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Minimal and Evolving Networks

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A Luca,
per l'amicizia.

Several results presented in this thesis appear in the following papers:

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- C. Mantegazza, M. Novaga, and A. Pluda, *Motion by curvature of networks with two triple junctions*, Geometric flows, accepted 2016.
- C. Mantegazza, M. Novaga, A. Pluda, and F. Schulze, *Evolution of networks with multiple junctions*, submitted 2016.

Contents

Introduction

This thesis is devoted to the study of minimal networks from both the static and the dynamic point of view and in particular we consider problems at the border of geometry and analysis.

In the first chapter we search for networks embedded in a given Riemannian surface with minimal length that satisfy some topological constraints, namely that one of being spines of the surface. Using standard techniques of the calculus of variation, we prove that such minimal networks exist for any closed Riemannian surfaces and then we focus on special cases (flat torus, hyperbolic surfaces) to obtain more information about their classification.

In the second chapter we let the networks evolve according to the "gradient flow" of the length. Intuitively this means that the curves which form the network evolves with normal velocity equal to the curvature. We consider solution in strong sense and in particular we discuss the short time existence and the singularity formation at the maximal time of existence, generalizing some results for the curve shortening flow of simple closed curves.

In differential topology, a *spine* of a closed smooth 2–dimensional surface S is a smooth finite simplicial complex $\Gamma \subset S$ such that $\dim \Gamma < \dim S$ and S minus a small open ball can be retracted onto Γ. In particular $S \setminus \Gamma$ is a disc.

We look for *spines of minimal length* (1-dimensional Hausdorff measure) in Riemannian 2–dimensional surfaces. As an example, the spheres S^2 equipped with any Riemannian metric have spines of minimal length since any point of S^2 is a spine.

In Section 2 we have the following *existence theorem*: every closed Riemannian surface has a spine of minimal length, moreover the spine is a point if the surface is a sphere, a closed geodesic if the surface is diffeomorphic to the projective plane and in the remaining cases a network consisting of finitely many geodesic arcs meeting at 3–points with angle $\frac{2}{3}\pi$.

The rest of the chapter is devoted to classification of minimal spines for surfaces that are neither diffeomorphic to a sphere nor to the projective plane. We say that a spine is *minimal* if it composed by finitely many geodesic arcs meeting with angle $\frac{2}{3}\pi$ at 3–points (Definition 2.8), in other words minimal spines are critical points of the length functional (clearly a spine of minimal length is minimal, but the converse may not hold).

At this point we pass to study these geometric objects on constant curvature surfaces. All constant curvature metrics on a closed oriented surface S, considered up to orientation-preserving isometries and global rescalings, form the *moduli space* $\mathcal{M}(\mathrm{S})$ of S. The moduli space is not compact: the moduli space $\mathcal{M}(\mathrm{T})$ of the torus is the $(2, 3, \infty)$ – orbifold which we represent as (a quotient of) a subset of the complex plane (see Section 3.2 for definition and properties of the moduli space of the torus).

In Section 3 we completely classify all minimal spines of *flat* tori: every flat torus T contains finitely many minimal spines up to isometries and the number of such minimal spines is a proper function $c \colon \mathcal{M}(\mathrm{T}) \to \mathbb{N}$ (shown in Fig. 11), which means that this number tends to infinity as the flat metric tends to infinity in moduli space.

We see in particular that the "square" and the "hexagonal" tori (two tori that have additional isometries and are identified with the point $z = i$ and $z = e^{\frac{\pi}{3}i}$ of the moduli space of the torus, as explained in Section 3.2) are the only flat tori that contain a unique minimal spine up isometries. Moreover every oriented flat torus has a unique spine of minimal length up to isometries, unless it is a "rectangular, non–square" torus and in this case the minima are exactly two (Theorem 3.10).

Let

$$\mathrm{S}\colon \mathcal{M}(T) \to \mathbb{R}\,, \quad S : z \mapsto \sqrt{\frac{1+|z|^2-|\Re(z)|+\sqrt{3}\Im(z)}{\Im(z)}}$$

be a function that assigns to a (unit-area) flat torus $z \in \mathcal{M}(T)$ the minimal length $\mathrm{S}(z)$ of a spine in T, we prove in Section 3.7 that this function has a unique global minimum at the "hexagonal" torus. The function S may be called the *spine systole* by analogy with the (geodesic) systole that measures the length of the shortest closed geodesic.

In Section 4 we turn to higher genus *hyperbolic surfaces* and we ask whether minimal spines have the same qualitative behavior there. Let S_g be a closed orientable surface of genus $g \geq 2$ and let $\mathcal{M}(S_g)$ be the moduli space of S_g. Every hyperbolic surface in $\mathcal{M}(S_g)$ has a spine of minimal length and its length defines a spine systole $\mathrm{S}\colon \mathcal{M}(S_g) \to \mathbb{R}$.

An *extremal surface* is a hyperbolic surface that contains a disc of the maximum possible radius in genus g. Such surfaces were defined and studied by Bavard [**17**] and various other authors, see [**40**] and the references therein. We can prove the following: the function $\mathrm{S}\colon \mathcal{M}(S_g) \to \mathbb{R}$ is continuous and proper and its global minima are precisely the extremal surfaces.

A full classification of all minimal spines on all hyperbolic surfaces would of course be desirable; for the moment, we content ourselves with Theorem 4.2: every closed hyperbolic surface has finitely many minimal spines of bounded length.

In particular, we do not know if a closed hyperbolic surface has finitely many minimal spines overall (counting spines only up to isometry does not modify the problem, since the isometry group of the surface is finite).

For non positive *constant curvature surfaces* we are also able to say that minimal spines (stationary point for the lengths functional) are local minima for the length functional among spines, with respect to the Hausdorff distance (Section 5).

We notice that the notion of spine is well defined not only for surface but also for manifold of any dimension $n \geq 2$. The existence of spines with minimal area for all closed irreducible 3-manifolds has already been proved by Choe [**24**]. The techniques used in that paper are much more elaborate than the ones we use here.

In the second chapter of the thesis we study the *motion by curvature* of a network of non–intersecting curves in the plane. This problem was proposed by Mullins [**16**] and discussed in [**16, 18, 19, 42, 56**], and attracted in recent years the interest of several authors, see [**32, 53, 67, 46, 15, 69, 71, 76, 81, 83, 84, 90, 55, 23**].

One strong motivation to study this flow is the analysis of models of two–dimensional multiphase systems, where the problem of the dynamics of the interfaces between different phases arises naturally. As an example, the model where the energy of a configuration is simply the total length of the interfaces has proven useful in the analysis of the growth of grain boundaries in a polycrystalline material, see [**16, 42, 56**]. A second motivation is more theoretical: the evolution of such a network of curves is the simplest example of motion by mean curvature of a set which is *essentially* singular. There are indeed in the literature various generalized definitions of flow by mean curvature for non–regular sets (see [**5, 18, 27, 33, 52**], for instance).

Despite the mean curvature flow of a smooth submanifold is deeply, even if not completely, understood, the evolution of generalized submanifolds, possibly singular, for instance *varifolds*, has not been studied too much in detail after the seminal work by K. Brakke [**18**], where it is proved the existence of a global (very) weak solution in the *geometric measure theory* context, called "Brakke flow". In this direction, we also mention the works by T. Ilmanen [**51**], K. Kasai and Y. Tonegawa [**54**], and S. Esedoglu and F. Otto [**32**] (see also [**21, 28**] for an approach based on the implicit variational scheme introduced in [**5, 62**]). In particular, in [**55**], L. Kim and Y. Tonegawa obtain a global existence result for the evolution of grain boundaries in $\mathbb{R}^n$ (which reduces to the evolution of networks when $n = 2$), showing a regularity result when the "density" is less than two. Moreover, in [**90**] Y. Tonegawa and N. Wickramasekera adapt the parabolic blow–up method to study the singularities of a Brakke flow of networks and obtain an estimate on the

Hausdorff dimension of the singular times.

The definition of the flow is the first problem one has to face, due to the contrast between the intrinsic singular nature of the involved objects and the reasonable desire to have something as "smooth" as possible. Consider, for instance the network described by two curves crossing each other and forming a 4–point. There are actually several possible candidates for the flow, even excluding a priori the possibility of "fattening" phenomena (which can happen for instance in the "level sets" approach of L. C. Evans and J. Spruck [**33**]).

Actually, one would like that a good/robust definition of curvature flow should give uniqueness of the motion (at least for "generic" initial data) and forces the network, by a instantaneous regularization, to have only triple junctions, with the three angles between the concurring curves of 120 degrees, with the possible exception of some discrete set of times. This last property is suggested by the variational nature of the problem, since the flow can be considered as the "gradient flow" in the space of networks of the *length functional*. It must anyway be said that such space does not have a natural linear structure and such a "gradient" is not actually a well defined "velocity vector" driving the motion. However the heuristic idea of gradient flow suggests that normal component of the velocity at every point of the network (except the multi–points) is equal to the curvature. Moreover for all positive time the network should contains only regular triple junctions, that is every junction has order three with all angles equal to 120 degrees, except for the initial time and a discrete set of times (this property is also suggested by numerical simulations and physical experiments, see [**16, 19, 42, 56**] and the *grain growth* movies at *http://facstaff.susqu.edu/brakke*).

The notion of Brakke flow [**18**] is satisfactory from this point of view, but is anyway too weak in dealing with curves (see however a recent improvement obtained by K. Kasai and Y. Tonegawa [**54**]).

Another approach to existence is the one based on *minimizing movements* (see [**5**] and [**62**]), that is, a time discretization and iterated minimization procedure (see also the recent work [**59**] by T. Laux and F. Otto).

By their clear importance in this problem, we then call *regular* the networks having only multi–points with three concurrent curves (3–points) forming angles of 120 degrees.

What we did (following the "energetic" and experimental hints) is simply to put such *regularity* condition in the definition of curvature flow for every positive time (but not at the initial time). If the initial network is regular and smooth enough, we will see that this definition leads to an almost satisfactory (in a way "classical") short time existence theory of a flow by curvature. On the other hand if the initial network is non–regular networks, various complications arise.

In any case, even if the initial network is regular and smooth, we cannot avoid to deal also with non–regular networks, if we want to consider the long time behavior of the flow. Indeed, during the flow some of the triple junctions could "collapse" together when the length of one curve goes to zero (hence modifying the topological structure of the network and generating a 4–point). In this case one of the goal is to "restart" the evolution with a non–regular initial datum. A suitable short time existence (hence, restarting) result, has been proposed in [**53**].

The existence problem of a curvature flow for a regular initial network with only one 3–point (called *triod*) was first considered by L. Bronsard and F. Reitich in [**19**], where they proved the short time existence of the flow, and by D. Kinderlehrer and C. Liu in [**56**], who showed the global existence of a smooth solution if the initial regular triod is sufficiently close to a minimal Steiner configuration.

In Sections 1 and 2, we define regular networks and their flow by curvature and describe their basic properties. In Section 3, we present two extension of the short time existence theorem of L. Bronsard and F. Reitich to general regular initial networks satisfying some compatibility conditions at the triple junctions (Theorems 3.8 and 3.18).

Using some estimates proved in in Section 4, in Section 5 we prove a further short time existence result (Theorem 5.8) which holds with lower regularity assumption on the initial network.

The problem of uniqueness of the flow (which is also discussed in Sections 3 and 5) is actually quite delicate and does not have at the moment a clear answer. Even for an initial smooth regular network, there is uniqueness in the same special class of networks, while one should expect uniqueness in the natural class of curvature flows which are simply C^2 in space and C^1 in time (as it happens for the motion by curvature of a closed curve). The difficulty in getting such conclusion is due to the lack of possibility to use easily the *maximum principle*, which is the main tool to get estimates on the geometric quantities during the flow.

After such discussion of the existence and uniqueness of a curvature flow on some maximal time interval, we pass to the analysis of the long time behavior of the motion by curvature, employing a mix of PDE's, variational and differential geometry techniques. To this aim, in Section 6 we recall Huisken's monotonicity formula for mean curvature flow, which holds also for these evolutions. We introduce also the rescaling procedures used to get blow–up limits at the maximal time of smooth existence (discussed in Section 7), in order to describe the singularities of the flow and possibly exclude them.

One can reasonably expect, for instance, that an embedded regular network does not actually develop singularities during the flow if its "topological structure" does not change, due to the "collision" of two or more 3– points.

Under the assumption that the lengths of the curves are bounded away from zero and no multiplicities larger than one occur (and moreover in some special cases such "bad" multiplicities can be ruled out, see Section 9), this expectation is indeed true. Essentially, one needs to classify the possible blow–up limits at the singular time, in order to exclude them by means of geometric arguments. Under the previous hypotheses the only possible blow–up limits are given by a straight line, a halfline with multiplicity one, or a flat unbounded regular triod (called "standard triod") composed by three halflines for the origin of $\mathbb{R}^2$ forming angles of 120 degrees (see Proposition 7.22 and Section 8). Then, extending the argument in [**67**], together with such classification, one excludes the presence of singularities. Some key references for this method in the case of a single smooth curve are [**6, 45, 48, 49**]. The most relevant difference in dealing with networks is the lack of the maximum principle, which is the main tool to get point–wise estimates on the geometric quantities that describe the flow. For this reason, some important estimates which are almost trivial applications of the maximum principle in the smooth case, are here more complicated to prove (and sometimes we do not know if they actually hold) and one has to resort to integral estimates (see Section 4).

Section 8 is devoted to analyze the behavior of the flow of a general network approaching a singular time: we discuss the properties of possible limit networks with some lengths not bounded away from zero. The case in which the curvature is not a priori bounded is clearly the the most complicate (see Section 8.4), we are anyway able to show that if only two triple junctions collide along a curve (which collapses), forming a 4–point, the curvature remains bounded. The situation in which an entire region collapses is instead much more difficult to treat.

The major open problem is the so–called multiplicity–one conjecture: if the initial network $\mathbb{S}_0$ is embedded, not only $\mathbb{S}_t$ remains embedded for all the times, but also that every possible C^1_{loc}–limit of rescalings of networks of the flow is an embedded network with multiplicity one. This is a crucial ingredient in classifying blow–up limits of the flow, which, as we have already said, is the main method to understand the singularity formation. In Section 9 we discuss a scaling invariant, geometric quantity associated to a network, first proposed in [**44**] (see also [**49**]), later extended in [**71**] to the case of a triod, consisting in a sort of "embeddedness measure" which is positive when no self–intersections are present. By a monotonicity argument, we show that such quantity is bounded below along the flow, under the assumption that the number of 3– points of the network is not greater than two. As a consequence, in this case we have that every possible C^1_{loc}– limit of rescalings of networks of the flow is an embedded network with multiplicity one.

We underline that it is not clear to us how to obtain a similar bound/conclusion for a general network (with several triple junctions), since the analogous quantity (apparently), if there are more than two 3–points, does not share the above monotonicity property.

In Section 10 we collect all the examples in which we are able to treat completely the onset of the first singularity or in which we are able to establish a global existence result. We conclude with Section 11 where it is explained how, in some cases, one can restart the flow after a singularity by means of the existence theorem of a flow for non–regular networks by Ilmanen, Neves and Schulze [**53**].

CHAPTER 1

Spines of minimal length

In this chapter we raise the question whether every closed Riemannian manifold has a spine of minimal area, and we answer affirmatively in the case of surface.

The structure of the chapter is the following: in Section 1 we introduce the definition of spine giving several examples and we state the minimizing problem. Then we list some basic tools used in the sequel.

In Section 2 we present an existence result for spines of minimal length of 2–dimensional closed Riemannian surfaces, and we give the definition of *minimal* spine.

In Section 3 we completely classify minimal spines on tori $T = \mathbb{S}^1 \times \mathbb{S}^1$ endowed with a flat structure, proving that the number of such spines is a proper function on moduli space.

On hyperbolic surfaces we introduce the *spine systole*, a continuous real function on moduli space that measures the minimal length of a spine in each surface. We show that the spine systole is a proper function and has its global minima precisely on the extremal surfaces (those containing the biggest possible discs). We also show that the number of minimal spines of uniformly bounded length is finite on hyperbolic surfaces (Section 4).

To conclude, in Section 5, we prove that minimal spines of closed surfaces of non positive constant curvature are local minima for the length functional.

1. Definitions and preliminaries

1.1. Spines. Before giving the definition of spine, we recall some notions that apply both in the piecewise-linear and in the smooth category of manifolds.

DEFINITION 1.1. A *smooth finite simplicial complex* (or a *finite polyhedron*) in a smooth n–dimensional manifold M with (possibly empty) boundary is a subset $P \subset \mathrm{int}(M)$ homeomorphic to a simplicial triangulation, such that every simplex is diffeomorphic through some chart to a standard simplex in $\mathbb{R}^n$.

The subset P has a well-defined *regular neighborhood*, unique up to isotopy: this is a piecewise-linear notion that also applies in the smooth category **[47]**.

DEFINITION 1.2. A regular neighborhood N of P is a compact smooth n-dimensional submanifold $N \subset M$ containing P in its interior which collapses simplicially onto P for some smooth triangulation of M; as a consequence $N \setminus P$ is an open collar of ∂N and in particular $\partial N \neq \emptyset$. If the manifold M itself is a regular neighborhood of P we say that P is a *spine* of M.

This definition of course may apply only when $\partial M \neq \emptyset$, so to define a spine P of a closed manifold M we priorly remove a small open ball from M.

In all cases, we suppose that $\dim P < \dim M$, and this is the only restriction we make on dimensions.

REMARK 1.3. In particular, if M is a manifold without boundary, we get that $M \setminus P$ is an open ball. One can prove that this actually a characterization of spines for such manifolds.

EXAMPLE 1.4.

- Any point is a spine of the sphere S^n for all $n \geq 1$;
- any complex hyperplane is a spine of the complex projective space $\mathbb{CP}^n$;
- a spine may have strata of mixed dimensions: a natural spine for $S^m \times S^n$ is the transverse union of two spheres $S^m \times q$ and $p \times S^n$;

- if M is a Riemannian manifold, one could construct P as the cut locus of a point **[20]**.

The notion of spine is widely employed in topology: for instance, it may be used to define a *complexity* on manifolds **[74, 75, 72]**, to study group actions **[14]** and properties of Riemannian manifolds **[2]**. In dimensions 2 and 3 spines (with generic singularities) arise naturally and frequently as the dual of 1-vertex triangulations. Topologists usually consider spines only up to isotopy, and relate different spines (or triangulations) via some moves like "flips" on surfaces (see for instance **[13, 35, 57, 86]**) and the Matveev-Piergallini moves **[73, 79]** on 3-manifolds.

However, it seems that spines have not been much studied from a geometric measure theory point of view, and this is our main purpose.

We are interested here in the following problem:

QUESTION 1.5. Does every closed Riemannian manifold M of dimension $n \geq 2$ have a spine P of minimal area?

We notice that we consider by assumption only spines P with $\dim P < \dim M$, so that if M is endowed with a Riemannian metric, it is natural to consider the *area* of P, defined as its $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}(P)$.

There are examples in which an affirmative answer to Question 1.5 is trivial, for instance the area of P will be zero in the cases when the dimension of P is strictly smaller than $n-1$.

EXAMPLE 1.6.

- The sphere S^n has zero area spines (single points);
- the complex projective space $\mathbb{CP}^n$ has zero area spines (complex hyperplanes);
- if one considers the above described spine of $S^2 \times S^1$, (see Example 1.4), it has the same area 4π of S^2.

In the following we consider only the case in which the dimension of the manifold M is two, its spines have dimension at most 1 and it is of course reasonable to employ the word *length* to indicate their area. As in the general case stated before, there are trivial examples, in which the answer to Question 1.5 is *yes* : for instance for the sphere S^2 equipped with any Riemannian metric, and for the projective plane. In Section 2 we give a positive answer for all closed Riemannian surfaces.

We would like to mention that in higher dimensions, Question 1.5 has already been partially addressed by Choe **[24]**, who has provided a positive answer for all closed irreducible 3-manifolds. The techniques used in that paper are much more elaborated than the ones we use here.

1.2. Networks in Riemannian surfaces.

DEFINITION 1.7. An *network* $\Gamma \subset S$ is a union of a finite number of supports of simple smooth curves $\gamma^i : [0,1] \to S$, intersecting only at their endpoints. A point in which two or more curves concur is called *multi–point*.

Each curve γ^i of the network has length $\mathcal{H}^1\left(\mathrm{Im}\left(\gamma^i\right)\right)$ and the length of Γ is the sum of the lengths of all the curves, that is,

$$L(\Gamma) = \mathcal{H}^1\left(\bigcup_{i=1}^{n}\left(\mathrm{Im}\left(\gamma^i\right)\right)\right) = \sum_{i=1}^{n}\mathcal{H}^1\left(\mathrm{Im}\left(\gamma^i\right)\right). \tag{1.1}$$

In the following, we will search the minima of this functional L, restricted to the set of spines of a closed Riemannian surface S.

1.3. First variation of the length functional. Let Γ be a network in a closed Riemannian surface S. Let Φ_t with $t \in [0,T]$ be a smooth family of diffeomorphisms of S, with $\Phi_0(x) = x$ for all $x \in \Gamma$ and let $\Gamma_t = \Phi_t(\Gamma)$. Consider the vector field X on S defined by

$$X(x) = \frac{d}{dt}\Phi_t(x)\Big|_{t=0}.$$

The first variation formula for the length functional L for a network Γ is (see for instance **[65]**):

$$\frac{d}{dt}L(\Gamma_t)\Big|_{t=0} = \int_\Gamma \mathrm{div}_\tau X \, d\mathcal{H}^1 \,, \tag{1.2}$$

where with div_τ we denote the tangential divergence.

Let $x_1, \dots, x_m$ be the multi–points of Γ. At every multi–point x_j, l_j curves concur. For $k = 1, \dots l_j$ denote with τ_j^k the unit tangent vector to the k–th curves at x_j. Therefore, an integration by parts gives

$$\frac{d}{dt}L(\Gamma_t)\Big|_{t=0} = -\int_\Gamma H \cdot X \, d\mathcal{H}^1 + \sum_{j=1}^{m} \left(\sum_{k=1}^{l_j} \tau_j^k \cdot X \right) \,, \tag{1.3}$$

where H is the curvature of Γ (defined everywhere except at the multi–points). A network Γ is *stationary* if there holds

$$\int_\Gamma \mathrm{div}_\tau X \, d\mathcal{H}^1 = 0 \quad \text{for every vector field } X \text{ on } S \,, \tag{1.4}$$

Thanks to formula (1.3), condition (1.4) is equivalent to require that each curve of the network Γ is a geodesic arc and the sum of the unit tangent vectors of concurring curves at a common endpoint is equal to zero.

1.4. Rectifiable sets and stationary varifolds. Let S be a Riemannian surface. We recall that a set $\Gamma \subset S$ is *countably* 1*-rectifiable* if it can be covered by countably many images of Lipschitz maps from $\mathbb{R}$ to S, except a subset $\mathcal{H}^1$–negligible.

Let Γ be a countably 1-rectifiable, $\mathcal{H}^1$-measurable subset of S and let θ be a positive locally $\mathcal{H}^1$-integrable function on Γ. Following **[3, 4]** we define the *rectifiable* 1*-varifold* (Γ, θ) to be the equivalence class of all pairs $(\widetilde{\Gamma}, \widetilde{\theta})$, where $\widetilde{\Gamma}$ is countably 1-rectifiable with $\mathcal{H}^1\left((\Gamma \setminus \widetilde{\Gamma}) \cup (\widetilde{\Gamma} \setminus \Gamma)\right) = 0$ and where $\widetilde{\theta} = \theta$ $\mathcal{H}^1$-a.e. on $\Gamma \cap \widetilde{\Gamma}$. The function θ is called *multiplicity* of the varifold.

A varifold (Γ, θ) is *stationary* if there holds

$$\int_\Gamma \theta \, \mathrm{div}_\tau X \, d\mathcal{H}^1 = 0, \tag{1.5}$$

for any vector field X on S. Notice that, if θ is constant and Γ is a network, condition (1.5) is consistent with condition (1.4).

2. Existence of minimal spines of a Riemannian surfaces

In this section we prove an existence result for the spines of minimal length of any closed Riemannian surface:

THEOREM 2.1. *Every closed Riemannian surface S has a spine Γ of minimal length.*
The spine Γ is:

(1) *a point if S is diffeomorphic to a sphere,*
(2) *a closed geodesic if S is diffeomorphic to a projective plane,*
(3) *finitely many geodesic arcs meeting at 3–points with angle $\frac{2}{3}\pi$, otherwise.*

REMARK 2.2. The homotopy type of a spine Γ of a closed 2–surface S is completely determined by that of S. Indeed,

- A spine Γ is homotopically equivalent to a point if and only if S is diffeomorphic to a sphere. Indeed, a regular neighbourhood of a point must be a disc.
- A spine Γ is homotopically equivalent to a circle if and only if S is diffeomorphic to a projective plane. Indeed, a regular neighbourhood of a circle could be nothing but a Möbius strip, because an annulus has too many boundary components.

- Finally, assume that S is diffeomorphic neither to a sphere, nor to a projective plane. A spine of S is a network $\Gamma \subset S$. Moreover denoting by e the number of edges of Γ, by v the number of vertices and by χ the Euler characteristic, a necessary condition for Γ to be a spine of S is
$$\chi(S) - 1 = \chi(S - B^2) = \chi(\Gamma) = v - e\,. \tag{2.1}$$

PROPOSITION 2.3. *Consider a closed Riemannian surface S not diffeomorphic to a sphere. Then there is a constant $K > 0$ such that $\mathcal{H}^1(\Gamma) \geq K$ for every spine Γ of S.*

PROOF. Let $r > 0$ be the injectivity radius of S. Every homotopically non–trivial closed curve in S has length at least $2r$. Every spine Γ contains at least one homotopically non-trivial embedded closed curve and hence has length at least $K = 2r$.

To verify the last fact, recall that $S \setminus \Gamma$ is an open 2-disc and hence the inclusion map $i\colon \Gamma \hookrightarrow S$ induces a surjection $i_*\colon \pi_1(\Gamma) \to \pi_1(S)$. Since $\pi_1(S) \neq \{e\}$ the spine Γ contains some homotopically non-trivial loop, and this easily implies that it also contains an embedded one. □

Thanks to the topological observations we made in Remark 2.2, in Theorem 2.1 the case of the sphere is trivial and that of the projective plane is well known. Therefore, in the rest of the section we will suppose that S is neither diffeomorphic to a sphere nor to a projective plane.

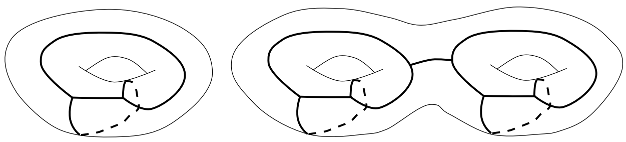


FIGURE 1. A spine of a surface of genus 1 and a surface of genus 2.

DEFINITION 2.4. Let S be a closed Riemannian surface, we say that $\Lambda \subset S$ is a quasi–spine of S if:

- Λ is a closed, connected set with finite length (in particular is rectifiable and connected by embedded rectifiable curves, see [**9**, Theorems 4.4.7 and 4.4.8]);
- Λ intersects every homotopically non–trivial closed curve in S.

REMARK 2.5. Let Γ be a spine of a closed Riemannian surface S. Then Γ intersect every homotopically non–trivial closed curve in S. In particular Γ is a quasi–spine of S.

PROOF. If γ is a homotopically non-trivial closed curve in S, then Γ intersects γ, because γ cannot be contained in the open disc $S \setminus \Gamma$, otherwise it would be homotopically non-trivial. □

REMARK 2.6. Reasoning as in Proposition 2.3, if Λ is a quasi–spine of a closed Riemannian surface S, then $\mathcal{H}^1(\Lambda) \geq c > 0$.

REMARK 2.7. Consider a closed Riemannian surface S, then there exists at least one spine Γ of S of finite length.

We call $\mathcal{A}$ the class of the spines of S, that is networks $\Gamma \subset S$ such that $S \setminus \Gamma$ is homeomorphic to the disc (see Remark 1.3). To prove Theorem 2.1 is more convenient to find first the minimum of the length functional in the larger class $\mathcal{A}'$ of the quasi–spines of S and then prove that the network $\Lambda \in \mathcal{A}'$ that realizes the minimum is a spine.

Proof of Theorem 2.1.

Step 1:: *Existence of quasi–spines of minimal length.*
We call
$$\ell = \inf\left\{\mathcal{H}^1(\Lambda) \mid \Lambda \in \mathcal{A}'\right\}. \tag{2.2}$$
We want to prove that ℓ is actually a minumum.

Let Λ be a *minimizing sequence* of quasi–spines, that is a sequence of closed, connected and rectifiable sets that intersect every homotopically non–trivial closed curve in S such that $\mathcal{H}^1(\Lambda) \to \ell$.

Thanks to Blaschke Theorem [**9**, Theorem 4.4.15], up to passing to a subsequence, the sequence Λ converges to a compact set Λ_∞ in the Hausdorff distance, and by Golab Theorem (see [**78**] for the generalization in a metric setting) we get that Λ_∞ is connected and

$$\mathcal{H}^1(\Lambda_\infty) \leq \liminf_n \mathcal{H}^1(\Lambda_n) = \ell\,. \tag{2.3}$$

The set Λ_∞ intersects every homotopically non-trivial closed curve γ of S: indeed for every n there exists $x_n \in \gamma \cap \Lambda_n$ (because Λ_n belongs to $\mathcal{A}'$). By the compactness of γ, possibly passing to subsequence, x_n converge to some $x_\infty \in \gamma$. On the other hand, since Λ_n converge to Λ_∞ in the Hausdorff distance and x_n belongs to Λ_n, then, by a well–known property of Hausdorff convergence, $x_\infty \in \Lambda_\infty$ and this proves that Λ_∞ intersects γ.

Hence we have proved that $\Lambda_\infty \in \mathcal{A}'$ and ℓ is a minimum.

Step 2: *Regularity of* Λ_∞.
We can associate to Λ_∞ a rectifiable varifold with multiplicity 1, and the minimality property implies that this varifold is stationary. Indeed it suffices to notice that $\mathcal{A}'$ is invariant under diffeomorphisms of S, and therefore, for every one parameter family of diffeomorphisms Φ_t there holds $\mathcal{H}^1(\Phi_t(\Lambda_\infty)) \geq \mathcal{H}^1(\Lambda_\infty)$, which implies the stationarity of Λ_∞.

Using the stationarity of Λ_∞ and regularity of 1–dimensional stationary varifolds (see [**4**]) we obtain that Λ_∞ is composed by a finite number of geodesic segments, joining finitely many multiple junctions where the sum of the concurring unit tangent vectors is equal zero. By standard arguments, the minimality (and not the stationarity) of Λ_∞ also implies that each junction has order three: this can be easily proved by reduction to the planar case (e.g., by using the exponential map) where this result is well known.

Step 3: *The set* Λ_∞ *is in the class* $\mathcal{A}$.
Let $\mathcal{S}$ be the set of all closed subsets of Λ_∞ that intersect every homotopically non-trivial closed curve in S. By Zorn's lemma there is a $\Lambda \in \mathcal{S}$ which is minimal with respect to inclusion. We now prove that Λ is a spine.

Let us divide the proof in different parts:

(1) *Let U be a connected componet of $S \setminus \Lambda$, then U is simply connected.*
Suppose by contradiction that U is not simply connected. Then U contains a simple, closed curve $\bar\gamma$ homotopically non-trivial in U, but homotopically trivial in S because $\bar\gamma$ does not intersect Λ. Hence $\bar\gamma$ is a boundary of a disc $D \subset S$ and $D \not\subset U$. We claim that $D \cap \Lambda \neq \emptyset$, indeed if $D \cap \Lambda = \emptyset$, then $U \cup D$ should be a connected set in $S \setminus \Lambda$, contradicting the fact that U is a connected component of $S \setminus \Lambda$. Let us consider $\widetilde\Lambda = \Lambda \setminus D$, to conclude we show that $\widetilde\Lambda$ is an element of $\mathcal{S}$ we obtain a contradiction ($\widetilde\Lambda$ is an element of $\mathcal{S}$ strictly contained in Λ, that was minimal for the inclusion). Indeed it suffices to show that every curve γ homotopically non–trivial in S intersects Λ out of D. Suppose not, then we can deform with an homotopy γ into a new curve which does not intersect D and agrees with γ out of D and therefore does not intersect Λ at all.

(2) *The connected component U is not a sphere.*
Suppose by contradiction that U is a sphere.
If A and B are two topological manifolds, non empty, closed and connected, and $e : A \to B$ is an embedding of A in B, then e is an homeomphism. In our case one may consider U as A and S as B and conclude that U is homeomorphic to S, a contradiction.

(3) *The connected component U is a disc.*
By the fact that any open, simply connected subset of a surface is homeomorphism to a sphere or to a disc and we have just excluded that U is a sphere, then U is a disc.

(4) *The set $S \setminus \Lambda$ is connected, that is is a single disc.*

If $S \setminus \Lambda$ consists of at least two open discs, there is an arc in Λ which lies in the boundary of both connected component. By removing from Λ a suitable open sub-arc in this arc we get again another element of $\mathcal{S}$ strictly contained in Λ (by an argument similar to the one in (1)).

(5) *The set Λ coincide with Λ_∞, hence $\Lambda_\infty \in \mathcal{A}$.*

We know that $\Lambda \subset \Lambda_\infty$ is a spine, and hence Λ coincide with Λ_∞, otherwise the length of Λ would be strictly smaller than that of Λ_∞, a contradiction.

□

DEFINITION 2.8. A spine of a Riemannian surface S is *minimal* if it is a point, a closed geodesic, or if it is composed by finitely many geodesic arcs, meeting with angle $\frac{2\pi}{3}$ at 3–points.

We have shown that a spine of minimal length is minimal. Of course, the converse may not hold. However, a minimal spine is a stationary point of the length functional thanks to (1.3).

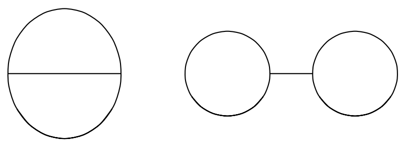
REMARK 2.9. If the surface S is neither diffeomorphic to a sphere nor to a projective plane, we have shown that minimal spines are trivalent graphs. Hence, adding the equation $3v = 2e$ to (2.1), we get that the number of edges and that of vertices of a minimal spine are completely determined by the topology of S.

REMARK 2.10. Notice that the extension of the existence result in higher dimension present several difficulties: there is no higher dimensional version of Golab Theorem, because of the lack of semicontinuity of the Hausdorff measure. Also, it is not clear if a limit of spines is still a spine. However, as we already observed, the existence of a spine of minimal area in a closed irreducible 3-manifold has been proved in [**24**].

3. Flat tori

In this section, we analyze minimal spines of a closed surface of genus 1: the torus $T = S^1 \times S^1$. In particular, we will fully determine all the minimal spines on T, endowed with any Euclidean metric.

3.1. Minimal spines of Riemannian tori. From Remark 2.9, for any Riemannian metric on the torus T, minimal spines have exactly 2 vertices and 3 edges. There are only two kinds of graph satisfying these properties: the θ-graph and the eyeglasses (see Figure 2).



$a)$ θ-graph $\qquad$ $b)$ eyeglasses

FIGURE 2. The two trivalent graphs with three edges.

Both graphs can be embedded in a torus, but only the first one gives a spine.

Indeed, it is easy to find a θ-spine in the torus (see Figure 3-a), and actually infinitely many non isotopic ones. Consider, instead, the eyeglasses as an abstract graph. Thickening the interiors of its edges, we get three bands. To have a spine on T, it remains to attach them, getting a surface homeomorphic to $T - B^2$. This is impossible. The band corresponding to one of the two "lenses" of the eyeglasses has to be glued to itself. The are only two ways to do this: one would give an unorientable surface (see Figure 3-b), the other would have too many boundary components (see Figure 3-c).

We have shown the following:

PROPOSITION 3.1. *Trivalent spines on the torus are θ-graphs.*

Combined with Theorem 2.1, we get the following

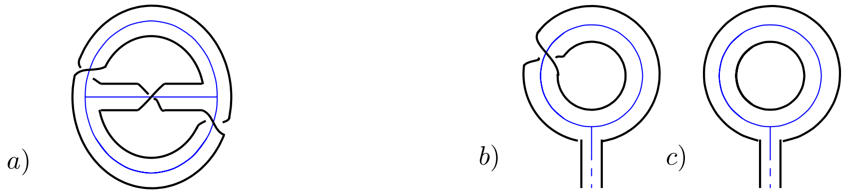


FIGURE 3. a) A regular neighbourhood of a θ-spine on T. $b) - c$) An eyeglasses spine on T does not exist.

THEOREM 3.2. *On every Riemannian torus, minimal spines (exist and) are embedded θ-graphs with geodesic arcs, forming angles of $\frac{2\pi}{3}$ at their two triple points.*

3.2. Basics on flat tori. From now, we consider only constant curvature Riemannian metrics on T. These correspond to flat (i.e., Euclidean) structures on T. We give a very informal introduction to the basic concepts to fix the notations, and refer the reader to **[34]** for more details.

Let us define

- the *Teichmüller space* $\mathcal{T}$ of the torus as the space of isotopy classes of unit-area flat structures on T,
- the *mapping class group* Mod of the torus as the group of isotopy classes of orientation preserving self-diffeomorphisms of T.
- the *moduli space* $\mathcal{M}$ of the torus as the space of oriented isometry classes of unit-area flat structures on T.

The group Mod acts on the set $\mathcal{T}$ and $\mathcal{M}$ is the quotient by this action.

The set $\mathcal{T}$ is endowed with a natural topology, which makes it homeomophic to a plane. With this topology, the group Mod acts on $\mathcal{T}$ properly discontinuously and with finite stabilizers. In this way, the projection $\pi : \mathcal{T} \to \mathcal{M}$ is an orbifold universal covering. The orbifold fundamental group $\pi_1\mathcal{M}$, acting on this covering by deck transformations, is isomorphic to the quotient of the group Mod by the kernel of the action (a normal subgroup of order two). In this way, the moduli space $\mathcal{M}$ is homeomorphic to the $(2, 3, \infty)$-orbifold. It has two singular points, one of order two and one of order three, and its underlying space is homeomorphic to a plane.

The set $\mathcal{T}$ is also endowed with a natural metric (the Teichmüller metric), which makes it isometric to hyperbolic plane $\mathbb{H}^2$. The action of the mapping class group can be identified with a discrete action by isometries.

Visualizing the hyperbolic plane with the upper-half space model $\{\Im(z) > 0\} \subset \mathbb{C}$, the action is given by integer Möbius transformations and the mapping class group is isomorphic to the group $SL_2(\mathbb{Z})$ of unit-determinant 2×2 integral matrices. The kernel of the action is $\{\pm I\}$, so that the moduli space $\mathcal{M}$ has the structure of a hyperbolic orbifold, with $\pi_1\mathcal{M} \simeq \mathrm{PSL}_2(\mathbb{Z})$. A fundamental domain for the action is the hyperbolic semi-ideal triangle

$$D = \big\{|z| \geq 1, |\Re(z)| \leq 1\big\},$$

with angles $\frac{\pi}{3}$, $\frac{\pi}{3}$ and 0 (in grey in Figure 4). The quotient $\mathcal{M}$ is the complete $(2, 3, \infty)$-hyperbolic orbifold of finite area, with one cusp and two conical singularities of angles π and $\frac{2\pi}{3}$. We see $\mathcal{M}$ as usual as D with the boundary curves appropriately identified.

From now, we identify the actions (groups, spaces and quotients):

$$Mod \curvearrowright \mathcal{T} \;\;\leftrightarrow\;\; \{\textit{integer Möbius transformations}\} \curvearrowright \mathbb{H}^2 \;\;\leftrightarrow\;\; SL_2(\mathbb{Z}) \curvearrowright \mathbb{C}^+ = \{Im(z) > 0\},$$

$$\mathcal{M} \equiv \mathbb{H}^2/PSL_2(\mathbb{Z}).$$

We will also be interested in the following space:

- the *non-oriented moduli space* $\mathcal{M}^{\mathrm{no}}$ is the space of all isometry classes of (unoriented) unit-area flat structures on T.

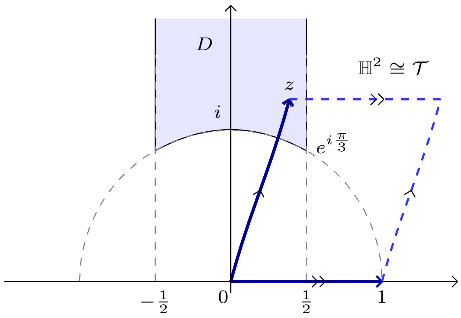


FIGURE 4. Every $z \in \mathbb{H}^2$ in the upper half-plane represents a flat torus obtained by identifying the opposite edges of the parallelogram with vertices $0, 1, z, z+1$. A fundamental domain D for the action of $\mathrm{PSL}_2(\mathbb{Z})$ is colored in grey.

We have $\mathcal{M}^{\mathrm{no}} = \mathcal{M}/_\iota$ where ι is the isometric involution that sends an oriented flat torus to the same torus with opposite orientation: the lift of ι to the fundamental domain D is the reflection with respect to the geodesic line $i\mathbb{R}^+$ and $\mathcal{M}^{\mathrm{no}}$ is the hyperbolic triangle orbifold

$$\mathcal{M}^{\mathrm{no}} = D \cap \{\Re(z) \geq 0\}$$

with angles $\frac{\pi}{2}$, $\frac{\pi}{3}$ and 0. We do not need to fix an orientation on T to talk about spines, hence we mainly work with $\mathcal{M}^{\mathrm{no}}$.

Every flat torus T has a continuum of isometries: the *translations* by any vector in $\mathbb{R}^2$ and the *reflections* with respect to any point $x \in T$. The tori lying in the mirror sides of $\mathcal{M}^{\mathrm{no}}$ have special names and enjoy some additional isometries:

- the *rectangular tori* are those lying in $i\mathbb{R}^+$,
- the *rhombic tori* are those in the other sides of $\mathcal{M}^{\mathrm{no}}$, namely $\{|z| = 1\} \cup \{\Re(z) = \frac{1}{2}\}$.

These flat tori are obtained by identifying the opposite sides of a rectangle and a rhombus, respectively. The tori in the cone points $z = i$ and $e^{\frac{\pi i}{3}}$ are the *square torus* and the *hexagonal torus*. On the hexagonal torus, the length d of the shortest diagonal of the rhombus equals the length l of any of its sides, while we have $d \geq l$ and $d \leq l$ on the sides $|z| = 1$ and $\Re = \frac{1}{2}$ respectively (we can call these rhombi *fat* and *thin*, repsectively).

The rectangular and rhombic tori are precisely the flat tori that admit orientation-reversing isometries.

Teichmüller and moduli spaces have natural Thurston and Mumford-Deligne compactifications. In the torus case, these are obtained respectively by adding the circle "at infinity" $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$ to $\mathcal{T} = \mathbb{H}^2$ and a single point to $\mathcal{M}$ or $\mathcal{M}^{\mathrm{no}}$. We denote the latter compactifications by $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}^{\mathrm{no}}}$.

REMARK 3.3. Let us consider $\mathbb{R}^2$ as the group of translations of the Euclidean plane. By a *lattice* of $\mathbb{R}^2$, we mean a subgroup $\Lambda \leq \mathbb{R}^2$ which is $\mathbb{Z}$-generated by two linearly independent vectors. If we consider the moduli space of the torus from this point of view, we could seen it as the space of homothety and isometry classes of lattices of $\mathbb{R}^2$:

$$\mathcal{M} \longleftrightarrow \{\text{lattices of } \mathbb{R}^2\}/\sim$$

3.3. Hexagons. Minimal spines on a flat torus are intimately related to a particular class of Euclidean hexagons.

DEFINITION 3.4. A *semi-regular hexagon* is a Euclidean hexagon with all internal angles equal to $\frac{2\pi}{3}$ and with congruent opposite sides.

We define the *moduli space* $\mathcal{H}$ as the space of all oriented semi-regular hexagons considered up to homotheties and orientation-preserving isometries. Similarly $\mathcal{H}^{\mathrm{no}}$ is defined by considering non-oriented hexagons and by quotienting by homotheties and all isometries. We get a map $\mathcal{H} \to \mathcal{H}^{\mathrm{no}}$ that is at most 2-to-1.

Two opposite sides of a semi-regular hexagon are parallel and congruent. A semi-regular hexagon is determined up to isometry by the lengths $a, b, c > 0$ of three successive sides, hence we get

$$\mathcal{H} = \big\{(a,b,c) \,|\, a,b,c > 0\big\}/_{\mathbb{R}_{>0}\times A_3}$$
$$\mathcal{H}^{\rm no} = \big\{(a,b,c) \,|\, a,b,c > 0\big\}/_{\mathbb{R}_{>0}\times S_3}$$

where the multiplicative group of positive real numbers $\mathbb{R}_{>0}$ acts on the triples by rescaling, while A_3 and S_3 act by permuting the components.

We can visualize the space $\mathcal{H}^{\rm no}$ in the positive orthant of $\mathbb{R}^3$ by normalizing (a,b,c) such that $a+b+c=1$ and $a \geq b \geq c$, and in this way $\mathcal{H}^{\rm no}$ is a triangle with one side removed (corresponding to $c=0$) as in Fig. 5. The other two sides parametrize the hexagons with $a=b\geq c$ and $a \geq b = c$. The regular hexagon is of course $(\frac13,\frac13,\frac13)$. The space $\mathcal{H}^{\rm no}$ is naturally an orbifold with mirror boundary made by two half-lines and one corner reflector of angle $\frac{\pi}{3}$.

The space $\mathcal{H}$ of oriented semi-regular hexagons is obtained by doubling the triangle $\mathcal{H}^{\rm no}$ along its two edges. Therefore $\mathcal{H}$ is topologically an open disc, and can be seen as an orbifold where the regular hexagon is a cone point of angle $\frac{2\pi}{3}$. The map $\mathcal{H} \to \mathcal{H}^{\rm no}$ may be interpreted as an orbifold cover of degree two.

The orbifold universal covering $\tilde{\mathcal{H}} \to \mathcal{H}$ is homeomorphic to the map $z \mapsto z^3$ from the complex plane to itself. The orbifold fundamental group $\pi_1\mathcal{H}$ is isomorphic to the cyclic group $\mathbb{Z}/3$, acting on the plane by rotations of angle $\frac{2\pi}{3}$.

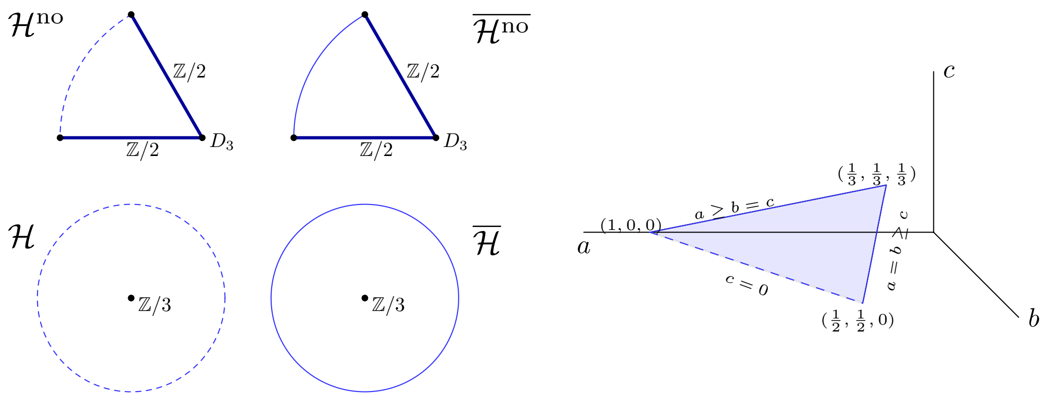


FIGURE 5. On the left, the orbifold structures on the moduli spaces of semi-regular hexagons, and their compactifications. On the right, a parametrization of $\mathcal{H}^{\rm no}$ by a triangle with one side removed in $\mathbb{R}^3$.

Both $\mathcal{H}$ and $\mathcal{H}^{\rm no}$ have natural compactifications, obtained by adding the side with $c=0$, which consists of points $(a, 1-a, 0)$ with $a \in [\frac12, 1]$. Each such point corresponds to some "degenerate" hexagon: the points with $a < 1$ may be interpreted as parallelograms with angles $\frac{\pi}{3}$ and $\frac{2\pi}{3}$, while $(1,0,0)$ should be interpreted as a segment: a *doubly degenerate* hexagon. The underlying spaces of the resulting compactifications $\overline{\mathcal{H}}$ and $\overline{\mathcal{H}^{\rm no}}$ are both homeomorphic to closed discs.

3.4. Spines. We now introduce two more moduli spaces $\mathcal{S}$ and $\mathcal{S}^{\rm no}$ which will turn out to be isomorphic to $\mathcal{H}$ and $\mathcal{H}^{\rm no}$.

Let the *moduli space* $\mathcal{S}$ be the set of all pairs (T,Γ), where T is a flat oriented torus and $\Gamma \subset T$ a minimal spine, considered up to orientation-preserving isometries (that is, $(T,\Gamma) = (T',\Gamma')$ if there is an orientation-preserving isometry $\psi\colon T \to T'$ such that $\psi(\Gamma) = \Gamma'$). We define analogously $\mathcal{S}^{\rm no}$ as the set of all pairs (T,Γ) where T is unoriented, quotiented by all isometries. Again we get a degree two orbifold covering $\mathcal{S} \to \mathcal{S}^{\rm no}$.

The opposite edges of a semi-regular hexagon H are congruent, and by identifying them we get a flat torus T. The boundary ∂H of the hexagon transforms into a minimal spine $\Gamma \subset T$ in the gluing process. This simple operation define two maps

$$\mathcal{H} \longrightarrow \mathcal{S}, \qquad \mathcal{H}^{\mathrm{no}} \longrightarrow \mathcal{S}^{\mathrm{no}}.$$

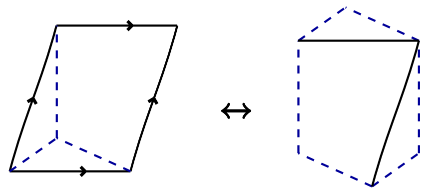


FIGURE 6. The bijection $\mathcal{H} \longleftrightarrow \mathcal{S}$.

PROPOSITION 3.5. *Both maps are bijections.*

PROOF. The inverse map is the following: given (T, Γ), we cut T along Γ and get the original semi-regular hexagon H. □

We will therefore henceforth identify these moduli spaces and use the symbols $\mathcal{H}$ and $\mathcal{H}^{\mathrm{no}}$ to denote the moduli spaces of both semi-regular hexagons and pairs (T, Γ). Of course our aim is to use the first (hexagons) to study the second (spines in flat tori).

3.5. The forgetful maps. The main object of this section is the characterization of the forgetful maps

$$p\colon \mathcal{S} \longrightarrow \mathcal{M}, \qquad p\colon \mathcal{S}^{\mathrm{no}} \longrightarrow \mathcal{M}^{\mathrm{no}}$$

that send (T, Γ) to T forgetting the minimal spine Γ. The fiber $p^{-1}(T)$ over an oriented flat torus $T \in \mathcal{M}$ can be interpreted as the set of all minimal spines in T, considered up to orientation-preserving isometries of T. Likewise the fiber over an unoriented torus $T \in \mathcal{M}^{\mathrm{no}}$ is the set of minimal spines in T, considered up to all isometries.

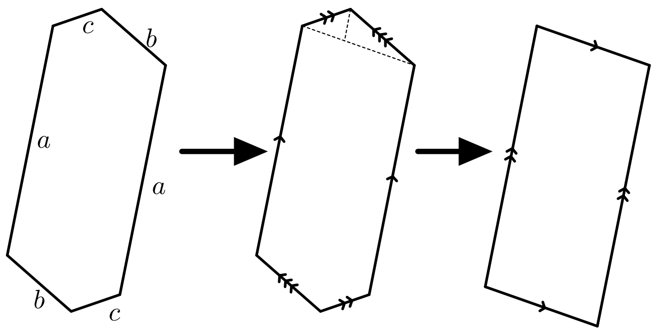


FIGURE 7. The composition $p\colon \mathcal{H} \to \mathcal{S} \to \mathcal{M}$.

As we said above, we identify $\mathcal{H}$, $\mathcal{H}^{\mathrm{no}}$ with $\mathcal{S}$, $\mathcal{S}^{\mathrm{no}}$ and consider the compositions (which we still name by p)

$$p\colon \mathcal{H} \xrightarrow{\sim} \mathcal{S} \longrightarrow \mathcal{M}, \qquad p\colon \mathcal{H}^{\mathrm{no}} \xrightarrow{\sim} \mathcal{S}^{\mathrm{no}} \longrightarrow \mathcal{M}^{\mathrm{no}}.$$

The map p is described geometrically in Fig. 7.

As we want to compute the cardinality of $p^{-1}(T)$, we need an explicit expression for the map $p\colon \mathcal{S}^{\mathrm{no}} \longrightarrow \mathcal{M}^{\mathrm{no}}$, to this aim we introduce in Proposition 3.6 the lift $\widetilde{p}\colon \mathcal{H}^{no} \longrightarrow \mathbb{H}^2$.

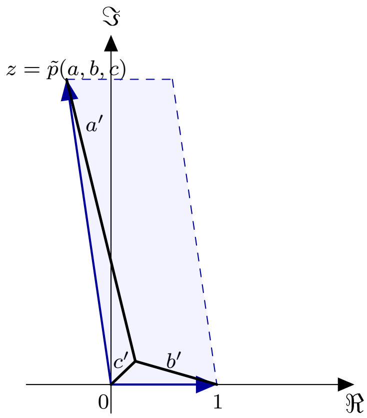


FIGURE 8. How to construct z from (a', b', c'): we pick the triod with angles $\frac{2\pi}{3}$ and lengths a', b', c', and place it in the upper half-plane so that the endpoints of the edges c' and b' lie in 0 and 1. The point z is the endpoint of the edge a'.

$$\begin{tikzcd}
\mathcal{S}^{no} \arrow[r, leftrightarrow, "\sim"] & \mathcal{H}^{no} \arrow[r, "\tilde{p}"] \arrow[dr, "p"'] & \mathbb{H}^2 \equiv \mathbb{C}^+ \arrow[d, "\pi"] \\
& & \mathcal{M}^{no}
\end{tikzcd}$$

PROPOSITION 3.6. *The map*

$$\tilde{p} \colon \mathcal{H}^{\text{no}} \longrightarrow \mathbb{H}^2$$
$$(a, b, c) \longmapsto \frac{2c^2 - ab + ac + bc}{2(b^2 + c^2 + bc)} + i\frac{\sqrt{3}}{2}\frac{ab + bc + ac}{b^2 + c^2 + bc}$$

is a lift of $p \colon \mathcal{H}^{\text{no}} \to \mathcal{M}^{\text{no}}$. *The map* $\tilde{p}$ *is injective and sends homeomorphically the triangle* $\mathcal{H}^{\text{no}}$ *onto the light grey domain drawn in Fig. 9.*

PROOF. The map $\tilde{p}$ is defined in Fig. 8. We first rescale the triple (a, b, c), which by hypothesis satisfies $a \geq b \geq c$ and $a + b + c = 1$, to (a', b', c'), by multiplying each term by $1/\sqrt{b^2 + c^2 + bc}$. Now we pick the *triod* with one vertex and three edges of length a', b', c' and with angles $\frac{2\pi}{3}$, and we place it in the half-space with two vertices in 0 and 1 as shown in the figure (we can do this thanks to the rescaling). The third vertex goes to some $z \in \mathbb{H}^2$ and we set $\tilde{p}(a, b, c) = z$.

The map $\tilde{p}$ is clearly a lift of p and it only remains to determine an explicit expression for $\tilde{p}$. Applying repeatedly the Carnot Theorem, we get:

$$\operatorname{Arg}(z) = \cos^{-1} \frac{2c^2 - ab + ac + bc}{2\sqrt{(b^2 + c^2 + bc)\,(a^2 + c^2 + ac)}},$$
$$|z| = \sqrt{\frac{a^2 + c^2 + ac}{b^2 + c^2 + bc}}.$$

Finally,

$$\tilde{p}(a, b, c) = z = \frac{2c^2 - ab + ac + bc}{2(b^2 + c^2 + bc)} + i\frac{\sqrt{3}}{2}\frac{ab + bc + ac}{b^2 + c^2 + bc}.$$

The proof is complete. □

Fig. 9 shows that the lift $\tilde{p}$ sends the two sides $a = b \geq c$ and $a \geq b = c$ of the triangle $\mathcal{H}^{\text{no}}$ to geodesic arcs in $\mathbb{H}^2$. Recall that the compactification $\overline{\mathcal{H}^{\text{no}}}$ is obtained by adding the singly-degenerate parallelograms $(a, 1 - a, 0)$ with $a \in [\frac{1}{2}, 1)$ and the doubly-degenerate $P = (1, 0, 0)$. The map $\tilde{p}$ also extends to the parallelograms, and sends them to the dashed line in Fig. 8, but it does *not* extend continuously to P, not even as a map from $\overline{\mathcal{H}^{\text{no}}}$ to $\overline{\mathbb{H}^2}$. However, the map p from moduli spaces does extend.

PROPOSITION 3.7. *The map* $p \colon \mathcal{H}^{\text{no}} \to \mathcal{M}^{\text{no}}$ *extends continuously to a map* $p \colon \overline{\mathcal{H}^{\text{no}}} \to \overline{\mathcal{M}^{\text{no}}}$.

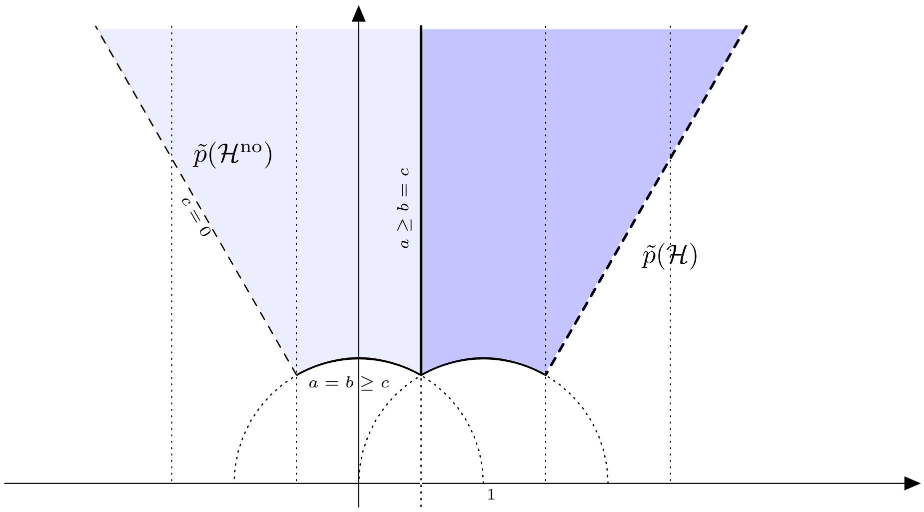


FIGURE 9. The lift $\tilde{p}$ sends $\mathcal{H}^{\mathrm{no}}$ homeomorphically to the light blue domain on the left. It sends the two sides $a \geq b = c$ and $a = b \geq c$ of $\mathcal{H}^{\mathrm{no}}$ to two geodesic lines in $\mathbb{H}^2$, and sends the line at infinity $c = 0$ to the constant-curvature line dashed in the figure. In the oriented setting, the lift $\tilde{p}$ sends $\mathcal{H}$ to the whole bigger blue domain but is discontinuous on the segment s of hexagons of type $a = b \geq c$.

PROOF. Send the doubly degenerate point P to the point at infinity in $\overline{\mathcal{M}}$. $\square$

The oriented picture is easily deduced from the non-oriented one. We can lift the map $p\colon \mathcal{H} \to \mathcal{M}$ to a map $\tilde{p}\colon \mathcal{H} \to \mathbb{H}^2$ in Teichmüller space whose image is the bigger (both light and dark) blue domain in Fig. 9, however the map $\tilde{p}$ is discontinuous at the segment consisting of all hexagons with $a = b \geq c$, which is sent to one of the two curved geodesic arcs in the picture. The map $p\colon \mathcal{H} \to \mathcal{M}$ extends continuously to a map $p\colon \overline{\mathcal{H}} \to \overline{\mathcal{M}}$.

The Möbius transformation $z \mapsto \frac{2iz+\sqrt{3}-i}{2z-1+\sqrt{3}i}$ gives a more symmetric picture of the image $\tilde{p}(\mathcal{H})$ inside the hyperbolic plane in the Poincaré disc model, as can be seen in Figure 10.

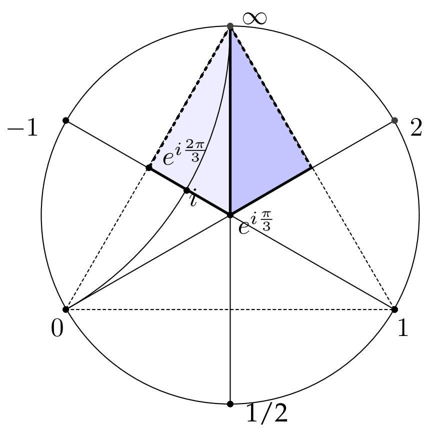


FIGURE 10. A picture of $\tilde{p}(\mathcal{H})$ in the Poincaré disc model.

REMARK 3.8. Now, it is possible to quantify the "distance" between two spines on different flat tori, in a natural way. Indeed, thanks to the map p and the Teichmüller metric, the moduli space of spines can be endowed with a structure of hyperbolic orbifold.

For, identify $-$ by a representation ρ $-$ the action of the orbifold fundamental group $\pi_1\mathcal{H}$ on the universal covering $\tilde{\mathcal{H}}$ with the action on the hyperbolic plane of the group Γ, generated by an order-three elliptic rotation about the point $e^{i\frac{\pi}{3}}$. To get a developing map $d : \tilde{\mathcal{H}} \to \mathbb{H}^2$ for the hyperbolic structure on $\mathcal{H}$, lift ρ-equivariantly the map p.

The developed image $d(\tilde{\mathcal{H}})$ of $\mathcal{H}$ is the interior in $\mathbb{H}^2$ of the regular non-geodesic ideal triangle joining the points ∞, 0, 1 in Figure 10. The set $\tilde{p}(\mathcal{H})$ is a fundamental domain for the action of Γ and the space $\mathcal{H}$ is identified with the quotient $d(\tilde{\mathcal{H}})/\Gamma$.

