hand side of Eq. (29) can be interpreted as outflow and inflow of probabilities.

For the mean photon number, we find from Eq. (29)

$$\langle \dot{n} \rangle = r\langle a_n\rangle - \mathcal{C}(\langle n\rangle - n_T). \quad (31)$$

Here we have introduced the notation $\langle f_n\rangle = \Sigma_{n=0}^{\infty}(f_n)P_n$ for any arbitrary function f_n. Clearly, the term $r\langle a_n\rangle$ determines the gain in the field while $\mathcal{C}(\langle n\rangle - n_T)$ characterizes the loss.

In steady state, the function P_n is independent of time. Therefore, in Eq. (29), by setting the time derivative of P_n equal to zero, we obtain an equation for steady-state photon distribution. Under the detailed balance condition, the equation reduces to

$$(ra_{n-1} + \mathcal{C}n_T n)P_{n-1} = \mathcal{C}(n_T+1)nP_n. \quad (32)$$

Hence, the steady-state solution for the photon distribution is

$$P_n = P_0 \prod_{m=1}^{n} \frac{n_T + (r/\mathcal{C})a_{m-1}/m}{n_T+1}, \quad (33)$$

where P_0 is determined by the normalization condition $\Sigma_{n=0}^{\infty}P_n = 1$.

In order to study the peak structure of P_n, we approximate this distribution by the continuous function

$$\tilde{P}_n = P_0 \exp\left\{\int_0^n dm \ln\left[\frac{n_T + (r/\mathcal{C})a_{m-1}/m}{n_T+1}\right]\right\}. \quad (34)$$

Here, we have replaced $\Pi_{m=1}^{n}(f_m)$ in Eq. (33) by $\exp[\int_0^n dm \ln(f_m)]$. The peak n_0 of the photon distribution P_n is determined approximately by the equation

$$n_0 = (r/\mathcal{C})a_{n_0-1}. \quad (35)$$

Below, we calculate the emission probability a_n, and then study in detail the photon statistics of the micromaser field operating in the Raman-Nath regime and in the Bragg regime.

A. Raman-Nath regime

We first consider the case when the micromaser is operating in the Raman-Nath regime of the atomic scattering. In this regime, the amplitudes $C_l^{(b,n+1)}(\tau)$ are given by Eq. (11). On substituting the expression for probability amlitude $C_l^{(b,n+1)}$ from Eq. (11) in Eq. (30), and by using the relation [14]

$$\sum_{l=-\infty}^{\infty} J_{2l+1}^2(\xi) = \frac{1}{2}[1 - J_0(2\xi)], \quad (36)$$

we find the coefficent a_n as

$$a_n = \frac{1}{2}\left[1 - J_0\left(\frac{4g\sqrt{n+1}}{\Delta_0}\sin\frac{\Delta_0\tau}{2}\right)\right], \quad (37)$$

which describes emission probability. This expression together with Eq. (29) describes the evolution of the photon distribution in the micromaser operating in the Raman-Nath regime of the atomic scattering.

It is interesting to note that when $\Delta_0 \neq 0$, that is when $p_0 \neq 0$, the coefficients a_n and, therefore, the photon distribution P_n are periodic functions of the interaction time τ, with the period $2\pi/\Delta_0$. Furthermore, when τ is an integer multiple of $2\pi/\Delta_0$, we have $a_n = 0$ for any n. In this case, there is no gain in the field, and the evolution of the photon distribution P_n is solely determined by the loss mechanism.

When $\Delta_0 = 0$, that is, when the atoms hit the cavity field orthogonally, we have

$$a_n = \frac{1}{2}[1 - J_0(2g\tau\sqrt{n+1})]. \quad (38)$$

In this case, there is no periodicity of a_n and P_n in τ. Since $|J_0(\xi)| < 1$ for $\xi \neq 0$, we have $a_n > 0$ for any n and any nonzero τ. The fact that $a_n \neq 0$ for any n and any nonzero τ

shows that there exists no trapping state. Due to the decay of the Bessel function J_0 with increasing argument, we have $a_n \rightarrow 1/2$ when $g\tau \rightarrow \infty$. Therefore, when the normalized interaction time $g\tau$ is large and the mean number n_T of thermal photons in the bath is zero, the steady-state photon distribution, given in Eq. (33), approaches the Poisson distribution $\exp(-\langle n\rangle)\langle n\rangle^n/n!$ with the mean photon number $\langle n\rangle = r/2\mathcal{C}$. This feature was not observed in the standard micromaser [16,17], where the oscillations of the emission probability do not decay.

