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A family of quotients of the Rees algebra and rigidity properties of local cohomology modules

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Contents

Introduction

This thesis is concerned with some problems in Commutative Algebra and Numerical Semigroup Theory. Classically, Commutative Algebra was developed together with Algebraic Geometry and several important geometric invariants such as dimension, multiplicity and embedding dimension have become central also in the algebraic context. Furthermore, the notions of local and graded ring play an important role in Commutative Algebra also because they represent the coordinate rings of affine and projective varieties. Analogously, some rings which arise in a natural way in Commutative Algebra turned out to be of great importance in Algebraic Geometry, for instance Cohen-Macaulay and Gorenstein rings. The first definition is strictly connected to the concept of regular elements; for example a one-dimensional Cohen-Macaulay local ring is simply a ring containing a regular element. These rings play an important role in Commutative Algebra and this is due also to the fact that the coordinate rings of many interesting algebraic varieties are Cohen-Macaulay. We refer the reader to the fundamental book of W. Bruns and J. Herzog [17] for more details. An important class of Cohen-Macaulay rings is represented by Gorenstein rings, see [30, Chapter 21], [17, Chapter 3] or [14]. They were introduced by A. Grothendieck, together with the notion of canonical module, in order to provide a duality theory; in fact their name is due to a duality property of singular plane curves studied by D. Gorenstein in his PhD thesis.

One of the main purposes of this thesis is to construct Gorenstein rings having particular properties. To this aim it is useful to introduce monomial curves, i.e. affine curves $\mathcal{C}$ of $\mathbb{A}^d_k$ defined parametrically by $x_1 = t^{n_1}$, $x_2 = t^{n_2}, \dots, x_d = t^{n_d}$, where $n_1, \dots, n_d$ are nonnegative integers. It is not difficult to see that the homogeneous coordinate ring of $\mathcal{C}$ is isomorphic to $k[t^{n_1}, \dots, t^{n_d}]$; particularly interesting are the completions of such rings, which are the one-dimensional local integral domains $k[[t^{n_1}, \dots, t^{n_d}]]$. These rings are often called numerical semigroup rings because, if $\gcd\{n_1, \dots, n_d\} = 1$, they are essentially determined by the numerical semigroup formed by the exponents of the monomials in it. A numerical semigroup S is a subsemigroup of the natural numbers containing zero and such that $\mathbb{N} \setminus S$ is finite. In other words, this means that there is an element $f(S)$ not in S such that every integer larger than $f(S)$ is in the semigroup; this element is known as the Frobenius number of S. In fact, Frobenius was the first to study numerical semigroups, in connection with the coin problem: "which is the largest monetary amount that cannot be obtained using only coins of specified denominations?". Clearly, the solution is the Frobenius number of the numerical semigroup generated by the denominations of the coins, see [69] for more details. In the last century were found many applications of Numerical Semigroup

Theory; we have already seen how it is possible to associate numerical semigroups to some particular rings and varieties, but there are more connections with Commutative Algebra and Algebraic Geometry, see e.g. [8, 12]. Algebraic Geometry also leads to applications of Numerical Semigroup Theory to Coding Theory via Weierstrass semigroups and AG-codes, see e.g. [27, 33] and the references therein. Again, in [8] it is possible to find other connections with Factorization Theory, whereas some applications to Number Theory can be found in the [77], that is also a very good introduction to Numerical Semigroup Theory.

Another central topic in Commutative Algebra and Algebraic Geometry is the theory of the Hilbert functions. In the graded case (with coefficients in a field) it is very natural, since associates to $n \in \mathbb{N}$ the dimension of the vector space of the homogeneous elements of degree n. The importance of Hilbert functions is due to the fact that it contains a great deal of information about the ring and the associated variety, such as dimension, multiplicity and arithmetical genus, see for instance [17, Chapter 4]. For a local ring R, one can consider the associated graded ring with respect to the maximal ideal of R, denoted by $\mathrm{gr}(R)$, that, roughly speaking, corresponds to the tangent cone at the origin of the variety associated with R; then the Hilbert function of R is defined as that of $\mathrm{gr}(R)$. Also in this case it is possible to obtain information about the ring from its Hilbert function, such as that provided by embedding dimension and multiplicity.

After the first preliminary chapter, this thesis is essentially divided into two parts. The first one, constituted by Chapters 2, 3 and 4, is concerned with local rings and numerical semigroup rings; more precisely in Chapter 2 we develop a new construction that we use in Chapters 3 and 4 to provide applications to Gorenstein local rings and Numerical Semigroup Theory, respectively. The second part, consisting of Chapter 5, is devoted to some problems regarding Hilbert function of local cohomology modules of graded rings.

In his famous book *Local Rings*, M. Nagata introduces a construction called *idealization* $R \ltimes M$ in order to generalize several results from the class of ideals to that of R-modules, where R is a commutative ring and M is an R-module. In the last fifty years many authors have studied this construction and have used it for several purposes, for instance for characterizing when a Cohen-Macaulay ring admits a canonical module, proving some versions of the Local Duality Theorem, and constructing new rings with particular properties. Quite a few generalizations and variations of idealization can be found in the literature; for example in [21] and [24], M. D'Anna and M. Fontana introduce a new ring that they call *amalgamated duplication* of R with respect to an ideal I; amalgamated duplication behaves in a similar fashion to idealization, but it has the advantage that it is a reduced ring if R is, whereas idealization is never reduced.

After recalling several preliminary results in the first chapter, we devote the second one to developing a new family of rings which generalizes both idealization and amalgamated duplication. Let I be an ideal of R and let $R[It] = \oplus_{i \geq 0} I^n t^n$ be the Rees algebra associated with R and I, where t is an indeterminate. Consider a monic polynomial $t^2 + at + b \in R[t]$ and let $(I^2(t^2 + at + b))$ be the contraction of the $R[t]$-ideal $(t^2 + at + b)$ to $R[It]$. We define

$$R(I)_{a,b} := \frac{R[It]}{(I^2(t^2 + at + b))}.$$

If we fix R and I, we get a family of rings depending on a and b and in Proposition 2.1.2

we prove that among the rings in this family there are always some which are isomorphic to the idealization and some to the amalgamated duplication or R with respect to I; for instance the members with $a = b = 0$ and $a + 1 = b = 0$ respectively. We also prove that in this family many properties are independent of a and b, e.g. dimension, noetherianity, embedding dimension, multiplicity, Cohen-Macaulayness; furthermore, most notable for us, we also prove that Gorensteinness and almost Gorensteinness depend only on R and I, see Corollary 2.2.3, Theorem 2.3.3, Proposition 2.3.8 and its corollary. These results explain on the one hand why idealization and amalgamated duplication have similar properties; on the other hand we can use them for producing other rings that share many properties with them. For instance, in Corollary 2.4.7 we show that, if R is a local integral domain, there are always infinitely many choices of b such that $R(I)_{0,b}$ is an integral domain, whereas it is well-known that neither idealization nor amalgamated duplication can be integral domains.

Surprisingly, this new family of rings has several applications also in Numerical Semigroup Theory. We recall that a *numerical semigroup* S is a submonoid of the natural numbers $\mathbb{N}$ such that $\mathbb{N} \setminus S$ is finite. If k is a field, we have already seen that we can associate a ring with S, called the *numerical semigroup ring* of S, that is defined as $k[[S]] := k[[t^s \mid s \in S]] \subseteq k[[t]]$ which is always a one-dimensional local integral domain. There is an extensive literature focusing on these rings; it is known that several properties of $k[[S]]$ can be read from those of S and, consequently, concepts like embedding dimension, multiplicity, type, ideals and other notions that come from Commutative Algebra and Algebraic Geometry have been introduced in numerical semigroup theory. For instance $E \subseteq S$ is said to be an ideal of S if $E + S \subseteq E$.

Given an ideal E of S and an odd integer $b \in S$, in Section 2.6 we introduce a new numerical semigroup $S \bowtie^b E$ called the *numerical duplication* of S with respect to E and b. Moreover, if R is a numerical semigroup ring, in Theorem 2.6.1 we show that there are infinitely many choices of b for which $R(I)_{0,b}$ is still a numerical semigroup ring, which is exactly the one associated to the numerical duplication of S with respect to suitable E and b. This fact turned out to be very useful for investigating the properties of the family, since all the other members often share the same properties.

Given a local ring $(R, \mathfrak{m})$, its associated graded ring is defined as $\mathrm{gr}(R) = \oplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$. The understanding of which properties $\mathrm{gr}(R)$ inherits from R or vice versa is a classic problem in local algebra. For example, it is well-known that if R is Cohen-Macaulay, its associated graded ring does not need to be Cohen-Macaulay. If R is a one-dimensional Cohen-Macaulay standard graded algebra it is well-known that its Hilbert function is non-decreasing, however in the local case, where the Hilbert function of R is equal to that one of $\mathrm{gr}(R)$ by definition, this is not true. In fact, in 1975 J. Herzog and R. Waldi found a numerical semigroup ring with embedding dimension 10 whose Hilbert function fails to be non-decreasing. After this example, there has been a great deal of interest in finding conditions for which the Hilbert function of a one-dimensional Cohen-Macaulay local ring is non-decreasing; the problem is far from being well-understood, but there is an extensive literature on this subject, see e.g. the survey [79, Section 4]. In this context M.E. Rossi posed the following problem: "Is the Hilbert function of a Gorenstein local ring of dimension one not decreasing?". In the last decade several authors have considered this problem giving a positive answer for some particular classes of numerical semigroup rings, see for instance [4, 5, 6, 7, 51, 63, 66]. However, in Chapter 3, we show that the answer is negative; this is probably the main result of the thesis. In fact, we prove that for any positive integers m and $h \geq 2$ such that $h \notin \{14+22k, 35+46k \mid k \in \mathbb{N}\}$ there exist infinitely many non-isomorphic one-dimensional Gorenstein local rings R such that $H_R(h-1) - H_R(h) > m$, where H_R denotes the Hilbert function of R, see Theorem 3.4.4. Its proof is constructive and, thus, we are able to produce infinitely many examples, see Section 3.4 for several explicit rings. In the proof we use the construction $R(I)_{a,b}$, which let us reduce the problem to finding some suitable almost Gorenstein rings. Moreover, since the almost Gorenstein rings that we find are actually numerical semigroup rings, the above result holds also in the numerical semigroup rings case, which is the most studied one. In fact, the notion of almost Gorenstein ring is a generalization of that of almost symmetric numerical semigroups. More precisely, if S is a numerical semigroup, we define $M := S \setminus \{0\}$, $f := \max(\mathbb{Z} \setminus S)$ and $K := \{f - x \mid x \in \mathbb{Z} \setminus S\}$, that are called the *maximal ideal*, the *Frobenius number* and the *canonical ideal* of S respectively; then S is said to be *symmetric* if $S = K$ and *almost symmetric* if $M + K \subseteq M$. If R is a one-dimensional Cohen-Macaulay local ring, a similar definition can be given replacing M with the maximal ideal of R and K with a canonical module of R included between R and its integral closure (when there exists); it is well-known that a numerical semigroup ring $k[[S]]$ is (almost) Gorenstein if and only if S is (almost) symmetric. Since they are the counterpart of Gorenstein rings, it is not surprising that symmetric numerical semigroups play an important role in numerical semigroup theory, but it is not limited to the connections with Commutative Algebra and Algebraic Geometry. In [78], given a numerical semigroup T and an integer d, the numerical semigroup $T/d := \{x \in \mathbb{N} \mid dx \in T\}$ is defined and called *one over* d of T in order to study the solutions of proportionally modular diophantine inequalities, which are inequalities of the form $ax \mod b \leq cx$, where a, b, and c are positive integers. In [75] and [76] J.C. Rosales and P.A. García-Sánchez prove that every numerical semigroup S is one half of infinitely many symmetric numerical semigroups and characterize all the symmetric "doubles" of S. It is easy to see that the numerical duplication $S \bowtie^b E$ is always a double of S and thus it turned out to be useful also in this context. A set $E \subseteq \mathbb{Z}$ is said to be a relative ideal of S if $E + S \subseteq E$ and there exists $x \in \mathbb{Z}$ such that $x + E \subseteq S$. It is possible to generalize the definition of the numerical duplication allowing that E is a relative ideal such that $E + E + b \subseteq S$; in this way in Proposition 2.6.6 we are able to prove that *every* numerical semigroup can be written as a numerical duplication. Using this new tool we reprove and generalize the results of J.C. Rosales and P.A. García-Sánchez; more precisely in Corollary 4.1.9 we prove that, if S has type t, then it is one half of infinitely many almost symmetric numerical semigroups with odd type included between 1 and $2t + 1$ and we characterize all of them in Corollary 4.1.5. Moreover, in Corollary 4.1.18 we prove that S is one half of an almost symmetric numerical semigroup with even type if and only if it is almost symmetric and also in this case we characterize all its almost symmetric doubles, see Corollary 4.1.14; these generalize the results of [73]. Moreover, given $d \geq 2$ and $x \leq 2t + 2$, we also prove in Corollary 4.2.4 that there exist infinitely many almost symmetric numerical semigroups T with type x and such that S is one over d of T; in this way we generalize the results of I. Swanson that proves that for $x \leq 2$ in [89].

The genus $g(S)$ of a numerical semigroup S is defined as the cardinality of $\mathbb{N} \setminus S$ and it is a very important number, especially for its connections with algebraic geometry and coding theory. In [71] A.M. Robles-Pérez, J.C. Rosales, and P. Vasco ask for "a formula, that depends on S, for computing $\min\{g(\overline{S}) \mid \overline{S}$ is a double of $S\}$". Here we give a complete answer in a more general context, in fact in Theorem 4.3.1 we prove that

$$\min\left\{g(\overline{S}) \mid S = \frac{\overline{S}}{d}\right\} = g(S) + \left\lceil \frac{(d-1)f}{2} \right\rceil.$$

Another important problem in this area is the searching for a formula for the Frobenius number of a quotient of a numerical semigroup; in Theorem 4.4.2 we give a formula for d-symmetric numerical semigroup and, as a consequence, also for symmetric and pseudo-symmetric, see Corollaries 4.4.4 and 4.4.6.

In the second part of the thesis we study the Hilbert function of certain graded modules. Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and let I be a homogeneous ideal of R. A monomial ideal L is said to be a *lex-ideal*, if for any monomial $u \in L$ and all monomials $v \in R$ with $\deg u = \deg v$ and $u \prec_{\text{lex}} v$ it follows that $v \in L$, where $\prec_{\text{lex}}$ denotes the lex-order. It is well-known that there is a one-to-one correspondence between Hilbert functions of homogeneous ideals of R and lex-ideals of R; i.e. it is possible to associate a unique lexicographic ideal I^{lex} with I. In [16, 49, 65] A.M. Bigatti, H.A. Hulett, and K. Pardue prove that the Betti numbers of I are always less than or equal to those of I^{lex}. Moreover, if $\operatorname{Gin}(I)$ is the generic initial ideal of I with respect to the reverse lexicographic order, it holds that $\beta_{i,j}(R/I) \leq \beta_{i,j}(R/\operatorname{Gin}(I)) \leq \beta_{i,j}(R/I^{\text{lex}})$ for all i, j. Several authors have studied when above equalities hold, but also when there are some *rigid behaviours* in the equalities; for instance in [19] A. Conca, J. Herzog and T. Hibi prove that if J is either a generic initial ideal of I or I^{lex} and $\beta_{i,j}(R/I) = \beta_{i,j}(R/J)$ for some fixed i and all j, then $\beta_{k,j}(R/I) = \beta_{k,j}(R/J)$ for all $k \geq i$ and all j. We are interested in similar properties of the Hilbert functions of the corresponding local cohomology modules with support on the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$. More precisely, if we set $h^i(R/I)_j := H^i_{\mathfrak{m}}(R/I)_j$, in [83] E. Sbarra proves that $h^i(R/I)_j \leq h^i(R/\operatorname{Gin}(I))_j \leq h^i(R/I^{\text{lex}})_j$ and therefore it is natural to ask whether there is results of rigidity type, like the theorem above for the Betti numbers. However, in Section 5.3 we introduce the notion of *i-partially sequentially Cohen-Macaulay module* proving that $h^k(R/I)_j = h^k(R/\operatorname{Gin}(I))_j$ for all $k \geq i$ and for all j if and only if R/I is i-partially sequentially Cohen-Macaulay; this generalizes a result of J. Herzog and E. Sbarra [47]. In Theorem 5.5.6 we use this characterization to prove the rigidity property between I and I^{lex} we are interested in, i.e. if $h^i(R/I)_j = h^i(R/I^{\text{lex}})_j$ for some fixed i and for all j, then the equalities hold also for all $k \geq i$; when $i = 0$ this results was already proved by E. Sbarra in [82]. Actually we prove that, if $h^i(R/\operatorname{Gin}(I))_j = h^i(R/I^{\text{lex}})_j$ for a fixed i and all j, then $h^k(R/I)_j = h^k(R/I^{\text{lex}})_j$ for all $k \geq i$ and all j; this is a stronger result and, in fact, it does not hold for Betti numbers.

Many results of this thesis are contained in [10, 11, 25, 26, 62, 84, 87, 88]. Several computations are performed by using the softwares Macaulay2 [41] and GAP [36], in particular the NumericalSgps package [28].

Chapter 1

Preliminaries

In this chapter we recall some preliminary results and fix the notation that we use throughout the thesis. All the rings considered are commutative and unitary and with $(R, \mathfrak{m}, k)$ or simply $(R, \mathfrak{m})$ we denote a local ring R with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. We write $Q(R)$ for the total ring of fractions of R, while $\overline{R}$ is the integral closure of R in $Q(R)$. The *length* of an R-module M will be indicated by $\ell_R(M)$.

Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. Of great importance is the *embedding dimension* of R which is defined as $\nu(R) := \mathfrak{m}/\mathfrak{m}^2$ and, by Nakayama's lemma, it is equal to the number of minimal generators of $\mathfrak{m}$. More generally, if M is an R-module, the embedding dimension of M is $\nu(M) := M/\mathfrak{m}M$ and it is equal to the minimal number of generators of M; we also denote by $G(I)$ this set of generators.

The notion of reduction of an ideal is of central importance in local algebra and we recall it here. Given an ideal I of R, a *reduction* of I is an ideal J such that $JI^n = I^{n+1}$ for some n; moreover J is said to be *minimal* if there are no other reductions of I contained in J. If k is infinite, minimal reductions always exist, see [50, Theorem 8.3.5]; in particular, if R is one-dimensional and I is *regular*, i.e. contains a regular element, every minimal reduction of I is principal, see [50, Proposition 8.3.7 and Corollary 8.3.9].

In the one-dimensional case we will need a generalization of the notion of reduction. Let M be a *fractional ideal* of R, i.e. there exists a non-zero-divisor $y \in R$ such that $yM \subset R$ and, if yM is regular, we consider an its minimal reduction xR. With some abuse of terminology, we call $xy^{-1}R$ a minimal reduction of M, where now $xy^{-1} \in Q(R)$. If $R \subseteq M$, then xy^{-1} is an invertible element of $Q(R)$: in fact yM is a regular ideal, since $y \in yM$, and a minimal reduction of a regular ideal has to be generated by a non-zero-divisor.

The notion of regular element naturally generalize to that of regular sequence. More precisely if M is an R-module, a sequence $x_1, \dots, x_n$ of not invertible elements of R is said to be an M-*regular sequence* if $M/(x_1, \dots, x_n)M \neq 0$ and x_i is an $M/(x_1, \dots, x_{i-1})$-regular element for $i = 1, \dots, n$. An M-regular sequence $x_1, \dots, x_n$ is maximal if there is no element x such that $x_1, \dots, x_n, x$ is M-regular. If R is a noetherian local ring and M is a finitely generated R-module, it is well-known that all maximal M-regular sequences have the same length, which is called the *depth* of M and is denoted by $\operatorname{depth} M$. It is not difficult to see that $\operatorname{depth} M \leq \dim M$

and M is said to be a *Cohen-Macaulay module*, if the equality holds. When R is not local, M is called Cohen-Macaulay if $M_{\mathfrak{m}}$ is either the zero module or a Cohen-Macaulay $R_{\mathfrak{m}}$-module for all maximal ideals $\mathfrak{m}$ of R. Furthermore, we say that R is a *Cohen-Macaulay ring* if it is a Cohen-Macaulay R-module. Given three R-modules M, N and K, we set $N :_K M := \{x \in K \,|\, xM \subseteq N\}$ or simply $N : M$, if K is the total ring of fractions of R; we also denote the annihilator of M by $\operatorname{Ann} M := 0 :_R M$, whereas the *dimension* of M is defined as $\dim M := \dim R/\operatorname{Ann} M$. A *maximal Cohen-Macaulay R-module* is a Cohen-Macaulay module satisfying $\dim M = \dim R$. See [17, Sections 1.1 and 1.2] for more detail about Cohen-Macaulay modules. If $R = k[x_1, \dots, x_n]$ is a polynomial ring over a field k, it is possible to give the same definitions above, considering only homogeneous regular elements.

Let M be a finitely generated module over a noetherian local ring $(R, \mathfrak{m}, k)$ and let $x_1, \dots, x_n$ be a maximal M-regular sequence. The *socle* of M is defined as $\operatorname{soc}(M) := 0 :_M \mathfrak{m} \cong \operatorname{Hom}_R(k, M)$ and the *type* $t(M)$ of M is defined to be the dimension as vector space of the socle of $M/(x_1, \dots, x_n)M$.

We will need a well-known result about maximal Cohen-Macaulay ideals, but we prefer include an easy proof, because we do not know an explicit reference.

Lemma 1.0.1. *Let R be a Cohen-Macaulay local ring and let I be a regular ideal. If I is a maximal Cohen-Macaulay R-module, then* $\operatorname{ht} I = 1$ *and R/I is a Cohen-Macaulay ring of dimension* $\dim R - 1$.

Proof. From the very definition of maximal Cohen-Macaulay module it follows that $\operatorname{depth} I = \operatorname{depth} R$ and, since I is regular, it has positive depth. Therefore the Depth Lemma [17, Proposition 1.2.9] yields that

$$\begin{gathered}\dim R - 1 \geq \dim R - \operatorname{ht} I = \dim R/I \geq \operatorname{depth} R/I \geq \\ \geq \min\{\operatorname{depth} I - 1, \operatorname{depth} R\} = \operatorname{depth} R - 1 = \dim R - 1.\end{gathered}$$

Hence, all the above inequalities are indeed equalities and the conclusion is now straightforward. $\square$

An important tool in local algebra are the superficial elements. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and let M be a finitely generated R-module. Given an ideal $\mathfrak{q}$ of R, a chain $\mathbb{M}$ of R-modules

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \dots$$

is a $\mathfrak{q}$-filtration if $\mathfrak{q} M_i \subseteq M_{i+1}$ for all i. We set $\mathbb{M} := \{M_i\}_{i \in \mathbb{N}}$. An element $x \in R$ is called $\mathbb{M}$-*superficial* for $\mathfrak{q}$ if there exists $c \in \mathbb{N}$ such that $(M_{n+1} :_M x) \cap M_c = M_n$ for every $n \geq c$. A sequence of elements $x_1, \dots, x_n$ is said to be an $\mathbb{M}$-*superficial sequence* for $\mathfrak{q}$ if x_i is a $\mathbb{M}/(x_1, \dots, x_{i-1})\mathbb{M}$-superficial element for $i = 1, \dots, n$, where $\mathbb{M}/(x_1, \dots, x_{i-1})\mathbb{M} := \{M_i/(x_1, \dots, x_{i-1})M_i\}_{i \in \mathbb{N}}$. More details on superficial elements can be found e.g. in [80, Chapter 1]; in the next proposition we collect the two properties useful for our purposes.

Proposition 1.0.2. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, let $\mathfrak{q}$ be an ideal of R and let $\mathbb{M}$ be a $\mathfrak{q}$-filtration of a finitely generated R-module M. Then:*

1. *If k is infinite, $\mathfrak{q}$ contains a $\mathbb{M}$-superficial element for $\mathfrak{q}$.*

2. *If $x_1, \ldots, x_r$ is a superficial sequence for $\mathfrak{m}$, then it is a M-regular sequence if and only if* $\operatorname{depth}(M) \geq r$.

1.1 Hilbert functions and lex ideals

In this section we recall several results about Hilbert function and lex-ideals; our main reference here is [17, Section 4.1]. A ring R is said to be Z-graded if $R = \oplus_{i \in \mathbb{Z}} R_i$ as abelian group and $R_i R_j \subseteq R_{i+j}$ for all i and j. In commutative algebra there are many natural graded rings, for instance the polynomial ring $R = k[x_1, \ldots, x_n]$ is standard graded, where R_i is the group of the homogeneous polynomial of degree i. More generally a graded R-module M is an R-module $M = \oplus_{i \in \mathbb{Z}} M_i$, where M_i are abelian groups, such that $R_i M_j \subseteq M_{i+j}$ for all i and j.

If M is a finitely generated graded R-module, the function $H_M : \mathbb{Z} \to \mathbb{N}$ defined as $H_M(i) := \ell_R(M_i)$ is called *Hilbert function* of M. Furthermore, its generating series $\operatorname{Hilb}(M) := \sum_{i \in Z} H_M(i) t^i$ is known as the *Hilbert series* of M.

If M has dimension d, there exists a unique polynomial $h(M; t) \in \mathbb{Z}[t, t^{-1}]$, called the *h-polynomial* of M, such that $h(M; 1) \neq 0$ and

$$\operatorname{Hilb}(M) = \frac{h(M; t)}{(1-t)^d}.$$

Moreover for $n \gg 0$ the Hilbert function is equal to a polynomial $P_M(t)$ called the *Hilbert polynomial* of M. There exist $e_0, e_1, \ldots, e_{d-1} \in \mathbb{Z}$ such that

$$P_M(t) = \sum_{i=0}^{d-1} (-1)^{d-1-i} \; e_{d-1-i} \binom{t+i}{i},$$

where $d = \dim M$. By definition the *multiplicity* $e(M)$ of M is equal to either e_0, if $d > 0$, or $\ell(M)$ otherwise.

Let $R = k[x_1, \ldots, x_n]$ be a standard graded polynomial ring over a field k and let I be a homogeneous ideal. The *saturation* of I is the ideal $I^{\,\text{sat}} := I : \mathfrak{m}^\infty = \cup_{k=1}^{\infty} (I : \mathfrak{m}^k)$ and it is well-known that $P_{R/I}(t) = P_{R/I^{\,\text{sat}}}(t)$.

A monomial ideal I is said to be a *lex-ideal*, lexicographic or lexsegment ideal, if for any monomial $u \in I_d$ and all monomials $v \in R_d$ with $u \prec_{\text{lex}} v$ one has $v \in I$, where $\prec_{\text{lex}}$ denotes the lex-order in which $x_1 \prec x_2 \prec \cdots \prec x_n$. It is well-known from Macaulay's work that, given an ideal I, there exists a unique lex-ideal with the same Hilbert function as I (cf. for instance [46, Theorem 6.3.1]) and we denote that by $I^{\,\text{lex}}$.

Among all lex-ideals we are interested in a special class. Following [56] we call an ideal *universal lex-ideal* if it is a lex-ideal of $k[x_1, \ldots, x_n]$ and has at most n generators. This notion was first introduced in [9] and the use of the word "universal" is due to the fact that the universal lex-ideals are exactly those lex-ideals whose extensions to any polynomial overring of R are still lex-ideals, see [56, Corollary 1.3]. Since there is a bijection between Hilbert functions and lex-ideals we can give a related definition: A numerical function $H : \mathbb{N} \longrightarrow \mathbb{N}$ is said to be *critical* if it is the Hilbert function of an universal lex-ideal and, accordingly, a homogeneous ideal is called *critical* if its Hilbert function is. We state some results about critical ideal proved in [56], while in the next section we will see another important property of the universal lex-ideals.

Theorem 1.1.1. *Let I be a homogeneous ideal of $R = k[x_1, \dots, x_n]$. Then:*

1. $\operatorname{depth} R/I^{\text{lex}} = \max\{n - |G(I^{\text{lex}})|, 0\}$*;*

2. *If I is critical, then* $\operatorname{depth} R/I = \operatorname{depth} R/I^{\text{lex}} = n - |G(I^{\text{lex}})|$*;*

3. *If R/I^{lex} has positive depth, then I is critical.*

Proof. The first two properties are proved in Corollary 1.4 and Theorem 1.6 of [56], while the last one is a consequence of 1. □

In the non-graded setting we cannot define the Hilbert function as above; on the other hand, in the local case it is possible to get information from the Hilbert function of a related ring. More precisely, if $(R, \mathfrak{m}, k)$ is a noetherian local ring, the *associated graded ring of R* is defined as $\operatorname{gr}(R) := \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$, where $R := \mathfrak{m}^0$. The *Hilbert function* of R is by definition that of its associated graded ring, that is the function $H_R : \mathbb{N} \to \mathbb{N}$ defined as $H_R(i) = \ell_R(\mathfrak{m}^i/\mathfrak{m}^{i+1})$. In the same way the multiplicity of R is by definition the multiplicity of the associated graded ring of R; it is clear that $H_R(1) = \nu(R)$.

1.2 Generic initial ideals

The notion of the generic initial ideal is essentially a variation of the initial ideal; this has the advantage that, if we do that with respect to the reverse lexicographic order, it preserves several properties of the ideal, like depth, projective dimension, and Castelnuovo-Mumford regularity, see e.g. [46, Section 4] or [42].

Let $R = k[x_1, \dots, x_n]$, where k is an infinite field and the variables $x_1 > x_2 > \dots > x_n$ have degree one. Let $\mathrm{GL}_n(k)$ denote the general linear group and consider the automorphism of R associated with an element $\alpha = (a_{ij})$ of $\mathrm{GL}_n(k)$, i.e.

$$\alpha : R \longrightarrow R, \qquad f(x_1, \dots, x_n) \mapsto f\left(\sum_{i=1}^{n} a_{1i}x_i, \dots, \sum_{i=1}^{n} a_{ni}x_i\right).$$

Given a monomial order $\prec$ on R, there exists a non-empty Zariski open subset $U \subseteq \mathrm{GL}_n(k)$ for which $\mathrm{in}_\prec \alpha I = \mathrm{in}_\prec \alpha' I$ for all $\alpha, \alpha' \in U$ (see [46, Theorem 4.1.2]). If $\alpha \in U$, the ideal $\mathrm{in}_\prec \alpha I$ is said to be the *generic initial ideal* of I and it is denoted by $\mathrm{gin}_\prec(I)$; if $\prec$ is the reverse lexicographic order we only write $\mathrm{Gin}(I)$. In the next proposition we collect some useful properties of the generic initial ideals, see [46, Chapter 4] for more details. The last property is proved in [58, Lemma 2.6].

Proposition 1.2.1. *Let I be an ideal of R and let $\prec$ be a monomial order. The following properties hold:*

1. *I and $\mathrm{gin}_\prec(I)$ have same Hilbert function and same depth;*

2. $\mathrm{gin}_\prec(\mathrm{gin}_\prec(I)) = \mathrm{gin}_\prec(I)$*;*

3. $\mathrm{gin}_\prec(I^{\,\mathrm{lex}}) = I^{\,\mathrm{lex}}$*;*

4. $\mathrm{Gin}(I)^{\,\mathrm{sat}} = \mathrm{Gin}(I^{\,\mathrm{sat}})$*;*

5. *If I is critical, then* $\mathrm{gin}_\prec(I) = I^{\,\mathrm{lex}}$*.*

Some other properties of the generic initial ideals are, indeed, shared by a larger class of ideals: weakly stable ideals. Given a monomial $u \in R$, let $m(u)$ denote the maximum integer for which $x_{m(u)}$ divides u and let l be the largest integer such that $x^l_{m(u)}$ divides u. We say that a monomial ideal I is *weakly stable* if for any monomial $u \in I$ and for all $j < m(u)$, there exists a positive integer k such that $x_j^k u / x^l_{m(u)} \in I$. These ideals are also called Borel-type ideals, quasi-stable ideals or monomial ideals of nested type. In the next proposition we summarize some properties of weakly stable ideals.

Proposition 1.2.2. *Let I be an ideal of R and let $\prec$ be a monomial order. The following properties hold:*

1. $I^{\,\mathrm{lex}}$ *is weakly stable;*

2. $\mathrm{gin}_\prec(I)$ *is weakly stable;*

3. $I^{\,\mathrm{sat}} = I : x_n^\infty = \cup_{k=1}^\infty (I : x_n^k)$*;*

4. $\operatorname{depth} R/I = n - \max\{m(u) \mid u \in G(I)\}$*.*

Proof. Since $I^{\,\mathrm{lex}} = \mathrm{gin}_\prec(I^{\,\mathrm{lex}})$, the first property is a particular case of 2, that is in turn implied by [46, Theorems 4.2.1 and 4.2.10]. The proof of 3 can be found in [46, Proposition 4.2.9], note also the definition above that proposition. Finally, the formula for the depth follows from the fact that, if $r = \max\{m(u) \mid u \in G(I)\}$, then $x_{r+1}, \ldots, x_n$ is a regular sequence of R/I and $(x_1, \ldots, x_r) \in \mathrm{Ass}(R/I)$; in fact, the image of x_r in R/I is a zero-divisor and the associated primes of a weakly stable ideals are of the form $(x_1, \ldots, x_j)$ for some j, see again [46, Proposition 4.2.9]. $\square$

Let M be an R-module. We set $\mathcal{M}_{-1} = 0$ and, for each non-negative integer k, we denote by $\mathcal{M}_k$ the maximum submodule of M of dimension less than or equal to k; we call $\{\mathcal{M}_k\}_{k \geq -1}$ the *dimension filtration* of M. The module M is said to be *sequentially Cohen-Macaulay*, briefly sCM, if $\mathcal{M}_k/\mathcal{M}_{k-1}$ is either zero or a k-dimensional Cohen-Macaulay module for all $k \geq 0$. If $M = R/I$ is a sCM module we simply say that I is a sCM ideal. The notion of sCM modules was introduced independently by P. Schenzel [85] and R.P. Stanley [86]. We notice that a weakly stable ideal is always sCM, see e.g. [18, Proposition 1.9].

1.3 Gorenstein and almost Gorenstein rings

In the first part of this section we recall the definitions and some properties of canonical modules and Gorenstein rings following [30, Chapter 21], but see also [17, Chapter 3]. Let R be a ring and let $M \subseteq E$ be two R-modules. We say that E is an essential extension of M if for all non-zero R-submodule N of E we have $N \cap M \neq 0$; if E is also an injective module, we call E

an *injective hull* of M. It is well-known that every module admits an injective hull and that it is unique up to isomorphism; we denote it by $E(M)$.

Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring of dimension d. If $d = 0$ the canonical module ω_R of R is defined as the injective hull of k, while if $d > 0$ a finitely generated R-module ω_R is said to be a *canonical module* of R if there exists a non-zero-divisor $x \in R$ such that $\omega_R/x\omega_R$ is a canonical module of R/xR. Two canonical modules of R are always isomorphic, however there exist Cohen-Macaulay local rings that do not admit canonical modules. Furthermore we recall that a canonical module of R is always a maximal Cohen-Macaulay R-module. If M is an $R-$module, we denote its *dual* $\mathrm{Hom}_R(M, \omega_R)$ by $M^\vee$; when R is one-dimensional and M is a fractional ideal it follows that $M^\vee \cong \omega_R : M$. We recall some useful properties:

Proposition 1.3.1. *Let ω_R a canonical module of R and let M be a maximal Cohen-Macaulay R-module, then:*
1. *ω_R has type 1 and $\nu(\omega_R) = t(R)$.*
2. *$R^\vee \cong \omega_R$ and $(\omega_R)^\vee \cong R$.*
3. *$(M^{\vee)^\vee} \cong M$.*
4. *If N is maximal Cohen-Macaulay of R that is also a submodule of M, then*

$$\ell\left(\frac{M}{N}\right) = \ell\left(\frac{N^\vee}{M^\vee}\right).$$

The notion of canonical module leads immediately to define a very important class of rings: Gorenstein rings; more precisely a Cohen-Macaulay local ring R is said to be a *Gorenstein ring* if it is isomorphic to its canonical module. This condition is equivalent to the fact that R has finite injective dimension or, most important for us, that R has type 1. In the non-local case we say that a Cohen-Macaulay ring is Gorenstein if its localizations at every maximal ideal are Gorenstein rings.

If an ideal is a canonical module of R is said to be a *canonical ideal* of R. We have already notice that not all the ring admits a canonical module, more precisely a Cohen-Macaulay local ring admits a canonical module if and only if it is a homomorphic image of a Gorenstein local ring. Even if a canonical module exists, it is not always true that there exists also a canonical ideal, in fact this happens if and only if R is generically Gorenstein, i.e. $R_\mathfrak{p}$ is Gorenstein for each minimal prime $\mathfrak{p}$ of R (see [17, Proposition 3.3.18]).

An important class of Gorenstein rings is that of the complete intersections. Following [30, Section 18.5], we recall that a local ring R is said to be a *complete intersection* if its completion with respect to the $\mathfrak{m}$-adic topology can be written as a regular local ring modulo a regular sequence.

Let now $(R, \mathfrak{m}, k)$ be a one-dimensional Cohen-Macaulay local ring. We say that R is an *almost Gorenstein* ring if its canonical module is isomorphic to a fractional ideal ω_R of R such that $R \subseteq \omega_R \subseteq \overline{R}$ and $\mathfrak{m} = \mathfrak{m}\omega_R$; this is equivalent to saying that $R \subseteq \omega_R \subseteq \mathfrak{m} : \mathfrak{m}$, in fact it is easy to see that $\mathfrak{m} : \mathfrak{m} \subseteq \overline{R}$ using the determinant trick. It follows that for a one-dimensional almost Gorenstein ring we have an exact sequence of R-modules

$$0 \to R \to \omega_R \to \omega_R/R \to 0$$

with $\mathfrak{m}\omega_R \subseteq \mathfrak{m}$. Moreover $1 \notin \mathfrak{m}\overline{R} \supseteq \mathfrak{m}\omega_R$, because $\mathfrak{m}\overline{R}$ is contained in all the maximal ideals of $\overline{R}$; it follows that $\mathfrak{m}\omega_R \subseteq \mathfrak{m}$ is equivalent to $\mathfrak{m}\omega_R \subseteq R$, i.e. $\mathfrak{m}(\omega_R/R) = 0$.

The notion of almost Gorenstein rings is introduced in [13] for one-dimensional analytically unramified rings and, if k is infinite, this definition is equivalent to the one given in [39]. Furthermore, in [40] the definition is generalized also in the case of arbitrary dimension. More precisely a Cohen-Macaulay local ring $(R, \mathfrak{m})$ of dimension d, admitting a canonical module ω_R, is said to be *almost Gorenstein* if there exists an exact sequence of R-modules

$$0 \to R \to \omega_R \to C \to 0$$

such that $\nu(C) = e(C)$; in particular it follows that $\dim C = d - 1$. Therefore, in general the condition $\mathfrak{m}C = 0$ given in dimension one, becomes $\mathfrak{m}C = (f_1, \ldots, f_{d-1})C$, for some $f_1, \ldots, f_{d-1} \in \mathfrak{m}$, that is equivalent to say $\nu(C) = \lambda_R(C/\mathfrak{m}C) = \lambda_R(C/(f_1, \ldots, f_{d-1})C) = e(C)$.

If we assume that R is one-dimensional and k is infinite, the definition of one-dimensional almost Gorenstein rings given above is equivalent to that given in [40], as proved in [40, Proposition 3.4].

Finally we notice that a zero-dimensional ring is almost Gorenstein if and only if is Gorenstein.

1.4 Local cohomology

Let $R = k[x_1, \ldots, x_n]$, $\mathfrak{m} = (x_1, \ldots, x_n)$ and let M be an R-module. Set

$$\Gamma_{\mathfrak{m}}(M) := \{x \in M \mid \mathfrak{m}^h x = 0 \text{ for some } h \in \mathbb{N}\}.$$

It is easy to see that $\Gamma_{\mathfrak{m}}(\cdot)$ is a left-exact additive R-linear functor and its right derived functors $H^i_{\mathfrak{m}}(\cdot)$ are called local cohomology functors. The module $H^i_{\mathfrak{m}}(M)$ is called *i-th local cohomology module* of M with support on $\mathfrak{m}$ and, if M is a graded modules, then also $H^i_{\mathfrak{m}}(M)$ is graded. From the definition it follows that $H^0_{\mathfrak{m}}(M) = \Gamma_{\mathfrak{m}}(M)$ and it is not difficult to see that $H^0_{\mathfrak{m}}(R/I) \cong I^{\text{sat}}/I$.

Local cohomology were introduced by A. Grothendieck for their applications in algebraic geometry. He also proved a famous vanishing theorem, i.e. if $t = \operatorname{depth} M$ and $d = \dim M$, then $H^i_{\mathfrak{m}}(M) = 0$ for $i < t$ and $i > d$, while $H^t_{\mathfrak{m}}(M) \neq 0$ and $H^d_{\mathfrak{m}}(M) \neq 0$. In particular the depth and dimension of a module M are respectively the minimum and the maximum index i for which $H^i_{\mathfrak{m}}(M) \neq 0$.

There exists a formula that links the Hilbert function of a module to the Hilbert function of the local cohomology modules. This was proved by Serre, but a proof can be found also in [17, Theorem 4.4.3 (b)].

Theorem 1.4.1. (Serre's formula) *If R is positively graded and M is a finitely generated R-module of dimension d, then for all $j \in \mathbb{Z}$ we have*

$$H_M(j) - P_M(j) = \sum_{i=0}^{d} (-1)^i \, \dim_k H^i_{\mathfrak{m}}(M)_j.$$

As in the local case, it is possible define the canonical module ω_R of R in the graded case. If M is a finitely generated graded R-module, using the canonical module Grothendieck also proved a kind of duality, known as *local duality theorem*:

$$\mathrm{Hom}_k(H^i_{\mathfrak{m}}(M), k) \cong \mathrm{Ext}^{n-i}_R(M, \omega_R),$$

see [17, Theorem 3.6.19].

1.5 Idealization and amalgamated duplication

Given a ring R and an R-module M, the *idealization* of R with respect to M, also called trivial extension, is a new ring defined as the abelian group $R \oplus M$ endowed with the multiplication $(r, m)(s, n) = (rs, rn + sm)$; we will denote it by $R \ltimes M$. It was introduced by M. Nagata in [59] in order to study an R-module as an ideal, in fact $N := \{(0, m) \mid m \in M\}$ is an ideal of $R \ltimes M$ and it is clearly isomorphic to M as R-module. On the other hand idealization is important also for other reasons, for example it can be used to prove that a Cohen-Macaulay ring possesses a canonical module if and only if is an image of a Gorenstein ring or to prove some versions of local duality theorem, but probably the most important application is the construction of new rings with particular properties. For more details about idealization see e.g. the survey [2].

It is easy to see that N^2 is equal to zero and thus idealization is never a reduced ring, if $M \neq 0$. For this reason, when the module is an ideal I, in [21] and [24] M. D'Anna and M. Fontana define the *amalgamated duplication* $R \bowtie I$ of R with respect to I as the abelian group $R \oplus I$ endowed with the multiplication $(r, i)(s, j) = (rs, rj + si + ij)$. This new ring is reduced if and only if R is and so it is possible to use other techniques to study amalgamated duplication, for example when R is an algebroid branch (see e.g. [21]). On the other hand $R \ltimes I$ and $R \bowtie I$ share a lot of properties; for instance they have the same dimension and $R \ltimes I$ is local, noetherian or Cohen-Macaulay if and only if $R \bowtie I$ is. In the next chapter we generalize these two constructions and re-prove all these properties in more generality. However we will need the following theorem; the part about idealization is proved by I. Reiten in [70], while the part regarding amalgamated duplication is proved by M. D'Anna in [21].