Hence, the moduli space of spines is an infinite-area incomplete hyperbolic orbifold. Its completion, supported on the complement of the doubly degenerate hexagon $\overline{\mathcal{H}} \setminus \{P\}$, is an orbifold with non-geodesic boundary. Its boundary is an infinite constant-curvature line whose points are (all the) four-valent geodesic spines with alternating angles $\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

Similarly, $\overline{\mathcal{H}^{\rm no}} \setminus \{P\}$ is a complete infinite-area hyperbolic orbifold with (unbounded, non-geodesic) boundary and two mirror edges.

3.6. The number of minimal spines. We are ready to determine the fiber $p^{-1}(T)$ for every flat torus T in $\mathcal{M}$ and $\mathcal{M}^{\rm no}$, which may be interpreted as the set of all minimal spines in T, considered up to (orientation-preservingly or all) isometries of T.

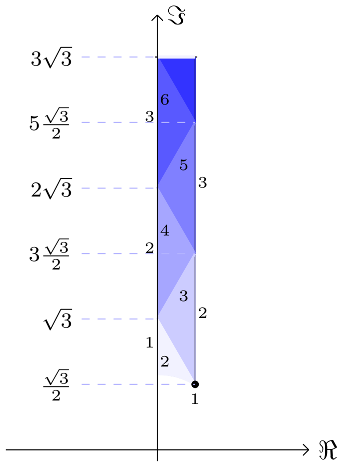


FIGURE 11. The number of minimal spines on each flat torus, in the unoriented and oriented setting. At each point z in $\mathcal{M}^{\rm no}$ or $\mathcal{M}$, the number $c^{\rm no}(z)$ or $c(z)$ is the smallest number among all numbers written on the adjacent strata (both functions are lower semi-continuous).

THEOREM 3.9. *Every unoriented flat torus T has a finite number $c^{\rm no}(T)$ of minimal spines up to isometries of T. Analogously, every oriented flat torus T has a finite number $c(T)$ of minimal spines up to orientation-preserving isometries. The proper functions $c^{\rm no}\colon \mathcal{M}^{\rm no} \to \mathbb{N}$ and $c\colon \mathcal{M} \to \mathbb{N}$ are shown in Fig. 11.*

PROOF. We can identify $\mathcal{H}^{\rm no}$ with its image $H = \tilde{p}(\mathcal{H}^{\rm no})$ and note that the restriction of the projection $\pi\colon \mathbb{H}^2 \to \mathcal{M}^{\rm no}$ to H is finite-to-one. Indeed, for $z, z' \in H$, $\pi(z) = \pi(z')$ if and only if $\Im z' = \Im z$ and $\Re z' = \pm\Re z + n$ for some integer n. The number $c^{\rm no}(z)$ is the cardinality of the fiber $p^{-1}(z)$ and is easily shown to be as in Fig. 11-(left).

The oriented case is treated analogously: in that case $\tilde{p}(\mathcal{H})$ is the bigger grey zone in $\mathbb{H}^2$ and $p(z) = p(z')$ if and only if $z' = z + n$ for some $n \in \mathbb{Z}$. We get the picture in Fig. 11-(right). $\square$

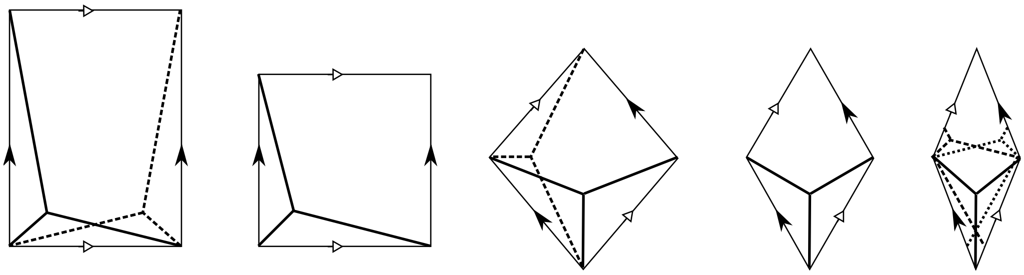


FIGURE 12. The minimal spines in some oriented tori having additional symmetries. As we pass from rectangular to thin rhombic, we find 2, 1, 2, 1, and then 3 spines up to orientation-preserving isometries. They reduce to 1, 1, 2, 1, 2 up to all isometries, including orientation-reversing ones. Rectangles and rhombi that are thinner (ie longer) than the ones shown here have additional minimal spines that wind around the thin part.

The subtler cases are those where T has additional symmetries: in Fig. 12 we show the minimal spines of T as the metric varies from rectangular to square, fat rhombic, hexagonal, and finally thin rhombic.

3.7. Length. Let us define the *length* of a union of curves on a flat torus in the moduli space $\mathcal{M}$ as its normalized length, that is its one dimensional Hausdorff measure on a unit-area representative torus. For example, if you consider the explicit representative given by the parallelogram $\mathbb{R}$-generated by 1 and $z \in \mathbb{C}$, you have to divide lengths by the square root of the area of that parallelogram, that is $\sqrt{\Im(z)}$.

Generalizing the well known concept for closed geodesics on surfaces (see [**34**]), we define the *length spectrum of minimal spines* of a flat torus T as the set of lengths of its minimal spines. Actually, we can define a more accurate spectrum, a set of triples corresponding to the lengths of the sides of any spine on T, in decreasing order, that is

$$\left\{ \frac{(a,b,c)}{\sqrt{(b^2+c^2+bc)\Im(\tilde{p}(a,b,c))}} \,\middle|\, (a,b,c) \in p^{-1}(T) \right\}.$$

We have shown that both sets are finite, for any flat torus T.

In the following, we determine the spines of minimal length.

THEOREM 3.10. *Every unoriented flat torus has a unique spine of minimal length up to isometry. In the oriented setting, instead, the same holds with the exception of the rectangular non square tori, for which there are exactly two.*

PROOF. Fix a torus $T \in \mathcal{M}^{\text{no}}$. In the proof of Theorem 3.9, we observed that all points of $\pi^{-1}(T) \cap H \subset \mathbb{H}^2$ have the same imaginary part, so the lengths of the tripods we found with the map $\tilde{p}$ can be compared without need of normalization. Clearly, only one is the shortest: that corresponding to $z_0 \in D \cap \{\Re(z) \geq 0\}$.

In the oriented case, consider the representative z_0 of $T \in \mathcal{M}$ with $z_0 \in D \setminus \{z \in \partial D \,|\, \Re(z) < 0\}$. There are three cases:

- if $\Re(z_0) > 0$, the unique shortest spine is that associated to z_0 itself;
- if $\Re(z_0) < 0$, it is associated to $z_0 + 1$;
- if $\Re(z_0) = 0$, that is T is a rectangular torus, both spines associated to z_0 and $z_0 + 1$ are of minimal length. These two spines are the same only for the square torus, indeed a rotation of $\frac{\pi}{2}$ sends one to the other.

These evident assertions can be verified by trigonometry, or by computing the lengths through the formula showed in the following . □

The explicit expression on $\tilde{p}(\mathcal{H}) \subset \mathbb{H}^2$ for the *length function* $L\colon \mathcal{H} \to (0, +\infty)$, which to a spine assigns its length, can be simply computed by the law of sines:

$$L(z) = \sqrt{\frac{1 + |z|^2 - \Re(z) + \sqrt{3}\Im(z)}{\Im(z)}}.$$

The formula for the *spine systole* $\mathrm{S}\colon \mathcal{M} \to (0, +\infty)$, a generalized systole which assigns to a flat torus the length of its shortest spines is

$$\mathrm{S}(z) = \sqrt{\frac{1 + |z|^2 - |\Re(z)| + \sqrt{3}\Im(z)}{\Im(z)}}.$$

Both functions are proper and almost everywhere smooth. They extend to continuous functions $L : \overline{\mathcal{H}} \to (0, +\infty]$ and $\mathrm{S} : \overline{\mathcal{M}} \to (0, +\infty]$.

For every continuous function $f : X \to \mathbb{R}$ on a topological 2-manifold X, recall that a point $p \in X$ is said to be *regular* if there is a (topological) chart around $p = 0$ in which $f(x, y) = \text{const}+x$. Otherwise, it is *critical*. A critical point $p \in X$ is said to be *non degenerate* if in a local (topological) chart around $p = 0$ we have either $f(x, y) = \text{const} - x^2 + y^2$, or $f(x, y) = \text{const} \pm (x^2 + y^2)$. The non degenerate critical points are necessarily isolated. The function f is *topologically Morse* if it is proper and all critical points are non degenerate. The classical Morse theory works also in the topological category.

REMARK 3.11. The functions $L\colon \mathcal{H} \to (0, +\infty)$ and $\mathrm{S}\colon \mathcal{M} \to (0, +\infty)$ are topologically Morse. For both functions, the set of sublevel k is empty if $k < \sqrt[4]{3}\sqrt{2}$, a point if $k = \sqrt[4]{3}\sqrt{2}$ and a topological disc otherwise. Hence, the unique critical point is a minimum. Therefore, among all minimal spines on all flat tori, exactly one is the shortest. As expected, it is the equilateral spine on the hexagonal torus. Its length is $\sqrt[4]{3}\sqrt{2} \approx 1.86$.

3.8. A direct computation of the number of minimal spines. Although we have already compute the number of minimal spines in any flat torus, up to translations and reflections, in this subsection we present our first proof. We restrict to the oriented setting, we construct directly some minimal θs, obtaining a lower bound for the number of minimal spines, and then we show that our construction is the only admissible one.

This method is less smart than the previous, because there is no way to generalize it to the case of other surfaces.

It could anyway be interesting because it is required a very poor knowledge of topology, for example we do not really need to introduce the definition of Teichmüller space and moduli space of the tori, but it is enough having an intuitive idea of them, that we state now.

A flat torus is the quotient of the Euclidean plane $\mathbb{R}^2 \equiv \mathbb{C}$ by the group G generated by two translations with respect to two $\mathbb{R}-$linearly independent vectors ($G \cong \mathbb{Z}^2$). A fundamental domain for such an action is an Euclidean parallelogram with sides corresponding to the two vectors. Up to Euclidean isometries and homotheties we can impose that one of the two vectors is $(1, 0) \in \mathbb{C}$ and the other vector $z \in \mathbb{C}$ has imaginary part greater than zero. The complex number z will parametrize isotopy classes of flat tori up to homothety, i.e. it is a point of the Teichmüller space of the torus $z \in \mathcal{T}$. Hence, up to translation, rotation and dilation we can identify a torus T, with a point $z \in \mathbb{C}^+$. The corresponding orientation-preserving isometry (and homothety) class of that flat torus will be denoted again by $z = \pi(z) \in \mathcal{M}$. Different representatives z of the same isometry class in $\mathcal{M}$ are obtained one from another by finite sequences of *Dehn twists*, that is to say, the mapping class group is generated by Dehn twists.

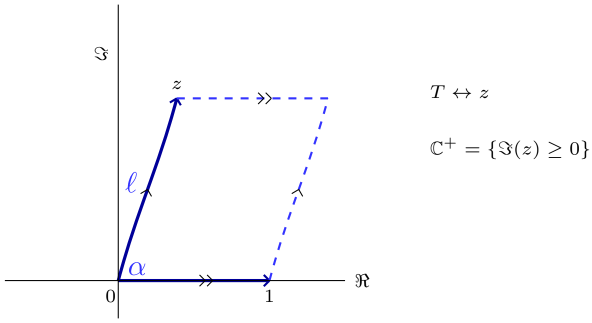


FIGURE 13.

To resume, every flat torus is thus described with two parameters, an angle α and a length ℓ, that uniquely identify a point $z \in \mathbb{C}^+$. The only thing that we have to keep in mind from the theory of moduli space of tori, is that it is exhaustive to consider $\frac{\pi}{3} \leq \alpha \leq \frac{2\pi}{3}$ and $\ell \geq 1$, because they parametrize all possible tori, up to rescaling and up to oriented isometries. To visualize the torus, we associated to the point $z = (\ell, \alpha) \in \mathbb{C}^+$ the parallelogram with vertices 0, 1, z and $z+1$. In the sequel we fix a torus, that is the associated parallelogram, and we compute the number of minimal spine of it. We remind that all the translations and reflections through a point of a minimal spines are still minimal spines, hence we will count the number of them up to these two types of transformation. Thanks to this fact we can always put one of the two vertices of the spines in the vertices of the parallelogram (that are all identified). We notice that if we look at the spines of a torus in the parallelogram that represent it and these spines have a vertex placed in the vertices of the parallelogram we can distinguish two type of them: the spines which "turn" around the parallelogram and the tight ones (see Figure 14). We first compute the number of tight spines and then we consider spines which turn around the parallelogram.

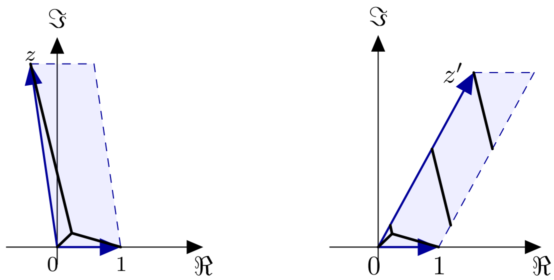


FIGURE 14. A tight spine of the torus z and a spine that "turns" around the torus z'

PROPOSITION 3.12. *Any flat torus, considered up to rescaling and up to oriented isometries, represented by the associated parallelogram, admits two tight θs, up to translations and reflections.*

PROOF. For simplicity, without lost of generality, we put one of the two triple junction in the vertices of the parallelogram. At this point find the tight minimal θs reduce to connect with the Steiner configuration three vertices. As we represent a flat torus with a parallelogram with angles of amplitude less or equal then $\frac{2\pi}{3}$, we can find two different Steiner configurations (see Figure (15)), connecting the vertices 0, 1 and z and 0, 1 and $z+1$ respectively. Notice that the other possible Steiner configuration are reflections with respect to a point of the previous. $\square$

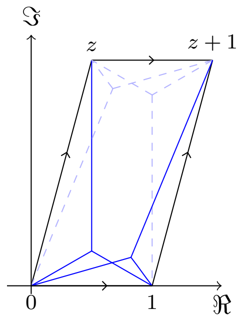
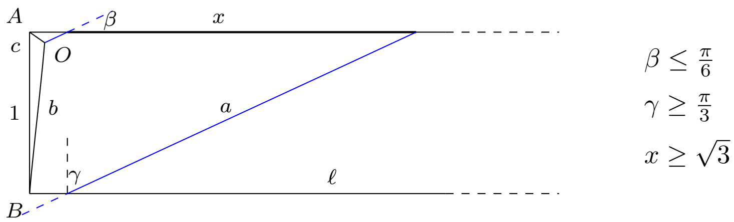


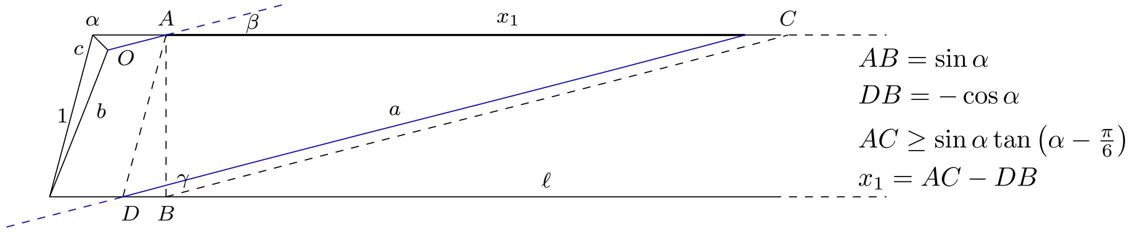
FIGURE 15. The possible Steiner configurations between three vertices of the parallelogram.

Now we pass to count the number of minimal spines "turning" around the parallelogram which represent the torus. First we consider the flat torus modeled by a rectangle ($\alpha = \frac{\pi}{2}$) with sides of lengths 1 and $\ell \geq 1$. We want to construct the minimal spine composed by three segments a, b and c with the segment a that makes the greatest number of "turns" around the torus; if we visualize the torus with a rectangle this means that we want to compute how many intersections there are between the segment a and the long side l of the rectangle. We put one of the two triple junctions in the vertices of the rectangle. In order to maximize the number of intersections, we put the other triple junction very near the vertex A and we connect it with the two nearest vertices with two segments b and c. As the angles between the three segments are of $\frac{2\pi}{3}$ (because the spine is minimal) the segment a intersects the boundary of the rectangle with a prescribed angle $\beta \leq \frac{\pi}{6}$.



We can easily compute the length between two consecutive intersection of the segment a with the same side of the parallelogram: $x \geq \sqrt{3}$.
Hence the maximum number of turns that the segment a can do around the torus, depending by the length l, that is the maximum number of intersections with the side ℓ, is $N_1 = \frac{\ell}{x} = \frac{\ell}{\sqrt{3}}$.
We can estimate the number of spines composed by only one segment that does turns around the torus with $\lfloor N_1 \rfloor$.
Now we notice that we can construct spines, which are different from the precedents, putting the second triple junction near the opposite vertex B of the rectangle. The number of these spines can again be estimated by $\lfloor N_2 \rfloor = \lfloor N_1 \rfloor = \lfloor \frac{\ell}{\sqrt{3}} \rfloor$
We can conclude that the number of minimal spines which turn around the torus represented by a rectangle is $N \geq \lfloor \frac{2\ell}{\sqrt{3}} \rfloor$.

Now we repeat the construction with a parallelogram instead of the rectangle. We put the second triple junction near the vertex of the angle $\frac{\pi}{2} \leq \alpha \leq \frac{2\pi}{3}$. As before, the segment is forced by the minimality condition to intersect the side of the parallelogram with an angle $\beta \leq \frac{2\pi}{3} - \alpha$, the complementary of β is $\gamma \geq \alpha - \frac{\pi}{6}$.



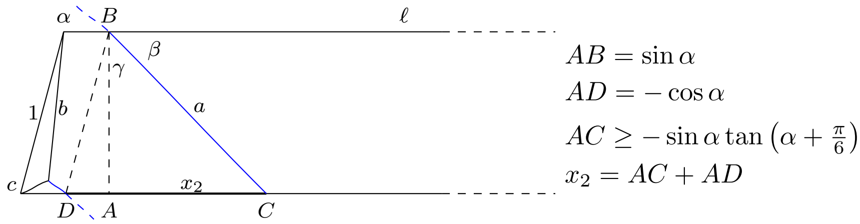
In this case the length between two different intersections is

$$x_1 \geq \sin\alpha \tan\left(\alpha - \frac{\pi}{6}\right) + \cos\alpha\,. \tag{3.1}$$

The maximum number of intersections is $N_1 = \frac{\ell}{x_1}$, and consequently, reasoning as previous, the number of spines is $\lfloor N_1 \rfloor$.

If instead we put the second triple junction near the acute angle of the parallelogram of size $(\pi - \alpha)$, the length we want to compute will change:

$$x_2 \geq \sin\alpha \tan\left(\frac{5\pi}{6} - \alpha\right) - \cos\alpha = -\sin\alpha\tan\left(\alpha + \frac{\pi}{6}\right) - \cos\alpha\,. \tag{3.2}$$



In this case the maximum number of intersections is $N_2 = \frac{1}{x_2}$.
Using simple trigonometric identities, we get

$$N_1 = \frac{1}{\sin\alpha\tan\left(\alpha - \frac{\pi}{6}\right) + \cos\alpha} = \frac{1}{\sqrt{3}}\left(\sqrt{3}\cos\alpha + \sin\alpha\right)\,,$$
$$N_2 = \frac{1}{-\sin\alpha\tan\left(\alpha + \frac{\pi}{6}\right) - \cos\alpha} = \frac{1}{\sqrt{3}}\left(-\sqrt{3}\cos\alpha + \sin\alpha\right)\,.$$

From this explicit construction for a torus with a flat metric described by a parallelogram with sides 1 and $\ell \geq 1$ and a angle $\frac{\pi}{3} \leq \alpha \leq \frac{2\pi}{3}$ we can conclude that the number of spines with only one segment which turns around a torus can be estimate by $N = \lfloor \frac{2\ell}{\sqrt{3}} \sin\alpha \rfloor$.

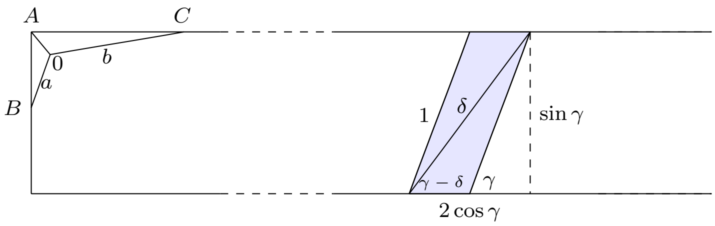
To conclude, we notice that we have found a lower bound for the number of minimal spines of a flat torus which depends only by the length l and by the angle α:

$$N \geq \lfloor \frac{2\ell}{\sqrt{3}} \sin\alpha \rfloor\,.$$

PROPOSITION 3.13. *If we represent a flat torus with a parallelogram, only one arc of geodesics of a minimal θ of the torus can intersect the side of the parallelogram. If two different arcs of geodesic of a spine of the torus intersect one of the sides of the parallelogram, then they eventually intersect each other.*

PROOF. First we prove our statement in a rectangle. We consider the case in which two segments of the θ intersect two consecutive sides of the rectangle.
We call O the triple junction, A the vertex of the rectangle, B the intersection between a segment (called a) and the short side of the rectangle and C the intersection between the second segment (called b) and the long side.
By minimality condition of the θ, AOB is a triangle with an angle of $\frac{2\pi}{3}$, hence the other two angles are less then $\frac{\pi}{3}$. In particular we are interested in the angle β in B between the segment a and the short side of the rectangle. As $0 \leq \beta \leq \frac{\pi}{3}$, its complementary γ is between $\frac{\pi}{6}$ and $\frac{\pi}{2}$. The

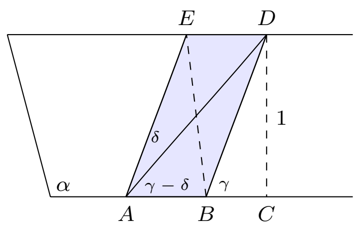
angle generated by the intersection of a and the long side is γ.
We consider a "strip" of the rectangle between two consecutive intersections of the segment a with the side of lenth l. In our rectangle this "strip" can be visualized by a parallelogram with angle γ and with a side of length 1 (up to dilation).



As we do not allow intersection between different segments composing the spine, a turn around the torus of the segment b have to stay in the "stripe" generated by the turns of a, without touch a. We call δ the amplitude of the angles between a and b which permit the lack of intersection. Now we compute how large can be δ at most:

$$\tan(\gamma-\delta) = \frac{\sin\gamma}{2\cos\gamma}\,,\text{that is}\,,$$
$$\tan\delta = \frac{\tan\gamma}{2+\tan^2\gamma} < \frac{1}{2}\,.$$

hence, we get $\delta < \arctan\left(\frac{1}{2}\right) < \frac{\pi}{6}$.
We kwon that the angle δ is $\frac{\pi}{3}$ because of the minimality condition, but we have just seen that if we want to avoid intersections, δ has to be stricly less than $\frac{\pi}{6}$.
The case in which the two segments interset two not consecutive sides can be reduced to the previous one.
These arguments prove that the two segment a and b of a minimal θ, both turning around the torus, that is two segments intersecting sides of the rectangle, intersect each other.
Now we repeat the argument for a more general case: a flat torus described by a parallelogram with angle $\frac{\pi}{3} \leq \alpha \leq \frac{2\pi}{3}$.
The angles β, γ and δ are defined as in the first situation, but the angle γ now has diffenrent bounds: $\frac{\pi}{6} \leq \gamma \leq \alpha \leq \frac{2\pi}{3}$.
In the figure we draw only the "strip" we are going to consider.



To simplify the sequent computation we impose at 1 the distance between the two long sides of the parallelogram.
We consider three different triangle:

- BCD has an angle equal to γ and the side CD equal to 1, so the side BC results equal to $\frac{1}{\tan\gamma}$.
- ABE with a angle equal to γ and a side parallel to the side of the parallelogram, the length of AB is $\frac{1}{\tan\gamma} + \frac{1}{\tan(\pi-\alpha)} = \frac{1}{\tan\gamma} - \frac{1}{\tan\alpha}$.
- ACD has an angle equal to $\gamma-\delta$ and the side CD equal to 1, so the side AC results equal to $\frac{1}{\tan\gamma-\delta}$.

Replacing in the equality $AC = AB+BC$ their values, we can express the tangent of δ in function of α and γ:

$$\begin{aligned}\tan\delta &= \frac{\tan\gamma\tan\alpha - \tan^2\gamma}{\tan^2\gamma\tan\alpha + 2\tan\alpha - \tan\gamma} \\ &= \frac{\tan\gamma\left(1-\frac{\tan\gamma}{\tan\alpha}\right)}{\tan^2\gamma + 2 - \frac{\tan\gamma}{\tan\alpha}}\end{aligned} \tag{3.3}$$

At this point we consider separately the case $\frac{\pi}{3} \leq \alpha \leq \frac{\pi}{2}$ and $\frac{\pi}{2} \leq \alpha \leq \frac{2\pi}{3}$.

- Case $\frac{\pi}{3} \leq \alpha \leq \frac{\pi}{2}$.
  The ratio $\frac{\tan\gamma}{\tan\alpha}$ is positive and less than 1 because $\frac{\pi}{6} \leq \gamma \leq \alpha \leq \frac{\pi}{2}$, so (3.3) can be estimated by
  $$\tan\delta \leq \frac{\tan\gamma}{1+\tan^2\gamma} \leq \frac{1}{2}\,.$$
- Case $\frac{\pi}{2} \leq \alpha \leq \frac{2\pi}{3}$.
  We observe that the situation in which δ is larger is when $\alpha = \frac{2\pi}{3}$.
  Sostituting this value in (3.3), we get
  $$\tan\delta = \frac{-\sqrt{3}\tan\gamma - \tan^2\gamma}{-\sqrt{3}\tan^2\gamma - 2\sqrt{3} - \tan\gamma} \leq \sqrt{3} \tag{3.4}$$

We have proved that, if $\frac{\pi}{3} \leq \alpha \leq \frac{2\pi}{3}$, if we want that two sides a and b of the minimal θ, turning around the torus, have not intersections, the angle δ between them has to be less than $\frac{\pi}{6}$, but this is in contradiction with the request of minimal θ.
Hence, in all the cases, it is not possible that two segments of the same minimal θ turn around the torus. □

PROPOSITION 3.14. *If we have a segment of a minimal θ which does turns around the torus and we represent the torus with a parallelogram of side 1 and $l \geq 1$, the segment cannot turns around the short side of the parallelogram.*

PROOF. Repeating the construction we have done at the beginning of the subsection (3.3), putting 1 instead of l and l instead of 1 (that is making the segment a turns around the short side of the parallelogram instead around the long one), we find that the length between two intersections of the short side should be

$$x \geq l\left(\sin\alpha\tan\left(\alpha - \frac{\pi}{6}\right) + \cos\alpha\right) \text{ with } \frac{\pi}{3} \leq \alpha \leq \frac{2\pi}{3}\,. \tag{3.5}$$

(Notice that sostituting α with $\pi - \alpha$ in (3.5) we find $x \geq -l\left(\sin\alpha\tan\left(\alpha+\frac{\pi}{6}\right) - \cos\alpha\right)$ and this is consistent with (3.2)).
We estimate the length x:

$$\begin{aligned}x &\geq l\left(\sin\alpha\tan\left(\alpha-\frac{\pi}{6}\right)+\cos\alpha\right) \geq \left(\sin\alpha\tan\left(\alpha-\frac{\pi}{6}\right)+\cos\alpha\right) \\ &= \sin\alpha\left(\frac{\sqrt{3}\tan\alpha - 1}{\sqrt{3}+\tan\alpha}\right) + \cos\alpha = \sin\alpha\frac{\sqrt{3}-\cos\alpha}{\sqrt{3}\cos\alpha+\sin\alpha} \\ &= \frac{\sqrt{3}}{\sqrt{3}\cos\alpha+\sin\alpha} = \frac{\sqrt{3}}{2\sin\left(\alpha+\frac{\pi}{3}\right)} \geq 1\,.\end{aligned} \tag{3.6}$$

With this computation, in particular, we find that the length between the firts intersection of the segment a with the side of length 1 of the parallelogram and the second intersection cannot be less than 1. Thus the segment intersects the long side l of the parallelogram before intersecting the short one, hence there is an intersection with the other segments of the spine. As intersection between different segment of the minimal θ are not allowed, we cannot construct a minimal spine with a segment turning around the short side of the parallelogram. □

To summarize, Proposition (3.13) and Proposition (3.14) tell us that there are no other possible minimal θ's in addition to those that we have constructed directly. Consideng also the two minimal θ's described in the proof of the Theorem (3.10), the number of minimal spine of a torus, up to dilation and reflection, is $N = \lfloor \frac{2l}{\sqrt{3}} \sin \alpha \rfloor + 2$.

4. Hyperbolic surfaces

Let now S_g be a closed orientable surface of genus $g \geq 2$. The oriented hyperbolic metrics on S_g form the *moduli space* $\mathcal{M}(S_g)$ and the minimum length of a spine furnishes the *spine systole*

$$\mathrm{S} \colon \mathcal{M}(S_g) \longrightarrow \mathbb{R}.$$

We now prove some facts on the function S.

THEOREM 4.1. *The function* S *is continuous and proper. Its global minima are precisely the extremal surfaces.*

PROOF. The function S is clearly continuous because the length of spines varies continuously in the metric. We now prove properness as an easy consequence of the Collar Lemma [**34**].

By Mumford's compactness theorem the subset $\mathcal{M}_\varepsilon(S_g) \subset \mathcal{M}(S_g)$ of all hyperbolic metrics with (closed geodesic) systole $\geq \varepsilon$ is compact for all $\varepsilon > 0$, and the Collar Lemma says that for sufficiently small $\varepsilon > 0$ a hyperbolic surface $S \in \mathcal{M}(S_g) \setminus \mathcal{M}_\varepsilon(S_g)$ has a simple closed curve γ of length $< \varepsilon$ with a collar of diameter $C(\varepsilon)$, for some function C such that $C(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Every spine Γ of S_g must intersect γ and cross the collar, hence $L(\Gamma) \geq C(\varepsilon)$ and therefore S is proper.

We now determine the global mimima of S. Let Γ be a spine in $S \in \mathcal{M}(S_g)$ of minimal length. The spine has $6g - 3$ edges. As in the Euclidean case, by cutting S along Γ we get a hyperbolic polygon P with $12g - 6$ edges and all interior angles $\frac{2\pi}{3}$. The length of Γ is half the perimeter of P.

Porti has shown [**80**] that, among all hyperbolic n-gons with fixed interior angles, the one with smaller perimeter is the unique one that has an inscribed circle. Therefore among all polygons P with angles $\frac{2\pi}{3}$ the one that minimizes the perimeter is precisely the regular one R, that is the one whose sides all have the same length. We deduce that the global minima for S are the hyperbolic surfaces that have R as a fundamental domain, and these are precisely the extremal surfaces, as proved by Bavard [**17**]. $\square$

It would be interesting to investigate the function S and check for instance whether it is a topological Morse function, see [**39**].

In the flat case we have shown that the number of minimal spines is finite. In the hyperbolic setting, we do not know if the same is true. To conclude the section, we prove a partial result:

THEOREM 4.2. *The number of minimal spines with bounded length of a closed hyperbolic surface S is finite.*

PROOF. In the proof of Theorem 2.1, it results clear that every set of minimal spines of equibounded length is compact. We now prove that for hyperbolic surfaces every such set is discrete, hence finite. By contradiction, let Γ_n be a sequence of distinct minimal spines of S of equibounded length, converging in the Hausdorff distance to the minimal spine Γ. Moreover $L(\Gamma_n) \to L(\Gamma)$. For every $\lambda \in [0, 1]$ and every n big enough, we construct, exactly as in the proof of Theorem 5.3, the (not necessarily minimal) spine Γ_n^λ and continuous function $F_n(\lambda) = L(\Gamma_n^\lambda)$. The surface S is hyperbolic, therefore, by Lemma 5.2, $F_n(\lambda)$ is strictly convex. Both Γ and Γ_n are minimal spines, that is stationary points of the length functional, hence, $F_n'(0) = F_n'(1) = 0$ and F_n is constant in λ: a contradiction. $\square$

5. Non positive constant curvature surfaces.

We restrict the attention to the case of non positive constant curvature surfaces. The goal is to show that minimal spines are local minimizers for the length functional, justifying our choice for the adjective "minimal". Let us begin with a definition and a well-known lemma about the convexity of the distance function in $\mathbb{H}^2$ that we take from [**34**].

DEFINITION 5.1. Let x_1, x_2 be points in $\mathbb{H}^2$, $\mathbb{R}^2$ or S^2 and $\lambda \in [0,1]$. The *convex combination* $x = \lambda x_1 + (1-\lambda)x_2$ is defined as follows:

$$\text{in } \mathbb{R}^2: \qquad x = \lambda x_1 + (1-\lambda)x_2$$
$$\text{in } \mathbb{H}^2, S^2: \qquad x = \frac{\lambda x_1 + (1-\lambda)x_2}{\|\lambda x_1 + (1-\lambda)x_2\|}$$

where in the $\mathbb{H}^2$ case we are considering the hyperboloid model in $\mathbb{R}^3$ with the Lorentzian scalar product $\langle \cdot, \cdot \rangle$ and $\|v\| = \sqrt{-\langle v, v\rangle}$.

LEMMA 5.2. *Let x_1, x_2, y_1, y_2 be points of the hyperbolic plane $\mathbb{H}^2$. For $\lambda \in (0,1)$, consider the convex combinations $x_\lambda = \lambda x_1 + (1-\lambda)x_2$ and $y_\lambda = \lambda y_1 + (1-\lambda)y_2$. Then, we have*

$$d(x_\lambda, y_\lambda) \leq \lambda d(x_1, y_1) + (1-\lambda) d(x_2, y_2)\,,$$

with equality only if x_1, x_2, y_1, y_2 belong to the same line. Here d denotes the distance in $\mathbb{H}^2$.

PROOF. Without loss of generality, for simplicity, we prove only the case $\lambda = \frac{1}{2}$, therefore x_λ (resp. y_λ) is the midpoint of x_1 and x_2 (resp. y_1 and y_2). If $x_\lambda = y_\lambda$ the theorem is trivial, hence we suppose $x_\lambda \neq y_\lambda$.

Let σ_p be the reflection at the point $p \in \mathbb{H}^2$. The map $\tau = \sigma_{y_\lambda} \circ \sigma_{x_\lambda}$ translates the line r containing the segment $x_\lambda y_\lambda$ by the quantity $2d(x_\lambda, y_\lambda)$: hence it is a hyperbolic transformation with axis r. We call $z_i = \tau(x_i)$ and note that $z_1 = \sigma_{y_\lambda}(x_2)$, hence $d(x_2, y_2) = d(z_1, y_1)$.

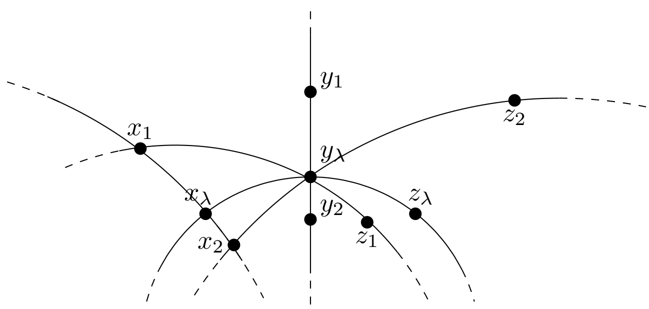


FIGURE 16.

The triangular inequality implies that

$$d(x_1, z_1) \leq d(x_1, y_1) + d(y_1, z_1) = d(x_1, y_1) + d(x_2, y_2)\,. \tag{5.1}$$

We notice that the equality holds only if x_1, y_1 and z_1 belong all to the same line.

A hyperbolic transformation has minimum displacement on its axis r, hence

$$2d(x_\lambda, y_\lambda) = d(x_\lambda, z_\lambda) = d(x_\lambda, \tau(x_\lambda)) \leq d(x_1, \tau(x_1)) = d(x_1, z_1)\,, \tag{5.2}$$

and the equality holds only if x_1 (and hence x_2) is in r. Finally we get $d(x_\lambda, y_\lambda) \leq \frac{1}{2}(d(x_1, y_1) + d(x_2, y_2))$ and hence d is convex.

Notice that the equality holds both in (5.1) and in (5.2) only if x_1, x_2, y_1, y_2 belong to the same line r. □

THEOREM 5.3. *Minimal spines of closed surfaces of non positive constant curvature are local minima for the length functional among spines, with respect to the Hausdorff distance.*

If the curvature is negative, these are strict local minima.

PROOF. We prove the first statement by contradiction.

Consider a sequence of spines Γ_n of the surface S converging in the Hausdorff distance to a minimal spine Γ, such that $L(\Gamma_n) < L(\Gamma)$. The minimal spine Γ of S is a network composed by geodesic arcs joining k triple junctions $x_1, \ldots, x_k$, where the number k depends only on the topology of S (see Remark 2.9). For n big enough, also Γ_n have k triple junctions $x_{1,n}, \dots, x_{k,n}$. Moreover, we can suppose that for n big enough Γ_n are composed only by geodesic segments. Indeed if Γ_n are not composed by geodesic segments, we can replace Γ_n with $\widetilde{\Gamma}_n$, union of geodesic arcs, with the same triple junctions of Γ_n, and the value of the length functional decreases $L(\widetilde{\Gamma}_n) \leq L(\Gamma_n) < L(\Gamma)$.

For n big enough, and for every $\lambda \in [0,1]$ and $i \in \{1, \dots k\}$, take the convex combination $x_{i,n}^\lambda = (1-\lambda)x_i + \lambda x_{i,n}$ and define Γ_n^λ as the spine obtained by joining the points $x_{i,n}^\lambda$ with geodesic segments in the same pattern of Γ. We get a continuous family of spines $\{\Gamma_n^\lambda\}_{\lambda\in[0,1]}$ such that $\Gamma_n^1 = \Gamma_n$ and $\Gamma_n^0 = \Gamma$. By Lemma 5.2, the continuous function $F_n(\lambda) = L(\Gamma_n^\lambda)$ is convex (convexity of the distance function is easily proved also in the Euclidean case) and $F_n(1) \leq F_n(0)$. We also have $F_n'(0) = 0$ because the minimal spine Γ is a stationary point of the length functional. This implies that $L(\Gamma_n) = F_n(1) \geq F_n(0) = L(\Gamma)$ and we have a contradiction.

In the hyperbolic case, Lemma 5.2 provides strict convexity and hence Γ is a strict local minimum. $\square$

REMARK 5.4. It is not restrictive to consider local minima of the length functional only among spines and not in the larger class of networks. Indeed, if we take a smooth family Φ_t of diffeomorphism of S with $t \in [0,T]$ and a spine Γ and we consider a small perturbation of Γ via these diffeomorphisms, $\Gamma_t = \Phi_t(\Gamma)$ is still a spine for t small enough. In particular, in the proof of Theorem 5.3, we show that $L(\Gamma) \leq L(\Phi_t(\Gamma))$, for all Φ_t and for t small enough.

CHAPTER 2

Evolution of networks with multiple junctions

1. Notation and definitions

Given a C^1 curve $\sigma : [0,1] \to \mathbb{R}^2$ we say that it is *regular* if $\sigma_x = \frac{d\sigma}{dx}$ is never zero. It is then well defined its unit tangent vector $\tau = \sigma_x/|\sigma_x|$. We define its unit normal vector as $\nu = R\tau = R\sigma_x/|\sigma_x|$, where $R : \mathbb{R}^2 \to \mathbb{R}^2$ is the counterclockwise rotation centred in the origin of $\mathbb{R}^2$ of angle $\pi/2$.
If the curve σ is C^2 and regular, its *curvature vector* is well defined as $\underline{k} = \tau_x/|\sigma_x| = \frac{d\tau}{dx}/|\sigma_x|$.
The arclength parameter of a curve σ is given by

$$s = s(x) = \int_0^x |\sigma_x(\xi)|\, d\xi\,.$$

Notice that $\partial_s = |\sigma_x|^{-1}\partial_x$, then $\tau = \partial_s\sigma$ and $\underline{k} = \partial_s\tau$, hence, the curvature of σ is given by $k = \langle \underline{k} \,|\, \nu\rangle$, as $\underline{k} = k\nu$.

DEFINITION 1.1. Let Ω be a smooth, convex, open set in $\mathbb{R}^2$. A *network* $\mathbb{S} = \bigcup_{i=1}^n \sigma^i([0,1])$ in Ω is a connected set in the plane described by a finite family of C^1, regular curves $\sigma^i : [0,1] \to \overline{\Omega}$ such that

(1) the "interior" of every curve σ^i, that is $\sigma^i(0,1)$, is embedded (hence, it has no self–intersections); a curve can self–intersect itself only possibly "closing" at its end–points;
(2) two different curves can intersect each other only at their end–points;
(3) the points where two or more curves intersect or a single curve self–intersects, have order at least three, considering $\mathbb{S}$ as a planar graph;
(4) any curve can "touch" the boundary of Ω only at one of its end–points;
(5) if a curve of the network touches the boundary of Ω at a point P, no other end–point of a curve can coincide with that point.

We call *multi–points* of the network the vertices $O^1, O^2, \dots, O^m \in \Omega$, seeing $\mathbb{S}$ as a planar graph, where the order is greater than one (actually it must be at least three, by the above Condition 3).

We call *end–points* of the network, the vertices $P^1, P^2, \dots, P^l \in \overline{\Omega}$ of $\mathbb{S}$ (on the boundary or not) with order one.

We say that a network is of class C^k or C^∞ if all the n curves are respectively C^k or C^∞.

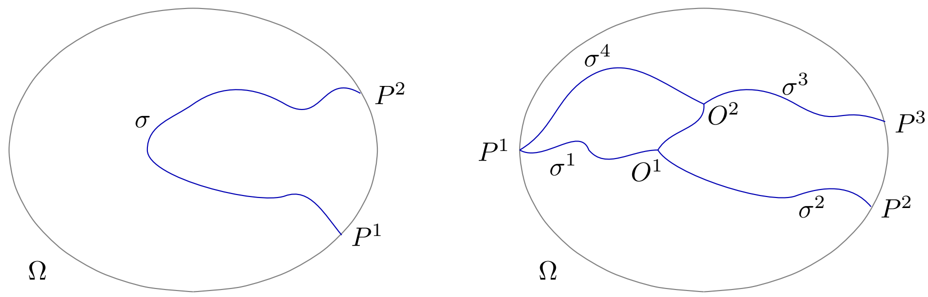


FIGURE 1. Two possible violations of the definition of network (Condition 4 and Condition 5, respectively).

REMARK 1.2. The conditions on the curves of a network in this definition have the following meanings:

- by the connectedness of the network, Condition 3 excludes the possibility of a network composed by a single embedded curve "closing" at its end–points. The evolution of a single smooth closed curve was widely studied by Gage, Hamilton, Grayson, Angenent et alt. (see **[10, 11, 12, 37, 36, 38, 41]**). Moreover also the case that such single curve form an angle or a cusps (the cusp is the most "delicate" situation) in "closing" can be dealt by means of the works of Angenent **[10, 11, 12]**, actually the curve becomes immediately smooth, flowing by curvature;

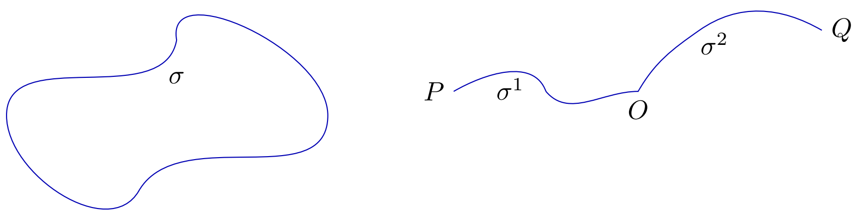


FIGURE 2. Two example of well known cases that are not networks: a single closed curve and two curves forming an angle at their junction.

- Condition 3 also exclude the possibility that only two curves concur at a vertex forming an angle (or a cusp, if they have the same tangent), situation that also can be analysed as above, by considering them as a single curve with a "singular" point that vanishes immediately under the flow;
- Condition 4 excludes the case of a single embedded curve with fixed ends. This evolution problem is discussed in **[49, 88, 89]**;
- the conditions on the boundary, in particular Condition 5, are to keep things simple and imply that the multi–points can be present only inside Ω, not on the boundary, while the end–points can be both inside or on $\partial\Omega$.

The curves σ^i have clearly nonzero finite lengths $L^i = \int_0^1 |\sigma^i_x(\xi)|\, d\xi$ and we will denote with $L = L^1 + \cdots + L^n$ the *global length* of the network.

DEFINITION 1.3. An *open network* $\mathbb{S} = \bigcup_{i=1}^n \sigma^i(I)$ in $\mathbb{R}^2$ is a connected set in the plane described by a finite family of C^1, regular curves $\sigma^i : I \to \mathbb{R}^2$, where I can be the interval $[0,1]$ or $[0,1)$, such that

(1) every "open" curve $\sigma^i : [0,1) \to \mathbb{R}^2$ is C^1–asymptotic to a halfline in $\mathbb{R}^2$ as $x \to 1$;
(2) the "interior" of every curve σ^i is embedded (hence, it has no self–intersections), only a curve of kind $\sigma^i : [0,1] \to \mathbb{R}^2$ can self–intersect itself and only "closing" at its end–points;
(3) two different curves can intersect each other only at their end–points;
(4) every end–point of a curve belongs to some multi–point of the network with order at least three, considering $\mathbb{S}$ as a planar graph;

As before, we say that an open network is of class C^k or C^∞ if all the n curves are respectively C^k or C^∞.

REMARK 1.4. Since we called these unbounded networks "open", we will adopt the word "closed" for the previous networks in Definition 1.1 which are bounded and possibly have some end–points.

Given a network composed by n curves with l end–points $P^1, P^2, \ldots, P^l \in \overline{\Omega}$ (if present) and m multi–points $O^1, O^2, \ldots O^m \in \Omega$, we will denote with σ^{pi} the curves of this network concurring at the multi–point O^p, with the index i varying from one to the order of the multi–point O^p (this is clearly redundant as some curves coincides, but useful for the notation). In the case of a network of n curves with only m 3–points, it is then composed by the family (with possible repetitions) of curves σ^{pi}, with $p \in \{1, 2, \dots, m\}$ and $i \in \{1, 2, 3\}$.

Our goal will be analysing the curvature flow of a network assuming either it is open or that all its end–points (if present), have to coincide with some points $P^1, P^2, \dots, P^l$ on the boundary of Ω (as we said, by Condition 5 in Definition 1.1, at most one curve of the network can arrive at any point P^r). We will discuss existence, uniqueness, regularity and asymptotic behavior of the evolution by curvature of such a network.
In the "closed" case we will ask that the end–points $P^r \in \partial\Omega$ stay "fixed" (*Dirichlet boundary conditions*) during the evolution. An analogous problem is to let such end–points "free" to move on the boundary of Ω but asking that the curves intersect orthogonally $\partial\Omega$ (*Neumann boundary conditions*).

We will define now a special class of networks that will play a key role in the analysis.

DEFINITION 1.5. We call a network (open or not) *regular* if its multi–points $O^1, O^2, \dots O^m \in \Omega$ have order three and at each of them the three concurring curves $\{\sigma^{pi}\}_{i=1,2,3}$ meet in such a way that the external unit tangents τ^{pi} satisfy $\tau^{p1} + \tau^{p2} + \tau^{p3} = 0$, which means that the three curves form three angles of 120 degrees at O^p (*Herring condition*).

We call a network *non–regular* if some multi–point has order different from three or if it has order three but the external unit tangents of the three concurring curves $\{\sigma^{pi}\}_{i=1,2,3}$ do not satisfy $\tau^{p1} + \tau^{p2} + \tau^{p3} = 0$. We will call such a point a *non–regular* multi–point.

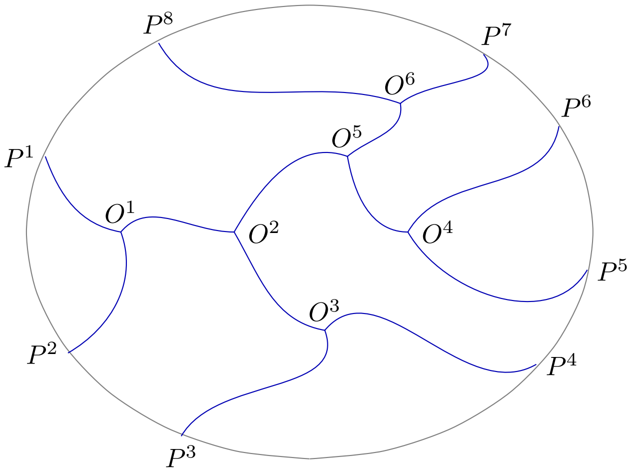


FIGURE 3. A regular network.

We are now ready to define the evolution by curvature of a C^2 regular network, which, in the "closed case", is the *geometric gradient flow* of the *Length functional*, that is, the sum of the lengths of all the curves of the network. Roughly speaking, a (solution of the) *flow by curvature* of a network is a smooth family of embedded, planar networks, such that the normal component of the velocity under the evolution, at every point of every curve of the evolving network, is given by the curvature vector of the curve at the point.

Given a time dependent family of regular C^2 networks of curves $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0,1], t)$, we let $\tau^i = \tau^i(x,t)$ the unit tangent vector to the curve γ^i, $\nu^i = \nu^i(x,t) = R\tau^i(x,t)$ the unit normal vector and $\underline{k}^i = \underline{k}^i(x,t) = k^i(x,t)\,\nu^i(x,t)$ its curvature vector, as previously defined.
Here and in all the sequel, we will denote with $\partial_x f$, $\partial_s f$ and $\partial_t f$ the derivatives of a function f along a curve γ^i with respect to the x variable, the arclength parameter s on such curve, defined by $s(x,t) = \int_0^x |\gamma^i_x(\xi,t)|\,d\xi$, and the time; $\partial^n_x f$, $\partial^n_s f$, $\partial^n_t f$ are the higher order partial derivatives which often we will also write as $f_x, f_{xx} \dots, f_s, f_{ss}, \dots$ and $f_t, f_{tt}, \dots$.

DEFINITION 1.6. We say that a family of homeomorphic, regular networks $\mathbb{S}_t$, each one composed by n curves $\gamma^i(\cdot,t) : I_i \to \overline{\Omega}$ (where I_i is the interval $[0,1]$ or $[0,1)$ in case of an open network), in a smooth convex, open set $\Omega \subset \mathbb{R}^2$, *moves by curvature* in the time interval $(0,T)$ if the functions $\gamma^i : I_i \times (0,T) \to \overline{\Omega}$ are of class C^2 in space and C^1 in time at least and satisfy

$$\begin{aligned}\gamma^i_t(x,t) &= k^i(x,t)\nu^i(x,t) + \lambda^i(x,t)\tau^i(x,t) \\ &= \frac{\langle \gamma^i_{xx}(x,t) \,|\, \nu^i(x,t)\rangle}{|\gamma^i_x(x,t)|^2}\nu^i(x,t) + \lambda^i(x,t)\tau^i(x,t)\end{aligned} \tag{1.1}$$

for some continuous functions λ^i, for every $x \in I_i$, $t \in (0,T)$, $i \in \{1,2,\dots,n\}$.
Another equivalent way to state this evolution equation is clearly

$$\gamma^i_t(x,t)^\perp = k^i(x,t)\nu^i(x,t) = \underline{k}^i(x,t) = \frac{\langle \gamma^i_{xx}(x,t) \,|\, \nu^i(x,t)\rangle}{|\gamma^i_x(x,t)|^2}\nu^i(x,t)\,. \tag{1.2}$$

We will call $\underline{v}^i = \gamma^i_t = k^i\nu^i + \lambda^i\tau^i$ and $\underline{\lambda}^i = \lambda^i\tau^i$ respectively the *velocity* and the *tangential velocity* of the curve γ^i, notice that the *normal velocity* is given by the curvature vector of the curve γ^i at every point. It is easy to see that $\underline{v}^i = \underline{k}^i + \underline{\lambda}^i$ and $|\underline{v}^i|^2 = |\underline{k}^i|^2 + |\underline{\lambda}^i|^2 = (k^i)^2 + (\lambda^i)^2$.

We underline that, in general, if there is no need to make explicit the curves composing a network, we simply write τ, ν, $\underline{v}$, $\underline{k}$, $\underline{\lambda}$, k, λ for the previous quantities, omitting the indices.
Moreover, we will adopt the following convention for integrals,

$$\int_{\mathbb{S}_t} f(t,\gamma,\tau,\nu,k,k_s,\dots,\lambda,\lambda_s\dots)\,ds = \sum_{i=1}^n \int_0^1 f(t,\gamma^i,\tau^i,\nu^i,k^i,k^i_s,\dots,\lambda^i,\lambda^i_s\dots)\,|\gamma^i_x|\,dx$$

as the arclength measure is given by $ds = |\gamma^i_x|\,dx$ on every curve γ^i.
Sometimes we will use also the following notation for the evolution of a network in $\Omega \subset \mathbb{R}^2$: we let $\mathbb{S} \subset \mathbb{R}^2$ a "referring" network homeomorphic to the all $\mathbb{S}_t$, and we consider a map $F : \mathbb{S} \times (0,T) \to \mathbb{R}^2$ given by the "union" of the maps $\gamma^i : I_i \times (0,T) \to \overline{\Omega}$ describing the curvature flow of the network in the time interval $(0,T)$, that is, $\mathbb{S}_t = F(\mathbb{S},t)$

REMARK 1.7. We spend some words on the above definition and evolution equation (1.1) which is not the usual way to describe the motion by curvature of a smooth curve, that is,

$$\gamma^i_t = \underline{k}^i = k^i\nu^i = \frac{\langle \gamma^i_{xx} \,|\, \nu^i\rangle}{|\gamma^i_x|^2}\,\nu^i\,. \tag{1.3}$$

Both motions are driven by a system of quasilinear partial differential equations, in our definition "admitting a correction" by a tangential term. Indeed, the two velocities differ only for a tangential component $\underline{\lambda}^i = \lambda^i\tau^i$. In the curvature evolution of a smooth curve it is well known that any tangential contribution to the velocity actually affects only the "inner motion" of the "single points" (Lagrangian point of view), but it does not affect the motion of a curve as a whole subset of $\mathbb{R}^2$, forgetting its parametrization (Eulerian point of view). Indeed, it can be shown that a flow of a closed curve satisfying equation (1.1) can be globally reparametrized (dynamically in time) in order it satisfies equation (1.3). However, in our situation such a global reparametrization is not possible due to the presence of the 3–points. It is necessary to consider such extra tangential terms in order to allow the motion of the 3–points also. Indeed, if the velocity would be in normal direction at every point of the three curves concurring at a 3–point, this latter should move in a direction which is normal to all of them, then the only possibility would be that it does not move at all (see also the discussions and examples in [**18, 19, 56**]).

REMARK 1.8. A very special case of an evolving curve γ^i satisfying equation (1.1) is a solution of the following system of quasilinear partial differential equations,

$$\gamma^i_t = \frac{\gamma^i_{xx}}{|\gamma^i_x|^2}\,. \tag{1.4}$$

In this case, it follows that

$$
\begin{array}{ll}
\underline{v}^i = \underline{v}^i(x,t) = \frac{\gamma^i_{xx}}{|\gamma^i_x|^2} & \text{velocity of the point } \gamma^i(x,t)\,, \\
\lambda^i = \lambda^i(x,t) = \frac{\langle \gamma^i_{xx}\,|\,\tau^i\rangle}{|\gamma^i_x|^2} = \frac{\langle \gamma^i_{xx}\,|\,\gamma^i_x\rangle}{|\gamma^i_x|^3} = -\partial_x \frac{1}{|\gamma^i_x|} & \text{tangential velocity of the point } \gamma^i(x,t)\,, \\
k^i = k^i(x,t) = \frac{\langle \gamma^i_{xx}\,|\,\nu^i\rangle}{|\gamma^i_x|^2} = \langle \partial_s\tau^i\,|\,\nu^i\rangle = -\langle \partial_s\nu^i\,|\,\tau^i\rangle & \text{curvature at the point } \gamma^i(x,t)\,.
\end{array}
$$

DEFINITION 1.9. A curvature flow γ^i for the initial, regular C^2 network $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$ which satisfies $\gamma^i_t = \frac{\gamma^i_{xx}}{|\gamma^i_x|^2}$ for every $t > 0$ will be called a *special curvature flow* of $\mathbb{S}_0$.

DEFINITION 1.10. Given an initial, regular, C^2 network $\mathbb{S}_0$, composed by n curves $\sigma^i : [0,1] \to \overline{\Omega}$, with m 3–points $O^1, O^2, \dots O^m \in \Omega$ and l end–points $P^1, P^2, \dots, P^l \in \partial\Omega$ in a smooth convex, open set $\Omega \subset \mathbb{R}^2$, we say that a family of homeomorphic networks $\mathbb{S}_t$, described by the family of time–dependent curves $\gamma^i(\cdot,t)$, is a solution of the motion by curvature problem with fixed end–points in the time interval $[0,T)$ if the functions $\gamma^i : [0,1]\times[0,T) \to \overline{\Omega}$ are continuous, there holds $\gamma^i(x,0) = \sigma^i(x)$ for every $x \in [0,1]$ and $i \in \{1,2,\dots,n\}$ (initial data), they are at least C^2 in space and C^1 in time in $[0,1]\times(0,T)$ and satisfy the following system of conditions for every $x \in [0,1]$, $t \in (0,T)$, $i \in \{1,2,\dots,n\}$,

$$
\begin{cases}
\gamma^i_x(x,t) \neq 0 & & \text{regularity} \\
\gamma^r(1,t) = P^r & \text{with } 0 \leq r \leq l & \text{end–points condition} \\
\sum_{j=1}^3 \tau^{pj}(O^p,t) = 0 & \text{at every 3–point } O^p & \text{angles of 120 degrees} \\
\gamma^i_t = k^i\nu^i + \lambda^i\tau^i & \text{for some continuous functions } \lambda^i & \text{motion by curvature}
\end{cases}
\tag{1.5}
$$

where we assumed conventionally (possibly reordering the family of curves and "inverting" their parametrization) that the end–point P^r of the network is given by $\gamma^r(1,t)$ (by Condition 4 in Definition 1.1 this can be always done).
Moreover, in the third equation we abused a little the notation, denoting with $\tau^{pj}(O^p,t)$ the respective unit normal vectors at O^p of the three curves $\gamma^{pj}(\cdot,t)$ in the family $\{\gamma^i(\cdot,t)\}$ concurring at the 3–point O^p.

We also state the same problem for regular, open networks.

DEFINITION 1.11. Given an initial, regular, C^2 open network $\mathbb{S}_0$, composed by n curves $\sigma^i : I_i \to \mathbb{R}^2$, we say that a family of homeomorphic open networks $\mathbb{S}_t$ with the same structure as $\mathbb{S}_0$ (in particular, same asymptotic halflines at infinity), described by the family of time–dependent curves $\gamma^i(\cdot,t)$, is a solution of the motion by curvature problem in the time interval $[0,T)$ if the functions $\gamma^i : I_i \times [0,T) \to \mathbb{R}^2$ are continuous, there holds $\gamma^i(x,0) = \sigma^i(x)$ for every $x \in I_i$ and $i \in \{1,2,\dots,n\}$ (initial data), they are of class at least C^2 in space and C^1 in time in $I_i \times (0,T)$ (here I_i denotes the interval $[0,1]$ or $[0,1)$ depending whether the curve is unbounded or not) and satisfy the following system for every $x \in I_i$, $t \in (0,T)$, $i \in \{1,2,\dots,n\}$,

$$
\begin{cases}
\gamma^i_x(x,t) \neq 0 & & \text{regularity} \\
\sum_{j=1}^3 \tau^{pj}(O^p,t) = 0 & \text{at every 3–point } O^p & \text{angles of 120 degrees} \\
\gamma^i_t = k^i\nu^i + \lambda^i\tau^i & \text{for some continuous functions } \lambda^i & \text{motion by curvature}
\end{cases}
\tag{1.6}
$$

where, in the second equation, we used the same notation as in Definition 1.10.

REMARK 1.12. In Definitions 1.10 and 1.11 the evolution equation (1.1) must be satisfied till the borders of the intervals $[0,1]$ and $[0,1)$, that is, at the 3–points and the end–points, for every positive time. This is not the usual way to state boundary conditions for parabolic problems (the parabolic nature of this evolution problem is clear by Definition 1.6 – see also Remark 1.7 and it will be even clearer in Section 3), where usually only continuity at the boundary is required. Anyway, as it is common in parabolic problems, at every positive time such boundary conditions are satisfied by any "natural solution".

This property of regularity at the boundary implies that

$$(k\nu + \lambda\tau)(P^r) = 0\,, \text{ for every } r \in \{1, 2, \dots, l\}\,,$$

and

$$(k^{pi}\nu^{pi} + \lambda^{pi}\tau^{pi})(O^p) = (k^{pj}\nu^{pj} + \lambda^{pj}\tau^{pj})(O^p)\,, \text{ for every } i, j \in \{1, 2, 3\}\,, p \in \{1, 2, \dots, m\}\,,$$

(where we abused a little the notation), obtained by simply requiring that the velocity is zero at every end–point and it is the same for any three curves at their concurrency 3–point.

Moreover, notice that in Definitions 1.10 and 1.11 the evolution equation (1.1) must be satisfied only for $t > 0$. If we want that the maps γ^i are C^2 in space and C^1 in time till the whole *parabolic boundary* (given by $[0, 1] \times \{0\} \cup \{0, 1\} \times [0, T)$ in Definition 1.10 and $[0, 1] \times \{0\} \cup \{0, 1\} \times [0, T)$ or $[0, 1) \times \{0\} \cup \{0\} \times [0, T)$ in Definition 1.11), the above conditions must be satisfied also by the initial regular network $\mathbb{S}_0$, for some functions λ_0 extending continuously the functions λ which are defined only for $t > 0$.