When the interaction time τ is small, so that

$$\Delta_0\tau \ll 1, \quad g\tau\langle n\rangle^{1/2} \ll 1, \tag{39}$$

we can expand a_n to the second order in τ, and obtain

$$a_n = \frac{1}{2}(g\tau)^2(n+1). \tag{40}$$

Apart from the prefactor 1/2, the expression (40) is the same as that for the probability of a one-photon transition in a two-level atom in the second-order perturbation theory. The prefactor 1/2 results from the spread of the atomic position along the standing-wave cavity field.

B. Bragg regime

We now consider the case when the micromaser is operating in the Bragg regime of the atomic scattering. In this regime the amplitudes $C_l^{(b,n+1)}(\tau)$ are given by Eq. (20). On substituting Eq. (20) into Eq. (30), we find

$$a_n = \sin^2[(n+1)^{l_0/2}\kappa\tau]. \tag{41}$$

This expression, together with Eq. (29), describes the evolution of the photon distribution in the micromaser operating on the Bragg scattering of the atoms. Apart from the definition, given in Eq. (18) for κ, Eq. (41) is similar to the probability for a l_0-photon transition in a two-level atom. Hence, the operation of the micromaser in the Bragg regime of the atomic scattering is a multiphoton action. Note that if the interaction time τ is chosen so that for integer numbers n_0 and q we have

$$(n_0+1)^{l_0/2}\kappa\tau = q\pi, \tag{42}$$

then we have $a_{n_0} = 0$. In this case the steady-state photon distribution of Eq. (33), for $n_T = 0$, is truncated at the value n_0, that is, the field trapping phenomenon [16] occurs.

When the interaction time τ is small so that

$$\kappa\tau\langle n\rangle^{l_0/2} \ll 1, \tag{43}$$

we can expand a_n to the second order in τ and obtain

$$a_n = (\kappa\tau)^2(n+1)^{l_0}. \tag{44}$$

V. DECAY OF OFF-DIAGONAL DENSITY MATRIX ELEMENTS AND THE LINEWIDTH

In this section we study the decay of off-diagonal density matrix elements and the linewidths of the field mode, following the approach of Ref. [15].

When we add and subtract appropriate terms, we can rewrite Eq. (28) for the matrix elements $\varrho_n^{(q)} \equiv \varrho_{n,n+q}$ in the form

$$\dot{\varrho}_n^{(q)} = -\mu_n^{(q)}\varrho_n^{(q)} + (\mathcal{O}\varrho)_n^{(q)}, \tag{45}$$

where

$$\mu_n^{(q)} = r u_n^{(q)} + c_n^{(q)} \tag{46}$$

and

$$\begin{aligned}(\mathcal{O}\varrho)_n^{(q)} = {} & r b_{n-1}^{(q)}\varrho_{n-1}^{(q)} - r b_n^{(q)}\varrho_n^{(q)} \\ & + \mathcal{C}(n_T+1)\sqrt{(n+1)(n+q+1)}\,\varrho_{n+1}^{(q)} \\ & - \mathcal{C}(n_T+1)\sqrt{n(n+q)}\,\varrho_n^{(q)} + \mathcal{C}n_T\sqrt{n(n+q)}\,\varrho_{n-1}^{(q)} \\ & - \mathcal{C}n_T\sqrt{(n+1)(n+q+1)}\,\varrho_n^{(q)}.\end{aligned} \tag{47}$$