Theorem 1.5.1. *Let R be a local ring and let I be a proper regular ideal. Then $R \ltimes I$ and $R \bowtie I$ are Gorenstein if and only if R is Cohen-Macaulay and I is a canonical ideal of R.*

We notice that $R \ltimes I$ and $R \bowtie I$ are never integral domains. Our first goal in this thesis will be to find a *similar* construction that can be an integral domain; in this way we will show that if R is a numerical semigroup ring, several properties of $R \ltimes I$ and $R \bowtie I$ are related to a suitable numerical semigroup, even if it is not possible to associate numerical semigroups with them in the standard ways.

1.6 Numerical semigroups

In this section we recall some definitions and results about numerical semigroups and numerical semigroup rings, we refer to the books [12] and [77] for the unproven facts stated here. A *numerical semigroup* S is a submonoid of the natural numbers such that its complement in $\mathbb{N}$ is finite. We called the elements of $\mathbb{N} \setminus S$ *gaps*, while their cardinality is said to be the *genus* of S. The greatest gap $f(S)$ plays a central role in numerical semigroup theory and is called *Frobenius number* of S. If $n_1, n_2, \ldots, n_\nu \in \mathbb{N}$, we set $\langle n_1, n_2, \ldots, n_\nu \rangle := \{\alpha_1 n_1 + \alpha_2 n_2 + \cdots + \alpha_\nu n_\nu \mid \alpha_i \in \mathbb{N}\}$; this is a numerical semigroup if and only if $\gcd(n_1, n_2, \ldots, n_\nu) = 1$ and in this case we say that $n_1, n_2, \ldots, n_\nu$ is a system of generators of $\langle n_1, n_2, \ldots, n_\nu \rangle$. It is well-known that every numerical semigroup has a unique minimal system of generators, in fact, if $M := S \setminus \{0\}$, it is the finite set $(M+M) \setminus M$. We often represent a numerical semigroup by $S = \{0, s_1, s_2, \ldots, f(S)+1 \rightarrow\}$, where $\rightarrow$ means that all the integers greater than $f(S)+1$ are in S.

We are interested in numerical semigroups because, if $S = \langle n_1, \ldots, n_\nu \rangle$ is a numerical semigroup and k is a field, the ring $k[[S]] := k[[t^{n_1}, \ldots, t^{n_\nu}]]$ is strictly connected to S, as we will see. We refer to $k[[S]]$ as the *numerical semigroup ring* associated with S or only semigroup ring for short. The integer ν is said to be the *embedding dimension* of S, while the smallest generator of S is the *multiplicity* of the numerical semigroup; we denote them by $\nu(S)$ and $e(S)$ respectively. We notice that S and $k[[S]]$ have the same embedding dimension and multiplicity.

Example 1.6.1. Consider the numerical semigroup S generated by 8,13,15 and 19, i.e.

$$S = \langle 8, 13, 15, 19 \rangle = \{0, 8, 13, 15, 16, 19, 21, 23, 24, 26, 27, 28, 29, 30, 31, 32, 34 \rightarrow\}.$$

In this case the Frobenius number is 33, whereas the multiplicity is 8. Moreover, it is easy to see that the generators are minimal and therefore the embedding dimension is 4. Finally, the genus of S, i.e. the number of its gaps, is 18.

The *Apéry set* of S is defined as $\mathrm{Ap}(S) := \{s \in S \mid s - e(S) \notin S\}$. Let ω_i be the minimum element of S that is congruent to i modulo $e(S)$; it is not difficult to see that $\mathrm{Ap}(S) = \{\omega_0 = 0, \omega_1, \omega_2, \ldots, \omega_{e(S)-1}\}$ and, consequently, the Apéry set has cardinality $e(S)$. It follows from the definition that $f(S) + s$ is in S for all non-zero $s \in S$; more generally the elements of

$$\mathrm{PF}(S) := \{x \in \mathbb{Z} \setminus S \mid x + s \in S \text{ for all } s \in S \setminus \{0\}\}$$

are called *pseudo-Frobenius numbers* of S and their cardinality $t(S)$, that is finite, is called the *type* of S. We notice that $t(S) = t(k[[S]])$, see e.g. [12, Proposition II.1.16] or [35, Theorem 23]. Moreover, if we order $\mathrm{Ap}(S)$ setting $n <_S m$ if and only if there exists $s \in S$ such that $m = n + s$, then the maximal elements are exactly $x + e(S)$ for any $x \in \mathrm{PF}(S)$.

Example 1.6.2. Let S be the numerical semigroup considered in Example 1.6.1. The Apéry set of S is $\mathrm{Ap}(S) = \{0, 41, 26, 19, 28, 13, 30, 15\}$. The minimal elements in $\mathrm{Ap}(S)$ are the minimal generators, whereas the maximal elements are 19,30 and 41. This means that $\mathrm{PF}(S) = \{11, 22, 33\}$ and consequently the type of S is 3.

As in ring theory, it is possible to develop an ideal theory for numerical semigroups. More precisely a set $E \subset \mathbb{Z}$ is said to be a *relative ideal* of S if $E + S \subseteq E$ and there exists $x \in S$ such that $x + E \in S$; if furthermore $x = 0$, i.e. $E \subseteq S$, we say that E is a *proper ideal* or, simply, an *ideal* of S. It is not difficult to see that the integral closure of $k[[S]]$ is $k[[t]]$, that is a DVR; if we consider the valuation v induced by $k[[t]]$ on $k[[S]]$, we have that $S = v(k[[S]]) := \{v(x) \mid x \in k[[S]] \setminus \{0\}\}$ and, for any ideal I of $k[[S]]$, we remark that $v(I)$ is an ideal of S.

An obvious example of proper ideal is the *maximal ideal* $M(S) := S \setminus \{0\}$, while an important relative ideal is the *standard canonical ideal* $K(S) := \{x \in \mathbb{N} \mid f(S) - x \notin S\}$; more generally, we call *canonical ideal* all relative ideal of the form $z + K(S)$ for some $z \in \mathbb{Z}$. If E and F are two relative ideals, it is not difficult to verify that also $E - F := \{x \in \mathbb{Z} \mid x + F \subseteq E\}$ is a relative ideal. The following lemma gives some useful properties of the standard canonical ideal.

Lemma 1.6.3. *The following properties hold:*

1. *$K(S)$ is finitely generated by the elements $f(S) - x$, where $x \in \mathrm{PF}(S)$;*

2. *For any relative ideal E, $K(S) - E = \{x \in \mathbb{Z} \mid f(S) - x \notin E\}$;*

3. *$E \subseteq F \Leftrightarrow K(S) - F \subseteq K(S) - E$ and $E = F \Leftrightarrow K(S) - F = K(S) - E$;*

4. *If $E \subseteq F$, then $|F \setminus E| = |(K(S) - E) \setminus (K(S) - F)|$;*

5. *$K(S) - (K(S) - E) = E$;*

6. *$K(S) - K(S) = S$.*

Proof. The first two properties follow from [74, Proposition 12] and [52, Hilfssatz 5] respectively, while the others are straightforward applications of the second one. ☐

A numerical semigroup is said to be *symmetric* if $f(S) - s \in S$ for all $s \notin S$, that is equivalent to say that $S = K(S)$. As the choice of the name can suggest, canonical ideals of S correspond exactly to the canonical ideals of $k[[S]]$ and S is symmetric if and only if $k[[S]]$ is a Gorenstein ring, as it was proved by E. Kunz in [53]. It is easy to see that if S is symmetric, its Frobenius number has to be odd, since otherwise $f(S)/2$ and $f - f(S)/2$ are not in S. Thus we say that S is *pseudo-symmetric* if $f(S) - s \in S$ for all $s \notin S$, except $f(S)/2$.

Since $t(S) = t(k[[S]])$ and $k[[S]]$ is Gorenstein exactly when has type 1, it follows that a numerical semigroup is symmetric if and only if it has type 1. Moreover it is easy to see that a pseudo-symmetric numerical semigroup has type 2, but there are numerical semigroups with type 2 that are not pseudo-symmetric. It is not difficult to prove that S is symmetric if and only if $g(S) = (f(S)+1)/2$ and S is pseudo-symmetric if and only if $g(S) = (f(S)+2)/2$. In [13], V. Barucci and R. Fröberg call a numerical semigroup *almost symmetric* if $g(S) = (f(S) + t(S))/2$ and they prove that this condition is equivalent to requiring that $M(S) + K(S) = M(S)$. For this reason, they also define almost Gorenstein rings as we did in Section 1.3 and, in fact, $k[[S]]$ is almost Gorenstein if and only if S is almost symmetric. The trivial observation that, if S is an almost symmetric semigroup, the type of S is odd if and only if $f(S)$ is odd, will be important in Chapter 3 and will result a great difference in behaviour between almost symmetric with odd and even type. We recall now a useful characterization of almost symmetric numerical semigroups which is due to H. Nari.

Theorem 1.6.4. [60, Theorem 2.4] *Let S be a numerical semigroup. Set $m = e(S) - t(S)$ and $\mathrm{Ap}(S) = A \sqcup B$, where $A = \{0 < \alpha_1 < \cdots < \alpha_m\}$, $B = \{\beta_1 < \cdots < \beta_{t(S)-1}\}$ and $\mathrm{PF}(S) = \{\beta_i - e(S) \mid 1 \leq i \leq t(S) - 1\} \cup \{\alpha_m - e(S) = f(S)\}$. Set also $f_i = \beta_i - e(S)$. Then the following conditions are equivalent:*

1. *S is almost symmetric;*

2. $\alpha_i + \alpha_{m-i} = \alpha_m$ *for all* $i \in \{1, 2, \dots, m-1\}$ *and* $\beta_j + \beta_{t(S)-j} = \alpha_m + e(S)$ *for all* $j \in \{1, 2, \dots, t(S)-1\}$*;*

3. $f_i + f_{t(S)-i} = f(S)$ *for all* $i \in \{1, \dots, t(S)-1\}$*.*

Using the theorem above it follows that the numerical semigroup considered in Example 1.6.2 is almost symmetric, since $11 + 22 = 33$.

If $x \in S$, it is clear that $f(S) - x \notin S$, because $f(S) \notin S$. These gaps are said to be *gaps of the first type* and represent the gaps that "have to be" in a numerical semigroup; it is straightforward to see that S is symmetric if and only if all its gaps are of the first type. This leads to define the set of *gaps of the second type* of S as $\mathrm{L}(S) = \{x \notin S \mid f(S) - x \notin S\}$; V. Barucci and R. Fröberg proved that S is almost symmetric if and only if $\mathrm{L}(S) \subseteq \mathrm{PF}(S)$, see [13]. As a consequence, we can give another characterization of almost symmetric numerical semigroups:

Lemma 1.6.5. *A numerical semigroup S is almost symmetric if and only if the following property holds for all $s \in \mathbb{Z} \setminus \{0\}$:*

$$s \in S \Longleftrightarrow f(S) - s \notin S \cup \mathrm{PF}(S) \tag{1.1}$$

Proof. If s is an element of S and $f(S)-s \in S \cup \mathrm{PF}(S)$, then we would have $f(S) = s + (f(S) - s) \in S$, that is a contradiction; as a consequence, (1.1) is equivalent to $f(S) - s \in S \cup \mathrm{PF}(S)$ for all $s \notin S$. Clearly, the set of the elements such that $f(S) - s \notin S$ for some $s \notin S$ is $\mathrm{L}(S)$, and then Condition 1.1 is equivalent to $\mathrm{L}(S) \subseteq \mathrm{PF}(S)$, i.e. S is almost symmetric. □

Let E be a relative ideal of S. We set $e(E) := \min E$, $f(E) := \max(\mathbb{Z} \setminus E)$ and $g(E) := |(\mathbb{Z} \setminus E) \cap \{e(E), e(E)+1, \dots, f(E)\}|$. Sometimes it is useful to shift a relative ideal E in order to get $f(S) = f(E)$, for this reason we define $\widetilde{E} := E + f(S) - f(E)$. The following hold:

Lemma 1.6.6. *Let E be a relative ideal of S. Then*

1. $f(\widetilde{E}) = f(S)$*;*

2. $\widetilde{E} \subseteq K(S)$*;*

3. $f(S) + 1 - g(S) \leq g(\widetilde{E}) + e(\widetilde{E})$ *and equality holds if and only if* $\widetilde{E} = K(S)$*, i.e. E is a canonical ideal.*

Proof. The first statement is obvious, whereas the second one follows easily from the definitions of relative ideal and of $K(S)$: if $x \in \widetilde{E} \setminus K(S)$, then $f(S) - x \in S$ and consequently $(f(S)-x)+x = f(S) = f(\widetilde{E}) \in \widetilde{E}$, a contradiction. As for the last assertion, we note first that the integer $g(\widetilde{E}) + e(\widetilde{E})$ is the number of elements in $\mathbb{N} \setminus \widetilde{E}$; on the other hand $f(S)+1-g(S)$ is the number of elements in S which are smaller than $f(S)+1$. Since $s \in S$ implies $f(S) - s \notin K(S) \supseteq \widetilde{E}$, the thesis follows from the definition of $K(S)$. □

We conclude this section with a lemma that will be useful in Chapter 4.

Lemma 1.6.7. *The Frobenius number of the relative ideal $K(S) - (M(S) - M(S))$ is equal to $f(S)$. Moreover, $K(S) + M(S) \subseteq K(S) - (M(S) - M(S)) \subseteq K(S)$.*

Proof. Let $E := K(S) - (M(S) - M(S))$. By Lemma 1.6.3.2 the Frobenius number of E is $f(S) - e(E)$, but since $0 \in M(S) - M(S) \subseteq \mathbb{N}$, it follows that $f(S) = f(E)$. Moreover, since $S \subseteq M(S) - M(S)$, we get $K(S) - (M(S) - M(S)) \subseteq K(S) - S = K(S)$. Finally, let $y \in K(S)$, $s \in M(S)$ and $z \in M(S) - M(S)$. Then $y + (s + z) \in K(S) + M(S) \subseteq K(S)$. □

Remark 1.6.8. It is possible to associate numerical semigroups with other type of rings that are useful for example in algebraic geometry or in coding theory. An *algebroid branch* is a local ring $(R, \mathfrak{m})$ of the form $k[[x_1, \dots x_n]]/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of height $n - 1$ and k is algebraically closed. Thus, R is a one-dimensional, noetherian, complete, local integral domain and its integral closure is isomorphic to $k[[t]]$. If we consider the valuation v induced by $k[[t]]$ on R, we get that $v(R) := \{v(r) \mid r \in R \setminus \{0\}\}$ is a numerical semigroup, called the *value semigroup* of R, and that $v(I) := \{v(i) \mid i \in I \setminus \{0\}\}$ is an ideal of $v(R)$.

We end this chapter by recalling a definition. Given a numerical semigroup T and a positive number $d \geq 2$ in [78] it is defined the set

$$\frac{T}{d} := \{s \mid ds \in T\}.$$

This is a numerical semigroup called one over d of T or, if $d = 2$, *one half of* T. If S is $1/d$ of T we also say that T is a *d-fold* of S or, when $d = 2$, a *double* of S. In general, if T is a d-fold of S for some d, we say that T is a *multiple* of S and that S is a *quotient* of T. It is clear that there are infinitely many d-folds of a given numerical semigroup. These semigroups were introduced in order to study the solutions of inequalities of the form $ax \mod b \leq cx$, where a, b, and c are positive integers; see [78] for more details.

Chapter 2

A family of quotients of the Rees algebra

Given a ring R and an ideal I, in this chapter we introduce a family of rings that includes idealization and amalgamated duplication among its members, giving a unified approach to them. We will prove that many properties of the family depend only on R and I and thus are the same for all the members. On the other hand, if R is an integral domain, there are integral domains in the family, even if it idealization and amalgamated duplication are never integral domains. In the last section we introduce the numerical duplication of a numerical semigroup, that is a particular case of the family and will play a crucial role in the following two chapters. Several results of this chapter are contained in [10, 11, 25, 26].

2.1 The family $R(I)_{a,b}$

Before to introduce the new family, we set some notations and prove some preparatory results. We recall that the Rees algebra associated with the ring R and its ideal I is the ring $R[It] = \bigoplus_{n\geq 0} I^n t^n \subseteq R[t]$, where t is an indeterminate.

Lemma 2.1.1. *Let $f(t) \in R[t]$ be a monic polynomial of positive degree k and let $(I^k f(t)) := f(t)R[t] \cap R[It]$. Then:*

1. *$(I^k f(t)) = \{f(t)g(t) \mid g(t) \in I^k R[It]\}$;*

2. *Each element of the ring $R[It]/(I^k f(t))$ is represented by an unique polynomial of $R[It]$ of degree less than k;*

3. *Both ring extensions $R \subseteq R[It]/(I^k f(t)) \subseteq R[t]/(f(t))$ are integral. In particular, the three rings have the same dimension.*

Proof. 1. Since $I^k R[It] = \{\sum_{i=0}^{n} b_i t^i \mid b_i \in I^{k+i}\}$, if $g(t) \in I^k R[It]$ then it is clear that $f(t)g(t)$ is in $(I^k f(t))$. Conversely, let $h(t) \in (I^k f(t))$, i.e. $h(t) = f(t)g(t) \in R[It]$ with $g(t) \in R[t]$; we want to prove that $g(t) \in I^k R[It]$ by induction on the degree of $g(t)$. If the degree of $g(t)$ is zero, i.e. $g(t) = r \in R$, the leading term of $f(t)r$ is rt^k and, since $f(t)r \in R[It]$, it follows that $r \in I^k \subset I^k R[It]$.

Assume now that the thesis is true for polynomial of degree $n-1$ and suppose that the leading term of $g(t)$ is $g_n t^n$; thus, the leading term of $f(t)g(t)$ is $g_n t^{k+n}$ and, since $f(t)g(t) \in R[It]$, then $g_n \in I^{k+n}$. Therefore $f(t)g_n t^n \in R[It]$. It follows that $f(t)g(t) - f(t)g_n t^n = f(t)\bar{g}(t) \in R[It]$, where $\deg \bar{g}(t) < n$. By inductive hypothesis $\bar{g}(t) \in I^k R[It]$, hence $g(t) = \bar{g}(t) + g_n t^n \in I^k R[It]$.

2. Let $h(t)$ be an element of $R[It]$ and consider the euclidean division by $f(t)$ that gives $h(t) = f(t)q(t)+r(t)$, with $\deg r(t) < k$. Moreover, an easy calculation shows that $q(t) \in I^k R[It]$ and $r(t) \in R[It]$. Thus $h(t) \equiv r(t) \mod (I^k f(t))$. Finally, if $r_1(t)$ and $r_2(t)$ are distinct polynomials of $R[It]$ with degree less than k, then $\deg(r_1(t) - r_2(t)) < k$ and therefore they represent different classes.

3. In the light of 2, the inclusions hold. Moreover, the class of t in $R[t]/(f(t))$ is integral over R and, a fortiori, over $R[It]/(I^k f(t))$, then the conlusion follows from immediately. $\square$

Now we can show that idealization and amalgamated duplication can be obtained as suitable quotients of the Rees algebra.

Proposition 2.1.2. *There are the following isomorphisms of rings:*

1. $R[It]/(I^2t^2) \cong R \ltimes I$;

2. $R[It]/(I^2(t^2 - t)) \cong R \bowtie I$.

Proof. 1. By the previous lemma we know that every element of $R[It]/(I^2t^2)$ is represented in an unique way by a polynomial $r + it$, where $r \in R$ and $i \in I$. The map $\alpha : R[It]/(I^2t^2) \to R \ltimes I$ defined by setting $\alpha(r + it + (I^2t^2)) = (r, i)$ is well-defined and it is a bijection. Moreover, it is straightforward that α preserves sums and, if $r, s \in R$, $i, j \in I$, we have

$$\alpha((r + it + (I^2t^2))(s + jt + (I^2t^2))) = \alpha(rs + (rj + si)t + ijt^2 + (I^2t^2)) =$$
$$= \alpha(rs + (rj + si)t + (I^2t^2)) = (rs, rj + si) = (r, i)(s, j).$$

2. As above, we claim that the map $\beta : R[It]/(I^2(t^2 - t)) \to R \bowtie I$ defined by setting $\beta(r + it + (I^2(t^2 - t))) = (r, r + i)$ is an isomorphism of rings. We only need to show that β preserves products: for any $r, s \in R$ and $i, j \in I$ we have that $\beta((r + it + (I^2(t^2 - t)))(s + jt + (I^2(t^2 - t))))$ is equal to

$$\beta(rs + (rj + si)t + ijt^2 + (I^2(t^2 - t))) = \beta(rs + (rj + si + ij)t + ij(t^2 - t) + (I^2(t^2 - t))) =$$
$$= \beta(rs + (rj + si + ij)t + (I^2(t^2 - t))) = (rs, rs + rj + si + ij) = (r, r + i)(s, s + j).$$

$\square$

In [1] it is introduced a generalization of idealization, called *n-trivial extension* of R by a family of n modules. There, the authors note that, using the same argument above, $R[It]/(I^{n+1}t^{n+1})$ is isomorphic to n-trivial extension of R by the family $\{I^j \mid j = 1, \dots, n\}$.

To provide a unified approach to idealization with respect to an ideal and amalgamated duplication, it is enough to consider the following family of rings.

Definition 2.1.3. Let R be a ring and let $I \neq 0$ be a proper ideal of R. We set

$$R(I)_{a,b} := R[It]/(I^2(t^2 + at + b)),$$

where $a, b \in R$. In the light of Lemma 2.1.1, we will denote the elements of $R(I)_{a,b}$ by polynomials $r + it$.

It is clear that, as an R-module, $R(I)_{a,b}$ is isomorphic to $R \oplus I$ and the natural injection $R \hookrightarrow R(I)_{a,b}$ is a ring homomorphism; it must be noted that in general $\{(0, i) \mid i \in I\}$ is not an ideal of $R(I)_{a,b}$, although this happens in the special cases of idealization and amalgamated duplication. We start our investigation studying the total ring of fractions and the integral closure of $R(I)_{a,b}$.

Proposition 2.1.4. *Let Q be the total ring of fractions of $R(I)_{a,b}$. Then the elements of Q are of the form $\frac{r+it}{u}$, where u is a regular element of R.*

Proof. Let $s + jt$ be a regular element of $R(I)_{a,b}$ and assume $(r + it)/(s + jt) \in Q$. Then for any $x \in R \setminus \{0\}$, one has $x(s + jt) \neq 0$.

Consider, now, the element $(ja - s + jt)$. It is enough to show, and we do, that it is a regular element of $R(I)_{a,b}$ and that the product $u = (s + jt)(ja - s + jt)$ is a regular element of R; since, if this is case, we can write $(r + it)/(s + jt) = (r + it)(ja - s - jt)/u$.

Since $-at - t^2 = b \in R$, we have $u = s(ja - s) - j^2 b \in R$. If $x(ja - s + jt) = 0$, for some $x \in R \setminus \{0\}$, then $xj = 0$ and consequently $x(ja - s + jt) = -xs = 0$, that is a contradiction because it implies that $x(s + jt) = 0$. It follows that u is regular in R, because, otherwise, there would exist $x \in R \setminus \{0\}$ such that $ux = 0$ and this would imply that $(s + jt)$ is not regular in $R(I)_{a,b}$, since $(ja - s + jt)x \neq 0$.

Moreover if $(ja - s + jt)$ is not regular in $R(I)_{a,b}$, there exists $(h + kt) \neq 0$ such that $(ja - s + jt)(h + kt) = 0$. Hence, $u(h + kt) = 0$ and therefore u is not regular in $R(I)_{a,b}$. But this would imply that u is not regular in R, that is a contradiction. $\square$

Corollary 2.1.5. *Let I be a regular ideal. The rings $R(I)_{a,b}$ and $R[t]/(t^2 + at + b)$ have the same total ring of fractions and the same integral closure.*

Proof. Let Q, Q' denote the total rings of fractions of $R(I)_{a,b}$ and $R[t]/(t^2 + at + b)$ respectively and let $i \in I$ be a regular element of R. It is clear that i is regular also as element of the ring $R(I)_{a,b}$ and then, if $r + r_1 t$ is an element of $R[t]/(t^2 + at + b)$, we have $r + r_1 t = (ir + ir_1 t)/i \in Q$. Moreover if $r + r_1 t$ is regular in $R[t]/(t^2 + at + b)$, it is straightforward to see that it is regular also in Q. Consequently, if $(r + r_1 t)/(s + s_1 t)$ is in Q', the elements $r + r_1 t$, $s + s_1 t$ are also in Q and $s + s_1 t$ is regular in Q, therefore $(r + r_1 t)/(s + s_1 t) \in Q$. On the other hand, if $(r + it)/u \in Q$, where u is R-regular, it is clear that u is regular also in $R[t]/(t^2 + at + b)$ and, thus, $(r + it)/u \in Q'$. Finally, since the $R(I)_{a,b} \subseteq R[t]/(t^2 + at + b)$ is integral, it follows that they have the same integral closure. $\square$

Remark 2.1.6. The previous corollary implies that the integral closure of $R(I)_{a,b}$ contains $\overline{R}[t]/(t^2 + at + b)$, where $\overline{R}$ is the integral closure of R. If I is a regular ideal, in the particular cases of idealization and amalgamated duplication the equality holds, as easily follows from [2, Theorem 4.2] and [24, Corollary 3.3] respectively, but this does not happen in general: for instance $\mathbb{Z}[t]/(t^2+4)$ is not integrally closed, since $(t/2)^2+1=0$.

Now we are going to show that the noetherianity of $R(I)_{a,b}$ does not depend on a and b.

Proposition 2.1.7. *The ring $R(I)_{a,b}$ is noetherian if and only if R is noetherian.*

Proof. If R is noetherian, then the Rees algebra $R[It]$ is noetherian; hence, also $R(I)_{a,b}$ is noetherian as a quotient of a noetherian ring.

Conversely, assume by contradiction that R is not noetherian; then there exists an ideal $J=(f_1,f_2,\dots)$ of R that is not finitely generated, where we may assume that $f_{i+1}\notin(f_1,\dots f_i)$ for all i. By hypothesis, the ideal $JR(I)_{a,b}$ is finitely generated; thus, its generators can be chosen among those of J (regarded as elements of $R(I)_{a,b}$), i.e. $JR(I)_{a,b}=(f_1,\dots,f_s)$ for some s. Consequently, $f_{s+1}=\sum_{k=1}^{s}f_k(r_k+i_kt)$ for some $r_k\in R$ and $i_k\in I$ and therefore $f_{s+1}=\sum_{k=1}^{s}f_kr_k$, that is a contradiction. $\square$

In this and next chapter we are mainly interested to the local case, i.e. when $R(I)_{a,b}$ is local. In the next proposition we show how the maximal ideal of $R(I)_{a,b}$ is linked to R.

Proposition 2.1.8. *The ring $R(I)_{a,b}$ is local if and only if R is local. In this case, if $\mathfrak{m}$ is the maximal ideal of R, then $\mathfrak{m}_{a,b}:=\{m+it\,|\,m\in\mathfrak{m},i\in I\}$ is the maximal ideal of $R(I)_{a,b}$.*

Proof. If $R(I)_{a,b}$ is local, also R is local, because $R\subseteq R(I)_{a,b}$ is an integral extension by Lemma 2.1.1. Conversely if $(R,\mathfrak{m})$ is local, it is enough to show that all the elements $r+it$ with $r\notin\mathfrak{m}$ are invertible in $R(I)_{a,b}$. From the equation $(r+it)(s+jt)=1$, we obtain the linear system

$$\begin{cases} rs-ibj=1\\ is+(r-ia)j=0\end{cases}$$

whose determinant is $\delta=r^2-iar+i^2b\in r^2+\mathfrak{m}$, that is invertible in R. Moreover, it is easy to check that if (s,j) is the solution of the system, then $j\in I$. Hence, $s+jt\in R(I)_{a,b}$ is the inverse of $r+it$. $\square$

As in the previous proposition, if $(R,\mathfrak{m},k)$ is local we will denote by $\mathfrak{m}_{a,b}$ the maximal ideal of $R(I)_{a,b}$. It is easy to see that $k\cong R(I)_{a,b}/\mathfrak{m}_{a,b}$ for all $a,b\in R$.

Remark 2.1.9. Since $R(I)_{a,b}$ is an R-algebra, every $R(I)_{a,b}$-module N is also an R-module by restriction and then $\ell_{R(I)_{a,b}}(N)\le\ell_R(N)$. On the other hand, if we consider a $R(I)_{a,b}$-module N annihilated by $\mathfrak{m}_{a,b}$, we have that, as an R-module, N is annihilated by $\mathfrak{m}$ and, hence, it is naturally an $(R(I)_{a,b}/\mathfrak{m}_{a,b})-$ and an $(R/\mathfrak{m})-$vector space; in particular, since $k=R/\mathfrak{m}\cong R(I)_{a,b}/\mathfrak{m}_{a,b}$, we get $\ell_{R(I)_{a,b}}(N)=\dim_k(N)=\ell_R(N)$.

Lemma 2.1.10. *Let $(R,\mathfrak{m})$ be a local ring. Then, for any $a,b\in R$ and $n>1$,*

$$\mathfrak{m}_{a,b}^n=\{m_n+m_{n-1}it\mid m_n\in\mathfrak{m}^n,\ m_{n-1}\in\mathfrak{m}^{n-1},\ i\in I\}.$$

In particular, $\mathfrak{m}_{a,b}^n\cong\mathfrak{m}^n\oplus\mathfrak{m}^{n-1}I$ as R-modules.

Proof. Let $\mathfrak{m}^n+\mathfrak{m}^{n-1}It$ denote the ideal on the right side. Let us first consider $n=2$. If $(r+it)$ and $(s+jt)$ are in $\mathfrak{m}_{a,b}$, then their product $rs-bij+(rj+si-aij)t\in\mathfrak{m}^2+\mathfrak{m}It$. Conversely, if $rs+uit$ is an element of $\mathfrak{m}^2+\mathfrak{m}It$, we have $rs+uit=rs+u(it)\in\mathfrak{m}_{a,b}^2$. Arguing similarly for all $n\geq 2$, we immediately obtain that $\mathfrak{m}_{a,b}^n=\mathfrak{m}^n+\mathfrak{m}^{n-1}It$. $\square$

The previous lemma allows us to prove that the Hilbert function and the completion of $R(I)_{a,b}$ are independent of a,b. This is shown, together with some other consequences, in the next result.

Corollary 2.1.11. *Let R be a noetherian local ring and $I\neq 0$ a proper ideal of R. Then, for every $a,b\in R$, we have:*

1. *The Hilbert function of $R(I)_{a,b}$ is*

$$H_{R(I)_{a,b}}(h)=H_R(h)+\ell_R(\mathfrak{m}^{h-1}I/\mathfrak{m}^hI)$$

for all $h\geq 1$;

2. *The embedding dimension of $R(I)_{a,b}$ is $\nu(R(I)_{a,b})=\nu(R)+\nu(I)$;*

3. *The multiplicity of $R(I)_{a,b}$ is independent of a and b;*

4. *$R(I)_{a,b}$ is never a regular ring;*

5. *$\widehat{R(I)_{a,b}}\cong\hat{R}(\hat{I})_{a,b}$.*

Proof. By Remark 2.1.9, the length of $\mathfrak{m}_{a,b}^n/\mathfrak{m}_{a,b}^{n+1}$ as $(R(I)_{a,b})-$module coincides with its length as R-module and this, together with the previous lemma, implies 1. Furthermore 2 and 3 follow from 1, since $\nu(R(I)_{a,b})=H_{R(I)_{a,b}}(1)=H_R(1)+\ell_R(I/\mathfrak{m}I)=\nu(R)+\nu(I)$ and the multiplicity of $R(I)_{a,b}$ are determined by the Hilbert function. Moreover, $\dim R(I)_{a,b}=\dim R\leq\nu(R)<\nu(R)+\nu(I)=\nu(R(I)_{a,b})$ and consequently $R(I)_{a,b}$ is never regular.

Finally, it is straightforward to see that the $\mathfrak{m}_{a,b}$-adic topology on $R(I)_{a,b}$ coincides with the $\mathfrak{m}$-adic topology on $R(I)_{a,b}$ induced by its structure as an R-module. Hence, as R-module, $\widehat{R(I)_{a,b}}\cong\widehat{R\oplus I}\cong\hat{R}\oplus\hat{I}$. Moreover, we can assume $R\subset\hat{R}$ by noetherianity and then $a,b\in\hat{R}$; consequently it is clear that $\widehat{R(I)_{a,b}}\cong\hat{R}(\hat{I})_{a,b}$. $\square$

Remark 2.1.12. In [23, Corollary 5.8] M. D'Anna, M. Fontana and C. Finocchiaro prove that, if $(R,\mathfrak{m},k)$ is a Cohen-Macaulay local ring and k is infinite, then $e(R\bowtie I)=e(R)+\ell_R(I/IJ)$, where J is a minimal reduction of $\mathfrak{m}$; in particular, if $\dim R=1$, this implies that $e(R\bowtie I)=2e(R)$.

Under the same assumptions, Proposition 2.1.11 implies that

$$e(R\bowtie I)=e(R)+\ell_R(I/IJ)$$

for every $a,b\in R$, where J is a minimal reduction of $\mathfrak{m}$.

We conclude this section by proving that, in the local case, also Cohen-Macaulayness is independent of a and b.

Proposition 2.1.13. *Let R be a local ring. The following conditions are equivalent:*

1. *R is a Cohen-Macaulay ring and I is a maximal Cohen-Macaulay R-module;*

2. *$R(I)_{a,b}$ is a Cohen-Macaulay R-module;*

3. *$R(I)_{a,b}$ is a Cohen-Macaulay ring;*

4. *R is a Cohen-Macaulay ring and each regular R-sequence of elements of R is also an $R(I)_{a,b}$-regular sequence.*

Proof. We set $d = \dim R = \dim R(I)_{a,b}$ and observe that it is also equal to the dimension of $R(I)_{a,b}$ as an R-module, because $\mathrm{Ann}_R(R(I)_{a,b}) = 0$.

$1 \Leftrightarrow 2$. As R-module, $R(I)_{a,b}$ is isomorphic to $R \oplus I$ and, therefore, it follows that $\operatorname{depth} R(I)_{a,b} = \min\{\operatorname{depth} R, \operatorname{depth} I\}$. Thus, $R(I)_{a,b}$ is a maximal Cohen-Macaulay R-module if and only if R is a Cohen-Macaulay ring and $\operatorname{depth} I = d = \dim R$, i.e. I is a maximal Cohen-Macaulay R-module.

$2 \Leftrightarrow 3$. We have already noticed that the dimension of $R(I)_{a,b}$ as a ring and as an R-module coincide. Moreover, since the extension $R \subset R(I)_{a,b}$ is finite, the depth of $R(I)_{a,b}$ as an $R-$module is equal to the depth of $R(I)_{a,b}$ as an $R(I)_{a,b}-$module (see [17, Exercise 1.2.26]).

$3 \Rightarrow 4$. R is Cohen-Macaulay, because 3 is equivalent to 1. Moreover, since in a Cohen-Macaulay ring $\mathbf{x}$ is a regular sequence if and only if it is part of a system of parameters (cf. [17, Theorem 2.1.2 (d)]), it is enough to show that, if $\mathbf{x}$ is part of a system of parameters of R, then it is also part of a system of parameters of $R(I)_{a,b}$; this is true because the extension $R \subseteq R(I)_{a,b}$ is integral.

$4 \Rightarrow 2$. This is obvious. $\square$

2.2 The Gorenstein property

In this section we generalize Theorem 1.5.1 to all the members of our family. We start with the artinian case. In this case R has no proper canonical ideals and in fact in the next theorem we prove that $R(I)_{a,b}$ is never Gorenstein.

Theorem 2.2.1. *Let $(R, \mathfrak{m}, k)$ be a local artinian ring and let $I \neq 0$ be a proper ideal of R. Then $R(I)_{a,b}$ is not Gorenstein for all $a, b \in R$.*

Proof. Since an artinian ring is Cohen-Macaulay, $R(I)_{a,b}$ is Gorenstein if and only if its socle is a one-dimensional k-vector space. By definition, $r + it$ is an element of $\operatorname{soc} R(I)_{a,b}$ if and only if

$$\begin{cases} rm - ijb = 0, \\ rj + mi - aij = 0 \end{cases}$$

for all $m + jt \in \mathfrak{m}_{a,b}$, i.e. for all $m \in \mathfrak{m}$ and all $j \in I$. In particular, if $j = 0$ we get $r \in \operatorname{soc} R$ and $i \in I \cap \operatorname{soc} R$; thus

$$\operatorname{soc} R(I)_{a,b} \subseteq \{r + it \mid r \in \operatorname{soc} R,\ i \in I \cap \operatorname{soc} R\}.$$

It is straightforward to check that also the opposite inclusion holds, thus the above is an equality. Furthermore, we claim that $I \cap \operatorname{soc} R \neq (0)$. Indeed, if $0 \neq x \in I$, we have $x\mathfrak{m}^n = (0)$ for some $n \in \mathbb{N}$, because $\mathfrak{m}$ is nilpotent by artinianity. We can assume that $x\mathfrak{m}^{n-1} \neq (0)$ and clearly $x\mathfrak{m}^{n-1} \subseteq I \cap \operatorname{soc} R$.

Consequently, if $0 \neq i \in I \cap \operatorname{soc} R$, we have that i and it are elements of $\operatorname{soc} R(I)_{a,b}$ and they are linearly independent over k; hence $R(I)_{a,b}$ is not a Gorenstein ring. $\square$

To study the Gorenstein property of $R(I)_{a,b}$ in positive dimension, in the next theorem we will give a formula for its type.

Theorem 2.2.2. *Let $(R, \mathfrak{m}, k)$ be a local Cohen-Macaulay ring of positive dimension d and let I be a proper ideal of R that is also a maximal Cohen-Macaulay module. The type of $R(I)_{a,b}$ is*

$$t(R(I)_{a,b}) = \ell_R\left(\frac{(J : \mathfrak{m}) \cap (JI : I)}{J}\right) + \ell_R\left(\frac{JI : \mathfrak{m}}{JI}\right),$$

where $J = (x_1, \dots, x_d)$ is an ideal of R generated by an R-regular sequence. In particular, the type of $R(I)_{a,b}$ is independent of a, b.

Proof. The type of $R(I)_{a,b}$ is equal to $\ell_{R(I)_{a,b}}((H : \mathfrak{m}_{a,b})/H)$ for any ideal H generated by a maximal $R(I)_{a,b}$-regular sequence. By Proposition 2.1.13 we can choose H generated by an $R-$regular sequence $\mathbf{x} = x_1, \dots, x_d$. This means that $H = JR(I)_{a,b} = \{j_1 + j_2 t \mid j_1 \in J,\, j_2 \in JI,\, t^2 = -at - b\}$, where J is the ideal of R generated by $\mathbf{x}$. Moreover, since $(JR(I)_{a,b} : \mathfrak{m}_{a,b})/JR(I)_{a,b}$ is annihilated by $\mathfrak{m}$, Remark 2.1.9 implies that its length as an $R(I)_{a,b}-$module coincides with its length as an $R-$module. Hence,

$$t(R(I)_{a,b}) = \ell_R\left(\frac{JR(I)_{a,b} : \mathfrak{m}_{a,b}}{JR(I)_{a,b}}\right).$$

We claim that

$$JR(I)_{a,b} : \mathfrak{m}_{a,b} = \left\{\frac{r}{s} + \frac{i}{s}\, t \mid \frac{r}{s} \in (JI : I) \cap (J : \mathfrak{m}),\ \frac{i}{s} \in (JI : \mathfrak{m})\right\}.$$

By Proposition 2.1.4 an element of $Q(R(I)_{a,b})$ is of the form $r/s + (i/s)t$, where $r, s \in R, i \in I$ and s is regular. Therefore, this is an element of $JR(I)_{a,b} : \mathfrak{m}_{a,b}$ if and only if

$$\begin{aligned}(r/s + (i/s)t)(m + jt) &= rm/s + (im/s)t + (rj/s)t + (ij/s)t^2 = \\ &= rm/s - ijb/s + (im/s + rj/s - ija/s)t\end{aligned}$$

is an element of $JR(I)_{a,b}$, for all $m \in \mathfrak{m}$ and $j \in I$, that means $rm/s - ijb/s \in J$ and $im/s + rj/s - ija/s \in JI$.

Suppose that $r/s + (i/s)t \in (JR(I)_{a,b} : \mathfrak{m}_{a,b})$. In particular, if $j = 0$ we have $rm/s \in J$ and $im/s \in JI$, that is $r/s \in (J : \mathfrak{m})$ and $i/s \in (JI : \mathfrak{m})$. Moreover, since $ja \in I \subseteq \mathfrak{m}$, we have that im/s, ija/s and $im/s + rj/s - ija/s$ are elements of JI; therefore $rj/s \in JI$ for all $j \in I$ and then $r/s \in (JI : I)$.

Conversely, suppose that $i/s \in (JI : \mathfrak{m})$ and $r/s \in (JI : I) \cap (J : \mathfrak{m})$. Then $rm/s - ijb/s \in J + JI = J$ and $im/s + rj/s - ija/s \in JI + JI + JI = JI$ and consequently $r/s + (i/s)t \in (JR(I)_{a,b} : \mathfrak{m}_{a,b})$.

Now it is straightforward to see that the homomorphism of R-modules

$$JR(I)_{a,b} : \mathfrak{m}_{a,b} \longrightarrow \frac{(J : \mathfrak{m}) \cap (JI : I)}{J} \times \frac{JI : \mathfrak{m}}{JI}$$

defined by $r/s + (i/s)t \mapsto (r/s + J, i/s + JI)$ is surjective and its kernel is $JR(I)_{a,b}$. The thesis follows immediately. $\square$

Since in positive dimension the type of $R(I)_{a,b}$ does not depend on $a, b \in R$, this happens also for the Gorenstein property; therefore, Theorems 1.5.1 and 2.2.1 imply the following corollary:

Corollary 2.2.3. *Let R be a local ring and let I be a proper regular ideal of R. For every $a, b \in R$ the ring $R(I)_{a,b}$ is Gorenstein if and only if R is a Cohen-Macaulay ring and I is a canonical ideal of R.*

Corollary 2.2.4. *Let R be a regular local ring and let I be a proper ideal of R. The ring $R(I)_{a,b}$ is Cohen-Macaulay if and only if is Gorenstein.*

Proof. By Auslander-Buchsbaum Formula, the ideal I is a maximal Cohen-Macaulay module if and only if is free (cf. [17, Theorem 1.3.3]). Moreover, R is an integral domain and an ideal of an integral domain is free if and only if is principal, that, since R is Gorenstein, is equivalent to say that I is a canonical ideal. Therefore, it is enough to apply Proposition 2.1.13 and Corollary 2.2.3. $\square$

The next proposition characterizes when $R(I)_{a,b}$ is complete intersection.

Proposition 2.2.5. *Let R be a local ring and let I be a proper regular ideal of R. The ring $R(I)_{a,b}$ is a complete intersection if and only if R is a complete intersection and I is a canonical ideal of R.*

Proof. Let $i_1, \dots, i_p$ be a set of minimal generators of I. Cohen's Structure Theorem guarantees that there exists a complete regular local ring S such that $\hat{R} \cong S/J$ for some ideal J of S. Consequently, using Corollary 2.1.11.5, the ring $\widehat{R(I)_{a,b}}$ can be presented as $S[[y_1, \dots, y_p]]/\ker \varphi$, where $\varphi : S[[y_1, \dots, y_p]] \to \widehat{R(I)_{a,b}}$ is defined by $\varphi(s) = s + J$ and $\varphi(y_h) = i_h t$, for every $h = i, \dots, p$. It is well-known that also $S[[y_1, \dots, y_p]]$ is a regular local ring.

Since $(i_h t)^2 = -ai_h^2 t - bi_h^2$ and $ai_h, bi_h^2 \in \hat{R}$, if we choose $\alpha_h, \beta_h \in S$ such that $\varphi(\alpha_h) = ai_h$ and $\varphi(\beta_h) = bi_h^2$, it follows that $\ker \varphi$ contains the elements of the form $F_h := y_h^2 + \alpha_h y_h + \beta_h$. Consequently, $\ker \varphi \supseteq J + (F_1, \dots, F_p)$. For every h, up to multiplication by invertible elements, F_h has to be a generator of $\ker \varphi$, since it contains a pure power of y_h of the lowest possible degree. Moreover, since the restriction of φ to S gives the presentation of $\hat{R}$, it follows that $\ker \varphi \cap S = J$ and it is easy to see that the minimal generators of J have to be also minimal generators of $\ker \varphi$. Consequently, $\nu(\ker \varphi)$ is bigger than or equal to $\nu(J) + p$.