We concentrated on regular network for the moment since in studying problems (1.5) and (1.6) starting from a non–regular network several difficulties arise, related to the presence of general multi–points: if there are multi–points O^p of order greater than three, there can be several possible candidates for the flow. Considering, for example, the case of a network composed by two curves crossing each other (presence of 4–point); one cannot easily decide how the angle at the meeting point must behave, indeed one can allow the four concurrent curves to separate in two pairs of curves, moving independently each other and could even be taken into account the creation of new multi–points from a single one.
If there are several multi–points during the flow some of them can collapse together and the length of at least one curve of the network can go to zero.
In these cases, one must possibly restart the evolution with a different set of curves, the topology of the network change dramatically, forcing to change the "structure" of the system of equations governing the evolution.
Anyway a very natural conjecture is that the curvature flow of a general network (under a suitably good definition) should be non–regular only for a discrete set of times. We will get back on this in the following sections.

2. Basic computations

We work out now some basic relations and formulas holding for a regular network evolving by curvature, assuming that all the derivatives of the functions γ^i and λ^i that appear exist.

LEMMA 2.1. *If γ is a curve moving by*

$$\gamma_t = k\nu + \lambda\tau\,,$$

then the following commutation rule holds,

$$\partial_t\partial_s = \partial_s\partial_t + (k^2 - \lambda_s)\partial_s\,. \tag{2.1}$$

PROOF. Let $f : [0, 1] \times [0, T) \to \mathbb{R}$ be a smooth function, then

$$\begin{aligned}\partial_t\partial_s f - \partial_s\partial_t f &= \frac{f_{tx}}{|\gamma_x|} - \frac{\langle \gamma_x \,|\, \gamma_{xt}\rangle f_x}{|\gamma_x|^3} - \frac{f_{tx}}{|\gamma_x|} = -\langle \tau \,|\, \partial_s\gamma_t\rangle \partial_s f\\ &= -\langle \tau \,|\, \partial_s(\lambda\tau + k\nu)\rangle \partial_s f = (k^2 - \lambda_s)\partial_s f\end{aligned}$$

and the formula is proved. □

Then we can compute, for an evolving curve as in the previous lemma,

$$\begin{aligned}\partial_t\tau &= \partial_t\partial_s\gamma = \partial_s\partial_t\gamma + (k^2-\lambda_s)\partial_s\gamma = \partial_s(\lambda\tau+k\nu)+(k^2-\lambda_s)\tau = (k_s+k\lambda)\nu\,, &(2.2)\\ \partial_t\nu &= \partial_t(\mathrm{R}\tau) = \mathrm{R}\,\partial_t\tau = -(k_s+k\lambda)\tau\,, &(2.3)\\ \partial_t k &= \partial_t\langle\partial_s\tau\,|\,\nu\rangle = \langle\partial_t\partial_s\tau\,|\,\nu\rangle = \langle\partial_s\partial_t\tau\,|\,\nu\rangle + (k^2-\lambda_s)\langle\partial_s\tau\,|\,\nu\rangle &(2.4)\\ &= \partial_s\langle\partial_t\tau\,|\,\nu\rangle + k^3 - k\lambda_s = \partial_s(k_s+k\lambda)+k^3-k\lambda_s\\ &= k_{ss}+k_s\lambda+k^3\,.\end{aligned}$$

Moreover, in the special case that $\lambda = \frac{\langle\gamma_{xx}\,|\,\gamma_x\rangle}{|\gamma_x|^3}$, when the curve γ evolves according to

$$\gamma_t = \frac{\gamma_{xx}}{|\gamma_x|^2} = k\nu + \lambda\tau$$

(see Remark 1.8), we can also compute

$$\begin{aligned}\partial_t\lambda &= -\,\partial_t\partial_x\frac{1}{|\gamma_x|} = \partial_x\frac{\langle\gamma_x\,|\,\gamma_{tx}\rangle}{|\gamma_x|^3} = \partial_x\frac{\langle\tau\,|\,\partial_s(\lambda\tau+k\nu)\rangle}{|\gamma_x|} = \partial_x\frac{(\lambda_s-k^2)}{|\gamma_x|} &(2.5)\\ &= \partial_s(\lambda_s-k^2) - \lambda(\lambda_s-k^2) = \lambda_{ss}-\lambda\lambda_s-2kk_s+\lambda k^2\,.\end{aligned}$$

We consider the curvature flow of a family of regular, C^∞ networks $\mathbb{S}_t$, composed by n curves γ^i with m 3–points $O^1, O^2, \dots, O^m$ and l end–points $P^1, P^2, \dots, P^l$.
Then, at every 3–point O^p, with $p \in \{1,2,\dots,m\}$, differentiating in time the concurrency condition

$$\gamma^{pi}(0,t) = \gamma^{pj}(0,t) \qquad \text{for every } i \text{ and } j,$$

where γ^{pi} denotes the i–th curve concurrent at the 3–point O^p and we supposed for simplicity that they are parametrized such that they all concur for $x=0$ at O^p, we get

$$\lambda^{pi}\tau^{pi} + k^{pi}\nu^{pi} = \lambda^{pj}\tau^{pj} + k^{pj}\nu^{pj}$$

at every 3–point O^p, with $p \in \{1,2,\dots,m\}$, for every $i,j \in \{1,2,3\}$.
Multiplying these vector equalities for τ^{pl} and ν^{pl} and varying i,j,l, thanks to the conditions $\sum_{i=1}^3 \tau^{pi} = \sum_{i=1}^3 \nu^{pi} = 0$, we get the relations

$$\begin{aligned}\lambda^{pi} &= -\lambda^{p(i+1)}/2 - \sqrt{3}k^{p(i+1)}/2\\ \lambda^{pi} &= -\lambda^{p(i-1)}/2 + \sqrt{3}k^{p(i-1)}/2\\ k^{pi} &= -k^{p(i+1)}/2 + \sqrt{3}\lambda^{p(i+1)}/2\\ k^{pi} &= -k^{p(i-1)}/2 - \sqrt{3}\lambda^{p(i-1)}/2\end{aligned}$$

with the convention that the second superscripts are to be considered "modulo 3". Solving this system we get

$$\begin{aligned}\lambda^{pi} &= \frac{k^{p(i-1)}-k^{p(i+1)}}{\sqrt{3}}\\ k^{pi} &= \frac{\lambda^{p(i+1)}-\lambda^{p(i-1)}}{\sqrt{3}}\end{aligned}$$

which implies

$$\sum_{i=1}^3 k^{pi} = \sum_{i=1}^3 \lambda^{pi} = 0 \tag{2.6}$$

at any 3–point O^p of the network $\mathbb{S}_t$.
Moreover, considering $\mathrm{K}^p = (k^{p1}, k^{p2}, k^{p3})$ and $\Lambda^p = (\lambda^{p1}, \lambda^{p2}, \lambda^{p3})$ as vectors in $\mathbb{R}^3$, we have seen that K^p and Λ^p belong to the plane orthogonal to the vector $(1,1,1)$ and

$$\mathrm{K}^p = \Lambda^p \wedge (1,1,1)/\sqrt{3}\,, \qquad\qquad \Lambda^p = -\mathrm{K}^p \wedge (1,1,1)/\sqrt{3}\,, \tag{2.7}$$

that is, $\mathrm{K}^p = \mathrm{S}\Lambda^p$ and $\Lambda^p = -\mathrm{S}\mathrm{K}^p$ where S is the rotation in $\mathbb{R}^3$ of an angle of $\pi/2$ around the axis $\mathrm{I} = \langle(1,1,1)\rangle$. Hence, it also follows that

$$\sum_{i=1}^{3}(k^{pi})^2 = \sum_{i=1}^{3}(\lambda^{pi})^2 \qquad \text{and} \qquad \sum_{i=1}^{3} k^{pi}\lambda^{pi} = 0\,.$$

at any 3–point O^p of the network $\mathbb{S}_t$.

Now we differentiate in time the angular condition $\sum_{i=1}^{3}\tau^{pi} = 0$ at every 3–point O^p, with $p \in \{1,2,\dots,m\}$, by equation (2.2) we get

$$k^{pi}_s + \lambda^{pi}k^{pi} = k^{pj}_s + \lambda^{pj}k^{pj}\,,$$

for every pair i,j. In terms of vectors in $\mathbb{R}^3$, as before, we can write

$$\mathrm{K}^p_s + \Lambda^p\mathrm{K}^p = (k^{p1}_s + \lambda^{p1}k^{p1}, k^{p2}_s + \lambda^{p2}k^{p2}, k^{p3}_s + \lambda^{p3}k^{p3}) \in \mathrm{I}\,.$$

Differentiating repeatedly in time all these vector relations we have

$$\begin{gathered}\partial_t^l\mathrm{K}^p\,,\,\partial_t^l\Lambda^p \perp \mathrm{I} \quad \text{and} \quad \partial_t^l\langle\mathrm{K}^p\,|\,\Lambda^p\rangle = 0\,,\\ \partial_t^l\Lambda^p = -\partial_t^l\mathrm{S}\mathrm{K}^p = -\mathrm{S}\partial_t^l\mathrm{K}^p\,,\\ \partial_t^m(\mathrm{K}^p_s + \Lambda^p\mathrm{K}^p) \in \mathrm{I}\,,\end{gathered} \tag{2.8}$$

which, making explicit the indices, give the following equalities at every 3–point O^p, with $p \in \{1,2,\dots,m\}$,

$$\begin{gathered}\partial_t^l\sum_{i=1}^{3}k^{pi} = \sum_{i=1}^{3}\partial_t^l k^{pi} = \partial_t^l\sum_{i=1}^{3}\lambda^{pi} = \sum_{i=1}^{3}\partial_t^l\lambda^{pi} = \partial_t\sum_{i=1}^{3}k^{pi}\lambda^{pi} = 0\,,\\ \sum_{i=1}^{3}(\partial_t^l k^{pi})^2 = \sum_{i=1}^{3}(\partial_t^l\lambda^{pi})^2 \quad \text{for every } l \in \mathbb{N},\\ \partial_t^m(k^{pi}_s + \lambda^{pi}k^{pi}) = \partial_t^m(k^{pj}_s + \lambda^{pj}k^{pj}) \;\; \text{for every pair } i,j \text{ and } m \in \mathbb{N}.\end{gathered}$$

Moreover, by the orthogonality relations with respect to the axis I, we get also

$$\partial_t^l\mathrm{K}^p\partial_t^m(\mathrm{K}^p_s + \Lambda^p\mathrm{K}^p) = \partial_t^l\Lambda^p\partial_t^m(\mathrm{K}^p_s + \Lambda^p\mathrm{K}^p) = 0\,,$$

that is,

$$\sum_{i=1}^{3}\partial_t^l k^{pi}\,\partial_t^m(k^{pi}_s + \lambda^{pi}k^{pi}) = \sum_{i=1}^{3}\partial_t^l\lambda^{pi}\,\partial_t^m(k^{pi}_s + \lambda^{pi}k^{pi}) = 0 \;\; \text{for every } l,m \in \mathbb{N}. \tag{2.9}$$

REMARK 2.2. By the previous computations, for every solution in Definitions 1.10, 1.11, at $t > 0$ the curvature at the end–points and the sum of the three curvatures at every 3–point have to be zero and the same for the functions λ.
Then, a necessary condition for the maps γ^i to be C^2 in space and C^1 in time till the whole *parabolic boundary* (given by $[0,1]\times\{0\}\cup\{0,1\}\times[0,T)$ in Definition 1.10 and $[0,1]\times\{0\}\cup\{0,1\}\times[0,T)$ or $[0,1)\times\{0\}\cup\{0\}\times[0,T)$ in Definition 1.11), is that these conditions are satisfied also by the initial regular network $\mathbb{S}_0$, for some functions λ_0 (see Remark 1.12) extending continuously the functions λ which are actually defined only for $t > 0$. That is, for the initial regular network $\mathbb{S}_0$, there must hold

$$(k\nu + \lambda_0\tau)(P^r) = 0, \;\text{ for every } r \in \{1,2,\dots,l\}$$

and

$$(k^{pi}\nu^{pi} + \lambda_0^{pi}\tau^{pi})(O^p) = (k^{pj}\nu^{pj} + \lambda_0^{pj}\tau^{pj})(O^p), \;\text{ for every } i,j \in \{1,2,3\}.$$

In particular, for the initial network $\mathbb{S}_0 = \bigcup_{i=1}^{n}\sigma^i(I_i)$, the curvature at the end–points and the sum of the three curvatures at every 3–point have to be zero.
These conditions on the curvatures of $\mathbb{S}_0 = \bigcup_{i=1}^{n}\sigma^i(I_i)$ are clearly *geometric*, independent of the parametrizations of the curves σ^i but intrinsic to the *set* $\mathbb{S}_0$ and they are not satisfied by a generic regular, C^2 network

3. Short time existence I

In this section we start dealing with the problem of existence/uniqueness for short time of a solution of evolution Problem (1.5) for an initial regular network $\mathbb{S}_0$, with fixed end–points on the boundary of a smooth, convex, open set $\Omega \subset \mathbb{R}^2$.

We will establish a short time existence theorem of a curvature flow for a special class of $C^{2+2\alpha}$ (second derivative in 2α–Hölder space, for some $\alpha \in (0, 1/2)$) regular initial networks, satisfying some "compatibility conditions" at the end–points and at the 3–points. We analyse more general regular networks in Section 5.
Then we will discuss the problem of the uniqueness of the curvature flow of a network.

For sake of simplicity, we will deal in some detail with the case of the simplest possible network, a *triod* and we will explain how the same line works for a regular network with general structure.

DEFINITION 3.1. A *triod* $\mathbb{T} = \bigcup_{i=1}^{3} \sigma^i([0,1])$ is a network composed only of three C^1 regular curves $\sigma^i : [0,1] \to \overline{\Omega}$ where Ω is a smooth, convex, open subset of $\mathbb{R}^2$. These three curves intersect each other only at a single 3–point O and have the other three end–points coinciding with three distinct points $P^i = \sigma^i(1) \in \overline{\Omega}$.

An *open triod* $\mathbb{T} = \bigcup_{i=1}^{3} \sigma^i([0,1))$ in $\mathbb{R}^2$ is given by three C^1 regular curves $\sigma^i : [0,1) \to \mathbb{R}^2$ which intersect each other only at a single 3–point O and each one of them is C^1–asymptotic to a halfline in $\mathbb{R}^2$, as $x \to 1$.

As before, the triod is regular if the tangents of the three curves form angles of 120 degrees at the 3–point O.

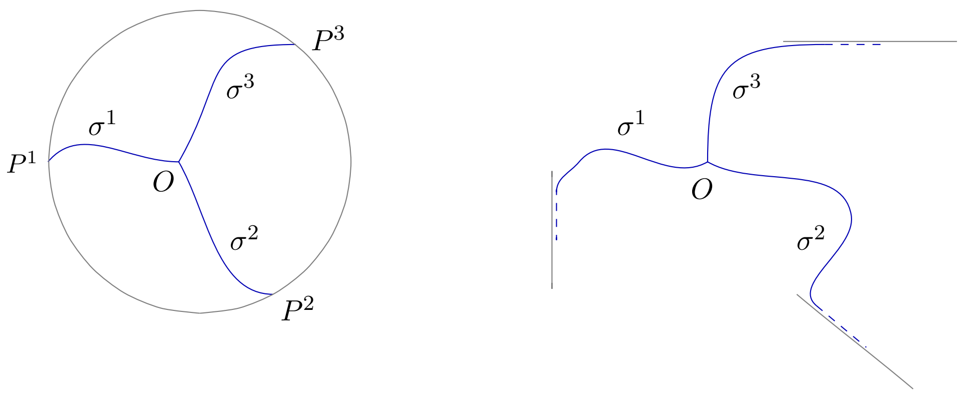


FIGURE 4. A regular triod on the left and an open regular triod on the right.

We restate Problem (1.5) for a triod.
The one parameter family of triods $\mathbb{T}_t = \bigcup_{i=1}^{3} \gamma^i([0,1],t)$ is a flow by curvature in the time interval $[0,T)$ of the initial, regular triod $\mathbb{T}_0 = \bigcup_{i=1}^{3} \sigma^i([0,1])$ in a smooth convex, open set $\Omega \subset \mathbb{R}^2$, if the three maps $\gamma^i : [0,1] \times [0,T) \to \overline{\Omega}$ are continuous, there holds $\gamma^i(x,0) = \sigma^i(x)$ for every $x \in [0,1]$ and $i \in \{1,2,3\}$ (initial data), they are at least C^2 in space and C^1 in time in $[0,1] \times (0,T)$ and satisfy the following system of conditions for every $x \in [0,1]$, $t \in (0,T)$, $i \in \{1,2,3\}$,

$$\begin{cases} \gamma^i_x(x,t) \neq 0 & & \text{regularity} \\ \gamma^i(1,t) = P^i & & \text{fixed end–points condition} \\ \sum_{i=1}^{3} \frac{\gamma^i_x(0,t)}{|\gamma^i_x(0,t)|} = 0 & & \text{angles of 120 degrees} \\ \gamma^i_t = k^i \nu^i + \lambda^i \tau^i & \text{for some continuous functions } \lambda^i & \text{motion by curvature} \end{cases} \tag{3.1}$$

In order to have a short time existence theorem, we make a special choice for the functions λ^i, by considering the system of quasilinear PDE's:

$$\begin{cases} \gamma^i_x(x,t) \neq 0 & \text{regularity} \\ \gamma^i(1,t) = P^i & \text{fixed end–points condition} \\ \gamma^i(x,0) = \sigma^i(x) & \text{initial data} \\ \sum_{i=1}^3 \frac{\gamma^i_x(0,t)}{|\gamma^i_x(0,t)|} = 0 & \text{angles of 120 degrees} \\ \gamma^i_t(x,t) = \frac{\gamma^i_{xx}(x,t)}{|\gamma^i_x(x,t)|^2} & \text{motion by curvature} \end{cases} \tag{3.2}$$

where we substituted $\gamma^i_t = k^i\nu^i + \lambda^i\tau^i$ with $\gamma^i_t = \frac{\gamma^i_{xx}}{|\gamma^i_x|^2}$ for every $x \in [0,1]\,, t \in [0,T)$ and $i \in \{1,2,3\}$ (see Remark 1.8).

By means of a method of Bronsard and Reitich in [**19**], based on Solonnikov theory [**87**] (see also [**61**]), as the system (3.2) satisfies that the so–called *complementary conditions* (see [**87**, p. 11]), which are a sort of algebraic relations between the evolution equation and the boundary constraints at the 3–point and at the end–points of the triod (see [**19**, Section 3], for more detail), there exists a unique solution $\gamma^i \in C^{2+2\alpha,1+\alpha}\left([0,1]\times[0,T)\right)$ of system (3.2), for some maximal time $T>0$, given any initial regular $C^{2+2\alpha}$ triod $\mathbb{T}_0 = \bigcup_{i=1}^3 \sigma^i([0,1])$, with $\alpha \in (0,1/2)$ provided it satisfies the so–called *compatibility conditions* of order 2.

DEFINITION 3.2. We say that for system (3.2) the *compatibility conditions of order 2* are satisfied by the initial triod $\mathbb{T}_0 = \bigcup_{i=1}^3 \sigma^i\left([0,1]\right)$ if at the end–points and at the 3–point, there hold all the relations on the space derivatives, up to second order, of the functions σ^i given by the boundary conditions and their time derivatives, assuming that the evolution equation holds also at such points.

Explicitly, the compatibility conditions of order 0 at the 3–point are

$$\sigma^i(0) = \sigma^j(0) \quad \text{for every } i,j \in \{1,2,3\}$$

and

$$\sigma^i(1) = P^i \quad \text{for every } i \in \{1,2,3\},$$

that is, simply the concurrency and fixed end–points conditions.
The compatibility condition of order 1 is given by

$$\sum_{i=1}^3 \frac{\sigma^i_x(0)}{|\sigma^i_x(0)|} = 0\,,$$

that is, the 120 degrees condition at the 3–point.
To get the second order conditions, one has to differentiate in time the first ones, getting

$$\frac{\sigma^i_{xx}(1)}{|\sigma^i_x(1)|^2} = 0 \quad \text{for every } i \in \{1,2,3\}\,,$$

and

$$\frac{\sigma^i_{xx}(0)}{|\sigma^i_x(0)|^2} = \frac{\sigma^j_{xx}(0)}{|\sigma^j_x(0)|^2} \quad \text{for every } i{,}j \in \{1,2,3\}\,.$$

THEOREM 3.3 (Bronsard and Reitich [**19**]). *For any initial, regular triod $\mathbb{T}_0 = \bigcup_{i=1}^3 \sigma^i([0,1])$, of class $C^{2+2\alpha}$ with $\alpha \in (0,1/2)$, satisfying the compatibility conditions of order 2, there exists a unique solution in $C^{2+2\alpha,1+\alpha}\left([0,1]\times[0,T)\right)$ of system* (3.2). *Moreover, every triod $\mathbb{T}_t = \bigcup_{i=1}^3 \gamma^i([0,1],t)$ satisfies the compatibility conditions of order* 2.

REMARK 3.4. Actually, in [**19**] the authors do not consider exactly Problem 3.1, but the analogous "Neumann problem". That is, they assign an angle conditions at the 3–point and require that the end–points of the three curve meet the boundary of Ω with a prescribed angle (respectively, 120 and 90 degrees in the case explicitly proved in detail).

A solution of system (3.2) clearly provides a solution to Problem (3.1).

THEOREM 3.5. *For any initial, regular $C^{2+2\alpha}$ triod $\mathbb{T}_0 = \bigcup_{i=1}^3 \sigma^i([0,1])$, with $\alpha \in (0,1/2)$, in a smooth, convex, open set $\Omega \subset \mathbb{R}^2$, satisfying the compatibility conditions of order 2, there exists a $C^{2+2\alpha,1+\alpha}([0,1]\times[0,T))$ curvature flow of $\mathbb{T}_0$ in a maximal positive time interval $[0,T)$. Moreover, every triod $\mathbb{T}_t = \bigcup_{i=1}^3 \gamma^i([0,1],t)$ satisfies the compatibility conditions of order 2.*

PROOF. If $\gamma^i \in C^{2+2\alpha,1+\alpha}([0,1]\times[0,T))$ is a solution of system (3.2), then it solves Problem (3.1) with

$$\lambda^i(x,t) = \frac{\langle \gamma^i_{xx}(x,t)\,|\,\tau^i(x,t)\rangle}{|\gamma^i_x(x,t)|^2} = \frac{\langle \gamma^i_{xx}(x,t)\,|\,\gamma^i_x(x,t)\rangle}{|\gamma^i_x(x,t)|^3}\,.$$

Indeed, it follows immediately by the regularity properties of this flow that the relative functions λ^i belong to the parabolic Hölder space $C^{2\alpha,\alpha}([0,1]\times[0,T))$ (hence, in $C^\alpha([0,1]\times[0,T))$, thus continuous) and all the triods $\mathbb{T}_t$ are in $C^{2+2\alpha}$, satisfying the compatibility conditions of order 2. The property that these evolving triods are regular follows by the standard fact that the maps γ^i_x are continuous, belonging to $C^{1+2\alpha,1/2+\alpha}([0,1]\times[0,T])$ (see **[58**, Section 8.8]), hence, being σ^i regular curves, $\gamma^i_x(x,t)\neq 0$ continues to hold for every $x\in[0,1]$ and for some positive interval of time.
The fact that a curve cannot self–intersects or two curve cannot intersect each other can be ruled out by noticing that such intersection cannot happen at the 3–point by geometric reasons, as the curvature is locally bounded and the curves are regular, then it is well known for the motion by curvature that strong maximum principle prevents such intersections for the flow of two embedded curves (or two distinct parts of the same curve). A similar argument and again the strong maximum principle also prevent that a curve "hits" the boundary of Ω at a point different from a fixed end–point of the triod. ☐

The method of Bronsard and Reitich extends to the case of an initial regular network $\mathbb{S}_0$ in a smooth, convex, open set $\Omega\subset\mathbb{R}^2$, with several 3–points and end–points. Indeed, as we said, such method relies on the uniform parabolicity of the system (which is the same) and on the fact that the complementary and compatibility conditions are satisfied.

DEFINITION 3.6. We say that a regular C^2 network $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$ is *2–compatible* if the maps σ^i satisfy the *compatibility conditions of order 2* for system (3.3), that is, $\sigma^i_{xx} = 0$ at every end–point and

$$\frac{\sigma^{pi}_{xx}(O^p)}{|\sigma^{pi}_x(O^p)|^2} = \frac{\sigma^{pj}_{xx}(O^p)}{|\sigma^{pj}_x(O^p)|^2}$$

for every pair of curves σ^{pi} and σ^{pj} concurring at any 3–point O^p (where we abused a little the notation like in Definition 1.10).

THEOREM 3.7. *For any initial, regular $C^{2+2\alpha}$ network $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$, with $\alpha\in(0,1/2)$, which is 2–compatible, there exists a unique solution in $C^{2+2\alpha,1+\alpha}([0,1]\times[0,T))$ of the following quasilinear system of PDE's (where we used the notation of Definition 1.10)*

$$\begin{cases} \gamma^i_x(x,t)\neq 0 & & \textit{regularity}\\ \gamma^r(1,t) = P^r & \textit{with } 0\leq r\leq l & \textit{fixed end–points condition}\\ \gamma^i(x,0) = \sigma^i(x) & & \textit{initial data}\\ \sum_{j=1}^3 \frac{\gamma^{pj}_x(O^p,t)}{|\gamma^{pj}_x(O^p,t)|} = 0 & \textit{at every 3–point } O^p & \textit{angles of } 120 \textit{ degrees}\\ \gamma^i_t(x,t) = \frac{\gamma^i_{xx}(x,t)}{|\gamma^i_x(x,t)|^2} & & \textit{motion by curvature} \end{cases} \tag{3.3}$$

for every $x\in[0,1]$, $t\in[0,T)$ and $i\in\{1,2,\dots,n\}$, in a maximal positive time interval $[0,T)$. Moreover, every network $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0,1],t)$ is 2–compatible.

As before, a solution of system (3.3) provides a solution to Problem (1.5), since the same geometric considerations in the proof of Theorem 3.5 hold also in this general case.

THEOREM 3.8. *For any initial, regular $C^{2+2\alpha}$ network $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$, with $\alpha\in(0,1/2)$, in a smooth, convex, open set $\Omega\subset\mathbb{R}^2$, which is 2–compatible, there exists a $C^{2+2\alpha,1+\alpha}([0,1]\times[0,T))$*

curvature flow of $\mathbb{S}_0$ *in a maximal positive time interval* $[0,T)$.
Moreover, every network $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0,1],t)$ *is 2–compatible.*

We recall that a special curvature flow of $\mathbb{S}_0$ is a curvature flow γ^i for the initial, regular C^2 network $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$ which satisfies $\gamma^i_t = \frac{\gamma^i_{xx}}{|\gamma^i_x|^2}$ for every $t > 0$ (see Definition 1.9).

By the very definition, every network of a special curvature flow is 2–compatible, for $t > 0$. Clearly, the solution given by Theorem 3.7 (which is the curvature flow mentioned in Theorem 3.8) is a special curvature flow.

REMARK 3.9. Notice that if we have a $C^{2,1}$ curvature flow $\mathbb{S}_t$, it is not necessarily 2–compatible for every time. It only have to satisfy $k\nu + \lambda\tau = 0$ at every end–point and

$$(k^{pi}\nu^{pi} + \lambda^{pi}\tau^{pi})(O^p) = (k^{pj}\nu^{pj} + \lambda^{pj}\tau^{pj})(O^p)\,, \text{ for every} i,j \in \{1,2,3\}$$

at every 3–point O^p (see Remark 2.2).
These relations implies anyway that for every evolving network $\mathbb{S}_t$ the curvature is zero at every end–point and the sum of the three curvatures at every 3–point is zero. We see now that this implies that reparametrizing $\mathbb{S}_t$ by a C^∞ map we obtain a 2–compatible network.

DEFINITION 3.10. We say that a regular C^2 network $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$ is *geometrically 2–compatible* if it admits a regular reparametrization by a C^∞ map such that it becomes 2–compatible.

By this definition, it is trivial that the property to be geometrically 2–compatible is invariant by reparametrization of the curves of a network. Moreover, it is a geometric property of a network since it involves only the curvature, by the following lemma.

LEMMA 3.11. *If for a regular* C^2 *network* $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$ *the curvature is zero at every end–point and the sum of the three curvatures at every 3–point is zero, then* $\mathbb{S}_0$ *is geometrically 2–compatible.*

PROOF. We look for some C^∞ maps $\theta^i : [0,1] \to [0,1]$, with $\theta^i_x(x) \neq 0$ for every $x \in [0,1]$ and $\theta^i(0) = 0$, $\theta^i(1) = 1$ such that the reparametrized curves $\widetilde{\sigma}^i = \sigma^i \circ \theta^i$ satisfy

$$\frac{\widetilde{\sigma}^i_{xx}}{|\widetilde{\sigma}^i_x|^2} = \frac{\widetilde{\sigma}^j_{xx}}{|\widetilde{\sigma}^j_x|^2}$$

for every pair of concurring curves $\widetilde{\sigma}^i$ and $\widetilde{\sigma}^j$ at any 3–point and $\widetilde{\sigma}^i_{xx} = 0$ at every end–point of the network. Setting $\widetilde{\lambda}^i_0 = \frac{\langle \widetilde{\sigma}^i_{xx} \,|\, \widetilde{\sigma}^i_x\rangle}{|\widetilde{\sigma}^i_x|^3}$ this means

$$\widetilde{k}^i\widetilde{\nu}^i + \widetilde{\lambda}^i_0\widetilde{\tau}^i = \widetilde{k}^j\widetilde{\nu}^j + \widetilde{\lambda}^j_0\widetilde{\tau}^j$$

for every pair of concurring curves $\widetilde{\sigma}^i$ and $\widetilde{\sigma}^j$ at any 3–point and $\widetilde{k}^i\widetilde{\nu}^i + \widetilde{\lambda}^i_0\widetilde{\tau}^i = 0$ at every end–point of the network. Since the curvature is invariant by reparametrization, by means of computations of Section 2 and the hypotheses on the curvature, these two conditions are satisfied if and only if $\widetilde{\lambda}^i_0 = 0$ at every end–point of the network and

$$\widetilde{\lambda}^i_0 = \frac{k^{i-1} - k^{i+1}}{\sqrt{3}}$$

at every 3–point of the network, for $i \in \{1,2,3\}$ (modulus 3).
Hence, we only need to find C^∞ reparametrizations θ^i such that at the borders of $[0,1]$ the values of $\widetilde{\lambda}^i_0 = \frac{\langle \widetilde{\sigma}^i_{xx} \,|\, \widetilde{\sigma}^i_x\rangle}{|\widetilde{\sigma}^i_x|^3}$ are given by these relations. This can be easily done since at the borders of the interval $[0,1]$ we have $\theta^i(0) = 0$ and $\theta^i(1) = 1$, hence

$$\widetilde{\lambda}^i_0 = \frac{\langle \widetilde{\sigma}^i_{xx} \,|\, \widetilde{\sigma}^i_x\rangle}{|\widetilde{\sigma}^i_x|^3} = -\partal_x \frac{1}{|\widetilde{\sigma}^i_x|} = -\partial_x \frac{1}{|\sigma^i_x \circ \theta^i|\theta^i_x} = \frac{\langle \sigma^i_{xx} \,|\, \sigma^i_x\rangle}{|\sigma^i_x|^3} + \frac{\theta^i_{xx}}{|\sigma^i_x||\theta^i_x|^2} = \lambda^i_0 + \frac{\theta^i_{xx}}{|\sigma^i_x||\theta^i_x|^2}$$

where $\lambda^i_0 = \frac{\langle \sigma^i_{xx} \,|\, \sigma^i_x\rangle}{|\sigma^i_x|^3}$, then we can simply choose any C^∞ functions θ^i with $\theta^i_x(0) = \theta^i_x(1) = 1$, $\theta^i_{xx} = -\lambda^i_0|\sigma^i_x||\theta^i_x|^2$ at every end–point and

$$\theta^i_{xx} = \left(\frac{k^{i-1} - k^{i+1}}{\sqrt{3}} - \lambda^i_0\right)\, |\sigma^i_x||\theta^i_x|^2$$

at every 3–point of the network (for instance, one can use a polynomial function). It follows that the reparametrized network $\widetilde{\mathbb{S}}_0 = \bigcup_{i=1}^n (\sigma^i \circ \theta^i)([0,1])$ is 2–compatible. $\square$

By this lemma and Remark 3.9, we immediately have the following proposition.

PROPOSITION 3.12. *Given a curvature flow $\mathbb{S}_t$ of an initial regular network $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$ of class C^2, all the networks $\mathbb{S}_t$, for $t > 0$, are geometrically 2–compatible.*

We now extend the short time existence result to regular, $C^{2+2\alpha}$ initial networks which are geometrically 2–compatible.

THEOREM 3.13. *For any initial regular $C^{2+2\alpha}$ network $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$ which is geometrically 2–compatible, with $\alpha \in (0, 1/2)$, in a smooth, convex, open set $\Omega \subset \mathbb{R}^2$, there exists a curvature flow which is in $C^{2+2\alpha,1+\alpha}([0,1] \times [0,T))$ for a maximal positive time interval $[0,T)$.*

PROOF. By the hypothesis, we can reparametrize the network $\mathbb{S}_0$ with some C^∞ maps θ^i in order it is 2–compatible. Clearly, if the network $\mathbb{S}_0$ belongs to $C^{2+2\alpha}$ the reparametrized one $\widetilde{\mathbb{S}}_0$ is still in $C^{2+2\alpha}$, hence, we can apply Theorem 3.8, to get the unique special curvature flow $\widetilde{\gamma}^i$ for $\widetilde{\mathbb{S}}_0 = \bigcup_{i=1}^n \widetilde{\sigma}^i([0,1]) = \bigcup_{i=1}^n (\sigma^i \circ \theta^i)([0,1])$ which is in $C^{2+2\alpha,1+\alpha}([0,1] \times [0,T))$ for a maximal positive time interval $[0,T)$. Moreover, every network $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0,1],t)$ is 2–compatible.

If now we consider the maps γ^i given by $\gamma^i(x,t) = \widetilde{\gamma}^i([\theta^i]^{-1}(x),t)$, we have that they still belong to $C^{2+2\alpha,1+\alpha}([0,1] \times [0,T))$ (as the maps $[\theta^i]^{-1}$ are in C^∞), $\gamma^i(\cdot,0) = \sigma^i$ and

$$
\begin{aligned}
\gamma^i_t(x,t) &= \partial_t[\widetilde{\gamma}^i([\theta^i]^{-1}(x),t)] \\
&= \widetilde{\gamma}^i_t([\theta^i]^{-1}(x),t) \\
&= \underline{\widetilde{k}}^i([\theta^i]^{-1}(x),t) + \widetilde{\lambda}^i([\theta^i]^{-1}(x),t)\widetilde{\tau}^i([\theta^i]^{-1}(x),t) \\
&= \underline{k}^i(x,t) + \lambda^i(x,t)\tau^i(x,t)\,,
\end{aligned}
$$

with $\lambda^i(x,t) = \widetilde{\lambda}^i([\theta^i]^{-1}(x),t)\widetilde{\tau}^i([\theta^i]^{-1}(x),t)$. Hence, γ^i is a flow by curvature of the network $\mathbb{S}_0$ in $C^{2+2\alpha,1+\alpha}([0,1] \times [0,T))$ $\square$

Clearly, it would be desirable to have an existence result for the flow on an initial $C^{2+2\alpha}$ network which is not necessarily geometrically 2–compatible or simply a C^2 network. We will try to address this problem in Section 5.

Anyway, suppose that the (only) C^2 network $\mathbb{S}_0$ is geometrically 2–compatible and we can find some (only) C^2 regular reparametrization φ^i turning it in a $C^{2+2\alpha}$ network, then, being this latter still geometrically 2–compatible we have a curvature flow by the previous theorem. Hence, composing this flow with the maps $[\varphi^i]^{-1}$, which are in C^2, we get a curvature flow for $\mathbb{S}_0$ (this situation happens, for instance, considering a $C^{2+2\alpha}$ network and reparametrizing it with maps which are C^2 but not $C^{2+2\alpha}$, obtaining a network $\mathbb{S}_0$).

This fact is related to the geometric nature of this evolution problem, indeed, in general, given any curvature flow γ^i of an initial network $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$, setting $\widetilde{\sigma}^i = \sigma^i \circ \theta^i$ for some orientation preserving C^2 functions $\theta^i : [0,1] \to [0,1]$, with $\theta_x(x) \neq 0$ for every $x \in [0,1]$, $\theta^i(0) = 0$ and $\theta^i(1) = 1$, we have, setting $\widetilde{\gamma}^i(x,t) = \gamma^i(\theta^i(x),t)$,

$$
\widetilde{\gamma}^i_t(x,t) = \partial_t[\gamma^i(\theta^i(x),t)] = \gamma^i_t(\theta^i(x),t) = \underline{k}^i(\theta^i(x),t) + \underline{\lambda}^i(\theta^i(x),t) = \underline{\widetilde{k}}^i(x,t) + \underline{\widetilde{\lambda}}^i(x,t)\,,
$$

with $\underline{\widetilde{\lambda}}^i(x,t) = \underline{\lambda}^i(\theta^i(x),t)$. Hence, $\widetilde{\gamma}^i = \gamma^i \circ \theta^i$ is a flow by curvature of the network $\widetilde{\mathbb{S}}_0 = \bigcup_{i=1}^n \widetilde{\sigma}^i([0,1])$ which is nothing more than a C^2 reparametrization of the network initial $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$. It follows easily that if $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i_0([0,1])$ and $\widetilde{\mathbb{S}}_0 = \bigcup_{i=1}^n \xi^i_0([0,1])$ describe the same initial regular C^2 network parametrized in two different ways, all the possible curvature flows of $\widetilde{\mathbb{S}}_0$ can be obtained by reparametrizazions of the curvature flows of $\mathbb{S}_0$ and viceversa.

Even more in general, considering the time–depending reparametrizations $\widetilde{\gamma}^i(x,t) = \gamma^i(\varphi^i(x,t),t)$ with some maps $\varphi^i : [0,1] \times [0,T) \to [0,1]$ in $C^0([0,1] \times [0,T)) \cap C^2([0,1] \times (0,T))$ such that

$\varphi^i(0,t)=0$, $\varphi^i(1,t)=1$ and $\varphi_x(x,t)\neq 0$ for every $(x,t)\in[0,1]\times[0,T)$, we compute

$$\begin{aligned}\widetilde{\gamma}^i_t(x,t) &= \partial_t[\gamma^i(\varphi^i(x,t),t)]\\ &= \gamma^i_x(\varphi^i(x,t),t)\varphi^i_t(x,t)+\gamma^i_t(\varphi^i(x,t),t)\\ &= \gamma^i_x(\varphi^i(x,t),t)\varphi^i_t(x,t)+\underline{k}^i(\varphi^i(x,t),t)+\underline{\lambda}^i(\varphi^i(x,t),t)\\ &= \widetilde{\underline{k}}^i(x,t)+\widetilde{\underline{\lambda}}^i(x,t)\,,\end{aligned}$$

with $\widetilde{\underline{\lambda}}^i(x,t)=\underline{\lambda}^i(\varphi^i(x,t),t)+\gamma^i_x(\varphi^i(x,t),t)\varphi^i_t(x,t)$. Hence, the reparametrized evolving network composed by the curves $\widetilde{\gamma}^i$ is a curvature flow for the initial network $\widetilde{\mathbb{S}}_0=\bigcup_{i=1}^n(\sigma^i\circ\varphi^i(\cdot,0))([0,1])$.
In particular, choosing special maps φ^i such that $\varphi^i(x,0)=x$ also holds, we have $\widetilde{\gamma}^i(x,0)=\gamma^i(x,0)=\sigma^i(x)$, hence, $\widetilde{\gamma}^i$ is another curvature flow for the initial network $\mathbb{S}_0=\bigcup_{i=1}^n(\sigma^i)([0,1])$.

REMARK 3.14. All this discussion suggests that the natural concept of uniqueness for the curvature flow of an initial network $\mathbb{S}_0=\bigcup_{i=1}^n\sigma^i([0,1])$, in our framework, is to consider uniqueness up to (dynamic) C^2 reparametrization of the curves of the network, we will get back to this at the end of this section.
Notice, moreover, that from what we saw above, we could also have considered networks simply as sets, forgetting their parametrization, and their curvature flows as flows of sets that could be parametrized in order to satisfy Definition 1.10.

We discuss now the higher regularity of the flow when the initial network is C^∞.

DEFINITION 3.15. We say that for system (3.3) the *compatibility conditions of every order* are satisfied by an initial regular C^∞ network $\mathbb{S}_0=\bigcup_{i=1}^n\sigma^i\,([0,1])$, and we call such a network *smooth*, if at every end–point and every 3–point, there hold all the relations on the space derivatives of the functions σ^i, obtained repeatedly differentiating in time the boundary conditions and using the evolution equation $\gamma^i_t(x,t)=\frac{\gamma^i_{xx}(x,t)}{|\gamma^i_x(x,t)|^2}$ to substitute time derivatives with space derivatives.
We say that a C^∞ flow by curvature $\mathbb{S}_t$ is *smooth* if all the networks $\mathbb{S}_t$ are *smooth*.

It is immediate by this definition that every network $\mathbb{S}_t$ of a C^∞ special curvature flow of an initial regular network $\mathbb{S}_0$, is smooth for every $t>0$.

REMARK 3.16. Notice that (compare with Remark 3.9) for the curvature flow of a network being smooth is quite more than being simply C^∞ up to the parabolic boundary. Anyway, analogously as before (Proposition 3.12), every network of a C^∞ curvature flow can reparametrized to be smooth.

If we assume that the initial regular network is smooth, we have the following higher regularity result.

THEOREM 3.17. *For any initial smooth, regular network $\mathbb{S}_0$ in a smooth, convex, open set $\Omega\subset\mathbb{R}^2$ there exists a unique C^∞ solution of system* (3.3) *in a maximal time interval* $[0,T)$.

PROOF. Since the initial network $\mathbb{S}_0$ satisfies the compatibility condition at every order, the method of Bronsard and Reitich actually provides, for every $n\in\mathbb{N}$, a unique solution of system (3.3), in $C^{2n+2\alpha,n+\alpha}([0,1]\times[0,T_n])$, satisfying at every time the compatibility conditions of order $0,1,\dots,2n$.
So, if we have a solution $\gamma^i\in C^{2n+2\alpha,n+\alpha}([0,1]\times[0,T_n])$ for $n\geq 1$, then the functions γ^i_x belong to $C^{2n-1+2\alpha,n-1/2+\alpha}([0,1]\times[0,T_n])$ (see **[58**, Section 8.8**]**). Considering the parabolic system satisfied by $v^i(x,t)=\gamma^i_t(x,t)$ (see **[71**, p. 250**]**), by Solonnikov results in **[87]**, $v^i=\gamma^i_t$ belongs to $C^{2n+2\alpha,n+\alpha}([0,1]\times[0,T_n])$ and since $\gamma^i_{xx}=\gamma^i_t\,|\gamma^i_x|^2$ with $|\gamma^i_x|^2\in C^{2n-1+2\alpha,n-1/2+\alpha}([0,1]\times[0,T_n])$, we get also $\gamma^i_{xx}\in C^{2n-1+2\alpha,n-1/2+\alpha}([0,1]\times[0,T_n])$.
Following **[64]**, we can then conclude that $\gamma^i\in C^{2n+1+2\alpha,n+1/2+\alpha}([0,1]\times[0,T_n])$.
Iterating this argument, we see that $\gamma^i\in C^\infty([0,1]\times[0,T_n])$, moreover, since for every $n\in\mathbb{N}$ the

solution obtained via the method of Bronsard and Reitich is unique, it must coincide with γ^i and we can choose all the T_n to be the same positive value T.
It follows that the solution is in C^∞ till the parabolic boundary, hence, all the compatibility conditions are satisfied at every time $t \in [0, T)$. $\square$

As a consequence, we have the following theorem.

THEOREM 3.18. *For any initial smooth, regular network $\mathbb{S}_0$ in a smooth, convex, open set $\Omega \subset \mathbb{R}^2$ there exists a smooth special curvature flow of $\mathbb{S}_0$ in a maximal positive time interval $[0, T)$.*

Also for C^∞ networks one can introduce the concept of *geometrically smoothness*. A network $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$ is geometrically smooth if it can be reparametrized to be smooth.

REMARK 3.19. By arguments similar to the ones of Lemma 3.11, it can be shown (by means of the computations of the next section) that, like for geometrical 2–compatibility, this property depends only on (some relations on) the curvature and its derivatives at the end–points and at the 3–points of a C^∞ network (see **[71]** for more details), that is, geometrical smoothness is again a geometric property (obviously invariant by C^∞ reparametrizations, by the definition).
Moreover, analogously as before (see Proposition 3.12), every C^∞ curvature flow of an initial regular network $\mathbb{S}_0$ is actually composed of *geometrically smooth* networks $\mathbb{S}_t$, for every $t > 0$.

The following analogous short time existence theorem holds, essentially with the same proof of Theorem 3.13.

THEOREM 3.20. *For any initial geometrically smooth, regular network $\mathbb{S}_0$ in a smooth, convex, open set $\Omega \subset \mathbb{R}^2$ there exists a C^∞ curvature flow of $\mathbb{S}_0$ in a maximal positive time interval $[0, T)$.*

We want to discuss now the concept of uniqueness of the flow. Even if the solution of system (3.3) is unique we have seen that there are anyway several solutions of Problem (1.5) for the same initial data, by dynamically reparametrizing it with maps $\varphi^i : [0,1] \times [0,T) \to [0,1]$ in $C^0([0,1] \times [0,T)) \cap C^2([0,1] \times (0,T))$ such that $\varphi^i(0,t) = 0$, $\varphi^i(1,t) = 1$, $\varphi^i(x,0) = x$ and $\varphi_x(x,t) \neq 0$ for every $(x,t) \in [0,1] \times [0,T)$. Hence, we consider uniqueness of the flow of an initial network, up to these reparametrizations.

DEFINITION 3.21. We say that the curvature flow of an initial C^2 network $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$ is *geometrically unique* (in some regularity class), if all the curvature flows satisfying Definition 1.10 can be obtained each other by means of time–depending reparametrizations.
To be precise, this means that if $\mathbb{S}_t$ and $\widetilde{\mathbb{S}}_t$ are two curvature flows of $\mathbb{S}_0$, described by some maps γ^i and $\widetilde{\gamma}^i$, there exists a family of maps $\varphi^i : [0,1] \times [0,T) \to [0,1]$ in $C^0([0,1] \times [0,T)) \cap C^2([0,1] \times (0,T))$ such that $\varphi^i(0,t) = 0$, $\varphi^i(1,t) = 1$, $\varphi^i(x,0) = x$, $\varphi^i_x(x,t) \neq 0$ and $\widetilde{\gamma}^i(x,t) = \gamma^i(\varphi^i(x,t),t)$ for every $(x,t) \in [0,1] \times [0,T)$.

It is obvious that if there is geometric uniqueness, any curvature flow gives a unique evolved network as a subset of $\mathbb{R}^2$, for every time $t \in [0,T)$, which is still the same set also if we change the parametrization of the initial network by the previous discussion.

Unfortunately, at the moment, the problem of geometric uniqueness of the curvature flow of a regular network in the class $C^{2,1}$ is open (even if the initial network is smooth). It is quite natural to conjecture that it holds, but the only available partial result, up to our knowledge, is given by the following proposition, consequence of the uniqueness part of Theorem 3.7.

PROPOSITION 3.22. *For any initial, regular $C^{2+2\alpha}$, with $\alpha \in (0, 1/2)$, network $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$ which is geometrically 2–compatible, in a smooth, convex, open set $\Omega \subset \mathbb{R}^2$, there exists a* geometrically unique *solution γ^i of Problem* (1.5) *in the class of maps $C^{2+2\alpha,1+\alpha}([0,1] \times [0,T))$ in a maximal positive time interval $[0,T)$.*

PROOF. We showed the existence of a special solution γ^i in Theorem 3.13.
We first show that if $\mathbb{S}_0$ satisfies the compatibility conditions of order 2 then the solution given by Theorem 3.7 is geometrically unique among the curvature flows in the class $C^{2+2\alpha,1+\alpha}([0,1] \times [0,T))$.

Suppose that $\widetilde{\gamma}^i : [0,1]\times[0,T') \to \overline{\Omega}$ is another maximal solution in $C^{2+2\alpha,1+\alpha}([0,1]\times[0,T'))$ satisfying $\widetilde{\gamma}^i_t = \widetilde{k}^i\widetilde{\nu}^i + \widetilde{\lambda}^i\widetilde{\tau}^i$ for some functions $\widetilde{\lambda}^i$ in $C^{2\alpha}([0,1]\times[0,T'))$, we want to see that it coincides to γ^i up to a reparametrization of the curves $\widetilde{\gamma}^i(\cdot,t)$ for every $t\in[0,\min\{T,T'\})$.
If we consider functions $\varphi^i : [0,1]\times[0,\min\{T,T'\}) \to [0,1]$ belonging to $C^{2+2\alpha,1+\alpha}([0,1]\times[0,\min\{T,T'\}))$ and the reparametrizations $\overline{\gamma}^i(x,t) = \widetilde{\gamma}^i(\varphi^i(x,t),t)$,
we have that $\overline{\gamma}^i\in C^{2+2\alpha,1+\alpha}([0,1]\times[0,\min\{T,T'\}))$ and

$$
\begin{aligned}
\overline{\gamma}^i_t(x,t) =& \,\partial_t[\widetilde{\gamma}^i(\varphi^i(x,t),t)]\\
=& \,\widetilde{\gamma}^i_x(\varphi^i(x,t),t)\varphi^i_t(x,t) + \widetilde{\gamma}^i_t(\varphi^i(x,t),t)\\
=& \,\widetilde{\gamma}^i_x(\varphi^i(x,t),t)\varphi^i_t(x,t) + \underline{\widetilde{k}}^i(\varphi^i(x,t),t) + \underline{\widetilde{\lambda}}^i(\varphi^i(x,t),t)\\
=& \,\widetilde{\gamma}^i_x(\varphi^i(x,t),t)\varphi^i_t(x,t) + \frac{\left\langle \widetilde{\gamma}^i_{xx}\left(\varphi^i(x,t),t\right) \,\middle|\, \widetilde{\nu}^i(\varphi^i(x,t),t)\right\rangle}{\left|\widetilde{\gamma}^i_x\left(\varphi^i(x,t),t\right)\right|^2}\widetilde{\nu}^i(\varphi^i(x,t),t)\\
&+ \widetilde{\lambda}^i(\varphi^i(x,t),t)\frac{\widetilde{\gamma}^i_x(\varphi^i(x,t),t)}{\left|\widetilde{\gamma}^i_x\left(\varphi^i(x,t),t\right)\right|}\,.
\end{aligned}
$$

We choose now maps $\varphi^i\in C^{2+2\alpha,1+\alpha}([0,1]\times[0,T''))$ which are solutions for some positive interval of time $[0,T'')$ of the following quasilinear PDE's

$$
\varphi^i_t(x,t) = \frac{\left\langle \widetilde{\gamma}^i_{xx}\left(\varphi^i(x,t),t\right) \,\middle|\, \widetilde{\gamma}^i_x(\varphi^i(x,t),t)\right\rangle}{\left|\widetilde{\gamma}^i_x\left(\varphi^i(x,t),t\right)\right|^4} - \frac{\widetilde{\lambda}^i(\varphi^i(x,t),t)}{\left|\widetilde{\gamma}^i_x\left(\varphi^i(x,t),t\right)\right|} + \frac{\varphi^i_{xx}(x,t)}{\left|\widetilde{\gamma}^i_x\left(\varphi^i(x,t),t\right)\right|^2\left|\varphi^i_x(x,t)\right|^2}\,, \tag{3.4}
$$

with $\varphi^i(0,t)=0$, $\varphi^i(1,t)=1$ and $\varphi^i(x,0)=x$ (hence, $\overline{\gamma}^i(x,0)=\gamma^i(x,0)=\sigma^i(x)$). The existence of such solutions follows by standard theory of quasilinear parabolic equations (see **[61, 63]**), noticing that the initial data $\varphi^i(x,0)=x$ satisfies the compatibility conditions of order 2 for equation (3.4)). Moreover, it is not difficult, by pushing a little the analysis, to show that $\varphi_x(x,t)\neq 0$ and that T'' can be taken equal to T'.

Then, it follows

$$
\begin{aligned}
\overline{\gamma}^i_t(x,t) =& \frac{\left\langle \widetilde{\gamma}^i_{xx}\left(\varphi^i(x,t),t\right) \,\middle|\, \widetilde{\gamma}^i_x(\varphi^i(x,t),t)\right\rangle}{\left|\widetilde{\gamma}^i_x\left(\varphi^i(x,t),t\right)\right|^4}\widetilde{\gamma}^i_x(\varphi^i(x,t),t) + \frac{\varphi^i_{xx}(x,t)\widetilde{\gamma}^i_x\left(\varphi^i(x,t),t\right)}{\left|\widetilde{\gamma}^i_x\left(\varphi^i(x,t),t\right)\right|^2\left|\varphi^i_x(x,t)\right|^2}\\
&+ \frac{\left\langle \widetilde{\gamma}^i_{xx}\left(\varphi^i(x,t),t\right) \,\middle|\, \widetilde{\nu}^i(\varphi^i(x,t),t)\right\rangle}{\left|\widetilde{\gamma}^i_x\left(\varphi^i(x,t),t\right)\right|^2}\widetilde{\nu}^i(\varphi^i(x,t),t)\\
=& \frac{\left\langle \widetilde{\gamma}^i_{xx}\left(\varphi^i(x,t),t\right) \,\middle|\, \widetilde{\tau}^i(\varphi^i(x,t),t)\right\rangle}{\left|\widetilde{\gamma}^i_x\left(\varphi^i(x,t),t\right)\right|^2}\widetilde{\tau}^i(\varphi^i(x,t),t) + \frac{\varphi^i_{xx}(x,t)\widetilde{\gamma}^i_x\left(\varphi^i(x,t),t\right)}{\left|\widetilde{\gamma}^i_x\left(\varphi^i(x,t),t\right)\right|^2\left|\varphi^i_x(x,t)\right|^2}\\
&+ \frac{\left\langle \widetilde{\gamma}^i_{xx}\left(\varphi^i(x,t),t\right) \,\middle|\, \widetilde{\nu}^i(\varphi^i(x,t),t)\right\rangle}{\left|\widetilde{\gamma}^i_x\left(\varphi^i(x,t),t\right)\right|^2}\widetilde{\nu}^i(\varphi^i(x,t),t)\\
=& \frac{\widetilde{\gamma}^i_{xx}\left(\varphi^i(x,t),t\right)}{\left|\widetilde{\gamma}^i_x\left(\varphi^i(x,t),t\right)\right|^2} + \frac{\varphi^i_{xx}(x,t)\widetilde{\gamma}^i_x\left(\varphi^i(x,t),t\right)}{\left|\widetilde{\gamma}^i_x\left(\varphi^i(x,t),t\right)\right|^2\left|\varphi^i_x(x,t)\right|^2}\\
=& \frac{\overline{\gamma}^i_{xx}(x,t)}{|\overline{\gamma}^i_x(x,t)|^2}\,.
\end{aligned}
$$

We can then conclude that, by the uniqueness part of Theorem 3.7, $\overline{\gamma}^i=\gamma^i$ for every $i\in\{1,2,\dots,n\}$, hence $\gamma^i(x,t)=\widetilde{\gamma}^i(\varphi^i(x,t),t)$ in the time interval $[0,\min\{T,T'\})$. Since this "repara - metrization relation" between any two maximal solutions of Problem (1.5) is symmetric (by means of the maps $[\varphi^i]^{-1}$), it follows that $T'=T$ and we are done.

Assume now that the network $\mathbb{S}_0$ is only geometrically 2–compatible, then the proof of Theorem 3.13 gives a special solution γ^i given by $\gamma^i(x,t)=\widetilde{\gamma}^i([\theta^i]^{-1}(x),t)$ where θ^i are smooth maps and $\widetilde{\gamma}^i$ is a special solution as above for the 2–compatible network $\widetilde{\mathbb{S}}_0=\bigcup_{i=1}^n\widetilde{\sigma}^i([0,1]) = \bigcup_{i=1}^n(\sigma^i\circ\theta^i)([0,1])$ which is in $C^{2+2\alpha,1+\alpha}([0,1]\times[0,T))$ for a maximal positive time interval $[0,T)$.

Suppose that $\overline{\gamma}^i : [0,1]\times[0,T')\to\overline{\Omega}$ is another maximal flow for $\mathbb{S}_0$ in $C^{2+2\alpha,1+\alpha}([0,1]\times[0,T'))$ satisfying $\overline{\gamma}^i_t = \overline{k}^i\overline{\nu}^i + \overline{\lambda}^i\overline{\tau}^i$ for some functions $\overline{\lambda}^i$ in $C^{2\alpha}([0,1]\times[0,T'))$. If we consider the maps

$\widetilde{\overline{\gamma}}^i(x,t) = \overline{\gamma}^i(\theta^i(x),t)$ these gives a $C^{2+2\alpha,1+\alpha}([0,1]\times[0,T'))$ curvature flow of the initial network $\widetilde{\mathbb{S}}_0$ which satisfies the compatibility conditions of order 2, hence, by the above argument, $T'=T$ and the maps $\widetilde{\overline{\gamma}}^i$ and $\widetilde{\gamma}^i$ only differ by reparametrization by some maps $\varphi^i \in C^{2+2\alpha,1+\alpha}([0,1]\times[0,T))$ with $\varphi^i(x,0)=x$, that is

$$\widetilde{\overline{\gamma}}^i(x,t) = \widetilde{\gamma}^i(\varphi^i(x,t),t)\,.$$

It follows that

$$\overline{\gamma}^i(x,t) = \widetilde{\overline{\gamma}}^i([\theta^i]^{-1}(x),t) = \widetilde{\gamma}^i(\varphi^i([\theta^i]^{-1}(x),t),t) = \gamma^i(\theta^i(\varphi^i([\theta^i]^{-1}(x),t)),t)$$

which shows that the two flows $\overline{\gamma}^i$ and γ^i of the initial network $\mathbb{S}_0$ coincide up to the time–dependent reparametrizations $(x,t)\mapsto(\theta^i(\varphi^i([\theta^i]^{-1}(x),t)),t)$. □

An immediate consequence is the following.

COROLLARY 3.23. *For any initial, regular geometrically smooth network $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$ in a smooth, convex, open set $\Omega\subset\mathbb{R}^2$, there exists a* geometrically unique *solution of Problem* (1.5) *in the class of maps $C^{2+2\alpha,1+\alpha}([0,1]\times[0,T))$ in a maximal positive time interval $[0,T)$. Moreover, such solution is C^∞.*

REMARK 3.24. Notice that it follows that any curvature flow as in the hypotheses of the above theorem and corollary is a reparametrization (of class $C^{2+2\alpha,1+\alpha}$ in the first case and C^∞ in the latter) of the special curvature flow given by Theorem 3.7 (which is C^∞ under the hypotheses of the corollary, by Theorem 3.17).

We conclude this section stating the following natural open problem, that we already mentioned, related to geometric uniqueness of the flow.

OPEN PROBLEM 3.25. Show that for any initial, regular $C^{2+2\alpha}$ network $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$, with $\alpha\in(0,1/2)$, which is 2–compatible, the solution given by Theorem 3.8 is the *geometrically unique* curvature flow of $\mathbb{S}_0$. That is, the maps γ^i give the geometrically unique solution of Problem (1.5) in the class of continuous maps $\gamma^i:[0,1]\times[0,T)\to\mathbb{R}^2$ which are of class at least C^2 in space and C^1 in time in $[0,1]\times(0,T)$.

The difficulty in getting such a uniqueness result is connected to the lack of some sort of (possibly geometric) maximum principle for this evolution problem.

4. Integral estimates

In this section we work out some integral estimates for a special flow by curvature of a smooth regular network. These estimates were previously proved for the case of the special curvature flow of a regular smooth triod with fixed end–points, in **[71]**. We extend now them to the case of a regular smooth network with "controlled" behavior of its end–points. We advise the reader that when the computations are exactly the same we will refer directly to [**71**, Section 3], where it is possible to find every detail.

In all this section we will assume that the special flow by curvature is given by a C^∞ solution γ^i of system (3.3), that is, there holds

$$\gamma^i_t(x,t) = \frac{\gamma^i_{xx}(x,t)}{|\gamma^i_x(x,t)|^2}\,,$$

(see Remark 1.8 and Definition 1.9 for the case of an initial C^2 network). Some of the estimates hold also for any smooth flow (the ones where we do not use the special form of the functions λ^i given by this equation), not necessarily special. To use these estimates for a general smooth flow, because of geometric uniqueness (see Corollary 3.23 and Remark 3.24), one must reparametrize such flow in order it becomes special, then carry back the geometric (invariant by reparametrization) estimates to the original flow.

We will see that such special flow of a regular smooth network with "controlled" end–points exists smooth as long as the curvature stays bounded and none of the lengths of the curves goes to zero (Theorem 4.14).

We suppose that the special solution maps γ^i above exist and are of class C^∞ in the time interval $[0,T)$ and that they describe the flow of a regular C^∞ network $\mathbb{S}_t$ in Ω, composed by n curves $\gamma^i(\cdot,t):[0,1]\to\overline{\Omega}$ with m 3–points $O^1,O^2,\dots,O^m$ and l end–points $P^1,P^2,\dots,P^l$. We will assume that either such end–points are fixed or that there exist uniform (in time) constants C_j, for every $j\in\mathbb{N}$, such that

$$|\partial_s^j k(P^r,t)|+|\partial_s^j\lambda(P^r,t)|\leq C_j\,, \tag{4.1}$$

for every $t\in[0,T)$ and $r\in 1,2,\dots,l$.

The very first computation we are going to show is the evolution in time of the total length of a network under the curvature flow.

PROPOSITION 4.1. *The time derivative of the measure ds on any curve γ^i of the network is given by the measure $(\lambda^i_s-(k^i)^2)\,ds$. As a consequence, we have*

$$\frac{dL^i(t)}{dt}=\lambda^i(1,t)-\lambda^i(0,t)-\int_{\gamma^i(\cdot,t)}(k^i)^2\,ds$$

and

$$\frac{dL(t)}{dt}=\sum_{r=1}^{l}\lambda(P^r,t)-\int_{\mathbb{S}_t}k^2\,ds\,,$$

where, with a little abuse of notation, $\lambda(P^r,t)$ is the tangential velocity at the end–point P^r of the curve of the network getting at such point, for any $r\in\{1,2,\dots,l\}$.
In particular, if the end–points P^r of the network are fixed during the evolution, we have

$$\frac{dL(t)}{dt}=-\int_{\mathbb{S}_t}k^2\,ds\,,$$

thus, in such case, the total length $L(t)$ is decreasing in time and uniformly bounded above by the length of the initial network $\mathbb{S}_0$.