Here, we have introduced the notation

$$u_n^{(q)} = a_{n,n+q} - b_{n,n+q}, \tag{48}$$

$$b_n^{(q)} = b_{n,n+q}, \tag{49}$$

and

$$\begin{aligned}c_n^{(q)} = {} & \mathcal{C}(n_T+1)(n+q/2) + \mathcal{C}n_T(n+q/2+1) \\ & - \mathcal{C}(n_T+1)\sqrt{n(n+q)} - \mathcal{C}n_T\sqrt{(n+1)(n+q+1)}.\end{aligned} \tag{50}$$

It is clear that the expression given in Eq. (47) for $(\mathcal{O}\varrho)_n^{(q)}$ consists of the terms that can be interpreted as outflows and inflows of $\varrho_n^{(q)}$. Therefore, the operator $\mathcal{O}$ can be considered as an operator for the right-hand side of a kinetic equation, $\dot{\tilde{\varrho}}_n^{(q)} = (\mathcal{O}\tilde{\varrho})_n^{(q)}$. We call $\overline{\varrho}_n^{(q)}$ a steady-state solution of this kinetic equation,

$$(\mathcal{O}\overline{\varrho})_n^{(q)} = 0. \tag{51}$$

We assume that the initial density matrix of the field is $\overline{\varrho}_n^{(q)}$. We study the decay of the off-diagonal matrix elements during the time when the field state is still near to the initial state $\overline{\varrho}_n^{(q)}$. When we substitute the ansatz

$$\varrho_n^{(q)}(\tau) = \exp[-\mathcal{D}_n^{(q)}(\tau)]\,\overline{\varrho}_n^{(q)} \tag{52}$$

into Eq. (45) and assume that

$$\mathcal{D}_{n\pm 1}^{(q)}(\tau) \cong \mathcal{D}_n^{(q)}(\tau), \tag{53}$$

we find

$$\dot{\mathcal{D}}_n^{(q)}(\tau)=\mu_n^{(q)}. \tag{54}$$

Taking into account the initial condition $\mathcal{D}_n^{(q)}(0)=0$, we find from Eq. (54)

$$\mathcal{D}_n^{(q)}(\tau)=\mu_n^{(q)}\tau. \tag{55}$$

The condition (53) is fulfilled when

$$\left|\frac{\partial}{\partial n}\mu_n^{(q)}\right|\tau\ll 1, \tag{56}$$

that is, (i) for short times t and (ii) when $\mu_n^{(q)}$, given in Eq. (46), is a slowly varying function of n.

According to Eqs. (52) and (55), the coefficient $\mu_n^{(q)}$ characterizes the phase diffusion of the matrix element $\varrho_n^{(q)}$ from the initial value $\overline{\varrho_n^{(q)}}$. Its real part is associated with the exponential decay of the field. In contrast to the standard micromaser [15] the coefficient $\mu_n^{(q)}$ here may have an imaginary part that which corresponds to the frequency shift of the field. We, therefore, present the coefficient $\mu_n^{(q)}$ as

$$\mu_n^{(q)}=\gamma_n^{(q)}+i\nu_n^{(q)}, \tag{57}$$

where $\gamma_n^{(q)}$ and $\nu_n^{(q)}$ are the real and imaginary parts, respectively. Then, from Eq. (46), we find

$$\gamma_n^{(q)}=r\,\mathrm{Re}\,u_n^{(q)}+c_n^{(q)}, \tag{58}$$

$$\nu_n^{(q)}=r\,\mathrm{Im}\,u_n^{(q)}. \tag{59}$$

On substituting Eq. (25) into Eq. (48), we obtain

$$u_n^{(q)}=1-\sum_{l=-\infty}^{\infty}[C_l^{(a,n)}(\tau)C_l^{*(a,n+q)}(\tau)+C_l^{(b,n+1)}(\tau)C_l^{*(b,n+q+1)}(\tau)]. \tag{60}$$

By using the normalization condition given in Eq. (8), we find that Eq. (60) reduces to $u_n^{(0)}=0$. On the other hand, Eq. (50) yields $c_n^{(0)}=0$. Hence, we have

$$\gamma_n^{(0)}=\nu_n^{(0)}=0. \tag{61}$$

This formula indicates that the diagonal matrix elements $\varrho_n^{(0)}$ do not decay.