Assume now that $R(I)_{a,b}$ is a complete intersection, i.e. that $\dim S + p - \dim \widehat{R(I)_{a,b}} = \nu(\ker \varphi)$, therefore we get:

$$\dim S + p - \dim \widehat{R(I)_{a,b}} = \nu(\ker \varphi) \geq \nu(J) + p \geq \dim S - \dim \hat{R} + p.$$

Since $\dim \hat{R} = \dim \widehat{R(I)_{a,b}}$, all the above are equalities and, in particular, $\nu(J) = \dim S - \dim \hat{R}$, i.e. R is a complete intersection. Since complete intersection are Gorenstein, Corollary 2.2.3 yields that I is a canonical ideal of R.

Conversely, assume that R is a complete intersection and that I is a canonical ideal of R. It follows that $\nu(I) = t(R) = 1$ and then, with the above notation, we get $\ker \varphi \supseteq J + (F_1)$. The other inclusion is also true, since, if $g(y_1) \in \ker \varphi$, its class modulo $J+(F_1)$ is of the form $g_0+g_1y_1$ (with $g_0, g_1 \in S$) and it belongs to $\ker \varphi$ if and only if $g_0 \in J$ and $\varphi(g_1)i_1t = 0$; since i_1 is a non-zero-divisor, the last equality implies that also $g_1 \in J$. This proves that $\nu(\ker \varphi) = \nu(J)+1$ and, since $\nu(J) = \dim S - \dim \hat{R}$, it follows that $R(I)_{a,b}$ is a complete intersection. □

2.3 The almost Gorenstein property

In this section we will study when $R(I)_{a,b}$ is almost Gorenstein and will prove that also this property is independent of a and b. We start with the classical case, i.e. the one-dimensional case, and then we use that to say something about the higher dimensional case. In this section all canonical ideals will be fractional ideals.

The one-dimensional case

Throughout this section we assume that $(R(I)_{a,b}, \mathfrak{m}_{a,b}, k)$, and consequently $(R, \mathfrak{m}, k)$, are Cohen-Macaulay one-dimensional local rings. We also assume that k is infinite and that R has a canonical module ω_R that is a fractional ideal such that $R \subseteq \omega_R \subseteq \overline{R}$. We know that there exists a minimal reduction zR of the fractional ideal $(\omega_R : I)$; recall that z is an element of $Q(R)$.

Since the inclusion $R \subseteq R(I)_{a,b}$ is a local homomorphism and $R(I)_{a,b}$ is a finitely generated R-module, it follows from [17, Theorem 3.3.7 (b)] that the canonical module of $R(I)_{a,b}$ is $\omega_{R(I)_{a,b}} = \mathrm{Hom}_R(R(I)_{a,b}, \omega_R)$, where the structure as an $R(I)_{a,b}-$module is given by $((r + it)\varphi)(s + jt) := \varphi((r + it)(s + jt))$, for $\varphi \in \mathrm{Hom}_R(R(I)_{a,b}, \omega_R)$. Clearly, as R-modules,

$$\begin{aligned}\omega_{R(I)_{a,b}} &\cong \mathrm{Hom}_R(R \oplus I, \omega_R) \cong \mathrm{Hom}_R(R, \omega_R) \oplus \mathrm{Hom}_R(I, \omega_R) \cong \\ &\cong \omega_R \oplus (\omega_R : I) \cong \frac{1}{z}(\omega_R : I) \oplus \frac{1}{z}\omega_R.\end{aligned}$$

We want to show that $\frac{1}{z}(\omega_R : I) \oplus \frac{1}{z}\omega_R$ and $\omega_{R(I)_{a,b}}$ are isomorphic as $R(I)_{a,b}$-module as well. More precisely, we define another $R-$module that is isomorphic to $\frac{1}{z}(\omega_R : I) \oplus \frac{1}{z}\omega_R$, that is

$$K := \left\{ \frac{x}{z} + \frac{y}{z}t \,|\, x \in (\omega_R : I), y \in \omega_R \right\}$$

and, for all $r+it \in R(I)_{a,b}$ and $\frac{x}{z}+\frac{y}{z}t \in K$ we define

$$(r+it)\left(\frac{x}{z}+\frac{y}{z}t\right) := \left(\frac{rx}{z}-\frac{biy}{z}+\left(\frac{ry}{z}+\frac{ix}{z}-\frac{aiy}{z}\right)t\right) \in K.$$

It is easy to see that K is now an $R(I)_{a,b}$-module.

Proposition 2.3.1. *The $R(I)_{a,b}$-module K defined above is a canonical ideal of $R(I)_{a,b}$ such that $R(I)_{a,b} \subseteq K \subseteq \overline{R(I)_{a,b}}$.*

Proof. Let $\varphi : K \to \mathrm{Hom}_R(R(I)_{a,b}, \omega_R)$ be the map that associates $(\frac{x}{z}+\frac{y}{z}t)$ with the homomorphism

$$f_{x,y} : s+jt \mapsto xj + y(s-ja).$$

Clearly φ is well defined and, since if $r+it, \;\; s+jt \in R(I)_{a,b}$ and $\frac{x}{z}+\frac{y}{z}t \in K$, one has

$$\begin{aligned}
&\left((r+it)\varphi\left(\frac{x}{z}+\frac{y}{z}t\right)\right)(s+jt) = (r+it)f_{x,y}(s+jt) = \\
&= f_{x,y}((r+it)(s+jt)) = f_{x,y}(rs-bij+(rj+is-aij)t) = \\
&= xrj+xis-aijx+yrs-bijy-arjy-aisy+a^2ijy = \\
&= f_{rx-biy,\, ix+ry-aiy}(s+jt) = \varphi\left((r+it)\left(\frac{x}{z}+\frac{y}{z}t\right)\right)(s+jt).
\end{aligned}$$

This shows that φ is a homomorphism of $R(I)_{a,b}$-modules. Moreover, if $f_{x,y}(s+jt)=0$ for all $s+jt \in R(I)_{a,b}$ and $\lambda \in I$ is regular, one has

$$\begin{cases} y = f_{x,y}(1) = 0 \\ \lambda x = f_{x,y}(\lambda a + \lambda t) = 0, \end{cases}$$

then $(x,y)=(0,0)$ and therefore φ is injective.

As for the surjectivity, let g be an element of $\mathrm{Hom}_R(R(I)_{a,b}, \omega_R)$. Consider a regular element $\lambda \in I$ and let

$$\begin{cases} x = \frac{g(\lambda t)}{\lambda} + g(a) \\ y = g(1) \end{cases}$$

Clearly, $y \in \omega_R$ and we claim that $x \in (\omega_R : I)$; indeed, for all $i \in I$,

$$ix = \frac{ig(\lambda t)}{\lambda} + ig(a) = \frac{\lambda g(it)}{\lambda} + g(ai) = g(ai+it) \in \omega_R$$

and thus $\frac{x}{z}+\frac{y}{z}t \in K$. Finally, for all $s+jt \in R(I)_{a,b}$, we get

$$\begin{aligned}
f_{x,y}(s+jt) &= xj + y(s-ja) = \frac{g(\lambda t)}{\lambda}j + g(aj) + g(s) - g(aj) = \\
&= \frac{\lambda g(jt)}{\lambda} + g(s) = g(s+jt)
\end{aligned}$$

and consequently φ is surjective.

We have $R \subseteq \frac{1}{z}(\omega_R : I)$ and, since $z \in (\omega_R : I)$, also $I \subseteq \frac{1}{z}\omega_R$; this means that $R(I)_{a,b} \subseteq K$. Furthermore, since z is a minimal reduction of $(\omega_R : I)$, it follows from [13, Proposition 16] that $\omega_R \subseteq (\omega_R : I) \subseteq z\overline{R}$. Therefore, $R(I)_{a,b} \subseteq K \subseteq \overline{R}[t]/(t^2 + at + b)$. We have already noticed in Remark 2.1.6 that the integral closure of $R(I)_{a,b}$ contains $\overline{R}[t]/(t^2+at+b)$, thus we have proven the desired inclusions.

Finally, it is easy to see that K is a fractional ideal of $R(I)_{a,b}$: by choosing two regular elements $i \in I$ and $r \in R$ such that $r\omega_R \subseteq R$, the element $riz \in R \subseteq R(I)_{a,b}$ is such that $rizK \subseteq R(I)_{a,b}$. $\square$

The following lemma is proved in the proof of [39, Proposition 6.1], but we include an alternative and easy proof here for the sake of clarity.

Lemma 2.3.2. *Let $\mathfrak{a}, \mathfrak{b}$, and $\mathfrak{c}$ be fractional ideals of R. The following hold:*

1. $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{c}$ *if and only if* $\mathfrak{a}\mathfrak{c}^\vee \subseteq \mathfrak{b}^\vee$;

2. $\mathfrak{m}I^\vee \subseteq zR$ *if and only if* $\mathfrak{m}\omega_R \subseteq zI$;

3. $II^\vee = zI$ *if and only if* $zI^\vee = (I^\vee)^2$.

Proof. From the properties of colon ideals we get

$$\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{c} \Leftrightarrow \mathfrak{c}^\vee \subseteq (\mathfrak{a}\mathfrak{b})^\vee \Leftrightarrow \mathfrak{c}^\vee \subseteq \mathfrak{b}^\vee : \mathfrak{a} \Leftrightarrow \mathfrak{a}\mathfrak{c}^\vee \subseteq \mathfrak{b}^\vee$$

and 1 is proven. Moreover, 2 and 3 follow from 1 with $\mathfrak{a} = \mathfrak{m}$, $\mathfrak{b} = I^\vee$, $\mathfrak{c} = zR$ and $\mathfrak{a} = I^\vee$, $\mathfrak{b} = I$, $\mathfrak{c} = zI$ respectively. $\square$

In the following theorem we characterize when $R(I)_{a,b}$ is almost Gorenstein. The equivalence between 1 and 2 was already proved by S. Goto, N. Matsuoka and T.T. Phuong in [39, Proposition 6.1] in the case of idealization.

Theorem 2.3.3. *The following conditions are equivalent:*

1. *The ring $R(I)_{a,b}$ is almost Gorenstein;*

2. $II^\vee = zI$ *and* $z\mathfrak{m} = \mathfrak{m}I^\vee$;

3. $R \subseteq z^{-1}I^\vee \subseteq (\mathfrak{m} : \mathfrak{m})$ *and* $z^{-1}I^\vee$ *is a ring.*

In particular, the almost Gorenstein property is independent of a and b.

Proof. $1 \Leftrightarrow 2$ Let K be the canonical ideal defined before Proposition 2.3.1. By definition $R(I)_{a,b}$ is almost Gorenstein if and only if $K\mathfrak{m}_{a,b} \subseteq \mathfrak{m}_{a,b}$ or, equivalently, $zK\mathfrak{m}_{a,b} \subseteq z\mathfrak{m}_{a,b}$. Given $m + it \in \mathfrak{m}_{a,b}$ and $x + yt \in zK$, where $m \in \mathfrak{m}$, $i \in I$, $x \in I^\vee$ and $y \in \omega_R$, this in turn means that $(m + it)(x + yt) = mx - biy + (my + ix - aiy)t \in z\mathfrak{m}_{a,b}$, that is

$$\begin{cases} mx - biy \in z\mathfrak{m} \\ my + ix - aiy \in zI. \end{cases}$$

Assume now that $R(I)_{a,b}$ is almost Gorenstein. If we choose $i = 0$, the first equation becomes $\mathfrak{m}I^\vee \subseteq z\mathfrak{m}$, i.e. $\mathfrak{m}I^\vee = z\mathfrak{m}$. Moreover, if we let $y = 0$, the second equation yields $II^\vee \subseteq zI$, i.e. $II^\vee = zI$.

Conversely, if 2 holds, in the light of the previous lemma we have

$$\begin{aligned} mx - biy \ &\in \ \mathfrak{m}I^\vee + I\omega_R \ \subseteq \ z\mathfrak{m} + \mathfrak{m}\omega_R \ \subseteq \ z\mathfrak{m} + zI \ \subseteq \ z\mathfrak{m} + z\mathfrak{m} \subseteq \ z\mathfrak{m} \\ my + ix - aiy \ &\in \ \mathfrak{m}\omega_R + II^\vee + I\omega_R \ \subseteq \ zI + zI + \mathfrak{m}\omega_R \ \subseteq \ zI. \end{aligned}$$

$2 \Leftrightarrow 3$ By Lemma 2.3.2, the condition $II^\vee = zI$ is equivalent to $zI^\vee = (I^\vee)^2$ and we claim that this holds if and only if $z^{-1}I^\vee$ is a ring. In fact $zI^\vee = (I^\vee)^2$ if and only if for all $x, y \in I^\vee$ one has $xy \in zI^\vee$; this is in turn equivalent to $z^{-1}xz^{-1}y \in z^{-1}I^\vee$ for all $z^{-1}x$, $z^{-1}y \in z^{-1}I^\vee$, i.e. $z^{-1}I^\vee$ is a subring of $Q(R)$.

Moreover, the condition $z\mathfrak{m} = \mathfrak{m}I^\vee$ is equivalent to $\mathfrak{m} : \mathfrak{m} \supseteq z^{-1}I^\vee$, i.e. $(z\mathfrak{m} : \mathfrak{m}) \supseteq I^\vee$, because the inclusion $\mathfrak{m}I^\vee \supseteq z\mathfrak{m}$ is always true, since $z \in I^\vee$. Finally, since $zI \subseteq \omega_R$, it follows that $R = (\omega_R : \omega_R) \subseteq (\omega_R : zI) = z^{-1}I^\vee$. $\square$

The last point of the previous theorem gives a way to construct a large class of one-dimensional almost Gorenstein rings. Consider an overring A of R such that $A \subseteq (\mathfrak{m} : \mathfrak{m})$; it follows that $A^\vee$ is a fractional ideal of R. Let $r \in R$ be a regular element such that $rA^\vee \subseteq R$ and set $I := rA^\vee$. Since a minimal reduction of $I^\vee = r^{-1}A$ is $z = r^{-1}$ and $z^{-1}I^\vee = rr^{-1}A = A$, it is clear that I satisfies the conditions of the last point of the previous theorem. For instance, if $A = R$, we get $A^\vee = (\omega_R : A) = \omega_R$; therefore, any ideal $I = rA^\vee$ is a canonical ideal and thus $R(I)_{a,b}$ is Gorenstein. The other extremal case is $A = (\mathfrak{m} : \mathfrak{m})$, that is a ring; then $A^\vee = (\omega_R : (\mathfrak{m} : \mathfrak{m}))$ and every ideal I of the form $r(\omega_R : (\mathfrak{m} : \mathfrak{m}))$ is such that $R(I)_{a,b}$ is almost Gorenstein, see [39, Corollary 6.2] for idealization's case.

On the other hand, if R is Gorenstein but not a DVR, $R(I)_{a,b}$ is almost Gorenstein only in the above two cases. In fact, from the fact that R is not a DVR it follows that $R : \mathfrak{m} = \mathfrak{m} : \mathfrak{m}$, since $x \in (R : \mathfrak{m}) \setminus (\mathfrak{m} : \mathfrak{m})$ means that $x\mathfrak{m} = R$ and it is not possible, because R and $\mathfrak{m}$ are not isomorphic; consequently, if $t(R) = 1$, there are no proper overrings between R and $\mathfrak{m} : \mathfrak{m}$. See [39, Corollary 6.4] for idealization's case.

When $R(I)_{a,b}$ is almost Gorenstein there is a simpler formula for its type.

Proposition 2.3.4. *Assume that $R(I)_{a,b}$ is almost Gorenstein, then*

$$t(R(I)_{a,b}) = 2\ell_R\left(\frac{z^{-1}I^\vee}{R}\right) + 1 = 2\ell_R\left(\frac{\omega_R}{zI}\right) + 1.$$

Proof. In the one-dimensional case the formula of Theorem 2.2.2 becomes

$$t(R(I)_{a,b}) = \ell_R\left(\frac{(R : \mathfrak{m}) \cap (I : I)}{R}\right) + \ell_R\left(\frac{I : \mathfrak{m}}{I}\right).$$

We note that $I : I = (\omega_R : (\omega_R : I)) : I = (\omega_R : I^\vee) : I = \omega_R : II^\vee = z^{-1}I^\vee$ by Theorem 2.3.3.2. Moreover $I : I \subseteq R : \mathfrak{m}$. Indeed, since $z\mathfrak{m} = \mathfrak{m}I^\vee$ by Theorem 2.3.3, it is easy to see that $z(I :$

$I) = \omega_R : I \subseteq z(R : \mathfrak{m})$. Finally, what we need to prove is that $\ell_R((I : \mathfrak{m})/I) = \ell_R(z^{-1}I^\vee/\mathfrak{m})$:

$$\begin{aligned}\ell_R\left(\frac{I:\mathfrak{m}}{I}\right) = \ell_R\left(\frac{(\omega_R:(\omega_R:I)):\mathfrak{m}}{I}\right) = \ell_R\left(\frac{\omega_R:\mathfrak{m}I^\vee}{I}\right) = \\ = \ell_R\left(\frac{\omega_R:I}{\mathfrak{m}I^\vee}\right) = \ell_R\left(\frac{I^\vee}{z\mathfrak{m}}\right) = \ell_R\left(\frac{z^{-1}I^\vee}{\mathfrak{m}}\right),\end{aligned}$$

where, since $R(I)_{a,b}$ is almost Gorenstein, we used $\mathfrak{m}I^\vee = z\mathfrak{m}$. □

If R is a DVR, Corollary 2.2.4 implies that $R(I)_{a,b}$ is almost Gorenstein if and only if is Gorenstein. On the other hand, if R is not a DVR, we have already noted before the previous proposition that its type is the length of $(\mathfrak{m} : \mathfrak{m})/R$ and then, since in the almost Gorenstein case $z^{-1}I^\vee \subseteq (\mathfrak{m} : \mathfrak{m})$, Proposition 2.3.4 implies the following corollary:

Corollary 2.3.5. *Let $R(I)_{a,b}$ be an almost Gorenstein ring. The type of $R(I)_{a,b}$ is odd and $1 \leq t(R(I)_{a,b}) \leq 2t(R) + 1$.*

As we proved above, the almost Gorenstein property and the type are independent of a and b; the next proposition can be thus derived from [39, Theorem 6.5], whereas the same statement is proven in idealization's case. We prefer to include a simpler and self-contained proof here.

Proposition 2.3.6. *The ring R is almost Gorenstein if and only if $R(\mathfrak{m})_{a,b}$ is almost Gorenstein. In this case, if R is not a DVR, the type of $R(\mathfrak{m})_{a,b}$ is $2t(R) + 1$.*

Proof. Using Corollary 2.2.3, it is clear that if R is a DVR, then $R(\mathfrak{m})_{a,b}$ is Gorenstein. Conversely, if $R(\mathfrak{m})_{a,b}$ is Gorenstein, then $\mathfrak{m}$ is a canonical ideal and $\mathfrak{m} : \mathfrak{m} = R$; this is possible only if R is a DVR, otherwise $t(R) = \ell((\mathfrak{m} : \mathfrak{m})/R) = 1$.

If R is almost Gorenstein and not a DVR, we claim that $\omega_R : \mathfrak{m} = \mathfrak{m} : \mathfrak{m}$. Indeed we have

$$\begin{aligned}t(R)+1 &= \ell_R\left(\frac{\mathfrak{m}:\mathfrak{m}}{\mathfrak{m}}\right) = \ell_R\left(\frac{\mathfrak{m}:\mathfrak{m}}{\omega_R}\right) + \ell_R\left(\frac{\omega_R}{\mathfrak{m}}\right) = \\ &= \ell_R\left(\frac{\mathfrak{m}:\mathfrak{m}}{\omega_R}\right) + \ell_R\left(\frac{\omega_R}{\mathfrak{m}\omega_R}\right) = \ell_R\left(\frac{\mathfrak{m}:\mathfrak{m}}{\omega_R}\right) + t(R),\end{aligned}$$

that implies $\ell_R((\mathfrak{m} : \mathfrak{m})/\omega_R) = 1$ and, since $\ell_R((\omega_R : \mathfrak{m})/\omega_R) = \ell_R(R/\mathfrak{m}) = 1$, the claim follows (see also [13, Definition/Proposition 20]). Moreover, it is straightforward to see that a minimal reduction of $\mathfrak{m} : \mathfrak{m}$ is 1, in fact if $x \in \mathfrak{m} : \mathfrak{m}$, then $x = x \cdot 1 \in (\mathfrak{m} : \mathfrak{m})^2$; it follows that $z^{-1}\mathfrak{m}^\vee = \mathfrak{m} : \mathfrak{m}$ and, by Theorem 2.3.3 and Proposition 2.3.4, $R(\mathfrak{m})_{a,b}$ is almost Gorenstein of type $2t(R) + 1$.

Conversely, if $R(\mathfrak{m})_{a,b}$ is almost Gorenstein, but not Gorenstein, then Theorem 2.3.3 implies that $z^{-1}(\omega_R : \mathfrak{m}) \subseteq \mathfrak{m} : \mathfrak{m}$. Moreover, it is well-known that in dimension one ω_R is an irreducible fractional ideal and $\ell_R((\omega_R : \mathfrak{m})/\omega_R) = 1$, thus $\omega_R : \mathfrak{m} \subset \bar{R}$ since, otherwise, if $x \in \bar{R} \setminus (\omega_R : \mathfrak{m})$, we have $\omega_R = (\omega_R : \mathfrak{m}) \cap (\omega_R, x)$. Consequently, by [13, Proposition 16], $z = 1$ is a minimal reduction of $\omega_R : \mathfrak{m}$ and then $\omega_R \subseteq \omega_R : \mathfrak{m} \subseteq \mathfrak{m} : \mathfrak{m}$. This implies that R is an almost Gorenstein ring. □

The general case

We will prove now that the almost Gorenstein property is independent of a and b also in the higher dimensional case, by reducing the problem to the one-dimensional case by means of the following lemma.

Lemma 2.3.7. *Let x be an element of the ring R that determines a non-zero-divisor on R/I, i.e. $(I : x) = I$. Then,*

$$\frac{R(I)_{a,b}}{xR(I)_{a,b}} \cong \frac{R}{xR}\left(\frac{I+xR}{xR}\right)_{\overline{a},\overline{b}}$$

where $\overline{a}$ and $\overline{b}$ are the images of a and b in R/xR.

Proof. It is not difficult to check that the ring homomorphism

$$\alpha : R(I)_{a,b} \to \frac{R}{xR}\left(\frac{I+xR}{xR}\right)_{\overline{a},\overline{b}}, \qquad r+it \mapsto (r+xR)+(i+xR)t$$

is surjective. The assumption on x implies that $I \cap xR = xI$ and, therefore, $i \in I \cap xR$ if and only if $i = xj$ with $j \in I$. Hence, $\mathrm{Ker}(\alpha) = xR(I)_{a,b}$, as desired. □

Proposition 2.3.8. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with positive dimension d and let I be a regular ideal of R. The almost Gorenstein property of $R(I)_{a,b}$ is independent of a and b.*

Proof. In the light of Theorem 2.3.3, we may assume that $d > 1$. Suppose that there exist two elements $a', b' \in R$ such that $R(I)_{a',b'}$ is almost Gorenstein, i.e. there exists an exact sequence of $R(I)_{a',b'}-$modules

$$0 \to R(I)_{a',b'} \to \omega_{R(I)_{a',b'}} \to C \to 0,$$

where the number of elements of a minimal system of generators of C equals its multiplicity. It is enough to show that $R(I)_{a,b}$ is almost Gorenstein for all $a, b \in R$. By Corollary 2.2.3, we may also assume that $R(I)_{a',b'}$ is not Gorenstein. Consider the filtration $\mathbb{M}$ of C induced by $\mathfrak{m}_{a',b'}$:

$$C \supseteq \mathfrak{m}_{a',b'}C \supseteq \mathfrak{m}^2_{a',b'}C \supseteq \cdots \supseteq \mathfrak{m}^i_{a',b'}C \supseteq \dots$$

This is a $\mathfrak{m}_{a',b'}-$filtration of the $R(I)_{a',b'}-$module C, but, if we consider C as an R-module, it is also a $\mathfrak{m}$-filtration. Therefore, an iterated use of Proposition 1.0.2 implies that there exists a $\mathbb{M}$-superficial sequence $\mathbf{f} = f_1, \dots, f_{d-1}$ for $\mathfrak{m}$ in R; it is easy to verify that $\mathbf{f}$ is also a $\mathbb{M}-$superficial sequence for $\mathfrak{m}_{a',b'}$. Moreover, we can choose $\mathbf{f}$ also R-regular and, since I has height one by Lemma 1.0.1, such that $I + \mathbf{f}R$ is $\mathfrak{m}$-primary. Consequently, $\mathbf{f}$ is also a $R(I)_{a,b}$-regular sequence for all $a, b \in R$ and the ideal of R/I generated by the classes $\overline{f}_1, \dots, \overline{f}_{d-1}$ is $\mathfrak{m}/I$-primary. Therefore $\overline{\mathbf{f}}$ is a R-regular sequence, because R/I is a Cohen-Macaulay ring of dimension $d-1$ by Lemma 1.0.1. Hence, by the previous lemma and [40, Theorem 3.7 (2)], it follows that

$$\frac{R(I)_{a',b'}}{\mathbf{f}R(I)_{a',b'}} \cong \frac{R}{\mathbf{f}R}\left(\frac{I+\mathbf{f}R}{\mathbf{f}R}\right)_{\overline{a'},\overline{b'}}$$

is almost Gorenstein of dimension 1. Therefore, by Theorem 2.3.3

$$\frac{R}{\mathbf{f}R}\left(\frac{I+\mathbf{f}R}{\mathbf{f}R}\right)_{\overline{a},\overline{b}}$$

is an almost Gorenstein ring for all $\overline{a},\overline{b} \in R/\mathbf{f}R$. Observe also that the ideal $(I+\mathbf{f}R)/\mathbf{f}R$ is $\mathfrak{m}/\mathbf{f}R$-primary. Finally, since $\mathbf{f}$ is an $R(I)_{a,b}$-regular sequence by Proposition 2.1.13, it follows that $R(I)_{a,b}$ is almost Gorenstein for all $a,b \in R$, by [40, Theorem 3.7 (1)]. $\square$

Remark 2.3.9. It is easy to see that if N_1 and N_2 are two isomorphic R-modules, then $R \ltimes N_1 \cong R \ltimes N_2$; however if I and J are two isomorphic ideals, the rings $R(I)_{a,b}$ and $R(J)_{a,b}$ do not need to be isomorphic. On the other hand, the previous proposition implies that $R(I)_{a,b}$ is almost Gorenstein if and only if $R(I)_{0,0} \cong R \ltimes I$ is and, as consequence, if I and J are two isomorphic regular ideals of R, then $R(I)_{a,b}$ is almost Gorenstein if and only if $R(J)_{a,b}$ is almost Gorenstein.

In [40] S. Goto, T. Takahashi, and N. Taniguchi characterize when idealization is almost Gorenstein; using their result we can prove a similar statement for all rings $R(I)_{a,b}$.

Corollary 2.3.10. *Let I be a regular ideal of R and assume that $I^\vee$ is isomorphic to a regular ideal of R. Then the following are equivalent:*

1. *$R(I)_{a,b}$ is almost Gorenstein for some $a,b \in R$;*

2. *$R(I)_{a,b}$ is almost Gorenstein for all $a,b \in R$;*

3. *I is a maximal Cohen-Macaulay R-module and any proper ideal J of R isomorphic to $I^\vee$ is such that $f_1 \in J$, $\mathfrak{m}(J+Q) = \mathfrak{m}Q$, and $(J+Q)^2 = Q(J+Q)$, for every parameter ideal $Q = (f_1, \dots, f_d)$ of R.*

Proof. The equivalence between 1 and 2 is just a reformulation of Proposition 2.3.8.

$2 \Rightarrow 3$ By Proposition 2.1.13, I is a maximal Cohen-Macaulay R-module, therefore also $J \cong I^\vee$ is a maximal Cohen-Macaulay R-module by [17, Theorem 3.3.10] and is also isomorphic to a regular ideal; moreover $J^\vee \cong I$, thus the idealizations $R \ltimes I$ and $R \ltimes J^\vee$ are isomorphic. Furthermore, by Lemma 1.0.1 R/J is a Cohen-Macaulay ring of dimension $d-1$; thus $R(I)_{0,0} \cong R \ltimes I \cong R \ltimes J^\vee$ is almost Gorenstein and the thesis follows from [40, Theorem 6.1].

$3 \Rightarrow 1$ Using again [40, Theorem 6.1] we immediately get that $R \ltimes J^\vee \cong R(I)_{0,0}$ is almost Gorenstein. $\square$

2.4 Spectra and localizations

In the previous sections we saw how many important properties of $R(I)_{a,b}$ are independent of a and b, whereas in this section we will show that this is not the case of the spectrum of $R(I)_{a,b}$. It is already known that idealization and amalgamated duplication have different spectra, as, for instance, idealization is never a reduced ring, whereas amalgamated duplication can be. The knowledge of the spectrum of $R(I)_{a,b}$ will allow us to find suitable a and b such that $R(I)_{a,b}$ is an integral domain, provided that R is a domain (note that $R \subseteq R(I)_{a,b}$).

We recall that $R \subseteq R(I)_{a,b} \subseteq R[t]/(t^2+at+b)$ are integral extensions, cf. Lemma 2.1.1, so that we can get some information on the spectrum of $R(I)_{a,b}$ from the knowledge of the spectrum of $R[t]/(t^2+at+b)$.

Proposition 2.4.1. *For every prime ideal $\mathfrak{p}$ of R there are at most two prime ideals of $R(I)_{a,b}$ lying over $\mathfrak{p}$. Moreover, if t^2+at+b is irreducible over $R/\mathfrak{m}$ for all maximal ideal $\mathfrak{m}$ of R, then there is exactly one prime ideal of $R(I)_{a,b}$ lying over $\mathfrak{p}$ for every $\mathfrak{p}$.*

Proof. Every prime ideal of $R(I)_{a,b}$ lying over $\mathfrak{p}$ has to be the contraction of a prime ideal of $R[t]/(t^2+at+b)$. Furthermore, given a prime ideal $\mathfrak{p}$ of R, it is well known (see e.g. [37, Chapter 6]) that $\mathfrak{p}[t]$ is a prime of $R[t]$ lying over $\mathfrak{p}$ and there exist infinitely many other prime ideals in $R[t]$ lying over $\mathfrak{p}$, all of them containing $\mathfrak{p}[t]$ and with no inclusions among them. More precisely, there is a bijection between these ideals and the non-zero prime ideals of $Q(R/\mathfrak{p})[t]$; therefore if J is one of these ideals, its image in $Q(R/\mathfrak{p})[t]$ is of the form $(f(t))$, for some irreducible polynomial $f(t)$; hence $J = \varphi_{\mathfrak{p}}^{-1}((f(t)))$, where $\varphi_{\mathfrak{p}}$ is the composition of the canonical homomorphisms $R[t] \to R/\mathfrak{p}[t] \hookrightarrow Q(R/\mathfrak{p})[t]$. It follows that the prime ideals of $R[t]/(t^2+at+b)$ lying over $\mathfrak{p}$ are of the form $J/(t^2+at+b)$, with $J \supseteq (t^2+at+b)$. This means that the polynomial $f(t)$, corresponding to J, divides the image of t^2+at+b in $Q(R/\mathfrak{p})[t]$. Hence, if t^2+at+b is irreducible in $Q(R/\mathfrak{p})[t]$, there is only one prime of $R[t]/(t^2+at+b)$ lying over $\mathfrak{p}$; on the other hand, if t^2+at+b has two distinct irreducible factors in $Q(R/\mathfrak{p})[t]$, there exist exactly two prime ideals in $R[t]/(t^2+at+b)$ lying over $\mathfrak{p}$. In all cases we proved that there are at most two prime ideals in $R(I)_{a,b}$ lying over $\mathfrak{p}$.

Suppose now that $J/(t^2+at+b)$ is in the spectrum of $R[t]/(t^2+at+b)$ and $(J/(t^2+at+b)) \cap R = \mathfrak{p}$. We know that $J = \varphi_{\mathfrak{p}}^{-1}((f(t)))$, where $f(t)$ is an irreducible factor of t^2+at+b in $Q(R/\mathfrak{p})[t]$. If $\mathfrak{p}' \in \operatorname{Spec} R$, $\mathfrak{p}' \subset \mathfrak{p}$, then the prime ideals of $R[t]/(t^2+at+b)$ lying over $\mathfrak{p}'$ correspond to the irreducible factors of t^2+at+b in $Q(R/\mathfrak{p}')[t]$. Since the factorization of t^2+at+b in $Q(R/\mathfrak{p}')[t]$ induces a factorization in $Q(R/\mathfrak{p})[t]$, $f(t)$ is irreducible also in $Q(R/\mathfrak{p}')[t]$ and we have a prime ideal of $R[t]/(t^2+at+b)$ lying over $\mathfrak{p}'$ of the form $J'/(t^2+at+b)$, with $J' = \varphi_{\mathfrak{p}'}^{-1}((f(t))) \subset J$. In particular, if $\mathfrak{m}$ is a maximal ideal of R containing $\mathfrak{p}$ and t^2+at+b is irreducible on $R/\mathfrak{m}$, then there is one and only one prime ideal of $R[t]/(t^2+at+b)$ lying over $\mathfrak{p}$ and the same happens for $R(I)_{a,b}$, because the extension $R(I)_{a,b} \subseteq R[t]/(t^2+at+b)$ is integral. $\square$

Remark 2.4.2. 1. For particular a and b the factorization of t^2+at+b in $Q(R/\mathfrak{p})[t]$ may not depend on $\mathfrak{p}$. For example, in the case of idealization, the equality $t^2 = t \cdot t$ implies that there is only one prime ideal lying over $\mathfrak{p}$ both in $R[t]/(t^2)$ and in idealization. As for the case of amalgamated duplication, the equality $t^2 - t = t \cdot (t-1)$ implies that there are two primes in $R[t]/(t^2-t)$ lying over $\mathfrak{p}$, namely $(\mathfrak{p}, t)$ and $(\mathfrak{p}, t-1)$. Contracting these ideals to amalgamated duplication we get the same ideal if and only if $\mathfrak{p} \supseteq I$ (see e.g. [24]).

2. By the proof of Proposition 2.4.1 we see that the extension $R \subseteq R[t]/(t^2+at+b)$ and the extension $R \subseteq R(I)_{a,b}$ as well fulfill the going down property. In particular a minimal prime of $R(I)_{a,b}$ lies over a minimal prime of R.

Now we want to show explicitly the prime ideals of the ring $R(I)_{a,b}$ with respect to those of R. In the previous proposition we showed that these depend on the reducibility of t^2+at+b in

$Q(R/\mathfrak{p})[t]$. If it is reducible and its roots are α/γ and β/δ, we can assume that $\gamma = \delta$. In this case it is easy to see that in $Q(R/\mathfrak{p})$ one has $\gamma a = -\alpha - \beta$ and $\gamma^2 b = \alpha\beta$ and, clearly, the same equalities hold in $R/\mathfrak{p}$; in this section we will use these equalities several times. We start with a technical lemma.

Lemma 2.4.3. *Let $\mathfrak{p}$ be a prime ideal of R and suppose that $t^2 + at + b = (t - \alpha/\gamma)(t - \beta/\gamma)$ in $Q(R/\mathfrak{p})[t]$. The two sets*

$$\mathfrak{p}_1 := \{r + it \,|\, r \in R, i \in I, \gamma r + \alpha i \in \mathfrak{p}\},$$
$$\mathfrak{p}_2 := \{r + it \,|\, r \in R, i \in I, \gamma r + \beta i \in \mathfrak{p}\}$$

are prime ideals of $R(I)_{a,b}$. Moreover $\mathfrak{p}_1 = \mathfrak{p}_2$ if and only if $(\alpha - \beta)I \subseteq \mathfrak{p}$.

Proof. First of all, we notice that these sets do not depend on the choice of α, β, and γ and then they are well defined. Cleary it is enough to consider $\mathfrak{p}_1$. We notice that, since $\gamma \notin \mathfrak{p}$ by definition, the condition $\gamma r + \alpha i$ is equivalent to $\gamma(\gamma r + \alpha i)$. Let $r + it \in \mathfrak{p}_1$ and $s + jt \in R(I)_{a,b}$. Then $(r + it)(s + jt) = rs - ijb + (rj + si - aij)t$ and, since $\gamma a = -\alpha - \beta, \gamma^2 b = \alpha\beta$ and $\gamma r + \alpha i \in \mathfrak{p}$, in $R/\mathfrak{p}$ we have

$$\gamma^2 rs - ij\gamma^2 b + \gamma\alpha rj + \gamma\alpha si - \gamma\alpha aij =$$
$$= \gamma^2 rs - ij\alpha\beta + \gamma\alpha rj - \gamma^2 rs + \gamma^2 arj = jr\gamma\beta + \gamma\alpha rj - \gamma\alpha rj - \gamma rj\beta = 0$$

and this means that $(r + it)(s + jt)$ is in $\mathfrak{p}_1$. Now we need to prove that $\mathfrak{p}_1$ is prime. Suppose that $(r + it)(s + jt) \in \mathfrak{p}_1$, then $\gamma rs - ijb\gamma + \alpha si + \alpha rj - \alpha ija \in \mathfrak{p}$ and, mupltiplying by γ, in $R/\mathfrak{p}$ we have

$$0 = \gamma^2 rs - ij\alpha\beta + \gamma\alpha si + \gamma\alpha rj + \alpha^2 ij + ij\alpha\beta =$$
$$\gamma s(\gamma r + \alpha i) + \alpha j(\gamma r + \alpha i) = (\gamma r + \alpha i)(\gamma s + \alpha j).$$

Since $R/\mathfrak{p}$ is a domain this implies that either $(\gamma r + \alpha i) = 0$ or $(\gamma s + \alpha j) = 0$, i.e. either $r + it$ or $s + jt$ is in $\mathfrak{p}_1$.

As for the last statement suppose first that $(\alpha - \beta)I \subseteq \mathfrak{p}$. Then if $r + it \in \mathfrak{p}_1$ we have $\gamma r + \beta i = \gamma r + \alpha i - (\alpha - \beta)i \in \mathfrak{p}$; therefore $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ and the other inclusion is clear for the same reason. Conversely, given an element $i \in I$, one has $\alpha i - \gamma it \in \mathfrak{p}_1$. Since $\mathfrak{p}_1 = \mathfrak{p}_2$, it follows that $\gamma\alpha i - \beta i\gamma \in \mathfrak{p}$ and therefore $\gamma(\alpha - \beta)i \in \mathfrak{p}$. Hence $(\alpha - \beta)i \in \mathfrak{p}$ because by definition $\gamma \notin \mathfrak{p}$ and $\mathfrak{p}$ is a prime ideal. □

Proposition 2.4.4. *Let $\mathfrak{p}$ be a prime ideal of R.*

1. *If $t^2 + at + b$ is irreducible in $Q(R/\mathfrak{p})[t]$, then the only prime ideal of $R(I)_{a,b}$ lying over $\mathfrak{p}$ is $\mathfrak{q} := \{p + it \,|\, p \in \mathfrak{p}, i \in I \cap \mathfrak{p}\}$.*

2. *If $t^2 + at + b = (t - \alpha/\gamma)(t - \beta/\gamma)$ in $Q(R/\mathfrak{p})[t]$, then the ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of the previous lemma are the only prime ideals of $R(I)_{a,b}$ lying over $\mathfrak{p}$.*

Proof. The first case is straightforward, because the prime ideal of $R(I)_{a,b}$ lying over $\mathfrak{p}$ is

$$\mathfrak{p}\frac{R[t]}{(t^2+at+b)} \cap R(I)_{a,b} = \{p+it \,|\, p \in \mathfrak{p}, i \in I \cap \mathfrak{p}\}.$$

As for the second case we clearly have that $\mathfrak{p}_1 \cap R = \mathfrak{p} = \mathfrak{p}_2 \cap R$, hence $\mathfrak{p}_1$ and $\mathfrak{p}_2$ lying over $\mathfrak{p}$. In general we know that they can be coincide and in this case it could be another prime ideal lying over $\mathfrak{p}$, but actually this can not happen. In fact, as we proved in the proof of Proposition 2.4.1, we know that one prime ideal of $R(I)_{a,b}$ lying over $\mathfrak{p}$ is the contraction to $R(I)_{a,b}$ of the ideal $J = \varphi_{\mathfrak{p}}^{-1}((t-\alpha/\gamma))$, see the proof of Proposition 2.4.1 for the notation. Since J contains $\mathfrak{p}$, it is easy to see that $\mathfrak{p}_1$ is contained in J. In fact if $r+it \in \mathfrak{p}_1$, then

$$r+it = \frac{r\gamma}{\gamma} + it = -\frac{\alpha i}{\gamma} + it + \frac{\gamma r + \alpha i}{\gamma} = i\left(-\frac{\alpha}{\gamma} + t\right) + \frac{\gamma r + \alpha i}{\gamma} \in \left(t - \frac{\alpha}{\gamma}\right) Q\left(\frac{R}{\mathfrak{p}}\right)$$

and therefore $r+it \in J$. Consequently by Incomparability, see [30, Corollary 4.18], we get that $\mathfrak{p}_1 = J$ and, using the same argument for $\mathfrak{p}_2$, this concludes the proof. □

Proposition 2.4.5. *$R(I)_{a,b}$ is an integral domain if and only if R is an integral domain and t^2+at+b is irreducible in $Q(R)[t]$.*

Proof. Clearly we can assume that R is an integral domain. If t^2+at+b is irreducible in $Q(R)[t]$, the ideal $\mathfrak{q} = \{p+it \,|\, p \in (0)R, i \in I \cap (0)R\} = (0)R(I)_{a,b}$ is prime and thus $R(I)_{a,b}$ is an integral domain. Conversely, suppose by contradiction that t^2+at+b is reducible in $Q(R)[t]$ and let $\mathfrak{p}_1, \mathfrak{p}_2$ be the prime ideals of $R(I)_{a,b}$ lying over (0). These are the minimal primes of $R(I)_{a,b}$ and, since it is a domain, they are equal to $(0)R(I)_{a,b}$. On the other hand it is easy to see that for all non-zero $i \in I$ the element $i\gamma - i\alpha t$ is in $\mathfrak{p}_1$ and it is different to zero, because R is a domain, and this is a contradiction. □

Our next goal is to show that, if R is a local noetherian integral domain, there are always integral domains $R(I)_{a,b}$ for some choices of a and b. To do this we need the following proposition that is due to M. D'Anna and R. Re.

Proposition 2.4.6. *Let R be a local noetherian integral domain with positive dimension. For all integers $n > 1$, not multiple of 4, there exist infinitely many elements $b \in R$ such that the polynomial $t^n - b$ is irreducible over $Q(R)$.*

Proof. We will use the following well-known criterion of irreducibility: if b is not a p-th power for all primes $p|n$ and $b \notin -4Q(R)^4$ if $4|n$, then $t^n - b$ is irreducible (see [54, Chapter VI, Theorem 9.1]). In particular, if 4 does not divide n and b is not a d-th power for all integers $d > 1$ such that $d|n$, then $t^n - b$ is irreducible.