PROOF. The formula for the time derivative of the measure ds follows easily by the commutation formula (2.1). Then,

$$\frac{dL^i(t)}{dt}=\frac{d}{dt}\int_{\gamma^i(\cdot,t)}1\,ds=\int_{\gamma^i(\cdot,t)}(\lambda^i_s-(k^i)^2)\,ds=\lambda^i(1,t)-\lambda^i(0,t)-\int_{\gamma^i(\cdot,t)}(k^i)^2\,ds\,.$$

Adding these relations for all the curves, the contributions of λ^{pi} at every 3–point O^p vanish, by relation (2.6), and the formula of the statement follows. If the end–points are fixed all the terms $\lambda(P^r,t)$ are zero and the last formula follows. ☐

The following notation will be very useful for the next computations in this section.

DEFINITION 4.2. We will denote with $\mathfrak{p}_\sigma(\partial_s^j\lambda,\partial_s^h k)$ a polynomial with constant coefficients in $\lambda,\dots,\partial_s^j\lambda$ and $k,\dots,\partial_s^h k$ such that every monomial it contains is of the form

$$C\prod_{l=0}^{j}(\partial_s^l\lambda)^{\alpha_l}\cdot\prod_{l=0}^{h}(\partial_s^l k)^{\beta_l}\quad\text{with}\quad\sum_{l=0}^{j}(l+1)\alpha_l+\sum_{l=0}^{h}(l+1)\beta_l=\sigma,$$

we will call σ the *geometric order* of $\mathfrak{p}_\sigma$.
Moreover, if one of the two arguments of $\mathfrak{p}_\sigma$ does not appear, it means that the polynomial does not contain it, for instance, $\mathfrak{p}_\sigma(\partial_s^h k)$ does not contain neither λ nor its derivatives.
We will denote with $\mathfrak{q}_\sigma(\partial_t^j\lambda,\partial_s^h k)$ a polynomial as before in $\lambda,\dots,\partial_t^j\lambda$ and $k,\dots,\partial_s^h k$ such that all its monomials are of the form

$$C\prod_{l=0}^{j}(\partial_t^l\lambda)^{\alpha_l}\cdot\prod_{l=0}^{h}(\partial_s^l k)^{\beta_l}\quad\text{with}\quad\sum_{l=0}^{j}(2l+1)\alpha_l+\sum_{l=0}^{h}(l+1)\beta_l=\sigma.$$

Finally, when we will write $\mathfrak{p}_\sigma(|\partial_s^j\lambda|, |\partial_s^h k|)$ (or $\mathfrak{q}_\sigma(|\partial_t^j\lambda|, |\partial_s^h k|)$) we will mean a finite sum of terms like

$$C\prod_{l=0}^{j}|\partial_s^l\lambda|^{\alpha_l}\cdot\prod_{l=0}^{h}|\partial_s^l k|^{\beta_l}\quad\text{with}\quad\sum_{l=0}^{j}(l+1)\alpha_l+\sum_{l=0}^{h}(l+1)\beta_l=\sigma,$$

where C is a positive constant and the exponents α_l, β_l are non negative *real* values (analogously for $\mathfrak{q}_\sigma$).
Clearly we have $\mathfrak{p}_\sigma(\partial_s^j\lambda, \partial_s^h k)\leq\mathfrak{p}_\sigma(|\partial_s^j\lambda|, |\partial_s^h k|)$.

By means of the commutation rule (2.1), the relations in the next lemma are easily proved by induction (Lemmas 3.7 and 3.8 in **[71]**).

LEMMA 4.3. *The following formulas hold for every curve of the evolving network* $\mathbb{S}_t$*:*

$$\begin{aligned}
&\partial_t\partial_s^j k=\partial_s^{j+2}k+\lambda\partial_s^{j+1}k+\mathfrak{p}_{j+3}(\partial_s^j k) &&\text{for every } j\in\mathbb{N},\\
&\partial_s^j k=\partial_t^{j/2}k+\mathfrak{q}_{j+1}(\partial_t^{j/2-1}\lambda, \partial_s^{j-1}k) &&\text{if } j\geq 2 \text{ is even},\\
&\partial_s^j k=\partial_t^{(j-1)/2}k_s+\mathfrak{q}_{j+1}(\partial_t^{(j-3)/2}\lambda, \partial_s^{j-1}k) &&\text{if } j\geq 1 \text{ is odd},\\
&\partial_t\partial_s^j\lambda=\partial_s^{j+2}\lambda-\lambda\partial_s^{j+1}\lambda-2k\partial_s^{j+1}k+\mathfrak{p}_{j+3}(\partial_s^j\lambda, \partial_s^j k) &&\text{for every } j\in\mathbb{N},\\
&\partial_s^j\lambda=\partial_t^{j/2}\lambda+\mathfrak{p}_{j+1}(\partial_s^{j-1}\lambda, \partial_s^{j-1}k) &&\text{if } j\geq 2 \text{ is even},\\
&\partial_s^j\lambda=\partial_t^{(j-1)/2}\lambda_s+\mathfrak{p}_{j+1}(\partial_s^{j-1}\lambda, \partial_s^{j-1}k) &&\text{if } j\geq 1 \text{ is odd}.
\end{aligned}$$

REMARK 4.4. Notice that, by relations (2.8) at any 3–point O^p of the network there holds $\partial_t^j\lambda^{pi}=(\mathrm{S}\partial_t^j\mathrm{K})^{pi}$, that is, the time derivatives of λ^{pi} are expressible as time derivatives of the functions k^{pi}. Then, by using repeatedly such relation and the first formula of Lemma 4.3, we can express these latter as space derivatives of k^{pi}. Hence, we will have the relation

$$\sum_{i=1}^{3}\mathfrak{q}_\sigma(\partial_t^j\lambda^{pi}, \partial_s^h k^{pi})\,\Big|_{\text{at the 3–point } O^p}=\mathfrak{p}_\sigma(\partial_s^{\max\{2j,h\}}\mathrm{K}^p)\,\Big|_{\text{at the 3–point } O^p}$$

with the meaning that this last polynomial contains also product of derivatives of different k^{pi}'s, because of the action of the linear operator S.
We will often make use of this identity in the computations in the sequel in the following form,

$$\sum_{i=1}^{3}\mathfrak{q}_\sigma(\partial_t^j\lambda^{pi}, \partial_s^h k^{pi})\,\Big|_{\text{at the 3–point } O^p}\leq\|\mathfrak{p}_\sigma(|\partial_s^{\max\{2j,h\}}k|)\|_{L^\infty}\,.$$

REMARK 4.5. We state the following *calculus rules* which will be used extensively in the sequel,

$$\mathfrak{p}_\alpha(\partial_s^j\lambda, \partial_s^h k)\cdot\mathfrak{p}_\beta(\partial_s^l\lambda, \partial_s^m k)=\mathfrak{p}_{\alpha+\beta}(\partial_s^{\max\{j,l\}}\lambda, \partial_s^{\max\{h,m\}}k)\,,$$
$$\mathfrak{q}_\alpha(\partial_t^j\lambda, \partial_s^h k)\cdot\mathfrak{q}_\beta(\partial_t^l\lambda, \partial_s^m k)=\mathfrak{q}_{\alpha+\beta}(\partial_t^{\max\{j,l\}}\lambda, \partial_s^{\max\{h,m\}}k)\,.$$

We already saw that the time derivatives of k and λ can be expressed in terms of space derivatives of k at any 3–point, the same holds for the space derivatives of λ, arguing by induction using the last two formulas in Lemma 4.3. Hence, it follows that

$$\partial_s^l\mathfrak{p}_\alpha(\partial_s^j\lambda, \partial_s^h k)=\mathfrak{p}_{\alpha+l}(\partial_s^{j+l}\lambda, \partial_s^{h+l}k)\,,\qquad\partial_t^l\mathfrak{p}_\alpha(\partial_s^j\lambda, \partial_s^h k)=\mathfrak{p}_{\alpha+2l}(\partial_s^{j+2l}\lambda, \partial_s^{h+2l}k)$$
$$\partial_t^l\mathfrak{q}_\alpha(\partial_t^j\lambda, \partial_s^h k)=\mathfrak{q}_{\alpha+2l}(\partial_t^{j+l}\lambda, \partial_s^{h+2l}k)\,,\qquad\mathfrak{q}_\alpha(\partial_t^j\lambda, \partial_s^h k)=\mathfrak{p}_\alpha(\partial_s^{2j}\lambda, \partial_s^{\max\{h,2j-1\}}k)\,.$$

We are now ready to compute, for $j\in\mathbb{N}$,

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{S}_t}|\partial_s^j k|^2\,ds =&\, 2\int_{\mathbb{S}_t}\partial_s^j k\,\partial_t\partial_s^j k\,ds+\int_{\mathbb{S}_t}|\partial_s^j k|^2(\lambda_s-k^2)\,ds\\
=&\, 2\int_{\mathbb{S}_t}\partial_s^j k\,\partial_s^{j+2}k+\lambda\partial_s^{j+1}k\,\partial_s^j k+\mathfrak{p}_{j+3}(\partial_s^j k)\,\partial_s^j k\,ds+\int_{\mathbb{S}_t}|\partial_s^j k|^2(\lambda_s-k^2)\,ds\\
=&\,-2\int_{\mathbb{S}_t}|\partial_s^{j+1}k|^2\,ds+\int_{\mathbb{S}_t}\partial_s(\lambda|\partial_s^j k|^2)\,ds+\int_{\mathbb{S}_t}\mathfrak{p}_{2j+4}(\partial_s^j k)\,ds\\
&-2\sum_{p=1}^{m}\sum_{i=1}^{3}\partial_s^j k^{pi}\,\partial_s^{j+1}k^{pi}\,\Big|_{\text{at the 3–point } O^p}+2\sum_{r=1}^{l}\partial_s^j k\,\partial_s^{j+1}k\,\Big|_{\text{at the end–point } P^r}\\
=&\,-2\int_{\mathbb{S}_t}|\partial_s^{j+1}k|^2\,ds+\int_{\mathbb{S}_t}\mathfrak{p}_{2j+4}(\partial_s^j k)\,ds+lC_jC_{j+1}\\
&-\sum_{p=1}^{m}\sum_{i=1}^{3}2\partial_s^j k^{pi}\,\partial_s^{j+1}k^{pi}+\lambda^{pi}|\partial_s^j k^{pi}|^2\,\Big|_{\text{at the 3–point } O^p}
\end{aligned}
\tag{4.2}
$$

where we integrated by parts the first term on the second line and we estimated the contributions given by the end–points P^r by means of assumption (4.1).

In the case that we consider the end–points $P^1, P^2, \dots, P^l$ to be fixed, we can assume that the terms C_jC_{j+1} are all zero in the above conclusion, by the following lemma.

LEMMA 4.6. *If the end–points P^r of the network are fixed, then there holds $\partial_s^j k = \partial_s^j \lambda = 0$, for every even $j \in \mathbb{N}$.*

PROOF. The first case $j = 0$ simply follows from the fact that the velocity $\underline{v} = \lambda\tau + k\nu$ is always zero at the fixed end–points P^r.
We argue by induction, we suppose that for every even natural $l \leq j-2$ we have $\partial_s^l k = \partial_s^l \lambda = 0$, then, by using the first equation in Lemma 4.3, we get

$$\partial_s^j k = \partial_t\partial_s^{j-2}k - \lambda\partial_s^{j-1}k - \mathfrak{p}_{j+1}(\partial_s^{j-2}k)$$

at every end–point P^r.
We already know that $\lambda = 0$ and by the inductive hypothesis $\partial_s^{j-2}k = 0$, thus $\partial_t\partial_s^{j-2}k = 0$. Since $\mathfrak{p}_{j+1}(\partial_s^{j-2}k)$ is a sum of terms like $C\prod_{l=0}^{j-2}(\partial_s^l k)^{\alpha_l}$ with $\sum_{l=0}^{j-2}(l+1)\alpha_l = j+1$ which is odd, at least one of the terms of this sum has to be odd, hence at least for one index l, the product $(l+1)\alpha_l$ is odd. It follows that at least for one even l the exponent α_l is nonzero. Hence, at least one even derivatives is present in every monomial of $\mathfrak{p}_{j+1}(\partial_s^{j-2}k)$, which contains only derivatives up to the order $(j-2)$.
Again, by the inductive hypothesis we then conclude that at the end–points $\partial_s^j k = 0$.
We can deal with λ similarly, by means of the relations in Lemma 4.3. □

In the very special case $j = 0$ we get explicitly

$$\frac{d}{dt}\int_{\mathbb{S}_t}k^2\,ds = -2\int_{\mathbb{S}_t}|k_s|^2\,ds+\int_{\mathbb{S}_t}k^4\,ds-\sum_{p=1}^{m}\sum_{i=1}^{3}2k^{pi}k_s^{pi}+\lambda^{pi}|k^{pi}|^2\,\Big|_{\text{at the 3–point } O^p}+lC_0C_1 \tag{4.3}$$

where the two constants C_0 and C_1 come from assumption (4.1).

Then, recalling relation (2.9), we have $\sum_{i=1}^{3}k^{pi}k_s^{pi}+\lambda^{pi}|k^{pi}|^2\,\big|_{\text{at the 3–point } O^p} = 0$, and substituting above,

$$\frac{d}{dt}\int_{\mathbb{S}_t}k^2\,ds = -2\int_{\mathbb{S}_t}|k_s|^2\,ds+\int_{\mathbb{S}_t}k^4\,ds+\sum_{p=1}^{m}\sum_{i=1}^{3}\lambda^{pi}|k^{pi}|^2\,\Big|_{\text{at the 3–point } O^p}+lC_0C_1\,, \tag{4.4}$$

hence, we lowered the maximum order of the space derivatives of the curvature in the 3–point terms, particular now it is lower than the one of the "nice" negative integral.

As we have just seen for the case $j = 0$, also for the general case we want to simplify the term $\sum_{i=1}^{3} 2\partial_s^j k^{pi} \partial_s^{j+1} k^{pi} + \lambda^{pi} |\partial_s^j k^{pi}|^2 \,\big|_{\text{at the 3–point } O^p}$, in order to control it.

Using formulas in Lemma 4.3, we have (see [**71**, pp. 258–259], for details)

$$\begin{aligned}
&2\partial_s^j k\ \partial_s^{j+1} k + \lambda |\partial_s^j k|^2 \\
&\quad = 2\partial_t^{j/2} k \cdot \partial_t^{j/2}(k_s + k\lambda) + \mathfrak{q}_{j+1}(\partial_t^{j/2-1}\lambda, \partial_s^{j-1} k) \cdot \partial_t^{j/2} k_s + \mathfrak{q}_{2j+3}(\partial_t^{j/2}\lambda, \partial_s^j k)\,.
\end{aligned}$$

We now examine the term $\mathfrak{q}_{j+1}(\partial_t^{j/2-1}\lambda, \partial_s^{j-1} k) \cdot \partial_t^{j/2} k_s$, which , by using Lemma 4.3, can be written as $\partial_t \mathfrak{q}_{2j+1}(\partial_t^{j/2-1}\lambda, \partial_s^{j-1} k) + \mathfrak{q}_{2j+3}(\partial_t^{j/2}\lambda, \partial_s^j k)$ (see [**71**, pp. 258–259], for details). It follows that

$$\begin{aligned}
&\sum_{p=1}^{m}\sum_{i=1}^{3} 2\partial_s^j k^{pi}\ \partial_s^{j+1} k^{pi}\ + \lambda^{pi} |\partial_s^j k^{pi}|^2 \lambda\,\Big|_{\text{at the 3–point } O^p} \\
&\qquad = \sum_{p=1}^{m}\sum_{i=1}^{3} \partial_t \mathfrak{q}_{2j+1}(\partial_t^{j/2-1}\lambda^{pi}, \partial_s^{j-1} k^{pi}) + \mathfrak{q}_{2j+3}(\partial_t^{j/2}\lambda^{pi}, \partial_s^j k^{pi})\,\Big|_{\text{at the 3–point } O^p}
\end{aligned}$$

Resuming, if $j \geq 2$ is even, we have

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{S}_t} |\partial_s^j k|^2\, ds = &- 2\int_{\mathbb{S}_t} |\partial_s^{j+1} k|^2\, ds + \int_{\mathbb{S}_t} \mathfrak{p}_{2j+4}(\partial_s^j k)\, ds + lC_jC_{j+1} \\
&+ \sum_{p=1}^{m}\sum_{i=1}^{3} \partial_t \mathfrak{q}_{2j+1}(\partial_t^{j/2-1}\lambda^{pi}, \partial_s^{j-1} k^{pi}) + \mathfrak{q}_{2j+3}(\partial_t^{j/2}\lambda^{pi}, \partial_s^j k^{pi})\,\Big|_{\text{at the 3–point } O^p}\,.
\end{aligned}$$

Now, the key tool to estimate $\int_{\mathbb{S}_t} \mathfrak{p}_{2j+4}(\partial_s^j k)\, ds$ and $\sum_{i=1}^{3} \mathfrak{q}_{2j+3}(\partial_t^{j/2}\lambda^{pi}, \partial_s^j k^{pi})\,\big|_{\text{at the 3–point } O^p}$ will be the following Gagliardo–Nirenberg interpolation inequalities (see [**77**, Section 3], pp. 257–263).

PROPOSITION 4.7. *Let γ be a C^∞, regular curve in $\mathbb{R}^2$ with finite length* L. *If u is a C^∞ function defined on γ and $m \geq 1$, $p \in [2, +\infty]$, we have the estimates*

$$\|\partial_s^n u\|_{L^p} \leq C_{n,m,p} \|\partial_s^m u\|_{L^2}^{\sigma} \|u\|_{L^2}^{1-\sigma} + \frac{B_{n,m,p}}{\mathrm{L}^{m\sigma}} \|u\|_{L^2} \tag{4.5}$$

for every $n \in \{0, \dots, m-1\}$ where

$$\sigma = \frac{n + 1/2 - 1/p}{m}$$

and the constants $C_{n,m,p}$ and $B_{n,m,p}$ are independent of γ. In particular, if $p = +\infty$,

$$\|\partial_s^n u\|_{L^\infty} \leq C_{n,m} \|\partial_s^m u\|_{L^2}^{\sigma} \|u\|_{L^2}^{1-\sigma} + \frac{B_{n,m}}{\mathrm{L}^{m\sigma}} \|u\|_{L^2} \qquad \textit{with} \qquad \sigma = \frac{n + 1/2}{m}. \tag{4.6}$$

After estimating the integral of every monomial of $\mathfrak{p}_{2j+4}(\partial_s^j k)$ by means of the Hölder inequality, one uses the Gagliardo–Nirenberg estimates on the result, concluding that

$$\int_{\mathbb{S}_t} \mathfrak{p}_{2j+4}(\partial_s^j k)\, ds \leq 1/4 \int_{\mathbb{S}_t} |\partial_s^{j+1} k|^2\, ds + C\left(\int_{\mathbb{S}_t} k^2\, ds\right)^{2j+3} + C\,,$$

where the constant C depends only on $j \in \mathbb{N}$ and the lengths of the curves of the network (see [**71**, pp. 260–262], for details).
Any term $\sum_{i=1}^{3} \mathfrak{q}_{2j+3}(\partial_t^{j/2}\lambda^{pi}, \partial_s^j k^{pi})\,\big|_{\text{at the 3–point } O^p}$ can be estimated similarly.

Hence, for every even $j \geq 2$ we can finally write

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{S}_t} |\partial_s^j k|^2\,ds \leq & -\int_{\mathbb{S}_t} |\partial_s^{j+1} k|^2\,ds + C\left(\int_{\mathbb{S}_t} k^2\,ds\right)^{2j+3} + C + lC_jC_{j+1}\\
& + \partial_t \sum_{p=1}^{m}\sum_{i=1}^{3} \mathfrak{q}_{2j+1}(\partial_t^{j/2-1}\lambda^{pi}, \partial_s^{j-1}k^{pi})\Big|_{\text{at the 3–point } O^p}\\
\leq & \, C\left(\int_{\mathbb{S}_t} k^2\,ds\right)^{2j+3} + \partial_t \sum_{p=1}^{m}\sum_{i=1}^{3} \mathfrak{q}_{2j+1}(\partial_t^{j/2-1}\lambda^{pi}, \partial_s^{j-1}k^{pi})\Big|_{\text{at the 3–point } O^p}\\
& + C + lC_jC_{j+1}\,.
\end{aligned}
\tag{4.7}
$$

Recalling the computation in the special case $j = 0$, this argument gives the same final estimate without the contributions coming from the 3–points:

$$
\left|\frac{d}{dt}\int_{\mathbb{S}_t} k^2\,ds\right| \leq C\left(\int_{\mathbb{S}_t} k^2\,ds\right)^3 + C + lC_0C_1\,. \tag{4.8}
$$

Integrating (4.7) in time on $[0,t]$ and estimating we get

$$
\begin{aligned}
\int_{\mathbb{S}_t} |\partial_s^j k|^2\,ds \leq & \int_{\mathbb{S}_0} |\partial_s^j k|^2\,ds + C\int_0^t \left(\int_{\mathbb{S}_\xi} k^2\,ds\right)^{2j+3} d\xi + Ct + lC_jC_{j+1}t\\
& + \sum_{p=1}^{m}\sum_{i=1}^{3} \mathfrak{q}_{2j+1}(\partial_t^{j/2-1}\lambda^{pi}(0,t), \partial_s^{j-1}k^{pi}(0,t))\\
& - \mathfrak{q}_{2j+1}(\partial_t^{j/2-1}\lambda^{pi}(0,0), \partial_s^{j-1}k^{pi}(0,0))\\
\leq & \, C\int_0^t \left(\int_{\mathbb{S}_\xi} k^2\,ds\right)^{2j+3} d\xi + \|\mathfrak{p}_{2j+1}(|\partial_s^{j-1}k|)\|_{L^\infty} + Ct + lC_jC_{j+1}t + C\,,
\end{aligned}
$$

where in the last passage we used Remark 4.4. The constant C depends only on $j \in \mathbb{N}$ and on the network $\mathbb{S}_0$.
Interpolating again by means of inequalities (4.6), one gets

$$
\|\mathfrak{p}_{2j+1}(|\partial_s^{j-1}k|)\|_{L^\infty} \leq 1/2\|\partial_s^j k\|_{L^2}^2 + C\|k\|_{L^2}^{4j+2}\,.
$$

Hence, putting all together, for every even $j \in \mathbb{N}$, we conclude

$$
\int_{\mathbb{S}_t} |\partial_s^j k|^2\,ds \leq C\int_0^t \left(\int_{\mathbb{S}_\xi} k^2\,ds\right)^{2j+3} d\xi + C\left(\int_{\mathbb{S}_t} k^2\,ds\right)^{2j+1} + Ct + lC_jC_{j+1}t + C\,.
$$

Passing from integral to L^∞ estimates, by using inequalities (4.6), we have the following proposition.

PROPOSITION 4.8. *If assumption* (4.1) *holds, the lengths of all the curves are uniformly positively bounded from below and the L^2 norm of k is uniformly bounded on $[0,T)$, then the curvature of $\mathbb{S}_t$ and all its space derivatives are uniformly bounded in the same time interval by some constants depending only on the L^2 integrals of the space derivatives of k on the initial network $\mathbb{S}_0$.*

By using the relations in Lemma 4.3 , one then gets also estimates for every time and space derivatives of λ which finally imply estimates on all the derivatives of the maps γ^i, stated in the next proposition (see **[71**, pp. 263–266] for details).

PROPOSITION 4.9. *If $\mathbb{S}_t$ is a C^∞ special evolution of the initial network $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i$, satisfying assumption* (4.1)*, such that the lengths of the n curves are uniformly bounded away from zero and the L^2 norm of the curvature is uniformly bounded by some constants in the time interval $[0,T)$, then*

- *all the derivatives in space and time of k and λ are uniformly bounded in $[0,1]\times[0,T)$,*
- *all the derivatives in space and time of the curves $\gamma^i(x,t)$ are uniformly bounded in $[0,1]\times[0,T)$,*

- *the quantities $|\gamma^i_x(x,t)|$ are uniformly bounded from above and away from zero in $[0,1]\times[0,T)$.*

All the bounds depend only on the uniform controls on the L^2 norm of k, on the lengths of the curves of the network from below, on the constants C_j in assumption (4.1)*, on the L^∞ norms of the derivatives of the curves σ^i and on the bound from above and below on $|\sigma^i_x(x,t)|$, for the curves describing the initial network $\mathbb{S}_0$.*

Before proceeding with another set of estimates we want to stress a particular case of (4.7) (we consider a single triod with moving end-points and $j=0$) putting it into a separate lemma. This result crucial in proving Theorem 8.20.

LEMMA 4.10. *Let $F:\mathbb{T}\times[0,T)\to\mathbb{R}^2$, with $T<\infty$, be a triod moving by curvature* with moving end–points $Q^i:[0,T)\to\Omega$ *such that the lengths of the three curves are uniformly bounded from below away from zero by $L>0$.*
Then, for some constants $C>0$, $\widetilde{C}>0$, independent of the triod, the following estimate holds:

$$\frac{d}{dt}\int_{\mathbb{T}_t}k^2\,ds\leq C\Big(\int_{\mathbb{T}_t}k^2\,ds\Big)^3+\frac{\widetilde{C}}{L}\Big(\int_{\mathbb{T}_t}k^2\,ds\Big)^2+\sum_{i=1}^3 k^i(2k^i_s+\lambda^i k^i)\,\Big\vert_{\text{at the point }Q^i(t)}.$$

PROOF. We start writing explicitly Equation (4.2) for $j=0$ and for a network with only one triple junction with moving end–points:

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}_t}k^2\,ds=&-2\int_{\mathbb{T}_t}k_s^2\,ds+\int_{\mathbb{T}_t}k^4\,ds-\sum_{i=1}^3 k^i(2k^i_s+\lambda^i k^i)\,\Big\vert_{\text{at the 3–point}}\\&+\sum_{i=1}^3 k^i(2k^i_s+\lambda^i k^i)\,\Big\vert_{\text{at the point }Q^i(t)}\\=&-2\int_{\mathbb{T}_t}k_s^2\,ds+\int_{\mathbb{T}_t}k^4\,ds+\sum_{i=1}^3\lambda^i|k^i|^2\,\Big\vert_{\text{at the 3–point}}\\&+\sum_{i=1}^3 k^i(2k^i_s+\lambda^i k^i)\,\Big\vert_{\text{at the point }Q^i(t)},\end{aligned}$$

where we applied the "orthogonality" relation 2.9.
Letting L to be the minimum of the length of the three curves of the triod, by Proposition 4.7 (applied to $u=k$ and having set $p=4$, $n=0$, $m=1$, $\sigma=1/4$) and Peter–Paul inequality, for any $\varepsilon>0$ we have the interpolation estimate

$$\begin{aligned}\int_{\mathbb{T}_t}k^4\,ds\leq&\Big[C\Big(\int_{\mathbb{T}_t}k_s^2\,ds\Big)^{1/8}\Big(\int_{\mathbb{T}_t}k^2ds\Big)^{3/8}+\frac{C}{L^{1/4}}\Big(\int_{\mathbb{T}_t}k^2\,ds\Big)^{1/2}\Big]^4\\\leq&\, C\Big(\int_{\mathbb{T}_t}k_s^2\,ds\Big)^{1/2}\Big(\int_{\mathbb{T}_t}k^2ds\Big)^{3/2}+\frac{C}{L}\Big(\int_{\mathbb{T}_t}k^2\,ds\Big)^2\\\leq&\,\varepsilon\int_{\mathbb{T}_t}k_s^2\,ds+C_1\Big(\int_{\mathbb{T}_t}k^2\,ds\Big)^3+\frac{C_2}{L}\Big(\int_{\mathbb{T}_t}k^2\,ds\Big)^2,\end{aligned}$$

where the constants C_1, C_2 depend on ε.
Moreover

$$\sum_{i=1}^3\lambda^i|k^i|^2\,\Big\vert_{\text{at the 3–point}}\leq C\|k\|^3_{L^\infty}$$

and again by Peter–Paul inequality, we obtain

$$\|k\|^3_{L^\infty}\leq\varepsilon\int_{\mathbb{T}_t}k_s^2\,ds+C_3\Big(\int_{\mathbb{T}_t}k^2\,ds\Big)^3$$

where the constant C_3 depend on ε.
Substituting in the equation above, after taking $\varepsilon<1$, we get the thesis. ☐

Now, we work out a second set of estimates where everything is controlled – still under the assumption (4.1) – only by the L^2 norm of the curvature and the inverses of the lengths of the curves at time zero.

As before we consider the C^∞ special curvature flow $\mathbb{S}_t$ of a smooth network $\mathbb{S}_0$ in the time interval $[0,T)$, composed by n curves $\gamma^i(\cdot,t):[0,1]\to\overline{\Omega}$ with m 3–points $O^1,O^2,\dots,O^m$ and l end–points $P^1,P^2,\dots,P^l$, satisfying assumption (4.1).

PROPOSITION 4.11. *For every $M>0$ there exists a time $T_M\in(0,T)$, depending only on the* structure *of the network and on the constants C_0 and C_1 in assumption* (4.1)*, such that if the square of the L^2 norm of the curvature and the inverses of the lengths of the curves of $\mathbb{S}_0$ are bounded by M, then the square of the L^2 norm of k and the inverses of the lengths of the curves of $\mathbb{S}_t$ are smaller than $2(n+1)M+1$, for every time $t\in[0,T_M]$.*

PROOF. The evolution equations for the lengths of the n curves are given by

$$\frac{dL^i(t)}{dt}=\lambda^i(1,t)-\lambda^i(0,t)-\int_{\gamma^i(\cdot,t)}k^2\,ds\,,$$

then, recalling computation (4.4), we have

$$\begin{aligned}
\frac{d}{dt}\left(\int_{\mathbb{S}_t}k^2\,ds+\sum_{i=1}^n\frac{1}{L^i}\right)\leq&-2\int_{\mathbb{S}_t}k_s^2\,ds+\int_{\mathbb{S}_t}k^4\,ds+6m\|k\|^3_{L^\infty}+lC_0C_1-\sum_{i=1}^n\frac{1}{(L^i)^2}\frac{dL^i}{dt}\\
=&-2\int_{\mathbb{S}_t}k_s^2\,ds+\int_{\mathbb{S}_t}k^4\,ds+6m\|k\|^3_{L^\infty}+lC_0C_1\\
&-\sum_{i=1}^n\frac{\lambda^i(1,0)-\lambda^i(0,t)+\int_{\gamma^i(\cdot,t)}k^2\,ds}{(L^i)^2}\\
\leq&-2\int_{\mathbb{S}_t}k_s^2\,ds+\int_{\mathbb{S}_t}k^4\,ds+6m\|k\|^3_{L^\infty}+lC_0C_1\\
&+2\sum_{i=1}^n\frac{\|k\|_{L^\infty}+C_0}{(L^i)^2}+\sum_{i=1}^n\frac{\int_{\mathbb{S}_t}k^2\,ds}{(L^i)^2}\\
\leq&-2\int_{\mathbb{S}_t}k_s^2\,ds+\int_{\mathbb{S}_t}k^4\,ds+(6m+2n/3)\|k\|^3_{L^\infty}+lC_0C_1+2nC_0^3/3\\
&+\frac{n}{3}\left(\int_{\mathbb{S}_t}k^2\,ds\right)^3+\frac{2}{3}\sum_{i=1}^n\frac{1}{(L^i)^3}
\end{aligned}$$

where we used Young inequality in the last passage.
Interpolating as before (and applying again Young inequality) but keeping now in evidence the terms depending on L^i in inequalities (4.5), we obtain

$$\begin{aligned}
\frac{d}{dt}\left(\int_{\mathbb{S}_t}k^2\,ds+\sum_{i=1}^n\frac{1}{L^i}\right)\leq&-\int_{\mathbb{S}_t}k_s^2\,ds+C\left(\int_{\mathbb{S}_t}k^2\,ds\right)^3+C\sum_{i=1}^n\frac{\left(\int_{\mathbb{S}_t}k^2\,ds\right)^2}{L^i}\\
&+C\sum_{i=1}^n\frac{\left(\int_{\mathbb{S}_t}k^2\,ds\right)^{3/2}}{(L^i)^{3/2}}+C\sum_{i=1}^n\frac{1}{(L^i)^3}+C\\
\leq&\,C\left(\int_{\mathbb{S}_t}k^2\,ds\right)^3+C\sum_{i=1}^n\frac{1}{(L^i)^3}+C\\
\leq&\,C\left(\int_{\mathbb{S}_t}k^2\,ds+\sum_{i=1}^n\frac{1}{L^i}+1\right)^3,
\end{aligned}$$

with a constant C depending only on the *structure* of the network and on the constants C_0 and C_1 in assumption (4.1).

This means that the positive function $f(t) = \int_{\mathbb{S}_t} k^2\, ds + \sum_{i=1}^n \frac{1}{L^i(t)} + 1$ satisfies the differential inequality $f' \leq Cf^3$, hence, after integration

$$f^2(t) \leq \frac{f^2(0)}{1 - 2Ctf^2(0)} \leq \frac{f^2(0)}{1 - 2Ct[(n+1)M + 1]}$$

then, if $t \leq T_M = \frac{3}{8C[(n+1)M+1]}$, we get $f(t) \leq 2f(0)$. Hence,

$$\int_{\mathbb{S}_t} k^2\, ds + \sum_{i=1}^n \frac{1}{L^i(t)} \leq 2\int_{\mathbb{S}_0} k^2\, ds + 2\sum_{i=1}^n \frac{1}{L^i(0)} + 1 \leq 2[(n+1)M] + 1\,.$$

□

By means of this proposition we can strengthen the conclusion of Proposition 4.9.

COROLLARY 4.12. *In the hypothesis of the previous proposition, in the time interval $[0, T_M]$ all the bounds in Proposition 4.9 depend only on the L^2 norm of k on $\mathbb{S}_0$, on the constants C_j in assumption* (4.1)*, on the L^∞ norms of the derivatives of the curves σ^i, on the bound from above and below on $|\sigma^i_x(x,t)|$ and on the lengths of the curves of the initial network $\mathbb{S}_0$.*

From now on we assume that the L^2 norm of the curvature and the inverses of the lengths of the curves are bounded in the interval $[0, T_M]$.

Considering $j \in \mathbb{N}$ even, if we differentiate the function

$$\int_{\mathbb{S}_t} k^2 + tk_s^2 + \frac{t^2k_{ss}^2}{2!} + \cdots + \frac{t^j|\partial_s^j k|^2}{j!}\, ds\,,$$

and we estimate with interpolation inequalities as before (see [**71**, pp. 268–269], for details), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{S}_t} k^2 + tk_s^2 + \frac{t^2k_{ss}^2}{2!} + \cdots + \frac{t^j|\partial_s^j k|^2}{j!}\, ds \qquad (4.9)\\
&\leq -\,\varepsilon\int_{\mathbb{S}_t} k_s^2 + tk_{ss}^2 + t^2k_{sss}^2 + \cdots + t^j|\partial_s^{j+1}k|^2\, ds + C\\
&\quad + \partial_t\sum_{p=1}^m\sum_{i=1}^3 t^2\mathfrak{q}_5(\lambda^{pi}, k_s^{pi}) + t^4\mathfrak{q}_9(\partial_t\lambda^{pi}, k_{sss}^{pi}) + \cdots + t^j\mathfrak{q}_{2j+1}(\partial_t^{j/2-1}\lambda^{pi}, \partial_s^{j-1}k^{pi})\,\Big|_{\text{at the 3–point } O^p}\\
&\quad + C\sum_{p=1}^m\sum_{i=1}^3 tk_s^{pi}k_{ss}^{pi} + t^3k_{sss}^{pi}k_{ssss}^{pi} + \cdots + t^{j-1}\partial_s^{j-1}k^{pi}\,\partial_s^j k^{pi}\,\Big|_{\text{at the 3–point } O^p}
\end{aligned}$$

in the time interval $[0, T_M]$, where $\varepsilon > 0$ and C are two constants depending only on the L^2 norm of the curvature, the constants in assumption (4.1) and the inverses of the lengths of the n curves of $\mathbb{S}_0$.

We proceed as we did before for the computation of $\frac{d}{dt}\int_{\mathbb{S}_t} |\partial_s^j k|^2\, ds$.

First we deal with the last line,

$$\sum_{i=1}^3 tk_s^{pi}k_{ss}^{pi} + t^3k_{sss}^{pi}k_{ssss}^{pi} + \cdots + t^{j-1}\partial_s^{j-1}k^{pi}\,\partial_s^j k^{pi}\,\Big|_{\text{at the 3–point}}\,.$$

By formulas in Lemma 4.3 and by Remark 4.4, for any term $\sum_{i=1}^{3} t^{h-1}\partial_s^{h-1}k^i\partial_s^h k^i\Big|_{\text{at the 3–point}}$ we can write

$$\begin{aligned}\sum_{i=1}^{3} t^{h-1}\partial_s^{h-1}k^i\partial_s^h k^i\Big|_{\text{at the 3–point}} &= \sum_{i=1}^{3} t^{h-1}\mathfrak{q}_{2h+1}(\partial_t^{h/2-1}\lambda^i, \partial_s^{h-1}k^i)\\ &\quad + t^{h-1}\partial_s^h k^i\cdot\mathfrak{q}_h(\partial_t^{h/2-1}\lambda^i, \partial_s^{h-2}k^i)\Big|_{\text{at the 3–point}}\\ &\leq t^{h-1}\|\mathfrak{p}_{2h+1}(|\partial_s^{h-1}k|)\|_{L^\infty} + t^{h-1}\|\partial_s^h k\|_{L^\infty}\|\mathfrak{p}_h(|\partial_s^{h-2}k|)\|_{L^\infty}\end{aligned}$$

(see [**71**, p. 270], for details).
The term $t^{h-1}\|\mathfrak{p}_{2h+1}(|\partial_s^{h-1}k|)\|_{L^\infty}$ is controlled as before by a small fraction of $t^{h-1}\int_{\mathbb{S}_t}|\partial_s^h k|^2\,ds$ and a possibly large multiple of t^{h-1} times some power of the L^2 norm of k (which is bounded), whereas $t^{h-1}\|\partial_s^h k\|_{L^\infty}\|\mathfrak{p}_h(|\partial_s^{h-2}k|)\|_{L^\infty}$ is the critical term.
Again by means of interpolation inequalities (4.6) one estimates $\|\partial_s^h k\|_{L^\infty}$, $\|\mathfrak{p}_h(\partial_s^{h-2}k)\|_{L^\infty}$ and $\|\partial_s^h k\|_{L^2}$ with the L^2 norm of k and its derivatives. After some computation (see [**71**, pp. 270–271], for details), one gets

$$\sum_{i=1}^{3} t^{h-1}\partial_s^{h-1}k^i\partial_s^h k^i\Big|_{\text{at the 3–point}} \leq \varepsilon_h/2\left(t^h\int_{\mathbb{S}_t}|\partial_s^{h+1}k|^2\,ds + t^{h-1}\int_{\mathbb{S}_t}|\partial_s^h k|^2\,ds + Ct^h\right) + C/t^{\theta_h}$$

with $\theta_h < 1$ and some small $\varepsilon_h > 0$.

We apply this argument for every even h from 2 to j, choosing accurately small values ε_j. Hence, we can continue estimate (4.9) as follows,

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{S}_t} k^2 + tk_s^2 + \frac{t^2k_{ss}^2}{2!} + \dots + \frac{t^j|\partial_s^j k|^2}{j!}\,ds\\ &\leq -\varepsilon/2\int_{\mathbb{S}_t} k_s^2 + tk_{ss}^2 + t^2k_{sss}^2 + \dots + t^j|\partial_s^{j+1}k|^2\,ds + C + C/t^{\theta_2} + \dots + C/t^{\theta_j}\\ &\quad + \partial_t\sum_{i=1}^{3} t^2\mathfrak{q}_5(\lambda^i, k_s^i) + t^4\mathfrak{q}_9(\partial_t\lambda^i, k_{sss}^i) + \dots + t^j\mathfrak{q}_{2j+1}(\partial_t^{j/2-1}\lambda^i, \partial_s^{j-1}k^i)\Big|_{\text{at the 3–point}}\\ &\leq C + C/t^\theta + \partial_t\sum_{i=1}^{3} t^2\mathfrak{q}_5(\lambda^i, k_s^i) + t^4\mathfrak{q}_9(\partial_t\lambda^i, k_{sss}^i) + \dots + t^j\mathfrak{q}_{2j+1}(\partial_t^{j/2-1}\lambda^i, \partial_s^{j-1}k^i)\Big|_{\text{at the 3–point}}\end{aligned}$$

for some $\theta < 1$.
Integrating this inequality in time on $[0, t]$ with $t \leq T_M$ and taking into account Remark 4.4, we get

$$\begin{aligned}&\int_{\mathbb{S}_t} k^2 + tk_s^2 + \frac{t^2k_{ss}^2}{2!} + \dots + \frac{t^j|\partial_s^j k|^2}{j!}\,ds\\ &\quad\leq \int_{\mathbb{S}_0} k^2\,ds + CT_M + CT_M^{(1-\theta)}\\ &\quad\quad + \sum_{i=1}^{3} t^2\mathfrak{q}_5(\lambda^i, k_s^i) + t^4\mathfrak{q}_9(\partial_t\lambda^i, k_{sss}^i) + \dots + t^j\mathfrak{q}_{2j+1}(\partial_t^{j/2-1}\lambda^i, \partial_s^{j-1}k^i)\Big|_{\text{at the 3–point}}\\ &\quad\leq \int_{\mathbb{S}_0} k^2\,ds + C + t^2\|\mathfrak{p}_5(|k_s|)\|_{L^\infty} + t^4\|\mathfrak{p}_9(|k_{sss}|)\|_{L^\infty} + \dots + t^j\|\mathfrak{p}_{2j+1}(|\partial_s^{j-1}k|)\|_{L^\infty}\,.\end{aligned}$$

Now we absorb all the polynomial terms, after interpolating each one of them between the corresponding "good" integral in the left member and some power of the L^2 norm of k, as we did in showing Proposition 4.8, hence we finally obtain for every even $j \in \mathbb{N}$,

$$\int_{\mathbb{S}_t} k^2 + tk_s^2 + \frac{t^2k_{ss}^2}{2!} + \dots + \frac{t^j|\partial_s^j k|^2}{j!}\,ds \leq \overline{C}_j$$

with $t \in [0, T_M]$ and a constant $\overline{C}_j$ depending only on the constants in assumption (4.1) and the bounds on $\int_{\mathbb{S}_0} k^2\,ds$ and on the inverses of the lengths of the curves of the initial network $\mathbb{S}_0$. This family of inequalities clearly implies

$$\int_{\mathbb{S}_t} |\partial_s^j k|^2\,ds \leq \frac{C_j j!}{t^j} \qquad \text{for every even } j \in \mathbb{N}. \tag{4.10}$$

Then, passing as before from integral to L^∞ estimates by means of inequalities (4.6), we have the following proposition.

PROPOSITION 4.13. *For every $\mu > 0$ the curvature and all its space derivatives of $\mathbb{S}_t$ are uniformly bounded in the time interval $[\mu, T_M]$ (where T_M is given by Proposition 4.11) by some constants depending only on μ, the constants in assumption* (4.1) *and the bounds on $\int_{\mathbb{S}_0} k^2\,ds$ and on the inverses of the lengths of the curves of the initial network $\mathbb{S}_0$.*

By means of these a priori estimates we can now work out some results about the smooth flow of an initial regular geometrically smooth network $\mathbb{S}_0$. Notice that these are examples of how to use the previous estimates on special smooth flows in order to get conclusion on general flows or even only C^∞ flows, as we mentioned at the beginning of this section.

THEOREM 4.14. *If $[0, T)$ is the maximal time interval of existence of a C^∞ curvature flow of an initial geometrically smooth network $\mathbb{S}_0$, with $T < +\infty$, then*

(1) *either the inferior limit of the length of at least one curve of $\mathbb{S}_t$ goes to zero when $t \to T$,*
(2) *or $\limsup_{t\to T} \int_{\mathbb{S}_t} k^2\,ds = +\infty$.*

Moreover, if the lengths of the n curves are uniformly positively bounded from below, then this superior limit is actually a limit and there exists a positive constant C such that

$$\int_{\mathbb{S}_t} k^2\,ds \geq \frac{C}{\sqrt{T-t}}\,,$$

for every $t \in [0, T)$.

PROOF. We can C^∞ reparametrize the flow $\mathbb{S}_t$ in order that it becomes a special smooth flow $\widetilde{\mathbb{S}}_t$ in $[0, T)$.
If the lengths of the curves of $\mathbb{S}_t$ are uniformly bounded away from zero and the L^2 norm of k is bounded, the same holds for the networks $\widetilde{\mathbb{S}}_t$, then, by Proposition 4.9 and Ascoli–Arzelà Theorem, the network $\widetilde{\mathbb{S}}_t$ converges in C^∞ to a smooth network $\widetilde{\mathbb{S}}_T$ as $t \to T$. Then, applying Theorem 3.18 to $\widetilde{\mathbb{S}}_T$ we could restart the flow obtaining a C^∞ special curvature flow in a longer time interval. Reparametrizing back this last flow, we get a C^∞ "extension" in time of the flow $\mathbb{S}_t$, hence contradicting the maximality of the interval $[0, T)$.
Now, considering again the flow $\widetilde{\mathbb{S}}_t$, by means of differential inequality (4.8), we have

$$\frac{d}{dt}\int_{\widetilde{\mathbb{S}}_t} \widetilde{k}^2\,ds \leq C\left(\int_{\widetilde{\mathbb{S}}_t} \widetilde{k}^2\,ds\right)^3 + C \leq C\left(1 + \int_{\widetilde{\mathbb{S}}_t} \widetilde{k}^2\,ds\right)^3\,,$$

which, after integration between $t, r \in [0, T)$ with $t < r$, gives

$$\frac{1}{\left(1 + \int_{\widetilde{\mathbb{S}}_t} \widetilde{k}^2\,ds\right)^2} - \frac{1}{\left(1 + \int_{\widetilde{\mathbb{S}}_r} \widetilde{k}^2\,ds\right)^2} \leq C(r-t)\,.$$

Then, if case (1) does not hold, we can choose a sequence of times $r_j \to T$ such that $\int_{\widetilde{\mathbb{S}}_{r_j}} \widetilde{k}^2\,ds \to +\infty$. Putting $r = r_j$ in the inequality above and passing to the limit $r \to T$ we get

$$\frac{1}{\left(1 + \int_{\widetilde{\mathbb{S}}_t} \widetilde{k}^2\,ds\right)^2} \leq C(T-t)$$

hence, for every $t \in [0, T)$

$$\int_{\widetilde{\mathbb{S}}_t} \widetilde{k}^2\,ds \geq \frac{C}{\sqrt{T-t}} - 1 \geq \frac{C}{\sqrt{T-t}}\,,$$

for some positive constant C and $\lim_{t\to T}\int_{\widetilde{\mathbb{S}}_t} k^2\,ds = +\infty$.
By the invariance of the curvature by reparametrization, this last estimate implies the same estimate for the flow $\mathbb{S}_t$. $\square$

This theorem obviously implies the following corollary.

COROLLARY 4.15. *If $[0,T)$, with $T < +\infty$, is the maximal time interval of existence of a C^∞ curvature flow of an initial geometrically smooth network $\mathbb{S}_0$ and the lengths of the curves are uniformly bounded away from zero, then*

$$\max_{\mathbb{S}_t} k^2 \geq \frac{C}{\sqrt{T-t}} \to +\infty\,, \tag{4.11}$$

as $t \to T$.

REMARK 4.16. In the case of the evolution γ_t of a single closed curve in the plane there exists a constant $C > 0$ such that if at time $T > 0$ a singularity develops, then

$$\max_{\gamma_t} k^2 \geq \frac{C}{T-t}$$

for every $t \in [0,T)$ (see **[48]**).
If this lower bound on the rate of blowing up of the curvature (which is clearly stronger than the one in inequality (4.11)) holds also in the case of the evolution of a network is an open problem (even if the network is a triod).

We conclude this section with following estimate from below on the maximal time of smooth existence.

PROPOSITION 4.17. *For every $M > 0$ there exists a positive time T_M such that if the L^2 norm of the curvature and the inverses of the lengths of the geometrically smooth network $\mathbb{S}_0$ are bounded by M, then the maximal time of existence $T > 0$ of a C^∞ curvature flow of $\mathbb{S}_0$ is larger than T_M.*

PROOF. As before, considering again the reparametrized special curvature flow $\widetilde{\mathbb{S}}_t$, by Proposition 4.11 in the interval $[0, \min\{T_M, T\})$ the L^2 norm of $\widetilde{k}$ and the inverses of the lengths of the curves of $\widetilde{\mathbb{S}}_t$ are bounded by $2M^2 + 6M$.
Then, by Theorem 4.14, the value $\min\{T_M, T\}$ cannot coincide with the maximal time of existence of $\widetilde{\mathbb{S}}_t$ (hence of $\mathbb{S}_t$), so it must be $T > T_M$. $\square$

The last part of the section is devoted to estimate the L^∞–norm of the curvature of the network trying to avoid as much as possible interpolation inequalities that introduce in the estimates coefficients depending on the length of the curves of the network.

LEMMA 4.18. *Let Ω be a convex open regular set and $\mathbb{S}_0$ a tree with end–points $P^1, P^2, \dots, P^l$ (not necessarily fixed during its motion) on $\partial\Omega$. Let $\mathbb{S}_t$ be a smooth evolution by curvature for $t \in [0,T)$ of the network $\mathbb{S}_0$ such that the square of the curvature at the end–points of $\mathbb{S}_t$ is uniformly bounded in time by some constant C. Then,*

$$\|k\|^2_{L^\infty} \leq 4^{n-1}C + D_n\|k\|_{L^2}\|k_s\|_{L^2}\,, \tag{4.12}$$

where $n \in \mathbb{N}$ is such that for every point $Q \in \mathbb{S}_0$ there is a path to get from Q to an end–point passing by at most n curves (clearly, n is smaller than the total number of curves of $\mathbb{S}_0$) and the constant D_n depends only on n.

PROOF. Let us first consider a network $\mathbb{S}_0$ with five curves, two triple junctions O^1, O^2 and four end–points P^1, P^2, P^3, P^4. In this case n is clearly equal to two. We call γ^i, for $i \leq 4$, the curve connecting P^i with one of the two triple junctions and γ^5 the curve connecting the two triple junctions (see the following Figure 5).

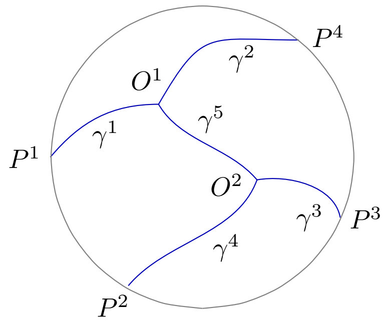


FIGURE 5. A tree with five curves.

Fixed a time $t \in [0,T)$, let $Q \in \gamma^i \subset \mathbb{S}_t$, for some $i \leq 4$. We compute

$$[k^i(Q)]^2 = [k^i(P^i)]^2 + 2\int_{P^i}^{Q} kk_s ds \leq C + 2\|k\|_{L^2}\|k_s\|_{L^2}\,,$$

hence, for every $Q \in \mathbb{S}_t \setminus \gamma^5$ we have

$$[k^i(Q)]^2 \leq C + 2\|k\|_{L^2}\|k_s\|_{L^2}\,.$$

Assume now instead that $Q \in \gamma^5$. Recalling that $\sum_{i=1}^{3} k^i = 0$ at each triple junction, by the previous argument we have $[k^i(O^1)]^2, [k^i(O^2)]^2 \leq C + 2\|k\|_{L^2}\|k_s\|_{L^2}$, for all $i \in \{1,2,3,4\}$, then it follows that $[k^5(O^1)]^2, [k^5(O^2)]^2 \leq 4C + 8\|k\|_{L^2}\|k_s\|_{L^2}$. Hence, arguing as before, we get

$$[k^5(Q)]^2 = [k^5(O^1)]^2 + 2\int_{O^1}^{Q} kk_s\, ds \leq 4C + 8\|k\|_{L^2}\|k_s\|_{L^2} + 2\int_{O^1}^{Q} kk_s\, ds\,,$$

In conclusion, we get the uniform in time inequality for $\mathbb{S}_t$

$$\|k\|^2_{L^\infty} \leq 4C + 10\|k\|_{L^2}\|k_s\|_{L^2}.$$

In the general case, since $\mathbb{S}_t$ are all trees homeomorphic to $\mathbb{S}_0$, we can argue similarly to get the conclusion by induction on n. □

LEMMA 4.19. *Let $\Omega \subset \mathbb{R}^2$ be open, convex and regular, let $\mathbb{S}_0$ be a tree with end–points $P^1, P^2, \dots, P^l$ on $\partial\Omega$ that satisfy assumptions* (4.1) *and let $\mathbb{S}_t$ for $t \in [0,T)$ be a smooth evolution by curvature of the network $\mathbb{S}_0$. Then $\|k\|^2_{L^2}$ is uniformly bounded on $[0,\widetilde{T})$ by $\sqrt{2}\big[\|k(\cdot,0)\|^2_{L^2}+1\big]$, where*

$$\widetilde{T} = \min\left\{T, 1/8C\left(\|k(\cdot,0)\|^2_{L^2}+1\right)^2\right\}.$$

Here the constant C depends only on the number $n \in \mathbb{N}$ of Lemma 4.18 and the constants in assumptions (4.1).

PROOF. By inequality (4.4) we have

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{S}_t} k^2\, ds \leq &-2\int_{\mathbb{S}_t} k_s^2\, ds + \int_{\mathbb{S}_t} k^4\, ds + \sum_{p=1}^{m}\sum_{i=1}^{3} \lambda^{pi}\left(k^{pi}\right)^2\Big|_{\text{at the 3–point } O^p} + C\\
\leq &-2\int_{\mathbb{S}_t} k_s^2\, ds + \|k\|^2_{L^\infty}\int_{\mathbb{S}_t} k^2\, ds + C\|k\|^3_{L^\infty} + C\,.
\end{aligned} \tag{4.13}$$

By estimate (4.12) and the Young inequality, we then obtain

$$\begin{aligned}
\|k\|^3_{L^\infty} &\leq C_n + C_n\|k\|^{\frac{3}{2}}_{L^2}\|k_s\|^{\frac{3}{2}}_{L^2} \leq C_n + \varepsilon\|k_s\|^2_{L^2} + C_{n,\varepsilon}\|k\|^6_{L^2}\,,\\
\|k\|^2_{L^\infty}\|k\|^2_{L^2} &\leq C_n\|k\|^2_{L^2} + D_n\|k\|^3_{L^2}\|k_s\|_{L^2} \leq C_n\|k\|^2_{L^2} + \varepsilon\|k_s\|^2_{L^2} + C_{n,\varepsilon}\|k\|^6_{L^2}\,,
\end{aligned}$$

for every small $\varepsilon > 0$ and a suitable constant $C_{n,\varepsilon}$.
Plugging these estimates into inequality (4.13) we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}_t} k^2 ds \leq & -2\|k_s\|^2 + \|k\|^2_{L^\infty}\|k\|^2 + C\|k\|^3_{L^\infty} + C \\ \leq & -2\|k_s\|^2 + C_n\|k\|^2_{L^2} + \varepsilon\|k_s\|^2_{L^2} + C_{n,\varepsilon}\|k\|^6_{L^2} + C_n + \varepsilon\|k_s\|^2_{L^2} + C_{n,\varepsilon}\|k\|^6_{L^2} + C_n \\ \leq & \, C\Big(\int_{\mathbb{S}_t} k^2 ds\Big)^3 + C\,, \end{aligned} \tag{4.14}$$

Where we chose $\varepsilon = 1/2$ and the constant C depends only on the number $n \in \mathbb{N}$ of Lemma 4.18 and the constants in conditions (4.1).

Calling $y(t) = \int_{\mathbb{S}_t} k^2\, ds + 1$, we can rewrite inequality (4.14) as the differential ODE

$$y'(t) \leq 2Cy^3(t)\,,$$

hence, after integration, we get

$$y(t) \leq \frac{1}{\sqrt{\frac{1}{y^2(0)} - 4Ct}}$$

and, choosing $\widetilde{T}$ as in the statement, the conclusion is straightforward. □

LEMMA 4.20. *Let $\Omega \subset \mathbb{R}^2$ be open, convex and regular, let $\mathbb{S}_0$ be a tree with five curves, two triple junctions O^1, O^2 and four end–points P^1, P^2, P^3, P^4 on $\partial\Omega$, as in Figure 5, satisfying assumptions* (4.1) *and assume that $\mathbb{S}_t$, for $t \in [0,T)$, is a smooth evolution by curvature of the network $\mathbb{S}_0$ such that $\|k\|_{L^2}$ is uniformly bounded on $[0,T)$.*
If the lengths of the curves of the network arriving at the end–points are uniformly bounded below by some constant $L > 0$, then $\|k_s\|_{L^2}$ is uniformly bounded on $[0,T)$.