The coefficients $\gamma_n^{(1)}$ determine the decay rates of the first-off-diagonal matrix elements $\varrho_n^{(1)}$ and, consequently, the linewidth of the field mode [6,7,13]. The coefficients $\nu_n^{(1)}$ characterize the frequency shift. In general, the coefficients $\gamma_n^{(1)}$ and $\nu_n^{(1)}$ depend on n. Therefore, the spectrum of the field is not a Lorentzian line but a sum of Lorentzian distributions. It is not easy to find the exact expressions for the linewidth and the frequency shift of such a spectrum. Following Refs. [13,6,7,15], we estimate the linewidth and the frequency shift by

$$D\equiv 2\gamma_{\langle n\rangle}^{(1)}=2[r\,\mathrm{Re}\,u_{\langle n\rangle}^{(1)}+c_{\langle n\rangle}^{(1)}] \tag{62}$$

and

$$\Omega\equiv-\nu_{\langle n\rangle}^{(1)}=-r\,\mathrm{Im}\,u_{\langle n\rangle}^{(1)}, \tag{63}$$

respectively. Here $\langle n\rangle$ is the steady-state mean number of photons.

Below, we examine in more detail the linewidth and the frequency shift of the field in the Raman-Nath and Bragg regimes.

A. Raman-Nath regime

In the Raman-Nath regime, the probability amplitudes $C_l^{(a,n)}(\tau)$ and $C_l^{(b,n+1)}(\tau)$ are given by Eqs. (10) and (11), respectively. We substitute these probability amplitudes in Eq. (60) and perform the summation. With the help of the relation [14]

$$\sum_{l=-\infty}^{\infty}J_l(\xi)J_l(\xi')=J_0(\xi-\xi'), \tag{64}$$

we obtain

$$u_n^{(q)}=1-J_0[(2g/\Delta_0)\sin(\Delta_0\tau/2)(\sqrt{n+q+1}-\sqrt{n+1})]. \tag{65}$$

Since the above expression for $u_n^{(q)}$ is real, we find $\nu_n^{(q)}=0$ and $\Omega=0$, that is, there is no frequency shift of the field in the microlaser operating in the Raman-Nath regime.

We now assume that $\langle n\rangle\gg 1$. In this case, we find from Eq. (65) the approximate expression

$$u_{\langle n\rangle}^{(1)}\cong 1-J_0[(g/\Delta_0\sqrt{\langle n\rangle})\sin(\Delta_0\tau/2)]. \tag{66}$$

The corresponding evaluation for $c_{\langle n\rangle}^{(1)}$ is found from Eq. (50) to be

$$c_{\langle n\rangle}^{(1)}\cong\frac{\mathcal{C}(2n_T+1)}{8\langle n\rangle}. \tag{67}$$

When we substitute Eqs. (66) and (67) into Eq. (62), we arrive at the expression

$$D\cong 2r\left[1-J_0\left(\frac{g}{\Delta_0\sqrt{\langle n\rangle}}\sin\frac{\Delta_0\tau}{2}\right)\right]+\frac{\mathcal{C}(2n_T+1)}{4\langle n\rangle} \tag{68}$$

for the linewidth of the field.

B. Bragg regime

In the Bragg regime the amplitudes $C_l^{(a,n)}(\tau)$ and $C_l^{(b,n+1)}(\tau)$ are given by Eqs. (16) and (20), respectively. When we insert these equations in Eq. (60), we find

$$u_n^{(q)}=1-e^{iq\nu\tau}\cos\{[(n+q+1)^{l_0/2}-(n+1)^{l_0/2}]\kappa\tau\}. \tag{69}$$

In the case $\langle n\rangle\gg 1$, we find the approximate expression

$$u_{\langle n\rangle}^{(1)}\cong 1-e^{i\nu\tau}\cos[(l_0/2)\langle n\rangle^{l_0/2-1}\kappa\tau]. \tag{70}$$