Let $\mathfrak{p}$ be a prime ideal of R with height 1. Then $\dim R_{\mathfrak{p}} = 1$ and, by the Krull-Akizuki Theorem, its integral closure in $Q(R_P) = Q(R)$ is noetherian (see, e.g. [50, Theorem 4.9.2]), hence it is a Dedekind ring. So there is a discrete valuation $v : Q(R)^* \to \mathbb{Z}$ with $v((\overline{R_{\mathfrak{p}}})_M) = \mathbb{N}$, where M is the maximal ideal of $\overline{R_{\mathfrak{p}}}$. Since $R \subseteq R_{\mathfrak{p}} \subseteq (\overline{R_{\mathfrak{p}}})_M$ have the same field of fractions, it follows that $v(R) \subseteq \mathbb{N}$ is a semigroup and it is easy to see that it is a numerical semigroup. So any $x > f(v(R))$ belongs to $v(R)$. In particular, there exist infinitely many elements $b \in R$ such that $v(b)$ is prime to n, so b cannot be a d-th power in $Q(R)$ for all $d > 1$ such that $d|n$. Hence we can find infinitely many $b \in R$ such that $(t^n - b) \subset Q(R)[t]$ is irreducible. $\square$

By the previous proposition we can find infinitely many b such that $t^2 - b$ is irreducible in $Q(R)[t]$ and by Proposition 2.4.5 we immediately get the following corollary.

Corollary 2.4.7. *Let R be a local noetherian integral domain with positive dimension and let I be an ideal of R. Then, there exist infinitely many elements $b \in R$ such that $R(I)_{0,-b}$ is an integral domain.*

In Section 2.6 we will prove that if R is a numerical semigroup ring, the rings $R(I)_{0,-b}$ of the previous corollary are still numerical semigroups. This will lead us to define the numerical duplication of a numerical semigroup that will be of crucial importance in the next two chapters.

Proposition 2.4.8. *Let $\mathfrak{p}$ be a prime ideal of R.*

1. *Suppose that $t^2 + at + b$ is irreducible in $Q(R/\mathfrak{p})[t]$ and that $\mathfrak{q}$ is the prime ideal of R lying over $\mathfrak{p}$. Then $(R(I)_{a,b})_{\mathfrak{q}} \cong R_{\mathfrak{p}}(I_{\mathfrak{p}})_{a,b}$.*

2. *Suppose that $t^2 + at + b = (t - \alpha/\gamma)(t - \beta/\gamma)$ in $Q(R/\mathfrak{p})[t]$ and let $\mathfrak{p}_1, \mathfrak{p}_2$ be the prime ideals of $R(I)_{a,b}$ lying over $\mathfrak{p}$.*

- **(a)** *If $(\alpha - \beta)I \subseteq \mathfrak{p}$, then $(R(I)_{a,b})_{\mathfrak{p}_i} \cong R_{\mathfrak{p}}(I_{\mathfrak{p}})_{a,b}$ for $i = 1, 2$.*
- **(b)** *If $(\alpha - \beta)I \nsubseteq \mathfrak{p}$, then $(R(I)_{a,b})_{\mathfrak{p}_i} \cong R_{\mathfrak{p}}$ for $i = 1, 2$.*

Proof. 1. We have $s + jt \in R(I)_{a,b} \setminus \mathfrak{q}$ if and only if at least one between s and j is in $R \setminus \mathfrak{p}$. Given an element $(r + it)/(s + jt) \in (R(I)_{a,b})_{\mathfrak{q}}$, we can multiply it by $(s - aj - jt)/(s - aj - jt)$; in fact clearly $s - aj - jt \in R(I)_{a,b} \setminus \mathfrak{q}$, if $j \in R \setminus \mathfrak{p}$, but this also happens if $j \in \mathfrak{p}$ and $s \in R \setminus \mathfrak{p}$, because in this case $s - aj \in R \setminus \mathfrak{p}$. Hence we get an injective homomorphism between $(R(I)_{a,b})_{\mathfrak{q}}$ and $R_{\mathfrak{p}}(I_{\mathfrak{p}})_{a,b}$ given by

$$\frac{r + it}{s + jt} \mapsto \frac{r + it}{s + jt} \cdot \frac{(s - aj - jt)}{(s - aj - jt)} = \frac{rs - ajr + ijb}{s^2 - ajs + bj^2} + \frac{si - rj}{s^2 - ajs + bj^2} t.$$

Moreover this is surjective because a generic element $r/s + (i/s')t$ comes from $(rs' + ist)/(ss') \in (R(I)_{a,b})_{\mathfrak{q}}$.

2. (a) We recall that in this case $\mathfrak{p}_1 = \mathfrak{p}_2$. Consider an element $r + it \in R(I)_{a,b} \setminus \mathfrak{p}_1$, then $ia - r + it \notin \mathfrak{p}_1$, because $\gamma ia - \gamma r + \alpha i = -i\alpha - i\beta - \gamma r + \alpha i \notin \mathfrak{p}$ since $r + it \notin \mathfrak{p}_2$. Therefore, given an element $(s + jt)(r + it) \in (R(I)_{a,b})_{\mathfrak{p}_1}$, one has

$$\frac{s + jt}{r + it} \cdot \frac{ia - r + it}{ia - r + it} = \frac{sia - rs - bij}{ria - r^2 - bi^2} + \frac{si - rj}{ria - r^2 - bi^2} t.$$

Clearly $ria - r^2 - bi^2 \in R \setminus \mathfrak{p}$, because $\mathfrak{p}_1 \cap R = \mathfrak{p}$, and therefore we get a well-defined ring homomorphism

$$f : (R(I)_{a,b})_{\mathfrak{p}_1} \to R_{\mathfrak{p}}(I_{\mathfrak{p}})_{a,b}$$

$$f\left(\frac{s+jt}{r+it}\right) = \frac{sia-rs-bij}{ria-r^2-bi^2} + \frac{si-rj}{ria-r^2-bi^2}t$$

that is injective by construction. As for surjectivity, if $\frac{r}{s_1} + \frac{i}{s_2}t$ is an element of $R_\mathfrak{p}(I_\mathfrak{p})_{a,b}$, it is easy to see that it is equal to $f(\frac{rs_2+is_1t}{s_1s_2})$. Hence f is an isomorphism and therefore $(R(I)_{a,b})_{\mathfrak{p}_i} \cong R_\mathfrak{p}(I_\mathfrak{p})_{a,b}$ for $i = 1, 2$.
(b) Consider the map $g_1 : R_\mathfrak{p} \to (R(I)_{a,b})_{\mathfrak{p}_1}$, $g_1\left(\frac{r}{s}\right) = \frac{r}{s}$. Clearly this is well defined and is an injective ring homomorphism. As for surjectivity consider a generic $\frac{r+it}{s+jt} \in (R(I)_{a,b})_{\mathfrak{p}_1}$ and let λ be an element of I such that $\lambda(\alpha-\beta) \notin \mathfrak{p}$. Then $-\beta\lambda\gamma + \gamma^2\lambda t \notin \mathfrak{p}_1$ and it is easy to see that $(r+it)(-\beta\lambda\gamma+\gamma^2\lambda t) = (r\gamma+i\alpha)(-\beta\lambda+\gamma\lambda t)$. It follows that

$$\frac{r+it}{s+jt} \cdot \frac{-\beta\lambda\gamma+\gamma^2\lambda t}{-\beta\lambda\gamma+\gamma^2\lambda t} = \frac{r\gamma+i\alpha}{s\gamma+j\alpha}.$$

Hence $g_1\left(\frac{r\gamma+i\alpha}{s\gamma+j\alpha}\right) = \frac{r+it}{s+jt}$ and $(R(I)_{a,b})_{\mathfrak{p}_1} \cong R_\mathfrak{p}$. For $(R(I)_{a,b})_{\mathfrak{p}_2}$ we can use the same argument. □

In the light of the previous proposition we can immediately state the following corollary.

Corollary 2.4.9. *Let R be a ring, let I be an ideal of R and let $a, b \in R$. Denote by $\mathbb{M}$ the set of all the maximal ideals $\mathfrak{m}$ of R except those for which $t^2+at+b = (t-\alpha/\gamma)(t-\beta/\gamma)$ in $R/\mathfrak{m}[t]$ and $(\alpha-\beta)I \nsubseteq \mathfrak{m}$. Then:*

1. *The ring $R(I)_{a,b}$ is Cohen-Macaulay if and only if R is Cohen-Macaulay and $I_\mathfrak{m}$ is a maximal Cohen-Macaulay $R_\mathfrak{m}$-module for all $\mathfrak{m} \in \mathbb{M}$;*

2. *Assume that $I_\mathfrak{m}$ is regular for all $\mathfrak{m} \in \mathbb{M}$. The ring $R(I)_{a,b}$ is Gorenstein if and only if R is Cohen-Macaulay and $I_\mathfrak{m}$ is a canonical ideal of $R_\mathfrak{m}$ for all $\mathfrak{m} \in \mathbb{M}$.*

2.5 Idealization and amalgamated duplication

In Proposition 2.1.2 we proved that idealization and amalgamated duplication are isomorphic to $R(I)_{0,0}$ and $R(I)_{-1,0}$ respectively. It is natural to ask if there are other choices of a and b for which we get a ring isomorphic either idealization or amalgamated duplication and when we find *new rings*. In this section we will restrict us to the case in which the polynomial t^2+at+b is reducible in $R[t]$ and our first goal is to understand when $R(I)_{a,b}$ is reduced.

Clearly we can assume that R is a reduced ring. Since idealization is never reduced and in this case amalgamated duplication is always a reduced ring, it is clear that this property depends on a and b, but in both cases does not depend on the ideal I. In the next example we will show that for other choices of a and b this is not true.

Example 2.5.1. Let k be a field with characteristic 2 and set $R := k[X,Y]/(XY)$, that is a reduced ring. Denote by x, y the images of X, Y in R and consider $R(I)_{x,y^2} := R[It]/I^2(t^2 + xt + y^2)$. If $I := (y)$, it follows that $(y)^2(t^2+xt+y^2) = (y(t+y))^2$ in $R[t]$ and then $R(I)_{x,y^2}$ is not a reduced ring.

On the other hand if $I := (x)$ one has $(x)^2(t^2+xt+y^2) = (x^2)(t^2+xt)$ in $R[t]$ and we claim that this ring is reduced. In fact suppose that there exists a nilpotent element $r+\lambda xt$ in $R(I)_{x,y^2} = R(I)_{x,0}$, i.e. $0 = (r+\lambda xt)^n = r^n + t(\dots)$. Since R is reduced, it follows that $r=0$; moreover $0 = (\lambda x)^n t^n = \lambda^n x^{2n-1} t$ implies that $Y|\lambda$ in $k[X,Y]$, that is $\lambda x = \lambda_1 xy = 0$ in R. This proves that $R(I)_{x,y^2}$ is a reduced ring.

Actually using Corollary 2.5.5 it will be possible to see that the two rings of the previous example are isomorphic to $R \ltimes (y)$ and $R \bowtie (x^2)$ respectively. However in Example 2.5.7 we will show a ring of our subfamily that is isomorphic to neither idealization nor amalgamated duplication.

Proposition 2.5.2. *Let R be a ring, let I be a proper ideal of R and suppose that $t^2+at+b = (t-\alpha)(t-\beta)$ in $R[t]$. The ring $R(I)_{a,b}$ is reduced if and only if R is reduced and $I \cap \operatorname{Ann}(\alpha-\beta) = (0)$.*

Proof. According to Proposition 2.4.4, the set of minimal primes of $R(I)_{a,b}$ is $A = \{\mathfrak{p}_1, \mathfrak{p}_2 \,|\, \mathfrak{p} \in \operatorname{Min}(R)\}$, where $\operatorname{Min}(R)$ denotes the set of the minimal primes of R, and therefore $R(I)_{a,b}$ is reduced if and only if $N := \bigcap A = (0)$.

Assume that R is reduced and $I \cap \operatorname{Ann}(\alpha-\beta) = (0)$. Fix $\mathfrak{p} \in \operatorname{Min}(R)$ and let $r+it$ be an element of N. Since $r+it \in \mathfrak{p}_1 \cap \mathfrak{p}_2$, it follows that $r+\alpha i$ and $r+\beta i$ are in $\mathfrak{p}$ and then $(\alpha-\beta)i \in \mathfrak{p}$. This implies that we get $i(\alpha-\beta) \in \bigcap\limits_{\mathfrak{p}\in\operatorname{Min}(R)} \mathfrak{p} = (0)$ and then $i=0$, because $I \cap \operatorname{Ann}(\alpha-\beta) = (0)$ by hypothesis. Finally $r=0$, since R is reduced.

Conversely it is clear that R is reduced, because it is contained in $R(I)_{a,b}$. Moreover, if $i \in I \cap \operatorname{Ann}(\alpha-\beta)$, the equalities

$$(-\beta i+it)^2 = \beta^2 i^2 - bi^2 + (-2\beta i^2 - ai^2)t = (\beta-\alpha)i^2\beta + (\alpha-\beta)i^2 t = 0,$$

imply that $-\beta i + it = 0$ and consequently $i=0$. □

Corollary 2.5.3. *Let R be a reduced ring, let I be a proper ideal of R and suppose that $t^2+at+b = (t-\alpha)(t-\beta)$ in $R[t]$. If $\alpha-\beta$ is regular, then $R(I)_{a,b}$ is reduced. In particular this happens when R is an integral domain and $\alpha \neq \beta$.*

Proposition 2.5.4. *Let R be a ring, let I be a proper ideal of R and suppose that $t^2+at+b = (t-\alpha)(t-\beta)$ in $R[t]$. The following statements hold:*

1. *$R(I)_{a,b} \cong R \ltimes I$, if $\alpha-\beta \in \operatorname{Ann}(I^2)$.*

2. *$R(I)_{a,b} \cong R \bowtie (\alpha-\beta)I$, if $\operatorname{Ann}(\alpha-\beta) \cap I = (0)$. In particular if $\alpha-\beta$ is invertible, then $R(I)_{a,b} \cong R \bowtie I$.*

Proof. Consider the ring automorphism of $R[t]$ given by $t \mapsto t+\alpha$. Then

$$R(I)_{a,b} = \frac{R[It]}{I^2((t-\alpha)(t-\beta))} \cong \frac{R[It]}{I^2(t^2+(\alpha-\beta)t)} = R(I)_{\alpha-\beta,0}.$$

1. If $\alpha-\beta \in \operatorname{Ann}(I^2)$, then Proposition 2.1.2 implies that

$$R(I)_{a,b} \cong \frac{R[It]}{I^2(t^2+(\alpha-\beta)t)} = \frac{R[It]}{I^2(t^2)} \cong R \ltimes I$$

2. Consider the map $\varphi : R(I)_{\alpha-\beta,0} \to R((\alpha-\beta)I)_{-1,0}$ given by $\varphi(r+it) = r-(\alpha-\beta)it$. This is a ring homomorphism, in fact

$$\varphi((r+it)(s+jt)) = \varphi(rs+(rj+si-ij(\alpha-\beta))t) = rs-(\alpha-\beta)(rj+si-ij(\alpha-\beta))t$$

$$\varphi(r+it)\varphi(s+jt) = (r-(\alpha-\beta)it)(s-(\alpha-\beta)jt) = rs-(\alpha-\beta)(rj+si-ij(\alpha-\beta))t.$$

Moreover φ is clearly surjective and, since $\mathrm{Ann}(\alpha-\beta)\cap I = (0)$, it is also injective. Hence φ is an isomorphism and the thesis follows, since $R((\alpha-\beta)I)_{-1,0} \cong R \bowtie (\alpha-\beta)I$ by Proposition 2.1.2. $\square$

Corollary 2.5.5. *Let R be a reduced ring. The following statements hold:*

1. $R(I)_{a,b} \cong R \ltimes I$ *if and only if* $\alpha-\beta \in \mathrm{Ann}(I)$.

2. $R(I)_{a,b} \cong R \bowtie (\alpha-\beta)I$ *if and only if* $\mathrm{Ann}(\alpha-\beta)\cap I = (0)$.

Proof. 1. If $\alpha-\beta \in \mathrm{Ann}(I)$, $R(I)_{a,b} \cong R \ltimes I$ by previous proposition. Conversely suppose that $R(I)_{a,b} \cong R \ltimes I$. Then, over any prime ideal of R lies exactly one prime ideal of $R(I)_{a,b}$ and by Lemma 2.4.3 and Proposition 2.4.4 this happens if and only if $(\alpha-\beta)I \subseteq \bigcap_{\mathfrak{p}\text{ prime}} \mathfrak{p} = (0)$, because R is reduced. Hence $\alpha-\beta \in \mathrm{Ann}(I)$.
2. We need to prove that if $R(I)_{a,b} \cong R \bowtie (\alpha-\beta)I$, then $\mathrm{Ann}(\alpha-\beta)\cap I = (0)$. If this does not happen $R(I)_{a,b}$ is not reduced by Proposition 2.5.2 and this is a contradiction because amalgamated duplication is reduced, if R is reduced (see [24, Theorem 3.5]). $\square$

If R is not reduced the first point of previous corollary does not hold as shown in the next example.

Example 2.5.6. Consider the ring $R := \mathbb{Z}/2^n\mathbb{Z}$. The non units of R are the classes represented by $2^\alpha m$ with $\alpha \geq 1$ and m odd. It follows that the square of any ideal is annihilated by 2^{n-2}. This means that, for any ideal I, one has $R[It]/I^2(t^2+2^{n-2}t) = R[It]/I^2(t^2) \cong R \ltimes I$. Moreover if we choice $I = (2)$, it follows that $2^{n-2} \notin \mathrm{Ann}(I)$.

The next example shows that there are choices of a and b for which $R \bowtie I$ is not isomorphic to idealization or amalgamated duplication.

Example 2.5.7. Consider $R := k[X,Y]/(XY)$ and $I = (x,y)$, where k is a field with $\mathrm{char}\, k \neq 2$ and x and y denote the image of X and Y in R. Defining $\alpha := y-r$ and $\beta := -y-r$, with $r \in R$, it follows that $(t-\alpha)(t-\beta) = t^2+2rt+r^2-y^2$. We have $\alpha-\beta = 2y \notin \mathrm{Ann}(I)$ and $x \in \mathrm{Ann}(\alpha-\beta)\cap I$. It follows from Corollary 2.5.5 that $R(I)_{2r,r^2-y^2}$ is not isomorphic to $R \ltimes I$; moreover, since $\mathrm{Ann}(\alpha-\beta)\cap I \neq (0)$, $R(I)_{2r,r^2-y^2}$ is not reduced and then is not isomorphic to $R \bowtie I$.

2.6 Numerical duplication of a numerical semigroup

Given a numerical semigroup S and an ideal of S, in this section we introduce a construction that gives another numerical semigroup. We will show that this is a particular case of the general construction considered in this chapter, while in the next two chapters we will show some applications.

More precisely let $E \subseteq S$ be a proper ideal of S and let $b \in S$ be an odd integer. We set $2 \cdot X := \{2x \mid x \in X\}$, where X is either S or E (note that $2 \cdot X \neq 2X := X + X$). Then we define the *numerical duplication* of S with respect to E and b as the following subset of $\mathbb{N}$:

$$S \bowtie^b E := 2 \cdot S \cup (2 \cdot E + b).$$

It is easy to check that $S \bowtie^b E$ is a numerical semigroup. The next theorem shows that this construction is a particular case of the rings $R(I)_{a,b}$, moreover it gives a simply way to get explicitly examples that are not isomorphic to both idealization and amalgamated duplication.

Theorem 2.6.1. *Let $R := k[[S]]$ be a numerical semigroup ring, let I be a proper ideal of R and let $b := x^m \in R$, with m odd. Then $R(I)_{0,-b}$ is isomorphic to the semigroup ring $k[[T]]$, where $T := S \bowtie^m v(I)$.*

Proof. If $S = \langle n_1, \ldots, n_\nu \rangle$, an element of $R(I)_{0,-b}$ is of the form $r(x) + i(x)t$, where $r(x) = r(x^{n_1}, \ldots, x^{n_\nu}) \in k[[S]]$ and $i(x) = i(x^{n_1}, \ldots, x^{n_\nu}) \in I$. Now it is easy to check that the map $\Phi : R(I)_{0,-b} \to k[[T]]$, defined by $\Phi(r(x) + i(x)t) = r(x^2) + i(x^2)x^m$, is an isomorphism of rings. $\square$

Example 2.6.2. If $R := k[[x^6, x^7, x^9, x^{17}]]$, $b := x^7$ and $I := (x^{13}, x^{14}, x^{16})$, then we get $R(I)_{0,-b} \cong k[[x^{12}, x^{14}, x^{18}, x^{33}, x^{34}, x^{35}, x^{39}]]$. Since I is a canonical ideal of R, according to Corollary 2.2.3, the ring $R(I)_{0,-b}$ is Gorenstein. In fact it is easy to see that the numerical semigroup $\langle 12, 14, 18, 33, 34, 35, 39 \rangle$ is symmetric.

In the next remark we show an immediate consequence of the previous theorem.

Remark 2.6.3. It is well-known that if the embedding dimension of an almost symmetric numerical semigroup S is less than 4, then $t(S) < \nu(S)$. In [61] the authors ask if this holds also when $\nu(S) = 4$ and indeed A. Moscariello proves this fact in [55]. In the latter paper the author computes all the almost symmetric numerical semigroups whose genus is not greater than 32, that are more or less 10^6, and finds that in all these examples $t(S) \leq \nu(S) + 1$, consequently he asks if $t(S)$ is always bounded by a function of $\nu(S)$. On the other hand if we take an almost symmetric semigroup S such that $t(S) \geq \nu(S)$, from Theorem 2.6.1, Corollary 2.1.11 and Proposition 2.3.6 follows that $T = S \bowtie^b M(S)$ is still almost symmetric for all odd $b \in S$ and $t(T) - \nu(T) > t(S) - \nu(S)$. This implies that the inequality $t(S) \leq \nu(S) + x$ does not hold for any $x \in \mathbb{N}$.

In the next example we show how the construction of the previous remark works.

Example 2.6.4. All the following numerical semigroups are almost symmetric:

- $S_1 := \langle 17, 18, 22, 23, 25, 49 \rangle$;
- $S_2 := S_1 \bowtie^{17} M(S_1) = \langle 34, 36, 44, 46, 50, 51, 53, 61, 63, 67, 98, 115 \rangle$;
- $S_3 := S_2 \bowtie^{51} M(S_2) = \langle 68, 72, 88, 92, 100, 102, 106, 119, 122, 123, 126, 134, 139, 143, 151, 153, 157, 173, 177, 185, 196, 230, 247, 281 \rangle$;
- $S_4 := S_3 \bowtie^{119} M(S_3) = \langle 136, 144, 176, 184, 200, 204, 212, 238, 244, 246, 252, 255, 263, 268, 278, 286, 295, 302, 303, 306, 314, 319, 323, 331, 346, 354, 357, 363, 365, 370, 371, 387, 392, 397, 405, 421, 425, 433, 460, 465, 473, 489, 494, 511, 562, 579, 613, 681 \rangle$;

A direct computation shows that $t(S_1) = 7$ and $\nu(S_1) = 6$, while according to the previous remark we have $t(S_4) = 2t(S_3) + 1 = 4t(S_2) + 3 = 8t(S_1) + 7$ and $\nu(S_4) = 2\nu(S_3) = 4\nu(S_2) = 8\nu(S_1)$. Consequently we get $t(S_1) - \nu(S_1) = 1$, $t(S_2) - \nu(S_2) = 3$, $t(S_3) - \nu(S_3) = 7$ and $t(S_4) - \nu(S_4) = 15$. Clearly if we continue in this way we can arbitrarily increase the difference between the type and the embedding dimension.

Even if we are interested especially in the case of Theorem 2.6.1, in the following we show that the numerical duplication arises also when R is an algebroid branch.

Theorem 2.6.5. *Let R be an algebroid branch, let I be a proper ideal of R and let $b \in R$ such that $m := v(b)$ is odd. Then $R(I)_{0,-b}$ is an algebroid branch and its value semigroup is $v(R) \bowtie^m v(I)$.*

Proof. Let $R = k[[x_1, \dots, x_n]]/\mathfrak{p}$. In this first section of this chapter we saw that $R(I)_{0,-b}$ is a local, noetherian and complete one-dimensional ring, because R is. Moreover, since $v(b)$ is odd, from the proof of Proposition 2.4.6 it follows that $t^2 - b$ is irreducible in $Q(R)[t]$ and $R(I)_{0,-b}$ is an integral domain. Then, since $R(I)_{0,-b}$ contains its residue field k, Cohen's structure theorem implies that it is isomorphic to a ring of the form $k[[y_1, \dots, y_l]]/\mathfrak{q}$, for some prime ideal $\mathfrak{q}$ of height $l - 1$, i.e. $R(I)_{0,-b}$ is an algebroid branch.

Let $k[[y]]$ the integral closure of $R(I)_{0,-b}$ in its quotient field $Q(R(I)_{0,-b}) = Q(R)(t) = k((y))$. We denote by v' the valuation associated with $k[[y]]$, in particular $v'(y) = 1$. Since $Q(R) = k((x))$, we have $k((y)) = k((x))(t)$. Moreover, $t^2 = b$ implies that $2v'(t) = v'(b) = mv'(x)$. In order to obtain $v'(y) = 1$ it is necessary that $v'(t) = m$ and $v'(x) = 2$. Consequently it easily follows that $v'(R(I)_{0,-b}) = v(R) \bowtie^m v(I)$. $\square$

In the case of the numerical duplication it is possible to generalize the definition. In fact given a numerical semigroup S, an odd integer $b \in S$ and a relative ideal E, it is still possible to define the numerical duplication of S with respect to E and b in the same way:

$$S \bowtie^b E := 2 \cdot S \cup (2 \cdot E + b).$$

This time $S \bowtie^b E$ is a numerical semigroup if and only if $E + E + b \subseteq S$. We will see that the numerical duplication has a better behaviour when E is a proper ideal, but working with relative ideals we have a huge advantage, as the next theorem shows.

Proposition 2.6.6. *Every numerical semigroup T can be realized as a numerical duplication $S \bowtie^b E$, where $S = \frac{T}{2}$, b is an odd element of S and E is a relative ideal of S such that $b + E + E \subseteq S$.*

Proof. Let b be an odd element of S and set $E := \{x \in \mathbb{Z} \,|\, 2x \in T - b\}$. It follows that E is a relative ideal of S, in fact if there exist $s \in S$ and $e \in E$ such that $s+e \notin E$, then $2(s+e)+b \notin T$, but $2s + (2e + b) \in T + T \subseteq T$, since $s \in S$ and $e \in E$; contradiction.

Moreover if e, e' are two elements of E, then $b+2e$ and $b+2e'$ is in T; therefore $2b+2e+2e' \in T$ and it is equivalent to $b + e + e' \in S$. Hence $b + E + E \subseteq S$. Finally, by construction, it is clear that $T = S \bowtie^b E$. □

Note that in the previous proof we have not determined b, so there exist infinitely many ways to obtain a numerical semigroup as a numerical duplication.

Proposition 2.6.7. *Let S be a numerical semigroup, let E be a relative ideal of S and let $b \in S$ be an odd integer such that $E + E + b \subseteq S$. The following properties hold:*

1. $f(S \bowtie^b E) = \max\{2f(S), 2f(E) + b\}$. *Furthermore* $f(S \bowtie^b E) = 2f(E) + b$, *if E is proper;*

2. $g(S \bowtie^b E) = g(S) + g(E) + m(E) + \frac{b-1}{2}$*;*

3. $S \bowtie^b E$ *is symmetric if and only if E is a canonical ideal of S.*

Proof. 1. This is straightforward because $2f(S)$ is the greatest even gap, while $2f(E) + b$ is greatest odd gap. Furthermore, if E is proper, it is clear that $f(E) \geq f(S)$ and then $2f(E)+b \geq 2f(S)$.
2. The even gaps of $S \bowtie^b E$ correspond bijectively to the gaps of S. Moreover every odd integer smaller than $2m(E) + b$ (that is positive) is not in $S \bowtie^b E$ and if $x \notin S \bowtie^b E$ is an odd integer such that $2m(E) + b \leq x \leq 2f(E) + b$, then $x = 2y + b$, with $y \notin E$ and $m(E) \leq y \leq f(E)$. Now the thesis is clear.
3. We recall that $S \bowtie^b E$ is symmetric if and only if $f(S \bowtie^b E) + 1 = 2g(S \bowtie^b E)$. Moreover a symmetric semigroup has odd Frobenius number and then $f(S \bowtie^b E)$ has to be $2f(E) + b$. Using this and 2 we get that $S \bowtie^b E$ is symmetric if and only if

$$\begin{aligned} 2f(E) + b + 1 &= 2g(S) + 2g(E) + 2m(E) + b - 1 \\ \iff f(E) + 1 &= g(S) + g(E) + m(E). \end{aligned}$$

With the notation of Section 1.6, we have $m(E) = m(\widetilde{E}) + f(S) - f(E)$ and $g(E) = g(\widetilde{E})$. Therefore the last equality is equivalent to $f(S) + 1 - g(S) = g(\widetilde{E}) + m(\widetilde{E})$ and, by Lemma 1.6.6, S is symmetric if and only if E is a canonical ideal of S. □

Corollary 2.6.8. *Every symmetric numerical semigroup T can be written as $S \bowtie^b K(S)$ for some odd $b \in S$, where S is one half of T. In particular the set of the symmetric doubles of a numerical semigroup S is $\{S \bowtie^b K(S) \,|\, K(S) + K(S) + b \subseteq S\}$.*

Proof. By Theorem 2.6.6 and Proposition 2.6.7, every symmetric numerical semigroup T is of the form $S \bowtie^b E$ for some canonical ideal E and some odd $b \in S$. Then $E = K(S) + x$ and, since

$0 \in K(S)$ and $E+E+b \subseteq S$, it follows that $b' := 2x+b \in S$. Consequently $S \bowtie^b E = S \bowtie^{b'} K(S)$ by definition and, since the numerical duplication of S is always a double of S, the thesis follows immediately. $\square$

The previous corollary gives an different proof of the results of [75] and [76], where it is stated that every numerical semigroup is one half of infinitely many symmetric numerical semigroups, and gives another way to construct all of them.

Remark 2.6.9. By definition $K(S \bowtie^b E) = \{z \in \mathbb{Z} \mid f(S \bowtie^b E) - z \notin S \bowtie^b E\}$ and we have that $a := f(S \bowtie^b E) - z \notin S \bowtie^b E$ if and only if either a is even and $a/2 \notin S$ or a is odd and $\frac{a-b}{2} \notin E$. Hence it follows that

$$z \in K(S \bowtie^b E) \iff z = f(S \bowtie^b E) - a \quad \text{with} \quad \begin{cases} \frac{a}{2} \notin S, & a \text{ even}, \\ \frac{a-b}{2} \notin E, & a \text{ odd}. \end{cases}$$

If E is a proper ideal, using Theorems 2.2.2 and 2.6.1 it is possible to find a formula for the type of $S \bowtie^b E$. In particular it follows that $t(S \bowtie^b E)$ does not depend on b. However in the next proposition we easily generalize this formula when E is relative.

Proposition 2.6.10. *Let S be a numerical semigroup, let E be a relative ideal of S and let $b \in S$ be an odd integer. Then the number of the even pseudo-Frobenius numbers of S is $|((M(S) - M(S)) \cap (E - E)) \setminus S|$ and*

$$t(S \bowtie^b E) = |((M(S) - M(S)) \cap (E - E)) \setminus S| + |((E - M(S)) \cap (M(S) - (b + E))) \setminus E|.$$

Proof. Denote $T = S \bowtie^b E$. If $x = 2h$ be an even integer not belonging to T, i.e. $h \notin S$, then $x \in M(T) - M(T)$ if and only if $2h + 2s \in M(T)$ for every $s \in M(S)$ and $2h + 2t + b \in M(T)$ for every $t \in E$. These two conditions are equivalent to $h \in M(S) - M(S)$ and $h \in E(S) - E(S)$, respectively. Hence we get the first part of the statement and consequently the first summand of the formula.

Let now $x = 2h + b$ be an odd integer not belonging to T, i.e. $h \notin E$. In this case $x \in M(T) - M(T)$ if and only if $2h + b + 2s \in M(T)$ for every $s \in M(S)$ and $2h + b + 2t + b \in M(T)$ for every $t \in E$. In this case the two conditions are equivalent to $h \in E - M(S)$ and $h \in M(S) - (b + E)$, respectively. $\square$

If E is a proper ideal, it follows that $E - M(S) \subseteq M(S) - (b + E)$. To prove this, we first notice that if $E = S$, then given $h \in S - M(S) = M(S) - M(S)$ we get $h + b + S \subseteq M(S) + S = M(S)$. Otherwise if $E \subseteq M(S)$, since $b \in M(S)$, the condition $h \in E - M(S)$ implies $h + b + E \subseteq E + E \subseteq M(S)$. Hence $E - M(S) \subseteq M(S) - (b + E)$ and clearly the formula of the previous proposition is independent of b. However we remark that it is not true if E is not proper, see for instance Example 4.1.20.

Chapter 3

One-dimensional Gorenstein local rings with decreasing Hilbert function

A problem posed by M.E. Rossi is the question whether the Hilbert function of a one-dimensional Gorenstein local ring is non-decreasing. In this chapter we solve this problem constructing infinitely many one-dimensional Gorenstein local rings, including numerical semigroup rings, whose Hilbert function decreases at some levels. Using the construction of the previous chapter, we first reduce the problem to find some suitable almost Gorenstein rings and then we give an explicit construction to find them. We also prove that for any positive integers m and $h \geq 2$ such that $h \notin \{14+22k, 35+46k \mid k \in \mathbb{N}\}$ there exist infinitely many non-isomorphic one-dimensional Gorenstein local rings such that $H_R(h-1) - H_R(h) > m$. In the last section we show several explicit examples. This chapter is based on [62].

3.1 Historical background

The study of the Hilbert function is a very classic problem in commutative algebra and algebraic geometry, both in the graded and local settings. Even if we restrict our attention to the context of one-dimensional Cohen-Macaulay local rings, we will find out an extensive literature regarding this topic (see e.g. the survey of M.E. Rossi [79]). One very natural question is to understand when the Hilbert function is non-decreasing; in fact it it is well-known that this is always true in the graded case. Clearly, this is the case when the associated graded ring is Cohen-Macaulay. However, in [48] J. Herzog and R. Waldi show that in the numerical semigroup ring $R = k[[t^{30}, t^{35}, t^{42}, t^{47}, t^{148}, t^{153}, t^{157}, t^{169}, t^{181}, t^{193}]]$ one has $H_R(1) = 10 > 9 = H_R(2)$; we also notice that the associated graded ring of R is Buchsbaum, which is another generalization of the Cohen-Macaulay property. In 1978 J. Sally conjectured in [81] that a one-dimensional Cohen-Macaulay local ring with embedding dimension at most three has non-decreasing Hilbert function. This conjecture was proved in the equicharacteristic case by J. Elias in 1993 and in general by J. Elias and J. Martínez-Borruel in 2011, see [31] and [32] respectively. Furthermore, if the embedding dimension is greater than four, in [64] F. Orecchia constructs reduced one-dimensional local rings with decreasing Hilbert function, whereas in [43] S.K. Gupta and L.G. Roberts find a one-dimensional Cohen-Macaulay local ring with embedding dimension 4 and decreasing Hilbert function; here, and in the rest of the chapter, decreasing Hilbert function means that $H_R(h-1) > H_R(h)$ for some h. We remark that it is not known if there exist numerical semigroup rings R with $4 \leq v(R) \leq 9$ and decreasing Hilbert function, in fact the example with smallest embedding dimension is that of Herzog and Waldi above (but there are also several other examples with embedding dimension 10, see for instance [63]).

M.E. Rossi formulated the following problem, which was later written in [79, Problem 4.9]: "Is the Hilbert function of a Gorenstein local ring of dimension one not decreasing?". In the last ten years several authors provided some partial positive answers, most of all confined in the semigroup case. We recall some of these in chronological order:

- in [5] F. Arslan and P. Mete, for large families of complete intersection rings and the Gorenstein semigroup rings with embedding dimension 4, under some arithmetical conditions;
- in [6] F. Arslan, P. Mete and M. Şahin, for infinitely many families of Gorenstein rings obtained by introducing the notion of nice gluing of numerical semigroups;
- in [66] D.P. Patil and G. Tamone, for the rings associated with balanced numerical semigroups with embedding dimension 4;
- in [7] F. Arslan, N. Sipahi and N. Şahin, for other 4-generated Gorenstein semigroup rings constructed by non-nice gluing;
- in [51] R. Jafari and S. Zarzuela Armengou, for some families of Gorenstein semigroup rings through the concept of extension;
- in [4] F. Arslan, A. Katsabekis, and M. Nalbandiyan, for other families of Gorenstein 4-generated semigroup rings;
- in [63] A. Oneto and G. Tamone, for semigroup rings R for which $\nu(R) \geq e(R) - 4$.

However, we will see that in general the question has a negative answer. We notice that the Hilbert function of a numerical semigrup S is simply the function $H_S : \mathbb{N} \to \mathbb{N}$ defined as $H_S(h) := |hM(S) \setminus (h+1)M(S)|$. In the sequel we will denote the Hilbert function of S by $H_S = [H_S(0), H_S(1), \dots, H_S(r) \to]$, where $\to$ means that $H_S(h) = H_S(r)$ for all $h \geq r$. We also say that S has *decreasing Hilbert function at level* h when $H_S(h-1) > H_S(h)$.

3.2 Reduction to the almost Gorenstein case

Let R be a one-dimensional Cohen-Macaulay local ring and let I be a canonical ideal of R. In the previous chapter we proved that in this case $R(I)_{a,b}$ is a one-dimensional Gorenstein local ring for all $a, b \in R$. Moreover, by Corollary 2.1.11 we know that $H_{R(I)_{a,b}}(h) = H_R(h) + \ell_R(\mathfrak{m}^{h-1}I/\mathfrak{m}^h I)$ for all $h \geq 1$.

Suppose now that R is almost Gorenstein and let ω_R be a canonical module for which $R \subseteq \omega_R \subseteq \overline{R}$. If $x \in R$ is a regular element such that $x\omega_R \subset R$, the ideal $I = x\omega_R$ is a canonical ideal of R; actually, all canonical ideals of R can be obtained in this way, see e.g. [39, Corollary

2.8]. Therefore, for all $h \geq 2$ we get

$$\ell_R\left(\frac{I\mathfrak{m}^{h-1}}{I\mathfrak{m}^h}\right) = \ell_R\left(\frac{x\omega_R\mathfrak{m}^{h-1}}{x\omega_R\mathfrak{m}^h}\right) = \ell_R\left(\frac{x\mathfrak{m}^{h-1}}{x\mathfrak{m}^h}\right) = \ell_R\left(\frac{\mathfrak{m}^{h-1}}{\mathfrak{m}^h}\right) = H_R(h-1). \tag{3.1}$$

Therefore, using Corollary 2.1.11 and the fact that the embedding dimension of a canonical ideal is the type of R (cf. [17, Proposition 3.3.11]), we get the following proposition:

Proposition 3.2.1. *Let R be an one-dimensional almost Gorenstein ring and let I be a canonical ideal of R. Then the Hilbert function of $R(I)_{a,b}$ is:*

$$\begin{aligned} &H_{R(I)_{a,b}}(0) = 1 \\ &H_{R(I)_{a,b}}(1) = \nu(R) + t(R) \\ &H_{R(I)_{a,b}}(h) = H_R(h) + H_R(h-1) \qquad \text{if } h \geq 2. \end{aligned}$$

This easy proposition has a corollary of great importance for us.

Corollary 3.2.2. *Let R be a one-dimensional almost Gorenstein ring and let I be a canonical ideal of R. Then $R(I)_{a,b}$ is a one-dimensional Gorenstein local ring for all choices of $a, b \in R$. Moreover*

$$\begin{aligned} &H_{R(I)_{a,b}}(1) - H_{R(I)_{a,b}}(2) = t(R) - H_R(2) \\ &H_{R(I)_{a,b}}(h-1) - H_{R(I)_{a,b}}(h) = H_R(h-2) - H_R(h) \qquad \text{if } h \geq 3. \end{aligned}$$

Proof. We only need to prove the formulas for the Hilbert function. It follows from the previous proposition that for all $h \geq 3$

$$H_{R(I)_{a,b}}(h-1) - H_{R(I)_{a,b}}(h) = H_R(h-1) + H_R(h-2) - H_R(h) - H_R(h-1) = H_R(h-2) - H_R(h)$$

and the first formula can be found in the same way, since $H_R(1) = \nu(R)$. $\square$

This corollary gives us a method to find one-dimensional Gorenstein local rings with decreasing Hilbert function, in fact it is enough to find one-dimensional almost Gorenstein rings R such that $H_R(h-2) > H_R(h)$ for some $h \geq 3$. This clearly implies that R has decreasing Hilbert function as well, but as far as we know there are not almost Gorenstein rings with decreasing Hilbert function to be found in the literature; on the other hand it is not surprising that having decreasing Hilbert function is a weaker condition, as the next examples show.

Example 3.2.3. 1. The ring $R = k[[t^{30}, t^{35}, t^{42}, t^{47}, t^{108}, t^{110}, t^{113}, t^{118}, t^{122}, t^{127}, t^{134}, t^{139}]]$ is almost Gorenstein and its Hilbert function is $H_R = [1, 12, 17, 16, 25, 30 \rightarrow]$. Therefore H_R decreases, but $H_R(h-2) \leq H_R(h)$ for all $h \geq 2$.
2. The almost Gorenstein semigroup ring

$$k[[t^{56}, t^{63}, t^{72}, t^{79}, t^{271}, t^{273}, t^{275}, t^{278}, t^{282}, t^{285}, t^{289}, t^{291}, t^{298}, t^{304}, t^{305}, t^{307}, t^{311}, t^{314}, t^{318}, t^{320}, t^{321}, t^{322}, t^{325}, t^{332}]]$$

has Hilbert function $[1, 24, 23, 27, 25, 36, 49, 56 \rightarrow]$. It is also interesting to notice that it has two "valleys", as far as we know this is the unique example of a such semigroup ring in literature.

In the next section we give a procedure to find almost Gorenstein semigroup rings such that $H_R(h-2) > H_R(h)$ for any fixed $h \geq 3$. Moreover in Proposition 3.3.5 we will show a particular case in which having decreasing Hilbert function is equivalent to have $H_R(h-2) > H_R(h)$ for a fixed $h \geq 3$.

Obviously Corollary 2.1.11 can be also used to construct family of rings with non-decreasing Hilbert function. For instance, let S be a two-dimensional Cohen-Macaulay local ring with minimal multiplicity, i.e. having multiplicity $1 + \operatorname{codim} S$. In [68, Theorem 1.1] it is proved that if $(R, \mathfrak{m})$ is a one-dimensional Cohen-Macaulay local ring which is a quotient of S, then every maximal Cohen-Macaulay R-module M has non-decreasing Hilbert function, where the Hilbert function of M is defined as $H_M(h) = \ell(M\mathfrak{m}^h/M\mathfrak{m}^{h+1})$. Therefore Corollary 2.1.11 implies that $R(I)_{a,b}$ has non-decreasing Hilbert function for any maximal Cohen-Macaulay ideal I; in particular, if I is a canonical ideal, $R(I)_{a,b}$ is a one-dimensional Gorenstein local ring with non-decreasing Hilbert function for all $a, b \in R$. See [68] for several classes of rings verifying the above hypotheses.

3.3 Construction of almost Gorenstein rings

In order to obtain almost Gorenstein local rings R such that $H_R(h-2) > H_R(h)$ for some $h \geq 3$, we focus on semigroup rings. First we give some definitions.

Definition 3.3.1. Let $S = \langle n_1, \dots, n_\nu \rangle$ be a numerical semigroup with maximal ideal M and multiplicity e.

1. If s is an element of S, the *order* of s is $\operatorname{ord}(s) := \max\{i \mid s \in iM\}$.

2. A *maximal representation* of $s \in S$ is $s = \sum_{j=1}^{\nu} a_j n_j$, where $a_j \in \mathbb{N}$ and $\sum_{j=1}^{\nu} a_j = \operatorname{ord}(s)$.

3. $\operatorname{Ap}_k(S) := \{s \in \operatorname{Ap}(S) \mid \operatorname{ord}(s) = k\}$.