PROOF. We first estimate $\|k_s\|^2_{L^\infty}$ in terms of $\|k_s\|_{L^2}$ and $\|k_{ss}\|_{L^2}$.
Fixed a time $t \in [0,T)$, let $Q \in \gamma^i \subset \mathbb{S}_t$, for some $i \leq 4$. We compute

$$[k^i_s(Q)]^2 \;=\; [k^i_s(P^i)]^2 + 2\int_{P^i}^{Q} k_s k_{ss}\, ds \leq C + 2\|k_s\|_{L^2}\|k_{ss}\|_{L^2}\,,$$

hence, in this case,

$$[k^i_s(Q)]^2 \leq C + 2\|k_s\|_{L^2}\|k_{ss}\|_{L^2}\,,$$

for every $Q \in \mathbb{S}_t \setminus \gamma^5$.
Assume now instead that $Q \in \gamma^5$. Recalling that $k^i_s + \lambda^i k^i = k^j_s + \lambda^j k^j$ at each triple junction, we get

$$k^5_s(O^1) = k^i_s(O^1) + \lambda^i(O^1)k^i(O^1) - \lambda^5(O^1)k^5(O^1)\,,$$

hence,

$$\begin{aligned}|k^5_s(O^1)| &\leq |k^i_s(O^1)| + C\|k\|^2_{L^\infty} \\ &\leq |k^i_s(O^1)| + C\|k\|_{L^2}\|k_s\|_{L^2} + C \\ &\leq |k^i_s(O^1)| + C\left(1 + \|k_s\|_{L^2}\right)\,,\end{aligned}$$

by Lemma 4.19. Then,

$$[k^5_s(O^1)]^2 \leq 2[k^i_s(O^1)]^2 + C\left(1 + \|k_s\|^2_{L^2}\right) \tag{4.15}$$

and it follows

$$\begin{aligned}[k^5_s(Q)]^2 \;&=\; [k^5_s(O^1)]^2 + 2\int_{O^1}^{Q} k_s k_{ss}\, ds \\ &\leq\; 2[k^i_s(O^1)]^2 + C\left(1 + \|k_s\|^2_{L^2}\right) + 2\int_{O^1}^{Q} k_s k_{ss}\, ds \\ &\leq\; C + C\|k_s\|^2_{L^2} + 2\|k_s\|_{L^2}\|k_{ss}\|_{L^2}\,,\end{aligned}$$

since, by the previous argument, we have $[k_s^i(O^1)]^2, [k_s^i(O^2)]^2 \leq C + 2\|k_s\|_{L^2}\|k_{ss}\|_{L^2}$, for all $i \in \{1,2,3,4\}$. Hence, we conclude

$$\|k_s\|_{L^\infty}^2 \leq C + C\|k_s\|_{L^2}^2 + 2\|k_s\|_{L^2}\|k_{ss}\|_{L^2}\,. \tag{4.16}$$

We now pass to estimate $\|k_s\|_{L^2}$. Making computation (4.2) explicit for $j=1$, we have

$$\partial_t \int_{\mathbb{S}_t} k_s^2\,ds \leq -2\int_{\mathbb{S}_t} k_{ss}^2\,ds + 7\int_{\mathbb{S}_t} k^2k_s^2\,ds - \sum_{p=1}^{2}\sum_{i=1}^{3} 2k_s^{pi}k_{ss}^{pi} + \lambda^{pi}\left(k_s^{pi}\right)^2\Big|_{\text{at the 3–point } O^p} + C\,. \tag{4.17}$$

Then, as in Section 4 we work to lower the differentiation order of the boundary term $\sum_{i=1}^3 k_s^ik_{ss}^i$ at each 3–point.
We claim that the following equality holds at each 3–point,

$$3\sum_{i=1}^{3}\lambda^ik^ik_t^i = \partial_t\sum_{i=1}^{3}\lambda^i\left(k^i\right)^2\,. \tag{4.18}$$

Keeping in mind that, at every 3–point, we have $\sum_{i=1}^3 k^i = 0$ and $\lambda^i = \frac{k^{i-1}-k^{i+1}}{\sqrt{3}}$, with the convention that the superscripts are considered modulus three (see Section 2), we obtains

$$\begin{aligned}
\sqrt{3}\sum_{i=1}^{3}\lambda^ik^ik_t^i &= \sum_{i=1}^{3}\left(k^{i-1}-k^{i+1}\right)k^ik_t^i\\
&= \sum_{i=1}^{3}k^{i+1}\left(k^{i+1}+k^{i-1}\right)k_t^i - k^{i-1}\left(k^{i+1}+k^{i-1}\right)k_t^i\\
&= \sum_{i=1}^{3}\left[\left(k^{i+1}\right)^2-\left(k^{i-1}\right)^2\right]k_t^i\,,
\end{aligned}$$

and

$$\begin{aligned}
\sqrt{3}\partial_t\sum_{i=1}^{3}\lambda^i\left(k^i\right)^2 &= \sqrt{3}\sum_{i=1}^{3}\lambda_t^i\left(k^i\right)^2 + 2\lambda^ik^ik_t^i\\
&= \sum_{i=1}^{3}\left(k_t^{i-1}-k_t^{i+1}\right)\left(k^i\right)^2 + 2\sum_{i=1}^{3}\left(k^{i-1}-k^{i+1}\right)k^ik_t^i\\
&= \sum_{i=1}^{3}\left[\left(k^{i+1}\right)^2-\left(k^{i-1}\right)^2+2k^ik^{i-1}-2k^ik^{i+1}\right]k_t^i\\
&= \sum_{i=1}^{3}\left[\left(k^{i+1}\right)^2-\left(k^{i-1}\right)^2-2(k^{i-1}+k^{i+1})k^{i-1}+2(k^{i-1}+k^{i+1})k^{i+1}\right]k_t^i\\
&= 3\sum_{i=1}^{3}\left[\left(k^{i+1}\right)^2-\left(k^{i-1}\right)^2\right]k_t^i\,,
\end{aligned}$$

thus, equality (4.18) is proved.
Now we use such equality to lower the differentiation order of the term $\sum_{i=1}^3 k_s^ik_{ss}^i$. Recalling

the formula $\partial_t k = k_{ss} + k_s\lambda + k^3$ and that $\sum_{i=1}^3 k_t^i = \partial_t \sum_{i=1}^3 k^i = 0$, we get

$$\begin{aligned}
\sum_{i=1}^{3} k_s^i k_{ss}^i &= \sum_{i=1}^{3} k_s^i \big[k_t^i - \lambda^i k_s^i - \left(k^i\right)^3\big] \\
&= \sum_{i=1}^{3} \big(k_s^i + \lambda^i k^i - \lambda^i k^i\big) k_t^i - \sum_{i=1}^{3} \lambda^i \left(k_s^i\right)^2 + \left(k^i\right)^3 k_s^i \\
&= \sum_{i=1}^{3} \big(k_s^i + \lambda^i k^i\big) k_t^i - \sum_{i=1}^{3} \lambda^i k^i k_t^i - \sum_{i=1}^{3} \lambda^i \left(k_s^i\right)^2 + \left(k^i\right)^3 k_s^i \\
&= -\partial_t \sum_{i=1}^{3} \lambda^i \left(k^i\right)^2/3 - \sum_{i=1}^{3} \lambda^i \left(k_s^i\right)^2 + \left(k^i\right)^3 k_s^i \,, \qquad (4.19)
\end{aligned}$$

at the triple junctions O^1 and O^2, where we used the fact that $k_s^i + \lambda^i k^i$ is independent of $i \in \{1, 2, 3\}$.
Substituting this equality into estimate (4.17), we obtain

$$\begin{aligned}
\partial_t \int_{\mathbb{S}_t} k_s^2 \, ds \leq & -2 \int_{\mathbb{S}_t} k_{ss}^2 \, ds + 7 \int_{\mathbb{S}_t} k^2 k_s^2 \, ds + \sum_{p=1}^{2} \sum_{i=1}^{3} 2 \left(k^{pi}\right)^3 k_s^{pi} + \lambda^{pi} \left(k_s^{pi}\right)^2 \Big|_{\text{at the 3–point } O^p} + C \\
& + 2\partial_t \sum_{p=1}^{2} \sum_{i=1}^{3} \lambda^{pi} \left(k^{pi}\right)^2/3 \Big|_{\text{at the 3–point } O^p} \\
\leq & -2 \int_{\mathbb{S}_t} k_{ss}^2 \, ds + C\|k\|_{L^2}^2 \|k_s\|_{L^\infty}^2 + \sum_{p=1}^{2} \sum_{i=1}^{3} 2 \left(k^{pi}\right)^3 k_s^{pi} + \lambda^{pi} \left(k_s^{pi}\right)^2 \Big|_{\text{at the 3–point } O^p} \\
& + 2\partial_t \sum_{p=1}^{2} \sum_{i=1}^{3} \lambda^{pi} \left(k^{pi}\right)^2/3 \Big|_{\text{at the 3–point } O^p} + C \,. \qquad (4.20)
\end{aligned}$$

Using the previous estimate on $\|k_s\|_{L^\infty}$, the hypothesis of uniform boundedness of $\|k\|_{L^2}$ and Young inequality, we get

$$\begin{aligned}
\|k\|_{L^2}^2 \|k_s\|_{L^\infty}^2 &\leq C + C\|k_s\|_{L^2}^2 + C\|k_s\|_{L^2} \|k_{ss}\|_{L^2} \\
&\leq C + C\|k_s\|_{L^2}^2 + C_\varepsilon \|k_s\|_{L^2}^2 + \varepsilon \|k_{ss}\|_{L^2}^2 \\
&= C + C_\varepsilon \|k_s\|_{L^2}^2 + \varepsilon \|k_{ss}\|_{L^2}^2 \,,
\end{aligned}$$

for any small value $\varepsilon > 0$ and a suitable constant C_ε.
We deal now with the boundary term $\sum_{i=1}^3 2 \left(k^i\right)^3 k_s^i + \lambda^i \left(k_s^i\right)^2$.
By the fact that $k_s^i + \lambda^i k^i = k_s^j + \lambda^j k^j$, for every pair i, j, it follows that $(k_s + \lambda k)^2 \sum_{i=1}^3 \lambda^i = 0$, hence,

$$\sum_{i=1}^{3} \lambda^i \left(k_s^i\right)^2 = -\sum_{i=1}^{3} \left(\lambda^i\right)^3 \left(k^i\right)^2 + 2 \left(\lambda^i\right)^2 k^i k_s^i \,,$$

then, we can write

$$\begin{aligned}
\sum_{i=1}^{3} 2 \left(k^i\right)^3 k_s^i + \lambda^i \left(k_s^i\right)^2 &= \sum_{i=1}^{3} 2 \left(k^i\right)^3 k_s^i - \left(\lambda^i\right)^3 \left(k^i\right)^2 - 2 \left(\lambda^i\right)^2 k^i k_s^i \\
&= \sum_{i=1}^{3} 2 \big[\left(k^i\right)^3 - \left(\lambda^i\right)^2 k^i\big] k_s^i - \sum_{i=1}^{3} \left(\lambda^i\right)^3 \left(k^i\right)^2 \\
&= 2(k_s + \lambda k) \sum_{i=1}^{3} \left(k^i\right)^3 - \left(\lambda^i\right)^2 k^i + \sum_{i=1}^{3} \left(\lambda^i\right)^3 \left(k^i\right)^2 - 2\lambda^i \left(k^i\right)^4 \,. \qquad (4.21)
\end{aligned}$$

At the triple junction O^1, where the curves γ^1, γ^2 and γ^5 concur, there exists $i \in \{1,2\}$ such that $|k^i(O^1)| \geq \frac{K}{2}$, where $K := \max_{j\in\{1,2,3\}} |k^j(O^1)|$, hence at the 3–point O^1

$$
\begin{aligned}
2(k_s + \lambda k) \sum_{i=1}^{3} \left(k^i\right)^3 - \left(\lambda^i\right)^2 k^i + \sum_{i=1}^{3} \left(\lambda^i\right)^3 \left(k^i\right)^2 - 2\lambda^i \left(k^i\right)^4 \qquad&\\
\leq CK^5 + C|k^i_s(O^1)|K^3&\\
\leq C|k^i(O^1)|^5 + C|k^i_s(O^1)||k^i(O^1)|^3&\\
\leq C\|k^i\|^5_{L^\infty(\gamma^i)} + C\|k^i_s\|_{L^\infty(\gamma^i)}\|k^i\|^3_{L^\infty(\gamma^i)}\,.&
\end{aligned}
$$

We estimate now $C\|k\|^5_{L^\infty(\gamma^i)} + C\|k_s\|_{L^\infty(\gamma^i)}\|k\|^3_{L^\infty(\gamma^i)}$ via the Gagliardo–Nirenberg interpolation inequalities in Proposition 4.7. Letting $u = k^i$, $p = +\infty$, $m = 2$ and $n = 0, 1$ in formula (4.5), we get

$$
\|k^i\|_{L^\infty(\gamma^i)} \leq C\|k^i_{ss}\|^{\frac14}_{L^2(\gamma^i)}\|k^i\|^{\frac34}_{L^2(\gamma^i)} + \frac{B}{L^{\frac12}}\|k^i\|_{L^2(\gamma^i)} \leq C\|k^i_{ss}\|^{\frac14}_{L^2(\gamma^i)} + C_L \tag{4.22}
$$

$$
\|k^i_s\|_{L^\infty(\gamma^i)} \leq C\|k^i_{ss}\|^{\frac34}_{L^2(\gamma^i)}\|k^i\|^{\frac14}_{L^2(\gamma^i)} + \frac{B}{L^{\frac32}}\|k^i\|_{L^2(\gamma^i)} \leq C\|k^i_{ss}\|^{\frac34}_{L^2(\gamma^i)} + C_L\,, \tag{4.23}
$$

hence,

$$
C\|k^i\|^5_{L^\infty(\gamma^i)} + C\|k^i\|^3_{L^\infty(\gamma^i)}\|k^i_s\|_{L^\infty(\gamma^i)} \leq C\|k^i_{ss}\|^{\frac54}_{L^2(\gamma^i)} + C\|k^i_{ss}\|^{\frac32}_{L^2(\gamma^i)} + C_L \leq \varepsilon\|k^i_{ss}\|^2_{L^2(\gamma^i)} + C_{L,\varepsilon}\,.
$$

Thus, finally,

$$
2(k_s + \lambda k) \sum_{i=1}^{3} \left(k^i\right)^3 - \left(\lambda^i\right)^2 k^i + \sum_{i=1}^{3} \left(\lambda^i\right)^3 \left(k^i\right)^2 - 2\lambda^i \left(k^i\right)^4 \leq \varepsilon\|k^i_{ss}\|^2_{L^2(\gamma^i)} + C_{L,\varepsilon} \leq \varepsilon\|k_{ss}\|^2_{L^2} + C_{L,\varepsilon}\,.
$$

Coming back to computation (4.20), we have

$$
\begin{aligned}
&\partial_t\left(\int_{\mathbb{S}_t} k_s^2\,ds - 2\sum_{p=1}^{2}\sum_{i=1}^{3} \lambda^{pi}\left(k^{pi}\right)^2 /3 \,\Big|_{\text{at the 3–point } O^p}\right)\\
&\leq -2\int_{\mathbb{S}_t} k^2_{ss} ds + C\|k_s\|^2_{L^2} + \varepsilon\|k_{ss}\|^2_{L^2} + C_{L,\varepsilon}\\
&\leq -2\int_{\mathbb{S}_t} k^2_{ss} ds + C\|k_s\|^2_{L^2} + 2\varepsilon\|k_{ss}\|^2_{L^2} - C_{L,\varepsilon}\|k^i\|^3_{L^\infty(\gamma^i)} + C_{L,\varepsilon}\\
&\leq C_{L,\varepsilon}\left(\int_{\mathbb{S}_t} k_s^2\,ds - 2\sum_{p=1}^{2}\sum_{i=1}^{3} \lambda^{pi}\left(k^{pi}\right)^2 /3 \,\Big|_{\text{at the 3–point } O^p}\right) + C_{L,\varepsilon}\,,
\end{aligned}
$$

where we chose $\varepsilon < 1$.
By Gronwall's Lemma, it follows that $\|k_s\|^2_{L^2} - 2\sum_{p=1}^{2}\sum_{i=1}^{3} \lambda^{pi}\left(k^{pi}\right)^2 /3 \,\Big|_{\text{at the 3–point } O^p}$ is uniformly bounded, for $t \in [0, T)$, by a constant depending on L and its value on the initial network $\mathbb{S}_0$. Then, applying Young inequality to estimate (4.12) of Lemma 4.18, there holds

$$
\|k\|^3_{L^\infty} \leq C + C\|k\|^{3/2}_{L^2}\|k_s\|^{3/2}_{L^2} \leq C + C_\varepsilon\|k\|^6_{L^2} + \varepsilon\|k_s\|^2_{L^2} \leq C_\varepsilon + \varepsilon\|k_s\|^2_{L^2}\,,
$$

as $\|k\|_{L^2}$ is uniformly bounded in $[0, T)$. Choosing $\varepsilon > 0$ small enough, we conclude that also $\|k_s\|_{L^2}$ is uniformly bounded in $[0, T)$. ☐

5. Short time existence II

First we consider a C^∞ flow by curvature $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0,1], t)$ and we discuss what happens if we reparametrize every curve of the network proportionally to arclength.

If we consider smooth functions $\varphi^i : [0,1] \times [0,T) \to [0,1]$ and the reparametrizations $\widetilde{\gamma}^i(x,t) = \gamma^i(\varphi^i(x,t),t)$, imposing that $|\widetilde{\gamma}^i_x|$ is constant, we must have that $|\gamma^i_x(\varphi^i(x,t),t)|\varphi^i_x(x,t) = L^i(t)$, where $L^i(t)$ is the length of the curve γ^i at time t.

It follows that $\varphi^i(x,t)$ can be obtained by integrating the ODE

$$\varphi^i_x(x,t) = L^i(t)/|\gamma^i_x(\varphi^i(x,t),t)|$$

with initial data $\varphi^i(0,t)=0$, and that it is C^∞ as L^i and γ^i are C^∞.

Being a reparametrization, $\widetilde{\gamma}^i$ is still a C^∞ curvature flow, that is, $\widetilde{\gamma}^i_t = \widetilde{k}^i\widetilde{\nu}^i + \widetilde{\lambda}^i\widetilde{\tau}^i$. We want to determine the functions $\widetilde{\lambda}^i = \langle \widetilde{\gamma}^i_t \,|\, \widetilde{\tau}^i\rangle$, differentiating this equation in arclength and keeping into account that $\widetilde{\gamma}_x(x,t) = L^i(t)\widetilde{\tau}^i(x,t)$, we get

$$\widetilde{\lambda}^i_s = \frac{\langle \widetilde{\gamma}^i_{tx} \,|\, \widetilde{\tau}^i\rangle}{|\widetilde{\gamma}^i_x|} + \langle \widetilde{\gamma}^i_t \,|\, \partial_s\widetilde{\tau}^i\rangle = \frac{\langle \partial_t(L^i\widetilde{\tau}^i) \,|\, \widetilde{\tau}^i\rangle}{L^i} + \langle \widetilde{k}^i\widetilde{\nu}^i + \widetilde{\lambda}^i\widetilde{\tau}^i \,|\, \widetilde{k}^i\widetilde{\nu}^i\rangle = \frac{\partial_t L^i}{L^i} + (\widetilde{k}^i)^2\,.$$

This equation immediately says that $\widetilde{\lambda}^i_s - (\widetilde{k}^i)^2$ is constant in space. Moreover, we know that $\partial_t L^i(t) = \widetilde{\lambda}^i(1,t) - \widetilde{\lambda}^i(0,t) - \int_{\gamma^i(\cdot,t)} (\widetilde{k}^i)^2\,ds$ (see Proposition 4.1) and that the values of $\widetilde{\lambda}^i$ at the end–points or 3–points of the network are (uniformly) linearly related (hence, also bounded) to the values of $\widetilde{k}^i$; hence, we can conclude that $\widetilde{\lambda}^i_s$ is bounded by $L^i(t)$ and a quadratic expression in $\|\widetilde{k}(\cdot,t)\|_\infty$.

We show now that the geometrically unique solution starting from an initial $C^{2+2\alpha}$ network which is geometrically 2–compatible (see Proposition 3.22) can be actually reparametrized to be a C^∞ curvature flow for every positive time (so that the geometric estimates of Section 4 can be applied). This clearly can be seen as a (geometric) parabolic regularization property.

THEOREM 5.1. *For any initial, regular $C^{2+2\alpha}$ network $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$, with $\alpha \in (0,1/2)$, which is geometrically 2–compatible, the geometrically unique solution γ^i found in Proposition 3.22 can be reparametrized to be a C^∞ curvature flow on $(0,T)$, that is, the networks $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0,1],t)$ are geometrically smooth for every positive time.*

PROOF. We first assume that $\mathbb{S}_0$ is 2–compatible.
By inspecting the proof of Theorem 3.3 in **[19]** one can see that the solution to system (3.3) actually depends continuously in $C^{2+2\alpha,1+\alpha}$ on the initial data σ^i in the $C^{2+\alpha}$ norm. Then, we approximate the network $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$ in $C^{2+2\alpha}$ with a family of smooth networks $\mathbb{S}_j$ with the same end–points, composed of C^∞ curves $\sigma^i_j \to \sigma^i$, as $j\to\infty$. Hence, for every $\varepsilon>0$, the smooth solutions of system (3.3) for these approximating initial networks, given by the curves $\gamma^i_j(x,t) : [0,1]\times[0,T-\varepsilon] \to \overline{\Omega}$, converge as $j\to\infty$ in $C^{2+2\alpha,1+\alpha}([0,1]\times[0,T-\varepsilon])$ to the solution γ^i for the initial network $\mathbb{S}_0$. By the $C^{2+2\alpha}$ convergence, the inverses of the lengths of the initial curves, the integrals $\int_{\mathbb{S}_j} k_j^2\,ds$ and $|\partial_x\sigma^i_j(x)|$ (from above and away from zero) for all the approximating networks are equibounded, thus Proposition 4.13 gives uniform estimates on the L^∞ norms of the curvature and of all its derivatives in every rectangle $[0,1]\times[\mu,T_M)$, with $\mu>0$ and $T_M \le T$.
We now reparametrize every curve $\gamma^i_j(\cdot,t)$ and $\gamma^i(\cdot,t)$ proportionally to arclength by some maps φ^i_j and φ^i as above. Notice that, since γ^i_j and γ^i are uniformly bounded in $C^{2+2\alpha,1+\alpha}$, we have that the maps $\partial_x\gamma^i_j$ and $\partial_x\gamma^i$ uniformly bounded $C^{1+2\alpha,1/2+\alpha}$. Hence, by a standard ODE argument, the reparametrizing maps φ^i_j and φ^i above are also uniformly bounded in $C^{1+2\alpha,1/2+\alpha}$, in particular, they are uniformly Hölder continuous in space and time. This means that the reparametrized maps $\widetilde{\gamma}^i_j$ converge uniformly to $\widetilde{\gamma}^i$ which is a (only continuous in t) reparametrization of the original flow. It is easy to see that these latter gives a curvature flow of the arclength reparametrized network $\widetilde{\mathbb{S}}_0 = \bigcup_{i=1}^n(\sigma^i\circ\varphi^i(\cdot,0))[0,1]$ which then still belongs to $C^{2+2\alpha}$.

As the curvature and all its arclength derivatives are invariant under reparametrization and the equibounded lengths of the curves also, the above uniform estimates hold also for the reparametrized maps $\widetilde{\gamma}^i_j$ in every rectangle $[0,1]\times[\mu,T_M)$. Moreover, by the discussion about reparametrizing these curves proportional to arclength, it follows that we have uniform estimates also on $\widetilde{\lambda}^i_j$ and all their arclength derivatives for these flows in every rectangle $[0,1]\times[\mu,T_M)$. Hence, the curves $\widetilde{\gamma}^i_j$, possibly passing to a subsequence, actually converge in $C^\infty([0,1]\times[\mu,T_M))$, for every $\mu>0$, to the limit flow $\widetilde{\gamma}^i$ which then belongs to $C^\infty([0,1]\times(0,T))\cap C^0([0,1]\times[0,T))$.

If $\mathbb{S}_0$ were only geometrically 2–compatible, this procedure could have been done for the flow of its 2–compatible reparametrization, giving the same resulting flow, as the arclength reparametrized flow is the same for any two flows differing only for a reparametrization (the fact that the flow of a $C^{2+2\alpha}$ geometrically 2–compatible initial network is a reparametrization of the flow of a 2–compatible $C^{2+2\alpha}$ initial network is stated in Remark 3.24).

The last step is to find extensions $\theta^i : [0,1]\times[0,T)\to[0,1]$ of the arclength reparametrizing maps $\varphi^i(\cdot,0)\in C^{2+2\alpha}$ which are in $C^\infty([0,1]\times(0,T))$ and satisfy $\theta^i(x,0)=\varphi^i(x,0)$, $\theta^i(0,t)=0$, $\theta^i(1,t)=1$ and $\theta^i_x(x,t)\neq 0$ for every x and t. This can be done, for instance, by means of time–dependent convolutions with smooth kernels. Then, the maps $\overline{\gamma}^i(\cdot,t)=\widetilde{\gamma}^i([\theta^i(\cdot,t)]^{-1},t)$ give a curvature flow of the network $\mathbb{S}_0=\bigcup_{i=1}^n\sigma^i([0,1])$ which becomes immediately C^∞ for every positive time $t>0$. □

As for every positive time, the flow obtained by this theorem is C^∞, hence, every network $\mathbb{S}_t$ is geometrically smooth, again by Remark 3.24 this flow can be reparametrized, from any positive time on, to be a C^∞ special smooth flow.
This argument can clearly be applied to any $C^{2+2\alpha,1+\alpha}$ curvature flow $\mathbb{S}_t$ in a time interval $(0,T)$, being every network of this flow geometrically 2–compatible (Proposition 3.12), simply considering as initial network any $\mathbb{S}_{t_0}$ with $t_0>0$.

COROLLARY 5.2. *Given any $C^{2+2\alpha,1+\alpha}$ curvature flow in an interval of time $(0,T)$, for every $\mu>0$, the restricted flow $\mathbb{S}_t$ for $t\in[\mu,T)$ can be reparametrized to be a C^∞ special curvature flow in $[\mu,T)$.*
In particular, this applies to any $C^{2+2\alpha,1+\alpha}$ curvature flow of an initial, regular $C^{2+2\alpha}$ geometrically 2–compatible network $\mathbb{S}_0=\bigcup_{i=1}^n\sigma^i([0,1])$.

REMARK 5.3. Even if this theorem and the corollary are sufficient for our purpose to study the singularity formation in the next sections, one would expect that by the usual standard parabolic regularization, the unique solution γ^i of system (3.3) for every initial $C^{2+2\alpha}$ network, at least if it is 2–compatible, is actually C^∞ for every positive time, hence a special curvature flow. Another question is whether *any* curvature flow, hence only in $C^{2,1}$, can be reparametrized to be a C^∞ (special) curvature flow in $[\mu,T)$. These problem are actually open at the moment.
Also open is what is the largest class of initial networks admitting a special curvature flow.

OPEN PROBLEM 5.4. The unique solution γ^i of system (3.3) for an initial $C^{2+2\alpha}$ network $\mathbb{S}_0$, at least if it is 2–compatible, is C^∞ for every positive time?

OPEN PROBLEM 5.5. Every curvature flow of a regular network can be reparametrized to be a C^∞ (special) curvature flow for every positive time?

OPEN PROBLEM 5.6. What are the minimal regularity hypotheses on an initial network $\mathbb{S}_0$ such that it admits a special curvature flow?

A consequence of these "geometric" parabolic results is the extension of Theorem 4.14 and Corollary 4.15 to any $C^{2+\alpha,1+\alpha}$ curvature flow. As before, we apply such results to the reparametrized C^∞ special curvature flow given by Corollary 5.2, then it is clear that the conclusions holds also for the original flow since they are concerned only with the curvature and the lengths of the curves, which are invariant by reparametrization.

THEOREM 5.7. *If $T<+\infty$ is the maximal time interval of existence of a $C^{2+\alpha,1+\alpha}$ curvature flow $\mathbb{S}_t$, then*

(1) *either the inferior limit of the length of at least one curve of $\mathbb{S}_t$ goes to zero when $t\to T$,*
(2) *or $\limsup_{t\to T}\int_{\mathbb{S}_t}k^2\,ds=+\infty$, hence, the curvature is not bounded as $t\to T$.*

Moreover, if the lengths of the n curves are uniformly positively bounded from below, then this superior limit is actually a limit and there exists a positive constant C such that

$$\int_{\mathbb{S}_t}k^2\,ds\geq\frac{C}{\sqrt{T-t}}\quad\text{and}\quad\max_{\mathbb{S}_t}k^2\geq\frac{C}{\sqrt{T-t}}$$

for every $t\in[0,T)$.

Thanks to Proposition 4.13, we can now improve Theorems 3.13 and 3.20 to show the existence of a curvature flow for a regular initial network $\mathbb{S}_0$ which is only C^2.

THEOREM 5.8. *For any initial C^2 regular network $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$ there exists a solution γ^i of Problem* (1.5) *in a maximal time interval $[0,T)$.*
Such curvature flow $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0,1],t)$ is a smooth flow for every time $t > 0$, moreover, the unit tangents τ^i are continuous in $[0,1]\times[0,T)$, the functions $k(\cdot,t)$ converge weakly in $L^2(ds)$ to $k(\cdot,0)$, as $t \to 0$, and the function $\int_{\mathbb{S}_t} k^2\,ds$ is continuous on $[0,T)$.

PROOF. We can approximate in $W^{2,2}(0,1)$ (hence in $C^1([0,1])$) the network $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$ with a family of smooth networks $\mathbb{S}_j$, composed of C^∞ curves $\sigma^i_j \to \sigma^i$, as $j\to\infty$ with the same end–points and satisfying $\partial_x\sigma^i_j(0) = \partial_x\sigma^i(1)$, $\partial_x\sigma^i_j(1) = \partial_x\sigma^i(1)$.
By the convergence in $W^{2,2}$ and in C^1, the inverses of the lengths of the initial curves, the integrals $\int_{\mathbb{S}_j} k^2\,ds$ and $|\partial_x\sigma^i_j(x)|$ (from above and away from zero) for all the approximating networks are equibounded, thus Proposition 4.17 assures the existence of a uniform interval $[0,T)$ of existence of smooth evolutions given by the curves $\gamma^i_j(x,t) : [0,1]\times[0,T) \to \overline{\Omega}$.
Now, by the same reason, Proposition 4.13 gives uniform estimates on the L^∞ norms of the curvature and of all its derivatives in every rectangle $[0,1]\times[\mu,T_M)$, with $\mu>0$.
This means that if we reparametrize at every time all the curves γ^i_j proportional to their arclength, by means of a diagonal argument, we can find a subsequence of the family of reparametrized flows $\widetilde{\gamma}^i_j$ which converges in $C^\infty_{\mathrm{loc}}([0,1]\times(0,T))$ to some flow, parametrized proportional to its arclength, $\widetilde{\gamma}^i$ in the time interval $(0,T)$. Moreover, by the hypotheses, the curves of the initial networks $\widetilde{\sigma}^i_j$ converge in $W^{2,2}(0,1)$ to $\widetilde{\sigma}^i$ which are the reparametrizations, proportional to their arclength, of the curves σ^i of the initial network $\mathbb{S}_0$. If we show that the maps $\widetilde{\gamma}^i$ are continuous up to the time $t=0$ we have a curvature flow for the network $\widetilde{\mathbb{S}}_0 = \bigcup_{i=1}^n \widetilde{\sigma}^i([0,1])$ which then gives a curvature flow for the original network $\mathbb{S}_0$ in $C^\infty([0,1]\times(0,T))$, reparametrizing it back with some family of continuous maps $\theta^i : [0,1]\times[0,T)\to[0,1]$ with $\theta^i_x \neq 0$ everywhere, $\theta^i\in C^\infty([0,1]\times(0,T))$ and $\widetilde{\sigma}^i(\theta^i(\cdot,0)) = \sigma^i$ (this can be easily done as the maps $\theta^i(\cdot,0)$ are of class C^2, since in general, the arclength reparametrization maps have the same regularity of the network).
Hence, we deal with the continuity up to $t=0$ of the maps $\widetilde{\gamma}^i$. By the uniform L^2 bound on the curvature and the parametrization proportional to the arclength, the theorem of Ascoli–Arzelà implies that for every sequence of times $t_l\to0$, the curves $\widetilde{\gamma}^i(\cdot,t_l)$ have a converging subsequence in $C^1([0,1])$ to some family of limit curves $\zeta^i : [0,1]\to\overline{\Omega}$, still parametrized proportionally to arclength, by the C^1 convergence. Moreover, we can also assume that $k(\cdot,t_l)$ converge weakly in $L^2(ds)$ to the curvature function associated to the family of curves ζ^i. We want to see that actually $\zeta^i = \widetilde{\sigma}^i$, hence showing that the flow $\widetilde{\gamma}^i : [0,1]\times[0,T)\to\overline{\Omega}$ is continuous and that the unit tangent vector $\tau : [0,1]\times[0,T)\to\mathbb{R}^2$ is a continuous map up to the time $t=0$ (this property is stable under the above reparametrization so it then will hold also for the final curvature flow γ^i).
We consider a function $\varphi\in C^\infty(\mathbb{R}^2)$ and the time derivative of its integral on the evolving networks $\widetilde{\gamma}^i_j$, that is,

$$\begin{aligned}
\frac{d}{dt}\int_{\widetilde{\mathbb{S}}_j(t)} \varphi\,ds &= \int_{\widetilde{\mathbb{S}}_j(t)} \varphi(\widetilde{\lambda}_s - \widetilde{k}^2)\,ds + \int_{\widetilde{\mathbb{S}}_j(t)} \langle\nabla\varphi\,|\,\widetilde{\underline{k}} + \widetilde{\underline{\lambda}}\rangle\,ds\\
&= -\int_{\widetilde{\mathbb{S}}_j(t)} \varphi\widetilde{k}^2\,ds - \int_{\widetilde{\mathbb{S}}_j(t)} \langle\nabla\varphi\,|\,\widetilde{\tau}\rangle\widetilde{\lambda}\,ds + \int_{\widetilde{\mathbb{S}}_j(t)} \langle\nabla\varphi\,|\,\widetilde{\underline{k}} + \widetilde{\underline{\lambda}}\rangle\,ds\\
&= -\int_{\widetilde{\mathbb{S}}_j(t)} \varphi\widetilde{k}^2\,ds + \int_{\widetilde{\mathbb{S}}_j(t)} \langle\nabla\varphi\,|\,\widetilde{\underline{k}}\rangle\,ds\,,
\end{aligned}$$

where we integrated by parts, passing from first to second line.
Let us consider now any sequence of times t_l converging to zero as above, such that the curves $\widetilde{\gamma}^i(\cdot,t_l)$ converge in $C^1([0,1])$ to some family of limit curves $\zeta^i : [0,1]\to\overline{\Omega}$ (still parametrized proportionally to arclength) as above, describing some regular network $\overline{\mathbb{S}}$, and $k(\cdot,t_l)$ converge weakly in $L^2(ds)$ to the curvature function associated to the family of curves ζ^i. Integrating this

equality in the time interval $[0, t_l]$ we get

$$\int_{\widetilde{\mathbb{S}}_j(t_l)} \varphi\, ds - \int_{\widetilde{\mathbb{S}}_j(0)} \varphi\, ds = -\int_0^{t_l}\int_{\widetilde{\mathbb{S}}_j(t)} \varphi \widetilde{k}^2\, ds\, dt + \int_0^{t_l}\int_{\widetilde{\mathbb{S}}_j(t)} \langle \nabla\varphi \,|\, \underline{\widetilde{k}}\rangle\, ds\, dt$$

which clearly passes to the limit as $j \to \infty$, by the smooth convergence of the flows $\widetilde{\gamma}^i_j$ to the flow $\widetilde{\gamma}^i$ (and the uniform bound on $\int_{\widetilde{\mathbb{S}}_j(t)} \widetilde{k}^2\, ds$) and of the initial networks $\widetilde{\mathbb{S}}_j(0) = \bigcup_{i=1}^n \widetilde{\sigma}^i_j([0,1])$ to $\widetilde{\mathbb{S}}_0 = \bigcup_{i=1}^n \widetilde{\sigma}^i([0,1])$, hence,

$$\int_{\widetilde{\mathbb{S}}_{t_l}} \varphi\, ds - \int_{\widetilde{\mathbb{S}}_0} \varphi\, ds = -\int_0^{t_l}\int_{\widetilde{\mathbb{S}}_t} \varphi \widetilde{k}^2\, ds\, dt + \int_0^{t_l}\int_{\widetilde{\mathbb{S}}_t} \langle \nabla\varphi \,|\, \underline{\widetilde{k}}\rangle\, ds\, dt$$

By the uniform bound on the L^2 norm of the curvature, we then get

$$\left|\int_{\widetilde{\mathbb{S}}_{t_l}} \varphi(\widetilde{\gamma}(\cdot, t_l))\, ds - \int_{\widetilde{\mathbb{S}}_0} \varphi(\widetilde{\sigma})\, ds\right| \leq Ct_l\,,$$

where we made explicit the integrands, for sake of clarity. Sending $l \to \infty$ we finally obtain

$$\left|\int_{\overline{\mathbb{S}}} \varphi(\zeta)\, ds - \int_{\widetilde{\mathbb{S}}_0} \varphi(\widetilde{\sigma})\, ds\right| = 0\,,$$

that is,

$$\int_{\overline{\mathbb{S}}} \varphi\, ds = \int_{\widetilde{\mathbb{S}}_0} \varphi\, ds$$

for every function $\varphi \in C^\infty(\mathbb{R}^2)$.
Since, both the networks $\widetilde{\mathbb{S}}_0 = \bigcup_{i=1}^n \widetilde{\sigma}^i([0,1])$ and $\overline{\mathbb{S}} = \bigcup_{i=1}^n \zeta^i([0,1])$ are regular, parametrized proportionally to their arclength and of class C^1, this equality for every $\varphi \in C^\infty(\mathbb{R}^2)$ implies that $\widetilde{\sigma}^i = \zeta^i$, which is what we wanted.

Notice that, the continuity of γ^i and τ also implies that the measures $\mathcal{H}^1 \llcorner \mathbb{S}_t$ weakly* converge to $\mathcal{H}^1 \llcorner \mathbb{S}_0$, where $\mathcal{H}^1$ is the Hausdorff one–dimensional measure, as $t \to 0$.

Finally, integrating on $[0, t)$ inequality (4.8) (forgetting the absolute value and the contributions from the end–points), for the approximating flows $\widetilde{\gamma}^i_j$, and passing to the limit as $j \to \infty$, we see that the function $\int_{\widetilde{\mathbb{S}}_t} k^2\, ds$ is *continuous* on $[0, T)$ (also at $t = 0$), by the uniform bound on the L^2 norm of the curvature of the networks. Being such integral invariant by reparametrization, this also holds for the flow γ^i. The same for the weak convergence in $L^2(ds)$ of the functions $k(\cdot, t)$ to $k(\cdot, 0)$ as $t \to 0$. □

REMARK 5.9.

(1) The relevance of this theorem is that the initial network is not required to satisfy any compatibility condition, but only to have angles of 120 degrees between the concurring curves at every 3–point, that is, to be regular. In particular, it is not necessary that the sum of the three curvatures at a 3–point is zero.
(2) The geometric uniqueness of the solution γ^i found in this theorem is an open problem.
(3) As for every positive time the flow obtained by this theorem is C^∞, hence every network $\mathbb{S}_t$ is geometrically smooth, arguing as before (by means of Remark 3.24), the same conclusions of Corollary 5.2 apply, that is, this flow can be reparametrized, from any positive time on, to be a C^∞ special smooth flow.
(4) It should be noticed that if the initial curves σ^i are C^∞, the flow $\mathbb{S}_t$ is smooth till $t = 0$ far from the 3–points, that is, in any closed rectangle included in $(0,1) \times [0, T)$ we can locally reparametrize the curves γ^i to get a smooth flow up to $t = 0$. This follows from the local estimates for the motion by curvature (see **[31]**).
(5) It is easy to see that, pushing a little the argument in the proof of this theorem, one can find a curvature flow with the same properties also if the initial network $\mathbb{S}_0$ is regular and composed of regular curves of class $W^{2,2}(0,1)$ only.

Arguing as for Theorem 5.7, we have the following corollary.

COROLLARY 5.10. *If $T < +\infty$ is the maximal time interval of existence of the curvature flow $\mathbb{S}_t$ of an initial regular C^2 network given by the previous theorem, then*

(1) *either the inferior limit of the length of at least one curve of $\mathbb{S}_t$ goes to zero when $t \to T$,*
(2) *or $\limsup_{t\to T} \int_{\mathbb{S}_t} k^2\, ds = +\infty$, hence, the curvature is not bounded as $t \to T$.*

Moreover, if the lengths of the n curves are uniformly positively bounded from below, then this superior limit is actually a limit and there exists a positive constant C such that

$$\int_{\mathbb{S}_t} k^2\, ds \geq \frac{C}{\sqrt{T-t}} \quad \textit{and} \quad \max_{\mathbb{S}_t} k^2 \geq \frac{C}{\sqrt{T-t}}$$

for every $t \in [0, T)$.

OPEN PROBLEM 5.11. Every curvature flow of a regular network, hence only $C^{2,1}$, shares the properties stated in this corollary?
Notice that it would follow by a positive answer to Problem 5.5.

5.1. Smooth flows are Brakke flows. We introduce now the concept of *Brakke flow* (with equality) of a network.

DEFINITION 5.12. A *regular Brakke flow* is a family of $W^{2,2}_{\mathrm{loc}}$ networks $\mathbb{S}_t$ in Ω, satisfying the inequality

$$\overline{\frac{d}{dt}} \int_{\mathbb{S}_t} \varphi(\gamma, t)\, ds \leq -\int_{\mathbb{S}_t} \varphi(\gamma, t) k^2\, ds + \int_{\mathbb{S}_t} \langle \nabla\varphi(\gamma, t)\,|\,\underline{k}\rangle\, ds + \int_{\mathbb{S}_t} \varphi_t(\gamma, t)\, ds\,, \tag{5.1}$$

for every *non negative* smooth function with compact support $\varphi : \Omega \times [0, T) \to \mathbb{R}$ and $t \in [0, T)$, where $\overline{\frac{d}{dt}}$ is the *upper* derivative (the $\overline{\lim}$ of the incremental ratios).

If the time derivative at the left hand side exists and the inequality is an equality, for every smooth function with compact support $\varphi : \Omega \times [0, T) \to \mathbb{R}$ and $t \in [0, T)$, that is,

$$\frac{d}{dt} \int_{\mathbb{S}_t} \varphi(\gamma, t)\, ds = -\int_{\mathbb{S}_t} \varphi(\gamma, t) k^2\, ds + \int_{\mathbb{S}_t} \langle \nabla\varphi(\gamma, t)\,|\,\underline{k}\rangle\, ds + \int_{\mathbb{S}_t} \varphi_t(\gamma, t)\, ds\,, \tag{5.2}$$

we say that $\mathbb{S}_t$ is a regular Brakke flow *with equality*.

REMARK 5.13. Actually, the original definition of Brakke flow given in [**18**, Section 3.3] (in any dimension and codimension) allows the networks $\mathbb{S}_t$ to be simply one–dimensional countably rectifiable subsets of $\mathbb{R}^2$, with possible integer multiplicity $\theta_t : \mathbb{S}_t \to \mathbb{N}$, and with a distributional notion of tangent space and (mean) curvature, called *rectifiable varifolds* (see [**85**]). With such a general definition, the networks are identified with the associated Radon measures $\mu_t = \theta_t \mathcal{H}^1 \llcorner \mathbb{S}_t$.
More precisely, the inequality

$$\begin{aligned}\overline{\frac{d}{dt}} \int_{\mathbb{S}_t} \varphi(x, t)\theta_t(x)\, d\mathcal{H}^1(x) \leq &-\int_{\mathbb{S}_t} \varphi(x, t) k^2(x, t)\theta_t(x)\, d\mathcal{H}^1(x) + \int_{\mathbb{S}_t} \langle \nabla\varphi(x, t)\,|\,\underline{k}(x, t)\rangle \theta_t(x)\, d\mathcal{H}^1(x) \\ &+ \int_{\mathbb{S}_t} \varphi_t(x, t)\theta_t(x)\, d\mathcal{H}^1(x)\,,\end{aligned} \tag{5.3}$$

must hold for every non negative smooth function with compact support $\varphi : \Omega \times [0, T) \to \mathbb{R}$ and $t \in [0, T)$, where $\mathcal{H}^1$ is the Hausdorff one–dimensional measure in $\mathbb{R}^2$.
These weak conditions were introduced by Brakke in order to prove an existence result [**18**, Section 4.13] for a family of initial sets much wider than networks of curves, but, on the other hand, it lets open the possibility of instantaneous vanishing of some parts of the sets during the evolution.

A big difference between Brakke flows and the evolutions obtained as solutions of Problem (1.5) is that the former networks are simply considered as *subsets* of $\mathbb{R}^2$ without any mention to their parametrization (that clearly is not unique). This means that actually a Brakke flow can be a family of networks given by the maps $\gamma^i(x, t)$ which are C^2 in space, but possibly do not have absolutely any regularity with respect to the time variable t.

An open question is whether any Brakke flow with equality, possibly under some extra hypotheses, admits a reparametrization such that it becomes a solution of Problem (1.5).
This problem is clearly also related to the uniqueness of the Brakke flows with equality (maybe further restricting the candidates to a special class with extra geometric properties).

PROPOSITION 5.14. *Any solution of Problem* (1.5) *in* $C^{2,1}([0,1]\times[0,T))$ *is a regular Brakke flow with equality.*
In particular, for every curve $\gamma^i(\cdot,t)$ *and for every time* $t\in[0,T)$ *we have*

$$\frac{dL^i(t)}{dt}=\lambda^i(1,t)-\lambda^i(0,t)-\int_{\gamma^i(\cdot,t)}k^2\,ds \tag{5.4}$$

and

$$\frac{dL(t)}{dt}=-\int_{\mathbb{S}_t}k^2\,ds\,,$$

that is, the total length $L(t)$ *is decreasing in time and it is uniformly bounded by the length of the initial network* $\mathbb{S}_0$.

PROOF. If the flow γ^i is in $C^\infty([0,1]\times[0,T))$, we have

$$\begin{aligned}\frac{dL^i(t)}{dt}&=\frac{d}{dt}\int_0^1|\gamma^i_x|\,dx\\&=\int_0^1\frac{\langle\gamma^i_{xt}\,|\,\gamma^i_x\rangle}{|\gamma^i_x|}\,dx\\&=\int_0^1\left\langle\partial_x\gamma^i_t\,\middle|\,\frac{\gamma^i_x}{|\gamma^i_x|}\right\rangle dx\\&=\int_0^1\langle\partial_x\gamma^i_t\,|\,\tau^i\rangle\,dx\\&=\langle\gamma^i_t(1,t)\,|\,\tau^i(1,t)\rangle-\langle\gamma^i_t(0,t)\,|\,\tau^i(0,t)\rangle-\int_0^1\langle\gamma^i_t\,|\,\partial_x\tau^i\rangle\,dx\,.\end{aligned}$$

Then, approximating the maps γ^i with a family of maps $\gamma^{i\epsilon}\in C^\infty$ such that $\gamma^{i\epsilon}\to\gamma^i$ in C^1 and $\gamma^{i\epsilon}_{xx}\to\gamma^i_{xx}$ in C^0, as $\epsilon\to0$, we see that we can pass to the limit in this formula and conclude that it holds for the original flow which is only in $C^{2,1}([0,1]\times[0,T))$. Finally, since $\partial_x\tau^i=k^i\nu^i|\gamma^i_x|$, we get

$$\frac{dL^i(t)}{dt}=\lambda^i(1,t)-\lambda^i(0,t)-\int_{\gamma^i(\cdot,t)}k^2\,ds$$

as $\gamma^i_t=k^i\nu^i+\lambda^i\tau^i$.
The formula for the derivative of the total length of the evolving network then follows by the zero–sum property of the functions λ^i at every 3–point at the fact that all the λ^i are zero at the end–points.
A similar argument shows that formula (5.2) defining a Brakke flow with equality also holds. □

THEOREM 5.15. *If* $\mathbb{S}_t$ *is a curvature flow of a* C^2 *initial network such that*

- *the unit tangents* τ^i *are continuous in* $[0,1]\times[0,T)$,
- *the functions* $k(\cdot,t)$ *converge weakly in* L^2 *to* $k(\cdot,0)$, *as* $t\to0$,
- *the function* $\int_{\mathbb{S}_t}k^2\,ds$ *is continuous on* $[0,T)$,

then $\mathbb{S}_t$ *is a regular Brakke flow with equality.*

PROOF. By the previous Theorem 5.14, we only need to check Brakke equality (5.2) at $t=0$. For every positive time and for every smooth test function $\varphi:\overline{\Omega}\times[0,T)\to\mathbb{R}$, we have

$$\frac{d}{dt}\int_{\mathbb{S}_t}\varphi\,ds=-\int_{\mathbb{S}_t}\varphi\,k^2\,ds+\int_{\mathbb{S}_t}\langle\nabla\varphi\,|\,\underline{k}\rangle\,ds\,d+\int_{\mathbb{S}_t}\varphi_t\,ds\,,$$

hence, it suffices to show that the right member is continuous at $t=0$. By the hypotheses, the only term that really need to be checked is $\int_{\mathbb{S}_t}\varphi\, k^2\, ds$, we separate it as the sum of $\int_{\mathbb{S}_t}\varphi^+\, k^2\, ds$ and $\int_{\mathbb{S}_t}\varphi^-\, k^2\, ds$ and we show the continuity of these two terms separately (here $\varphi^+ = \varphi\wedge 0$ and $\varphi^- = \varphi\vee 0$). Thus, we assume that $0\leq\varphi\leq 1$, then, by the weak convergence in $L^2(ds)$ of $k(\cdot,t)$ to $k(\cdot,0)$, the integral $\int_{\mathbb{S}_t}\varphi\, k^2\, ds$ is lower semicontinuous in t, that is, $\int_{\mathbb{S}_0}\varphi\, k^2\, ds\leq\liminf_{t_l\to 0}\int_{\mathbb{S}_t}\varphi\, k^2\, ds$ for every $t_l\to 0$, but if this is not an equality for some sequence of times, it cannot happen that $\int_{\mathbb{S}_t} k^2\, ds$ is continuous at $t=0$, indeed, we would have

$$\begin{aligned}\lim_{t_l\to 0}\int_{\mathbb{S}_t} k^2\, ds &\geq \liminf_{t_l\to 0}\int_{\mathbb{S}_t}\varphi\, k^2\, ds + \liminf_{t_l\to 0}\int_{\mathbb{S}_t}(1-\varphi)\, k^2\, ds\\ &> \int_{\mathbb{S}_0}\varphi\, k^2\, ds + \int_{\mathbb{S}_0}(1-\varphi)\, k^2\, ds = \int_{\mathbb{S}_t} k^2\, ds\,.\end{aligned}$$

This concludes the proof. □

COROLLARY 5.16. *The curvature flows whose short time existence is proved in Theorems 3.13 and 3.20 are Brakke flows with equality. The curvature flow of an initial C^2 regular network obtained in Theorem 5.8 is also a Brakke flow with equality. Any curvature flow of a regular network is a Brakke flow with equality for every positive time.*

We conclude this section with the following property of Brakke flows.

PROPOSITION 5.17. *For any regular Brakke flow with equality (hence, for every curvature flow of a regular network) such that the curvature is uniformly bounded in a time interval $[0,T)$, the lengths of the curves of the network $L^i(t)$ converge to some limit, as $t\to T$.*
In particular, if the flow satisfies the conclusion of Theorem 5.7 or Corollary 5.10 at the maximal time of existence T, there must be at least one curve such that $L^i(t)\to 0$, as $t\to T$.

PROOF. If the curvature is bounded, by formula (5.4), any function L^i as a uniformly bounded derivative, as k controls λ at the end–points and 3–points of the network, thus the conclusion follows. □

6. The monotonicity formula and rescaling procedures

Let $F:\mathbb{S}\times[0,T)\to\mathbb{R}^2$ be the curvature flow of a regular network in its maximal time interval of existence. As before, with a little abuse of notation, we will write $\tau(P^r,t)$ and $\lambda(P^r,t)$ respectively for the unit tangent vector and the tangential velocity at the end–point P^r of the curve of the network getting at such point, for any $r\in\{1,2,\dots,l\}$.

A modified form of Huisken's monotonicity formula for smooth hypersurfaces moving by mean curvature (see **[48]**), holds. It can be proved starting by formula (5.2) and with a slight modification of the computation in the proof of Lemma 6.3 in **[71]**.
Let $x_0\in\mathbb{R}^2, t_0\in(0,\infty)$ and $\rho_{x_0,t_0}:\mathbb{R}^2\times[0,t_0)$ be the one–dimensional *backward heat kernel* in $\mathbb{R}^2$ relative to (x_0,t_0), that is,

$$\rho_{x_0,t_0}(x,t)=\frac{e^{-\frac{|x-x_0|^2}{4(t_0-t)}}}{\sqrt{4\pi(t_0-t)}}\,.$$

We will often write $\rho_{x_0}(x,t)$ to denote $\rho_{x_0,T}(x,t)$ (or ρ_{x_0} to denote $\rho_{x_0,T}$), when T is the maximal (singular) time of existence of a smooth curvature flow.

PROPOSITION 6.1 (Monotonicity formula). *Assume $t_0>0$. For every $x_0\in\mathbb{R}^2$ and for every $t\in[0,\min\{t_0,T\})$ the following identity holds*

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}_t}\rho_{x_0,t_0}(x,t)\, ds = &-\int_{\mathbb{S}_t}\left|\underline{k}+\frac{(x-x_0)^\perp}{2(t_0-t)}\right|^2\rho_{x_0,t_0}(x,t)\, ds\\ &+\sum_{r=1}^{l}\left[\left\langle\frac{P^r-x_0}{2(t_0-t)}\,\middle|\,\tau(P^r,t)\right\rangle-\lambda(P^r,t)\right]\rho_{x_0,t_0}(P^r,t)\,.\end{aligned}\tag{6.1}$$

Integrating between t_1 and t_2 with $0 \leq t_1 \leq t_2 < t_0$ we get

$$\int_{t_1}^{t_2}\int_{\mathbb{S}_t}\left|\underline{k}+\frac{(x-x_0)^\perp}{2(t_0-t)}\right|^2\rho_{x_0,t_0}(x,t)\,ds\,dt=\int_{\mathbb{S}_{t_1}}\rho_{x_0,t_0}(x,t_1)\,ds-\int_{\mathbb{S}_{t_2}}\rho_{x_0,t_0}(x,t_2)\,ds$$
$$+\sum_{r=1}^{l}\int_{t_1}^{t_2}\left[\left\langle\frac{P^r-x_0}{2(t_0-t)}\,\middle|\,\tau(P^r,t)\right\rangle-\lambda(P^r,t)\right]\rho_{x_0,t_0}(P^r,t)\,dt\,.$$

We need the following lemma in order to estimate the end–points contribution in this formula (see Lemma 6.5 in [**71**]).

LEMMA 6.2. *For every $r \in \{1,2,\dots,l\}$ and $x_0 \in \mathbb{R}^2$, the following estimate holds*

$$\left|\int_t^{t_0}\left[\left\langle\frac{P^r-x_0}{2(t_0-\xi)}\,\middle|\,\tau(P^r,\xi)\right\rangle-\lambda(P^r,\xi)\right]\rho_{x_0,t_0}(P^r,\xi)\,d\xi\right|\leq C\,,$$

where C is a constant depending only on the constants C_l in assumption (4.1).
As a consequence, for every point $x_0 \in \mathbb{R}^2$, we have

$$\lim_{t\to t_0}\sum_{r=1}^{l}\int_t^{t_0}\left[\left\langle\frac{P^r-x_0}{2(t_0-\xi)}\,\middle|\,\tau(P^r,\xi)\right\rangle-\lambda(P^r,\xi)\right]\rho_{x_0,t_0}(P^r,\xi)\,d\xi=0\,.$$

As a consequence, the following definition is well posed.

DEFINITION 6.3 (Gaussian densities). For every $x_0 \in \mathbb{R}^2, t_0 \in (0,\infty)$ we define the *Gaussian density function* $\Theta_{x_0,t_0} : [0,\min\{t_0,T\}) \to \mathbb{R}$ as

$$\Theta_{x_0,t_0}(t)=\int_{\mathbb{S}_t}\rho_{x_0,t_0}(x,t)\,ds$$

and provided $t_0 \leq T$, the *limit density function* $\widehat{\Theta} : \mathbb{R}^2 \times (0,\infty) \to \mathbb{R}$ as

$$\widehat{\Theta}(x_0,t_0)=\lim_{t\to t_0}\Theta_{x_0,t_0}(t)\,.$$

Moreover, we will often write $\Theta_{x_0}(t)$ to denote $\Theta(x_0,T)$ and $\widehat{\Theta}(x_0)$ for $\widehat{\Theta}(x_0,T)$.

The limit $\widehat{\Theta}$ exists and it is finite. Moreover, the map $\widehat{\Theta} : \mathbb{R}^2 \to \mathbb{R}$ is upper semicontinuous (see [**67**, Proposition 2.12]).

6.1. Parabolic rescaling of the flow. For a fixed $\mu > 0$ the standard parabolic rescaling of a curvature flow given by the map F above, around a space–time point (x_0,t_0), is defined as the family of maps

$$F^\mu_{\mathfrak{t}}=\mu\big(F(\cdot,\mu^{-2}\mathfrak{t}+t_0)-x_0\big)\,, \tag{6.2}$$

where $\mathfrak{t} \in [-\mu^2 t_0, \mu^2(T-t_0))$. Note that this is again a curvature flow in the domain $\mu(\Omega - x_0)$ with new time parameter $\mathfrak{t}$.

Given a sequence $\mu_i \nearrow +\infty$ and a space–time point (x_0,t_0), where $0 < t_0 \leq T$, we then consider the sequence of curvature flows $F^{\mu_i}_{\mathfrak{t}}$ in the whole $\mathbb{R}^2$ that we denote with $\mathbb{S}^{\mu_i}_{\mathfrak{t}}$.
Recall that the monotonicity formula implies

$$\Theta_{x_0,t_0}(t)-\widehat{\Theta}(x_0,t_0)=\int_t^{t_0}\int_{\mathbb{S}_\sigma}\left|\underline{k}+\frac{(x-x_0)^\perp}{2(t_0-\sigma)}\right|^2\rho_{x_0,t_0}(\cdot,\sigma)\,ds\,d\sigma$$
$$-\sum_{r=1}^{l}\int_t^{t_0}\left[\left\langle\frac{P^r-x_0}{2(t_0-\sigma)}\,\middle|\,\tau(P^r,\sigma)\right\rangle-\lambda(P^r,\sigma)\right]\rho_{x_0,t_0}(P^r,\sigma)\,d\sigma\,.$$

Changing variables according to the parabolic rescaling, we obtain

$$\Theta_{x_0,t_0}(t_0+\mu_i^{-2}\mathfrak{t})-\widehat{\Theta}(x_0,t_0)=\int_{\mathfrak{t}}^{0}\int_{\mathbb{S}^{\mu_i}_{\mathfrak{s}}}\Big|\underline{k}-\frac{x^\perp}{2\mathfrak{s}}\Big|^2\rho_{0,0}(\cdot,\mathfrak{s})\,ds\,d\mathfrak{s}$$
$$+\sum_{r=1}^{l}\int_{\mathfrak{t}}^{0}\Big[\Big\langle\frac{P^r_i}{2\mathfrak{s}}\,\Big|\,\tau(P^r_i,\mathfrak{s})\Big\rangle+\lambda(P^r_i,\mathfrak{s})\Big]\rho_{0,0}(P^r_i,\mathfrak{s})\,d\mathfrak{s}\,,$$

where $P^r_i=\mu_i(P^r-x_0)$.

Hence, sending $i\to\infty$, by Lemma 6.2, for every $\mathfrak{t}\in(-\infty,0)$ we get

$$\lim_{i\to\infty}\int_{\mathfrak{t}}^{0}\int_{\mathbb{S}^{\mu_i}_{\mathfrak{s}}}\Big|\underline{k}-\frac{x^\perp}{2\mathfrak{s}}\Big|^2\rho_{0,0}(\cdot,\mathfrak{s})\,ds\,d\mathfrak{s}=0\,.$$

6.2. Huisken's dynamical rescaling. Next we introduce the rescaling procedure of Huisken in **[48]** at the maximal time T.
Fixed $x_0\in\mathbb{R}^2$, let $\widetilde{F}_{x_0}:\mathbb{S}\times[-1/2\log T,+\infty)\to\mathbb{R}^2$ be the map

$$\widetilde{F}_{x_0}(p,\mathfrak{t})=\frac{F(p,t)-x_0}{\sqrt{2(T-t)}}\qquad \mathfrak{t}(t)=-\frac{1}{2}\log{(T-t)}$$

then, the rescaled networks are given by

$$\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}}=\frac{\mathbb{S}_t-x_0}{\sqrt{2(T-t)}}\tag{6.3}$$

and they evolve according to the equation

$$\frac{\partial}{\partial\mathfrak{t}}\widetilde{F}_{x_0}(p,\mathfrak{t})=\underline{\widetilde{v}}(p,\mathfrak{t})+\widetilde{F}_{x_0}(p,\mathfrak{t})$$

where

$$\underline{\widetilde{v}}(p,\mathfrak{t})=\sqrt{2(T-t(\mathfrak{t}))}\cdot\underline{v}(p,t(\mathfrak{t}))=\underline{\widetilde{k}}+\underline{\widetilde{\lambda}}=\widetilde{k}\nu+\widetilde{\lambda}\tau\qquad\text{and}\qquad t(\mathfrak{t})=T-e^{-2\mathfrak{t}}\,.$$

Notice that we did not put the sign $\widetilde{\ }$ over the unit tangent and normal, since they remain the same after the rescaling.
We will write $\widetilde{O}^p(\mathfrak{t})=\widetilde{F}_{x_0}(O^p,\mathfrak{t})$ for the 3–points of the rescaled network $\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}}$ and $\widetilde{P}^r(\mathfrak{t})=\widetilde{F}_{x_0}(P^r,\mathfrak{t})$ for the end–points, when there is no ambiguity on the point x_0.
The rescaled curvature evolves according to the following equation,

$$\partial_{\mathfrak{t}}\widetilde{k}=\widetilde{k}_{\sigma\sigma}+\widetilde{k}_\sigma\widetilde{\lambda}+\widetilde{k}^3-\widetilde{k}$$

which can be obtained by means of the commutation law

$$\partial_{\mathfrak{t}}\partial_\sigma=\partial_\sigma\partial_{\mathfrak{t}}+(\widetilde{k}^2-\widetilde{\lambda}_\sigma-1)\partial_\sigma\,,$$

where we denoted with σ the arclength parameter for $\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}}$.

By a straightforward computation (see **[48]**) we have the following rescaled version of the monotonicity formula.

PROPOSITION 6.4 (Rescaled monotonicity formula). *Let $x_0\in\mathbb{R}^2$ and set*

$$\widetilde{\rho}(x)=e^{-\frac{|x|^2}{2}}$$

For every $\mathfrak{t}\in[-1/2\log T,+\infty)$ the following identity holds

$$\frac{d}{d\mathfrak{t}}\int_{\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}}}\widetilde{\rho}(x)\,d\sigma=-\int_{\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}}}|\,\underline{\widetilde{k}}+x^\perp|^2\widetilde{\rho}(x)\,d\sigma+\sum_{r=1}^{l}\Big[\Big\langle\widetilde{P}^r(\mathfrak{t})\,\Big|\,\tau(P^r,t(\mathfrak{t}))\Big\rangle-\widetilde{\lambda}(P^r,\mathfrak{t})\Big]\widetilde{\rho}(\widetilde{P}^r(\mathfrak{t}))$$

where $\widetilde{P}^r(\mathfrak{t}) = \frac{P^r - x_0}{\sqrt{2(T-t(\mathfrak{t}))}}$.
Integrating between $\mathfrak{t}_1$ *and* $\mathfrak{t}_2$ *with* $-1/2 \log T \leq \mathfrak{t}_1 \leq \mathfrak{t}_2 < +\infty$ *we get*

$$\begin{aligned}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2} \int_{\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}}} |\,\underline{\widetilde{k}} + x^\perp|^2 \widetilde{\rho}(x)\, d\sigma\, d\mathfrak{t} = &\int_{\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_1}} \widetilde{\rho}(x)\, d\sigma - \int_{\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_2}} \widetilde{\rho}(x)\, d\sigma \\ &+ \sum_{r=1}^{l} \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} \Big[\Big\langle\, \widetilde{P}^r(\mathfrak{t}) \,\Big|\, \tau(P^r, t(\mathfrak{t})) \Big\rangle - \widetilde{\lambda}(P^r, \mathfrak{t}) \Big]\, \widetilde{\rho}(\widetilde{P}^r(\mathfrak{t})\, d\mathfrak{t}\,.\end{aligned} \tag{6.4}$$

We have also the analog of Lemma 6.2 (see Lemma 6.7 in **[71]**).

LEMMA 6.5. *For every* $r \in \{1, 2, \dots, l\}$ *and* $x_0 \in \mathbb{R}^2$, *the following estimate holds*

$$\left| \int_{\mathfrak{t}}^{+\infty} \Big[\Big\langle\, \widetilde{P}^r(\xi) \,\Big|\, \tau(P^r, t(\xi)) \Big\rangle - \widetilde{\lambda}(P^r, \xi) \Big]\, d\xi \right| \leq C\,,$$

where C *is a constant depending only on the constants* C_l *in assumption* (4.1).
As a consequence, for every point $x_0 \in \mathbb{R}^2$, *we have*

$$\lim_{\mathfrak{t} \to +\infty} \sum_{r=1}^{l} \int_{\mathfrak{t}}^{+\infty} \Big[\Big\langle\, \widetilde{P}^r(\xi) \,\Big|\, \tau(P^r, t(\xi)) \Big\rangle - \widetilde{\lambda}(P^r, \xi) \Big]\, d\xi = 0\,.$$

7. Classification of possible blow–up limits

In this section we want to discuss the possible limits of an evolving network at the maximal time of existence. When the curvature does not remain bounded, we are interested in the possible blow–up limit networks after Huisken's rescaling procedure, using the rescaled monotonicity formula (see Section 6). In some cases, such limit *sets* are no more regular networks, so we introduce the following definition.

DEFINITION 7.1 (Degenerate regular network). Consider a couple $(G, \mathbb{S})$ with the following properties:

- $G = \bigcup_{i=1}^n E^i$ is an oriented graph with possible unbounded edges E^i, such that every vertex has only one or three concurring edges (we call end–points of G the vertices with order one);
- we have a family of C^1 curves $\sigma^i : I^i \to \mathbb{R}^2$, where I^i is the interval $(0,1)$, $[0,1)$, $(0,1]$ or $[0,1]$, and orientation preserving homeomorphisms $\varphi^i : E^i \to I^i$, then $\mathbb{S}$ is the union of the images of I^i through the curves σ^i, that is, $\mathbb{S} = \bigcup_{i=1}^n \sigma^i(I^i)$ (notice that the interval $(0,1)$ can only appear if it is associated to an unbounded edge E^i without vertices, which is clearly a single connected component of G);
- in the case that I^i is $(0,1)$, $[0,1)$ or $(0,1]$, the map σ^i is a regular C^1 curve with unit tangent vector field τ^i;
- in the case that $I^i = [0,1]$, the map σ^i is either a regular C^1 curve with unit tangent vector field τ^i, or a constant map and in this case it is "assigned" also a *constant* unit vector $\tau^i : I^i \to \mathbb{R}^2$, that we still call unit tangent vector of σ^i (we call these maps σ^i "degenerate curves");
- for every degenerate curve $\sigma^i : I^i \to \mathbb{R}^2$ with assigned unit vector $\tau^i : I^i \to \mathbb{R}^2$, we call "assigned exterior unit tangents" of the curve σ^i at the points 0 and 1 of I^i, respectively the unit vectors $-\tau^i$ and τ^i.
- the map $\Gamma : G \to \mathbb{R}^2$ given by the union $\Gamma = \bigcup_{i=1}^n (\sigma^i \circ \varphi^i)$ is well defined and continuous;
- for every 3–point of the graph G, where the edges E^i, E^j, E^k concur, the exterior unit tangent vectors (real or "assigned") at the relative borders of the intervals I^i, I^j, I^k of the concurring curves σ^i, σ^j σ^k have zero sum ("*degenerate* 120 *degrees condition*").