When we substitute this equation into Eqs. (62) and (63) and use Eq. (67), we obtain the linewidth as

$$D \cong 2r\left[1-\cos(\nu\tau)\cos\left(\frac{1}{2}l_0\langle n\rangle^{l_0/2-1}\kappa\tau\right)\right]+\frac{\mathcal{C}(2n_T+1)}{4\langle n\rangle} \tag{71}$$

and the frequency shift as

$$\Omega \cong r\,\sin(\nu\tau)\cos\left(\frac{1}{2}l_0\langle n\rangle^{l_0/2-1}\kappa\tau\right), \tag{72}$$

of the field in the microlaser operating on the atomic Bragg scattering.

VI. DISCUSSION

In this paper we have studied the microlasers in two optical regimes, namely, the Raman-Nath regime and the Bragg regime. The set of basic equations is obtained by considering the preparation of an atom in a specific momentum state, and taken to be in resonant with the field of the cavity.

In the Raman-Nath regime, the cavity field remains uniform on the dimension of the cavity, and the recoil frequency is very low compared with the Rabi frequency. Therefore, there is no appreciable energy transfer from the cavity to the atom, hence, the momentum of the atom is required to be conserved. The cavity mode function contributes in the form of Bessel behavior to the probability amplitudes.

In the case of zero transversal momentum, that is when the atom is incident perpendicularly at the field, the probability amplitudes are no more periodic functions of interaction time. This implies that for a large interaction time the gain of the medium becomes constant which indicates a steady state, and if the mean thermal photon numbers are zero, the photon distribution becomes just Poissonian with the mean photon numbers $\langle n\rangle = r/2C$. This feature is again to the credit of the microlaser. At the onset of the experiment the photon distribution inside the cavity is thermal, which changes as the atoms pass through the cavity with fluctuations in mean photon number depending upon the Rabi frequency and after a few oscillations it gets a constant value of mean photons with a Poisson distribution of photons. This feature can be seen from Fig. 1(a), where the average number of photons is plotted as a function of scaled interaction time.

In case the initial transversal momentum of the atom is nonzero, the probability amplitudes are having periodic dependence on interaction time with a period $T=2\pi/\Delta_0$. This interesting behavior adds up more properties to the system. Initially the average number of photons observes the same behavior as in the case of zero transversal momentum; however, after half of the period, $T/2$, it just reverses and comes to the initial value when the interaction time becomes equal to one period. Physically, for these values of interaction time the field inside the cavity becomes transparent to the atom and displays, therefore, thermal distribution. The atom leaves the cavity without contributing to photon statistics. A complete cycle of the photon distribution function resulting from this phenomenon is shown in Fig. 3.