4. $D_k := \{s \in S \mid \operatorname{ord}(s) = k-1 \text{ and } \operatorname{ord}(s+e) > k\}$, $D_k^t := \{s \in D_k \mid \operatorname{ord}(s+e) = t\}$.

5. $C_k := \{s \in S \mid \operatorname{ord}(s) = k \text{ and } s - e \notin (k-1)M\}$.

Remark 3.3.2. The importance of the sets C_k and D_k lies in the formula $H_S(k-1) - H_S(k) = |D_k| - |C_k|$. Moreover, it holds that $C_k = \operatorname{Ap}_k(S) \bigcup \{\cup_h (D_h^k + e(S)) \mid 2 \leq h \leq k-1\}$. For the proof of this results see for instance [20] or [66].

In [22, Corollary 3.11] it is proved that if a numerical semigroup S has decreasing Hilbert function, then $|\operatorname{Ap}_2(S)| \geq 3$. Therefore we start by considering the simpler case $|\operatorname{Ap}_2(S)| = 3$.

Proposition 3.3.3. *Let S be a numerical semigroup and assume that $|\operatorname{Ap}_2(S)| = 3$, $\operatorname{Ap}_k(S) = \emptyset$ for all $k \geq 3$ and H_S is decreasing. Then S is not almost symmetric.*

Proof. Since $|\operatorname{Ap}(S)| = e(S)$ and $|\operatorname{Ap}_1(S)| = \nu(S)$, it follows that $\nu(S) = e(S) - 3$. Moreover, [63, Theorem 4.2.3] implies that there exist $n_1 < n_2 \in \operatorname{Ap}_1(S)$ such that $\operatorname{Ap}_2(S) = \{2n_1, n_1 + n_2, 2n_2\}$. Since $2n_1 - e \notin S$, the element $n_1 - e$ is not a pseudo-Frobenius number of S; therefore, if S is almost symmetric, Theorem 1.6.4 implies that $\operatorname{ord}(f(S)+e(S)) > 1$ and $f(S)+e(S) = 2n_2$. On the other hand, $3n_2 - e(S) \in \operatorname{Ap}(S)$ by [63, Proposition 4.3.1] and this is a contradiction because $3n_2 - e(S) > 2n_2$. $\square$

In the light of the previous proposition we focus on the case $|\operatorname{Ap}_3| = 1$. Under this assumption the following proposition holds:

Proposition 3.3.4. [63, Proposition 3.4] *Let S be a numerical semigroup and assume that $|\operatorname{Ap}_2(S)| = 3, |\operatorname{Ap}_3(S)| = 1$ and H_S is decreasing. Let $\ell = \min\{h \mid H_S \text{ decreases at level } h\}$ and let $d = \max\{\operatorname{ord}(\sigma) \mid \sigma \in \operatorname{Ap}(S)\}$. Then $\ell \leq d$ and there exist $n_1, n_2 \in \operatorname{Ap}_1(S)$ such that for $2 \leq h \leq \ell$*

$$C_h = \{hn_1, (h-1)n_1 + n_2, \dots, n_1 + (h-1)n_2, hn_2\} = \operatorname{Ap}_h(S) \cup (D_{h-1} + e(S))$$
$$D_\ell + e(S) = \{(d+1)n_1, \ell n_1 + n_2, (\ell-1)n_1 + 2n_2, \dots, (\ell+1)n_2\}.$$

Furthermore, if $(\ell, d) \neq (3, 3)$, then $\operatorname{Ap}_k(S) = kn_1$, for $3 \leq k \leq d$.

Proposition 3.3.5. *Let S be a numerical semigroup and assume that $|\operatorname{Ap}_2(S)| = 3, |\operatorname{Ap}_3(S)| = 1$ and H_S is decreasing. Let ℓ be the minimum level in which the Hilbert function of S decreases.*

1. *If $\ell \geq 3$, then $H_S(h) = H_S(\ell - 1)$ for all $h \in [1, \ell - 1]$. Furthermore $H_S(\ell - 2) - H_S(\ell) = 1$.*

2. *If S is almost symmetric, then $f(S) = dn_1 - e(S)$ and $kn_1 - e(S)$ is not a pseudo-Frobenius number for any $k \leq d$.*

3. *If S is almost symmetric, then $\ell \geq 3$.*

Proof. 1. If $2 < h < \ell$, it follows from Proposition 3.3.4 that $C_h = (D_{h-1} + e(S)) \cup \operatorname{Ap}_h(S)$ and then $|D_{h-1}| = |C_h| - 1$, since $|\operatorname{Ap}_h(S)| = 1$. Moreover it also implies that $|C_h| = |C_{h-1}| + 1$. Hence for all $h = 2, \dots, \ell - 1$ we have

$$H_S(h-1) - H_S(h) = |D_h| - |C_h| = |C_{h+1}| - 1 - (|C_{h+1}| - 1) = 0.$$

Consequently $H_S(1) = H_S(2) = \cdots = H_S(\ell - 1)$. As for the last part of the statement it is enough to note that, again by the previous proposition, we have

$$H_S(\ell - 1) - H_S(\ell) = |D_\ell| - |C_\ell| = \ell + 2 - (\ell + 1) = 1.$$

2. Of course $f(S) + e(S)$ is the greatest element of the Apéry set and in our case it can be either $2n_2$ or dn_1. If $f(S) + e(S) = 2n_2$, then by Theorem 1.6.4 there would exist $n \in \operatorname{Ap}(S)$ such that $(d-1)n_1 + n = 2n_2$, but this is impossible, since $d \geq 3$ implies $\operatorname{ord}(2n_2) \geq 3$. Finally if follows from Theorem 1.6.4 that kn_1 is not a pseudo-Frobenius number, because $kn_1 + (d-k)n_1 = dn_1$.
3. Suppose by contradiction that $\ell = 2$. By Proposition 3.3.4, $(d+1)n_1 - e \in D_2$ and so $\operatorname{ord}((d+1)n_1 - e) = 1$. Consequently by Theorem 1.6.4 there exists $n \in \operatorname{Ap}(S)$ such that $(d+1)n_1 - e(S) + n = dn_1 + ke$, with either $k = 0$ or $k = 1$. It follows that $n_1 + n = (1+k)e(S) \leq 2e(S)$, which is impossible. □

Inspired by Theorem 1.6.4 and Proposition 3.3.4 we can construct a new family of numerical semigroups. We will prove that it defines almost symmetric numerical semigroups with decreasing Hilbert functions satisfying the assumptions of Proposition 3.3.4.

Construction 3.3.6. Let $\ell \geq 4$ be an integer such that $\ell \notin \{14+22k, 35+46k \mid k \in \mathbb{N}\}$ and let $e := \ell^2 + 3\ell + 4$. We also set

$$\left[\begin{array}{lll} n_1 := \ell^2+5\ell+3 = e+(2\ell-1), & n_2 := 2\ell^2+3\ell-2 = e+(\ell^2-6), & \text{if } \ell \text{ is odd} \\ n_1 := \ell^2+4\ell+1 = e+(\ell-3), & n_2 := 2\ell^2+2\ell-2 = e+(\ell^2-\ell-6), & \text{if } \ell \text{ is even} \end{array}\right.$$

Let S be the numerical semigroup generated by the subset $\Gamma \subseteq \mathbb{N}$

$$\Gamma := (\{e, n_1, n_2, t_1, t_2\} \cup \{s_{p,q}\} \cup \{r_{p,q}\}) \setminus \{n_1+n_2, 2n_2\}$$

where:

$$\begin{aligned} \{s_{p,q}\} &:= \{pn_1 + qn_2 - (p+q-2)e \mid 0 \leq p \leq \ell, 1 \leq q \leq \ell+1, 2 \leq p+q \leq \ell+1\} \\ \{r_{p,q}\} &:= \{\ell n_1 + e - s_{p,q} \mid 2 \leq p+q \leq \ell+1, p \geq 1, q \geq 1\} \\ t_1 &:= (\ell+1)n_1 - (\ell-1)e, \\ t_2 &:= \ell n_1 + e - t_1 = \ell e - n_1. \end{aligned}$$

The idea of the previous construction is the following. We want to construct a numerical semigroup S satisfying the hypothesis of Proposition 3.3.4 with a fixed $d = \ell \geq 4$. Suppose first that we already know $e := e(S)$, n_1 and n_2; then we know all the elements of the Apéry set with order greater than 1. Moreover, since the elements of D_k have order $k-1$, the elements of $D_k - (k-2)e$ have order 1 or are not in S. The latter case means that some elements of $D_k - he$ are in Ap_q for some $q \geq 2$ and it is possible to exclude this with a suitable choice of e, n_1 and n_2: for this and other technical reasons we require that $\ell n_1 = (\ell+2)n_2 - (\ell-1)e$. We get the following generators:

$$s_{p,q} := pn_1 + qn_2 - (p+q-2)e,$$
$$t_1 := (\ell+1)n_1 - (\ell-1)e,$$

where $0 \leq p \leq \ell$, $1 \leq q \leq \ell+1$ and $2 \leq p+q \leq \ell+1$.
Moreover, to force that S is almost symmetric we use Theorem 1.6.4, where we recall that in our case $\alpha_m = \ell n_1$. Clearly the elements $\{pn_1 \mid 1 \leq p \leq \ell\}$ satisfy the first condition of Theorem 1.6.4.2 and we will show that also the elements of $\{n_2\} \cup \{qn_2 - (q-2)e \mid 2 \leq q \leq \ell+1\}$ satisfy the one of the conditions of Theorem 1.6.4.2. On the other hand, if $2 \leq p+q \leq \ell+1$, $p \geq 1$ and $q \geq 1$, we require that among our generators there are also

$$r_{p,q} := \ell n_1 + e - s_{p,q},$$
$$t_2 := \ell n_1 + e - t_1 = \ell e - n_1.$$

Since $\mathrm{Ap}(S) = \mathrm{Ap}_1(S) \cup \{2n_1, n_1+n_2, 2n_2\} \cup \{kn_1 \mid 3 \leq k \leq \ell\}$, it follows that $|\mathrm{Ap}_1(S)|$ has to be equal to $e - \ell - 1$. Moreover, the elements $\{e, n_1, n_2, t_1, t_2\}$, $\{s_{p,q}\}$ and $\{r_{p,q}\} \setminus \{n_1+n_2, 2n_2\}$ are in $\mathrm{Ap}_1(S)$. Thus, if we require that these elements are distinct, we get

$$e \geq 5 + \frac{\ell^2+3\ell}{2} + \frac{\ell^2+\ell}{2} - 2 + \ell + 1 = \ell^2 + 3\ell + 4,$$

and therefore, if we set $e = \ell^2 + 3\ell + 4$, we can expect that in the Apéry set there are only the elements described above; in fact this happens with the values of n_1 and n_2 given in the previous construction. Clearly, many details have to be checked, but, before we deal with this, it is preferable to show the validity of the construction by giving some examples.

Example 3.3.7. In this example we show the almost symmetric semigroups of Construction 3.3.6 for $\ell \in [4, 7]$.

1. If $\ell = 4$ we get the numerical semigroup

$$S = \langle 32, 33, 38, 69, 72, 73, 74, 75, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95\rangle.$$

In this case we have $\mathrm{Ap}_2(S) = \{66, 71, 76\}$, $\mathrm{Ap}_3(S) = \{99\}$, $\mathrm{Ap}_4(S) = \{132\}$. Furthermore, $\mathrm{PF}(S) = \{37, 39, \ldots, 61, 63, 100\}$ and $H_S = [1, 27, 27, 27, 26, 27, 29, 30, 31, 32 \rightarrow]$.

2. If $\ell = 5$ the numerical semigroup S is

$$S = \langle 44, 53, 63, 117, 125, 127, 134, 135, 136, 137, 142, 143, 144, 145, 146, 147, 152, 153, 154, 155, 156, 157, 162, 163, 164, 165, 166, 167, 172, 173, 174, 175, 182, 183, 184, 192, 193, 202\rangle.$$

Moreover, we have $\mathrm{Ap}_2(S) = \{106, 116, 126\}$, $\mathrm{Ap}_3(S) = \{159\}$, $\mathrm{Ap}_4(S) = \{212\}$, $\mathrm{Ap}_5 = \{265\}$, $\mathrm{PF}(S) = \{72,\ 73,\ 81,\ 82,\ 83,\ 90,\ 91,\ 92,\ 93,\ 98,\ 99,\ 100,\ 101,\ 102,\ 103,\ 108,\ 109,\ 110,\ 111,\ 112,\ 113,\ 118,\ 119,\ 120,\ 121,\ 122,\ 123,\ 128,\ 129,\ 130,\ 131,\ 138,\ 139,\ 140,\ 148,\ 149,\ 221\ \}$ and $H_S = [1, 38, 38, 38, 38, 37, 44 \rightarrow]$.

3. If $\ell = 6$ we get

$$S = \langle 58, 61, 82, 137, 146, 149, 152, 155, 158, 161, 167, 170, 173, 176, 179, 182, 185, 188, 191, 194, 197, 200, 203, 206, 209, 212, 215, 218, 221, 224, 227, 230, 233, 236, 239, 242, 245, 248, 251, 254, 257, 260, 263, 266, 269, 272, 275, 278, 281, 284, 287\rangle.$$

Moreover, $H_S = [1, 51, 51, 51, 51, 51, 50, 51, 53, 54, 55, 56, 57, 58 \rightarrow]$.

4. Finally if $\ell = 7$ we have

$$S = \langle 74, 87, 117, 217, 221, 230, 243, 247, 251, 252, 256, 260, 264, 269, 273, 277, 281, 282, 286, 290, 294, 299, 303, 307, 311, 312, 316, 320, 324, 329, 333, 337, 341, 342, 346, 350, 354, 359, 363, 367, 371, 372, 376, 380, 384, 389, 393, 397, 401, 402, 406, 410, 414, 419, 423, 427, 431, 432, 436, 440, 449, 453, 462, 466, 479, 492\rangle$$

and its Hilbert function is $[1, 66, 66, 66, 66, 66, 66, 65, 74 \rightarrow]$.

To validate Construction 3.3.6, we need some technical lemmata. The coming results are not difficult, but they require a lot of work before we can move on to the proof of our main result here, Theorem 3.3.13. In the following results S, e, n_1, n_2 and ℓ will be as defined in Construction 3.3.6.

Lemma 3.3.8. *We have:*

1. *If ℓ is odd, then $\gcd(e, n_1, n_2) = 1$ if and only if $\ell \notin \{35 + 46k \mid k \in \mathbb{N}\}$.*

2. *If ℓ is even, then $\gcd(e, n_1, n_2) = 1$ if and only if $\ell \notin \{14 + 22k \mid k \in \mathbb{N}\}$.*

In particular S is a numerical semigroup.

Proof. 1. If we suppose that $\gcd(e, n_1, n_2) \neq 1$, it follows that

$$\begin{cases} 2\ell - 1 = ab \\ \ell^2 - 6 = ac, \;\; a, b, c \in \mathbb{N} \end{cases} \iff \begin{cases} 2\ell = ab + 1 \\ 4\ell^2 - 24 = 4ac = a^2b^2 + 2ab - 23 \end{cases}$$

that implies $a(ab^2 + 2b - 4c) = 23$. Therefore, if $a > 1$, we get $a = 23$ and $23b^2 + 2b - 4c = 1$. Thus $b = 2q + 1$, for some $q \in \mathbb{N}$ and the last equation becomes $92q^2 + 96q + 24 = 4c$, that is $c = 23q^2 + 24q + 6$. Hence

$$\begin{cases} \ell = 23q + 12 \\ b = 2q + 1 \\ c = 23q^2 + 24q + 6 \end{cases} \Longrightarrow q = 2k + 1, k \in \mathbb{N} \Longrightarrow \begin{cases} \ell = 46k + 35 \\ a = 23, \quad b = 2q + 1 \\ c = 23q^2 + 24q + 6. \end{cases}$$

Furthermore, since $e = (\ell + 2)(2\ell - 1) - (\ell^2 - 6)$, it follows that $\ell \in \{35 + 46k \,|\, k \in \mathbb{N}\}$ implies $\gcd(e, n_1, n_2) \neq 1$.
2. Since $(\ell^2 - \ell - 6) = (\ell - 3)(\ell + 2)$, as above we get

$$\begin{cases} \ell - 3 = ab \\ \ell^2 + 3\ell + 4 = ac, \;\; a, b \text{ odd}, c \text{ even} \end{cases} \iff \begin{cases} \ell = ab + 3 \\ a(ab^2 + 9b - c) = -22, \end{cases}$$

therefore $a = 11$ and $\ell = 11b + 3 = 14 + 22k$ for some $k \in \mathbb{N}$. The thesis follows as above. □

Lemma 3.3.9. *If ℓ is odd, set $F = (2\ell - 1)$ and $G = (\ell^2 - 6)$; set $F = (\ell - 3)$ and $G = (\ell^2 - \ell - 6)$ otherwise. Then:*

1. *If $n \in \{s_{p,q}\} \cup \{r_{p,q}\}$, it follows that $n = 2e + aF + bG$ with $a + b \in [1, \ell + 1], a \in [-\ell, \ell]$ and $b \in [1, \ell + 1]$.*

2. *If $n, n' \in \{s_{p,q}\} \cup \{r_{p,q}\}$, then $n - n' = aF + bG$ with $-\ell \leq a + b, b \leq \ell$ and $-2\ell \leq a \leq 2\ell$.*

3. *If $aF + bG = he$, with $a, a + b \in [-2\ell - 3, 2\ell + 3]$ and $b \in [-\ell - 1, 2\ell + 2]$, then a, b and h verify one of the following systems:*

$$(1)\begin{cases} a = \nu(\ell + 2) \\ b = -\nu \\ a + b = \nu(\ell + 1) \\ -1 \leq \nu \leq 1 \\ h = \nu, \text{ if } \ell \text{ odd} \\ h = 0, \text{ if } \ell \text{ even} \end{cases} \quad (2)\begin{cases} a = 2 \\ b = \ell + 1 \\ a + b = \ell + 3 \\ h = \ell - 2, \text{ if } \ell \text{ odd} \\ h = \ell - 3, \text{ if } \ell \text{ even} \end{cases} \quad (3)\begin{cases} a = -\ell \\ b = \ell + 2 \\ a + b = 2 \\ h = \ell - 3 \end{cases} \quad (4)\begin{cases} a = -2\ell - 2 \\ b = \ell + 3 \\ a + b = 1 - \ell \\ h = \ell - 4, \text{ if } \ell \text{ odd} \\ h = \ell - 3, \text{ if } \ell \text{ even} \end{cases}$$

If $h < 0$, in addition to the case (1) there is also the case (2′) obtained from (2) by substituting a, b, h with their opposites.

Proof. 1. If ℓ is odd, using $(\ell - 3)e = -\ell(2\ell - 1) + (\ell^2 - 6)(\ell + 2)$, we get

$$\begin{aligned} s_{p,q} &= pn_1 + qn_2 - (p + q - 2)e &&= 2e + (2\ell - 1)\,p + (\ell^2 - 6)\,q \\ r_{p,q} &= \ell n_1 + e - s_{p,q} &&= 2e - (2\ell - 1)\,p + (\ell^2 - 6)(\ell + 2 - q) = \\ & &&= 2e + (2\ell - 1)\,p' + (\ell^2 - 6)\,q'. \end{aligned}$$

Otherwise, if ℓ is even, since $(\ell-3)e = -\ell(\ell-3) + (\ell^2-\ell-6)(\ell+2)$, we get

$$\begin{array}{rclcl} s_{p,q} & = & pn_1 + qn_2 - (p+q-2)e & = & 2e + (\ell-3)\,p + (\ell^2-\ell-6)\,q \\ r_{p,q} & = & \ell n_1 + e - s_{p,q} & = & 2e - (\ell-3)\,p + (\ell^2-\ell-6)(\ell+2-q) \\ & & & = & 2e + (\ell-3)\,p' + (\ell^2-\ell-6)\,q', \end{array}$$

where in both cases we have

$$\begin{array}{lll} 2 \le p+q \le \ell+1, & 0 \le p \le \ell, & 1 \le q \le \ell+1 \\ 1 \le p'+q' \le \ell, & -\ell \le p' \le 0, & 1 \le q' \le \ell+1 \end{array}$$

2. It immediately follows from 1.
3. We can assume that $h \ge 0$, while the other case can be obtained by changing the values of a and b to their opposites, but we need to be careful that b is in the prescribed range. If ℓ is odd and $a(2\ell-1) + b(\ell^2-6) = he = h((\ell+2)(2\ell-1) - (\ell^2-6))$, then for all $\mu \in \mathbb{Z}$ we get $(a - h(\ell+2) + \mu(\ell^2-6))(2\ell-1) + (b + h - \mu(2\ell-1))(\ell^2-6) = 0$ and therefore

$$\begin{cases} a - h(\ell+2) = -\mu(\ell^2-6) \\ b + h = \mu(2\ell-1), \end{cases} \Longrightarrow \begin{cases} h = \mu(2\ell-1) - b \\ a - (\mu(2\ell-1) - b)(\ell+2) = -\mu(\ell^2-6) \end{cases}$$

that implies $a + b(\ell+2) = \mu e$. On the other hand, when ℓ is even, we have $a(\ell-3) + b(\ell^2-\ell-6) = (\ell-3)(a + b(\ell+2)) = he$. Therefore, since $(\ell-3, e) = 1$ by the previous lemma, it follows that $h = (\ell-3)\mu$ for some μ and then $a + b(\ell+2) = \mu e$. In both cases we obtain the equality $a + b(\ell+2) = \mu e$. Moreover, since $e = -\ell + (\ell+2)(\ell+2)$, we get:

$$\begin{cases} a = -\mu\ell + \nu(\ell+2) \\ b = \mu(\ell+2) - \nu \\ a + b = 2\mu + \nu(\ell+1) \end{cases}$$

Since $h \ge 0$, we can assume that also μ is non-negative. If $\mu = 0$ we get $\nu \in [-1,1]$ and

$$(1) \begin{cases} a = \nu(\ell+2), \\ b = -\nu, \\ a + b = \nu(\ell+1) \\ h = \nu, \text{ if } \ell \text{ odd}, \\ h = 0, \text{ if } \ell \text{ even}. \end{cases}$$

Otherwise if $\mu > 0$, we get again $\nu \in [-1,1]$. Indeed $a + b \in [-2\ell-3, 2\ell+3]$ implies that $\nu \le 1$, while $\nu \ge -1$ because $-\ell + \nu(\ell+2) \ge a = -\ell + \nu(\ell+2) - (\mu-1)\ell \ge -2\ell-3$; furthermore, if $\nu = -1$, then $\mu = 1$. Moreover if $\nu = 0, 1$, it follows from $b = \mu(\ell+2) - \nu = \mu(\ell+1) + \mu - \nu \le 2\ell+2$ that $\mu \le 1$. Therefore for $\mu = 1$ and $\nu \in [-1,1]$ we get the systems $(2), (3), (4)$ of the thesis. $\square$

Lemma 3.3.10. *Let* $\Gamma' := \{kn_1 \mid k \in [1,\ell]\} \cup \{n_2, t_1, t_2\} \cup \{s_{p,q}\} \cup \{r_{p,q}\}$. *Then:*
1. $s_{0,\ell+1} = \ell n_1 - n_2$ *and* $\ell n_1 + e - s_{0,q} \in \{s_{0,q'} \mid 2 \le q' \le \ell\}$, *for each* $2 \le q \le \ell$.
2. $e < n_1 < n_2$ *are the lowest elements in* Γ', *while* ℓn_1 *is the greatest element in* Γ'.

Proof. 1. The first equality follows from $\ell n_1 = (\ell+2)n_2 - (\ell-1)e$. Moreover, if $q' = \ell+2-q \in [2, \ell]$, by definition and by formula above we get

$$\ell n_1 + e - s_{0,q} = (\ell + 2 - q)n_2 - (\ell - q)e = q' n_2 - (q' - 2)e = s_{0,q'}.$$

2. The first statement is easy to check. To show that ℓn_1 is the greatest element, it is straightforward to see that it is greater than n_2, t_1, t_2, and kn_1 for $k \in [1, \ell - 1]$. Moreover according to the previous lemma we have $n = 2e + aF + bG \in \{s_{p,q}\} \cup \{r_{p',q'}\}$, where b and $a + b$ are in $[1, \ell + 1]$. If ℓ is odd we get

$$\begin{aligned}\ell n_1 - n = \ell(e + F) - 2e - aF - bG = (\ell - 2)e + (\ell - a - b)F - b(G - F) \geq \\ \geq (\ell - 2)(\ell^2 + 3\ell + 4) - (2\ell - 1) - (\ell + 1)(\ell^2 - 2\ell - 5) = 2\ell^2 + 3\ell - 2 = n_2 > 0.\end{aligned}$$

If ℓ is even, we can use the same argument. □

Lemma 3.3.11. *As above, let* $\Gamma' := \{kn_1 \,|\, k \in [1, \ell]\} \cup \{n_2, t_1, t_2\} \cup \{s_{p,q}\} \cup \{r_{p,q}\}$*. Then:*

1. *The elements of* Γ' *are all non-zero modulo* e*.*

2. *The elements of* Γ' *have distinct residues modulo* e*.*

3. *If* $m, n, n' \in \Gamma'$*, the equality* $m + n = n' + \alpha e$ *implies either* $\alpha > 0$ *or* $\alpha = 0$ *and* $n' \in \Gamma'' := \{n_1 + n_2, 2n_2, kn_1 \,|\, 2 \leq k \leq \ell\}$*.*

Proof. 1. For any element $s \in \Gamma'$, following the notation of Lemma 3.3.9 we have $s = aF + bG + ke$. Therefore if $s = \lambda e$ for some $\lambda \geq 0$, it follows that $aF + bG = he$ with $h = \lambda - k$. Then we get four cases:

$s \in \{s_{p,q}\} \cup \{r_{p',q'}\}$	$aF + bG = (\lambda - 2)e$	$a + b \in [1, \ell + 1]$	$b \in [1, \ell + 1], a \in [-\ell, \ell]$
$s = t_1$	$(\ell + 1)F = (\lambda - 2)e$	$a + b = a = \ell + 1$	$b = 0$
$s = t_2$	$F = (\ell - \lambda - 1)e$	$a + b = a = 1$	$b = 0$
$s = kn_1$	$kF = (\lambda - k)e$	$a + b = a = k \leq \ell$	$b = 0$

Every case verifies the assumptions of Lemma 3.3.9.3. In the cases $(1), \ldots, (4)$ of Lemma 3.3.9.3, either $b = -\nu \in [-1, 1]$ or $b \geq \ell + 1$; moreover, $b = \ell + 1$ if and only if $a + b = \ell + 3$, while $b = -\nu$ if and only if $a + b = \nu(\ell + 1)$. It is easy to see that there are no possible cases.

2. Let $m, n \in \Gamma'$ and suppose that $m = n + he$ with $h > 0$. If $m = pF + qG + ke$ and $n = p'F + q'G + k'e$, we get $he = m - n = (p - p')F + (q - q')G + (k - k')e$ and then $aF + bG = (h - k + k')e$. Thus, recalling the table above and Lemma 3.3.9, the possible cases

are:

	$a+b$	b
$m,n \in \{s_{p,q}, r_{p',q'}\}$	$[-\ell,\ell]$	$[-\ell,\ell]$
$m \in \{s_{p,q}, r_{p',q'}\}, n = kn_1$	$[1-k, \ell-k+1]$	$[1,\ell+1]$
$m \in \{s_{p,q}, r_{p',q'}\}, n = n_2$	$[0,\ell]$	$[0,\ell]$
$m \in \{s_{p,q}, r_{p',q'}\}, n = t_1$	$[-\ell,0]$	$[1,\ell+1]$
$m \in \{s_{p,q}, r_{p',q'}\}, n = t_2$	$[2,\ell+2]$	$[1,\ell+1]$
$m = kn_1, n = n_2$	$k-1$	-1
$m = kn_1, n = t_1$	$k-\ell-1$	0
$m = kn_1, n = t_2$	$k+1$	0
$m = t_1, n = n_2$	ℓ	-1
$m = t_1, n = t_2$	$\ell+2$	0
$m = t_2, n = n_2$	-2	-1

As in 1, we can easily see that there are no possible cases using Lemma 3.3.9.3.

3. We set:

$$p_1 := \begin{cases} kn_1 & = kF + ke \quad a+b = a = k \leq \ell \quad b = 0 \quad \textit{or} \\ t_1 & = (\ell+1)F + 2e \quad a+b = a = \ell+1 \quad b = 0 \quad \textit{or} \\ t_2 & = -F + (\ell-1)e \quad a+b = a = -1 \quad b = 0 \end{cases}$$

Then we can write $p_1 = aF + \delta e$, with $\delta \in [1,\ell]$ and $a \in [-1, \ell+1] \setminus \{0\}$. Consequently we can divide the elements of Γ' in three types:

$$\begin{array}{llll} p_1 & = aF + \delta e \quad \delta \in [1,\ell], & a+b \in [-1,\ell+1] & b = 0 \\ n_2 & = G + e & a+b = 1 & b = 1,\ a = 0 \\ p_3 & := aF + bG + 2e \in \{s_{p,q}, r_{p',q'}\} & a+b \in [1,\ell+1] & a \in [-\ell,\ell],\ b \in [1,\ell+1] \end{array}.$$

Denote by σ_1 any sum $p_1 + p_1'$:

$$\sigma_1 := \begin{cases} kn_1 + hn_1 & = (k+h)F + (k+h)e & a+b \in [2, 2\ell] & b = 0 \\ kn_1 + t_1 & = (\ell+1+k)F + (2+k)e & a+b = a = (\ell+1+k) & b = 0 \\ kn_1 + t_2 & = (k-1)F + (k+\ell-1)e & a+b = a = k-1 & b = 0 \\ 2t_1 & = (2\ell+2)F + 4e & a+b = 2\ell+2 & b = 0 \\ t_1 + t_2 & = \ell F + (\ell+1)e & a+b = \ell & b = 0 \\ 2t_2 & = -2F + (2\ell-2)e & a+b = -2 & b = 0 \end{cases}$$

Denote by σ_2 the sum $p_1 + n_2$:

$$\sigma_2 := \begin{cases} kn_1 + n_2 & = kF + 1G + (k+1)e & a+b = k+1 & a = k & b = 1 \\ t_1 + n_2 & = (\ell+1)F + G + 3e & a+b = \ell+2 & a = \ell+1 & b = 1 \\ t_2 + n_2 & = -F + G + \ell e & a+b = 0 & a = -1 & b = 1 \end{cases}$$

Moreover, let $p_3, p_3' \in \{s_{p,q}, r_{p,q}\}$, where $p_3 = a'F + b'G + 2e$ and $p_3' = a''F + b''G + 2e$, and denote by $\sigma_3 := aF + bG + \beta e$ any sum $p_3 + p_1, p_3 + n_2, p_3 + p_3'$:

$$\sigma_3 = \begin{cases} p_3 + kn_1 & = (k+a')F + b'G + (k+2)e & a+b \in [k+1, \ell+k+1] & b \in [1, \ell+1] \\ p_3 + t_1 & = (\ell+1+a')F + b'G + 4e & a+b \in [\ell+2, 2\ell+2] & b \in [1, \ell+1] \\ p_3 + t_2 & = (a'-1)F + b'G + (\ell+1)e & a+b \in [0, \ell] & b \in [1, \ell+1] \\ p_3 + n_2 & = a'F + (b'+1)G + 3e & a+b \in [2, \ell+2] & b \in [2, \ell+2] \\ p_3 + p_3' & = (a'+a'')F + (b'+b'')G + 4e & a+b \in [2, 2\ell+2] & b \in [2, 2\ell+2] \end{cases}$$

In conclusion we can write:

$$\begin{array}{llllll} \sigma_1 & = aF + \lambda e, & \lambda \in [2, 2\ell] & a = a+b \in [-2, 2\ell+2] & b = 0 \\ \sigma_2 & = aF + G + \mu e, & \mu \in [2, \ell+1] & a+1 = a+b \in [0, \ell+2] & b = 1 \\ 2n_2 & = 2G + 2e, & \mu = 2 & a = 0 & b = 2 \\ \sigma_3 & = aF + bG + \nu e, & \nu \in [3, \ell+2] & a \in [-2\ell, 2\ell],\ a+b \in [0, 2\ell+2] & b \in [1, 2\ell+2] \end{array}$$

We notice that, since $2n_1 > n_2$, if $\sigma_i = n_2 + \alpha e$ it follows that $\alpha > 0$ by Lemma 3.3.10.2.
Let $m + n = \sigma_i$ and $n' = p_j$, where $1 \leq i, j \leq 3$. Assume $\sigma_i = p_j + \alpha e$ and consider the following table:

$$\begin{array}{llll} \sigma_1 - p_1 = aF + ue & u \in [2-\ell, 2\ell-1] & a+b \in [-\ell-3, 2\ell+3] & b = 0 \\ \sigma_1 - p_3 = aF + bG + ue, & u \in [0, 2\ell-2] & a+b \in [-\ell-3, 2\ell+1] & b \in [-\ell-1, -1] \\ \sigma_2 - p_1 = aF + G + ue, & u \in [2-\ell, \ell] & a+b \in [-\ell-1, \ell+3] & b = 1 \\ \sigma_2 - p_3 = aF + bG + ue, & u \in [0, \ell-1] & a+b \in [-\ell-1, \ell+1] & b \in [-\ell, 0] \\ 2n_2 - p_1 = aF + 2G + ue, & u \in [2-\ell, 1] & a+b \in [-\ell+1, 3] & b = 2 \\ 2n_2 - p_3 = aF + bG, & u = 0 & a+b \in [-\ell+1, 1] & b \in [-\ell+1, 1] \\ \sigma_3 - p_1 = aF + bG + ue, & u \in [3-\ell, \ell+1] & a+b \in [-\ell-1, 2\ell+3] & b \in [1, 2\ell+2] \\ \sigma_3 - p_3 = aF + bG + ue, & u \in [1, \ell] & a+b \in [-\ell-1, 2\ell+1] & b \in [-\ell, 2\ell+1] \end{array}$$

Setting $h = \alpha - u$, we have that $\sigma_i - p_j = aF + bG + ue = \alpha e$ if and only if $aF + bG = he$. Therefore to prove that, in all possible cases, either $\alpha > 0$ or $\alpha = 0$ and $p_j \in \Gamma''$, we can apply Lemma 3.3.9.3, in fact we notice that the integers a, b and $a+b$ verify the required assumptions. It is straightforward to see that the cases $2n_2 - p_1$ and $2n_2 - p_3$ are impossible, except when $2n_2 = p_3$. Now we listed all possible cases.

• Case $2n_2 = p_3$: this equality means $2G + 2e = a'F + b'G + 2e$, that is $a'F + (b'-2)G = 0$, where $a' \in [-\ell, \ell], b' \in [1, \ell+1]$. Hence we are in case (1) of Lemma 3.3.9.3 with $h = 0$ and so either $b' - 2 = 0, a = 0, p_3 = 2n_2$, or ℓ even, $b' - 2 = \pm 1$ and $a = \pm(\ell+2)$, that are impossible.

• Case $\sigma_1 - p_1$: we have $b = 0$ and then $a + b = a = h = 0$ by Lemma 3.3.9.3. If $p_1 \notin \Gamma''$, it is easy to see that $p_1 = t_1$ and thus $\sigma_1 = (\ell+1)F + (\ell+1)e$, because $a = 0$. In this case $\alpha = u = \ell + 1 - 2 > 0$.

• Case $\sigma_1 - p_3$: we have $u \geq 0$ and $b \in [-\ell-1, -1]$, thus $h \geq 0$ and α is always positive, except when either $h = u = 0, b = -1, a = \ell+2$, and ℓ is even or $a = -2, b = -\ell-1$, and $h \in [2-\ell, 3-\ell]$. In the first case, it is easy to see that $u = h = 0$ implies $\sigma_1 = 2n_1$. From this case follows that $a + b \geq \ell + 1$, but this is impossible when $\sigma_1 = 2n_1$. In the second one $p_3 = (\ell+1)G + 2e$ implies that $\sigma_1 = 2t_2 = -2F + (2\ell-2)e$, then $u = 2\ell - 4$ and $\alpha \geq \ell - 2 > 0$.

• Case $\sigma_2 - p_1$: we have $b = 1, a + b = -\ell - 1$, and $h \in \{-1, 0\}$ by Lemma 3.3.9.3. It follows that $\sigma_2 = -F + G + (\ell+1)e$ and $p_1 = (\ell+1)F + 2e$; then $u = \ell + 1 - 2 \geq 2$ and $\alpha > 0$.

• Case $\sigma_2 - p_3$: in this case $u \geq 0$ and, according to Lemma 3.3.9.3, we have $b \in [-1, 0]$ that implies $h \geq 0$. Thus we always have $\alpha \geq 0$, except when $u = h = 0$ and it is easy to see that $u = 0$ implies $\sigma_2 = n_1 + n_2$. If $b = 0$, then $a + b = 0$ implies $p_3 = n_1 + n_2 \in \Gamma''$.

• Case $\sigma_3 - p_1 = aF + bG + ue = \alpha e$, where $\sigma_3 = a'F + b'G + \nu' e$ and $p_1 = a''F + \delta e$:
- if $h \geq \ell - 2$, then $\alpha > 0$, since $u \in [3 - \ell, \ell + 1]$.
- if $h = \ell - 3 = -u$, we have $\alpha = 0$ and $\nu' - \delta = 3 - \ell$, i.e. $\delta = \ell + \nu' - 3$. This implies $\nu' = 3$ and $\delta = \ell$, because $\nu' \leq 3$ and $\ell \geq \delta$; therefore $p_1 = \ell n_1 \in \Gamma''$.
- if $h < \ell - 3$ and ℓ is odd, the possible cases of Lemma 3.3.9.3 are (1) and (4). In case (1), since $b \geq 1$, we get $b = 1$ and thus $h = -1, b = 1$ and $a + b = -1 - \ell$. Since $a' + b' \geq 0$, we deduce that $a'' = \ell + 1$ and hence $p_1 = t_1 = (\ell + 1)F + 2e$, while $\sigma_3 = p_3' + t_2$ has $\nu' = \ell + 1$; therefore $u = \ell - 1 > 2$ and $\alpha = h + \ell - 1 > 0$. In case (4) we have $h = \ell - 4$ and then $\alpha > 0$ except when $u \in [4 - \ell, 3 - \ell]$. If $u = 4 - \ell$, then $\alpha = 0$ and $\delta = \ell + \nu' - 4$; thus $\nu' \in [3, 4]$, since $\delta \leq \ell$. Moreover $\nu' = 4$, implies $\delta = \ell$ and then $p_1 = \ell n_1 \in \Gamma''$, while from $\nu' = 3$ follows $\delta = \ell - 1$ that implies $p_1 = (\ell - 1)n_1 \in \Gamma''$, because, if $p_1 = t_2$, then $a + b \in [1, 2\ell - 1]$; this is a contradiction by (4). If $u = 3 - \ell$, then $\delta = \ell + \nu' - 3$ implies $\nu' = 3$ as above and the possible σ_3 are $p_3 + n_1$ and $p_3 + n_2$ and thus $b = b' \leq \ell + 2$, but this is incompatible with the value $b = \ell + 3$ in (4).
- $h < \ell - 3$ and ℓ is even, then $h = 0$, $b = -\nu$ and $a + b = \nu(1 + \ell)$. Since in $\sigma_3 - p_1$ we have $a + b \geq -\ell - 1$, then $\nu \geq -1$. Now $b \geq 1$ implies $\nu = -1, b = 1$ and $a + b = -\ell - 1$. Therefore we can proceed as in the case $h = -1, b = 1$ treated above, when ℓ is odd.

• Case $\sigma_3 - p_3$: we always have $\alpha > 0$, except if $b = 1 = u, a + b = -\ell - 1$ and ℓ is odd; in this case $\sigma_3 = a'F + b'G + \nu' e$, with $a' + b' \geq 0$. Hence $a + b = -\ell - 1$ implies that $p_3 = a''F + b''G + 2e$, with $a'' + b'' = \ell + 1$ and $a' + b' = 0$. Then $\sigma_3 = p_3' + t_2 = a'F + b'G + (\ell + 1)e$ and $u = \ell - 1 > 2$, that is a contradiction because $u = 1$. $\square$

Proposition 3.3.12. *The Apéry set of S is $\{0\} \cup \Gamma'$, where*

$$\Gamma' := \{kn_1 \mid k \in [1, \ell]\} \cup \{n_2, t_1, t_2\} \cup \{s_{p,q}\} \cup \{r_{p,q}\}.$$

Furthermore $\ell n_1 - e$ is the Frobenius number of S and, with the notation of Theorem 1.6.4,

$$\mathrm{Ap}_2(S) = \{2n_1, n_1 + n_2, 2n_2\} \qquad \mathrm{Ap}_k(S) = \{kn_1\} \text{ for } 3 \leq k \leq \ell$$
$$A = \{0, n_2, s_{0,\ell+1}\} \cup \{kn_1 \mid k \in [1, \ell]\} \qquad B = \{s_{p,q} \mid (p, q) \neq (0, \ell + 1)\} \cup \{r_{p,q}\} \cup \{t_1, t_2\}.$$

Proof. We notice that if $r \geq 2$, $\beta \geq 0$ and $s, s_1, \ldots, s_r \in \Gamma'$, the equality $s = s_1 + \cdots + s_r + \beta e$ is impossible or implies $\beta = 0$ and $s \in \Gamma'' := \{n_1 + n_2, 2n_2\} \cup \{kn_1 \mid k \geq 2\}$. In fact by the previous lemma there exists $s' \in \Gamma'$ such that $s_1 + s_2 = s' + \beta' e$, where $\beta' \geq 0$. Therefore if $r \geq 3$, we have $s = s' + s_3 + \cdots + s_r + (\beta + \beta')e$; by iterating and by Lemma 3.3.11, we deduce that the unique possible case is $\beta = 0$ and $s \in \Gamma''$. Since the minimal generators of S are in Γ', this implies that for every element $s \in \Gamma'$ we have $s - e \notin S$ and this means that $0 \cup \Gamma' \subseteq \mathrm{Ap}(S)$. From $|\{0\} \cup \Gamma'| = e$ and the fact that these elements are all distinct by the previous lemma, it follows that $\mathrm{Ap}(S) = 0 \cup \Gamma'$; moreover it is clear that the elements in Γ' of order greater than 1 are in Γ''.
Since by Lemma 3.3.10 $n_1 < n_2$ are the smallest elements in Γ', it follows that $\mathrm{ord}(2n_1) =$

$\operatorname{ord}(n_1+n_2)=2$. On the other hand $\operatorname{ord}(2n_2)=2$, because in the proof of Lemma 3.3.11.3 we proved that $\sigma_i=2n_2$ if and only if $\sigma_i=n_2+n_2$. Consequently $\operatorname{Ap}_2(S)=\{2n_1,n_1+n_2,2n_2\}$. Moreover $\operatorname{ord}(kn_1)=k$ because n_1 is the smallest element of S and then $\operatorname{Ap}_k(S)=\{kn_1\}$.
Finally, it follows from Lemma 3.3.10 that (ℓn_1-e) is the Frobenius number of S and, by construction and by Lemma 3.3.10, we get A and B as in the statement. $\square$

Theorem 3.3.13. *Let $R=K[[S]]$, where S is as in Construction 3.3.6. Then*

1. *The ring R is almost Gorenstein;*

2. *The embedding dimension of R is $\nu(R)=e-(\ell+1)=\ell^2+2\ell+3$;*

3. *The Hilbert function of R decreases at level ℓ, in particular*

$$H_R=[1,\nu,\nu,\dots,\nu,\nu-1,H_R(\ell+1),\dots];$$

4. *The type of R is $t(R)=\nu(R)-1=\ell^2+2\ell+2$.*

Proof. 1. This is a consequence of the previous proposition and Theorem 1.6.4.

2. In the previous proposition we showed that there are $\ell+1$ elements of the Apéry set of S with order greater than 1. Consequently $\nu(R)=\nu(S)=|\operatorname{Ap}_1(S)|+1=e-1-(\ell+1)+1=\ell^2+2\ell+3$.