Then, we call $\mathbb{S} = \bigcup_{i=1}^n \sigma^i(I^i)$ a *degenerate regular network*.

If one or several edges E^i of G are mapped under the map $\Gamma : G \to \mathbb{R}^2$ to a single point $p \in \mathbb{R}^2$, we call this sub–network given by the union G' of such edges E^i, the *core* of $\mathbb{S}$ at p.

We call multi–points of the degenerate regular network $\mathbb{S}$, the images of the vertices of multiplicity three of the graph G, by the map Γ.

We call end–points of the degenerate regular network $\mathbb{S}$, the images of the vertices of multiplicity one of the graph G, by the map Γ.

REMARK 7.2.

- A regular network is clearly a degenerate regular network.
- This definition will be useful to deal with the limit sets when at some time a curve of the network "collapses", that is, its length goes to zero (see Proposition 8.21).
- A degenerate regular network $\mathbb{S}$ with underlying graph G, seen as a subset in $\mathbb{R}^2$, is a C^1 network, not necessarily regular, that can have end–points and/or unbounded curves. Moreover, self–intersections and curves with integer multiplicities can be present. Anyway, by the degenerate 120 degrees condition at the last point of the definition, at every image of a multi–point of G the sum (possibly with multiplicities) of the exterior unit tangents (the "assigned" ones cancel each other in pairs) is zero. Notice that this implies that every multiplicity–one 3–point must satisfy the 120 degrees condition.

LEMMA 7.3. *Let $\mathbb{S} = \bigcup_{i=1}^n \sigma^i(I_i)$ a degenerate regular network in Ω and $X : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth vector field with compact support. Then, there holds*

$$\int_{\mathbb{S}} \partial_s \langle X(\sigma) \,|\tau\rangle \, d\overline{\mathcal{H}}^1 = -\sum_{r=1}^{l} \langle X(P^r) \,|\tau(P^r)\rangle \,,$$

where $P^1, P^2, \dots, P^l$ are the end–points of $\mathbb{S}$, $\tau(P^1), \tau(P^2), \dots, \tau(P^l)$ are the exterior unit tangents at P^r and $\overline{\mathcal{H}}^1$ is the 1–dimensional Hausdorff measure, counting multiplicities.

PROOF. This is a consequence of the degenerate 120 degrees condition, implying that the sum of all the contribution at a multi–point given by the boundary terms after the integration on each single curve, is zero (as the sum of the exterior unit tangents of the concurring curves). Thus, the only remaining terms are due to the end–points of the degenerate regular network. $\square$

DEFINITION 7.4. We say that a sequence of regular networks $\mathbb{S}_k = \bigcup_{i=1}^n \sigma_k^i(I_k^i)$ converges in C^1_{loc} to a degenerate regular network $\mathbb{S} = \bigcup_{j=1}^l \sigma_\infty^j(I_\infty^j)$ with underlying graph $G = \bigcup_{j=1}^l E^j$ if:

- letting $O^1, O^2, \dots, O^m$ the multi–points of $\mathbb{S}$, for every open set $\Omega \subset \mathbb{R}^2$ with compact closure in $\mathbb{R}^2 \setminus \{O^1, O^2, \dots, O^m\}$, the networks $\mathbb{S}_k$ restricted to Ω are definitely described by families of regular curves which, after possibly reparametrizing them, converge to the family of regular curves given by the restriction of $\mathbb{S}$ to Ω;
- for every multi–point O^p of $\mathbb{S}$, image of one or more vertices of the graph G (if a core is present), there is a sufficiently small $R > 0$ and a graph $\widetilde{G} = \bigcup_{r=1}^s F^r$, with edges F^r associated to intervals J^r, such that:
  - the restriction of $\mathbb{S}$ to B_R is a regular degenerate network described by a family of curves $\widetilde{\sigma}_\infty^r : J^r \to \mathbb{R}^2$ with (possibly "assigned", if the curve is degenerate) unit tangent $\widetilde{\tau}_\infty^r$,
  - definitely for every k, the restriction of $\mathbb{S}_k$ to B_R is a regular network with underlying graph $\widetilde{G}$, described by the family of regular curves $\widetilde{\sigma}_k^r : J^r \to \mathbb{R}^2$,
  - for every j, possibly after reparametrization of the curves, the sequence of maps $J^r \ni x \mapsto \big(\widetilde{\sigma}_k^r(x), \widetilde{\tau}_k^r(x)\big)$ converge in C^0_{loc} to the maps $J^r \ni x \mapsto \big(\widetilde{\sigma}_\infty^r(x), \widetilde{\tau}_\infty^r(x)\big)$, for every $r \in \{1, 2, \dots, s\}$.

We will say that $\mathbb{S}_k$ converges to $\mathbb{S}$ in $C^1_{\mathrm{loc}} \cap E$, where E is some function space, if the above curves also converge in the topology of E.

REMARK 7.5.

- It is easy to see that if a sequence of regular networks $\mathbb{S}_k$ converges in C^1_{loc} to a degenerate regular network $\mathbb{S}$, the associated one–dimensional Hausdorff measures, counting multiplicities, weakly–converge (as measures) to the one–dimensional Hausdorff measure associated to the set $\mathbb{S}$ seen as a subset of $\mathbb{R}^2$.

- If a degenerate regular network $\mathbb{S}$ is the limit of a sequence of regular network as above, being these embedded, it clearly can have only *tangent* self–intersections but not a "crossing" of two of its curves.
- If $\mathbb{S}$ is the limit of a sequence of "rescalings" of the networks of a curvature flow $\mathbb{S}_t$ with fixed end–points, it can have only one end–point at the origin of $\mathbb{R}^2$ and only if the center of the rescalings coincides with an end–point of $\mathbb{S}_t$, otherwise, it has no end–points at all (they go to ∞ in the rescaling).

7.1. Self–similarly shrinking networks.

DEFINITION 7.6. A regular C^2 open network $\mathbb{S} = \bigcup_{i=1}^n \sigma^i(I_i)$ is called a *regular shrinker* if at every point $x \in \mathbb{S}$ there holds

$$\underline{k} + x^\perp = 0. \tag{7.1}$$

This relation is called the *shrinkers equation*.

The name comes from the fact that if $\mathbb{S} = \bigcup_{i=1}^n \sigma^i(I_i)$ is a shrinker, then the evolution given by $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i(I_i, t)$ where $\gamma^i(x,t) = \sqrt{1-2t}\,\sigma^i(x)$ is a self–similarly shrinking curvature flow in the time interval $(-\infty, \frac{1}{2})$ with $\mathbb{S} = \mathbb{S}_0$. Viceversa, if $\mathbb{S}_t$ is a self–similarly shrinking curvature flow in the maximal time interval $(-\infty, \frac{1}{2})$, then $\mathbb{S}_0$ is a shrinker.

In general a shrinker is composed by lines through the origin, halfline pointing the origin or pieces of the so–called Abresch-Langer curves [**1**]. The only embedded shrinking curves in $\mathbb{R}^2$ are the lines through the origin and the unit circle. The embedded, connected regular shrinkers with one triple junction are exactly two (up to translation): the standard triod and the Brakke spoon (see [**23**]).

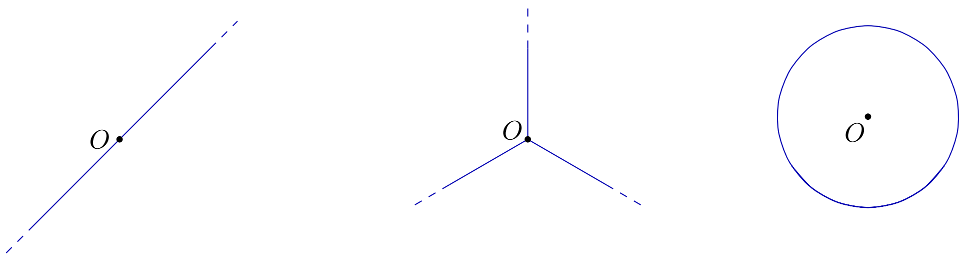


FIGURE 6. Simple examples of regular shrinkers are a line for the origin, an unbounded triod composed of three halflines from the origin meeting at 120 degrees, that we call *standard triod* and the unit circle $\mathbb{S}^1$.

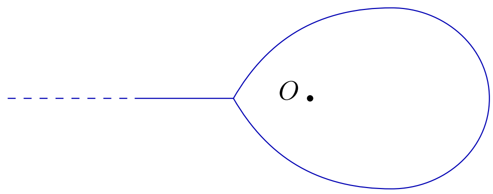


FIGURE 7. A less simple example of a regular shrinker: a *Brakke spoon* is a regular shrinker composed by a halfline which intersects a closed curve, forming angles of 120 degrees. It was first mentioned in [**18**] as an example of evolving network with a loop shrinking down to a point, leaving a halfline (that then vanishes instantaneously, in the contest of Brakke flows). Up to rotation, this particular spoon–shaped network is unique (see [**23**]).

About shrinkers with two triple junctions, it is not difficult to show that there are only two possible topological shapes for a complete embedded, regular shrinker: one is the "lens/fish" shape and the other is the shape of the Greek "Theta" letter (or "double cell"), as in the next figure.

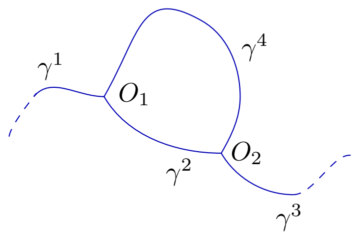


FIGURE 8. A lens/fish–shaped and a Θ–shaped network.

It is well known that there exist unique (up to a rotation) lens–shaped or fish–shaped, embedded, regular shrinkers which are symmetric with respect to a line through the origin of $\mathbb{R}^2$ (see **[23, 84]**). Instead, there are no regular Θ–shaped shrinkers (see **[15]**).

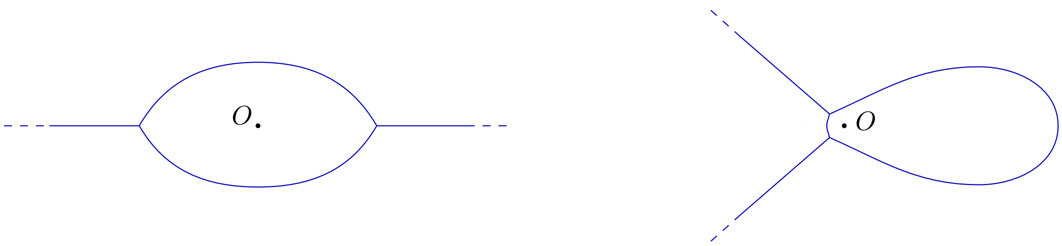


FIGURE 9. A lens–shaped and a fish–shaped shrinker.

DEFINITION 7.7. A *standard triod* is an unbounded shrinker triod composed of three halflines from the origin meeting at 120 degrees (Figure 6).
A *Brakke spoon* is a shrinker composed by a halfline which intersects a closed curve, forming angles of 120 degrees (first mentioned in **[18]**, Figure 7).
A *standard lens* is a shrinker with two triple junctions symmetric with respect to two perpendicular axes, composed by two halflines pointing the origin, posed on a symmetry axis and opposite with respect to the other. Each halfline intersects two equal curves forming an angle of 120 degrees (Figure 9).
A *fish* is a shrinker with the same topology of the standard lens, but symmetric with respect to only one axis. The two halfines, pointing the origin, intersect two different curves, forming angles of 120 degrees (Figure 9).

DEFINITION 7.8 (Degenerate shrinkers). We call a degenerate regular network $\mathbb{S} = \bigcup_{i=1}^{n} \sigma^i(I_i)$ a *degenerate regular shrinker* if at every point $x \in \mathbb{S}$ there holds

$$\underline{k} + x^{\perp} = 0\,.$$

Clearly, a regular shrinker is a degenerate regular shrinker and, as before, the maps $\gamma^i(x,t) = \sqrt{-2t}\,\sigma^i(x)$ describe the self–similarly shrinking evolution of a degenerate regular network $\mathbb{S}_t$ in the time interval $(-\infty, 0)$, with $\mathbb{S} = \mathbb{S}_0$.

REMARK 7.9. As every non–degenerate curve of a degenerate regular shrinker (or simply of a regular shrinker) satisfies the equation $\underline{k} + x^{\perp} = 0$, it must be a piece of a line though the origin or of the so called *Abresch–Langer curves*. Their classification results in **[1]** imply that any of these non straight pieces is compact, hence any unbounded curve of a shrinker must be a line or an halfline "pointing" the origin. Moreover, it also follows that if a curve contains the origin, then it is a straight line through the origin (if it is in the interior) or a halfline from the origin (if it is an end–point of the curve).

An example, that plays a role in the sequel, is the union of four halflines from the origin that form alternate angles of 120 and 60 degrees.

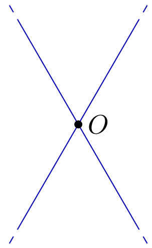


FIGURE 10. An example of degenerate regular shrinker: four halflines from the origin forming angles in pair of $120/60$ degrees.

For a degenerate regular shrinker $\mathbb{S}$, in analogy with Definition 6.3, we denote with

$$\Theta_{\mathbb{S}} = \Theta_{0,0}(-1/2) = \int_{\mathbb{S}} \rho_{0,0}(\cdot, -1/2)\, d\overline{s}$$

its Gaussian density (here $d\overline{s}$ denotes the integration with respect to the canonical measure on $\mathbb{S}$, counting multiplicities). Notice that the integral $\Theta_{0,0}(t) = \int_{\mathbb{S}_t} \rho_{0,0}(\cdot, t)\, d\overline{s}$ is constant for $t \in (-\infty, 0)$, hence equal to $\widehat{\Theta}(0)$ for the self–similarly shrinking curvature flow $\mathbb{S}_t = \sqrt{-2t}\,\mathbb{S}$ generated by $\mathbb{S}$, as above.

The Gaussian density of a straight line through the origin is 1, of a halfline from the origin is $1/2$, of a standard triod $\mathbb{T}$ is $3/2$. The Gaussian density of the unit circle $\mathbb{S}^1$ can be easily computed to be

$$\Theta_{\mathbb{S}^1} = \sqrt{\frac{2\pi}{e}} \approx 1{,}52\,.$$

Notice that $\Theta_{\mathbb{T}} = 3/2 < \Theta_{\mathbb{S}^1} < 2$.

We have the following two classification results for degenerate regular shrinkers, see [**53**, Lemma 8.3, 8.4].

LEMMA 7.10. *Let $\mathbb{S} = \bigcup_{i=1}^n \sigma^i(I_i)$ be a degenerate regular shrinker which is C^1_{loc}–limit of regular networks homeomorphic to the underlying graph G of $\mathbb{S}$ (as in Definition 7.1) and assume that G is a tree without end–points. Then $\mathbb{S}$ consists of halflines from the origin, with possibly a core at the origin.*
Moreover, if G is connected, without end–points and $\mathbb{S}$ is a network with unit multiplicity, this latter can only be

- *a line (no core),*
- *a standard triod (no core),*
- *two lines intersecting at the origin forming angles of $120/60$ degrees (the core is a collapsed segment in the origin with "assigned" unit tangent vector bisecting the angles of 120 degrees).*

PROOF. We assume that G is connected, otherwise we argue on every single connected component. By the hypothesis of approximation with regular (embedded) networks, G is a planar graph.

As we said in Remark 7.9, if a non–degenerate curve contains the origin, then it is a piece of a straight line. Otherwise, it is contained in a compact subset of $\mathbb{R}^2$ and has a constant winding direction with respect to the origin. Aside from the circle, any other solution has a countable, non–vanishing number of self–intersections (all these facts were shown in [**1**]).

Suppose that the network $\mathbb{S}$ has a core at some point $P \in \mathbb{S}$, then, at least an edge of G is mapped into P.

Being the graph G a tree, it can be seen easily by induction, that from P there must exit $N+2$ (not necessarily distinct) non–collapsed curves, where N is the number (greater than one) of 3–points contained in the core. Moreover, considering the the longest simple "path" in G which is mapped in the core at P of $\mathbb{S}$, orienting it and "following" its edges, the assigned unit tangent vector (possibly changed of sign on some edges in order to coincide with the orientation of the path) cannot "turn" of an angle of 60 degrees in the same "direction" for two consecutive times, otherwise, since G is a tree, the approximating networks must have definitely a self–intersection (see Figure 11 below).

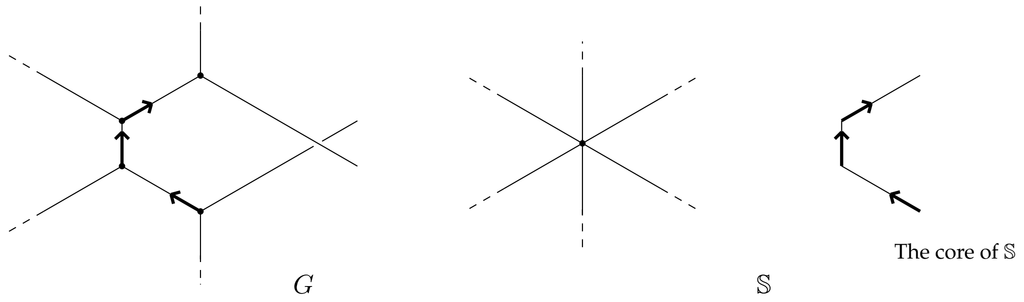


FIGURE 11. If the assigned unit tangent vector "turns" of an angle of 60 degrees in the same direction for two consecutive times, G has self–intersections. An example of such a pair $(G, \mathbb{S})$.

Hence, the assigned unit tangent vector "turns" of an angle of 60 degrees then it must "turn" back, in passing from an edge to another along such longest path. This means that at the initial/final point of such path, either the two assigned unit tangent vectors are the same (when the number of edges is odd) or they differ of 60 degrees (when the number of edges is even). By a simple check, we can then see that, in the first case the four curves images of the four non–collapsed edges exiting from such initial/final points of the path, have four different exterior unit tangent vectors at P (opposite in pairs), in the second case, they have three exterior unit tangent vectors at P which are non–proportional each other.

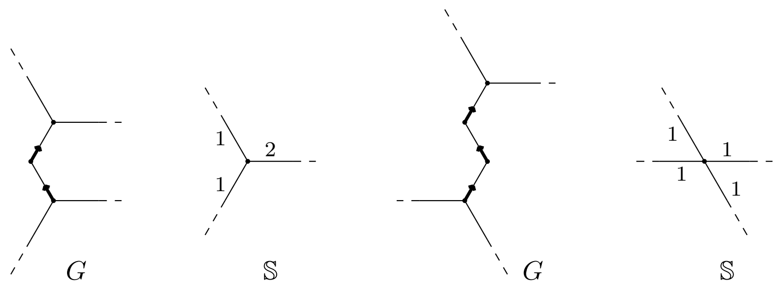


FIGURE 12. Examples of the edges at the initial/final points of the longest simple path in G and of the relative curves in $\mathbb{S}$, the numbers 1 and 2 denote their multiplicity.

If then there is a 3–point or a core at some point $P \neq 0$, since at most two of the four directions in the first case above and at most one of the three directions in the second case, can belong to the straight line for P and the origin, there are always at least two non–straight Abresch–Langer curves arriving/starting at P. Clearly, this property holds also if there is no core at P, but P is simply a 3–point.

Let us consider $\mathbb{S}' \subset \mathbb{S}$, which consists of $\mathbb{S}$ with the interior of all the pieces of straight lines removed and let σ^i one of the two curves above. We follow σ^i till its other end–point Q. At this end–point, even if there is a core at Q, there is always another different non–straight curve σ^j to continue moving in $\mathbb{S}$ avoiding the pieces of straight lines (hence staying far from the origin). Actually, either the underlying intervals I_i and I_j are concurrent at the vertex corresponding to Q in the graph G or there is a path in G ("collapsed" in the core at Q) joining I_i and I_j. We then go on with this path on $\mathbb{S}$ (and on G) till, looking at things on the graph G, we arrive at an already considered vertex, which happens since the number of vertices of G is finite, obtaining a closed loop, hence, a contradiction. Thus, $\mathbb{S}'$ cannot contain 3–points or cores outside the origin. If anyway $\mathbb{S}$ contains a non–straight Abresch–Langer curve, we can repeat this argument getting again a contradiction, hence, we are done with the first part of the lemma, since then $\mathbb{S}$ can only consist of halflines from the origin.

Now we assume that G is connected and $\mathbb{S}$ is a network with multiplicity one, composed of halflines from the origin.
If there is no core, $\mathbb{S}$ is homeomorphic to G and composed only by halflines for the origin, hence G has at most one vertex, by connectedness. If G has no vertices, then $\mathbb{S}$ must be a line, if it has a 3–point, $\mathbb{S}$ is a standard triod.
If there is a core in the origin, by the definition of degenerate regular network it follows that the halflines of $\mathbb{S}$ can only have six possible directions, by the 120 degrees condition, hence, by the unit multiplicity hypothesis, the graph G is a tree in the plane with at most six unbounded edges. Arguing as in the first part of the lemma, if N denotes the number (greater than one) of 3–points contained in the core, it follows that N can only assume the values $2, 3, 4$. Repeating the argument of the "longest path", we immediately also exclude the case $N = 3$, since there would be a pair of coincident halflines in $\mathbb{S}$, against the multiplicity–one hypothesis, while for $N = 4$ we have only two possible situations, described at the bottom of the following figure.

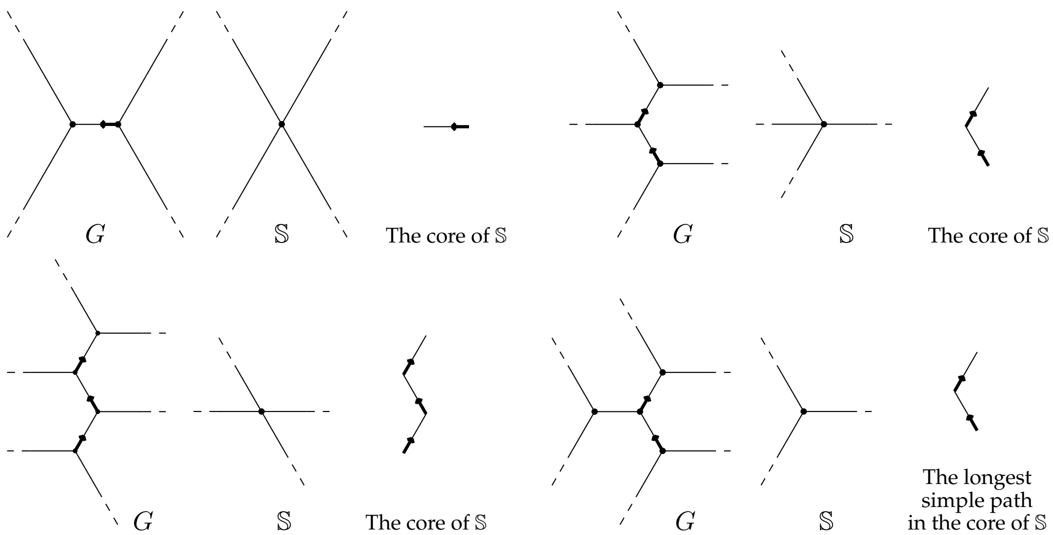


FIGURE 13. The possible local structure of the graphs G, with relative networks $\mathbb{S}$ and cores, for $N = 2, 3, 4$.

Hence, if $N = 4$, in both two situations above there is in $\mathbb{S}$ at least one halfline with multiplicity two, thus such case is also excluded.
Then, we conclude that the only possible network with a core is when $N = 2$ and $\mathbb{S}$ is given by two lines intersecting at the origin forming angles of $120/60$ degrees and the core consists of a collapsed segment which must have an "assigned" unit tangent vector bisecting the two angles of 120 degrees formed by the four halflines. ☐

LEMMA 7.11. *Let $\mathbb{S} = \bigcup_{i=1}^{n} \sigma^i(I_i)$ be a degenerate regular shrinker which is C^1_{loc}–limit of regular networks homeomorphic to the underlying graph G of $\mathbb{S}$ (as in Definition 7.1) and assume that $\Theta_{\mathbb{S}} < \Theta_{\mathbb{S}^1}$. Then, the graph G of $\mathbb{S}$ is a tree. Thus, $\mathbb{S}$ is either a multiplicity–one line or a standard triod.*

PROOF. By the hypotheses, we see that G is a planar graph. We assume that G is not a tree, that is, it contains a loop, then we can find a (possibly smaller) loop bounding a region. If such loop is in a core at some point P, it is easy to see, by the degenerate 120 degrees condition, that such region has six edges and, arguing as in Lemma 7.10, that there must always be at least two non–collapsed, non–straight Abresch–Langer curves arriving/starting at P in different directions.

Then, if we assume that the complement of $\mathbb{S}$ in $\mathbb{R}^2$ contains no bounded components, repeating the argument in the proof of the previous lemma, it follows that $\mathbb{S}$ consists of a union of halflines for the origin and the loops of G are all collapsed in the core. Then, by what we said above, there must be at least six halflines emanating from (the core at) the origin. This implies that $\Theta_{\mathbb{S}} \geq 3$, which is a contradiction.

Let now B be a bounded component of the complement of $\mathbb{S}$ and γ a connected component of the sub–network of $\mathbb{S}$ which bounds B, counted with unit multiplicity. Since γ is an embedded, closed curve, smooth with corners and no triple junctions, we can evolve it by "classical" curve shortening flow γ_t, for $t \in [-1/2, t_0)$ where we set $\gamma_{-1/2} = \gamma$, until it shrinks at some $t_0 > -1/2$ to a "round" point $x_0 \in \mathbb{R}^2$ (by the works of Angenent, Gage, Grayson, Hamilton [**10, 11, 12, 37, 36, 38, 41**], see Remark 1.2).
By the monotonicity formula, we have

$$\int_\gamma \rho_{x_0,t_0}(\cdot, -1/2)\, ds \geq \Theta_{\mathbb{S}^1}$$

and, by the work of Colding–Minicozzi [**26**, Section 7.2], there holds

$$\Theta_{\mathbb{S}} = \int_{\mathbb{S}} \rho_{0,0}(\cdot, -1/2)\, d\overline{s} = \sup_{x_0 \in \mathbb{R}^2, t_0 > -1/2} \int_{\mathbb{S}} \rho_{x_0,t_0}(\cdot, -1/2)\, d\overline{s}\,. \tag{7.2}$$

Then,

$$\Theta_{\mathbb{S}} \geq \int_{\mathbb{S}} \rho_{x_0,t_0}(\cdot, -1/2)\, d\overline{s} \geq \int_\gamma \rho_{x_0,t_0}(\cdot, -1/2)\, ds \geq \Theta_{\mathbb{S}^1}\,,$$

which is a contradiction and we are done. □

7.2. Some geometric properties of the network flow. Before proceeding, we show some geometric properties of the curvature flow of a network that we will need in the sequel.

PROPOSITION 7.12. *Let $\mathbb{S}_t$ be the curvature flow of a regular network in a smooth, convex, bounded, open set Ω, with fixed end–points on the boundary of Ω, for $t \in [0, T)$. Then, for every time $t \in [0, T)$, the network $\mathbb{S}_t$ intersects the boundary of Ω only at the end–points and such intersections are transversal for every positive time. Moreover, $\mathbb{S}_t$ remains embedded.*

PROOF. By continuity, the 3–points cannot hit the boundary of Ω at least for some time $T' > 0$. The convexity of Ω and the strong maximum principle (see [**82**]) imply that the network cannot intersect the boundary for the first time at an inner regular point. As a consequence, if $t_0 > 0$ is the "first time" when the $\mathbb{S}_t$ intersects the boundary at an inner point, this latter has to be a 3–point. The minimality of t_0 is then easily contradicted by the convexity of Ω, the 120 degrees condition and the nonzero length of the curves of $\mathbb{S}_{t_0}$.
Even if some of the curves of the initial network are tangent to $\partial\Omega$ at the end–points, by the strong maximum principle, as Ω is convex, the intersections become immediately transversal, and stay so for every subsequent time.
Finally, if the evolution $\mathbb{S}_t$ loses embeddedness for the first time, this cannot happen neither at a boundary point, by the argument above, nor at a 3–point, by the 120 degrees condition. Hence it must happen at interior regular points, but this contradicts the strong maximum principle. □

PROPOSITION 7.13. *In the same hypotheses of the previous proposition, if the smooth, bounded, open set Ω is strictly convex, for every fixed end–point P^r on the boundary of Ω, for $r \in \{1, 2, \dots, l\}$, there is a time $t_r \in (0, T)$ and an angle α_r smaller than $\pi/2$ such that the curve of the network arriving at P^r form an angle less that α_r with the inner normal to the boundary of Ω, for every time $t \in (t_r, T)$.*

PROOF. We observe that the evolving network $\mathbb{S}_t$ is contained in the convex set $\Omega_t \subset \Omega$, obtained by letting $\partial\Omega$ (which is a finite set of smooth curves with end–points P^r) move by curvature keeping fixed the end–points P^r (see **[49, 88, 89]**). By the strict convexity of Ω and strong maximum principle, for every positive $t > 0$, the two curves of the boundary of Ω concurring at P^r form an angle smaller that π which is not increasing in time. Hence, the statement of the proposition follows. □

We briefly discuss now the behavior of the area of regions enclosed by the evolving regular network $\mathbb{S}_t$. Let us suppose that a (moving) region $\mathcal{A}(t)$ is bounded by some curves $\gamma^1, \gamma^2, \dots, \gamma^m$ and let $A(t)$ its area. Possibly reparametrizing these curves which form the loop $\ell = \bigcup_{i=1}^m \gamma^i$ in the network, we can assume that ℓ is parametrized counterclockwise, hence, the curvature k is positive at the convexity points of the boundary of $\mathcal{A}(t)$. Then, we have

$$A'(t) = -\sum_{i=1}^m \int_{\gamma^i} \langle k\nu \,|\, \nu \rangle \, ds = -\sum_{i=1}^m \int_{\gamma^i} k \, ds = -\sum_{i=1}^m \Delta\theta_i$$

where $\Delta\theta_i$ is the difference in the angle between the unit tangent vector τ and the unit coordinate vector $e_1 \in \mathbb{R}^2$ at the final and initial point of the curve γ^i, indeed (supposing the unit tangent vector of the curve γ^i "lives" in the second quadrant of $\mathbb{R}^2$ – the other cases are analogous) there holds

$$\partial_s \theta_i = \partial_s \arccos\langle \tau \,|\, e_1 \rangle = -\frac{\langle \tau_s \,|\, e_1 \rangle}{\sqrt{1 - \langle \tau \,|\, e_1 \rangle^2}} = k \,,$$

so

$$A'(t) = -\sum_{i=1}^m \int_{\gamma^i} \partial_s \theta_i \, ds = -\sum_{i=1}^m \Delta\theta_i$$

Being ℓ a closed loop and considering that at all the end–points of the curves γ^i the angle of the unit tangent vector "jumps" of 120 degrees, we have

$$m\pi/3 + \sum_{i=1}^m \Delta\theta_i = 2\pi \,,$$

hence,

$$A'(t) = -(2 - m/3)\pi \,. \tag{7.3}$$

An immediate consequence is that the area of every region fully bounded by the curves of the network evolves linearly and, more precisely, it increases if the region has more than six edges, it is constant with six edges and it decreases if the edges are less than six. Moreover, this implies that if a region with less than six edges is present, with area A_0 at time $t = 0$, the maximal time T of existence of a smooth flow is finite and

$$T \leq \frac{A_0}{(2 - m/3)\pi} \leq \frac{3A_0}{\pi} \,. \tag{7.4}$$

REMARK 7.14. Since every bounded region contained in a shrinker must decrease its area during the curvature flow of such shrinker (since it is homothetically contracting), another consequence is that the only compact regions that can be present in a regular shrinker are bounded by less than six curves (actually, this conclusion also holds for the "visible" regions – not the cores – of any degenerate regular shrinker).
Moreover, letting a shrinker evolve, since every bounded region must collapse after a time interval of $1/2$, the area of such region is only dependent by the number m of its edges (less than 6), by equation (7.3), indeed

$$A(0) = A(0) - A(1/2) = -\int_0^{1/2} A'(t) \, dt = \int_0^{1/2} (2 - m/3)\pi \, dt = (2 - m/3)\pi/2 \,.$$

This implies that the possible structures (topology) of the shrinkers with equibounded diameter are finite.

It is actually conjectured in [**46**, Conjecture 3.26] that there is an upper bound for the possible number of bounded regions of a shrinker. This would imply that the possible topological structures of shrinkers are finite.

7.3. Limits of Huisken's dynamical procedure. We deal now with the possible blow–up limits arising from Huisken's dynamical procedure. We recall that

$$\widetilde{\rho}(x) = e^{-\frac{|x|^2}{2}}\,.$$

DEFINITION 7.15. We say that a (possibly degenerate and with multiplicity) network $\mathbb{S}$ has *bounded length ratios* by the constant $C > 0$, if

$$\overline{\mathcal{H}}^1(\mathbb{S} \cap B_R(\overline{x})) \leq CR\,, \tag{7.5}$$

for every $\overline{x} \in \mathbb{R}^2$ and $R > 0$ ($\overline{\mathcal{H}}^1$ is the 1–dimensional Hausdorff measure counting multiplicities).

Notice that this is a scaling invariant property, with the same constant C.

LEMMA 7.16. *Let $\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}}$ be the family of rescaled networks , obtained via Huisken's dynamical procedure around some $x_0 \in \mathbb{R}^2$, as defined in formula* (6.3).

(1) *There exists a constant $C = C(\mathbb{S}_0)$ such that, for every $\overline{x}, x_0 \in \mathbb{R}^2$, $\mathfrak{t} \in \big[-\frac{1}{2}\log T, +\infty\big)$ and $R > 0$ there holds*

$$\mathcal{H}^1(\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}} \cap B_R(\overline{x})) \leq CR\,. \tag{7.6}$$

(2) *For any $\varepsilon > 0$ there is a uniform radius $R = R(\varepsilon)$ such that*

$$\int_{\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}} \setminus B_R(\overline{x})} e^{-|x|^2/2}\,ds \leq \varepsilon\,,$$

that is, the family of measures $e^{-|x|^2/2}\,\mathcal{H}^1 \llcorner \widetilde{\mathbb{S}}_{x_0,\mathfrak{t}}$ is tight *(see* [**29**]*).*

PROOF. By Definition 1.3, if $\mathbb{S}_0$ is an open network, the number of unbounded curves (C^1–asymptotic to straight lines) is finite. Then, it is easy to see that, open or not, $\mathbb{S}_0$ has bounded length ratios, that is, there exists a constant $C > 0$ such that

$$\mathcal{H}^1(\mathbb{S}_0 \cap B_R(\overline{x})) \leq C'R\,, \tag{7.7}$$

for all $\overline{x} \in \mathbb{R}^2$ and $R > 0$. This implies that the *entropy* of $\mathbb{S}_0$ (see [**26, 66**]) is bounded, i.e.

$$E(\mathbb{S}_0) = \sup_{\overline{x}\in\mathbb{R}^2,\tau>0} \int_{\mathbb{S}_0} \frac{e^{-\frac{|x-\overline{x}|^2}{4\tau}}}{\sqrt{4\pi\tau}}\,ds = \sup_{\overline{x}\in\mathbb{R}^2,\tau>0} \int_{\mathbb{S}_0} \rho_{\overline{x},\tau}(\cdot,0)\,ds \leq C''\,.$$

Indeed, for any $\overline{x}\in\mathbb{R}^2$ and $\tau>0$, changing variable as $y=(x-\overline{x})/2\tau$, we have

$$\begin{aligned}
\int_{\mathbb{S}_0}\frac{e^{-\frac{|x-\overline{x}|^2}{4\tau}}}{\sqrt{4\pi\tau}}\,ds &= \int_{\frac{\mathbb{S}_0-\overline{x}}{2\tau}}\frac{e^{-\frac{|y|^2}{2}}}{\sqrt{2\pi}}\,ds\\
&=\sum_{n=0}^{\infty}\int_{\frac{\mathbb{S}_0-\overline{x}}{2\tau}\cap(B_{n+1}(0)\setminus B_n(0))}\frac{e^{-\frac{|y|^2}{2}}}{\sqrt{2\pi}}\,ds\\
&\leq\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}e^{-n^2/2}\mathcal{H}^1\Big(\frac{\mathbb{S}_0-\overline{x}}{2\tau}\cap B_{n+1}(0)\Big)\\
&=\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}e^{-n^2/2}\mathcal{H}^1\Big(\frac{1}{2\tau}\big(\mathbb{S}_0\cap B_{2\tau(n+1)}(\overline{x})-\overline{x}\big)\Big)\\
&=\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}e^{-n^2/2}\mathcal{H}^1\big(\mathbb{S}_0\cap B_{2\tau(n+1)}(\overline{x})\big)\frac{1}{2\tau}\\
&\leq\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}e^{-n^2/2}(n+1)C'\\
&=C'
\end{aligned}$$

since the series converges (in the last inequality we applied estimate (7.7)).
Then, by the monotonicity formula (6.1), for any $\overline{x}\in\mathbb{R}^2$, $t\in[0,T)$ and $R>0$, by setting $\tau=t+R^2$, we have

$$\int_{\mathbb{S}_t}\frac{e^{-\frac{|x-\overline{x}|^2}{4R^2}}}{\sqrt{4\pi}R}\,ds=\int_{\mathbb{S}_t}\rho_{\overline{x},t+R^2}(\cdot,t)\,ds\leq\int_{\mathbb{S}_0}\rho_{\overline{x},t+R^2}(\cdot,0)\,ds\leq C''\,,$$

hence,

$$\mathcal{H}^1(\mathbb{S}_t\cap B_R(\overline{x}))\leq\sqrt{4\pi}eR\int_{\mathbb{S}_t\cap B_R(\overline{x})}\frac{e^{-\frac{|x-\overline{x}|^2}{4R^2}}}{\sqrt{4\pi}R}\,ds\leq\sqrt{4\pi}C''eR\,.$$

Since this conclusion is scaling invariant, it also holds for all the rescaled networks $\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}}$ and the first point of the lemma follows with $C=\sqrt{4\pi}C''e$. The second point is a consequence of the first one, indeed, we have

$$\begin{aligned}
\int_{\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}}\setminus B_R(\overline{x})}e^{-\frac{|x^2|}{2}}\,ds &=\sum_{n=1}^{\infty}\int_{\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}}\cap(B_{(n+1)R}(\overline{x})\setminus B_{nR}(\overline{x}))}e^{-\frac{|x|^2}{2}}\,ds\\
&\leq\sum_{n=1}^{\infty}e^{-n^2R^2/2}\mathcal{H}^1\big(\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}}\cap B_{(n+1)R}(\overline{x})\big)\\
&\leq C\sum_{n=1}^{\infty}e^{-n^2R^2/2}(n+1)R\\
&=f(R)
\end{aligned}$$

and the function f satisfies $\lim_{R\to+\infty}f(R)=0$. □

PROPOSITION 7.17. *Let $\mathbb{S}_t=\bigcup_{i=1}^n\gamma^i([0,1],t)$ be a $C^{2,1}$ curvature flow of regular networks in the time interval $[0,T]$, then, for every $x_0\in\mathbb{R}^2$ and for every subset $\mathcal{I}$ of $[-1/2\log T,+\infty)$ with infinite Lebesgue measure, there exists a sequence of rescaled times $\mathfrak{t}_j\to+\infty$, with $\mathfrak{t}_j\in\mathcal{I}$, such that the sequence of rescaled networks $\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_j}$ (obtained via Huisken's dynamical procedure) converges in $C^{1,\alpha}_{\mathrm{loc}}\cap W^{2,2}_{\mathrm{loc}}$, for any $\alpha\in(0,1/2)$, to a (possibly empty) limit degenerate regular shrinker $\widetilde{\mathbb{S}}_\infty$ (possibly with multiplicity). Moreover, we have*

$$\lim_{j\to\infty}\frac{1}{\sqrt{2\pi}}\int_{\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_j}}\widetilde{\rho}\,d\sigma=\frac{1}{\sqrt{2\pi}}\int_{\widetilde{\mathbb{S}}_\infty}\widetilde{\rho}\,d\overline{\sigma}=\Theta_{\widetilde{\mathbb{S}}_\infty}=\widehat{\Theta}(x_0)\,.\tag{7.8}$$

where $d\overline{\sigma}$ denotes the integration with respect to the canonical measure on $\widetilde{\mathbb{S}}_\infty$, counting multiplicities.

PROOF. Letting $\mathfrak{t}_1 = -1/2 \log T$ and $\mathfrak{t}_2 \to +\infty$ in the rescaled monotonicity formula (6.4), by Lemma 6.5 we get

$$\int_{-1/2\log T}^{+\infty} \int_{\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}}} |\,\underline{\widetilde{k}} + x^\perp|^2 \widetilde{\rho}\, d\sigma\, d\mathfrak{t} < +\infty\,,$$

which implies

$$\int_{\mathcal{I}} \int_{\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}}} |\,\underline{\widetilde{k}} + x^\perp|^2 \widetilde{\rho}\, d\sigma\, d\mathfrak{t} < +\infty\,.$$

Being the last integral finite and being the integrand a nonnegative function on a set of infinite Lebesgue measure, we can extract within $\mathcal{I}$ a sequence of times $\mathfrak{t}_j \to +\infty$, such that

$$\lim_{j\to+\infty} \int_{\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_j}} |\,\underline{\widetilde{k}} + x^\perp|^2 \widetilde{\rho}\, d\sigma = 0\,. \tag{7.9}$$

It follows that, for every ball of radius R in $\mathbb{R}^2$, the networks $\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_j}$ have curvature uniformly bounded in $L^2(B_R)$. Moreover, by the first point of Lemma 7.16, for every ball B_R centered at the origin of $\mathbb{R}^2$ we have the uniform bound $\mathcal{H}^1(\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_j} \cap B_R) \leq CR$, for some constant C independent of $j \in \mathbb{N}$. Then, reparametrizing the rescaled networks by arclength, we obtain curves with uniformly bounded first derivatives and with second derivatives in L^2_{loc}.
By a standard compactness argument (see [**48, 60**]), the sequence $\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_j}$ of reparametrized networks admits a subsequence $\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_{j_l}}$ which converges, weakly in $W^{2,2}_{\mathrm{loc}}$ and strongly in $C^{1,\alpha}_{\mathrm{loc}}$, to a (possibly empty) limit regular degenerate C^1 network $\widetilde{\mathbb{S}}_\infty$ (possibly with multiplicity).
Since the integral functional

$$\widetilde{\mathbb{S}} \mapsto \int_{\widetilde{\mathbb{S}}} |\,\underline{\widetilde{k}} + x^\perp|^2 \widetilde{\rho}\, d\sigma$$

is lower semicontinuous with respect to this convergence (see [**85**], for instance), the limit $\widetilde{\mathbb{S}}_\infty$ satisfies $\underline{\widetilde{k}}_\infty + x^\perp = 0$ in the sense of distributions.
A priori, the limit network is composed by curves in $W^{2,2}_{\mathrm{loc}}$, but from the relation $\underline{\widetilde{k}}_\infty + x^\perp = 0$, it follows that the curvature $\underline{\widetilde{k}}_\infty$ is continuous. By a bootstrap argument, it is then easy to see that $\widetilde{\mathbb{S}}_\infty$ is actually composed by C^∞ curves.

By means of the second point of Lemma 7.16, we can pass to the limit in the Gaussian integral and we get

$$\lim_{j\to\infty} \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_j}} \widetilde{\rho}\, d\sigma = \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\mathbb{S}}_\infty} \widetilde{\rho}\, d\overline{\sigma} = \Theta_{\widetilde{\mathbb{S}}_\infty}\,.$$

Recalling that

$$\frac{1}{\sqrt{2\pi}} \int_{\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_j}} \widetilde{\rho}\, d\sigma = \int_{\mathbb{S}_{t(\mathfrak{t}_j)}} \rho_{x_0}(\cdot, t(\mathfrak{t}_j))\, ds = \Theta_{x_0}(t(\mathfrak{t}_j)) \to \widehat{\Theta}(x_0)$$

as $j \to \infty$, equality (7.8) follows.

The convergence in $W^{2,2}_{\mathrm{loc}}$ is implied by the weak convergence in $W^{2,2}_{\mathrm{loc}}$ and equation (7.9). □

REMARK 7.18.

(1) In the case of a special rate of blow–up of the curvature, the so called *Type I* singularities, when there exists a constant C such that

$$\max_{\mathbb{S}_t} k^2 \leq \frac{C}{T-t}$$

for every $t \in [0,T)$, the proof of this proposition gets easier and we get a stronger convergence to the limit network. This is due to the uniform pointwise bound on the curvature

(consequently on its derivatives) that we get after the rescaling by the Type I estimate, see [**71**, Section 6, Proposition 6.16], for instance.
(2) It can be shown, by the arguments in [**67**, Proposition 3.2 and Theorem 2.4], that the convergence of the rescaled networks $\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_j}$ to $\widetilde{\mathbb{S}}_\infty$ is locally smooth far from the cores and non multiplicity–one curves of $\widetilde{\mathbb{S}}_\infty$.

Notice that the blow–up limit shrinker $\widetilde{\mathbb{S}}_\infty$ obtained by this proposition a priori depends on the chosen sequence of rescaled times $\mathfrak{t}_j$. If such a limit is a multiplicity–one line (or a halfline if x_0 is an end–point of the network), by White's local regularity theorem for mean curvature flow in [**91**], hence $\widehat{\Theta}(x_0) = 1$ ($\widehat{\Theta}(x_0) = 1/2$ in the case of a halfline), then locally around x_0 the curvature is bounded and the limit is unique. In general, the uniqueness of such limit is actually unknown.

OPEN PROBLEM 7.19 (Uniqueness of the Blow-up Assumption – **U**). The limit degenerate regular shrinker $\widetilde{\mathbb{S}}_\infty$ is independent of the chosen converging sequence of rescaled networks $\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_j}$ in Proposition 7.17? The full family $\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}}$ converges to $\widetilde{\mathbb{S}}_\infty$, as $\mathfrak{t} \to +\infty$?

REMARK 7.20. The above uniqueness assumption, in case the limit degenerate regular shrinker $\widetilde{\mathbb{S}}_\infty$ is actually a multiplicity–one regular shrinker (or the same for the limit degenerate regular self–similarly shrinking flow $\mathbb{S}^\infty_\mathfrak{t}$), that is, there are no cores and multiplicities, implies that the singularity is of Type I. Indeed, by Lemma 8.7, the convergence of the rescaled networks to $\widetilde{\mathbb{S}}_\infty$ is smooth, which implies that the curvature is locally uniformly bounded by $C/\sqrt{T-t}$.

It is then natural in view of this remarks to state also the following open problem.

OPEN PROBLEM 7.21 (Type I Conjecture). Every singularity if of Type I?

7.4. Blow–up limits under hypotheses on the lengths of the curves of the network.

PROPOSITION 7.22. *Let $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0,1],t)$ be the curvature flow of a regular network with fixed end–points in a smooth, convex, bounded open set $\Omega \subset \mathbb{R}^2$, such that three end–points of the network are never aligned. Assume that the lengths $L^i(t)$ of the curves of the networks satisfy*

$$\lim_{t\to T} \frac{L^i(t)}{\sqrt{T-t}} = +\infty\,, \tag{7.10}$$

for every $i \in \{1,2,\dots,n\}$. Then, any limit degenerate regular shrinker $\widetilde{\mathbb{S}}_\infty$, obtained by Proposition 7.17, if non–empty, is one of the following networks.
If the rescaling point belongs to Ω:

- *a straight line through the origin with multiplicity $m \in \mathbb{N}$ (in this case $\widehat{\Theta}(x_0) = m$);*
- *a standard triod centered at the origin with multiplicity 1 (in this case $\widehat{\Theta}(x_0) = 3/2$).*

If the rescaling point is a fixed end–point of the evolving network (on the boundary of Ω):

- *a halfline from the origin with multiplicity 1 (in this case $\widehat{\Theta}(x_0) = 1/2$).*

Moreover, we have

$$\lim_{j\to\infty} \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_j}} \widetilde{\rho}\, d\sigma = \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\mathbb{S}}_\infty} \widetilde{\rho}\, d\overline{\sigma} = \Theta_{\widetilde{\mathbb{S}}_\infty} = \widehat{\Theta}(x_0)\,, \tag{7.11}$$

and the L^2 norm of the curvature of $\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_j}$ goes to zero in every ball $B_R \subset \mathbb{R}^2$, as $j \to \infty$.

PROOF. We assume, by Proposition 7.17, that the sequence $\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_j}$ of reparametrized networks converges in $C^1_{\mathrm{loc}} \cap W^{2,2}_{\mathrm{loc}}$ to the limit regular shrinker network $\widetilde{\mathbb{S}}_\infty$ composed by C^∞ curves (with possibly multiplicity), which are actually non–degenerate as the bound from below on their lengths prevents any "collapsing" along the rescaled sequence.
If the point $x_0 \in \mathbb{R}^2$ is distinct from all the end–points P^r, then $\widetilde{\mathbb{S}}_\infty$ has no end–points, since they go to infinity along the rescaled sequence. If $x_0 = P^r$ for some r, the set $\widetilde{\mathbb{S}}_\infty$ has a single end–point at the origin of $\mathbb{R}^2$.

Moreover, from the lower bound on the length of the curves it follows that all the curves of $\widetilde{\mathbb{S}}_\infty$ have infinite length, hence, by Remark 7.9, they must be pieces of straight lines from the origin, because of the uniform bound $\mathcal{H}^1(\mathbb{S}^{\mu_i}_{\mathfrak{t}} \cap B_R) \leq C_R$, for every ball B_R.
This implies that every connected component of the graph underlying $\widetilde{\mathbb{S}}_\infty$ can contain at most one 3–point and in such case such component must be mapped to a standard triod (the 120 degrees condition must satisfied) with multiplicity one since the sequence of converging networks are all embedded (to get in the C^1_{loc}–limit a triod with multiplicity higher than one it is necessary that the approximating networks have self–intersections). Moreover, again since the converging networks are all embedded, if a standard triod is present, a straight line or another triod cannot be there, since they would intersect transversally (see Remark 7.5). Viceversa, if a straight line is present, a triod cannot be present.

If an end–point is not present, that is, we are rescaling around a point in Ω (not on its boundary), if a 3–point is not present, the only other possibility is a straight line (possibly with multiplicity) for through the origin of $\mathbb{R}^2$.

If an end–point is present, we are rescaling around an end–point of the evolving network, hence, by the convexity of Ω (which contains all the networks) the limit $\widetilde{\mathbb{S}}_\infty$ must be contained in a halfplane with boundary a straight line H for the origin. This exclude the presence of a standard triod since it cannot be contained in any halfplane. Another halfline is obviously excluded, since they "come" only from end–points and they are all distinct. In order to exclude the presence of a straight line, we observe that the argument of Proposition 7.13 implies that, if $\Omega_t \subset \Omega$ is the evolution by curvature of $\partial\Omega$ keeping fixed the end–points P^r, the blow–up of Ω_t at an end–point must be a cone spanning angle strictly less then π (here we use the fact that three end–points are not aligned), and $\widetilde{\mathbb{S}}_\infty$ is contained in such a cone. It follows that $\widetilde{\mathbb{S}}_\infty$ cannot contain a straight line.

In every case the curvature of $\widetilde{\mathbb{S}}_\infty$ is zero everywhere and the last statement follows by the $W^{2,2}_{\mathrm{loc}}$–convergence.

Finally, formula (7.11) is a special case of equation (7.8). □

REMARK 7.23. If the two curves describing the boundary of Ω around an end–point P^r are actually segments of the same line, that is, the three end–points are P^{r-1}, P^r, P^{r+1} aligned, the argument of Proposition 7.13 does not work and we cannot conclude that taking a blow–up at P^r we only get a halfline with unit multiplicity. It could also be possible that a straight line (possibly with multiplicity) for the origin is present, coinciding with H. Moreover, in such special case, it forces also the halfline to be contained in H, since the only way to get a line, without self–intersections in the sequence of converging networks contained in Ω, is that the curves that are converging to the straight line "pushes" the curve getting to the end–point of the network, toward the boundary of Ω.

OPEN PROBLEM 7.24. Is it possible to classify in general all the possible limit degenerate shrinkers $\widetilde{\mathbb{S}}_\infty$ obtained by Huisken's dynamical procedure?

REMARK 7.25. If the evolving network is a tree, every connected component of a limit degenerate regular shrinker (possibly with multiplicities) is still a tree, hence, by Lemma 7.10 and the same argument of the proof of Proposition 7.22, such network has zero curvature and it is a union of halflines from the origin, possibly with multiplicity and a core.

REMARK 7.26. In Section 8 we will discuss under what hypotheses, the (unscaled) evolving networks $\mathbb{S}_t$ converge to some limit (well–behaved) *set* $\mathbb{S}_T \subset \mathbb{R}^2$, as $t \to T$, and what are the relations between such $\mathbb{S}_T$ and any limit degenerate shrinker $\widetilde{\mathbb{S}}_\infty$.

8. The behavior of the flow at a singular time

By means of the tools of the previous sections, we want to discuss now the behavior of the network approaching the singular time T.

We have seen in Corollary 5.10 (Theorem 5.7) that at the maximal time $T < +\infty$ of existence of the curvature flow $\mathbb{S}_t$ of an initial regular C^2 network with fixed end–points in a smooth, convex, bounded open set $\Omega \subset \mathbb{R}^2$, given by Theorem 5.8, either the curvature is not bounded, as $t \to T$,

or the inferior limit of the length $L^i(t)$ of at least one curve of $\mathbb{S}_t$ goes to zero when $t \to T$. Hence, if all the lengths of the curves of the network are uniformly positively bounded from below, the maximum of the modulus of the curvature goes to $+\infty$, as $t \to T$. By Proposition 5.17, we also know that if the curvature is uniformly bounded, all the lengths of the curves converge as $t \to T$, thus at least some $L^i(t)$ must go to zero, as $t \to T$.
We will then divide our analysis in the following three cases:

- all the lengths of the curves of the network are uniformly positively bounded from below and the maximum of the curvature goes to $+\infty$, as $t \to T$;
- the curvature is uniformly bounded along the flow and the length $L^i(t)$ of at least one curve of $\mathbb{S}_t$ goes to zero when $t \to T$;
- the curvature is not bounded and the length of at least one curve of the network is not positively bounded from below, as $t \to T$.

In all the three cases, the possible blow–up limits will play a key role, with the obvious consequence that the fewer possibilities we have, the easier we can get conclusions. In particular, like when studying the evolution of a single smooth closed curve along the analogous line (see **[49]**, for instance), it is fundamental to exclude to get blow–up limits of multiplicity larger than one, in particular "multiple lines". For curves this can be done by means of some "embeddedness" or "non–collapsing" quantities, for instance as in **[44, 49]** that actually inspired the analogous one that we will discuss in Section 9.
Unfortunately, for a general regular network, this is still conjectural and possibly the major open problem in the subject.

OPEN PROBLEM 8.1 (Multiplicity–One Conjecture – **M1**). Every possible C^1_{loc}–limit of rescalings of networks of the flow is an embedded network with multiplicity one.

This conjecture obviously implies the following one, but actually they are equivalent.

OPEN PROBLEM 8.2 (No Double–Line Conjecture – **L1**). A straight line with multiplicity larger than one cannot be obtained as a C^1_{loc}–limit of rescalings of networks of the flow.

Indeed, if **M1** does not hold, since the networks of the flow are all embedded, any limit of rescalings $\mathbb{S}_i$ can lose embeddedness only if two curves in the limit network "touch" each other at some point $x_0 \in \mathbb{R}^2$ with a common tangent (or they locally coincide, if they "produce" a piece of curve with multiplicity larger than one). Then, "slowly" dilating the networks $\mathbb{S}_i$ around x_0, in order that the distance between such two curves and x_0 still go to zero, we would get a multiplicity–two line, contradicting **L1**.

We will see in Section 9 some cases in which we are able to show that the multiplicity–one conjecture holds, that is,

- If during the flow the triple junctions stay uniformly far each other, then **M1** is true.
- If the initial network has at most two triple junctions, then **M1** is true.

REMARK 8.3. If **M1** holds, the limit network $\widetilde{\mathbb{S}}_\infty$ in Proposition 7.17 is composed of embedded, multiplicity–one networks. In particular, under the hypotheses of Proposition 7.22, any blow–up limit network at a point x_0 and singular time T, obtained by Huisken's procedure, or self–similarly shrinking network flow, obtained by the parabolic rescaling procedure, is (if not empty) a "static" straight line through the origin (then $\widehat{\Theta}(x_0) = 1$) or a standard triod (then $\widehat{\Theta}(x_0) = 3/2$), if the rescaling point belongs to Ω. If the rescaling point is instead a fixed end–point of the evolving network on the boundary of Ω, then such limit can only be a single halfline from the origin (and $\widehat{\Theta}(x_0) = 1/2$).

Before analyzing the three situations above, we set some notation and we show some general properties of the flow at the singular time.

We let $F : \mathbb{S} \times [0, T) \to \overline{\Omega}$, with $T < +\infty$, represent the curvature flow $\mathbb{S}_t$ of a regular network moving by curvature in its maximal time interval of smooth existence. We let $O^1, O^2, \dots, O^m$ the 3–points of $\mathbb{S}$.

We define the set of *reachable points* of the flow as follows:

$$\mathcal{R} = \big\{x \in \mathbb{R}^2 \,\big|\, \text{ there exist } p_i \in \mathbb{S} \text{ and } t_i \nearrow T \text{ such that } \lim_{i\to\infty} F(p_i, t_i) = x\big\}\,.$$

Such a set is easily seen to be closed, contained in $\overline{\Omega}$ (hence compact as Ω is bounded) and the following lemma holds.

LEMMA 8.4. *A point $x \in \mathbb{R}^2$ belongs to $\mathcal{R}$ if and only if for every time $t \in [0, T)$ the closed ball with center x and radius $\sqrt{2(T-t)}$ intersects $\mathbb{S}_t$.*

PROOF. One of the two implications is trivial. We have to prove that if $x \in \mathcal{R}$, then $F(\mathbb{S}, t) \cap \overline{B}_{\sqrt{2(T-t)}}(x) \neq \emptyset$. If x is one of the end–points, the result is obvious, otherwise we define the function $d_x(t) = \inf_{p\in\mathbb{S}} |F(p, t) - x|$, where, due to the compactness of $\mathbb{S}$ the infimum is actually a minimum and definitely, as $t \to T$, let us say for $t > t_x$, it cannot be taken definitely at an end–point, by the assumption $x \in \mathcal{R}$ and x is different from an end–point and it is not taken at a 3–point, by the 120 degrees angle condition. Since the function $d_x : [0, T) \to \mathbb{R}$ is locally Lipschitz, we can use Hamilton's trick (see **[43]** or **[68**, Lemma 2.1.3]), to compute its time derivative and get (for any point q, different by an end–point, where at time t the minimum of $|F(p, t) - x|$ is attained)

$$\begin{aligned}\partial_t d_x(t) = \partial_t |F(q, t) - x| &\geq \frac{\langle k(q, t)\nu(q, t) + \lambda(q, t)\tau(q, t), F(q, t) - x\rangle}{|F(q, t) - x|}\\ &= \frac{\langle k(q, t)\nu(q, t), F(q, t) - x\rangle}{|F(q, t) - x|} \geq -\frac{1}{d_x(t)}\,,\end{aligned}$$

since at a point of minimum distance the vector $\frac{F(q,t)-x}{|F(q,t)-x|}$ is parallel to $\nu(q, t)$. Integrating this inequality over time, we get

$$d_x^2(t) - d_x^2(s) \leq 2(s - t) \qquad \text{for } s > t > t_x\,.$$

We now use the hypothesis that x is reachable (i.e. $\lim_{t_i\to T} d_x(t_i) = 0$) and we conclude

$$d_x^2(t) = \lim_{i\to\infty} [d_x^2(t) - d_x^2(t_i)] \leq 2 \lim_{i\to\infty} (t_i - t) = 2(T - t)\,,$$

for every $t > t_x$. □

As a consequence, when we consider the blow–up limit by the Huisken's procedure of the evolving networks around points of $\overline{\Omega}$, we have a dichotomy among these latter. Either the limit of any sequence of rescaled networks is not empty and we are rescaling around a point in $\mathcal{R}$, or the blow–up limit is empty, since the distance of the evolving network from the point of blow–up is too large. Conversely, if the blow–up point belongs to $\mathcal{R}$, the above lemma ensures that any rescaled network contains at least one point of the closed unit ball of $\mathbb{R}^2$, so it cannot be empty.

We now show that, assuming the multiplicity–one conjecture, as $t \to T$, all the 3–points of the network $\mathbb{S}_t$ converge.

LEMMA 8.5. *There exists a radius $R = R(\mathbb{S}_t) > 0$, such that if a blow–up limit regular shrinker $\widetilde{\mathbb{S}}_\infty$ (or $\mathbb{S}^\infty_{-1/2}$) has no triple junctions in the ball $B_R(0)$, then it is a line for the origin of $\mathbb{R}^2$ or the unit circle.*

PROOF. Assume that the conclusion is false, then there is a sequence $R_i \to +\infty$ and blow–up limit regular shrinkers $\mathbb{S}_i$, all different by a line or circle, such that each $\mathbb{S}_i$ has no triple junctions in $B_{R_i}(0)$, for every $i \in \mathbb{N}$.