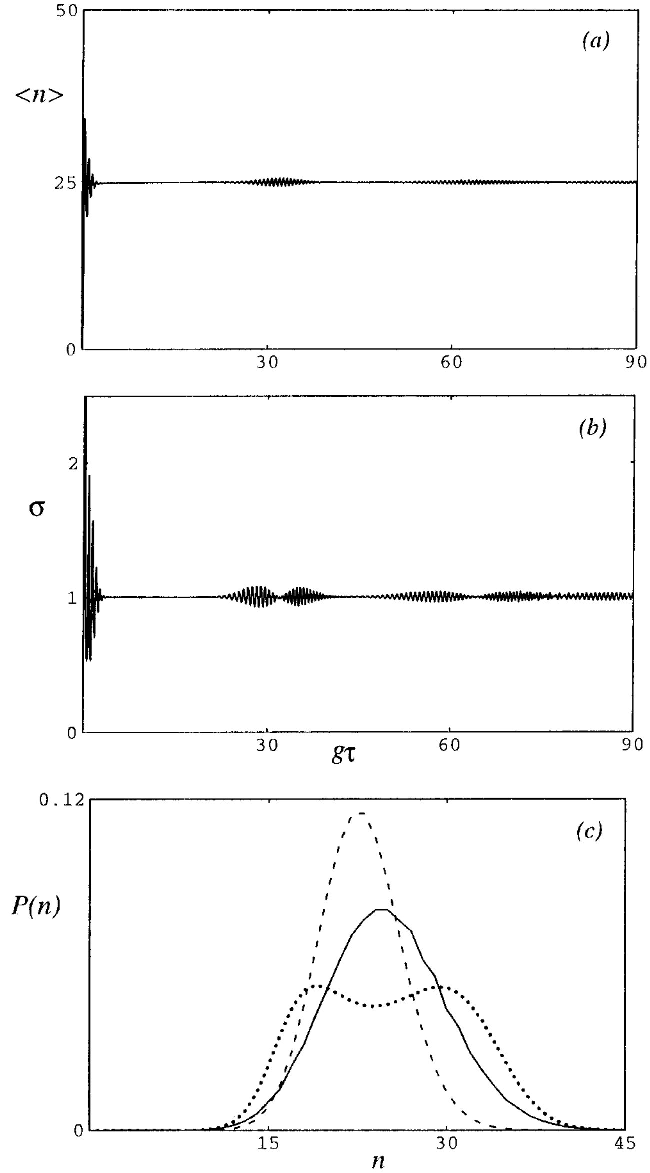

FIG. 1. We display (a) the mean number of photons and (b) the cavity Q factor of the emitted field as a function of the interaction time in a case of normal incidence of the atoms at the field. In (c) we display photon statistics for three different interaction times.

In the Bragg regime, the recoil frequency is much larger as compared with Rabi frequency and there is an appreciable transfer of energy from the cavity field to the incoming atom in the form of integral multiple of field quanta. Hence, in this case conservation of energy is the governing principle. As a result of energy conservation, there exists two values for the transferred field momenta corresponding to initial momentum of the atom, that is, the number of transferred quanta, l_0, can be zero or $2p_0/\hbar k$. The value $l_0=0$ stands for the incoming field whereas $l_0=2p_0/\hbar k$ indicates the scattered beam of atoms. The incident atom entering with an initial transversal momentum p_0 contributes l_0 photons to the field and leaves the cavity with a lesser momentum $p_{out}=p_0-l_0\hbar k$, hence conserving the momentum and energy. This phenomenon is analogous to Bragg scattering in solid state physics.

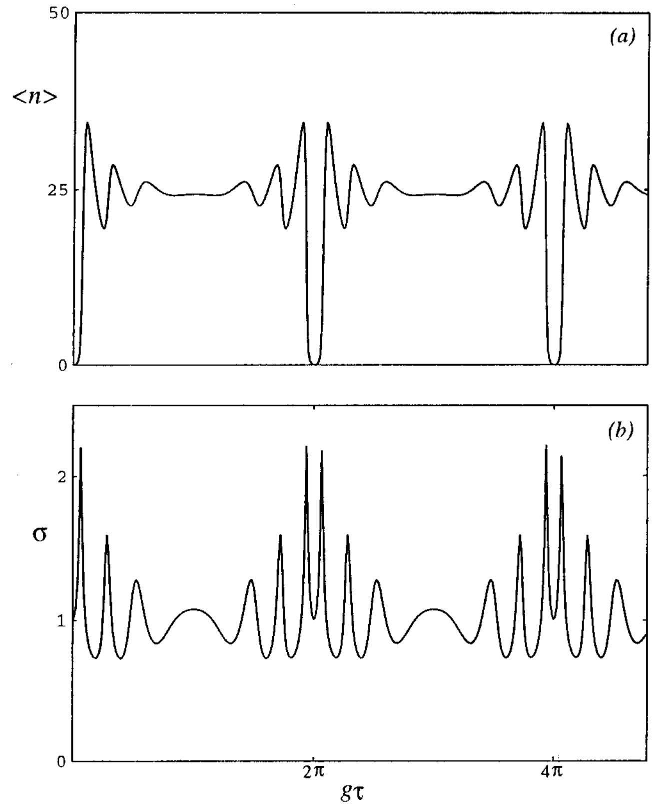

FIG. 2. We display (a) the mean number of photons and (b) the cavity Q factor of the emitted field as a function of the interaction time in the Raman-Nath regime.