3. First we show that if $\operatorname{ord}(d)=k-1$ and $d\in D_k$, then

$$d+e=an_1+bn_2 \text{ with } \operatorname{ord}(an_1+bn_2)=k+1. \qquad (1)$$

For any $x\in C_h$, where $h\geq 2$, consider a maximal representation $x=\sum a_in_i$, with $\sum a_i=h$ and $n_i\in\operatorname{Ap}_1(S)$. If $y=\sum b_in_i$, with $0\leq b_i\leq a_i$ and $\sum b_i=h'$, then [63, Proposition 1.4.1] implies that $y\in C_{h'}$. Since in our case $C_2=\{2n_1,n_1+n_2,2n_2\}$, it follows that $C_h\subseteq\{an_1+bn_2\mid a+b=h\}$ for every $h\geq 2$. In particular for each $k\in[2,\ell-1]$ and $d\in D_k$, we have the maximal representation $d+e=an_1+bn_2$. Now suppose by contradiction that $\operatorname{ord}(an_1+bn_2)=k'=k+p$, with $p\geq 2$. By [63, Proposition 2.2.1], for any $y=a'n_1+b'n_2$ with $0\leq a'\leq a, 0\leq b'\leq b$ and $a'+b'=q\leq p+1$, we have $y\in\operatorname{Ap}_q(S)$. Since $k'\geq 4$ and $\operatorname{Ap}_3(S)=\{3n_1\}$, if $k'\leq\ell$ it follows that $an_1+bn_2=k'n_1\in D_k+e\subseteq M(S)+e$, that is impossible because $k'n_1\in\operatorname{Ap}(S)$. If $k'=\ell+1+\alpha$, with $\alpha\geq 0$, we get $d+e=(\ell+1)n_1+\alpha n_1=t_1+(\ell-1)e+\alpha n_1$ and thus $\operatorname{ord}(d)\geq 1+(\ell-2)+\alpha=k'-2$. It follows that $\operatorname{ord}(d)=k'-2=k-1$, because $\operatorname{ord}(d)\leq k'-2$. Hence $an_1+bn_2\in C_{k+1}\cap(D_k+e)$.
Since $H(k)=H(k-1)+|C_k|-|D_k|$, to prove that the Hilbert function of R decreases al level ℓ it is enough to show that:

- $D_k+e=\{kn_1+n_2,(k-1)n_1+2n_2,\dots,(k+1)n_2\}=C_{k+1}\setminus\{(k+1)n_1\}$ for any $k\in[2,\ell-1]$, in particular $|D_k|=k+1$;

- $D_\ell+e=\{(\ell+1)n_1,\ell n_1+n_2,(\ell-1)n_1+2n_2,\dots,(\ell+1)n_2\}$, in particular $|D_\ell|=\ell+2$.

If $k=2$ we have $C_2=\operatorname{Ap}_2(S)$ and, by definition of $s_{p,q}$, we get $D_2\supseteq\{2n_1+n_2-e,n_1+2n_2-e,3n_2-e\}$. Then $D_2=\{2n_1+n_2-e,n_1+2n_2-e,3n_2-e\}$, otherwise the Hilbert function decreases at level 2, that is impossible by Proposition 3.3.5. Hence $C_3=(D_2+e)\cup\{3n_1\}$.
Now we proceed by induction. Let $3\leq k\leq\ell$ and assume the thesis true for $k-1$. This means

that we know the structures of $C_2, \dots, C_k, D_2, \dots, D_{k-1}$. Consider the element $an_1 + bn_2$ with $a+b = k+1 \in [4, \ell+1]$ and $a < k+1$, if $k \neq \ell$. We know that $s_{a,b} = an_1+bn_2-(k-1)e \in \mathrm{Ap}_1$ is such that $\mathrm{ord}(s_{a,b}+(k-1)e) \geq k+1$. Moreover, since $s_{a,b} \notin D_2$, it follows that $\mathrm{ord}(s_{a,b}+e) = 2$; hence there exists $r \in [1, k-2]$ such that $\mathrm{ord}(s_{a,b}+re) = r+1$ and $\mathrm{ord}(s_{a,b}+(r+1)e) > r+2$, i.e. $s_{a,b}+re \in D_{r+2}$, where $r+2 \leq k$. If $r+2 < k$, by induction there would exist $a'n_1+b'n_2-e \in D_{r+2}$ such that $s_{a,b}+re = a'n_1+b'n_2-e = s_{a',b'}+(a'+b'-3)e$, but this is impossible because $s_{a,b}$ and $s_{a',b'}$ have distinct residues modulo e. Hence, $r = k-2$, $an_1+bn_2-e \in D_k$ and $an_1+bn_2 \in C_{k+1}$ by (1). This proves 3.

4. The type of R is $t(R) = t(S) = |B| + 1 = \dfrac{\ell^2+3\ell}{2} - 1 + \dfrac{\ell^2+\ell}{2} + 3 = \ell^2 + 2\ell + 2.$ $\square$

Without using Construction 3.3.6, but by similar techniques, it is possible to construct other almost symmetric semigroups such that $H_S(h-1) > H_S(h)$. Since in Construction 3.3.6 we require $h \geq 4$, the following provides examples for $h = 2, 3$. Moreover, we exhibit also another almost symmetric semigroup with $|\,\mathrm{Ap}_2(S)| = 3$, $|\,\mathrm{Ap}_3(S)| = 1$ and decreasing Hilbert function, but with multiplicity less than that of the semigroups of Construction 3.3.6.

Example 3.3.14. **1**. The numerical semigroup

$$S = \langle 33, 41, 42, 46, 86, 90, 91, 95, 96, 97, 98, 100, 101, 103, 104, 105, 106, 109, 110, 111, \\ 113, 114, 118, 122\rangle$$

is almost symmetric and its Hilbert function $[1, 24, 23, 23, 31, 33 \rightarrow]$ decreases at level 2. Moreover $\mathrm{Ap}_2(S) = \{82, 83, 84, 87, 88, 92, 127\}$, $\mathrm{Ap}_3(S) = \{126\}$, $\mathrm{Ap}_4(S) = \{168\}$ and $\mathrm{Ap}_k(S) = \emptyset$, if $k \geq 5$.

2. The numerical semigroup

$$S = \langle 32, 33, 38, 58, 59, 60, 61, 62, 63, 67, 68, 69, 72, 73, 74, 75, 77, 78, 79, 80, 81, 82, 83, \\ 84, 85, 86, 87, 88\rangle$$

is almost symmetric with Hilbert function $[1, 28, 28, 27, 27, 29, 30, 31, 32 \rightarrow]$ decreasing at level 3. Furthermore $\mathrm{Ap}_2(S) = \{66, 71, 76, 121\}$ and $\mathrm{Ap}_k(S) = \emptyset$, if $k \geq 3$.

3. The numerical semigroup

$$S = \langle 30, 33, 37, 64, 68, 71, 73, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 91, \\ 92, 94, 95, 98, 101\rangle$$

is almost symmetric with decreasing Hilbert function $[1, 25, 25, 25, 24, 27, 28, 29, 30 \rightarrow]$. Moreover $\mathrm{Ap}_2(S) = \{66, 70, 74\}$, $\mathrm{Ap}_3(S) = \{99\}$, $\mathrm{Ap}_4(S) = \{132\}$ and $\mathrm{Ap}_k(S) = \emptyset$, if $k \geq 5$.

3.4 The Gorenstein case

In this section we consider the Gorenstein case and in particular we give many explicit examples of one-dimensional Gorenstein local rings with decreasing Hilbert function. To this purpose we use numerical duplication to produce suitable semigroup rings; in this case it is enough to show that numerical duplication is symmetric and has decreasing Hilbert function. Since the Gorenstein property and the hilbert function are independent of a and b, it is clear that we can use the same argument for every member of the family $R(I)_{a,b}$, for example idealization or amalgamated duplication.

We also notice that it is easy to compute the generators of $S \bowtie^b E$: if $\mathrm{G}(S) = \{n_1, \dots, n_r\}$ is the set of the minimal generators of S and E is generated as an ideal by $\{m_1, \dots, m_s\}$, then $S \bowtie^b E = \langle 2n_1, \dots, 2n_r, 2m_1 + b, \dots, 2m_s + b\rangle$. If $E = K(S) + z$, by Lemma 1.6.3 the numerical semigroup $S \bowtie^b E$ is generated by

$$\{2n_i, 2(f(S) - x_j + z) + b \,|\, n_i \in \mathrm{G}(S), x_j \in \mathrm{PF}(S)\}.$$

Moreover, if S is almost symmetric, Theorem 1.6.4 implies that $S \bowtie^b (K(S) + z)$ is minimally generated by

$$\{2n_i, 2z + b, 2x_j + 2z + b \,|\, n_i \in \mathrm{G}(S), x_j \in \mathrm{PF}(S) \setminus \{f(S)\}\}.$$

Now we are ready to show the first one-dimensional Gorenstein local ring with decreasing Hilbert function.

Example 3.4.1. Consider the first numerical semigroup S of Example 3.3.7. Set $b = 33$ and $E = K(S)+101 = K(S)+f(S)+1 \subseteq S$. Since we know the generators and the pseudo-Frobenius numbers of S, it follows from what we said above that $k[[S \bowtie^b E]]$ is equal to

$$\begin{aligned}k[[&t^{64}, t^{66}, t^{76}, t^{138}, t^{144}, t^{146}, t^{148}, t^{150}, t^{154}, t^{156}, t^{158}, t^{160}, t^{162}, t^{164}, t^{166}, t^{168}, t^{170}, t^{172}, t^{174}, t^{176},\\ &t^{178}, t^{180}, t^{182}, t^{184}, t^{186}, t^{188}, t^{190}, t^{235}, t^{309}, t^{313}, t^{315}, t^{317}, t^{319}, t^{321}, t^{323}, t^{325}, t^{327}, t^{329}, t^{331}, t^{333},\\ &t^{335}, t^{337}, t^{339}, t^{341}, t^{343}, t^{345}, t^{347}, t^{349}, t^{351}, t^{353}, t^{355}, t^{357}, t^{361}]]\end{aligned}$$

and this is a one-dimensional Gorenstein local ring. Moreover, Proposition 3.2.1 implies that its Hilbert function is $[1, 53, 54, 54, 53, 53, 56, 59, 61, 63, 64 \rightarrow]$.

The semigroups of Construction 3.3.6 allow us to produce Gorenstein rings whose Hilbert function decreases at level h, for any $h \geq 4$ and $h \notin \{14+22k, 35+46k \,|\, k \in \mathbb{N}\}$, while for $h = 3$ we can use Example 3.3.14.2. The next example will be useful to cover the case $h = 2$.

Example 3.4.2. Consider the numerical semigroup

$$\begin{aligned}S = \langle &68, 72, 78, 82, 107, 111, 117, 121, 158, 162, 166, 168, 170, 172, 174, 176, 178, 180, 182, 184,\\ &186, 188, 190, 192, 194, 196, 197, 198, 200, 201, 202, 205, 206, 207, 209, 210, 211, 213, 215,\\ &217, 219, 221, 223, 225, 227, 229, 231, 233, 235, 237, 239, 241, 245, 249\rangle.\end{aligned}$$

Using GAP [36], it is possible to see that S is almost symmetric, has type 53 and its Hilbert function is $[1, 54, 52, 50, 54, 64, 68 \rightarrow]$. Consequently Proposition 3.2.1 implies that $k[[S \bowtie^b K]]$ is a Gorenstein ring and has Hilbert function $[1, 107, 106, 102, 104, 118, 132, 136 \rightarrow]$ decreasing at levels 2 and 3 for all proper canonical ideals K and for all odd $b \in S$.

Now we want to show that there are no bounds for $H_R(h-1) - H_R(h)$ for any h, even if R is Gorenstein. We start with a lemma which is a straightforward consequence of Corollary 2.1.11 and Proposition 2.3.6.

Lemma 3.4.3. *Let $R^{(0)}$ be a local ring.*
1. *Consider the ring $R^{(i+1)} := R^{(i)}(\mathfrak{m}^{(i)})_{a^{(i)},b^{(i)}}$, where $\mathfrak{m}^{(i)}$ is the maximal ideal of $R^{(i)}$ and $a^{(i)}, b^{(i)}$ are two arbitrary elements of $R^{(i)}$. Then*

$$H_{R^{(i)}}(h) = 2H_{R^{(i-1)}}(h) = \cdots = 2^i H_{R^{(0)}}(h) \text{ for all } h > 0.$$

2. *If $R^{(0)}$ is a one-dimensional almost Gorenstein ring and has type t, then $R^{(i)}$ is almost Gorenstein and, if $R^{(0)}$ is not a DVR, has type $2^i t + 2^i - 1$.*

Theorem 3.4.4. *Let m and $h > 1$ be positive integers such that $h \notin \{14+22k, 35+46k \mid k \in \mathbb{N}\}$. There exist infinitely many non-isomorphic one-dimensional Gorenstein local rings R such that $H_R(h-1) - H_R(h) > m$.*

Proof. Let $h = 2$ and let $R^{(0)} = k[[S]]$ be the ring of Example 3.4.2. The previous lemma implies that $R^{(i)}$ is almost Gorenstein, $H_{R^{(i)}}(1) = 54 \cdot 2^i, H_{R^{(i)}}(2) = 52 \cdot 2^i$ and $t(R^{(i)}) = 54 \cdot 2^i - 1$; therefore, it is enough to apply Corollary 3.2.2 to $R = R^{(i)}$: if $i \geq i_0 = \lfloor \log_2(m+1) \rfloor$, then $H_{R(I)_{a,b}}(1) - H_{R(I)_{a,b}}(2) = t(R) - H_R(2) = 54 \cdot 2^i - 1 - 52 \cdot 2^i = 2^{i+1} - 1 > m$.

If $h \geq 3$, consider an almost Gorenstein ring R such that $H_R(h-2) - H_R(h) = n > 0$, whose existence we proved in Example 3.3.14.2 for $h = 3$ and in Theorem 3.3.13 for $h \geq 4$. The construction of the previous lemma with $i_0 = \lfloor \log_2(m/n) \rfloor + 1$ implies that $H_{R^{(i)}}(h-2) - H_{R^{(i)}}(h) = 2^i n > m$ for all $i \geq i_0$.

For each h of the statement, if $i \geq i_0$, let ω be a canonical ideal of $R^{(i)}$ and $a, b \in R^{(i)}$. It follows from Corollary 3.2.2 that the ring $R^{(i)}(\omega)_{a,b}$ has all the properties we are looking for. Clearly, for any $i \geq i_0$ we get infinitely many non-isomorphic rings, because their Hilbert functions are different. □

For any m and h as above, among the infinitely many rings provided by the previous theorem, there are always non-reduced rings (idealization), reduced rings that are not integral domains (amalgamated duplication), and semigroup rings (numerical duplication). As for semigroup rings, even if we do not require the Gorenstein property, as far as we know there are not examples of this kind in literature, in fact in all examples we know m is very small and $h \leq 4$; moreover, before the appearance of the very recent paper [63], in all examples $h = 2$.

In the next example we show how the construction of Lemma 3.4.3 and Theorem 3.4.4 works.

Example 3.4.5. Let $T^{(0)}$ be the second semigroup of Example 3.3.7 and construct the numerical semigroups of Lemma 3.4.3 applying numerical duplication, that is a particular case of the lemma. All the following semigroups are almost symmetric:

- $T^{(0)}$ has type 37 and $H_{T^{(0)}} = [1, 38, 38, 38, 38, 37, 44 \rightarrow]$;
- $T^{(1)} := T^{(0)} \bowtie^{53} M(T^{(0)})$ has type 75 and $H_{T^{(1)}} = [1, 76, 76, 76, 76, 74, 88 \rightarrow]$;
- $T^{(2)} := T^{(1)} \bowtie^{141} M(T^{(1)})$ has type 151 and $H_{T^{(2)}} = [1, 152, 152, 152, 152, 148, 176 \rightarrow]$;
- $T^{(3)} := T^{(2)} \bowtie^{317} M(T^{(2)})$ has type 303 and $H_{T^{(3)}} = [1, 304, 304, 304, 304, 296, 352 \rightarrow]$;

- $T^{(4)} := T^{(3)} \bowtie^{669} M(T^{(3)})$ has type 607 and $H_{T^{(4)}} = [1, 608, 608, 608, 608, 592, 704 \rightarrow]$;
- $T := T^{(4)} \bowtie^{1373} K$, where $K := K(T^{(4)}) + f(T^{(4)}) + 1 \subseteq T^{(4)}$, is symmetric and has Hilbert function
$$H_T = [1, 1215, 1216, 1216, 1216, 1200, 1296, 1408 \rightarrow].$$

If we want to find symmetric semigroups with bigger difference between $H(4)$ and $H(5)$, we can continue in this way and apply at the last step the numerical duplication with respect to a canonical ideal. We notice that in this example T has 1215 minimal generators included between 1408 and 23835.

Example 3.4.6. Consider the almost symmetric numerical semigroup

$$T_0 := \langle 30, 33, 37, 64, 68, 69, 71, 72, 73, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 91, 92 \rangle$$

that has Hilbert function $[1, 26, 26, 25, 24, 27, 28, 29, 30 \rightarrow]$. Let $K'(T_i)$ be a proper canonical ideal of T_i and b_i an arbitrary odd element of T_i. All the following numerical semigroups are symmetric and, since they are almost symmetric, their Hilbert functions can be computed by means of Proposition 3.2.1.

- The semigroup $T_1 := T_0 \bowtie^{b_0} K'(T_0)$ has Hilbert function
$$H_{T_1} = [1, 51, 52, 51, 49, 51, 55, 57, 59, 60 \rightarrow];$$
- The semigroup $T_2 := T_1 \bowtie^{b_1} K'(T_1)$ has Hilbert function
$$H_{T_2} = [1, 52, 103, 103, 100, 100, 106, 112, 116, 119, 120 \rightarrow];$$
- The semigroup $T_3 := T_2 \bowtie^{b_2} K'(T_2)$ has Hilbert function
$$H_{T_3} = [1, 53, 155, 206, 203, 200, 206, 218, 228, 235, 239, 240 \rightarrow];$$
- The semigroup $T_4 := T_3 \bowtie^{b_3} K'(T_3)$ has Hilbert function
$$H_{T_4} = [1, 54, 208, 361, 409, 403, 406, 424, 446, 463, 474, 479, 480 \rightarrow];$$
- The semigroup $T_5 := T_4 \bowtie^{b_4} K'(T_4)$ has Hilbert function
$$H_{T_5} = [1, 55, 262, 569, 770, 812, 809, 830, 870, 909, 937, 953, 959, 960 \rightarrow].$$

In the next example we show that if S is not almost symmetric, it is possible that its numerical duplication with respect to a canonical ideal has decreasing Hilbert function. Moreover it is proved in [63, Corollary 4.11] that in a symmetric semigroup with decreasing Hilbert function the difference between the multiplicity and the embedding dimension has to be greater or equal to 5: in the following example is 6.

Example 3.4.7. Consider $S := \langle 30, 33, 37, 64, 68, 69, 71, 72, 73, 75, \dots, 89, 91, 92, 95\rangle$ that has Hilbert function $[1, 27, 26, 25, 24, 27, 28, 29, 30, \rightarrow]$ and set $K := K(S) + 66 \subseteq S$. Then the semigroup $S \bowtie^{33} K$ is symmetric and has Hilbert function $[1, 54, 55, 55, 54, 57, 58, 59, 60 \rightarrow]$. Moreover, Proposition 3.2.1 implies that S is not almost symmetric.

When one considers a non-proper canonical ideal, the Hilbert function of its numerical duplication can be different from the expected one. On the other hand, the next examples show that also in this case it is possible to find symmetric semigroups with decreasing Hilbert function.

Example 3.4.8. In [63, Example 3.7] the following numerical semigroup is exhibited:

$$S := \langle 30, 33, 37, 73, 76, 77, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 91, 92, 94, 95, 98, 101, 108\rangle,$$

which has Hilbert function $[1, 24, 25, 24, 23, 25, 27, 29, 30 \rightarrow]$.
Set $K := K(S)$. It is easy to see that, for the following choices of b, one has $K + K + b \subseteq S$, but $S \bowtie^b K$ cannot be realized as a numerical duplication with respect to a proper ideal. We know that if the canonical ideal is proper the Hilbert function of its numerical duplication does not depend on b, while if it is not proper it is interesting to see how the Hilbert function can change for different choices of b. Moreover all of the following semigroups, but H_4, are symmetric and have decreasing Hilbert function:

- $H_1 := S \bowtie^{67} K$ has Hilbert function $[1, 43, 47, 45, 49, 51, 60 \rightarrow]$;
- $H_2 := S \bowtie^{73} K$ has Hilbert function $[1, 44, 43, 41, 49, 58, 60, \rightarrow]$;
- $H_3 := S \bowtie^{79} K$ has Hilbert function $[1, 44, 41, 40, 52, 58, 60, \rightarrow]$;
- $H_4 := S \bowtie^{81} K$ has Hilbert function $[1, 43, 45, 47, 52, 54, 56, 58, 60 \rightarrow]$;
- $H_5 := S \bowtie^{85} K$ has Hilbert function $[1, 44, 42, 45, 52, 54, 58, 60 \rightarrow]$;
- $H_6 := S \bowtie^{87} K$ has Hilbert function $[1, 46, 48, 47, 49, 51, 56, 58, 60 \rightarrow]$;
- $H_7 := S \bowtie^{93} K$ has Hilbert function $[1, 47, 49, 48, 48, 50, 55, 58, 60 \rightarrow]$.

Note that H_7 has the same Hilbert function of the numerical duplication with respect to a proper canonical ideal of S.

In many of our examples there are symmetric semigroups whose Hilbert function decreases at *more levels*. The next one decreases at 13 consecutive levels included level 14, suggesting that the restrictions of Theorem 3.4.4 could be removed.

Example 3.4.9. Let S be the semigroup of Construction 3.3.6 with $\ell = 15$, that has 258 minimal generators. According to GAP [36], the symmetric semigroup $S \bowtie^{957} K(S)$ has Hilbert function

$$[1, 514, 514, 513, 512, 511, 510, 509, 508, 507, 506, 505, 504, 503, 502, 500, 523, H_S(17), \dots].$$

With the last example we show that $S \bowtie^b K(S)$ can have decreasing Hilbert function, even if that of S is non-decreasing. We point out that, among all the symmetric semigroups with decreasing Hilbert function this is the semigroup with the smallest multiplicity and embedding dimension that we know.

Example 3.4.10. Consider the semigroup $S = \langle 19, 21, 24, 47, 49, 50, 51, 52, 53, 54, 55, 56, 58, 60 \rangle$, shown in [63, Example 3.2.1], and let $T := S \bowtie^{49} K(S)$. The Gorenstein local ring $k[[T]]$ is

$$k[[t^{38}, t^{42}, t^{48}, t^{49}, t^{94}, t^{100}, t^{101}, t^{102}, t^{104}, t^{105}, t^{106}, t^{107}, t^{108}, t^{109}, t^{110}, t^{111}, t^{112}, t^{113}, t^{115}, t^{116}, \\ t^{117}, t^{119}, t^{120}, t^{121}, t^{123}, t^{127}]].$$

Even if $k[[S]]$ has non-decreasing Hilbert function $[1, 14, 14, 14, 16, 18, 19 \rightarrow]$, the Hilbert function of $k[[T]]$ is $[1, 26, 25, 25, 32, 38 \rightarrow]$; we also note that its multiplicity is 38.

Chapter 4

Quotients and multiples of a numerical semigroup

In this chapter we study quotients and multiples of a numerical semigroup. Using numerical duplication we show that it is possible to study all the doubles of a fixed numerical semigroup and we prove that every numerical semigroup is one half of infinitely many almost symmetric numerical semigroups with odd type included between 1 and $2t(S) + 1$; we also prove that it is one half of an almost symmetric numerical semigroup with even type if and only if is almost symmetric. Moreover, we characterize all almost symmetric doubles of a numerical semigroup. If $d \geq 3$ we generalize the above results by proving that every numerical semigroup is one over d of infinitely many almost symmetric semigroups with type t, where $1 \leq t \leq 2t(S)+2$. Furthermore, we find a formula for the minimal genus of the multiples of a numerical semigroup, solving a problem settled by Robles-Pérez, Rosales, and Vasco in [71]. Finally, we prove a formula for the Frobenius number of the quotient of some families of numerical semigroups. Some results of this chapter are contained in [87] and [88].

4.1 Almost symmetric doubles

The notion of quotient of a numerical semigroup was introduced by in [78] in order to solve proportionally modular diophantine inequalities. Since its appearance several authors studied such quotients; for instance J.C. Rosales and P.A. García-Sánchez proved in [75, 76] that every numerical semigroup is one half of infinitely many symmetric semigroups, while J.C. Rosales proved in [72] that a numerical semigroup is one half of a pseudo-symmetric semigroup if and only if is either symmetric or pseudo-symmetric. In this section we will generalize these results; our main goal is to characterize all almost symmetric doubles of a numerical semigroup. Since we know that every double of a numerical semigroup can be obtained using the numerical duplication with respect to a relative ideal, we first focus on this construction. If we restrict to proper ideals, it is possible to use Theorem 2.3.3 to characterize when numerical duplication is almost symmetric. This result is not completely satisfactory, since there exist almost symmetric numerical semigroups that cannot be obtained as numerical duplication with respect to a proper ideal. In fact if such semigroup has even Frobenius number, we need to use a relative ideal, because numerical duplication with respect to a proper ideal has always odd Frobenius number. Moreover, the following example shows that we need relative ideals even if we want to construct all almost symmetric numerical semigroups with odd Frobenius number.

Example 4.1.1. Consider the numerical semigroup

$$T = \langle 9, 10, 14, 15 \rangle = \{0, 9, 10, 14, 15, 18, 19, 20, 23, 24, 25, 27, 28, 29, 30, 32 \rightarrow\}$$

which is almost symmetric and has odd Frobenius number. Its one half is

$$S = \frac{T}{2} = \{0, 5, 7, 9, 10, 12, 14 \rightarrow\}.$$

and we prove that there are not any $b \in S$ and ideals E such that $T = S \bowtie^b E$. If these exist, we should have $2 \cdot E + b = \{9, 15, 19, 23, 25, 27, 29, 33, 35, 37 \dots\}$ and therefore

$$\begin{array}{ll} E = \{2, 5, 7, 9, 10, 11, 12, 14 \rightarrow\} & \text{if } b = 5 \\ E = \{1, 4, 6, 8, 9, 10, 11, 13 \rightarrow\} & \text{if } b = 7 \\ E = \{0, 3, 5, 7, 8, 9, 10, 12 \rightarrow\} & \text{if } b = 9 \\ E \text{ contains a negative element} & \text{if } b > 9. \end{array}$$

In all cases E is not contained in S and therefore E cannot be a proper ideal of S.

It is convenient to consider the cases with odd and even Frobenius number separately. For simplicity, in this section S will be a numerical semigroup and $M = M(S)$, $K = K(S)$ and $\lambda = f(E) - f(S)$; recall that $\tilde{E} = E - \lambda$. We consider first the odd case.

Lemma 4.1.2. *Let E be a relative ideal of S and assume that $K - (M - M) \subseteq \widetilde{E}$. Then:*

1. $f(E) - x \in M - M$ *for all $x \notin E$;*

2. *If $K - \widetilde{E}$ is a numerical semigroup, then* $f(E) - x \in E - E$ *for all $x \notin E$.*

Proof. 1. Since $x \notin E$, it follows that $x - \lambda \notin \widetilde{E} \supseteq K - (M - M)$. This implies that there exists $y \in M - M$, such that $x - \lambda + y \notin K$, i.e. $f(S) + \lambda - x - y = f(E) - x - y \in S$. Since $y \in M - M$, which is a relative ideal, $f(E) - x = (f(E) - x - y) + y \in M - M$.

2. As above, there exists $y \in M - M$ such that $f(E) - x - y \in S$. If $f(E) - x - y \in M$, then $f(E) - x \in M$ and, since E is an ideal, $f(E) - x + t \in E$, for every $t \in E$, i.e. $f(E) - x \in E - E$.

It remains to consider the case $f(E) - x - y = 0$. We have $f(S) - y = x - \lambda \notin \widetilde{E}$; hence, applying Lemma 1.6.3, it follows that $y \in K - \widetilde{E}$. We need to show that, for every $t \in E$, $y + t \in E$, i.e. $y + t - \lambda \in \widetilde{E}$. Assume by contradiction that $y + t - \lambda \notin \widetilde{E}$ and applying again Lemma 1.6.3, we obtain $f(S) - (y + t - \lambda) \in K - \widetilde{E}$. Since $y \in K - \widetilde{E}$, which is a numerical semigroup, $f(S) - t + \lambda = (f(S) - (y + t - \lambda)) + y \in K - \widetilde{E}$ and then $f(S) \in K$, because $t - \lambda \in \widetilde{E}$; contradiction. $\square$

Theorem 4.1.3. *Let E be a relative ideal of S, $b \in S$ odd and assume that $2f(E)+b > 2f(S)$. Then $T = S \bowtie^b E$ is almost symmetric if and only if the following properties hold:*

1. $K - (M - M) \subseteq \widetilde{E} \subseteq K$*;*

2. $K - \widetilde{E}$ *is a numerical semigroup;*

3. $b + \lambda + E + K \subseteq M$*.*

Proof. Assume that the three conditions of the statement hold and prove that T is almost symmetric, i.e. $M(T) + K(T) \subseteq M(T)$. Using Remark 2.6.9, we get four cases:

(i) $2s \in M(T)$ and $2f(E) + b - a \in K(T)$, where $s \in M$, a is even and $\frac{a}{2} \notin S$;
(ii) $2s \in M(T)$ and $2f(E) + b - a \in K(T)$, where $s \in M$, a is odd and $\frac{a-b}{2} \notin E$;
(iii) $2t + b \in M(T)$ and $2f(E) + b - a \in K(T)$, where $t \in E$, a is even and $\frac{a}{2} \notin S$;
(iv) $2t + b \in M(T)$ and $2f(E) + b - a \in K(T)$, where $t \in E$, a is odd and $\frac{a-b}{2} \notin E$.

(i) Since $2s + 2f(E) + b - a$ is odd, it belongs to $M(T)$ if and only if $s + f(E) - \frac{a}{2} \in E$, i.e. $s + f(S) - \frac{a}{2} \in \widetilde{E}$. Since $\frac{a}{2} \notin S$, i.e. $f - \frac{a}{2} \in K$, we get $s + f(S) - \frac{a}{2} \in M + K \subseteq K - (M - M) \subseteq \widetilde{E}$, see Lemma 1.6.7.
(ii) Since $2s + 2f(E) + b - a$ is even, it belongs to $M(T)$ if and only if $s + f(E) - \frac{a-b}{2} \in M$. Since $\frac{a-b}{2} \notin E$, we can apply Lemma 4.1.2 to obtain $f(E) - \frac{a-b}{2} \in M - M$ that implies the thesis.
(iii) Since $2t + b + 2f(E) + b - a$ is even, it belongs to $M(T)$ if and only if $t + b + f(E) - \frac{a}{2} \in M$, i.e. $t + b + \lambda + f(S) - \frac{a}{2} \in M$. This follows from Condition 3, because $t \in E$ and $f - \frac{a}{2} \in K$.
(iv) Since $2t + b + 2f(E) + b - a$ is odd, it belongs to $M(T)$ if and only if $t + f(E) - \frac{a-b}{2} \in E$. Since $\frac{a-b}{2} \notin E$, the thesis follows immediately from Lemma 4.1.2.

Conversely assume that $K(T) + M(T) \subseteq M(T)$. Using the same argument of the case **(iii)** above, it is easy to see that the last condition is true. Moreover by Lemma 1.6.6 we know that $\widetilde{E} \subseteq K$. Pick now $y \in K - (M - M)$ and assume by contradiction that $y \notin \widetilde{E}$, that is equivalent to $y + \lambda \notin E$. By Remark 2.6.9 we get $2(f(S) - y) = 2f(E) + b - (2(y + \lambda) + b) \in K(T)$, because $2(y + e) + b$ is odd and $y + \lambda \notin E$. It follows that $2(f(S) - y) + 2s \in M(T)$ for every $s \in M$, that is $f(S) - y + s \in M$ for every $s \in M$ and thus $f(S) - y \in M - M$. On the other hand $y \in K - (M - M)$ and therefore $f(S) \in K$, that is a contradiction. Hence $K - (M - M) \subseteq \widetilde{E}$.

It remains to prove that $K - \widetilde{E}$ is a numerical semigroup. Since $\widetilde{E} \subseteq K$, it is clear that $0 \in K - \widetilde{E}$. Moreover from $K - (M - M) \subseteq \widetilde{E} \subseteq K$ and Lemma 1.6.3, we also get $S \subseteq K - \widetilde{E} \subseteq M - M$; in particular, $|\mathbb{N} \setminus (K - \widetilde{E})| < \infty$.

Let y and z be two elements of $K - \widetilde{E}$ and assume by contradiction that $y + z \notin K - \widetilde{E}$. By Lemma 1.6.3 $f(S) - (y + z) \in \widetilde{E}$ or, equivalently, $f(E) - (y + z) \in E$. It follows that $2f(E) - 2(y + z) + b \in M(T)$. Moreover, since $y \in K - \widetilde{E}$, applying again Lemma 1.6.3 we get $f(S) - y \notin \widetilde{E}$, that is $f(E) - y \notin E$; consequently $2f(E) + b - (2(f(E) - y) + b) = 2y \in K(T)$ (see Remark 2.6.9) and analogously $2z \in K(T)$. It follows that $f(T) = 2f(E) + b = 2y + (2z + (2f(E) - 2(y + z) + b)) \in K(T) + (K(T) + M(T)) = M(T)$. This is a contradiction and then $K - \widetilde{E}$ is a numerical semigroup. ☐

Remark 4.1.4. **1.** If the conditions of the previous theorem are satisfied, then $E = \lambda + \widetilde{E} \subseteq \lambda + K$; then the last condition yields that $E + E + b \subseteq S$.

2. If E is a proper ideal, then $2f(E)+b > 2f(S)$. Moreover, if $K-(M-M) \subseteq \widetilde{E}$, the last condition of the theorem is always satisfied. In fact, by definition of K, an element of $b+\lambda+E+K$ is of the form $z = b+\lambda+e+f(S)-x$, with $e \in E$ and $x \notin S \supseteq E$; then Lemma 4.1.2 implies that

$$z = b+f(E)-f(S)+e+f(S)-x = (f(E)-x)+b+e \in (M-M)+E\setminus\{0\} \subseteq (M-M)+M \subseteq M.$$

Thus, according to Theorem 2.3.3, $S \bowtie^b E$ is almost symmetric if and only if $K-(M-M) \subseteq \widetilde{E} \subseteq K$ and $K-\widetilde{E}$ is a numerical semigroup. In particular this property does not depend on b.

3. If E is not proper, the last condition is necessary as the following example shows. Consider the numerical semigroup $S = \{0,4,5,6,8 \rightarrow\}$ and the relative ideal $E = \{2,3,4,6 \rightarrow\}$. It is a straightforward verification that $K-(M-M) = M = \widetilde{E}$, $K = S$, $E+E+5 \subseteq S$, $K-\widetilde{E} = M-M$ and $2f(E)+5 > 2f(S)$; then $T = S \bowtie^5 E$ and $K-\widetilde{E}$ are numerical semigroups and $K-(M-M) \subseteq \widetilde{E} \subseteq K$. However, $T = \{0,8,9,10,11,12,13,16 \rightarrow\}$ is not almost symmetric because $1 \in K(T)$, but $1+M(T) \nsubseteq M(T)$.

The following corollary is an immediate consequence of Theorem 4.1.3.

Corollary 4.1.5. *The almost symmetric doubles of S with odd type are exactly the numerical semigroups $S \bowtie^b E$ such that:*

1. $2f(S) < 2f(E)+b$;

2. $K-(M-M) \subseteq \widetilde{E} \subseteq K$;

3. $K-\widetilde{E}$ *is a numerical semigroup;*

4. $b+\lambda+E+K \subseteq M$.

If E is a proper ideal and $S \bowtie^b E$ is almost symmetric, it is possible to use Proposition 2.3.4 to obtain a simpler formula for the type of $S \bowtie^b E$. Here we provide another proof that works also if E is a relative ideal.

Lemma 4.1.6. *Let E be a relative ideal of S. Assume that $K-(M-M) \subseteq \widetilde{E}$. Then the map $\varphi : y \mapsto f(S)-y$ induces a bijection between $(K-\widetilde{E})\setminus S$ and $(\widetilde{E}-M)\setminus(\widetilde{E}\cup\{f(S)\})$. In particular, $|(K-\widetilde{E})\setminus S| = |(\widetilde{E}-M)\setminus \widetilde{E}|-1$.*

Proof. We first prove that $\varphi : (K-\widetilde{E})\setminus S \longrightarrow (\widetilde{E}-M)\setminus(\widetilde{E}\cup\{f(S)\})$ is well defined. Let $y \in (K-\widetilde{E})\setminus S$. By Lemma 1.6.3, $f(S)-y \notin \widetilde{E}$, whereas $y \notin S$ implies that $y \neq 0$ and, thus, $f(S)-y \neq f(S)$. It remains to show that $f(S)-y \in \widetilde{E}-M$. Assume that there exists $s \in M$ such that $f(S)-y+s \notin \widetilde{E} = K-(K-\widetilde{E})$. Then there exists $z \in K-\widetilde{E}$, such that $f(S)-y+s+z \notin K$, i.e. $y-s-z \in S$. Since $z \in K-\widetilde{E} \subseteq M-M$, we get $s+z \in M$ and, therefore, $y = (y-s-z)+(s+z) \in S+M \subseteq S$ which contradicts the choice of y.

Since φ is clearly injective, we only need to show the surjectivity: for all $z \in (\widetilde{E}-M)\setminus(\widetilde{E}\cup\{f(S)\})$, $f(S)-z \in (K-\widetilde{E})\setminus S$. Assume that $f(S)-z \in S$; since $z \neq f(S)$, we then have $f(S)-z \in M$. Hence, $f(S) = z+(f(S)-z) \in (\widetilde{E}-M)+M \subseteq \widetilde{E}$, that is a contradiction. Finally, since $z \notin \widetilde{E}$, applying Lemma 1.6.3 we obtain $f(S)-z \in K-\widetilde{E}$. □

Proposition 4.1.7. *Let E be a relative ideal, $b \in S$ odd, and assume that $S \bowtie^b E$ is almost symmetric with odd Frobenius number. Then*

$$t(S \bowtie^b E) = 2|(E - M) \setminus E| - 1 = 2|(K - \widetilde{E}) \setminus S| + 1 = 2|K \setminus \widetilde{E}| + 1$$

In particular, $1 \leq t(S \bowtie^b E) \leq 2t(S) + 1$.

Proof. Since $S \bowtie^b E$ is almost symmetric with odd Frobenius number, if x is the number of the even pseudo-Frobenius numbers of $S \bowtie^b E$, Theorem 1.6.4 implies that $t(S \bowtie^b E) = 2x+1$. On the other hand, it follows from Proposition 2.6.10 that $x = |((M-M) \cap (E-E)) \setminus S|$, while Theorem 4.1.3 implies that $K-(M-M) \subseteq \widetilde{E} \subseteq K$. Therefore, we have $E-E = \widetilde{E}-\widetilde{E} \subseteq K-\widetilde{E} \subseteq M-M$ and

$$t(S \bowtie^b E) = 2|(\widetilde{E} - \widetilde{E}) \setminus S| + 1.$$

Moreover, we claim that $K - \widetilde{E} = \widetilde{E} - \widetilde{E}$. Assume by contradiction that there exists $y \in (K-\widetilde{E}) \setminus (\widetilde{E}-\widetilde{E})$, i.e. there exists $t \in \widetilde{E}$ such that $y+t \in K \setminus \widetilde{E}$. Lemma 1.6.3 now implies that $f(S) - y - t \in K - \widetilde{E}$ and, since $y \in K - \widetilde{E}$, which is a numerical semigroup by Theorem 4.1.3, we can conclude that $f(S) - t \in K - \widetilde{E}$. Thus, $f(S) = f(S) - t + t \in K$, that is a contradiction. It follows that

$$t(S \bowtie^b E) = 2|(K - \widetilde{E}) \setminus S| + 1 = |K \setminus \widetilde{E}| + 1.$$

The missing equalities of the statement follow from the previous lemma. As for the last part, it is enough to show that $t(S \bowtie^b E) \leq 2t(S) + 1$. We have that $t(S) = |(M - M) \setminus S|$ and, since $S \subseteq K - \widetilde{E} \subseteq M - M$, we get $|(K - \widetilde{E}) \setminus S| \leq t(S)$ as desired. □

Remark 4.1.8. The previous proposition implies that, if $S \bowtie^b E$ is almost symmetric with odd Frobenius number, its type is independent of b. It follows that, if $S \bowtie^b E$ and $S \bowtie^{b'} E$ are almost symmetric with odd Frobenius numbers, they have the same type; however, if one of them is almost symmetric, it is not true that the other one is and, in this case, it is possible for them to have different type. For example, if S and E are as in Remark 4.1.4.3, the numerical semigroup $S \bowtie^9 E$ is almost symmetric and, according to the previous proposition, has type 3, whereas $S \bowtie^5 E$ is not almost symmetric and has type 2.

Now we are ready to generalize some results of [75] and [76].

Corollary 4.1.9. *Let S be a numerical semigroup and x be an odd integer such that $1 \leq x \leq 2t(S) + 1$. Then, for every odd $b \in S$, there exist infinitely many ideals $E \subseteq S$ such that $S \bowtie^b E$ is almost symmetric and $t(S \bowtie^b E) = x$. In particular, S is one half of infinitely many almost symmetric numerical semigroups T with type x.*

Proof. Assume that $x = 2m + 1$ with $1 \leq m \leq t(S)$, let F be the relative ideal obtained by adding to S the m biggest elements of $(M-M) \setminus S$ and consider $\widetilde{E} = K-F$. It is straightforward to check that F is a numerical semigroup and, if we consider the ideal $E = \widetilde{E} + z$, for some $z \in \mathbb{N}$ such that $E \subseteq S$, we get that $S \bowtie^b E$ is almost symmetric. Then, its type $t(S \bowtie^b E)$ is equal to $2|(K - \widetilde{E}) \setminus S| + 1 = 2|F \setminus S| + 1 = 2m + 1$. The last part of the statement is clear. □

We now consider the case when the Frobenius number of $S \bowtie^b E$ is even. We start by proving an easy lemma.

Lemma 4.1.10. *Let T be a numerical semigroup and let $\mathrm{PF}(T) = \{f_1 < \cdots < f_t\}$. Set $S := \frac{T}{2}$.*
1. *If f_i is even, then $f_i/2 \in \mathrm{PF}(S)$. In particular, the type of S is greater than or equal to the number of even pseudo-Frobenius numbers of T.*
2. *If f_t is even, then $f(S) = f_t/2$.*

Proof. 1. If f_i is even, then $f_i/2 \in \mathbb{N} \setminus S$, since $f_i \notin T$. For all positive elements s of S, $2s \in T$ and $2s + f_i \in T$, since $f_i \in \mathrm{PF}(T)$. Hence, $s + f_i/2 \in S$ and then $f_i/2 \in \mathrm{PF}(S)$.
2. This follows immediately from the fact that f_t and $f(S)$ are the maximal gaps of T and S respectively. $\square$

It is proved in [72] that one half of a pseudo-symmetric numerical semigroup is either symmetric or pseudo-symmetric, i.e. an almost symmetric semigroup with type 1 or 2. In the following we generalize this result to all almost symmetric numerical semigroup with even Frobenius number.