As we said in the discussion above, any shrinker $\mathbb{S}_i$ must intersect the unit circle, hence, by the shrinkers equation (7.1), we can extract a subsequence of $\mathbb{S}_i$ locally converging in C^1 to a non empty limit shrinker $\overline{\mathbb{S}}$ without triple junctions at all. By the work of Abresch and Langer **[1]**, then $\overline{\mathbb{S}}$ must be a line for the origin or the unit circle and this latter case is excluded, since, definitely, also $\mathbb{S}_i$ would be a circle, which is a contradiction. If the limit $\overline{\mathbb{S}}$ is a line, by the multiplicity–one conjecture, its multiplicity must be one, being any limit of blow–up limits again a blow–up limit. Then, since the "topological complexity" of all $\mathbb{S}_i$ is uniformly bounded (the number of possible halflines going to infinity, regions, curves, triple junctions), as they are all blow–up limits of the same flow $\mathbb{S}_t$, the contribution of $\mathbb{S}_i \setminus B_{\overline{R}}(0)$ to the Gaussian density of the whole $\mathbb{S}_i$ is uniformly small as we want, if we choose a value $\overline{R}$ large enough, while the contribution of $\mathbb{S}_i \cap B_{\overline{R}}(0)$ is smaller than one, as $\mathbb{S}_i \to \overline{\mathbb{S}}$, which is a multiplicity–one line, in $B_{\overline{R}}(0)$. Hence, we conclude that the Gaussian density of $\mathbb{S}_i$ is definitely close to one, then, Lemma 7.11 implies that $\mathbb{S}_i$ is also definitely a line for the origin, which is again a contradiction and we are done. $\square$

LEMMA 8.6. *If* **M1** *holds, there exist the limits* $x_i = \lim_{t\to T} O^i(t)$, *for* $i \in \{1, 2, \dots, m\}$ *and the set* $\{x_i = \lim_{t\to T} O^i(t) \mid i = 1, 2, \dots, m\}$ *coincides with the union of the set of the points* x *in* Ω *where* $\widehat{\Theta}(x) > 1$ *with the set of the end–points of* $\mathbb{S}_t$ *such that the curve getting there "collapses" as* $t \to T$.

PROOF. Let $\mathcal{D} = \{x \in \Omega \mid \widehat{\Theta}(x) > 1\}$, $\mathcal{O}(t) = \{O^1(t), O^2(t), \dots, O^m(t)\}$ and $\mathcal{P} = \{P^1, P^2, \dots, P^l\}$. Let $R > 0$ be given by the previous lemma and consider a finite subset $\overline{\mathcal{D}} \subset \mathcal{D}$, supposing that the set

$$\mathcal{I}_{\overline{\mathcal{D}}} = \left\{ \mathfrak{t} \in [-1/2 \log T, +\infty) \mid \max_{x\in\overline{\mathcal{D}}} d(x, \mathcal{O}(t(\mathfrak{t}))) \geq R\sqrt{2(T - t(\mathfrak{t}))} \right\}$$

has infinite Lebesgue measure, there must be $x_0 \in \overline{\mathcal{D}}$ such that

$$\mathcal{I}_{x_0} = \left\{ \mathfrak{t} \in [-1/2 \log T, +\infty) \mid d(x_0, \mathcal{O}(t(\mathfrak{t}))) \geq R\sqrt{2(T - t(\mathfrak{t}))} \right\},$$

Hence, by rescaling with Huisken's procedure around x_0, by Proposition 7.17, we can extract a sequence of times $\mathfrak{t}_j \in \mathcal{I}_{x_0}$ such that the rescaled networks $\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_j}$ converge in the C^1_{loc} to a line for the origin of $\mathbb{R}^2$, since in any ball centered in the origin there cannot be 3–points, by construction of $\mathcal{I}_{x_0}$ and we assumed **M1**. This clearly implies that $\widehat{\Theta}(x_0) = 1$, contradicting the hypothesis $x_0 \in \mathcal{D}$, hence, $\mathcal{I}_{\overline{\mathcal{D}}}$ must have finite Lebesgue measure. It is easy to see that this implies that the points of $\overline{\mathcal{D}}$, and thus of $\mathcal{D}$, cannot be more than the number m of the 3–points of the evolving network $\mathbb{S}_t$.

If now we consider a small $\delta > 0$, every point x in the open set

$$\Omega_\delta = \Omega \setminus \left\{x \in \Omega \mid d(x, \mathcal{D} \cup \mathcal{P}) \leq \delta\right\}$$

satisfies $\widehat{\Theta}(x) = 1$, hence, by compactness and White's local regularity theorem implies the curvature of the evolving network is uniformly bounded in such set. Then, definitely, as $t \to T$ every 3–point $O^i(t)$, for every $i \in \{1, 2, \dots, m\}$, has to "choose" a point of $\mathcal{D} \cup \mathcal{P}$ to stay close (δ is small and $\mathcal{D} \cup \mathcal{P}$ is finite), otherwise it would be possible to find a subsequence of times $t_j \to T$ such that the networks $\mathbb{S}_{t_j}$ restricted to the set Ω_δ, converge (because of bounded curvature, see the proof of Proposition 8.21 for more details) to a network in Ω_δ with a multi–point $x_0 \in \Omega_\delta$ and this is not possible since it would imply that $\widehat{\Theta}(x_0) \geq 3/2 > 1$ which is a contradiction with the fact that the function $\widehat{\Theta}$ is equal to one at every reachable point of Ω_δ.

This argument clearly implies that for every $i \in \{1, 2, \dots, m\}$, $O^i(t)$ converge to some $x_i \in \mathcal{D} \cup \mathcal{P}$.

Finally, if $x \in \mathcal{D}$, there must be a multi–point in any blow–up limit shrinker, otherwise we can only have a line that would imply $\widehat{\Theta}(x) = 1$, against the assumption. Hence, for some $i \in \{1, 2, \dots, m\}$ and $t_n \to T$ there must hold $O^i(t_n) \to x_i$ that forces $\lim_{t\to T} O^i(t) = x_i$.
If the curve of $\mathbb{S}_t$ getting to an end–point P^r collapses, clearly, as before, for some $k \in \{1, 2, \dots, m\}$ and $t_j \to T$ there must hold $O^k(t_j) \to P^r = x_k$ and we have the same conclusion $\lim_{t\to T} O^k(t) = P^r = x_k$. $\square$

The following lemma is helpful in strengthening the convergence in Proposition 7.17.

LEMMA 8.7. *Given a sequence of smooth curvature flows of networks* $\mathbb{S}^i_t$ *in a time interval* (t_1, t_2) *with uniformly bounded length ratios, if in a dense subset of times* $t \in (t_1, t_2)$ *the networks* $\mathbb{S}^i_t$ *converge in a ball* $B \subset \mathbb{R}^2$ *in* C^1_{loc}, *as* $i \to \infty$, *to a multiplicity–one, embedded,* C^∞*–curve* γ_t *moving by curvature in* $B' \supset \overline{B}$, *for* $t \in (t_1, t_2]$ *(hence, the curvature of* γ_t *is uniformly bounded), then for every* $(x_0, t_0) \in B \times (t_1, t_2]$, *the curvature of* $\mathbb{S}^i_t$ *is uniformly bounded in a neighborhood of* (x_0, t_0) *in space–time. It follows that, for every* $(x_0, t_0) \in B \times (t_1, t_2]$, *we have* $\mathbb{S}^i_t \to \gamma_t$ *smoothly around* (x_0, t_0) *in space–time (possibly, up to local reparametrizations of the networks* $\mathbb{S}^i_t$*).*

PROOF. Being γ_t a smooth flow of an embedded curve in B, for (x,t) in a suitably small neighborhood of $(x_0,t_0) \in B \times (t_1,t_2]$ we have that $\Theta_{x,t}(\tau) \leq 1+\varepsilon/2 < 3/2$, for every $\tau \in (\tau_0,t)$ and some $\tau_0 > 0$, where $\varepsilon > 0$ is smaller than the "universal" constant given by White's local regularity theorem in **[91]**. Then, in a possibly smaller space–time neighborhood of (x_0,t_0), for a fixed time $\overline{\tau} \in (\tau_0,t)$ where the C^1_{loc}–convergence of the networks $\mathbb{S}^i_{\overline{\tau}} \to \gamma_{\overline{\tau}}$ holds (such a subset of times is dense), definitely the Gaussian density functions of $\mathbb{S}^i_{\overline{\tau}}$ satisfy $\Theta^i_{x,t}(\overline{\tau}) < 1+\varepsilon < 3/2$ (the Gaussian density functions are clearly continuous under the C^1_{loc} convergence with uniform length ratios estimate, by the exponential decay of backward heat kernel). Hence, by the monotonicity formula this also holds for *every* $\tau \in (\overline{\tau},t)$. In other words, $\Theta^i_{x,t}(t-r^2) < 1+\varepsilon < 3/2$, for every (x,t) in a space–time neighborhood of (x_0,t_0), $0 < r < r_0$ and $i > i_0$, for some $r_0 > 0$. Notice that this "forbids" the presence of a 3–point of $\mathbb{S}^i_t$ in such space–time neighborhood.
Then, White's theorem (see Theorem 3.5 in **[91]**) gives a uniform local (in space–time) estimate on the curvature of all $\mathbb{S}^i_t$, which actually implies uniform bounds on all its higher derivatives (for instance, by Ecker and Huisken interior estimates in **[30]**), around (x_0,t_0). Hence, the statement of the lemma follows. □

We conclude this section with a geometric construction that we will use several times in the following.
We consider the curvature flow of network $\mathbb{S}_t$ in a strictly convex set Ω, with fixed end–points $\{P^1,P^2,\dots,P^l\}$ on $\partial\Omega$, in a maximal time interval $[0,T)$. We recall that as the curves composing the network are at least C^2 and the boundary points are fixed, at each P^r both the velocity and the curvature are zero, namely, the compatibility conditions of order two (see definition 3.6) are satisfied.
For every end–point P^i, we define the "symmetrized" networks $\mathbb{H}^i_t$ each one obtained as the union of $\mathbb{S}_t$ with its reflection $\mathbb{S}^{R_i}_t$ with respect to P^i. As the domain Ω is strictly convex and $\mathbb{S}_t$ is inside Ω, this operation clearly does not introduce self–intersections in the union $\mathbb{H}^i_t = \mathbb{S}_t \cup \mathbb{S}^{R_i}_t$ and the number of triple junctions of $\mathbb{H}^i_t$ is exactly twice the number of $\mathbb{S}_t$. Every network $\mathbb{H}^i_t$ is a

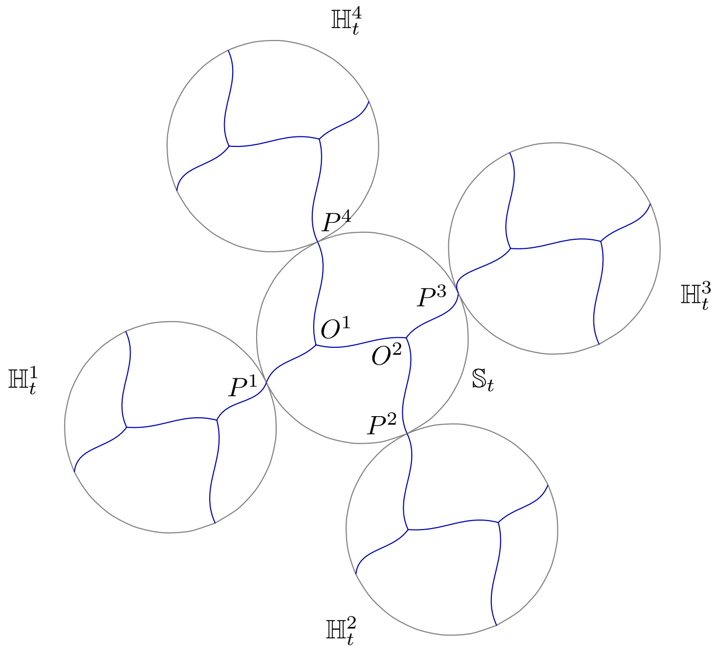


FIGURE 14. A network $\mathbb{S}_t$ with the associated networks $\mathbb{H}^i_t$.

regular network and the flow is still in $C^{2,1}$, thanks to the compatibility conditions of order two satisfied at P^i. The evolution is clearly symmetric with respect to P^i. If we have that the flow $\mathbb{S}_t$ is smooth then also all the flows $\mathbb{H}^i_t$ are smooth (see Definition 3.15).

8.1. Regularity without "vanishing" of curves. We assume that the lengths of all the curves of the network are uniformly positively bounded from below, hence the maximum of the curvature goes to $+\infty$, as $t \to T$. We are going to show that if **M1** holds, T cannot be a singular time, hence we conclude that this case simply cannot happen. This conclusion justifies the title of this section: to have a singularity (assuming the multiplicity–one conjecture) some curve must disappear.

Performing a Huisken's rescaling at any reachable, interior point $x_0 \in \Omega$ (at the other interior points of Ω the blow–up limits are empty), since we assumed that the multiplicity–one conjecture holds, by the discussion in Remark 8.3, we obtain as blow–up limit only a straight line with unit multiplicity or a standard triod. If we instead rescale at an end–point P^r we get a halfline.

THEOREM 8.8. *Let $\mathbb{S}_t$ be a smooth flow in the maximal time interval $[0,T)$ for the initial network $\mathbb{S}_0$. Let x_0 be a reachable point for the flow such that the sequence of rescaled networks $\widetilde{\mathbb{S}}_{x_0,\mathrm{t}_j}$ (introduced in Proposition 7.17) converges, as $j \to \infty$, in $C^{1,\alpha}_{\mathrm{loc}} \cap W^{2,2}_{\mathrm{loc}}$, for any $\alpha \in (0,1/2)$, to a limit $\widetilde{\mathbb{S}}_\infty$ that is one of the following:*

- *a straight line trough the origin;*
- *a halfline from the origin;*
- *a standard triod.*

Then,

$$|k(x,t)| \leq C < +\infty$$

for all x in a neighborhood of x_0 and $t \in [0,T)$.

Thanks Theorem 8.8, we conclude that the curvature is uniformly locally bounded along the flow, around every point $x_0 \in \mathcal{R}$. By the compactness of the set of reachable points $\mathcal{R}$, this argument clearly implies that the curvature of $\mathbb{S}_t$ is uniformly bounded, as $t \to T$, which is a contradiction. Then,

PROPOSITION 8.9. *Assuming* **M1**, *if $T < +\infty$ is the maximal time interval of existence of the curvature flow of a regular network with fixed end–points, given by Theorem 5.8, then the inferior limit of the length of at least one curve must go to zero, as $t \to T$.*

REMARK 8.10. As we conjecture (Problem 5.11) the general validity of Corollary 5.10, we expect that the conclusion of this proposition actually holds for *every* curvature flow of a regular network.
Obviously, it would follow by a positive answer to Problem 5.11.

REMARK 8.11. Proposition 8.9 can be seen as the global (in space) version of the result in **[67]**, dealing with the situation of a single 3–point. Usually, in analytic problems local and global (in space) regularity coincide, actually in this case the tool to pass from one to the other is the validity of the multiplicity–one conjecture.

PROPOSITION 8.12. *Let $\mathbb{S}_t$ be a tree. Suppose that **M1** holds and that $T = \infty$. Then for every sequence of times $t_i \to \infty$, it exists a subsequence (not relabeled) such that the evolving network $\mathbb{S}_{t_i}$ converges in $C^{1,\alpha} \cap W^{2,2}$, for every $\alpha \in (0,1/2)$ to a possibly degenerate regular network with zero curvature (hence "stationary" for the length functional) as $i \to \infty$.*

PROOF. If $\mathbb{S}_0$ is a tree and $T = +\infty$, then $\mathbb{S}_t$ converges, as $t \to +\infty$, to a regular network with zero curvature (a stationary point for the length functional). Indeed, as the total length of the network decreases, we have the estimate

$$\int_0^{+\infty} \int_{\mathbb{S}_t} k^2 \, ds \, dt \leq L(0) < +\infty \,, \tag{8.1}$$

by the first equation in Proposition 4.1. Then, suppose by contradiction that for a sequence of times $t_j \nearrow +\infty$ we have $\int_{\mathbb{S}_{t_j}} k^2 \, ds \geq \delta$ for some $\delta > 0$. By the following estimate, which is inequality (4.14) in Lemma 4.19,

$$\frac{d}{dt}\int_{\mathbb{S}_t} k^2\,ds \le C\Big(1+\Big(\int_{\mathbb{S}_t} k^2\Big)\Big)^3\,,$$

holding (in the case of fixed end–points) with a uniform constant C independent of time, we would have $\int_{\mathbb{S}_{\widetilde{t}}} k^2\,ds \ge \frac{\delta}{2}$, for every $\widetilde{t}$ in a uniform neighborhood of every t_j. This is clearly in contradiction with the estimate (8.1). Hence, $\lim_{t\to+\infty}\int_{\mathbb{S}_t} k^2\,ds = 0$ and, consequently, for every sequence of times $t_i \to +\infty$, there exists a subsequence (not relabeled) such that the evolving networks $\mathbb{S}_{t_i}$ converge in $C^{1,\alpha}\cap W^{2,2}$, for every $\alpha \in (0,1/2)$, to a possibly degenerate regular network with zero curvature (hence, "stationary" for the length functional), as $i\to\infty$. □

REMARK 8.13. We underline that, in taking the limit of $\mathbb{S}_{t_i}$, as $t_i \to T = +\infty$, one or more curves could collapse (possibly to an end–point).

8.2. Proof of Theorem 8.8. Without loss of generality we restrict to the case of a triod. We start considering the case when the blow–up limit is a straight line.

PROPOSITION 8.14. *If the sequence of rescaled triods $\widetilde{\mathbb{T}}_{x_0,t_j}$ converges to a straight line, then the curvature of the evolving triod is uniformly bounded for $t\in[0,T)$ in a ball around the point x_0.*

PROOF. Assume that there is a straight line L through the origin of $\mathbb{R}^2$ such that the sequence of rescaled triods $\widetilde{\mathbb{T}}_{x_0,t_j}$ converges to L as $j\to\infty$.
Recalling Lemma 8.6 this implies that the distance $|O(t)-x_0|$ is uniformly bounded from below, so that there exists $i\in\{1,2,3\}$ such that the rescaled curves $\frac{\gamma^i_{t_j}}{\sqrt{2(T-t_j)}}$ converge to L as $j\to\infty$.
In particular, for all $M>1$ there exists $j_M\in\mathbb{N}$ such that the curve $\gamma^i_{t_{j_M}}\cap B_{7M\sqrt{2(T-t_{j_M})}}(x_0)$ is a graph over the line x_0+L. By Corollary 8.17 it follows that $\gamma^i_t\cap B_{M\sqrt{2(T-t_{j_M})}}(x_0)$ is also a graph over the line x_0+L for all $t\in[t_{j_M}, t_{j_M}+M^2(T-t_{j_M}))\supset[t_{j_M},T)$, and its slope v^i (with respect to the line x_0+L) is uniformly bounded by a constant independent of M and t. Therefore, if $M>2$, from Proposition 8.18 (applied with $\theta=1/2$) it follows that the curvature of the curve $\gamma^i_t\cap B_{M\sqrt{2(T-t_{j_M})}}(x_0)$ and all its derivatives are bounded for $t\in[t_{j_M},T)$ and we are done. □

We then consider the case of a halfline.

PROPOSITION 8.15. *If the sequence of rescaled triods $\widetilde{\mathbb{T}}_{x_0,t_j}$ converges to a halfline, then the curvature of the evolving triod is uniformly bounded for $t\in[0,T)$ in a ball around the point x_0.*

PROOF. By the C^1_{loc} convergence of the rescaled flow to the halfline, we can see that the point x_0 must be one of the endpoints of the triod, which we will denote with P. We now perform a reflexion with center P of the triod and we consider the motion by curvature of the union of the two (mutually reflected through P) triods which is still a motion by curvature, now of a *network* of curves (see **[71]** for more details). Since at the endpoint P the curvature vanishes by construction, the point P stays fixed during the motion of the network and the sequence of rescaled networks around $P=x_0$ converges in the C^1_{loc} topology to a straight line. We can now repeat the proof of Proposition 8.14 to conclude. □

We introduce some preliminary lemmas useful to prove Proposition 8.20. In the following, given $\overline{x}\in\mathbb{R}^2$ and $R>0$, we denote by $Q_R(\overline{x})$ the square

$$Q_R(\overline{x}) := \big\{x\in\mathbb{R}^2:\ |x_1-\overline{x}_1|\le R,\ |x_2-\overline{x}_2|\le R\big\}\,.$$

PROPOSITION 8.16. *Suppose that the curve γ_0 is a graph over $\langle e_1\rangle$ in the square $Q_{2R}(x_0)$, and assume that the curve $\gamma_t\cap Q_{2R}(x_0)$ is contained in the horizontal strip $\{|x_2|\le\delta\}$ for any $t\in[0,\tau)$, with $\tau>0$ and $0<\delta<R$.*
Then $\gamma_t\cap Q_{2R}(x_0)$ is a graph over $\langle e_1\rangle$ for all $t\in[0,\tau)$.

PROOF. See **[67**, Proposition 2.21**]** □

COROLLARY 8.17. *Assume that $\gamma_0 \cap B_{7R}(x_0)$ is a graph over $\langle e_1 \rangle$, contained in the horizontal strip $\{|x_2| \leq R\}$. Then $\gamma_t \cap B_{2R}(x_0)$ is a graph over $\langle e_1 \rangle$ for all $t \in [0, \tau)$, with $\tau = R^2/2$. Moreover, letting $v = \langle \nu \,|\, e_2 \rangle^{-1}$, we have*

$$\sup_{t\in[0,\tau)} \sup_{\gamma_t \cap B_R(x_0)} v \leq C \sup_{\gamma_0 \cap B_{2R}(x_0)} v$$

for some $C > 0$ independent of R.

PROOF. See [**67**, Corollary 2.22] □

We recall the following result [**31**, Corollary 3.2 and Corollary 3.5].

PROPOSITION 8.18. *Suppose that γ_t is a graph over $\langle e_1 \rangle$ in $B_R(x_0)$ for all $t \in [0, \tau)$. Then letting $\theta \in (0,1)$ and $m \geq 0$, we have*

$$\sup_{\gamma_t \cap B_{\sqrt{\theta R^2 - 2t}}(x_0)} t^{m+1} |\partial_s^m k|^2 \leq C_{m,v}$$

for all $t \in [0, \tau)$, where the constant $C_{m,v}$ depends only on m, θ and $\sup_{t\in[0,\tau)} \sup_{\gamma_t \cap B_{\sqrt{R^2-2t}}(x_0)} v$.

PROPOSITION 8.19. *Let γ_t be as in Proposition 8.18. For all $\theta \in (0,1)$ we have*

$$\sup_{\gamma_t \cap B_{\sqrt{\theta R^2 - 2t}}(x_0)} |k|^2 \leq \frac{C_v}{(1-\theta)^2} \left(\frac{1}{R^2} + \sup_{\gamma_0 \cap B_R(x_0)} |k|^2 \right) \tag{8.2}$$

for all $t \in [0, \tau)$, where the constant C_v depends only on $\sup_{t\in[0,\tau)} \sup_{\gamma_t \cap B_R(x_0)} v$.

PROOF. See [**67**, Proposition 2.24] □

PROPOSITION 8.20. *If the sequence of rescaled triods $\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}_j}$ converges to a standard triod, then the curvature of the evolving triod is uniformly bounded for $t \in [0, T)$ in a ball around the point x_0.*

PROOF. Since the subset $\mathcal{I}$ of $[-1/2 \log T, +\infty)$ defined by $\mathcal{I} = \cup_{j=1}^{\infty} (\mathfrak{t}_j + \log\sqrt{3/2}, \mathfrak{t}_j + \log\sqrt{3})$ has obviously infinite Lebesgue measure, by Proposition 7.17, we can assume that there exists another sequence of rescaled triods $\widetilde{\mathbb{T}}_{x_0,\widetilde{\mathfrak{t}}_j}$, with $\widetilde{\mathfrak{t}}_j \in (\mathfrak{t}_j + \log\sqrt{3/2}, \mathfrak{t}_j + \log\sqrt{3})$ for every $j \in \mathbb{N}$, which is also C^1_{loc} converging to a flat triod (*a priori* not necessarily the same one) centered at the origin of $\mathbb{R}^2$ as $j \to \infty$. Indeed, even if the two blow–up limits are different, they both must be a flat triod, as equality (7.11) must hold for both of them. Moreover, the L^2 norm of the curvature of the modified sequence of rescaled triods, as well as the one of the original sequence of rescaled triods, converges to zero on every compact subset of $\mathbb{R}^2$.
Finally, passing to a subsequence, we can also assume that $\mathfrak{t}_j$ and $\widetilde{\mathfrak{t}}_j$ (hence, also t_j and $\widetilde{t_j}$) are increasing sequences.
Notice that, by means of the rescaling relation $\mathfrak{t}(t) = -\frac{1}{2}\log(T-t)$, the condition $\widetilde{\mathfrak{t}}_j \in (\mathfrak{t}_j + \log\sqrt{3/2}, \mathfrak{t}_j + \log\sqrt{3})$ reads, for the original time parameter, as $\widetilde{t_j} \in \left(\frac{2}{3}t_{j_M} + \frac{1}{3}T, \frac{1}{3}t_{j_M} + \frac{2}{3}T\right)$.

Repeating the argument in the proof of Proposition 8.14, for any M large enough there exists j_M such that for all $i \in \{1,2,3\}$ the curve $\gamma_t^i \cap B_{5M\sqrt{2(T-t_{j_M})}}(x_0) \setminus B_{M\sqrt{2(T-t_{j_M})}}(x_0)$ is a graph over $x_0 + L^i$ for all $t \in [t_{j_M}, T)$, with slope (with respect to the line $x_0 + L^i$) uniformly bounded by a constant C_v independent of M and $t \in [t_{j_M}, T)$ (here and in the sequel we denote by C_v a generic constant, depending on v, which may vary from line to line). Moreover, by Lemma 8.6, we can also assume that the 3–point $O(t)$ in this time interval does not get into the annulus $B_{5M\sqrt{2(T-t_{j_M})}}(x_0) \setminus B_{M\sqrt{2(T-t_{j_M})}}(x_0)$.
By Proposition 8.18, with $\theta < 1/2 < 9/16 + \frac{1}{2M^2}$, it follows that the subsequent evolution of the curves

$$\gamma_{t_M}^i \cap \left(B_{4M\sqrt{2(T-t_{j_M})}}(x_0) \setminus B_{2M\sqrt{2(T-t_{j_M})}}(x_0) \right),$$

that, with an abuse of notation as we cannot exclude that other parts of $\mathbb{T}_t$ get into the annulus $B_{4M\sqrt{2(T-t_{j_M})}}(x_0) \setminus B_{2M\sqrt{2(T-t_{j_M})}}(x_0)$, we still denote by

$$\gamma_t^i \cap \left(B_{4M\sqrt{2(T-t_{j_M})}}(x_0) \setminus B_{2M\sqrt{2(T-t_{j_M})}}(x_0) \right),$$

for $i \in \{1,2,3\}$, are smooth evolutions for all $t \in [t_{j_M}, T)$ and the following estimate holds

$$|k_s^i(t)|^2 \leq \frac{C_v}{(t-t_{j_M})^2} \leq \frac{C_v}{(\widetilde{t}_{j_M}-t_{j_M})^2} \leq \frac{9C_v}{(T-t_{j_M})^2}\,, \tag{8.3}$$

for all $t \in [\widetilde{t}_{j_M}, T)$, where the constant C_v depends only on the slope with respect to the line $x_0 + L^i$.

Since, by Proposition 7.22, the L^2 norm of the curvature (in the rescaled ball $\widetilde{B}_{5M}(0)$) of the sequence of rescaled triods $\widetilde{\mathbb{T}}_{x_0,\widetilde{\mathfrak{t}}_j}$, which is given by

$$\sqrt{2(T-\widetilde{t}_j)}\int_{\mathbb{T}_{\widetilde{t}_j}\cap B_{5M\sqrt{2(T-t_{j_M})}}(x_0)} k^2\,ds\,,$$

converges to zero as $j \to \infty$, the above estimate (8.3) on the derivative of the curvature, which for the sequence of rescaled triods becomes $|\widetilde{k}_s^i(\mathfrak{t}_j)| \leq 3\sqrt{C}$, implies that the L^∞ norm of the curvature of the rescalings of the curves

$$\gamma^i_{\widetilde{t}_j} \cap \Big(B_{4M\sqrt{2(T-t_{j_M})}}(x_0) \setminus B_{2M\sqrt{2(T-t_{j_M})}}(x_0)\Big)\,,$$

which is given by

$$\sqrt{2(T-\widetilde{t}_j)}\left(\sup_{\mathbb{T}_{\widetilde{t}_j}\cap\big(B_{4M\sqrt{2(T-t_{j_M})}}(x_0)\setminus B_{2M\sqrt{2(T-t_{j_M})}}(x_0)\big)} |k|\right)\,,$$

converges to zero as $j \to \infty$.

Since the above argument holds not only for j_M but for every $j \geq j_M$, fixed any $\varepsilon \in (0, 1/2)$, first considering an $M > 2$ large enough and then choosing a suitably large j_M, we can assume that

- $M > \max\{1/\sqrt{\varepsilon}, C_2/\varepsilon^{1/3}\}$, where the constant C_2 is the one appearing in Lemma 4.10,
- $$\int_{\mathbb{T}_{\widetilde{t}_{j_M}}\cap B_{5M\sqrt{2(T-t_{j_M})}}(x_0)} k^2\,ds \leq \frac{\varepsilon}{\sqrt{2(T-\widetilde{t}_{j_M})}} \leq \frac{\sqrt{3}\varepsilon}{\sqrt{2(T-t_{j_M})}}\,, \tag{8.4}$$
- $$\sup_{\mathbb{T}_{\widetilde{t}_{j_M}}\cap\big(B_{4M\sqrt{2(T-t_{j_M})}}(x_0)\setminus B_{2M\sqrt{2(T-t_{j_M})}}(x_0)\big)} k^2 \leq \frac{\varepsilon}{2(T-\widetilde{t}_{j_M})} \leq \frac{3\varepsilon}{2(T-t_{j_M})}\,. \tag{8.5}$$

By Proposition 8.19, as $M > 2$, at the points

$$\gamma^i_t \cap \Big(B_{\frac{7}{2}M\sqrt{2(T-t_{j_M})}}(x_0) \setminus B_{\frac{5}{2}M\sqrt{2(T-t_{j_M})}}(x_0)\Big)\,,$$

we have the estimate

$$|k^i(t)|^2 \leq C_v\left(\sup_{\gamma^i_{\widetilde{t}_{j_M}}\cap\big(B_{4M\sqrt{2(T-t_{j_M})}}(x_0)\setminus B_{2M\sqrt{2(T-t_{j_M})}}(x_0)\big)} |k^i|^2 + \frac{1}{M^2(T-t_{j_M})}\right)$$

for all $t \in [\widetilde{t}_{j_M}, T)$, with a constant C_v depending only on the slope of the curve with respect to the line $x_0 + L^i$, which is uniformly bounded. Thus, by the above estimate (8.5) we get

$$|k^i(t)|^2 \leq \frac{C_v}{T-t_{j_M}}\Big(\varepsilon + \frac{1}{M^2}\Big) \leq \frac{2C_v\varepsilon}{T-t_{j_M}} \tag{8.6}$$

as we already chose $M^2 > 1/\varepsilon$ above, for all the points of the curve $\gamma^i_t \cap \Big(B_{\frac{7}{2}M\sqrt{2(T-t_{j_M})}}(x_0) \setminus B_{\frac{5}{2}M\sqrt{2(T-t_{j_M})}}(x_0)\Big)$ and times $t \in [\widetilde{t}_{j_M}, T)$. We want to underline once more that the constant C depends only on the slope of the curve with respect to the line $x_0 + L^i$.

It follows that for every $t \in [\widetilde{t}_{j_M}, T)$, all the triods $\widehat{\mathbb{T}}_t$ determined by "cutting" $\mathbb{T}_t$ at the new (moving in time) endpoints $Q^i(t) = \gamma_t^i \cap \partial B_{3M\sqrt{2(T-t_{j_M})}}(x_0)$ have the lengths of their three curves uniformly bounded away from zero from below and unit tangent vectors at the endpoints $Q^i(t)$ which form angles with the respective velocity vectors $\partial_t Q^i(t)$ which are also bounded away from zero, uniformly in time, because of the uniform control on the slope of the curves with respect to the line $x_0 + L^i$. This implies that the norm of the curvature $|k^i(Q^i(t))|$ at any endpoint $Q^i(t)$ controls the norm of the tangential velocity $|\lambda^i(Q^i(t))|$, up to a multiplicative constant C_v (depending only on the slope), uniformly bounded in time for $t \in [\widetilde{t}_{j_M}, T)$.
Then, from estimates (8.3), (8.6), we conclude

$$\begin{aligned}\Big|k^i(Q^i(t))k^i_s(Q^i(t))\Big| &\leq \frac{C_v\varepsilon^{1/2}}{(T-t_{j_M})^{\frac{3}{2}}}\,,\\ \Big|[k^i(Q^i(t))]^2\lambda^i(Q^i(t))\Big| \leq C_v\Big|k^i(Q^i(t))\Big|^3 &\leq \frac{C_v\varepsilon^{3/2}}{(T-t_{j_M})^{\frac{3}{2}}}\,,\end{aligned}$$

for every $t \in [\widetilde{t}_{j_M}, T)$, where the constant C_v depends only on the slope of the curve with respect to the line $x_0 + L^i$. Moreover, we can clearly always increase j_M as we like without affecting C_v (this is actually true for every constant depends only on the slope of the curve with respect to the line $x_0 + L^i$), since, by Proposition 8.14, as $j \to \infty$, the three curves

$$\gamma^i_{t_j} \cap B_{5M\sqrt{2(T-t_{j_M})}}(x_0) \setminus B_{M\sqrt{2(T-t_{j_M})}}(x_0)$$

converge to a smooth limit. Hence, we can also assume that $2C_v\varepsilon^{1/6} < 1$ and $2(C_1+C_v+1)\varepsilon^{1/3} < 1$.

At this point we observe that the length of every curve of the triod (being all the curves graphs in the annulus $B_{3M\sqrt{2(T-t_{j_M})}}(x_0) \setminus B_{2M\sqrt{2(T-t_{j_M})}}(x_0)$) is bounded from below by a uniform factor (depending only on the slope v) times $M\sqrt{T-t_{j_M}}$. Then, by means of Lemma 4.10, we now prove an inequality for the time derivative of the L^2 norm of the curvature of the triods $\widehat{\mathbb{T}}_t$ which are determined by the three (moving in time) endpoints $Q^i(t)$, for $t \in [\widetilde{t}_{j_M}, T)$. Notice that here the constants C_1 and C_2 are "universal", C_v depends only on the slope of the curve with respect to the line $x_0 + L^i$ and we use the two previous inequalities to estimate the terms coming from the endpoints:

$$\begin{aligned}\frac{d}{dt}\int_{\widehat{\mathbb{T}}_t} k^2\,ds &\leq C_1\Big(\int_{\widehat{\mathbb{T}}_t} k^2\,ds\Big)^3 + \frac{C_2C_v}{M\sqrt{T-t_{j_M}}}\Big(\int_{\widehat{\mathbb{T}}_t} k^2\,ds\Big)^2 + \frac{C_v\varepsilon^{1/2}}{(T-t_{j_M})^{\frac{3}{2}}}\\ &\leq C_1\Big(\int_{\widehat{\mathbb{T}}_t} k^2\,ds\Big)^3 + \frac{C_v\varepsilon^{\frac{1}{3}}}{\sqrt{T-t_{j_M}}}\Big(\int_{\widehat{\mathbb{T}}_t} k^2\,ds\Big)^2 + \frac{C_v\varepsilon^{1/2}}{(T-t_{j_M})^{\frac{3}{2}}}\\ &\leq C_1\Big(\int_{\widehat{\mathbb{T}}_t} k^2\,ds\Big)^3 + \frac{\varepsilon^{\frac{1}{6}}}{\sqrt{T-t_{j_M}}}\Big(\int_{\widehat{\mathbb{T}}_t} k^2\,ds\Big)^2 + \frac{C_v\varepsilon^{1/2}}{(T-t_{j_M})^{\frac{3}{2}}}\,,\end{aligned}$$

as we chose $M > C_2/\varepsilon^{\frac{1}{3}}$ and $2C_v\varepsilon^{1/6} < 1$.
Then, letting

$$A(t) := \max\left\{\int_{\widehat{\mathbb{T}}_t} k^2\,ds\,,\ \frac{\varepsilon^{\frac{1}{6}}}{\sqrt{T-t_{j_M}}}\right\}\,,$$

it follows

$$A'(t) \leq \overline{C}_v A^3(t)$$

for almost every $t \in [\widetilde{t}_{j_M}, T)$, where the constant $\overline{C}_v$ is given by $C_1 + C_v + 1$.
Integrating this differential inequality and recalling estimate (8.4), implying that

$$A(\widetilde{t}_{j_M}) \leq \max\left\{\frac{\sqrt{3}\varepsilon}{\sqrt{2(T-t_{j_M})}}\,,\ \frac{\varepsilon^{\frac{1}{6}}}{\sqrt{T-t_{j_M}}}\right\} \leq \frac{\varepsilon^{\frac{1}{6}}}{\sqrt{T-t_{j_M}}}\,,$$

as $\varepsilon < 1/2$, we get

$$A(t) \leq \frac{1}{\sqrt{A(\widetilde{t}_{j_M})^{-2} - 2\overline{C}_v(t - \widetilde{t}_{j_M})}}\,,$$

hence,

$$A(t) \leq \frac{\varepsilon^{\frac{1}{6}}}{\sqrt{T - t_{j_M} - 2\overline{C}_v\varepsilon^{\frac{1}{3}}(t - \widetilde{t}_{j_M})}}\,,$$

for every $t \in [\widetilde{t}_{j_M}, T)$.
As $(t - \widetilde{t}_{j_M}) \leq (T - t_{j_M})$, it follows that the function $A(t)$ is uniformly bounded on $[\widetilde{t}_{j_M}, T)$ as soon as $2\overline{C}_v\varepsilon^{\frac{1}{3}} < 1$, which is satisfied by our previous assumption on $\varepsilon > 0$.

We now notice that the three curves of the triod $\mathbb{T}_t$, connecting respectively the points P^i and Q^i (determined by $\mathbb{T}_t \setminus \widehat{\mathbb{T}}_t$) cannot get too close to the point $x_0 = \lim_{t\to T} O(t)$ along the flow. Indeed, the parts of these curves in the annulus

$$B_{5M\sqrt{2(T-t_{j_M})}}(x_0) \setminus B_{3M\sqrt{2(T-t_{j_M})}}(x_0)$$

are graphs for every $t \in [\widetilde{t}_{j_M}, T)$, while the remaining pieces "outside" at time $t = \widetilde{t}_{j_M}$, by maximum principle, during their subsequent evolution can never get into the circle of radius $R(t) = \sqrt{16M^2(T - t_{j_M}) - 2(t - t_{j_M})}$ and center x_0, also moving by mean curvature in the time interval $[\widetilde{t}_{j_M}, T)$ and, as $t \to T$, converging to the circle of radius

$$\sqrt{16M^2(T - t_{j_M}) - 2(T - t_{j_M})} = \sqrt{(16M^2 - 2)(T - t_{j_M})}\,,$$

which is clearly positive as $M^2 > 2$, hence far from the point x_0.
Consequently, since the closed subset of the set of reachable points obtained as possible limit points of these three curves as $t \to T$ is contained in a closed set far from x_0, by Propositions 8.14 and 8.15, we can cover such a set by a finite number of balls where the curvature of the evolving triod is uniformly bounded during the flow. Being also the total length of the evolving triods uniformly bounded and being the L^2 norm of the curvature of the "subtriods" $\widehat{\mathbb{T}}_t$, given by the square root of the uniformly bounded function $A(t)$, we conclude that the full L^2 norm of the curvature of the evolving triods $\mathbb{T}_t$ is bounded, in contradiction with Proposition 4.14. This concludes the proof. □

8.3. Limit networks with bounded curvature. The analysis in this case consists in understanding the possible limit networks that can arise, as $t \to T$, under the assumption that the curvature is uniformly bounded along the flow. This in order to find out how to continue the flow (if possible), which will be the argument of the next section.

As we said, at least one curve of the network $\mathbb{S}_t$ has to "vanish", approaching the singular time T. Anyway, we are going to show that, as $t \to T$, there is a unique limit degenerate regular network in Ω, which can be *non–regular* seen as a subset of $\mathbb{R}^2$ since *a priori* multi–points can appear, but anyway the sum of the unit tangent vectors of the concurring curves at every multi–point must be zero, see Remark 7.2 (we recall that this implies that every "genuine" triple junction which is present still satisfies the 120 degrees condition).

PROPOSITION 8.21. *If $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0,1], t)$ is the curvature flow of a regular network in Ω with fixed end–points in a maximal time interval $[0, T)$ such that the curvature is uniformly bounded along the flow, the networks $\mathbb{S}_t$, up to reparametrization proportional to arclength, converge in C^1 to some degenerate regular network $\widehat{\mathbb{S}}_T = \bigcup_{i=1}^n \widehat{\gamma}^i_T([0,1])$ in Ω, as $t \to T$.*
Moreover, the non–degenerate curves of $\widehat{\mathbb{S}}_T$ belong to $C^1 \cap W^{2,\infty}$ and they are smooth outside the multi–points.

PROOF. As we said at the beginning of this section, by Proposition 5.17, since $\mathbb{S}_t$ is the curvature flow of a regular network, there exist the limits of the lengths of the curves $L^i(T) = \lim_{t\to T} L^i(t)$, for every $i \in \{1, 2, \ldots, n\}$. Moreover, every limit of $\mathbb{S}_t$ is a connected, bounded subset of $\mathbb{R}^2$.

As the curvature and the total length are bounded by some constant C, after reparametrizing the curves γ^i proportional to arclength getting the maps $\widehat{\gamma}^i$, these latter are a family of uniformly Lipschitz maps such that $\widehat{\gamma}^i_t$ and $\widehat{\gamma}^i_{xx}$ are uniformly bounded in space and time by some constant D.
Then, it is easy to see that, uniformly for $x \in [0, 1]$, we have

$$|\widehat{\gamma}^i(x,t) - \widehat{\gamma}^i(x,\bar{t})| \leq \int_t^{\bar{t}} |\widehat{\gamma}^i_t(x,\xi)|\, d\xi \leq D|t-\bar{t}|\,,$$

which clearly means that $\widehat{\gamma}^i(\cdot,t) : [0,1] \to \mathbb{R}^2$ is a Cauchy sequence in $C^0([0,1])$, hence the flow of reparametrized regular networks converges in C^1 to a limit family of C^1 curves $\widehat{\gamma}^i_T : [0,1] \to \mathbb{R}^2$, as $t \to T$, composing the degenerate regular network $\widehat{\mathbb{S}}_T = \bigcup_{i=1}^n \widehat{\gamma}^i_T([0,1])$. Clearly, by the bound on the curvature, these curves either are "constant" or belong to $W^{2,\infty}$, moreover, by Lemma 8.7, they are smooth outside the multi–points.
About the convergence of the unit tangent vectors, we observe that

$$\left|\frac{\partial \widehat{\tau}^i(x,t)}{\partial x}\right| = \left|\frac{\partial \tau^i(s,t)}{\partial s}\right| L^i(t) = |k(s,t)| L^i(t) \leq C L^i(t) \leq C^2\,, \tag{8.7}$$

hence, every sequence of times $t_j \to T$ have a – not relabeled – subsequence such that the maps $\widehat{\tau}^i(\cdot,t_j)$ converge uniformly to some maps $\widehat{\tau}^i_T$.
If the curve $\widehat{\gamma}^i_T$ is a regular curve (that is, $L^i(t)$ does not go to zero), it is easy to see that the limit maps $\widehat{\tau}^i_T$ must coincide with the unit tangent vector field $\widehat{\tau}^i_T$ to the curve $\widehat{\gamma}^i_T$, hence, the full sequence $\widehat{\tau}^i(\cdot,t)$ converges.
If $L^i(t)$ converges to zero, as $t \to T$, by inequality (8.7), the maps $\widehat{\tau}^i(\cdot,t_j)$ converge to a constant unit vector $\widehat{\tau}^i_T$ which, if independent of the subsequence t_j, will be the "assigned" constant unit vector to the degenerate constant curve $\widehat{\gamma}^i_T$ of the degenerate regular network $\mathbb{S}$, as in Definition 7.1.
We claim that $\widehat{\mathbb{S}}_T$ contains at least one regular non–degenerate curve, otherwise, as $t \to T$, the whole network $\mathbb{S}_t$ is contracting at a single point, this clearly can happen only if the network has no end–points and the radius of the smallest ball containing it is going to zero as $t \to T$. Being this ball tangent to the network $\mathbb{S}_t$ at some interior point of a curve, at such point the curvature of the network must be larger or equal to the inverse of the radius of such ball and this is a contradiction, by the uniform bound on the curvature.
If now we consider the set of the regular non–degenerate curves of $\widehat{\mathbb{S}}_T$, their end–points contain all the constant images of the degenerate curves and the Herring condition determines the "unit tangent vectors" of the (one or two) degenerate curves concurring there (the mutual position, if these are two, is uniquely determined by the embeddedness of the converging regular networks). This argument can be iterated, considering now the degenerate concurring curves with respect to the previous degenerate curves and so on, to determine uniquely the unit tangent vectors at all the 3–points of the limit degenerate regular network $\widehat{\mathbb{S}}_T$. Hence, the limit degenerate unit tangent vectors of $\widehat{\gamma}^i_T$ are independent of the chosen sequence of times $t_j \to T$ and we are done. □

REMARK 8.22. In the special situation that no curves collapse, as $t \to T$ (we actually conjecture that this cannot happen, that is, that Problem 5.11 has a positive answer and Corollary 5.10 applies), the limit network $\widehat{\mathbb{S}}_T$ is a regular network in $W^{2,\infty}$, hence, one can use the extension of Theorem 5.8 mentioned at point 5 of Remark 5.9, in order to continue the flow after the time T. In this very "strange" case, one should investigate if this "extended" curvature flow, which is $C^{2,1}$ with the exception of time $t = T$, is actually always $C^{2,1}$, getting a contradiction by the maximality of the interval of $C^{2,1}$ existence $[0, T]$.

If we consider the family of the non–degenerate curves of $\widehat{\mathbb{S}}_T$, they describe a C^1 network, that we call $\mathbb{S}_T$, which is not necessarily a regular network (it can have multi–points), but by Remark 7.2, the sum of the exterior unit tangent vectors of the concurring curves at every multi–point in Ω must still be zero.

REMARK 8.23. Notice that, even if $\mathbb{S}_T$ is smooth outside its multi–points and $W^{2,\infty}$, we cannot set that its curves are actually of class C^2.

PROPOSITION 8.24. *If* **M1** *is true, every vertex of the network* $\widehat{\mathbb{S}}_T$ *is either a regular triple junction or an end–point of* $\mathbb{S}_t$ *or*

- *a* 4*–point where the four concurring curves have opposite unit tangents in pairs and form angles of* $120/60$ *degrees between them* – collapse of a curve in the "interior" of $\mathbb{S}_t$,
- *a* 2*–point at an end–point of the network* $\mathbb{S}_t$ *where the two concurring curves form an angle of* 120 *degrees among them* – collapse of the curve getting to such end–point of $\mathbb{S}_t$.

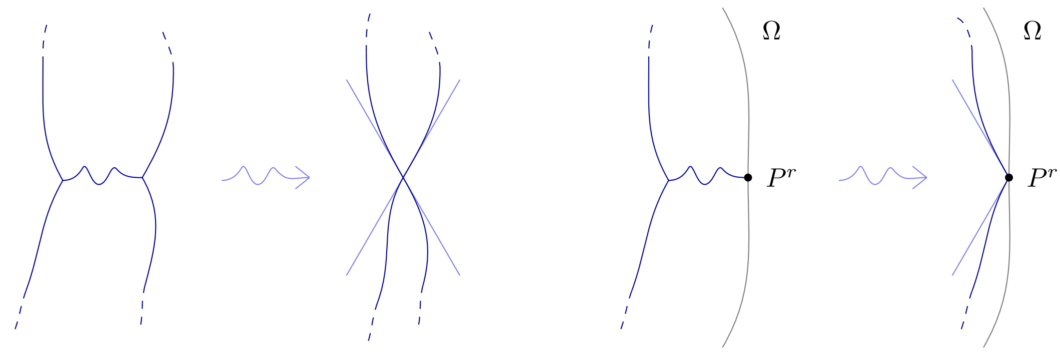


FIGURE 15. Collapse of a curve in the interior and at an end–point of $\mathbb{S}_t$.

PROOF. Since the curvature is bounded, no regions can collapse, by the computations in Section 7.2, hence, around every point the network is definitely locally a tree, as $t \to T$. Assuming that the vertex belongs to Ω, we can follow the argument in the proof of the second part of Lemma 7.10, to show that the core of $\mathbb{S}_T$ must be a single curve and we have in $\widehat{\mathbb{S}}_T$ a 4–point where the four concurring curves have opposite unit tangents in pairs and form angles of $120/60$ degrees between them. The only extra fact we have to show is that $\widehat{\mathbb{S}}_T$ cannot have a multi–point O_T with two concurring curves with the same inner unit tangent vector. Indeed, in such case, by embeddedness they must form a cusp and "slowly" dilating the networks $\mathbb{S}_t$ at suitable times around O_T, in order that the distance between such two curves and O_T still go to zero, we would get a multiplicity–two halfline, contradicting the multiplicity–one conjecture **M1**, that we assumed to hold.
In the case the vertex of $\widehat{\mathbb{S}}_T$ coincides with an end–point P^r of $\mathbb{S}_t$, we get the statement, by considering the network $\mathbb{H}^r_t$, obtained by the union of $\mathbb{S}_t$ with its reflection with respect to the point P^r (see the discussion just before Section 8.1) and applying the previous conclusion to such network. □

REMARK 8.25. It follows that every core (there could be more than one) of $\mathbb{S}_T$ is composed by a single collapsed curve.

REMARK 8.26. Notice that if at an end–point the two curves of the boundary of the convex set Ω form an angle (or the whole network is contained in an angle whose vertex is such end–point) with amplitude less than 120 degrees, then the collapse situation described in Proposition 8.24 cannot happen at such end–point. This is, for instance, the case of an initial triod contained in a triangle with angles less than 120 degrees and fixed end–points in the vertices.
The same conclusion holds, by the argument in the proof of Proposition 7.13, calling $\Omega_t \subset \Omega$ the evolution by curvature of $\partial\Omega$, keeping fixed the end–points of $\mathbb{S}_t$, if the angle formed by Ω_t at such end–point, becomes smaller than 120 degrees.

COROLLARY 8.27. *If* **M1** *holds and the curvature of* $\mathbb{S}_t$ *is uniformly bounded during the flow, the networks* $\mathbb{S}_t$*, up to reparametrization, converge in* C^1 *to some degenerate regular network* $\widehat{\mathbb{S}}_T$*, whose non–degenerate curves form a* C^1 *network* $\mathbb{S}_T$*, having all its multi–points which are among the ones described*

in Proposition 8.24.
Moreover, the curves of $\mathbb{S}_T$ belong to $C^1 \cap W^{2,\infty}$ and are smooth outside the multi–points.

All the previous arguments can be easily localized and we have the following conclusion.

PROPOSITION 8.28. *If* **M1** *holds and the curvature of $\mathbb{S}_t$ is locally uniformly bounded around a point $x_0 \in \overline{\Omega}$, as $t \to T$, the networks $\mathbb{S}_t$, up to reparametrization, converge in C^1_{loc} locally around x_0 to some degenerate regular network $\widehat{\mathbb{S}}_T$ whose non–degenerate curves form a C^1 network $\mathbb{S}_T$, having a possibly non–regular multi–point at x_0 which is among the ones described in Proposition 8.24.*
Moreover, the curves of $\mathbb{S}_T$ belong to $C^1 \cap W^{2,\infty}$, in a neighborhood of x_0, and are smooth outside the multi–point.

REMARK 8.29. We can call these singularities with bounded curvature *Type 0* singularities. They are peculiar of the network flow, as they cannot appear in the motion by curvature of a single curve.

8.4. Vanishing of curves with unbounded curvature. The last case, when the curvature is not bounded and the length of at least one curve of the evolving network $\mathbb{S}_t$ is not positively bounded from below, as $t \to T$, is the most delicate. Performing, as before, the blow–up procedure, even assuming the multiplicity–one conjecture, there can be several shrinkers as possible blow–up limits given by Proposition 7.17 and we need to classify them in order to understand the behavior of the flow $\mathbb{S}_t$ approaching the singular time T. In doing that the (local) structure (topology) of the evolving network plays an important role in the analysis, since it restricts the family of possible shrinkers obtained as blow–up limits of $\mathbb{S}_t$.
A very relevant case is when the evolving network is a tree, that is, it has no loops.

PROPOSITION 8.30. *If* **M1** *holds and the evolving regular network $\mathbb{S}_t$ is definitely locally a tree around some $x_0 \in \overline{\Omega}$, as $t \to T$, then the curvature of $\mathbb{S}_t$ is locally uniformly bounded around x_0, during the flow.*

PROOF. Let $\mathbb{S}_t$ be a smooth flow in the maximal time interval $[0, T)$ of the initial network $\mathbb{S}_0$ and let $x_0 \in \overline{\Omega}$ be a reachable point for the flow in B (we clearly only need to consider reachable points).
Let us consider the sequence of rescaled networks $\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_j}$, obtained via Huisken's dynamical procedure, as in Proposition 7.17. Then, as $j \to \infty$, it converges in $C^{1,\alpha}_{\mathrm{loc}} \cap W^{2,2}_{\mathrm{loc}}$, for any $\alpha \in (0, 1/2)$, to a limit degenerate regular shrinker $\widetilde{\mathbb{S}}_\infty$. Thanks to the multiplicity one hypothesis **M1** and to the topology of the network (locally a tree, see Lemma 7.10), if we suppose that $x_0 \notin \partial\Omega$, then $\widetilde{\mathbb{S}}_\infty$ can only be:

- a straight line;
- a standard triod;
- four concurring halflines with opposite unit tangent vectors in pairs, forming angles of $120/60$ degrees between them.

By White's local regularity theorem in **[91]** (or by Proposition 8.14), if the sequence of rescaled networks converges to a straight line, the curvature of the evolving network is uniformly bounded for $t \in [0, T)$ in a ball around the point x_0. Thanks to Proposition 8.20 the same holds in the case of the standard triod. Hence, the only situation we have to deal with to complete the proof in this case is the collapse of two triple junctions at a point of Ω, when the limit is the degenerate regular network composed by four concurring halflines with opposite unit tangents in pairs forming angles of $120/60$ degrees between them. We claim that also in this case the curvature is locally uniformly bounded during the flow, around the point x_0 (the next proposition and lemmas are devoted to prove this fact).

If instead $x_0 \in \partial\Omega$, the only two possibilities for $\widetilde{\mathbb{S}}_\infty$ are:

- a halfline;
- two concurring halflines forming an angle of 120 degrees.

For both these situation the thesis is obtained by going back to the case in which $x_0 \in \Omega$, with the "reflection construction" we described just before Section 8.1. □

PROPOSITION 8.31. *Let $\mathbb{S}_t$ be a smooth flow in the maximal time interval $[0,T)$ for the initial network $\mathbb{S}_0$. Let x_0 be a reachable point for the flow such that the sequence of rescaled networks $\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_j}$ (introduced in Proposition 7.17) converges, as $j\to\infty$, in $C^{1,\alpha}_{\mathrm{loc}}\cap W^{2,2}_{\mathrm{loc}}$, for any $\alpha\in(0,1/2)$, to a limit degenerate regular shrinker $\widetilde{\mathbb{S}}_\infty$ composed by four concurring halflines with opposite unit tangent vectors in pairs, forming angles of $120/60$ degrees between them. Then,*

$$|k(x,t)|\leq C<+\infty$$

for all x in a neighborhood of x_0 and $t\in[0,T)$.

PROOF. By hypothesis, the sequence of rescaled networks $\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_j}$ converges, as $j\to\infty$, in $C^{1,\alpha}_{\mathrm{loc}}\cap W^{2,2}_{\mathrm{loc}}$, for any $\alpha\in(0,1/2)$, to $\widetilde{\mathbb{S}}_\infty$ as in the statement of the proposition. By arguing as in [**67**, Theorem 2.4], we can assume that for $R>0$ large enough there exists $j_0\in\mathbb{N}$, such the flow $\mathbb{S}_t$ has equibounded curvature, no 3–points and an uniform bound from below on the lengths of the four curves in the annulus $B_{3R\sqrt{2(T-t_j)}}(x_0)\setminus B_{R\sqrt{2(T-t_j)}}(x_0)$, for every $t\in[t_j,T)$ and $j\geq j_0$. We can thus introduce four "artificial" moving boundary points $P^r(t)\in\mathbb{S}_t$ with $|P^r(t)-x_0|=2R\sqrt{2(T-t_j)}$, with $r\in\{1,2,3,4\}$ and $t\in[t_j,T)$ such that the estimates (4.1) are satisfied, that is, the hypotheses about the end–points $P^i(t)$ of Lemmas 4.18, 4.19 and 4.20.
As the sequence of rescaled networks $\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_j}$ converges, as $j\to\infty$, in $W^{2,2}_{\mathrm{loc}}$, to a limit network $\widetilde{\mathbb{S}}_\infty$ with zero curvature, we have

$$\lim_{j\to\infty}\|\widetilde{k}\|_{L^2(B_{3R}(0)\cap\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_j})}=0\,,\qquad\text{that is,}\qquad\int_{B_{3R}(0)\cap\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_j}}\widetilde{k}^2\,d\sigma\leq\varepsilon_j\,,$$

for a sequence $\varepsilon_j\to0$ as $j\to\infty$. Rewriting this condition for the non–rescaled networks, we have

$$\int_{B_{3R\sqrt{2(T-t_j)}}(x_0)\cap\mathbb{S}_{t_j}}k^2\,ds\leq\frac{\varepsilon_j}{\sqrt{2(T-t_j)}}\,. \tag{8.8}$$

Applying now Lemma 4.19 to the flow of networks $\mathbb{S}_t$ in the ball $B_{2R\sqrt{2(T-t_j)}}(x_0)$ in the time interval $[t_j,T)$, we have that $\|k\|_{L^2(B_{2R\sqrt{2(T-t_j)}}(x_0)\cap\mathbb{S}_t)}$ is uniformly bounded, up to time

$$T_j=t_j+\min\Big\{T,1/8C\left(\|k\|^2_{L^2(B_{2R\sqrt{2(T-t_j)}}(x_0)\cap\mathbb{S}_{t_j})}+1\right)^2\Big\}\,.$$

We want to see that actually $T_j>T$ definitely, hence, $\|k\|_{L^2(B_{2R}(x_0)\cap\mathbb{S}_t)}$ is uniformly bounded for $t\in[0,T)$. If this is not true, we have

$$\begin{aligned}T_j&=t_j+\frac{1}{8C\left(\|k\|^2_{L^2(B_{2R\sqrt{2(T-t_j)}}(x_0)\cap\mathbb{S}_{t_j})}+1\right)^2}\\&\geq t_j+\frac{1}{8C\left(\varepsilon_j/\sqrt{2(T-t_j)}+1\right)^2}\\&=t_j+\frac{2(T-t_j)}{8C\left(\varepsilon_j+\sqrt{2(T-t_j)}\right)^2}\\&=T+(2(T-t_j))\bigg(\frac{2}{8C\left(\varepsilon_j+\sqrt{2(T-t_j)}\right)^2}-1\bigg)\,,\end{aligned}$$

which is clearly definitely larger than T, as $\varepsilon_j\to0$, when $j\to\infty$.

Choosing then $j_1\geq j_0$ large enough, since $\|k\|_{L^2(B_{2R\sqrt{2(T-t_{j_1})}}(x_0)\cap\mathbb{S}_t)}$ is uniformly bounded for all times $[t_{j_1},T)$ and the length of the four curves that connect the junctions with the "artificial" boundary points $P^r(t)$ are bounded below by a uniform constant, Lemma 4.20 applies, hence, thanks to Lemma 4.18, we have a uniform bound on $\|k\|_{L^\infty(B_{2R\sqrt{2(T-t_{j_1})}}(x_0)\cap\mathbb{S}_t)}$ for every $t\in[0,T)$. ☐

Proposition 8.31 clearly implies Theorem 8.30. An obvious consequence is that evolving trees do not develop this kind of singularities, hence, their curvature flow is smooth till a curve collapses with uniformly bounded curvature. Moreover, it is easy to see that if no regions collapse, the network is locally a tree around every point of $\overline{\Omega}$, so Theorem 8.30 applies.

COROLLARY 8.32. *If* **M1** *holds and* $\mathbb{S}_0$ *is a tree, the curvature of* $\mathbb{S}_t$ *is uniformly bounded during the flow (hence, we are in the case of Corollary 8.27 in the previous section).*

Combining Propositions 8.28 and 8.30, we have the following local conclusion.

THEOREM 8.33. *If* **M1** *holds and* $\mathbb{S}_t$ *is definitely locally a tree around a point* $x_0 \in \overline{\Omega}$, *either the flow* $\mathbb{S}_t$ *is locally smooth, or up to reparametrization proportional to arclength, converge in* C^1_{loc} *locally around* x_0, *as* $t \to T$, *to some degenerate regular network* $\widehat{\mathbb{S}}_T$ *whose non–degenerate curves form a* C^1 *network* $\mathbb{S}_T$ *with a possibly non–regular multi–point which is among the ones described in Proposition 8.24. Moreover, the curves of* $\mathbb{S}_T$ *belong to* $C^1 \cap W^{2,\infty}$, *in a neighborhood of* x_0, *and are smooth outside the multi–point.*

REMARK 8.34. Bounded curvature is not actually the case in general if some loops are present, indeed we have seen that a region bounded by less than six curves possibly collapses and in such case the curvature cannot stay bounded.