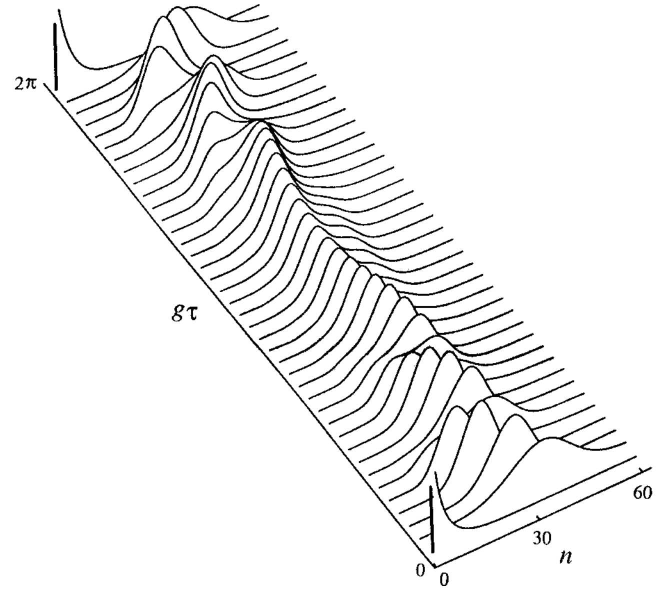

FIG. 3. A complete cycle of photon statistics inside the cavity for initial nonzero transversal momentum in Raman-Nath scattering.

Looking at the probability amplitudes in this case, given in Eq. (19) and Eq. (20), it comes out that the probability amplitudes have sinusoidal dependence on interaction time as in conventional micromasers. However, the dependence on the number of photons is governed by l_0 as $(n+1)^{l_0/2}$; moreover, a phase factor appears that depends upon the number of photons *n*, linearly. This phase shift provides a frequency shift to the emitted field spectra, as expressed in Eq. (72). A point to note is that this observed frequency shift is associated only with the microlasers.

When the initial momentum of the atom is such that they can support the transfer of just one photon from the atom to the field, that is, $l_0=1$, the probability amplitudes, apart from the phase factor, are exactly the same as in the case of micromasers. The only difference appears in the Rabi frequency which reduces to half as compared with conventional lasers and micromasers. This difference arises since an atom, on its passage through the node of the optical field, is pushed towards the antinodes and thus reduces the probability amplitude in either of the emitted directions to half. Apart from this slight difference, the microlaser for these values has an average number of photons $\langle N\rangle$ and cavity Q factor Q_f, exactly similar to conventional micromasers [16–18]. The spectrum of the field, as expressed in Eq. (71), is having an additional modulation which is again special to microlasers. This modulation indicates the behavior of noise in the cavity field as a function of the interaction time.

For all the odd values of l_0, the atom that initially comes in an excited state leaves the cavity after contributing photons to the cavity. The number of photons contributed to the cavity are proportional to the initial momentum of the atom, hence in this regime the microlaser observes a multiphoton transition. Hence, the gain of the cavity field also becomes nonlinear. The gain of the field can be expressed as

$$\langle \dot{n}\rangle = rg^2k^2\langle (n+1)^{l_0}\rangle - C(\langle n\rangle - n_T), \tag{73}$$

which indicates this phenomenon.

In order to realize our suggested scheme, we use the experimental setup of Hennrich *et al.* [19]. We consider a cloud of rubidium atoms, cooled and stored in a magneto-optic trap. By means of an atomic fountain we control the atomic dynamics in such a way that at one time there is only one atom in the cavity. We take the vacuum Rabi frequency as $g=2\pi\times 5$ MHz. We consider the atoms with a small velocity component from 1 m/s to 1 cm/s, parallel to the cavity. For a rubidium atom of mass 1.42×10^{-25} kg moving with a velocity 0.01 m/s out of the above-mentioned range, the de Broglie wavelength is 2 nm. In the Raman-Nath regime with these parameters at hand we may find the periodic behavior in an average photon number $\langle n\rangle$ with a period of 0.2 μs.

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