Theorem 4.1.11. *If T is almost symmetric with even Frobenius number, then $S := \frac{T}{2}$ is almost symmetric and its type is exactly the number of even pseudo-Frobenius numbers of T.*

Proof. Let $\mathrm{PF}(S) = \{f_1 < \cdots < f_t\}$. Since $f_i \notin S$, then $2f_i \notin T$ and by Lemmata 1.6.5 and 4.1.10, $2(f_t - f_i) = 2f_t - 2f_i = f(T) - 2f_i \in T \cup \mathrm{PF}(T)$. If $2(f_t - f_i) \in T$, then $s = f_t - f_i \in S$ and, therefore, $f_t = f_i + s$. If $s \neq 0$, then $f_t \in S$, since $f_i \in \mathrm{PF}(S)$; thus, $s = 0$ and $f_t = f_i$. Consequently, if $i \in \{1, \ldots, t-1\}$, it follows that $2f_{t-1} = 2(f_t - f_i) \in \mathrm{PF}(T)$. In this way we get $t-1$ even pseudo-Frobenius numbers. Therefore, since we have to add $2f_t(S)$, there are at least t even pseudo-Frobenius numbers in T and, by the previous lemma, they are exactly t. Finally, from Theorem 1.6.4 easily descends that S is almost symmetric. $\square$

As in the odd case, using numerical duplication we can construct all almost symmetric doubles with even Frobenius number of a numerical semigroup. Let us start with some preparatory facts.

Lemma 4.1.12. *Suppose that S is almost symmetric and that E is a relative ideal such that $E + E + b \subseteq S$, where $b \in S$ is odd. Assume also that $2f(S) \geq 2f(E) + b$. Then, the following conditions are equivalent:*
1. $\mathrm{PF}(S) \subseteq E - E$.
2. $M - M \subseteq E - E$.
3. $K \subseteq E - E$.

Proof. First, claim that $f(S) \in E - E$. Suppose that there exists $e \in E$ such that $f(S) + e \notin E$; then we clearly have $2e + b > 0$, since it is odd and $E + E + b \subseteq S$. Therefore, $2f(S) < 2(f(S) + e) + b \leq 2f(E) + b$, that is a contradiction. Now, by definition of almost symmetric semigroups, we have $M - M = S \cup \mathrm{PF}(S) = S \cup \mathrm{L}(S) \cup \{f(S)\} = K \cup \{f(S)\}$. Moreover, since E is a relative ideal, we have $S \subseteq E - E$. Thus

$$\mathrm{PF}(S) \subseteq E - E \Longleftrightarrow S \cup \mathrm{PF}(S) \subseteq E - E \Longleftrightarrow$$
$$\Longleftrightarrow M - M \subseteq E - E \Longleftrightarrow K \cup \{f(S)\} \subseteq E - E \Longleftrightarrow K \subseteq E - E,$$

as desired. $\square$

Lemma 4.1.13. *Let S be almost symmetric and let E be a relative ideal. If the equivalent conditions of the previous lemma hold, then $M - E = K - E$.*

Proof. Clearly $M - E \subseteq K - E$, because $M \subseteq K$. Suppose by contradiction that equality does not hold, i.e. there exists $x \in (K - E) \setminus (M - E)$. This means that there exists $e \in E$ such that $x + e \in K \setminus M$. Since $x + e \in K$, one has $f(S) - x - e \notin S$ and then $f - x - e \in \mathrm{L}(S) \cup \{f(S)\} = \mathrm{PF}(S) \subseteq E - E$. Hence, $f(S) - x = (f(S) - x - e) + e \in E$ and, since $x \in K - E$, we get $f(S) = (f(S) - x) + x \in K$, that is a contradiction. $\square$

We are now ready to characterize almost symmetric numerical semigroups with even Frobenius number.

Theorem 4.1.14. *Let E be a relative ideal of S and let $b \in S$ be an odd integer for which $E + E + b \subseteq S$ and $2f(S) > 2f(E) + b$. Then, the numerical semigroup $T := S \bowtie^b E$ is almost symmetric if and only if the following properties hold:*

1. *S is almost symmetric;*

2. *$M - E \subseteq (E - M) + b$;*

3. *$K \subseteq E - E$.*

Proof. By definition, T is almost symmetric if and only if $M(T) + K(T) \subseteq K(T)$ and by Lemma 2.6.9 this is equivalent to the following four conditions:

(i) $2m + 2f(S) - a \in M(T)$ for all $m \in M$ and a even such that $\frac{a}{2} \notin S$;
(ii) $2m + 2f(S) - a \in M(T)$ for all $m \in M$ and a odd such that $\frac{a-b}{2} \notin E$;
(iii) $2e + b + 2f(S) - a \in M(T)$ for all $e \in E$ and a even such that $\frac{a}{2} \notin S$;
(iv) $2e + b + 2f(S) - a \in M(T)$ for all $e \in E$ and a odd such that $\frac{a-b}{2} \notin E$.

Analyzing each of the above conditions, we will see that **(i), (ii), (iii)** are equivalent to the properties 1,2,3 respectively, whereas condition **(iv)** always holds, if **(i)** and **(iii)** do. The thesis will follow immediately from these facts.

(i) We have $2m + 2f(S) - a \in M(T)$ if and only if $m + f(S) - \frac{a}{2} \in M$, that is $f(S) - \frac{a}{2} \in M - M = S \cup \mathrm{PF}(S)$, for all $\frac{a}{2} \notin S$. By Lemma 1.6.5, thus is equivalent to say that S is almost symmetric.
(ii) In this case $2m + 2f(S) - a \in M(T)$ if and only if $\frac{2m+2f(S)-a-b}{2} \in E$, that is $m + f(S) - \frac{a-b}{2} - b \in E$. The latter is equivalent to $f(S) - x \in (E - M) + b$, for all $x \notin E$; thus, applying Lemmata 1.6.3 and 4.1.13, we get $M - E \subseteq (E - M) + b$.

(iii) The property $2e+b+2f(S)-a \in M(T)$ is equivalent to $e+f(S)-\frac{a}{2} \in E$, i.e. $f(S)-\frac{a}{2} \in E-E$. Recalling the definition of K, this is in turn equivalent to $K \subseteq E-E$.
(iv) We have $2e+b+2f(S)-a \in M(T)$ if and only if $e+f(S)-\frac{a-b}{2} \in M$, i.e. $f(S)-x \in M-E$ for all $x \notin E$. Lemma 1.6.3 yields that it is equivalent to say that $K-E \subseteq M-E$ and, if we assume **(i)** and **(iii)**, this is always the case by Lemma 4.1.13. □

Also, for an almost symmetric numerical semigroup with even Frobenius number, we provide a good formula for its type.

Corollary 4.1.15. *Under the assumptions of the previous theorem, if $S \bowtie^b E$ is almost symmetric, then*
$$t(S \bowtie^b E) = t(S) + |(M-(b+E)) \setminus E|.$$

Proof. Using Proposition 2.6.10, it is enough to observe that $M-M \subseteq E-E$ by Lemma 4.1.12 and that $M-E \subseteq (E-M)+b$ is equivalent to $M-(b+E) \subseteq E-M$. □

Corollary 4.1.16. *If S is almost symmetric, its almost symmetric doubles with even Frobenius number are exactly the numerical semigroups $S \bowtie^b E$ such that*

1. $M-E \subseteq (E-M)+b$*;*

2. $K \subseteq E-E$*;*

3. $2f(S) > 2f(E)+b$*.*

Moreover, if S is not almost symmetric, then it has not almost symmetric doubles with even Frobenius number.

Next, to complete the picture, we show that when S is almost symmetric, it always has almost symmetric doubles with even Frobenius number.

Lemma 4.1.17. *If $S \neq \mathbb{N}$ is almost symmetric, then there exists at least one relative ideal E and one odd integer $b \in S$ such that $S \bowtie^b E$ is almost symmetric with even Frobenius number.*

Proof. We set $E := \mathbb{N}$ and $b := f(S)+1$ if it is odd, or $b := f(S)+2$ otherwise. First of all, we notice that, if $e, e' \in E$, then $e+e'+b > f(S)$ and $E+E+b \subseteq S$. Moreover, $2f(E)+b = -2+b \leq -2+f(S)+2 < 2f(S)$. By Theorem 4.1.14, we have to prove that $K \subseteq E-E$ and $M-E \subseteq (E-M)+b = (E+b)-M$. It is easy to see that $E-E = \mathbb{N}$ and $M-E = \{f(S)+1 \rightarrow\}$; thus we clearly have $K \subseteq E-E$ and, if $m \in M$ and $x \in M-E$, we also have $m+x \geq f(S)+2$; therefore, $m+x \in \{f(S)+2 \rightarrow\} \subseteq \{b \rightarrow\} = E+b$ yields $M-E \subseteq ((E+b)-M)$, as desired. □

We have already seen that one half of an almost symmetric numerical semigroup with even Frobenius number is almost symmetric and the previous lemma proves the converse. We have then proven the following corollary, which generalizes [73, Theorem 15].

Corollary 4.1.18. *A numerical semigroup different from $\mathbb{N}$ is almost symmetric if and only if it is one half of an almost symmetric numerical semigroup with even Frobenius number.*

Notice that, if $\mathbb{N}$ is one half of a semigroup T, then T contains every even positive integer. Hence, $f(T)$ is odd and it is easy to see that in this case T is symmetric.

Remark 4.1.19. 1. Assume that $E+E+b \subseteq S$. We may assume without loss of generality that the smallest element of E is zero. If we define $E' := E - m(E)$ and $b' := b + 2m(E) \in S$, we have that $m(E') = 0$ and $E' + E' + b' = E - m(E) + E - m(E) + b + 2m(E) = E + E + b \subseteq S$. Moreover, if $e \in E$, then $2e + b = 2(e - m(E)) + b + 2m(E) \in 2E' + b'$ and vice versa; therefore, $S \bowtie^b E = S \bowtie^{b'} E'$.

2. If S is almost symmetric, we would like to find its almost symmetric doubles with even Frobenius number. By the previous observation we can assume that the smallest element of E is zero. According to Corollary 4.1.16, we need $2f(S) > 2f(E) + b$ which implies $b < 2f(S) - 2f(E) \leq 2f(S) + 2$. Hence, we have a finite number of possibilities for b; moreover, if b is fixed, we have $-1 \leq f(E) < f(S) - \frac{b}{2}$ and then there are only finitely many choices also for E. This fact is obvious because there is a finite number of semigroups with fixed Frobenius number, but this remark will be useful in the next example.

Example 4.1.20. Consider the pseudo-symmetric numerical semigroup $S = \{0, 3, 5 \rightarrow\}$. We want to construct all the almost symmetric doubles of S with even Frobenius number.

By the previous remark, we have $b < 10$ and, for a fixed b, we get $-1 \leq f(E) < 4 - \frac{b}{2}$; we are looking only for those ideals which contain 0 and for which $f(E) \neq 0$. We have four possibilities:

$$\begin{array}{lcl} b = 3 & \Longrightarrow & f(E) = -1, 1, 2; \\ b = 5 & \Longrightarrow & f(E) = -1, 1; \\ b = 7 & \Longrightarrow & f(E) = -1; \\ b = 9 & \Longrightarrow & f(E) = -1. \end{array}$$

The unique ideals with Frobenius number -1 and 1 are, respectively, $E_1 = \mathbb{N}$ and $E_2 = \{0, 2 \rightarrow\}$, whereas there are two ideals with Frobenius number 2: $E_3 = \{0, 3 \rightarrow\}$ and $E_4 = \{0, 1, 3 \rightarrow\}$. Now we notice that, if $b = 3$, E_1 and E_4 are not acceptable, because in this case $E + E + b \nsubseteq S$. It is also straightforward to check that $E_i - E_i = E_i$ for $i = 1, 2, 3$ and, thus, $K = \{0, 2, 3, 5 \rightarrow\}$ is contained in $E_i - E_i$ for $i = 1, 2$ but not for $i = 3$. Finally, we have,

$$\begin{array}{ll} M - E_1 = \{5 \rightarrow\}, & E_1 - M = \{-3 \rightarrow\}, \\ M - E_2 = \{3, 5 \rightarrow\}, & E_2 - M = \{-3, -1 \rightarrow\} \end{array}$$

and consequently we get:

$$\begin{array}{ll} b = 3 & M - E_2 \subseteq (E_2 - M) + b, \\ b = 5 & M - E_1 \subseteq (E_1 - M) + b, \\ & M - E_2 \nsubseteq (E_2 - M) + b, \\ b = 7 & M - E_1 \subseteq (E_1 - M) + b, \\ b = 9 & M - E_1 \nsubseteq (E_1 - M) + b. \end{array}$$

We conclude that there are three possibilities which give raise to the numerical semigroups

$$\begin{array}{l} S \bowtie^3 E_2 = \{0, 3, 6, 7, 9 \rightarrow\}, \\ S \bowtie^5 E_1 = \{0, 5, 6, 7, 9 \rightarrow\}, \\ S \bowtie^7 E_1 = \{0, 6, 7, 9 \rightarrow\}. \end{array}$$

The first two are pseudo-symmetric, whereas the last one is almost symmetric of type four. This example also shows that the type depends on b, even when numerical duplication is almost symmetric.

4.2 Almost symmetric multiples

In this section we construct some almost symmetric multiples of a numerical semigroup. In this way we generalize some results of the previous section and of [89]. Let S be a numerical semigroup and let $d \geq 2$ be an integer. Consider $b \in S$ which is not a multiple of d and such that $b > df(S)$; let also E be a relative ideal such that $K(S) - (M(S) - M(S)) \subseteq E \subseteq K(S)$, where the last condition implies that $f(S) = f(E)$. Then, we define the following sets:

$$A := \{d \cdot S\}, \quad B := \{d \cdot E + b\}, \quad C := \{z \in \mathbb{N} \mid z \not\equiv 0, b \mod d, \ z > \lfloor (df(S) + b)/2 \rfloor\}$$

and $S \bowtie^{b,d} E := A \cup B \cup C$. In the rest of the section we use the notation above.

Lemma 4.2.1. *The set $S \bowtie^{b,d} E$ is a d-fold of S and has Frobenius number $df(S) + b$.*

Proof. First of all we note that, by definition, all integers greater than $df(S) + b$ are in $S \bowtie^{b,d} E$. We need to show that $S \bowtie^{b,d} E$ is a numerical semigroup; for this we need to consider six cases:
◦ Let $dx, dx_1 \in A$. In this case $dx + dx_1 = d(x + x_1) \in A$, because S is a numerical semigroup;
◦ Let $dx \in A$ and $dy + b \in B$. Since E is a relative ideal, $dx + dy + b = d(x + y) + b \in B$;
◦ Let $dx \in A$ and $z \in C$. Since $dx + z \equiv z \mod d$ and $dx + z \geq z > \lfloor (df(S) + b)/2 \rfloor$, we have $dx + z \in C$;
◦ Let $dy + b, dy_1 + b \in B$. In this case $dy + b + dy_1 + b \geq b + b > df(S) + b$ and, therefore, the sum is in $S \bowtie^{b,d} E$;
◦ Let $dy + b \in B$ and $z \in C$. Clearly $dy + b + z$ cannot be equal to b modulo d; if it is not a multiple of d, then it belongs to C, since otherwise $dy + b + z > dy + df(S) + z > df(S)$ and, thus, is in A;
◦ Let $z, z_1 \in C$. If $df(S) + b$ is even, then $z + z_1 > (df(S) + b)/2 + (df(S) + b)/2 = df(S) + b$ and, thus, it belongs to $S \bowtie^{b,d} E$. Otherwise $z + z_1 \geq (df(S) + b + 1)/2 + (df(S) + b + 1)/2 = df(S) + b + 1$ and, again, this means that $z + z_1$ is in $S \bowtie^{b,d} E$. □

Lemma 4.2.2. *Let $\mathcal{A} = \{d(f(S) - x) \mid x \in K(S) \setminus E\}$ and $\mathcal{B} = \{dx + b \mid x \in K(S) \setminus E\}$. Then, the gaps of the second type of $S \bowtie^{b,d} E$ are*

$$\mathrm{L}(S \bowtie^{b,d} E) = \begin{cases} \mathcal{A} \cup \mathcal{B} & \text{if } df(S) + b \text{ is odd,} \\ \mathcal{A} \cup \mathcal{B} \cup \frac{df(S)+b}{2} & \text{if } df(S) + b \text{ is even.} \end{cases}$$

Proof. Let y be an integer. If $y \not\equiv 0, b \mod d$, then $df(S) + b - y \equiv b - y \not\equiv 0, b \mod d$. In this case, if y is a gap of the second type, we have

$$df(S) + b = (df(S) + b - y) + y \leq \left\lfloor \frac{df(S) + b}{2} \right\rfloor + \left\lfloor \frac{df(S) + b}{2} \right\rfloor \leq df(S) + b,$$

and this is only possible when $df(S)+b$ is even and $y=(df(S)+b)/2$. Suppose now that $y=dx+b$; it is clear that $dx+b \in \mathrm{L}(S \bowtie^{b,d} E)$ if and only if $df(S)+b-(dx+b)=d(f(S)-x) \in \mathrm{L}(S \bowtie^{b,d} E)$. Therefore, it is enough to consider the elements of this form. Moreover, $dx+b$ and $d(f(S)-x)$ are not in $S \bowtie^{b,d} E$ if and only if $x \notin E$ and $f(S)-x \notin S$, i.e. $x \in K(S) \setminus E$. □

Theorem 4.2.3. *The numerical semigroup $S \bowtie^{b,d} E$ is almost symmetric if and only if $K(S)-E$ is a numerical semigroup. In this case*

$$t(S \bowtie^{b,d} E) = \begin{cases} 2|K(S) \setminus E| + 1 & \text{if } df(S)+b \text{ is odd,} \\ 2|K(S) \setminus E| + 2 & \text{if } df(S)+b \text{ is even.} \end{cases}$$

Proof. We first assume that $S \bowtie^{b,d} E$ is almost symmetric and suppose by contradiction that $K(S)-E$ is not a numerical semigroup. Given $x,y \in K(S)-E$ such that $x+y$ is not in $K(S)-E$, it follows by Lemma 1.6.3 that $f(S)-x, f(S)-y \notin E$ and $f(S)-x-y \in E$. This means that $dx, dy \in \mathrm{L}(S \bowtie^{b,d} E) \subseteq \mathrm{PF}(S \bowtie^{b,d} E)$ and $d(f(S)-x-y)+b \in (S \bowtie^{b,d} E) \setminus \{0\}$; therefore, $df(S)$ is in $S \bowtie^{b,d} E$, that is a contradiction.

We assume now that $K(S)-E$ is a numerical semigroup and we will prove that $\mathrm{L}(S \bowtie^{b,d} E) \subseteq \mathrm{PF}(S \bowtie^{b,d} E)$. Let $\mathcal{A}$ and $\mathcal{B}$ be as in Lemma 4.2.2. We suppose first that $df(S)+b$ is odd and analyze the following six cases:

∘ Let $d(f(S)-x) \in \mathcal{A}$ and $dm \in A \setminus \{0\}$. Since $K(S)-(M(S)-M(S)) \subseteq E$, we have $K(S)-E \subseteq M(S)-M(S)$ and then $f(S)-x \in M(S)-M(S)$. Hence, $f(S)-x+m \in M(S)$ and $d(f(S)-x)+dm$ belongs to $S \bowtie^{b,d} E$.
∘ Let $d(f(S)-x) \in \mathcal{A}$ and $dy+b \in B$. Their sum belongs to $S \bowtie^{b,d} E$ if and only if $f(S)-x+y \in E$; suppose by contradiction that this does not hold. It follows from Lemma 1.6.3 that $x-y, f(S)-x \in K(S)-E$ and this implies that $f(S)-y \in K(S)-E$, because $K(S)-E$ is a numerical semigroup. Therefore, $f(S) \in K(S)$ and we are done.
∘ Let $d(f(S)-x) \in \mathcal{A}$ and $z \in C$. We have $d(f(S)-x)+z \equiv z \mod d$ and then it belongs to $S \bowtie^{b,d} E$, because it is greater than z.
∘ Let $dx+b \in \mathcal{B}$ and $dm \in A \setminus \{0\}$. Suppose by contradiction that $dx+b+dm$ does not belong to $S \bowtie^{b,d} E$. This means that $x+m$ does not belong to E and $f(S)-x-m \in K(S)-E$ by Lemma 1.6.3. As in the first case we have $K(S)-E \subseteq M(S)-M(S)$ and, thus, $f(S)-x-m \in M(S)-M(S)$. This implies that $f(S)-x = f(S)-x-m+m \in M(S)$ and we get a contradiction, because x is in $K(S)$.
∘ Let $dx+b \in \mathcal{B}$ and $dy+b \in B$. As in Lemma 4.2.1, $dx+b+dy+b \geq b+b > df(S)+b$ and then it belongs to $S \bowtie^{b,d} E$.
∘ Finally, if $dx+b \in \mathcal{B}$ and $z \in C$, we can use the same argument of Lemma 4.2.1.

Now suppose that $df(S)+b$ is even. According to Lemma 4.2.2, in this case there is another element in $\mathrm{L}(S \bowtie^{b,d} E)$. Since d is not a divisor of b, it is easy to see that $(df(S)+b)/2 \not\equiv 0, b \mod d$. Therefore, using the same argument as in Lemma 4.2.1, we easily get that this is a pseudo-Frobenius number of $S \bowtie^{b,d} E$. The formula for the type follows from the previous lemma. □

In [89, Theorems 5 and 6], I. Swanson proves that for all $d \geq 3$ there exist infinitely many symmetric and pseudo-symmetric d-folds of S. By means of the previous theorem we generalize these result as follows:

Corollary 4.2.4. *Let $d \geq 3$ be an integer. Every numerical semigroup S is one over d of infinitely many almost symmetric semigroups with type t, where $1 \leq t \leq 2t(S) + 2$.*

Proof. Let $t = 2v + 1$ be an odd integer included between 1 and $2t(S) + 2$. We only need to find an ideal E such that $|K(S) \setminus E| = v$ and $K(S) - E$ is a numerical semigroup, since we may choose infinitely many b such that $df(S) + b$ is either odd or even (note that this is not possible if $d = 2$). We have already found such an ideal in the proof of Corollary 4.1.9, so the conclusion follows. □

We notice that the previous corollary is only a partial result, in the sense that in general there also other possible values of t. Consider for instance the almost symmetric numerical semigroup $T = \{0, 9 \rightarrow\}$ that has type 8 and set $d = 3$. Then $\frac{T}{3} = \{0, 3 \rightarrow\}$ has type 2 and $t(T) = 3t\left(\frac{T}{3}\right) + 2$. Hence we can not find this multiple with the previous construction.

Moreover, if $d = 2^i$, using Corollary 4.1.9, we can construct an almost symmetric double S_1 of S that has type $2t(S) + 1$; in the same way we get an almost symmetric double S_2 of S_1 that has type $2t(S_1) + 1 = 4t(S) + 3$ and we eventually find an almost symmetric d-fold of S that has type $dt(S) + d - 1$.

4.3 Minimal genus of the multiples

In the previous sections our aim was to find and characterize some particular multiples of a numerical semigroup. It is natural to ask for some their properties; for example A.M. Robles-Pérez, J.C. Rosales, and P. Vasco posed the following problem:

Problem. [71, Problem 42] *Let S be a numerical semigroup. Find a formula, that depends on S, for computing* $\min\{g(\overline{S}) \mid \overline{S} \text{ is a double of } S\}$.

We will answer to this question in a more general context, indeed in the next theorem we find a formula for the minimal genus of a d-fold of S. We recall that $d \cdot X = \{dx \mid x \in X\}$.

Theorem 4.3.1. *Let S be a numerical semigroup with Frobenius number f and let $d \geq 2$ be an integer. Then*

$$\min\left\{g(\overline{S}) \mid S = \frac{\overline{S}}{d}\right\} = g(S) + \left\lceil \frac{(d-1)f}{2} \right\rceil.$$

Moreover every d-fold with minimal genus has Frobenius number df.

Proof. Let $\overline{S}$ be a d-fold of S. By definition, $\overline{S}$ has exactly $g(S)$ gaps that are multiples of d. In order to count the other gaps, we first notice that $f \notin S$ implies $df \notin \overline{S}$. Consider now $0 < x \leq df/2$, where x is not a multiple of d. If x is not a gap of $\overline{S}$, then $df - x \notin \overline{S}$, since df is not in $\overline{S}$. All these gaps are different and, since there are $\lfloor \frac{f}{2} \rfloor$ multiples of d smaller than $df/2$, they are $\lfloor \frac{df}{2} \rfloor - \lfloor \frac{f}{2} \rfloor = \lceil \frac{(d-1)f}{2} \rceil$. From this it follows that $g(\overline{S}) \geq= g(S) + \lceil \frac{(d-1)f}{2} \rceil$.

Now, the proof is complete if we can exhibit a d-fold of S with genus $g(S) + \lfloor \frac{df}{2} \rfloor - \lfloor \frac{f}{2} \rfloor$. It easy to see that an example of such a semigroup is

$$T = d \cdot S \cup \{b + i \,|\, i \in \mathbb{N},\ d \nmid b + i\},$$

where $b = \lfloor \frac{df}{2} \rfloor + 1$. Finally, from the proof it is clear that every d-fold with minimal genus has Frobenius number df. $\square$

Since the d-folds with minimal genus have all the same Frobenius number, there are a finite number of such semigroups. In the next example we see that in general a numerical semigroup has more than one d-fold with minimal genus.

Example 4.3.2. Consider the numerical semigroup $S = \{0, 5, 7, 8, 10, 12 \rightarrow\}$ and set $d = 3$. In the proof of the previous theorem we proved that a 3-fold of S with minimal genus is $\{0, 15, 17, 19, 20, 21, 22, 23, 24, 25, 26, 28, 29, 30, 31, 32, 34 \rightarrow\}$ and is genus is $g(S) + \lceil \frac{(3-1)11}{2} \rceil = 7 + 11 = 18$. It is easy to see that $\{0, 8, 14, 15, 16, 20, 21, 22, 23, 24, 26, 28, 29, 30, 31, 32, 34 \rightarrow\}$ is another such 3-fold.

Proposition 4.3.3. *Let S be a numerical semigroup with Frobenius number f and let $\overline{S}$ be a d-fold of S with minimal genus.*

1. *If either f is even or d is odd, $\overline{S}$ has type $t(S)$.*

2. *If f is odd and d is even, $\overline{S}$ has type either $t(S)$ or $t(S) + 1$. Moreover, if S is almost symmetric, then $\overline{S}$ has type $t(S) + 1$.*

Proof. Reasoning as in the proof of Theorem 4.3.1, we easily see that, if $x \notin \overline{S}$ is not a multiple of d, then either $df - x \in \overline{S}$ or $x = df/2$. Therefore, the pseudo-Frobenius numbers of $\overline{S}$ different from $df/2$ are multiples of d. It is easy to see that, if f is even or d is odd, it follows that $t(\overline{S}) \leq t(S)$, whereas $t(\overline{S}) \leq t(S) + 1$ otherwise, since in the first case $df/2$ is not an integer or is a multiple of d.

If x is a pseudo-Frobenius number of S, we claim that $dx \in \mathrm{PF}(\overline{S})$. Clearly if $dy \in \overline{S}$, it follows that $dx + dy = d(x + y) \in \overline{S}$, because $x \in \mathrm{PF}(S)$. Let z be a non-zero element of $\overline{S}$ such that d does not divide x and suppose that $dx + z \notin \overline{S}$. We have already noticed that $df - dx - z = d(f - x) - z \in \overline{S}$, then $d(f - x) = d(f - x) - z + z \in \overline{S}$ and this implies $f - x \in S$; this is a contradiction because $x \in \mathrm{PF}(S)$ and then $x + (f - x) = f \in S$. Hence, $t(\overline{S}) \geq t(S)$ and this proves 1 and the first part of 2.

We only need to prove that if f is odd, d is even and S is almost symmetric, then $df/2$ is a pseudo-Frobenius number of $\overline{S}$. Suppose by contradiction that there exists $x \in \overline{S} \setminus \{0\}$ such that $df/2 + x \notin \overline{S}$. If $df - (df/2 + x) = df/2 - x \in \overline{S}$, then $df/2 \in \overline{S}$ and, thus, $df \in \overline{S}$; this is a contradiction. Whereas, if $df/2 - x \notin \overline{S}$, since it is a multiple of d by the beginning of the proof, we have $\frac{df/2 - x}{d} \in \mathrm{L}(S) \subseteq \mathrm{PF}(S)$ that implies $df/2 - x \in \mathrm{PF}(\overline{S})$ by the first part of the proof and, thus, $df/2 = df/2 - x + x \in \overline{S}$; contradiction. $\square$

If f is odd and d is even, but S is not almost symmetric, the type of $\overline{S}$ can be $t(S)$ as the following example shows. We notice that in this case the semigroup constructed in the proof of Theorem 4.3.1 has always type $t(S) + 1$.

Example 4.3.4. Consider $S = \{0, 5, 6, 7, 10 \rightarrow\}$, which is not almost symmetric. The numerical semigroups

$$T = \{0, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 33, 34, 35, 37 \rightarrow\},$$

$$T' = \{0, 14, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 33, 34, 35, 37 \rightarrow\}$$

are 4-folds of S with minimal genus. It is easy to see that 18 is a pseudo-Frobenius number of T but not of T' and this implies that $t(T) = 3 = t(S) + 1$, whereas $t(T') = 2 = t(S)$.

Corollary 4.3.5. *Let S be an almost symmetric numerical semigroup with Frobenius number f and let $\overline{S}$ be a d-fold of S. Set $t := t(S)$, if either f is even or d is odd, and $t := t(S) + 1$ otherwise. Then $\overline{S}$ has minimal genus if and only if is almost symmetric and has type t.*

Proof. In the light of the previous proposition we may assume that $\overline{S}$ has type t. Suppose first that f is even or d is odd. Since S is almost symmetric, we have $t(S) = 2g(S) - f$ by definition and, thus,

$$g(\overline{S}) \geq \frac{df + t}{2} = \frac{df + 2g(S) - f}{2} = g(S) + \frac{(d-1)f}{2} = g(S) + \left\lceil \frac{(d-1)f}{2} \right\rceil.$$

Moreover, $\overline{S}$ has minimal genus if and only if the above equality holds and this happens exactly when $\overline{S}$ is almost symmetric. A similar argument proves the remaining case. $\square$

Since a d-fold of S with minimal genus has Frobenius number $df(S)$, we get the next corollary.

Corollary 4.3.6. *Let $S, \overline{S}$, and t be as in the previous corollary. If $\overline{S}$ is almost symmetric and has type t, then d divides $f(\overline{S})$ and $f(S) = f(\overline{S})/d$.*

Now let $\mathcal{D}(S)$ denote the set of the symmetric doubles of S. We proved in Corollary 2.6.8 that all the elements in this family are of the form $S \bowtie^b K(S)$ for some odd $b \in S$ such that $b + K(S) + K(S) \subseteq S$. Since $S \bowtie^b K$ is symmetric and has Frobenius number $2f(S) + b$, its genus is

$$g(S \bowtie^b K(S)) = \frac{f(S \bowtie^b K(S)) + 1}{2} = \frac{2f(S) + b + 1}{2} = f(S) + \frac{b+1}{2}$$

and, therefore, the genus is minimal when b is minimal. Hence, we have the following:

Proposition 4.3.7. *There exists an unique numerical semigroup with minimal genus among the members of $\mathcal{D}(S)$ and this is $S \bowtie^b K(S)$, where b is the smallest odd element of S such that $K(S) + K(S) + b \subseteq S$; its genus is $f(S) + \frac{b+1}{2}$.*

Observe that if $b > f(S)$, then $K(S) + K(S) + b \subseteq S$. This means that, in order to find the b of the previous proposition, we have to check only a finite number of elements.

Example 4.3.8. Consider $S = \langle 6, 7, 11\rangle = \{0, 6, 7, 11, 12, 13, 14, 17 \rightarrow\}$; in this case $K(S) = \{0, 1, 6, 7, 8, 11, 12, 13, 14, 15, 17 \rightarrow\}$. Since $0 + 1 + 7 \notin S$, it follows that $K(S) + K(S) + 7 \nsubseteq S$. On the other hand, $K(S) + K(S) + 11 \subseteq S$ and, thus, the symmetric double of S with minimal genus is

$$S \bowtie^{11} K(S) = \{0, 11, 12, 13, 14, 22, 23, 24, 25, 26, 27, 28, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 44 \rightarrow\},$$

that has genus $f(S) + \frac{b+1}{2} = 16 + 6 = 22$. We notice that the minimal genus of a double of S is $g(S) + \lceil \frac{f(S)}{2} \rceil = 10 + 8 = 18$ and, according to the proof of Theorem 4.3.1, it is obtained by the numerical semigroup $\{0, 12, 14, 17, 19, 21, 22, 23, 24, 25, 26, 27, 28, 29, 31, 33 \rightarrow\}$.

There is a particular case in which we can avoid to look for the "right" b.

Corollary 4.3.9. *Let S be an almost symmetric numerical semigroup. The element with minimal genus in the family $\mathcal{D}(S)$ is $S \bowtie^b K(S)$, where b is the smallest odd element of S. Its genus is $f(S) + \frac{b+1}{2}$.*

Proof. By definition $K(S) = S \cup \mathrm{L}(S)$ and this implies that $K(S) = (S \cup \mathrm{PF}(S)) \setminus \{f(S)\}$, since S is almost symmetric. Hence, since $b \in S \setminus \{0\}$, it follows that $K(S) + K(S) + b \subseteq S$ and the proof is complete by Theorem 4.3.7. □

4.4 Frobenius number of some quotients

In this section we study the Frobenius number of the quotients of some classes of numerical semigroups. This is a widely open question, see e.g. [29] and [77, Open Problem 6.20]; on the other hand, we recall that we have already found a formula in Corollary 4.3.6. We start with a definition due to I. Swanson, see [89].

Definition 4.4.1. Let d be a positive number. A numerical semigroup is said to be *d-symmetric* if for all integers $n \in \mathbb{Z}$, whenever d divides n, either n or $f(S) - n$ is in S.

A symmetric numerical semigroup is d-symmetric for all $d \in \mathbb{N}$ and a 1-symmetric semigroup is symmetric.

Theorem 4.4.2. *Let $d \geq 2$ be an integer and S be a d-symmetric numerical semigroup. If x is the smallest element of S such that $x \equiv f(S) \mod d$, then*

$$f\left(\frac{S}{d}\right) = \frac{f(S) - x}{d}.$$

Proof. Since $x \in S$, it follows that $f(S) - x$ is a gap of S and then $\frac{f(S)-x}{d} \notin \frac{S}{d}$. Suppose by contradiction that there exists a gap y of $\frac{S}{d}$ greater than $\frac{f(S)-x}{d}$. Therefore $dy \notin S$ and thus $f(S) - dy \in S$, since S is d-symmetric. Consequently we have $x \leq f(S) - dy$ by definition of x and this is a contradiction because $y > \frac{f(S)-x}{d}$, i.e. $x > f(S) - dy$. □

Example 4.4.3. Let $S = \{0, 11, 14, 16, 20, 21, 22, 25, 27 \rightarrow\}$. S is 5-symmetric but not 2, 3 or 4-symmetric. By the previous theorem, we get

$$f\left(\frac{S}{5}\right) = \frac{26 - 11}{5} = 3,$$

and in fact $\frac{S}{5} = \{0, 4 \rightarrow\}$. On the other hand, if $d = 3, 4$ we get $f(\frac{S}{d}) = 5, 3$ that is wrong. Finally we notice that, even if S is not 2-symmetric, the formula for $d = 2$ is correct, because 2 divides $f(S)$.

The notion of d-symmetric numerical semigroups is not much studied, but Theorem 4.4.2 yields interesting consequences for other classes of semigroups.

Corollary 4.4.4. *Let S be either a symmetric or a pseudo-symmetric numerical semigroup. If x is the smallest element of S such that $x \equiv f(S) \mod d$, then*

$$f\left(\frac{S}{d}\right) = \frac{f(S) - x}{d}$$

for all integers $d \geq 2$.

Proof. A symmetric numerical semigroup is d-symmetric, while a pseudo-symmetric numerical semigroup is d-symmetric if and only if $2d$ does not divide $f(S)$. On the other hand, if $f(S)$ is a multiple of d, we have $x = 0$ and $f(S)/d$ is the maximum gap of $\frac{S}{d}$. □

Symmetric and pseudo-symmetric numerical semigroups are the almost symmetric numerical semigroups with type one and two respectively, but the next example shows that the previous corollary does not hold in the almost symmetric case.

Example 4.4.5. Let $d \geq 2$ be an integer and consider the numerical semigroup $S = \{0, d+2 \rightarrow\}$. We have $\frac{S}{d} = \{0, 2 \rightarrow\}$, but the formula of the previous corollary would predict

$$f\left(\frac{S}{d}\right) = \frac{d + 1 - (2d + 1)}{d} = -1.$$

Unfortunately, in general, it is not easy to find the element x required in Theorem 4.4.2. However, since in a symmetric numerical semigroup the Frobenius number is odd, we get the following interesting corollary.

Corollary 4.4.6. *Let S be a symmetric numerical semigroup. Then*

$$f\left(\frac{S}{2}\right) = \frac{f(S) - x}{2},$$

where x is the smallest odd generator of S.

The next corollary was proved in a different way in [72, Proposition 7].

Corollary 4.4.7. *Let $a < b$ be two positive integers with* $\gcd(a,b) = 1$. *Then*

$$f\left(\frac{\langle a,b\rangle}{2}\right) = \begin{cases} \frac{ab-b}{2} - a & \text{if } a \text{ is odd,} \\ \frac{ab-a}{2} - b & \text{if } a \text{ is even.} \end{cases}$$

Proof. It is well-known that $\langle a,b\rangle$ is symmetric and that $f(\langle a,b\rangle) = ab - a - b$, see [77, Prop. 2.13 and Cor. 4.17]. If a is odd, we have $x = a$ and then

$$f\left(\frac{\langle a,b\rangle}{2}\right) = \frac{f(\langle a,b\rangle) - a}{2} = \frac{ab - a - b - a}{2} = \frac{ab-b}{2} - a.$$

If a is even, then b is odd because $\gcd(a,b) = 1$ and consequently $x = b$. The second formula can be found with a similar argument. □

In [77] and [29] the authors ask for a formula for $f\left(\frac{\langle a,b\rangle}{d}\right)$, at least when $b = a+1$. To this aim, Corollary 4.4.4 can be of some use, even though it might be difficult to find x. For instance, in the next corollary we give a formula for $d = 5$.

Corollary 4.4.8. *Let a be a positive integer. Then*

$$f\left(\frac{\langle a,a+1\rangle}{5}\right) = \begin{cases} \frac{a^2}{5} - a - 1 & \text{if } a \equiv 0 \mod 5, \\ \frac{a^2-3a-3}{5} & \text{if } a \equiv 1,2 \mod 5, \\ \frac{a^2-a-1}{5} & \text{if } a \equiv 3 \mod 5, \\ \frac{a^2-1}{5} - a & \text{if } a \equiv 4 \mod 5. \end{cases}$$

Proof. By [77, Proposition 2.13], the Frobenius number of $\langle a, a+1\rangle$ is $a^2 - a - 1$. It is enough to find the smallest $x \in S$ such that $x \equiv a^2 - a - 1 \mod 5$ and it is easy to see that

$$\begin{array}{ll} x = 4(a+1) & \text{if } a \equiv 0 \mod 5, \\ x = 2(a+1) & \text{if } a \equiv 1,2 \mod 5, \\ x = 0 & \text{if } a \equiv 3 \mod 5, \\ x = 4a & \text{if } a \equiv 4 \mod 5. \end{array}$$

□

Corollary 4.4.6 gives a formula for the Frobenius number of one half of S, provided that we know a formula for $f(S)$. In the next corollary we collect some cases using the formulas found in [44, Corollary 3.11], [34, Theorem 4], and [15, Théorème 2.3] respectively; for the first and the third case see also Remark 10.7 and Proposition 9.15 of [77].

Let $T = \langle n_1, \dots, n_\nu\rangle$ be a numerical semigroup. We set

$$x = \min\{n_i \mid n_i \text{ is odd}\},$$

$$c_i = \min\{k \in \mathbb{N} \setminus \{0\} \mid kn_i \in \langle n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_\nu\rangle\},$$

$$c_i n_i = \sum_{j \neq i} r_{ij} n_j.$$

Corollary 4.4.9. 1. *Let S be a numerical semigroup with three minimal generators. It is symmetric if and only if $S = \langle am_1, am_2, bm_1 + cm_2 \rangle$, where a, b, c, m_1, m_2 are natural numbers such that $a \geq 2, b + c \geq 2, \gcd(m_1, m_2) = 1$, and $\gcd(a, bm_1 + cm_2) = 1$. In this case*

$$f\left(\frac{S}{2}\right) = \frac{a(m_1 m_2 - m_1 - m_2) + (a-1)(bm_1 + cm_2) - x}{2};$$

2. *Let $S = \langle n_1, \ldots, n_4 \rangle$ be a symmetric numerical semigroup that is not complete intersection, then*

$$f\left(\frac{S}{2}\right) = \frac{n_2 c_2 + n_3 c_3 + n_4 r_{14} - (n_1 + n_2 + n_3 + n_4 + x)}{2};$$

3. *If S is a free numerical semigroup for the arrangement of its minimal generators $\{n_1, \ldots, n_\nu\}$, then*

$$f\left(\frac{S}{2}\right) = \frac{(c_2 - 1)n_2 + \cdots + (c_\nu - 1)n_\nu - n_1 - x}{2}.$$

Chapter 5

Rigidity properties of local cohomology modules

Given a homogeneous ideal I of $R = k[x_1, \dots, x_n]$, in this chapter we compare the Hilbert function of $H^i_{\mathfrak{m}}(R/I)$ with that of $H^i_{\mathfrak{m}}(R/I^{\,\mathrm{lex}})$, where $\mathfrak{m} = (x_1, \dots, x_n)$. Our main result says that if there exists i such that the Hilbert functions of $H^i_{\mathfrak{m}}(R/I)$ and $H^i_{\mathfrak{m}}(R/I^{\,\mathrm{lex}})$ are equal, then the same equality holds for all $h \geq i$. Moreover we give some characterizations of these ideals, especially when $i = 0$. We also introduce the notion of i-partially sequentially Cohen-Macaulay module that allows us to characterize the ideals for which $H^h_{\mathfrak{m}}(R/I)$ and $H^h_{\mathfrak{m}}(R/\operatorname{Gin}(I))$ have the same Hilbert function for all $h \geq i$, generalizing a result of J. Herzog and E. Sbarra [50]. This chapter is based on the article [84].

5.1 Inequalities and known results

In this chapter I will be a homogeneous ideal of $R = k[x_1, \dots, x_n]$, where k is a field, that we can assume infinite without loss of generality. In the recent literature there are several results concerning the Betti numbers of I, in connection with those of its generic initial ideals and its lex-ideal. For instance, the classical Bigatti-Hulett-Pardue theorem, proved in [16], [49] and [65], assure that the maximal Betti numbers of a given Hilbert function are achieved by its lex-ideal or, equivalently, that the Betti numbers of I are always less than or equal to those of $I^{\,\mathrm{lex}}$. Furthermore, it is well-known that, for all i and j, the following inequalities hold:

$$\beta_{ij}(R/I) \leq \beta_{ij}(R/\operatorname{Gin}(I)) \leq \beta_{ij}(R/I^{\,\mathrm{lex}}).$$

It is natural to ask when equalities hold above and in fact in [3] A. Aramova, J. Herzog and T. Hibi prove that in characteristic zero the first inequality holds for all i and j if and only if I is componentwise linear, while in [45] J. Herzog and T. Hibi prove that the both hold for all i and j if and only if I is a Gotzmann ideal, i.e. $\beta_{0j}(I) = \beta_{0j}(I^{\,\mathrm{lex}})$ for all j. Their result also provides an example of what we mean by rigidity property: if holds an equality for $i = 0$, then this forces all other equalities for $i \geq 0$.

Since the Betti numbers are defined as $\beta_{ij}(R/I) = \dim_k \operatorname{Tor}_i^R(R/I,k)_j$, it is also natural to ask whether similar results hold for $\dim_k \operatorname{Ext}_R^i(R/I,R)_j$ or, equivalently via local duality, for $h^i(R/I)_j := \dim_k H_{\mathfrak{m}}^i(R/I)_j$, i.e. the Hilbert function of the local cohomology modules with support on $\mathfrak{m} = (x_1,\dots,x_n)$. As happens in the case of the Betti numbers, in [83] E. Sbarra proves that for all i and j

$$h^i(R/I)_j \leq h^i(R/\operatorname{Gin}(I))_j \leq h^i(R/I^{\,\mathrm{lex}})_j. \tag{5.1}$$

The next two theorems show when above equalities hold for all i and j.

Theorem 5.1.1. [47, Theorem 3.1] *The equality $h^i(R/I)_j = h^i(R/\operatorname{Gin}(I))_j$ holds for all i and j if and only if I is a sCM ideal.*

Theorem 5.1.2. [82, Theorem 0.1] *The following conditions are equivalent:*
1. $(I^{\,\mathrm{sat}})^{\,\mathrm{lex}} = (I^{\,\mathrm{lex}})^{\,\mathrm{sat}}$*;*
2. $h^0(R/I)_j = h^0(R/I^{\,\mathrm{lex}})_j$*, for all j;*
3. $h^i(R/I)_j = h^i(R/I^{\,\mathrm{lex}})_j$*, for all i,j.*

These two theorems were proved when $\operatorname{char} k = 0$, where the additional hypothesis on the characteristic is assumed in order to use some properties of $\operatorname{Gin}(I)$ that were not known in positive characteristic. We observe that, since $\operatorname{Gin}(I)$ is weakly stable in any characteristic, it is easy to see that the original proofs work in general. Thus, throughout this chapter no assumption on the characteristic is required.

5.2 Maximality and rigidity results

We study by stating another condition that we prove to be equivalent to those of Theorem 5.1.2. Despite its simplicity, it has important consequences that will be useful in the rest of the chapter.

Proposition 5.2.1. *Conditions* 1-3 *of Theorem 5.1.2 hold if and only if the following condition holds:*
4. $\operatorname{Gin}(I)^{\,\mathrm{sat}} = (I^{\,\mathrm{lex}})^{\,\mathrm{sat}}$.

Proof. $1 \Rightarrow 4$ Since $(I^{\,\mathrm{sat}})^{\,\mathrm{lex}}$ is a saturated lex-ideal, it has positive depth and $I^{\,\mathrm{sat}}$ is a critical ideal by Theorem 1.1.1. Therefore, Proposition 1.2.1 implies that $\operatorname{Gin}(I)^{\,\mathrm{sat}} = \operatorname{Gin}(I^{\,\mathrm{sat}}) = (I^{\,\mathrm{sat}})^{\,\mathrm{lex}} = (I^{\,\mathrm{lex}})^{\,\mathrm{sat}}$.
$4 \Rightarrow 1$ Since the saturation of a lex-ideal is still a lex-ideal, we get that

$$(I^{\,\mathrm{lex}})^{\,\mathrm{sat}} = ((I^{\,\mathrm{lex}})^{\,\mathrm{sat}})^{\,\mathrm{lex}} = (\operatorname{Gin}(I)^{\,\mathrm{sat}})^{\,\mathrm{lex}} = (\operatorname{Gin}(I^{\,\mathrm{sat}}))^{\,\mathrm{lex}}.$$

It is now enough to recall that $I^{\,\mathrm{sat}}$ and $\operatorname{Gin}(I^{\,\mathrm{sat}})$ have the same Hilbert function. ☐

Remark 5.2.2. 1. An ideal I satisfying the equivalent conditions 1-4 is a sCM ideal, see [82, Proposition 1.9].

2. From (5.1) follows that if $h^i(R/I)_j = h^i(R/I^{\,\mathrm{lex}})_j$ for all i and j, then $h^i(R/\operatorname{Gin}(I))_j = h^i(R/I^{\,\mathrm{lex}})_j$. Actually also the converse holds. In fact if $h^i(R/\operatorname{Gin}(I))_j = h^i(R/I^{\,\mathrm{lex}})_j$ for all i and j, the previous proposition and Proposition 1.2.1 imply that

$$\operatorname{Gin}(I)^{\,\mathrm{sat}} = \operatorname{Gin}(\operatorname{Gin}(I))^{\,\mathrm{sat}} = (\operatorname{Gin}(I)^{\,\mathrm{lex}})^{\,\mathrm{sat}} = (I^{\,\mathrm{lex}})^{\,\mathrm{sat}}$$

and thus it is enough to use Condition 4 again.

3. If I is critical, then $\operatorname{Gin}(I) = I^{\,\mathrm{lex}}$ and hence $h^i(R/I)_j = h^i(R/I^{\,\mathrm{lex}})_j$, for all i, j. In particular, I is a sCM ideal by 1.

4. In [56, Theorem 1.6] it is proved that a critical ideal has the same depth of its lex-ideal. Then, the first point can be seen as a generalization of this fact, because the depth of R/I is the least integer i such that $h^i(R/I)_j \neq 0$ for some j.

We will give another equivalent condition to those of Theorem 5.1.2 and Proposition 5.2.1 using the Björner-Wachs polynomial, an invariant introduced by A. Goodarzi in [38] in order to characterize sCM ideals. We first introduce some notations and recall a few results, see [38] for more details.
Let $I = \bigcap_1^s \mathfrak{q}_l$ be a reduced primary decomposition of I and let $\mathfrak{p}_l = \sqrt{\mathfrak{q}_l}$ the radical of $\mathfrak{q}_l$ for all $1 \leq l \leq s$. Furthermore we denote by $I^{\langle i \rangle}$ the ideal

$$I^{\langle i \rangle} := \bigcap_{\dim R/\mathfrak{p}_l > i} \mathfrak{q}_l;$$

it follows that $I^{\langle -1 \rangle} = I$ and $I^{\langle 0 \rangle} = \bigcap_{\mathfrak{p}_i \neq \mathfrak{m}} \mathfrak{q}_i = I^{\,\mathrm{sat}}$. For $i = 0, \ldots, d = \dim R/I$, we also denote by $U_i(R/I)$ the R-module $I^{\langle i \rangle}/I^{\langle i-1 \rangle}$. These modules are said to be the *unmixed layers* of R/I; if they are non-zero, they have dimension i. We notice that, $0 \subseteq I^{\langle 0 \rangle}/I \subseteq I^{\langle 1 \rangle}/I \subseteq \cdots \subseteq I^{\langle d-1 \rangle}/I \subseteq R/I$ is the dimension filtration of R/I and then the modules $U_i(R/I)$ are the quotients that appear in the definition of sCM-module.

The *Björner-Wachs polynomial* of R/I, briefly *BW-polynomial*, is defined to be

$$\operatorname{BW}(R/I; t; w) := \sum_{k=0}^{\dim R/I} h(U_k(R/I); t) w^k,$$

where we recall that $h(U_k(R/I); t)$ is the h-polynomial of $U_k(R/I)$. One of the main results of [38] is that $\operatorname{BW}(R/I; t; w) = \operatorname{BW}(R/\operatorname{Gin}(I); t; w)$ if and only if R/I is sCM.

Proposition 5.2.3. *The equivalent conditions of Theorem 5.1.2 and Proposition 5.2.1 hold if and only if* $\operatorname{BW}(R/I; t; w) = \operatorname{BW}(R/I^{\,\mathrm{lex}}; t; w)$.

Proof. Suppose first that I and $I^{\,\mathrm{lex}}$ have the same BW-polynomial. For all $i \geq 0$, $U_i(R/I)$ and $U_i(R/I^{\,\mathrm{lex}})$ have the same Hilbert function and the same happens for $I^{\langle i \rangle}$ and $(I^{\,\mathrm{lex}})^{\langle i \rangle}$. Therefore, for all j we have

$$h^0(R/I)_j = H_{I^{\langle 0 \rangle}}(j) - H_I(j) = H_{(I^{\,\mathrm{lex}})^{\langle 0 \rangle}}(j) - H_{I^{\,\mathrm{lex}}}(j) = h^0(R/I^{\,\mathrm{lex}})_j.$$

Conversely let $I^{\text{lex}} = \cap_{j=1}^{s} \mathfrak{q}'_j$ be a reduced primary decomposition of I^{lex}. By assumption

$$\operatorname{Gin}(I)^{\langle 0\rangle} = \operatorname{Gin}(I)^{\text{sat}} = (I^{\text{lex}})^{\text{sat}} = (I^{\text{lex}})^{\langle 0\rangle} = \bigcap_{\sqrt{q'_i}\neq \mathfrak{m}} \mathfrak{q}'_i.$$

It follows that $\operatorname{Gin}(I)^{\langle i\rangle} = (I^{\text{lex}})^{\langle i\rangle}$ for all $i \geq 0$, $U_i(\operatorname{Gin}(I)) = U_i(I^{\text{lex}})$ for $i = 1, \ldots, d-1$ and, therefore, $\operatorname{Hilb}(U_i(\operatorname{Gin}(I))) = \operatorname{Hilb}(U_i(I^{\text{lex}}))$ for $i \geq 0$. Consequently $\operatorname{Gin}(I)$ and I^{lex} have the same BW-polynomial. Moreover, since I is a sCM ideal by Remark 5.2.2.1, I and $\operatorname{Gin}(I)$ have the same BW-polynomial by [38, Theorem 17]. □

The following corollary follows from the proof of the previous proposition.

Corollary 5.2.4. *If I and I^{lex} have the same Björner-Wachs polynomial* BW*, then*

1. $\operatorname{Gin}(I)^{\langle i\rangle} = (I^{\text{lex}})^{\langle i\rangle}$ *for all* $i = 0, \ldots, d-1$.

2. $\operatorname{BW}(R/\operatorname{Gin}(I); t; w) = \operatorname{BW}$*, i.e. I is a sCM ideal.*

We also notice that the only if part of Proposition 5.2.3 can be obtained also using [38, Theorem 20].

5.3 Partially sequentially Cohen-Macaulay modules

We recall another rigidity property of Betti numbers proved by A. Conca, J. Herzog and T. Hibi in [19, Corollary 2.7].

Theorem 5.3.1. *Assume* $\operatorname{char} k = 0$. *Let J be either the lex-ideal of I or a generic initial ideal of I and suppose that $\beta_i(R/I) = \beta_i(R/J)$ for some i. Then $\beta_k(R/I) = \beta_k(R/J)$ for all $k \geq i$.*

It follows from Bigatti-Hulett-Pardue theorem that this statement is equivalent to say that, if $\beta_{ij}(R/I) = \beta_{ij}(R/J)$ for some i and all j, then $\beta_{kj}(R/I) = \beta_{kj}(R/J)$ for all $k \geq i$ and all j.

It is natural to ask for a similar result for the Hilbert function of local cohomology modules. It is easy to see that in general the answer is negative for generic initial ideals. For instance if one considers a non-sCM ideal with positive depth t, then $h^i(R/I)_j = h^i(R/\operatorname{Gin}(I))_j = 0$ for all $0 \leq i < t$ and all j, but Theorem 5.1.1 guarantees that there exists at least one index i for which $h^i(R/I)_j \neq h^i(R/\operatorname{Gin}(I))_j$ for some j. Furthermore $\operatorname{Gin}(I^{\text{sat}}) = \operatorname{Gin}(I)^{\text{sat}}$ yields immediately that, for any ideal I, $h^0(R/I)_j = h^0(R/\operatorname{Gin}(I))_j$ for all j.

In this section we introduce the notion of *partially sequentially Cohen-Macaulay* module which naturally characterizes those ideals for which $h^k(R/I)_j = h^k(R/\operatorname{Gin}(I))_j$ for all k larger than some fixed i. In the last section we consider and positively solve the similar problem for lex-ideals. We start by introducing the new definition.

Definition 5.3.2. *Let i be a non-negative integer. A finitely generated R-module M with dimension filtration $\{\mathcal{M}_k\}_{k\geq -1}$ is called i-partially sequentially Cohen-Macaulay, i-sCM for short, if $U_k(M) := \mathcal{M}_k/\mathcal{M}_{k-1}$ is either zero or a k-dimensional Cohen-Macaulay module for all $i \leq k \leq \dim M$.*

Clearly, sequentially Cohen-Macaulay modules are exactly the 0-sCM modules. It is not difficult to see that several known results about sCM modules can be generalized to context of the above definition. In the following lemma we collect some properties of this kind; the proofs are easy generalizations of the original ones that can be found in [38] and [85], but we include them for the sake of the completeness. For the rest of the chapter we let $h^i(M) = \operatorname{Hilb}(H^i_{\mathfrak{m}}(M))$.

Lemma 5.3.3. *Let M be a finitely generated R-module with dimension filtration $\{\mathcal{M}_k\}_{k\geq -1}$.*

1. *If $x \in R$ is an M-regular element, then M is i-sCM if and only if M/xM is $(i-1)$-sCM.*

2. *R/I is i-sCM if and only if R/I^{sat} is i-sCM.*

3. *If M is i-sCM, then $H^k_{\mathfrak{m}}(M) \cong H^k_{\mathfrak{m}}(\mathcal{M}_k) \cong H^k_{\mathfrak{m}}(U_k(M))$ for all $k \geq i$.*

4. *If R/I is i-sCM, then $h^k(R/I) = \operatorname{Hilb}(U_k(R/I))$ for all $k \geq i$.*

5. *R/I is i-sCM if and only if $\operatorname{Hilb}(U_k(R/I)) = \operatorname{Hilb}(U_k(R/\operatorname{Gin}(I)))$ for all $k \geq i$.*

Proof. 1. First of all we note that $\mathcal{M}_0 = 0$, because R contains a regular element, and that the dimension filtration of M/xM is $\{\mathcal{M'}_k := (\mathcal{M}_{k+1} + xM)/xM\}_{k=0,\dots,d-1}$. Moreover, since x is M-regular, we claim that it is also $M/\mathcal{M}_k$-regular for all k. Indeed if $xm \in \mathcal{M}_k$ for some $m \in M$, then for all $y \in \operatorname{Ann}(\mathcal{M}_k)$ we get $xym = 0$ and, since x is M-regular, it follows $ym = 0$ that in turn implies $y \in \operatorname{Ann}(\mathcal{M}_k + \langle m\rangle)$. Consequently $\operatorname{Ann}(\mathcal{M}_k) = \operatorname{Ann}(\mathcal{M}_k + \langle m\rangle)$, then $\dim \mathcal{M}_k = \dim(\mathcal{M}_k + \langle m\rangle)$ and from the definition of $\mathcal{M}_k$ follows that $m \in \mathcal{M}_k$. Clearly this implies that x is also $U_k(M)$-regular for all k. Furthermore, it follows that $\mathcal{M}_k \cap xM = x\mathcal{M}_k$, in fact if $m \in M$ and $xm \in \mathcal{M}_k$, we have proved above that $m \in \mathcal{M}_k$. Now consider the surjective homomorphism $U_k(M) \to \mathcal{M'}_{k-1}/\mathcal{M'}_{k-2}$, defined as $m + \mathcal{M}_{k-1} \mapsto (m + xM) + \mathcal{M'}_{k-2}$. Then $m + \mathcal{M}_{k-1}$ is in the kernel if and only if $m \in \mathcal{M}_k \cap xM = x\mathcal{M}_k$ and so there is an isomorphism $U_k(M)/xU_{k-1}(M) \cong \mathcal{M'}_{k-1}/\mathcal{M'}_{k-2}$. Since x is a regular element, M is a i-sCM if and only if $U_k(M)/xU_k(M)$ is either zero or Cohen-Macaulay for all $k \geq i$; by the isomorphism above, this is equivalent to say that $\mathcal{M'}_k/\mathcal{M'}_{k-1}$ is either zero or Cohen-Macaulay for all $k \geq i-1$, i.e. M/xM is $(i-1)$-sCM.

2. This is trivial because $H^0_{\mathfrak{m}}(R/I) = I^{\text{sat}}/I$ and, since it has dimension zero, if we quotient by $H^0_{\mathfrak{m}}(R/I)$ the dimension filtration does not change, except for $k = 0$.

3. The short exact sequence $0 \to \mathcal{M}_{k-1} \to \mathcal{M}_k \to U_k(M) \to 0$ yields the exact sequence $H^k_{\mathfrak{m}}(\mathcal{M}_{k-1}) \to H^k_{\mathfrak{m}}(\mathcal{M}_k) \to H^k_{\mathfrak{m}}(U_k(M)) \to H^{k+1}_{\mathfrak{m}}(\mathcal{M}_{k-1})$ and, since $\dim \mathcal{M}_{k-1} \leq k-1$, it follows that $H^k_{\mathfrak{m}}(\mathcal{M}_k) \cong H^k_{\mathfrak{m}}(U_k(M))$. Moreover if $i \leq t < k$, we also get the exact sequence $H^{t-1}_{\mathfrak{m}}(U_{k-1}(M)) \to H^t_{\mathfrak{m}}(\mathcal{M}_{k-1}) \to H^t_{\mathfrak{m}}(\mathcal{M}_k) \to H^t_{\mathfrak{m}}(U_k(M))$ and, since $U_{k-1}(M)$ and $U_k(M)$ are Cohen-Macaulay of dimension $k-1$ and k respectively, it follows that $H^t_{\mathfrak{m}}(\mathcal{M}_k) \cong H^t_{\mathfrak{m}}(\mathcal{M}_{k-1})$. Consequently

$$H^k_{\mathfrak{m}}(\mathcal{M}) = H^k_{\mathfrak{m}}(\mathcal{M}_d) \cong H^k_{\mathfrak{m}}(\mathcal{M}_{d-1}) \cong \ldots \cong H^k_{\mathfrak{m}}(\mathcal{M}_{k+1}) \cong H^k_{\mathfrak{m}}(\mathcal{M}_k) \cong H^k_{\mathfrak{m}}(U_k(M)).$$

4. By 3 we can assume that $U_k(M)$ is Cohen-Macaulay of dimension k. If x is a $U_k(M)$-regular element of degree one, we have the exact sequence

$$0 \to H^{k-1}_{\mathfrak{m}}(U_k(M)/xU_k(M)) \to H^k_{\mathfrak{m}}(U_k(M))(-1) \to H^k_{\mathfrak{m}}(U_k(M)) \to 0$$

which, together with 3, implies that $h^{k-1}(U_k(M)/xU_k(M)) = (t-1)\,h^k(U_l(M)) = (t-1)\,h^k(M)$. In the same way, if $\mathbf{x}$ is a maximal regular sequence for $U_k(M)$, we get $h^0(U_k(M)/\mathbf{x}U_k(M)) = (t-1)^l\,h^k(M)$. On the other hand, since $U_k(M)/\mathbf{x}U_k(M)$ has dimension zero, it follows that $h^0(U_k(M)/\mathbf{x}U_k(M)) = \operatorname{Hilb}(U_k(M)/\mathbf{x}U_k(M))$ and then the thesis follows immediately.

5. The equality $\operatorname{Hilb}(U_k(R/I)) = \operatorname{Hilb}(U_k(R/\operatorname{Gin}(I)))$ holds for all $k \geq i$ if and only if $\operatorname{Hilb}(R/I^{\langle k\rangle}) = \operatorname{Hilb}(R/\operatorname{Gin}(I)^{\langle k\rangle})$ for all $k \geq i-1$ and, since $\operatorname{Gin}(I^{\langle k\rangle}) \subseteq \operatorname{Gin}(I)^{\langle k\rangle}$ by [38, Proposition 13], this condition is equivalent to $\operatorname{Gin}(I^{\langle k\rangle}) = \operatorname{Gin}(I)^{\langle k\rangle}$ for all $k \geq i-1$. On the other hand, if we consider the short exact sequence

$$0 \to U_k(M) \to R/I^{\langle k-1\rangle} \to R/I^{\langle k\rangle} \to 0,$$

using the Depth Lemma [17, Proposition 1.2.9], it is easy to see that R/I is i-sCM if and only if $\operatorname{depth}(R/\operatorname{Gin}(I^{\langle k\rangle})) = \operatorname{depth}(R/I^{\langle k\rangle}) \geq k+1$ for all $k \geq i-1$. Since for all ideals J the depth of $R/\operatorname{Gin}(J)$ is equal to the smallest integer t such that $\operatorname{Gin}(J)$ is a proper subset of $\operatorname{Gin}(J)^{\langle t\rangle}$ (see [38, Lemma 11]), the last condition is in turn equivalent to $\operatorname{Gin}(I^{\langle k\rangle}) = \operatorname{Gin}(I^{\langle k\rangle})^{\langle k\rangle}$ for all $k \geq i-1$. Finally to conclude the proof we only need to show that $\operatorname{Gin}(I^{\langle k\rangle})^{\langle k\rangle} = \operatorname{Gin}(I)^{\langle k\rangle}$. Since $I \subseteq I^{\langle k\rangle}$, it follows immediately that $\operatorname{Gin}(I) \subseteq \operatorname{Gin}(I^{\langle k\rangle})^{\langle k\rangle} \subseteq \operatorname{Gin}(I)^{\langle k\rangle}$. Then this implies that $\operatorname{Gin}(I)^{\langle k\rangle} \subseteq \operatorname{Gin}(I^{\langle k\rangle})^{\langle k\rangle} \subseteq (\operatorname{Gin}(I)^{\langle k\rangle})^{\langle k\rangle} = \operatorname{Gin}(I)^{\langle k\rangle}$, see [38, Corollary 15], and this is enough to conclude the proof. □

Remark 5.3.4. We define the *i-th truncated Björner-Wachs polynomial* of R/I as

$$\operatorname{BW}^i(R/I;t;w) := \sum_{k=i}^{\dim R/I} h(U_k(R/I);t)w^k.$$

A trivial consequence of the previous lemma is that R/I is i-sCM if and only if $\operatorname{BW}^i(R/I;t;w) = \operatorname{BW}^i(R/\operatorname{Gin}(I);t;w)$.

In the rest of the chapter we set $R_{[i]} = k[x_1,\ldots,x_i]$ and $J_i = J \cap R_i$ for any ideal J of R. Let $l \in R_1$ be a generic linear form which, without loss of generality we may write as $l = a_1x_1 + \cdots + a_{n-1}x_{n-1} - x_n$, where $a_i \in k$. Consider the map $g_n : R \to R_{[n-1]}$, defined by $x_i \mapsto x_i$ for $i = 1,\ldots,n-1$ and $x_n \mapsto a_1x_1 + \cdots + a_{n-1}x_{n-1}$. It follows that the homomorphism $\frac{R}{I} \to \frac{R_{[n-1]}}{g_n(I)}$ is surjective and its kernel is $(I+(l))/I$; therefore it induces the isomorphism

$$\frac{R}{I+(l)} \cong \frac{R_{[n-1]}}{g_n(I)}. \tag{5.2}$$

Since $\operatorname{Gin}(I)$ is an initial ideal, it is clearly a monomial ideal and consequently the image of $\operatorname{Gin}(I)$ in $R_{[n-1]}$ via the mapping $x_n \mapsto 0$ is $\operatorname{Gin}(I)_{[n-1]}$. With this notation, it follows from [42, Corollary 2.15] that

$$\operatorname{Gin}(g_n(I)) = \operatorname{Gin}(I)_{[n-1]}. \tag{5.3}$$

Now we are ready to prove a characterization of partially sequentially Cohen-Macaulay modules, that explains the reason why we introduced them.

Theorem 5.3.5. *The following conditions are equivalent:*

1. *R/I is i-sCM;*

2. *$h^k(R/I)_j = h^k(R/\operatorname{Gin}(I))_j$ for all $k \geq i$ and for all j.*

Proof. Since $R/\operatorname{Gin}(I)$ is sCM, it is always i-sCM and then $1 \Rightarrow 2$ is a consequence of the last two points of Lemma 5.3.3. As for the converse we use induction on $d = \dim R/I = \dim R/\operatorname{Gin}(I)$. If $d = 0$ the ring is Cohen-Macaulay and then sCM. If $d > 0$, by Lemma 5.3.3.2 we can substitute I with I^{sat} and then we can assume that R/I and $R/\operatorname{Gin}(I)$ have positive depth. It follows from [47, Theorem 1.4] that there exists a linear form l' which is $R/\operatorname{Gin}(I)$- and $\operatorname{Ext}_R^{n-k}(R/\operatorname{Gin}(I), \omega_R)$-regular for all k, where ω_R denotes the canonical module of R. Therefore, since a change of coordinates does not affect the computation of the generic initial ideal, we may assume that x_n is $R/\operatorname{Gin}(I)$- and $\operatorname{Ext}_R^{n-k}(R/\operatorname{Gin}(I), \omega_R)$-regular for all k. Consequently, for all k, the short exact sequence $0 \to R/\operatorname{Gin}(I)(-1) \to R/\operatorname{Gin}(I) \to R/(\operatorname{Gin}(I) + (x_n)) \to 0$, gives raise via Local Duality to short exact sequences

$$0 \to H_{\mathfrak{m}}^{k-1}(R/(\operatorname{Gin}(I) + (x_n))) \to H_{\mathfrak{m}}^k(R/\operatorname{Gin}(I))(-1) \to H_{\mathfrak{m}}^k(R/\operatorname{Gin}(I)) \to 0.$$

It follows that $h^{k-1}(R/(\operatorname{Gin}(I) + (x_n))) = (t-1)h^k(R/\operatorname{Gin}(I))$, for all k.

We also know that there exists a generic linear form l which is R/I-regular, because R/I has positive depth. Thus, for all k and some modules M_k, we get the exact sequences

$$0 \to M_k \to H_{\mathfrak{m}}^{k-1}(R/(I + (l))) \to H_{\mathfrak{m}}^k(R/I)(-1) \to H_{\mathfrak{m}}^k(R/I) \to C \to 0$$

and then $h^{k-1}(R/(I + (l))) \geq (t-1)h^k(R/I)$ for all k.

Using (5.2) and (5.3), we thus have

$$\begin{aligned}(t-1)h^k(R/I) &\leq h^{k-1}(R/(I + (l))) = h^{k-1}(R_{[n-1]}/g_n(I)) \\ &\leq h^{k-1}(R_{[n-1]}/\operatorname{Gin}(g_n(I))) = h^{k-1}(R_{[n-1]}/\operatorname{Gin}(I)_{[n-1]}) \\ &= h^{k-1}(R/(\operatorname{Gin}(I) + (x_n))) = (t-1)h^k(R/\operatorname{Gin}(I)),\end{aligned}$$

where all of the above inequalities are equalities for all $k \geq i$ by hypothesis. In particular, the equalities $h^k(R_{[n-1]}/g_n(I)) = h^k(R_{[n-1]}/\operatorname{Gin}(g_n(I)))$ for all $k \geq i-1$ and the inductive assumption imply that $R_{[n-1]}/g_n(I) \simeq R/(I + (l))$ is $(i-1)$-sCM. Consequently, R/I is i-sCM by Lemma 5.3.3.1, because l is R/I-regular. $\square$

5.4 Consecutive cancellations

In this section we prove a technical property of the Hilbert function of local cohomology modules, that we will use in the next section to prove the main result of the chapter. In [67, Theorem 1.1], using the proof of [65, Proposition 30], it is proven that the graded Betti numbers of a homogeneous ideal can be obtained from the graded Betti numbers of its associated lex-ideal by a sequence of consecutive cancellations. We will follow the same line of reasoning, but first introduce some definitions.

Let $\{c_{i,j}\}$ be a set of natural numbers, where $(i,j) \in \mathbb{N}^2$. Fix an index j and choose i and i' such that one is even and the other is odd; then we obtain a new set by a *cancellation* if we replace $c_{i,j}$ by $c_{i,j}-1$ and $c_{i',j}$ by $c_{i',j}-1$. Such a cancellation is said to be *consecutive* if $i' = i+1$. If I and J are two homogeneous ideal of R with the same Hilbert function and d is the dimension R/I, Theorem 1.4.1 implies that

$$\begin{array}{c} H_{R/I}(j) - P_{R/I}(j) = \sum_{i=0}^{d} (-1)^i \; h^i(R/I)_j \\ \| \\ H_{R/J}(j) - P_{R/J}(j) = \sum_{i=0}^{d} (-1)^i \; h^i(R/J)_j. \end{array} \tag{5.4}$$

In [83] E. Sbarra proves that for any monomial order $\prec$ one has $h^i(R/I)_j \leq h^i(R/\operatorname{in}_\prec(I))_j \leq h^i(R/I^{\,\mathrm{lex}})_j$ and then it follows from the equalities above that it is possible to obtain the set $\{h^i(R/I)_j\}$ from either $\{h^i(R/\operatorname{in}_\prec(I))_j\}$ or $\{h^i(R/I^{\,\mathrm{lex}})_j\}$ by a sequence of cancellations. In fact, we will show that the use of consecutive cancellations is enough.

In the following proposition we make use of a standard deformation argument by flat families, see [30, Section 15] for more details. We set some notations before proceeding with it. Consider $S = R[t]$ and let λ be an integral weight function on R such that $\operatorname{in}_\lambda(I) = \operatorname{in}_\prec(I)$. Given $f = \sum_i m_i \in R$ and $b = \max_i\{\lambda(m_i)\}$, we define $\tilde{f} = t^b f(t^{-\lambda(x_1)}x_1, \ldots, t^{-\lambda(x_n)}x_n)$ and consider $\widetilde{I} = (\tilde{f} \mid f \in I)$ that is an ideal of S.

Proposition 5.4.1. *Let $\prec$ be a monomial order. The set $\{h^i(R/I)_j\}$ can be obtained from the set $\{h^i(R/\operatorname{in}_\prec(I))_j\}$ by a sequence of consecutive cancellations.*

Proof. In the proof of [83, Lemma 2.2] it is proved that there exist $m_{l,j}, h_{l,j}, p_s \in \mathbb{N}$ such that

$$\begin{aligned} &\operatorname{Ext}^l_S(S/\widetilde{I}, S)_j \simeq \bigoplus_{s=1}^{m_{l,j}} k[t] \oplus \bigoplus_{s=1}^{h_{l,j}} k[t]/(t^{p_s}) \\ &\dim_k \operatorname{Ext}^l_R(R/I, R)_j = m_{l,j}; \\ &\dim_k \operatorname{Ext}^l_R(R/\operatorname{in}_\prec(I), R)_j = m_{l,j} + h_{l,j} + h_{l+1,j}, \end{aligned}$$

and the local duality theorem yields

$$\begin{aligned} h^0(R/\operatorname{in}_\prec(I))_j &= h^0(R/I)_j + h_{n,j} + h_{n-1,j}, \\ h^1(R/\operatorname{in}_\prec(I))_j &= h^1(R/I)_j + h_{n-1,j} + h_{n-2,j}, \\ &\vdots \\ h^{d-1}(R/\operatorname{in}_\prec(I))_j &= h^{d-1}(R/I)_j + h_{n-d+1,j} + h_{n-d,j}, \\ h^d(R/\operatorname{in}_\prec(I))_j &= h^d(R/I)_j + h_{n-d,j}. \end{aligned}$$

Therefore the thesis follows, because $0 = \sum_{i=0}^{d}(-1)^i \; h^i(R/\operatorname{in}_\prec(I))_j - \sum_{i=0}^{d}(-1)^i \; h^i(R/I)_j = h_{n,j}$. ☐

Corollary 5.4.2. *The set $\{h^i(R/I)_j\}$ can be obtained from the set $\{h^i(R/I^{\,\mathrm{lex}})_j\}$ by a sequence of consecutive cancellations.*

Proof. Following the proof of [67, Theorem 1.1], there is a finite sequence of homogeneous ideals which starts with the ideal I and finishes with the ideal $I^{\,\mathrm{lex}}$. This sequence is obtained by applying three types of basic operations: polarization, specialization by generic linear forms and taking initial ideals with respect to the lexicographic order. By the proof of [83, Theorem 5.4], we only need that $\{h^i(R/I)_j\}$ can be obtained from $\{h^i(R/\operatorname{in}_{\prec}(I))_j\}$ by a sequence of consecutive cancellations and then the thesis follows from the previous proposition. □

5.5 A rigidity property of lex-ideals

In this section we prove the main result of the chapter, which establishes an analogue of Theorem 5.3.1 and generalizes Theorem 5.1.2. We start with some preliminary results.

Lemma 5.5.1. [18, Lemma 1.4] *Let I be a weakly stable ideal. Then*

$$I_{[n-1]} : x_{n-1}^{\infty} = (I : x_n^{\infty})_{[n-1]} : x_{n-1}^{\infty}.$$

Lemma 5.5.2. [18, Lemma 1.5] *Let I be a weakly stable ideal, let $0 \leq i \leq n$ be an integer and set $J = (I_{[n-i+1]} : x_{n-i+1}^{\infty})_{[n-i]}$. Then*

$$\mathrm{Hilb}\left(H_{\mathfrak{m}}^{k+i}(R/I)\right) = \mathrm{Hilb}\left(H_{\mathfrak{m}_{[n-i]}}^{k}(R_{[n-i]}/J)\right) \cdot \left(\sum_{j<0} t^j\right)^i$$

for all $k = 0, \ldots, n-i$.

Lemma 5.5.3. *Let I and I' be two weakly stable ideals. If they have the same Hilbert polynomial, then $I_{[n-i]}$ and $I'_{[n-i]}$ have the same Hilbert polynomial for all $i = 0, \ldots, n$. Moreover the ideals $(I_{[n-i+1]} : x_{n-i+1}^{\infty})_{[n-i]}$ and $(I'_{[n-i+1]} : x_{n-i+1}^{\infty})_{[n-i]}$ have the same Hilbert polynomial.*

Proof. Since I and I' are weakly stable, $I : x_n^{\infty}$ and $I' : x_n^{\infty}$ are their saturation by Proposition 1.2.2 and then they have the same Hilbert polynomial; clearly this happens also for $(I : x_n^{\infty})_{[n-1]}$ and $(I' : x_n^{\infty})_{[n-1]}$. Saturating again these ideals and using Lemma 5.5.1, we get that $I_{[n-1]} : x_{n-1}^{\infty}$ and $I'_{[n-1]} : x_{n-1}^{\infty}$ have the same Hilbert polynomial; consequently this happens also to $I_{[n-1]}$ and $I'_{[n-1]}$. The last statement is clear. □

We prove first our rigidity property for weakly stable ideal, but we will remove this hypothesis in Theorem 5.5.6.

Proposition 5.5.4. *Let $I \subseteq R$ be a weakly stable ideal. If $h^i(R/I)_j = h^i(R/I^{\,\mathrm{lex}})_j$ for some $i \geq 0$ and all j, then $h^k(R/I)_j = h^k(R/I^{\,\mathrm{lex}})_j$ for all $k \geq i$ and all j.*

Proof. Clearly, by Theorem 5.1.2, we can assume $i \geq 1$. By Proposition 5.4.2, we know that the set $\{h^k(R/I)_j\}$ can be obtained from $\{h^k(R/I^{\,\mathrm{lex}})_j\}$ by means of a sequence of consecutive cancellations. Since at level i there is nothing to be cancelled, it follows that the set $\{h^k(R/I)_j\}_{k\geq i}$ can be obtained from $\{h^k(R/I^{\,\mathrm{lex}})_j\}_{k\geq i}$ by a sequence of consecutive cancellations. In particular this implies that

$$\sum_{k=i}^{n}(-1)^k\ h^k(R/I)_j = \sum_{k=i}^{n}(-1)^k\ h^k(R/I^{\,\mathrm{lex}})_j.$$

Set $J = (I_{[n-i+1]} : x_{n-i+1}^{\infty})_{[n-i]}$ and $J' = ((I^{\,\mathrm{lex}})_{[n-i+1]} : x_{n-i+1}^{\infty})_{[n-i]}$. Theorem 1.4.1 and Lemma 5.5.2 now imply that

$$H_{R_{[n-i]}/J}(j) - P_{R_{[n-i]}/J}(j) = \sum_{k=0}^{n-i}(-1)^k\ h^k(R_{[n-i]}/J)_j =$$

$$= \sum_{k=0}^{n-i}(-1)^k\ h^k(R_{[n-i]}/J')_j = H_{R_{[n-i]}/J'}(j) - P_{R_{[n-i]}/J'}(j).$$

Since J and J' have the same Hilbert polynomial by the previous lemma, they also have the same Hilbert function; moreover $J' = J^{\,\mathrm{lex}}$, because J' is a lex-ideal. Furthermore, the ideal $(I^{\,\mathrm{lex}})_{[n-i+1]} : x_{n-i+1}^{\infty}$ is a saturated lex-ideal of $R_{[n-i+1]}$ by Proposition 1.2.2.1 and, therefore, it has positive depth and at most $n-i$ generators, by Theorem 1.1.1.1. It follows that J' has at most $n-i$ minimal generators, i.e. is an universal lex-ideal, and thus J is a critical ideal. Consequently Remark 5.2.2.3 yields that $h^k(R_{[n-i]}/J)_j = h^k(R_{[n-i]}/J')_j$ for $k = 0, \ldots, n-i$ and, the conclusion follows using again Lemma 5.5.2. $\square$

It will be useful to re-state what we have just proved as follows.

Corollary 5.5.5. *Let I be a weakly stable ideal of R and let i be a positive integer. Set $J = (I_{[n-i+1]} : x_{n-i+1}^{\infty})_{[n-i]}$ and $J' = ((I^{\,\mathrm{lex}})_{[n-i+1]} : x_{n-i+1}^{\infty})_{[n-i]}$. The following are equivalent:*

1. $h^i(R/I)_j = h^i(R/I^{\,\mathrm{lex}})_j$ *for all j;*

2. $h^k(R/I)_j = h^k(R/I^{\,\mathrm{lex}})_j$ *for all $k \geq i$ and all j;*

3. *J and J' have the same Hilbert function;*

4. *J is a critical ideal;*

5. $\mathrm{Gin}(J) = J'$.

Proof. In the proof of the previous proposition we showed that $1 \Rightarrow 3 \Rightarrow 4 \Rightarrow 2$, which in turn obviously implies 1. Moreover Condition 3 descends immediately by 5, whereas, since $J' = J^{\,\mathrm{lex}}$, 4 implies 5 by Proposition 1.2.1.5. $\square$

We are now ready to prove the desired rigidity property.

Theorem 5.5.6. *Let i be a non-negative integer such that $h^i(R/\operatorname{Gin}(I))_j = h^i(R/I^{\,\mathrm{lex}})_j$ for all j. Then $h^k(R/I)_j = h^k(R/I^{\,\mathrm{lex}})_j$ for all $k \geq i$ and all j.*

Proof. We proved the case $i = 0$ in Remark 5.2.2.2, therefore we may assume that $i > 0$. Since $\operatorname{Gin}(I)$ is a weakly stable ideal, it follows from Proposition 5.5.4 that $h^k(R/\operatorname{Gin}(I))_j = h^k(R/I^{\text{lex}})_j$ for all $k \geq i$ and all j; hence, by Theorem 5.3.5, it is enough to prove that R/I is i-sCM.

Since R/I^{sat} has positive depth, there exists a generic linear form l_n which is R/I^{sat}-regular. Now, by (5.2), $R_{[n-1]}/g_n(I^{\text{sat}}) \simeq R/(I^{\text{sat}} + (l_n))$; thus, if we set $J_1 := g_n(I^{\text{sat}})$, by (5.3) it follows that $\operatorname{Gin}(J_1) = (\operatorname{Gin}(I)^{\text{sat}})_{[n-1]}$ and R/I is i-sCM if and only if $R_{[n-1]}/J_1$ is $(i-1)$-sCM. If $i - 1 > 0$ we can go on in this way: we saturate J_1 with respect to the maximal ideal of $R_{[n-1]}$, take a generic linear form $l_{n-1} \in R_{[n-1]}$, which is $R_{[n-1]}/{J_1}^{\text{sat}}$-regular, and apply (5.2). By letting J_2 be the ideal $g_{n-1}({J_1}^{\text{sat}})$, we have $R_{[n-2]}/J_2 \simeq R_{[n-1]}/({J_1}^{\text{sat}} + (l_{n-1}))$ and

$$\operatorname{Gin}(J_2) = (\operatorname{Gin}(J_1)^{\text{sat}})_{[n-2]} = (((\operatorname{Gin}(I)^{\text{sat}})_{[n-1]})^{\text{sat}})_{[n-2]} = ((\operatorname{Gin}(I)_{[n-1]})^{\text{sat}})_{[n-2]},$$

where the last equality holds by Lemma 5.5.1. After i steps, we get

$$\operatorname{Gin}(J_i) = ((\operatorname{Gin}(I)_{[n-i+1]})^{\text{sat}})_{[n-i]} \tag{5.5}$$

and R/I is i-sCM if and only if $R_{[n-i]}/J_i$ is 0-sCM, i.e. sCM. Since $\operatorname{Gin}(I)$ is a weakly stable ideal and $h^i(R/\operatorname{Gin}(I)) = h^i(R/I^{\text{lex}})$, it follows from Corollary 5.5.5 that $((\operatorname{Gin}(I)_{[n-i+1]})^{\text{sat}})_{[n-i]}$ is a critical ideal. By (5.5) also J_i is a critical ideal and Remark 5.2.2.3 implies that $R_{[n-i]}/J_i$ is sCM, as desired. $\square$

Corollary 5.5.7. *Set* $J = (\operatorname{Gin}(I)_{[n-i+1]} : x_{n-i+1}^{\infty})_{[n-i]}$ *and* $J' = ((I^{\text{lex}})_{[n-i+1]} : x_{n-i+1}^{\infty})_{[n-i]}$, *where i is a positive integer. The following are equivalent:*

1. $h^i(R/\operatorname{Gin}(I))_j = h^i(R/I^{\text{lex}})_j$ *for all* j;

2. $h^i(R/I)_j = h^i(R/I^{\text{lex}})_j$ *for all* j;

3. $h^k(R/I)_j = h^k(R/I^{\text{lex}})_j$ *for all* $k \geq i$ *for all* j;

4. J *and* J' *have the same Hilbert function;*

5. J *is a critical ideal;*

6. $\operatorname{Gin}(J) = J'$.

Remark 5.5.8. 1. Clearly Theorem 5.5.6 implies that if $h^i(R/\operatorname{Gin}(I))_j = h^i(R/I^{\text{lex}})_j$ for all j, then $h^i(R/I)_j = h^i(R/I^{\text{lex}})_j$ for all j. Even if the previous theorem is inspired by Theorem 5.3.1, we note that this is not true for the Betti numbers, see [57, Theorem 3.1].

2. As in Remark 5.3.4, it is easy to see that I and I^{lex} have the same i-th truncated BW-polynomial if and only if $h^i(R/\operatorname{Gin}(I))_j = h^i(R/I^{\text{lex}})_j$ for all j.

3. In [18] the authors introduce the notion of *zero-generic initial ideal* of an ideal, denoted by $\operatorname{Gin}_0(I)$. This shares many interesting properties with the usual one $\operatorname{Gin}(I)$ and the two notions coincide in characteristic 0. It is easy to see that all the equivalent conditions of the previous corollary are still valid for $\operatorname{Gin}_0(I)$. In fact, since $h^i(R/\operatorname{Gin}_0(I)) = h^i(R/\operatorname{Gin}(I))$ for all i, Theorem 5.5.6 clearly holds for $\operatorname{Gin}_0(I)$, and, since $\operatorname{Gin}_0(I)$ is weakly stable, the conclusions of Corollary 5.5.5 hold as well. Thus, we only need to show that $\operatorname{Gin}_0(J) = J'$ is equivalent to conditions 1-5. One direction is immediately seen, since if $\mathrm{Gin}_0(J) = J'$, they have the same Hilbert function; the ideal J' is a universal lex-ideal and, thus, J is critical. Conversely, if J is critical, so is $\mathrm{Gin}_0(J)$. Therefore, $\mathrm{Gin}(\mathrm{Gin}_0(J)) = \mathrm{Gin}_0(J)^{\,\mathrm{lex}}$. It is known that the ideal on the left is $\mathrm{Gin}_0(J)$, while the ideal on the right is $J' = J^{\,\mathrm{lex}}$ because $\mathrm{Gin}_0(J)$ and J have the same Hilbert function.

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