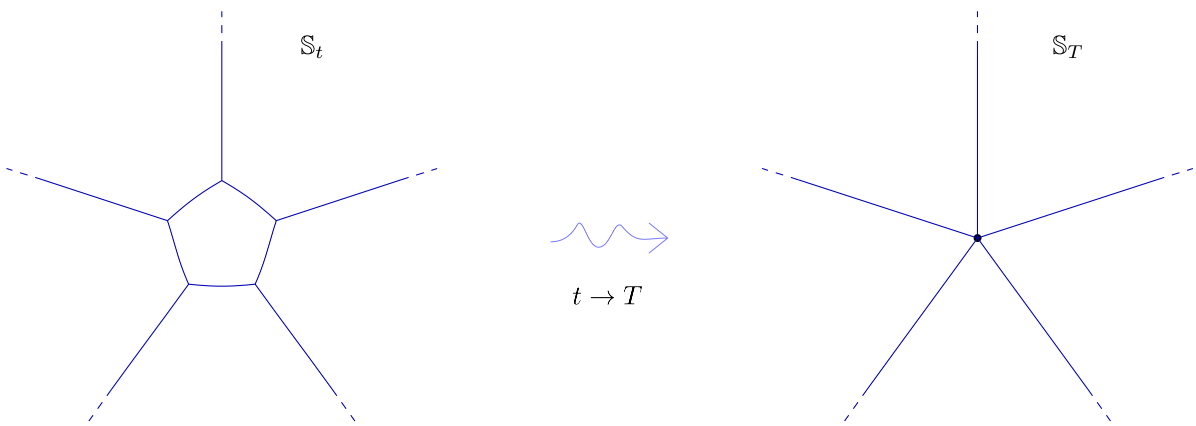


FIGURE 16. Homothetic collapse of a (symmetric) pentagonal region

PROPOSITION 8.35. *Let* $\mathbb{S}_0$ *be a network with a loop* ℓ *of length* L, *composed by less than six curves, enclosing a region of area* A *and let* $\mathbb{S}_t$ *be a smooth evolution by curvature of such network in the maximal time interval* $[0, T)$. *Then,* T *is finite and if* $\lim_{t\to T} L(t) = 0$, *there holds* $\lim_{t\to T} \int_{\mathbb{S}_t} k^2\, ds = +\infty$.

PROOF. If a loop composed of m curves, with $m < 6$, is present, integrating in time equation (7.3), we have

$$A(t) - A(0) = \left(-2\pi + m\left(\frac{\pi}{3}\right)\right) t\,,$$

therefore, $T \leq \frac{3A(0)}{(6-m)\pi}$, otherwise a region of the network collapses before the maximal time, which is impossible.

If $L(t) \to 0$ as $t \to T$, also the area $A(t)$ of the region enclosed in the loop must go to zero and $T = \frac{3A(0)}{(6-m)\pi}$. Then, combining equation (7.3) and Hölder inequality, one gets

$$\left| -2\pi + m\left(\frac{\pi}{3}\right)\right| = \left|\frac{dA(t)}{dt}\right| = \left|\int_{\ell_t} k\, ds\right| \leq (L(t))^{\frac{1}{2}} \left(\int_{\ell_t} k^2\, ds\right)^{\frac{1}{2}}\,,$$

hence,

$$\int_{\mathbb{S}_t} k^2\, ds \geq \int_{\ell_t} k^2\, ds \geq \frac{(6-m)^2\,\pi^2}{9L(t)}\,.$$

Then clearly, when $t \to T$, as $L(t) \to 0$, the L^2–norm of the curvature goes to infinity. $\square$

Determining what happens in the generic case can be quite complicated, because of the difficulty in classifying the shrinkers with loops, anyway, some special cases with "few" triple junctions can be fully understood. We will show some examples of this analysis in Section 10, considering networks with only one triple junction or with two 3–points. We underline that the interest in this latter case is due to the fact that the multiplicity–one conjecture holds for such networks (Corollary 9.11).

9. A geometric quantity

Given the smooth flow $\mathbb{S}_t = F(\mathbb{S}, t)$, we take two points $p = F(x,t)$ and $q = F(y,t)$ belonging to $\mathbb{S}_t$. A couple $(p = F(x,t), q = F(y,t))$ is in the class $\mathfrak{A}$ of the admissible ones if the segment joining p and q does not intersect the network $\mathbb{S}_t$ in other points. Moreover if the network $\mathbb{S}_t$ has more than one connected component, we take the two points p and q in the same connected component.
Given an admissible pair $(p = F(x,t), q = F(y,t))$ we consider the set of the embedded curves $\Gamma_{p,q}$ contained in $\mathbb{S}_t$ connecting p and q, forming with the segment $\overline{pq}$ a Jordan curve. Thus, it is well defined the area of the open region $\mathcal{A}_{p,q}$ enclosed by any Jordan curve constructed in this way and, for any pair (p,q), we call $A_{p,q}$ the smallest area of all such possible regions $\mathcal{A}_{p,q}$. If p and q are both points of a set of curves forming a loop, we define $\psi(A_{p,q})$ as

$$\psi(A_{p,q}) = \frac{A}{\pi}\sin\left(\frac{\pi}{A}A_{p,q}\right) \,,$$

where $A = A(t)$ is the area of the connected component of $\Omega \setminus \mathbb{S}_t$ which contains the open segment joining p and q.
We consider the function $\Phi_t : \mathbb{S} \times \mathbb{S} \to \mathbb{R} \cup \{+\infty\}$ as

$$\Phi_t(x,y) = \begin{cases} \frac{|p-q|^2}{\psi(A_{p,q})} & \text{if } x \neq y \text{ and } x, y \text{ are points of a loop;} \\ \frac{|p-q|^2}{A_{p,q}} & \text{if } x \neq y \text{ and } x, y \text{ are not both points of a loop;} \\ 4\sqrt{3} & \text{if } x \text{ and } y \text{ coincide with one of the 3–points } O^i \text{ of } \mathbb{S}; \\ +\infty & \text{if } x = y \neq O^i; \end{cases}$$

where $p = F(x,t)$ and $q = F(y,t)$.

REMARK 9.1. Following the argument of Huisken in [**49**], in the definition of the function Φ_t we introduce the function $\psi(A_{p,q})$, when the two points belong to a loop, because we want to maintain the function smooth also when $A_{p,q}$ is equal to $A/2$.

REMARK 9.2. As we have already said, if the network $\mathbb{S}_t$ has more than one connected component, we consider couple of points (p,q) belonging both to the same connected component. Indeed, if we take the points in two different connected component, the Jordan curve and the area enclosed in it are not defined and the function Φ_t has no meaning.

In the following, with a little abuse of notation, we consider the function Φ_t defined on $\mathbb{S}_t \times \mathbb{S}_t$ and we speak of admissible pair for the couple of points $(p,q) \in \mathbb{S}_t \times \mathbb{S}_t$ instead of $(x,y) \in \mathbb{S} \times \mathbb{S}$.
We define $E(t)$ as the infimum of Φ_t between all admissible couple of points $p = F(x,t)$ and $q = F(y,t)$:

$$E(t) = \inf_{(p,q) \in \mathfrak{A}} \Phi_t \tag{9.1}$$

for every $t \in [0,T)$.
We call $E(t)$ "embeddedness measure". We underline that similar geometric quantities have already been applied to similar problems in [**25, 44, 49**].
The following lemma holds, for its proof in the case of a compact network see [**25**, Theorem 2.1].

LEMMA 9.3. *The infimum of the function Φ_t between all admissible couples (p, q) is actually a minimum. Moreover, assuming that $0 < E(t) < 4\sqrt{3}$, for any minimizing pair (p, q) we have $p \neq q$ and neither p nor q coincides with one of the 3–points $O^i(t)$ of $\mathbb{S}_t$.*

REMARK 9.4. In the case of an open network without end–points, since the network is asymptotically C^1–close to a family of halflines (and during its curvature motion such halflines are fixed), there holds that if the infimum of Φ_t is less than a "structural" constant depending only on such halflines, then it is a minimum. By means of such modification to this lemma, all the rest of the analysis of this chapter also holds for the evolution of open networks, we let the details and the easy modifications of the arguments to the reader.

Notice that it follows that the network $\mathbb{S}_t$ is embedded *if and only if* $E(t) > 0$. Moreover, $E(t) \leq 4\sqrt{3}$ always holds, thus when $E(t) > 0$ the two points (p, q) of a minimizing pair can coincide if and only if $p = q = O^i(t)$.
Finally, since the evolution is smooth, it is easy to see that the function $E : [0, T) \to \mathbb{R}$ is locally Lipschitz, in particular, $\frac{dE(t)}{dt} > 0$ exists for almost every time $t \in [0, t)$.

If the network flow $\mathbb{S}_t$ has fixed end–points $\{P^1, P^2, \dots, P^l\}$ on the boundary of a strictly convex set Ω, we consider the flows $\mathbb{H}_t^i$ each obtained as the union of $\mathbb{S}_t$ with its reflection $\mathbb{S}_t^{R_i}$ with respect to the end–point P^i, as we described in the discussion just before Section 8.1.
We recall that this is still a smooth flow without self–intersections, where P^i is no more and end–point and the number of triple junctions of $\mathbb{H}_t^i$ is exactly twice the number of $\mathbb{S}_t$.

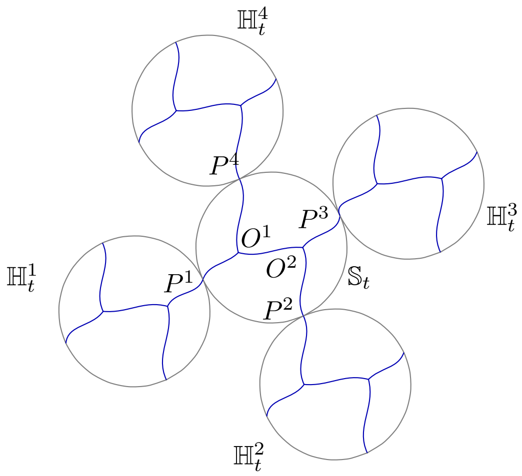


FIGURE 17. A tree–shaped network $\mathbb{S}_t$ with the associated networks $\mathbb{H}_t^i$.

We define for the networks $\mathbb{H}_t^i$ the functions $E^i : [0, T) \to \mathbb{R}$, analogous to the function $E : [0, T) \to \mathbb{R}$ of $\mathbb{S}_t$ and, for every $t \in [0, T)$, we call $\Pi(t)$ the minimum of the values $E^i(t)$. The function $\Pi : [0, T) \to \mathbb{R}$ is still a locally Lipschitz function (hence, differentiable for almost every time), clearly satisfying $\Pi(t) \leq E^i(t) \leq E(t)$ for all $t \in [0, T)$. Moreover, as there are no self–intersections, by construction, we have $\Pi(0) > 0$. If we prove that $\Pi(t) \geq C > 0$ for all $t \in [0, T)$, form some constant $C \in \mathbb{R}$, then, we can conclude that also $E(t) \geq C > 0$, for all $t \in [0, T)$.

THEOREM 9.5. *Let Ω be a open, bounded, strictly convex subset of $\mathbb{R}^2$. Let $\mathbb{S}_0$ be an initial regular network with at most two triple junctions and let the $\mathbb{S}_t$ be a smooth evolution by curvature of $\mathbb{S}_0$, defined in a maximal time interval $[0, T)$.*
Then, there exists a constant $C > 0$ depending only on $\mathbb{S}_0$ such that $E(t) \geq C > 0$, for every $t \in [0, T)$. In particular, the networks $\mathbb{S}_t$ remain embedded during the flow.

To prove this theorem we first show the next proposition and lemma.

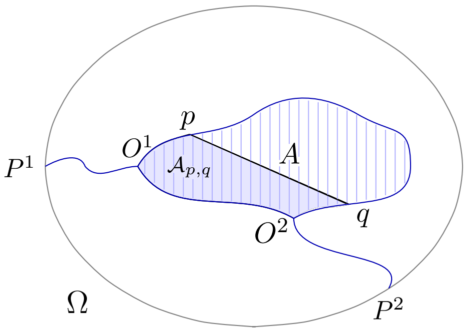


FIGURE 18. The situation considered in the following computations.

PROPOSITION 9.6. *At every point $t \in [0,T)$ such that $0 < E(t) < 1/4$ and for at least one minimizing pair (p,q) of Φ_t the curve $\Gamma_{p.q}$ contains at most two triple junction and neither p nor q coincides with one of the end–points P^i, if the derivative $\frac{dE(t)}{dt}$ exists, it is positive.*

PROOF. By simplicity, we consider in detail only the case shown in Figure 18. The computations in the other situations are analogous. Let $0 < E(t) < 1/4$ and let (p,q) a minimizing pair for Φ_t such that the two points are both distinct from the end–points P^i. We choose a value $\varepsilon > 0$ smaller than the "geodesic" distances of p and q from the 3–point O of $\mathbb{S}_t$ and between them, moreover if p and q both belong to the same curve we can also suppose that q is the closest to O. Possibly taking a smaller $\varepsilon > 0$, we fix an arclength coordinate $s \in (-\varepsilon,\varepsilon)$ and a local parametrization $p(s)$ of the curve containing p such that $p(0) = p$, with the same orientation of the original one. Let $\eta(s) = |p(s) - q|$, since

$$E(t) = \min_{s\in(-\varepsilon,\varepsilon)} \frac{\eta^2(s)}{\psi(A_{p(s),q})} = \frac{\eta^2(0)}{\psi(A_{p,q})},$$

if we differentiate in s we obtain

$$\frac{d\eta^2(0)}{ds}\psi(A_{p(0),q}) = \frac{d\psi(A_{p(0),q})}{ds}\eta^2(0). \tag{9.2}$$

We underline that we are considering the function ψ because we are doing all the computation for the case shown in Figure 18, where there is a loop. For a network without loops the computations are simpler: instead of formula (9.2), one has

$$\frac{d\eta^2(0)}{ds}A_{p(0),q} = \frac{dA_{p(0),q}}{ds}\eta^2(0),$$

see [**71**, Page 281], for instance.

As the intersection of the segment $\overline{pq}$ with the network is transversal, we have an angle $\alpha(p) \in (0,\pi)$ determined by the unit tangent $\tau(p)$ and the vector $q - p$.

We compute

$$
\begin{aligned}
\frac{d\eta^2(s)}{ds}\bigg|_{s=0} &= -2\langle \tau(p)\,|\,q-p\rangle = -2|p-q|\cos\alpha(p)\\
\frac{dA(s)}{ds}\bigg|_{s=0} &= 0\\
\frac{dA_{p(s),q}}{ds}\bigg|_{s=0} &= \frac{1}{2}|\tau(p)\wedge(q-p)| = \frac{1}{2}\langle \nu(p)\,|\,q-p\rangle = \frac{1}{2}|p-q|\sin\alpha(p)\\
\frac{d\psi(A_{p(s),q})}{ds}\bigg|_{s=0} &= \frac{dA_{p,q}}{ds}\cos\Big(\frac{\pi}{A}A_{p,q}\Big)\\
&= \frac{1}{2}|p-q|\sin\alpha(p)\cos\Big(\frac{\pi}{A}A_{p,q}\Big)\,.
\end{aligned}
$$

Putting these derivatives in equation (9.2) and recalling that $\eta^2(0)/\psi(A_{p,q}) = E(t)$, we get

$$
\cot\alpha(p) = -\frac{|p-q|^2}{4\psi(A_{p,q})}\cos\Big(\frac{\pi}{A}A_{p,q}\Big) = -\frac{E(t)}{4}\cos\Big(\frac{\pi}{A}A_{p,q}\Big)\,. \tag{9.3}
$$

Since $0 < E(t) < \frac{1}{4} < 4(2-\sqrt{3})$, we have $\sqrt{3}-2 < \cot\alpha(p) < 0$, which implies

$$
\frac{\pi}{2} < \alpha(p) < \frac{7\pi}{12}\,. \tag{9.4}
$$

The same argument clearly holds for the point q, hence defining $\alpha(q) \in (0,\pi)$ to be the angle determined by the unit tangent $\tau(q)$ and the vector $p-q$, by equation (9.3) it follows that $\alpha(p) = \alpha(q)$ and we simply write α for both.
We consider now a different variation, moving at the same time the points p and q, in such a way that $\frac{dp(s)}{ds} = \tau(p(s))$ and $\frac{dq(s)}{ds} = \tau(q(s))$.
As above, letting $\eta(s) = |p(s)-q(s)|$, by minimality we have

$$
\begin{aligned}
\frac{d\eta^2(0)}{ds}\,\psi(A_{p(s),q(s)})\big|_{s=0} &= \left(\frac{d\psi(A_{p(s),q(s)})}{ds}\bigg|_{s=0}\right)\eta^2(0) \quad \text{and}\\
\frac{d^2\eta^2(0)}{ds^2}\,\psi(A_{p(s),q(s)})\big|_{s=0} &\geq \left(\frac{d^2\psi(A_{p(s),q(s)})}{ds^2}\bigg|_{s=0}\right)\eta^2(0)\,.
\end{aligned}\tag{9.5}
$$

Computing as before,

$$
\begin{aligned}
\frac{d\eta^2(s)}{ds}\Big|_{s=0} &= 2\langle p-q\,|\,\tau(p)-\tau(q)\rangle = -4|p-q|\cos\alpha\\
\frac{dA_{p(s),q(s)}}{ds}\Big|_{s=0} &= -\frac{1}{2}\langle p-q\,|\,\nu(p)+\nu(q)\rangle = +|p-q|\sin\alpha\\
\frac{d^2\eta^2(s)}{ds^2}\Big|_{s=0} &= 2\langle \tau(p)-\tau(q)\,|\,\tau(p)-\tau(q)\rangle + 2\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle\\
&= 2|\tau(p)-\tau(q)|^2 + 2\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle\\
&= 8\cos^2\alpha + 2\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle\\
\frac{d^2A_{p(s),q(s)}}{ds^2}\Big|_{s=0} &= -\frac{1}{2}\langle \tau(p)-\tau(q)\,|\,\nu(p)+\nu(q)\rangle + \frac{1}{2}\langle p-q\,|\,k(p)\tau(p)+k(q)\tau(q)\rangle\\
&= -\frac{1}{2}\langle \tau(p)\,|\,\nu(q)\rangle + \frac{1}{2}\langle \tau(q)\,|\,\nu(p)\rangle\\
&\quad+\frac{1}{2}\langle p-q\,|\,k(p)\tau(p)+k(q)\tau(q)\rangle\\
&= -2\sin\alpha\cos\alpha - 1/2|p-q|(k(p)-k(q))\cos\alpha\\
\frac{d^2\psi(A_{p(s),q(s)})}{ds^2}\Big|_{s=0} &= \frac{d}{ds}\left\{\frac{dA_{p(s),q(s)}}{ds}\cos\Big(\frac{\pi}{A}A_{p(s),q(s)}\Big)\right\}\Big|_{s=0}\\
&= (-2\sin\alpha\cos\alpha - \frac{1}{2}|p-q|(k(p)-k(q))\cos\alpha)\cos\Big(\frac{\pi}{A}A_{p,q}\Big)\\
&\quad-\frac{\pi}{A}|p-q|^2\sin^2\alpha\sin\Big(\frac{\pi}{A}A_{p,q}\Big)\,.
\end{aligned}
$$

Substituting the last two relations in inequality (9.5), we get

$$
\begin{aligned}
&(8\cos^2\alpha + 2\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle)\psi(A_{p,q})\\
&\geq |p-q|^2\left\{(-2\sin\alpha\cos\alpha - \frac{1}{2}|p-q|(k(p)-k(q))\cos\alpha)\cos\Big(\frac{\pi}{A}A_{p,q}\Big)\right.\\
&\left.-\frac{\pi}{A}|p-q|^2\sin^2\alpha\sin\Big(\frac{\pi}{A}A_{p,q}\Big)\right\}\,,
\end{aligned}
$$

hence, keeping in mind that $\tan\alpha = \frac{-4}{E(t)\cos\left(\frac{\pi}{A}A_{p(s),q(s)}\right)}$, we obtain

$$
\begin{aligned}
2\psi(A_{p,q})\langle p-q\,|\; k(p)\nu(p)-k(q)\nu(q)\rangle &+ 1/2|p-q|^3(k(p)-k(q))\cos\alpha\cos\Big(\frac{\pi}{A}A_{p,q}\Big)\\
\geq&\, -2\sin\alpha\cos\alpha|p-q|^2\cos\Big(\frac{\pi}{A}A_{p,q}\Big)\\
&-8\psi(A_{p,q})\cos^2\alpha + |p-q|^4\sin^2\alpha\left[-\frac{\pi}{A}\sin\Big(\frac{\pi}{A}A_{p,q}\Big)\right]\\
=&\, -2\psi(A_{p,q})\cos^2\alpha\left(\tan\alpha\frac{|p-q|^2}{\psi(A_{p,q})}\cos\Big(\frac{\pi}{A}A_{p,q}\Big)+4\right)\\
&+|p-q|^4\sin^2\alpha\left[-\frac{\pi}{A}\sin\Big(\frac{\pi}{A}A_{p,q}\Big)\right]\\
=&\,|p-q|^4\sin^2\alpha\left[-\frac{\pi}{A}\sin\Big(\frac{\pi}{A}A_{p,q}\Big)\right]\,.
\end{aligned}
\tag{9.6}
$$

We now compute the derivative $\frac{dE(t)}{dt}$ by means of the Hamilton's trick (see **[43]** or **[68**, Lemma 2.1.3]), that is,

$$
\frac{dE(t)}{dt} = \frac{\partial}{\partial t}\Phi_t(\overline{p},\overline{q})\,,
$$

for *any* minimizing pair $(\overline{p},\overline{q})$ for Φ_t. In particular, $\frac{dE(t)}{dt} = \frac{\partial}{\partial t}\Phi_t(p,q)$ and, we recall, $\frac{|p-q|^2}{\psi(A_{p,q})} = E(t)$.

Notice that by minimality of the pair (p, q), we are free to choose the "motion" of the points $p(s)$, $q(s)$ "inside" the networks Γ_s in computing such partial derivative, that is,

$$\frac{dE(t)}{dt} = \frac{\partial}{\partial t}\Phi_t(p,q) = \frac{d}{ds}\Phi_t(p(s),q(s))\Big|_{s=t}.$$

Since locally the networks are moving by curvature and we know that neither p nor q coincides with the 3–point, we can find $\varepsilon > 0$ and two smooth curves $p(s), q(s) \in \Gamma_s$ for every $s \in (t-\varepsilon, t+\varepsilon)$ such that

$$\begin{aligned} p(t) &= p \qquad \text{and} \qquad \frac{dp(s)}{ds} = k(p(s),s)\,\nu(p(s),s)\,,\\ q(t) &= q \qquad \text{and} \qquad \frac{dq(s)}{ds} = k(q(s),s)\,\nu(q(s),s)\,. \end{aligned}$$

Then,

$$\frac{dE(t)}{dt} = \frac{\partial}{\partial t}\Phi_t(p,q) = \frac{1}{[\psi(A_{p,q})]^2}\left(\psi(A_{p,q})\frac{d|p(s)-q(s)|^2}{ds} - |p-q|^2\frac{d\psi(A_{p(s),q(s)})}{ds}\right)\Big|_{s=t}. \tag{9.7}$$

With a straightforward computation, we get the following equalities,

$$\begin{aligned} \frac{d|p(s)-q(s)|^2}{ds}\Big|_{s=t} &= 2\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle\\ \frac{dA(s)}{ds}\Big|_{s=t} &= -\frac{4\pi}{3}\\ \frac{dA_{p(s),q(s)}}{ds}\Big|_{s=t} &= \int_{\Gamma_{p,q}}\langle \underline{k}(s)\,|\nu_{\xi_{p,q}}\rangle\,ds + \frac{1}{2}|p-q|\langle\nu_{[p,q]}\,|\,k(p)\nu(p)+k(q)\nu(q)\rangle\\ &= 2\alpha - \frac{4\pi}{3} - \frac{1}{2}|p-q|(k(p)-k(q))\cos\alpha\\ \frac{d\psi(A_{p(s),q(s)})}{ds}\Big|_{s=t} &= -\frac{4\pi}{3}\left[\frac{1}{\pi}\sin\left(\frac{\pi}{A}A_{p,q}\right) - \frac{A_{p,q}}{A}\cos\left(\frac{\pi}{A}A_{p,q}\right)\right]\\ &\quad+\left(2\alpha - \frac{4\pi}{3} - \frac{1}{2}|p-q|(k(p)-k(q))\cos\alpha\right)\cos\left(\frac{\pi}{A}A_{p,q}\right) \end{aligned}$$

where we wrote $\nu_{\xi_{p,q}}$ and $\nu_{[p,q]}$ for the exterior unit normals to the region $A_{p,q}$, respectively at the points of the geodesic $\xi_{p,q}$ and of the segment $\overline{pq}$.
We remind that in general $\frac{dA(t)}{dt} = -(2-m/3)\pi$ where m is the number of triple junctions of the loop (see formula (7.3)), hence, we have $\frac{dA(t)}{dt} = -\frac{4\pi}{3}$, since we are referring to the situation in Figure 18, where there is a loop with exactly two triple junctions.
Substituting these derivatives in equation (9.7) we get

$$\begin{aligned} \frac{dE(t)}{dt} =& \frac{2\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle}{\psi(A_{p,q})}\\ &- \frac{|p-q|^2}{[\psi(A_{p,q})]^2}\left\{-\frac{4\pi}{3}\left[\frac{1}{\pi}\sin\left(\frac{\pi}{A}A_{p,q}\right) - \frac{A_{p,q}}{A}\cos\left(\frac{\pi}{A}A_{p,q}\right)\right]\right.\\ &\left.+\left(2\alpha - \frac{4\pi}{3} - \frac{1}{2}|p-q|(k(p)-k(q))\cos\alpha\right)\cos\left(\frac{\pi}{A}A_{p,q}\right)\right\} \end{aligned}$$

and, by equation (9.6),

$$\begin{aligned}\frac{dE(t)}{dt} \geq &- \frac{|p-q|^2}{[\psi(A_{p,q})]^2}\left\{-\frac{4}{3}\sin\left(\frac{\pi}{A}A_{p,q}\right)+\frac{4\pi}{3}\frac{A_{p,q}}{A}\cos\left(\frac{\pi}{A}A_{p,q}\right)\right.\\ &\left.+\left(2\alpha-\frac{4\pi}{3}\right)\cos\left(\frac{\pi}{A}A_{p,q}\right)+\frac{\pi}{A}|p-q|^2\sin^2(\alpha)\sin\left(\frac{\pi}{A}A_{p,q}\right)\right\}.\end{aligned}$$

It remains to prove that the quantity

$$\begin{aligned}&\frac{4}{3}\sin\left(\frac{\pi}{A}A_{p,q}\right)-\frac{4\pi}{3}\frac{A_{p,q}}{A}\cos\left(\frac{\pi}{A}A_{p,q}\right)+\left(\frac{4\pi}{3}-2\alpha\right)\cos\left(\frac{\pi}{A}A_{p,q}\right)\\ &-\frac{\pi}{A}|p-q|^2\sin^2(\alpha)\sin\left(\frac{\pi}{A}A_{p,q}\right)\end{aligned}$$

is positive.
As $E(t)=\frac{|p-q|^2}{\psi(A_{p,q})}=\frac{|p-q|^2}{\frac{A}{\pi}\sin(\frac{\pi}{A}A_{p,q})}$, we can write

$$\begin{aligned}&\frac{4}{3}\sin\left(\frac{\pi}{A}A_{p,q}\right)-\frac{4\pi}{3}\frac{A_{p,q}}{A}\cos\left(\frac{\pi}{A}A_{p,q}\right)+\left(\frac{4\pi}{3}-2\alpha\right)\cos\left(\frac{\pi}{A}A_{p,q}\right)\\ &-\frac{\pi}{A}|p-q|^2\sin^2(\alpha)\sin\left(\frac{\pi}{A}A_{p,q}\right)\\ =&\frac{4}{3}\sin\left(\frac{\pi}{A}A_{p,q}\right)-\frac{4\pi}{3}\frac{A_{p,q}}{A}\cos\left(\frac{\pi}{A}A_{p,q}\right)+\left(\frac{4\pi}{3}-2\alpha\right)\cos\left(\frac{\pi}{A}A_{p,q}\right)\\ &-E(t)\sin^2(\alpha)\sin^2\left(\frac{\pi}{A}A_{p,q}\right).\end{aligned}$$

Notice that using inequality (9.4), we can evaluate $\frac{4\pi}{3}-2\alpha\in(\pi/6,\pi/3)$, in particular, it is positive. We finally conclude the estimate of $\frac{dE(t)}{dt}$ and the proof of this proposition by separating the analysis in two cases, depending on the value of $\frac{A_{p,q}}{A}$.
If $0\leq\frac{A_{p,q}}{A}\leq\frac{1}{3}$, we have

$$\begin{aligned}\frac{dE(t)}{dt}\geq&\frac{4}{3}\sin\left(\frac{\pi}{A}A_{p,q}\right)-\frac{4\pi}{3}\frac{A_{p,q}}{A}\cos\left(\frac{\pi}{A}A_{p,q}\right)\\ &+\left(\frac{4\pi}{3}-2\alpha\right)\cos\left(\frac{\pi}{A}A_{p,q}\right)-E(t)\sin^2(\alpha)\sin^2\left(\frac{\pi}{A}A_{p,q}\right)\\ \geq&\left(\frac{4\pi}{3}-2\alpha\right)\cos\left(\frac{\pi}{A}A_{p,q}\right)-E(t)\sin^2(\alpha)\sin^2\left(\frac{\pi}{A}A_{p,q}\right)\\ \geq&\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{3}\right)-E(t)\sin^2\left(\frac{\pi}{3}\right)>0.\end{aligned}$$

If $\frac{1}{3}\leq\frac{A_{p,q}}{A}\leq\frac{1}{2}$, we get

$$\begin{aligned}\frac{dE(t)}{dt}\geq&\frac{4}{3}\sin\left(\frac{\pi}{A}A_{p,q}\right)-\frac{4\pi}{3}\frac{A_{p,q}}{A}\cos\left(\frac{\pi}{A}A_{p,q}\right)\\ &+\left(\frac{4\pi}{3}-2\alpha\right)\cos\left(\frac{\pi}{A}A_{p,q}\right)-E(t)\sin^2(\alpha)\sin^2\left(\frac{\pi}{A}A_{p,q}\right)\\ \geq&\frac{4}{3}\sin\left(\frac{\pi}{A}A_{p,q}\right)-\frac{4\pi}{3}\frac{A_{p,q}}{A}\cos\left(\frac{\pi}{A}A_{p,q}\right)-E(t)\sin^2(\alpha)\sin^2\left(\frac{\pi}{A}A_{p,q}\right)\\ \geq&\frac{4}{3}\left(\sin\left(\frac{\pi}{3}\right)-\frac{\pi}{3}\cos\left(\frac{\pi}{3}\right)\right)-E(t)>0.\end{aligned}$$

□

REMARK 9.7. We want to stress here the reason why we are able to prove Proposition 9.6 only when $\Gamma_{p,q}$ contains at most two triple junctions and so Theorem 9.5 only for networks with at most two 3–points. If we try to repeat the computations of the final part of this proof considering a situation such that $\Gamma_{p,q}$ contains more than two triple junctions, as the value of $\frac{dA(t)}{dt}$ changes according to $\frac{dA(t)}{dt} = -(2-m/3)\pi$, when $m \geq 3$, we only have $\frac{dA(t)}{dt} \geq -\pi$ (instead of being equal to $-4\pi/3$), which is not sufficient to get to the inequality $\frac{dE(t)}{dt} > 0$.

LEMMA 9.8. *Let Ω be a open, bounded, strictly convex subset of $\mathbb{R}^2$. Let $\mathbb{S}_0$ be an initial regular network with two triple junctions, and let the $\mathbb{S}_t$ be the evolution by curvature of $\mathbb{S}_0$ defined in a maximal time interval $[0,T)$. Then, there cannot be a sequence of times $t_j \to T$ such that, along such sequence, the two triple junctions converge to the same end–point of the network.*

PROOF. Let $O^1(t)$ and $O^2(t)$ be the two triple junctions of $\mathbb{S}_t$ and P^i the end–points on $\partial\Omega$. Suppose, by contradiction, that $\lim_{i\to\infty} O^j(t_i) = P^1$, for $j \in \{1,2\}$. Notice that if $\mathbb{S}_t$ is not a tree, then it has the structure either of a "lens/fish–shaped" network (see Figure 8) or of an "island–shaped" network (see Figure 26) . If we consider the sequence of rescaled networks $\widetilde{\mathbb{H}}^1_{P^1,\mathfrak{t}_j}$ obtained via Huisken's dynamical procedure applied to $\mathbb{H}^1_t$, as in Proposition 7.17, centered in P^1, it converges in $C^{1,\alpha}_{\mathrm{loc}} \cap W^{2,2}_{\mathrm{loc}}$, for any $\alpha \in (0,1/2)$ to a (not empty) limit degenerate regular shrinker $\widetilde{\mathbb{H}}_\infty$. We analyze the possible $\widetilde{\mathbb{H}}_\infty$ without using the multiplicity–one conjecture **M1**, to avoid a "circular argument". If the blow up limit $\widetilde{\mathbb{H}}_\infty$ is a line for the origin or four halflines forming angles of $120/60$ degrees, then, in both cases the curvature of the non–rescaled networks $\mathbb{H}_t$ (hence, of $\mathbb{S}_t$) is locally uniformly bounded around P^1 (by White's regularity theorem in **[91]** and Proposition 8.31, which are both independent of **M1**) and they (in the second case, by arguing as in Lemma 8.7) "forbid" the presence of another 3–point of $\mathbb{S}_t$ in a space–time neighborhood of (P^1, T), clearly contradicting the hypotheses.
If $\mathbb{S}_t$ contains a loop, it cannot vanish in the rescaling procedure, going to infinity (its curves would converge to two distinct halflines) or collapsing to a core (for instance in the origin), since the area of the region bounded by the rescaled loop in $\widetilde{\mathbb{S}}_{P^1,\mathfrak{t}}$ is constant in $\mathfrak{t}$ (see Section 7.2). Hence, such loop would be present in $\widetilde{\mathbb{H}}_\infty$ and actually could only have the structure of a "Brakke spoon" (see Figure 7) or of a "lens/fish–shaped" network (see Figure 9). It is then easy to see that, being part of a shrinker, it must contains the origin of $\mathbb{R}^2$ in its inside, which is clearly not possible in our situation. The last case we have to deal with is when $\mathbb{S}_t$ is a tree. It follows that also $\widetilde{\mathbb{S}}_\infty$ and so $\widetilde{\mathbb{H}}_\infty$ are trees and the same for the underlying graph is tree, then $\widetilde{\mathbb{H}}_\infty$ is a symmetric family of halflines from the origin, by Lemma 7.10.
The cases when $\widetilde{\mathbb{H}}_\infty$ has no core or the core is a single segment are the ones described above (we concluded that the other 3–points cannot tend to P^1, a contradiction). The remaining case is when the (symmetric) core of $\widetilde{\mathbb{H}}_\infty$ is given by three collapsed curves (and four 3–points) at the origin. In this case it is straightforward to see that $\widetilde{\mathbb{S}}_\infty$ contains a straight line for the origin, which is not possible since $\widetilde{\mathbb{S}}_\infty$ must be contained in an angle with opening less than π, by the strict convexity of Ω, as it is shown in Proposition 7.13. ☐

REMARK 9.9. We remark that the strictly convexity hypothesis on Ω can actually be weakened asking that Ω is convex and that there not exists three aligned end–points of the initial network $\mathbb{S}_0$ on $\partial\Omega$.

PROOF OF THEOREM 9.5. If $\mathbb{S}_t$ is the evolution of a network with only one triple junction, any of the evolving networks $\mathbb{H}^i_t$ has exactly two 3–points. Let $t \in [0,T)$ a time such that $0 < \Pi(t) < 1/4$ and Π and all embeddedness measures E^i, associated to the networks $\mathbb{H}^i_t$, are differentiable st t (this clearly holds for almost every time).
Let $E^i(t) = \Pi(t) < 1/4$ and $E^i(t)$ is realized by a pair of points p and q in $\mathbb{H}^i_t$, we separate the analysis in the following cases:

- If the points p and q of the minimizing pair are both end–points of $\mathbb{H}^i_t$, by construction $|p-q| \geq \varepsilon > 0$. Moreover, the area enclosed in the Jordan curve formed by the segment $\overline{pq}$ and by the geodesic curve $\Gamma_{p,q}$ can be uniformly bounded by above by a constant $\overline{C} > 0$, for instance, the area of a ball containing all the networks $\mathbb{H}^i_t$. Since $\varepsilon > 0$ and $\overline{C}$

depend only on Ω and on the structure of the initial network $\mathbb{S}_0$ (more precisely on the position of the end–points on the boundary of Ω, that stay fixed during the evolution and that do not coincide), the ratio $\frac{|p-q|^2}{\psi(A_{p,q})}$ (or $\frac{|p-q|^2}{A_{p,q}}$, if p, q do not belong to a loop) is greater of equal than some constant $C_\varepsilon = \frac{\varepsilon^2}{C} > 0$ uniformly, hence the same holds for $\Pi(t)$.

- If one point is internal and the other is an end–point of $\mathbb{H}^i_t$, we consider the following two situations. If one of the two point p and q is in $\mathbb{S}_t \subset \mathbb{H}^i_t$ and the other is in the reflected network $\mathbb{S}^{R_i}_t$, then, we obtain, by construction, a uniform bound from below on $\Pi(t)$ as in the case in which p and q are both boundary points of $\mathbb{H}^i_t$.
  Otherwise, if p and q are both in $\mathbb{S}_t$ and one of them coincides with P^j with $j \neq i$, either the other point coincides with P^i and we have again a uniform bound from below on $\Pi(t)$, as before, or both p and q are points of $\mathbb{H}^j_t$ both not coinciding with its end–points and $E^j(t) = E^i(t) = \Pi(t) < 1/4$, so we can apply the argument at the next point.
- If p and q are both "inside" $\mathbb{H}^i_t$, by Hamilton's trick (see [**43**] or [**68**, Lemma 2.1.3]), we have $\frac{d\Pi(t)}{dt} = \frac{dE^i(t)}{dt}$ and, by Proposition 9.6, $\frac{dE^i(t)}{dt} > 0$, hence $\frac{d\Pi(t)}{dt} > 0$.

All this discussion implies that at almost every point $t \in [0, T)$ such that $\Pi(t)$ is smaller than some uniform constant depending only on Ω and on the structure of the initial network $\mathbb{S}_0$, then $\frac{d\Pi(t)}{dt} > 0$, which clearly proves the theorem in the case a network with a single triple junction (see also [**71**, Section 4]).

Let now $\mathbb{S}_t$ be a flow of regular networks with *two* triple junctions. If there are no end–points, the conclusion follows immediately from Proposition 9.6. Hence, we assume that $\mathbb{S}_t$ has two or four end–points (in the first case there is a loop, in the second $\mathbb{S}_t$ is a tree), which are the only possibilities.
The analysis is the same as above, with only a delicate point to be addressed, that is, in the last case, when the two points p and q of the minimizing pair are "inside" $\mathbb{H}^i_t$ and we apply Proposition 9.6. Indeed, since $\mathbb{H}^i_t$ has *four* 3–points it can happen that the geodesic curve $\Gamma_{p.q}$ contains more than two 3–points, hence this case requires a special treatment. Notice that if the points p and q are both "inside" $\mathbb{S}_t \subset \mathbb{H}^i_t$, then Proposition 9.6 applies and we are done. We then assume that $p \in \mathbb{S}_t$, $q \in \mathbb{S}^{R_i}_t$, and $\Gamma_{p,q}$ contains more than two triple junctions.
We want to show that there exists a uniform positive constant ε such that $|p - q| \geq \varepsilon > 0$, which implies a uniform positive estimate from below on $E^i(t)$, as above. This will conclude the proof. Assume by contradiction that such a bound is not possible, then, for a sequence of times $t_j \to T$, the Euclidean distance between the two points p_j and q_j of the associated minimizing pair of Φ_{t_j} goes to zero, as $j \to \infty$, and this can happen only if $p_i, q_i \to P^i$. It follows, by the maximum principle that the two 3–points $O^1(t)$ and $O^2(t)$ converge to P^i on some sequence of times $t_k \to T$ (possibly different from t_j), which is forbidden by Lemma 9.8 and we are done. □

REMARK 9.10. Notice, by inspecting the previous proof, that in the case that $\mathbb{S}_t$ has a single 3–point, the strict convexity of Ω is not necessary, convexity is sufficient.

9.1. Consequences for the multiplicity–one conjecture. The quantity $E(t)$ considered in the previous section is clearly, by definition, dilation and translation invariant, moreover it is continuous under C^1_{loc}–convergence of networks. Hence, if $E(t) \geq C > 0$ for every $t \in [0, T)$, the same holds for every C^1_{loc}–limit of rescalings of networks of the flow $\mathbb{S}_t$. This clearly implies the multiplicity–one conjecture **M1**.

COROLLARY 9.11. *If Ω is strictly convex and the initial network $\mathbb{S}_0$ has at most two triple junctions, then the multiplicity–one conjecture* **M1** *is true for the flow $\mathbb{S}_t$.*

A by–product of the proofs of Proposition 9.6 and Theorem 9.5 is actually that also the function $\Pi(t)$ is positively uniformly bounded from below during the flow.

COROLLARY 9.12. *If Ω is strictly convex and the initial network $\mathbb{S}_0$ has at most two triple junctions, then the multiplicity–one conjecture* **M1** *is true for all the "symmetrized" flows $\mathbb{H}^i_t$.*

REMARK 9.13. Actually, in general, if we are able to show the multiplicity–one conjecture for a curvature flow $\mathbb{S}_t$ in a strictly convex open set Ω, then, by construction and Proposition 7.13, it also holds for all the "symmetrized" flows $\mathbb{H}^i_t$. This remark is in order since in the analysis of the flow $\mathbb{S}_t$ in the previous sections, we used the "reflection" argument at the end–points of the network $\mathbb{S}_t$, then we argued applying **M1** to the resulting networks $\mathbb{H}^i_t$ (to be precise, in Section 8.1 and in the proofs of Proposition 8.24 and of Theorem 8.30).

Another situation that can be analyzed by means of the ideas of this section is the following.

PROPOSITION 9.14. *If during the curvature flow of a tree $\mathbb{S}_t$ the triple junctions stay uniformly far from each other and from the end–points, then* **M1** *is true for the flows $\mathbb{S}_t$ and all $\mathbb{H}^i_t$. As a consequence, the evolution of $\mathbb{S}_t$ does not develop singularities at all.*

PROOF. We divide all the pairs of curves of the evolving tree $\mathbb{S}_t$ in two families, depending if the curve of a pair have a common 3–point or not. In the second case, by means of maximum principle and the assumption on the 3–points, there is a uniform constant $C > 0$ such that any couple of points, one on each curve of such pair, have distance bounded below by C. Then, if the pair of points of $\mathbb{S}_t$ realizing the quantity $E(t)$ stay on such curves it follows $E(t) \geq C' > 0$ for some uniform constant C'. In case $E(t) < C'$, it follows that such pair of points either stay on the same curve or on two curves with a common 3–point. Hence, *the* "geodesic" curve $\Gamma_{p,q}$ contains at most one 3–point, being $\mathbb{S}_t$ a tree. This implies that $\frac{dE(t)}{dt} > 0$, by Proposition 9.3. Then, the multiplicity–one conjecture follows for $\mathbb{S}_t$ and for all the "symmetrized" flows $\mathbb{H}^i_t$, by the same argument in the proof of Theorem 9.5, taking into account the hypothesis that the triple junctions stay uniformly far also from the end–points.
Then, the only possible singularities of the flow are given by the collapse of a curve of the network, but this is excluded by the hypotheses, hence the flow is smooth for all times. ☐

10. Examples

In this section we collect all the example of network for which either we are able to establish global existence or we have a complete characterization of the appearance of the first singularity. They are all the regular networks with at most two triple junctions in a regular, open and (strictly) convex set $\Omega \subset \mathbb{R}^2$.

As we have already seen in the previous sections a key point in the analysis of the onset of the singularities is the classification of the possible shrinkers. For networks with two triple junction the classification is complete thanks to the paper by Chen and Guo [**23**] and by the recent work by Baldi, Haus and Mantegazza [**15**], but unfortunately for more complicated topological structure (networks with more than two triple junctions) a classification of the possible shrinkers lacks.

Another great difficulty is to show the multiplicity–one conjecture (**M1**) (see Open Problem 8.1 and related discussions). In Section 9, borrowing ideas of Hamilton [**43**] and Huisken [**49**], we have introduced a geometric quantity which we used to prove **M1** for regular networks with at most two triple junctions in a strictly convex subset of $\mathbb{R}^2$. Indeed, thanks to the monotonicity of such quantity, we have excluded the presence of curves with multiplicity greater than one in the blow-up limits. Unfortunately, this line of proof cannot be extended to networks with more that two triple junctions, as this quantity is not monotone anymore.

In [**71, 67**] the authors study the evolution by curvature of a triod, that is, a network of three planar curves meeting at a triple junction. They prove that if the lengths of the three curves are bounded away from zero during the evolution, then the triod tends to the unique Steiner configuration connecting the three fixed end–points on $\partial\Omega$.

The simplest case of a network with a loop (a region bounded by one or more curves) is treat in [**81**]: consider a spoon, that is a network composed by two curves, one of them closed, meeting only at one triple junction; then in finite time either the closed curve shrinks to a point approaching the shape of a Brakke spoon (Figure 7) or the non–closed curve vanishes and the 3–point hit the boundary forming an angle of 120 degrees.

In **[69]** we consider networks with exactly two triple junctions and we obtain a complete description of the evolution till the appearance of the first singularity.

At the moment the triod and a tree-shaped network with two triple junctions are the only cases in which we are able to pass from local to global existence without requiring extra hypotheses.

10.1. Classification. If we consider the possible (topological) structures of regular networks with only one triple junction, we see that there are only two: the *triod* or the *spoon*. As the triod is the simplest configuration of an "essentially" singular one–dimensional set to let evolve by curvature, a spoon is the simplest case with a loop.

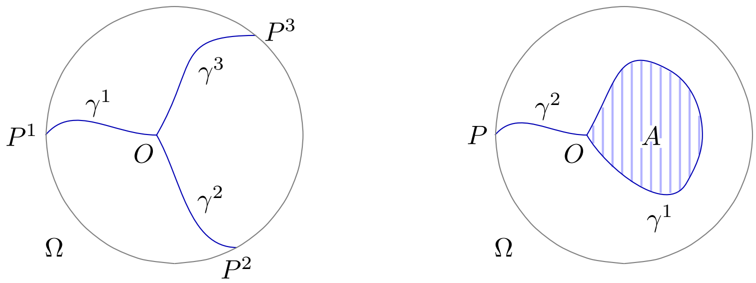


FIGURE 19. Networks with only one triple junction: the triod and the "spoon" network.

DEFINITION 10.1. Fixed a smooth, open, convex set $\Omega \subset \mathbb{R}^2$, a triod is a network (a tree) $\mathbb{T}$ composed only of three regular, embedded C^1 curves $\gamma^i : [0,1] \to \overline{\Omega}$. These curves intersects each other forming an angle of 120 degrees at a single 3–point O, that is, $\gamma^1(0) = \gamma^1(0) = \gamma^1(0) = O$, and have the other three end–points P^1, P^2, P^3 are fixed on the boundary of Ω with $\gamma^i(1) = P^i$, for $i \in \{1,2,3\}$.

DEFINITION 10.2. A spoon $\Gamma = \gamma^1([0,1]) \cup \gamma^2([0,1])$ is the union of two regular, embedded C^1 curves $\gamma^1, \gamma^2 : [0,1] \to \overline{\Omega}$ which intersect each other forming an angle of 120 degrees at a triple junction O, that is, $\gamma^1(0) = \gamma^1(1) = \gamma^2(0) = O \in \Omega$ and $\gamma^2(1) = P \in \partial\Omega$. We call γ^1 the "closed" curve and γ^2 the "open" curve of the spoon and we denote with A the area of the region enclosed in the loop given by γ^1.

We consider now regular networks with exactly two triple junctions and we focus on their topological classification.

We parametrize the curves composing the network with $\gamma^i : [0,1] \to \mathbb{R}^2$. In each 3–point either concur three different not closed curves (for instance $O^1 = \gamma^1(0) = \gamma^2(0) = \gamma^3(0)$) or two curves, one of which closed (that is $O^1 = \gamma^1(0) = \gamma^1(1) = \gamma^2(0)$). As now we do not consider open networks (networks with branches that go to infinity asymptotic to half-lines, see Definition 1.3), if a curve is not closed (hence $\gamma^1(0) \neq \gamma^1(1)$) there are only two possibilities for its end–point not concurring in O^1: either to be an end–point on the boundary of Ω, or to be in the other triple junction O^2. If we repeat the above reasoning for every end–point, we obtain all cases shown in Figure 20.

When we say that a network presents a loop ℓ in its structure, we mean that there is a Jordan curve in $\mathbb{S}$ that encloses an area A. For networks with two triple junctions, there are two cases (see Figure 20):

- the loop ℓ is composed by a single curve $\gamma : [0,1] \to \mathbb{R}^2$, $\gamma(0) = \gamma(1)$ and at this junction we have an angle of 120 degrees. The length L of ℓ coincides with the length of γ.
- the loop ℓ is composed by two curves $\gamma^1, \gamma^2 : [0,1] \to \mathbb{R}^2$, that meet each other at their end–points and at both junctions there is an angle of 120 degrees. The length L of ℓ is the sum of the lengths of the two curves.

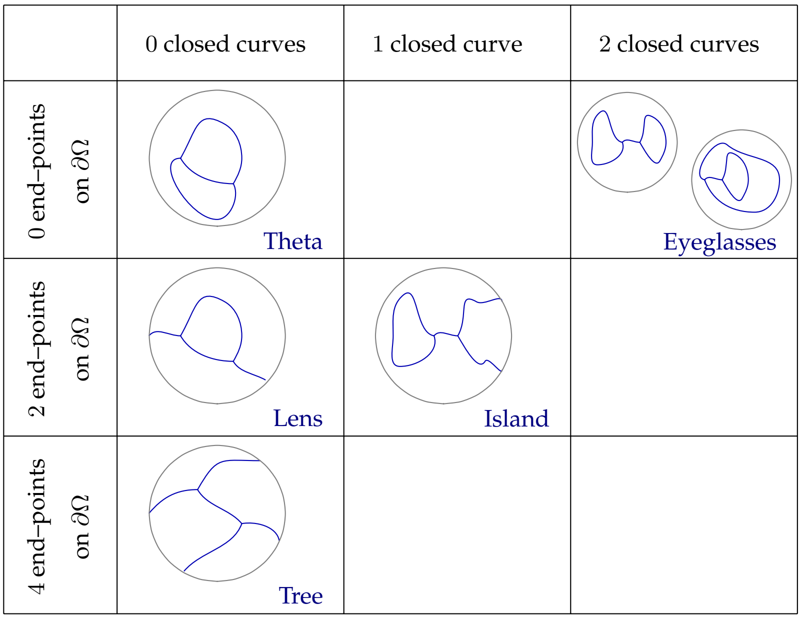


FIGURE 20. Networks with two triple junctions.

10.2. The triod. Suppose that $T < +\infty$, then, by Proposition 8.9 the lengths of the three curves cannot be uniformly positively bounded from below. Hence, as Ω is strictly convex, Corollary 8.32 imply that the curvature of $\mathbb{T}_t$ is uniformly bounded and there must be a "collapse" of a curve to a fixed end–point on $\partial\Omega$, when $t \to T$, as depicted in the right side of Figure 15.

Suppose instead that $T = +\infty$. Then, by Proposition 8.12, for every sequence of times $t_i \to \infty$, there exists a (not relabeled) subsequence such that the evolving triod $\mathbb{T}_{t_i}$ converge in C^1 to a possibly degenerate regular network with zero curvature, as $i \to \infty$, that is, a Steiner network connecting the three fixed points P^i on $\partial\Omega$ (which possibly have a zero–length degenerate curve, for instance if the three end–points are the vertices of a triangle with and angle of 120 degrees). Moreover, as the Steiner network connecting three points (and length minimizing) is unique, if it exists, for every subsequence of times we have the same limit network, hence, the *full* sequence of triods $\mathbb{T}_t$ converge to such limit, as $t \to +\infty$.

THEOREM 10.3. *For any smooth, embedded, regular initial triod $\mathbb{T}_0$ in a smooth, strictly convex open set $\Omega \subset \mathbb{R}^2$, with fixed end–points $P^1, P^2, P^3 \in \partial\Omega$, there exists a unique smooth evolution by curvature of $\mathbb{T}_0$ which at every time is a smooth embedded regular triod in Ω, in a maximal time interval $[0, T)$.*
If T is finite, then a curve collapses to an end–point, when $t \to T$ while the curvature remains bounded.
If $T = +\infty$, then the triods $\mathbb{T}_t$ tend, as $t \to +\infty$, to the unique Steiner (length minimizing) triod (possibly degenerate) connecting the three fixed end–points.

We notice that there is an obvious example where the length of one curve goes to zero in finite time: consider an initial triod $\mathbb{T}_0$ with the boundary points P^i on $\partial\Omega$ such that one angle of the triangle with vertices P^1, P^2, P^3 is greater than 120 degrees. In this case the Steiner triod does not exist, hence the maximal time of a smooth evolution must be finite. Instead if the angles of the triangle with vertices P^1, P^2, P^3 are smaller than 120 degrees and the initial triod $\mathbb{T}_0$ is contained in the convex envelop of P^1, P^2, P^3, then no lengths go to zero during the evolution, the maximal time of existence in ∞ and the the triods $\mathbb{T}_0$ tend, as $t \to +\infty$, to the unique Steiner configuration.

In the case that T is finite and a curve collapses to an end–point at the moment we are not able to restart the flow, even if the curvature is bounded.

10.3. The spoon. In Section 7.2 we discussed the behavior of the area of the bounded regions enclosed by an evolving regular network. In the case of the spoon, the loop is formed by only one curve and there is only one triple junction. Equation (7.3) then gives

$$A'(t) = -\frac{5\pi}{3}\,. \tag{10.1}$$

This implies that the maximal time T of existence of a smooth flow of a spoon is finite and

$$T \leq \frac{3A_0}{5\pi}\,, \tag{10.2}$$

where A_0 is the initial area enclosed in the loop (see Proposition 8.35).

As $t \to T$, the only possible limit regular shrinkers $\widetilde{\Gamma}_\infty$ arising from Huisken's rescaling procedure at a reachable point $x_0 \in \overline{\Omega}$ are given by

- a halfline from the origin,
- a straight line through the origin,
- a standard triod,
- a Brakke spoon (see Figure 7).

This follows by the simple topological structure of Γ_t and the uniqueness (up to rotation) of the Brakke spoon among the shrinkers in its topological class (see Section 7.6). We remind that all the possible blow–up limits are non–degenerate networks with multiplicity one, thank to Corollary 9.11.

We first notice that, if the curve γ^1 shrinks, then the curvature clearly cannot be bounded, hence, it is not possible that both lengths of γ^1 and γ^2 go to zero, as $t \to T$.

Suppose that the length of the "open" curve γ^2 is uniformly positively bounded from below, then the curve γ^1 must shrink and the maximum of the curvature goes to $+\infty$ as $t \to T$ (indeed, $\lim_{t\to T}\int_{\mathbb{S}_t} k^2\,ds = +\infty$, by Proposition 8.35). Then, if $x_0 = \lim_{t\to T} O(t)$, taking a blow–up $\widetilde{\Gamma}_\infty$ at $x_0 \in \Omega$ we can only get a Brakke spoon, since in the other cases (a halfline is obviously excluded) the curvature would be locally bounded and the flow regular. Hence, as $t \to T$, the length of the closed curve γ^1 goes to zero, the area $A(t)$ enclosed in the loop goes to zero at $T = \frac{3A_0}{5\pi}$, indeed $A(t) = A_0 - 5\pi t/3$ and Γ_t converges to a limit network composed only by the limit C^1 curve γ^2_T connecting P with x_0 (and curvature going as $o(1/d_{x_0})$, for more details see **[70]**).

If instead the length of the curve γ^2 is not positively bounded from below for hypothesis, then, as $t \to T$, such curve collapses to the end–point P, the curvature stays bounded and the network Γ_t is locally a tree around every point, uniformly in $t \in [0,T)$. Hence, the region enclosed by the curve γ^1 does not vanishes and the triple junction O has collapsed onto the boundary point P, maintaining the 120 degrees condition and bounded curvature (see Proposition 8.31). The networks Γ_t converge in C^1, as $t \to T$, to a limit network Γ_T.

THEOREM 10.4. *Consider a smooth, embedded, initial spoon Γ_0 in a smooth, strictly convex and open set $\Omega \subset \mathbb{R}^2$, with a fixed end–point $P \in \partial\Omega$, with initial area enclosed in the closed curve equal to A_0. Then there exists a smooth evolution by curvature Γ_t of Γ_0 in a maximal time interval $[0,T)$ with $T \leq \frac{3A_0}{5\pi}$, which at every time is a smooth embedded regular spoon in Ω.*
Moreover,

- *either the limit of the length of the curve that connects the 3–point the end–point P goes to zero, as $t \to T$, $T < \frac{3A_0}{5\pi}$, the curvature remains bounded and Γ_t converges to a limit C^1 network;*
- *or the lengths of the curve composing the loop goes to zero, as $t \to T$, in this case $T = \frac{3A_0}{5\pi}$, the area $A(t)$ of the bounded region goes to zero, $\lim_{t\to T}\int_{\mathbb{S}_t} k^2\,ds = +\infty$, the limit network Γ_t is composed by a single open C^1 curve γ^2_T.*

In the second case, at the "free" end–point $x_0 = \lim_{t\to T} O(t) \in \Omega$ of the limit curve γ^2_T, for a subsequence of rescaled times $\mathfrak{t}_j \to +\infty$ the associate rescaled networks $\widetilde{\Gamma}_{\mathfrak{t}_j}$ around x_0 tend in $C^1_{\mathrm{loc}} \cap W^{2,2}_{\mathrm{loc}}$ to a Brakke spoon, as $j \to \infty$.

At the moment we do not have a way to restart the flow in the first situation. In the second one, a natural "choice" is to assume that the flow ends and the whole network vanishes for $t > T$.

We conclude this example with a couple of open questions.

OPEN PROBLEM 10.5 (Special case of Problem 7.19). The limit Brakke spoon obtained in the previous theorem (in the second situation) is independent of the chosen sequence of times $t_k \to +\infty$? That is, the direction of its unbounded halfline is unique?

OPEN PROBLEM 10.6. Having in mind the "convexification" result for simple closed curves by Grayson (see **[41]**), a natural question is: if we consider an initial spoon moving by curvature with the length of the non–closed curve uniformly positively bounded below during the evolution, the closed curve becomes eventually convex and then remains convex?

These two open problems above are connected each other, since the uniqueness of the blow–up limit (which is a Brakke spoon, hence with a convex region) would imply that the region is definitely convex, by the smooth convergence of the rescaled networks to the Brakke spoon (this follows by the argument of Lemma 8.6 in **[53]**, see the discussion just after the proof of Lemma 8.7).

10.4. Analysis of singularities of networks with two triple junctions. In this section we first analyze the possible blow-up at a singular time of the evolution of a networks with two triple junctions of general topological type, then we discuss in detail the specific networks, case by case.

We consider the possible limits $\widetilde{\mathbb{S}}_\infty$ arising from Huisken's rescaling procedure. An important fact is that all the possible limits $\widetilde{\mathbb{S}}_\infty$ are embedded network with multiplicity–one by Corollary 9.11 in Section 9.

PROPOSITION 10.7. *If the rescaling point x_0 belongs to Ω, then the blow–up limit network $\widetilde{\mathbb{S}}_\infty$ (if not empty) is one of the following:*

- *a straight line through the origin;*
- *a standard triod centered at the origin;*
- *a Brakke spoon;*
- *four halflines from the origin forming angles in pair of* $120/60$ *degrees;*
- *a standard lens;*
- *a fish.*

If the rescaling point x_0 is a fixed end–point of the evolving network (on the boundary of Ω), then the blow–up limit network $\widetilde{\mathbb{S}}_\infty$ (if not empty) is one of the following:

- *a halfline from the origin;*
- *two halflines from the origin forming an angle of* 120 *degrees.*

PROOF. The limit (possibly degenerate) network $\widetilde{\mathbb{S}}_\infty$ has to satisfy the shrinkers equation $k_\infty + x^\perp = 0$ for all $x \in \widetilde{\mathbb{S}}_\infty$ (see the proof of Proposition 7.22).

If we assume that $\widetilde{\mathbb{S}}_\infty$ is a degenerate regular shrinkers, that is, a core is present, since there are only two 3–points, the only possibility is that a single curve (connecting the two triple junctions or a triple junction with an end–point, by Lemma 9.8) "collapses" in the limit forming such a core of $\widetilde{\mathbb{S}}_\infty$, which then must be composed by four halflines from the origin forming angles in pair of $120/60$ degrees, if $x_0 \in \Omega$, or by two halflines from the origin forming an angle of 120 degrees, when $x_0 \in \partial\Omega$.

If $\widetilde{\mathbb{S}}_\infty$ is not degenerate and the curvature k_∞ is constantly zero, the network is composed only by halflines or straight lines. Then, the possible flat regular shrinkers are either a straight line through the origin or a standard triod, if $x_0 \in \Omega$, or a halfline, if $x_0 \in \partial\Omega$.
If instead the curvature is not constantly zero and the network $\widetilde{\mathbb{S}}_\infty$ is not degenerate, by the classification of regular shrinkers with two triple junctions that we discussed in Section 7.1, we can only have either Brakke spoon, the standard lens or the fish. In all these three cases the center of the homothety is inside the enclosed region, hence x_0 cannot be an end–point on the boundary of Ω. $\square$

PROPOSITION 10.8. *Let $\mathbb{S}_0$ be a network with two triple junctions and with a loop ℓ of length L, enclosing a region of area A and let $\mathbb{S}_t$ be a smooth evolution by curvature of such network in the maximal time interval $[0,T)$. Then, T is finite and if $\lim_{t\to T} L(t)=0$, there holds $\lim_{t\to T}\int_{\mathbb{S}_t} k^2\,ds=+\infty$.*

PROOF. If a loop is present, by the classification of topological structures of the networks with two triple junctions, it must be composed of m curves, with $m<6$, hence, Proposition 8.35 applies. □

THEOREM 10.9. *Let $\Omega\subset\mathbb{R}^2$ be a smooth, strictly convex, open set. Let $\mathbb{S}_0$ be a compact initial network with two triple junctions and with possibly fixed end–points on $\partial\Omega$, and let $\mathbb{S}_t$ be the smooth evolution by curvature of $\mathbb{S}_0$ in a maximal time interval $[0,T)$.*

Then, if the network $\mathbb{S}_0$ has at least one loop, then the maximal time of existence T is finite and one of the following situations occurs:

(1) *the limit of the length of a curve that connects the two 3–points goes to zero as $t\to T$, and the curvature remains bounded;*
(2) *the limit of the length of a curve that connects the 3–point with an end–point goes to zero as $t\to T$, and the curvature remains bounded;*
(3) *the lengths of the curves composing the loop go to zero as $t\to T$, and $\lim_{t\to T}\int_{\mathbb{S}_t} k^2\,ds=+\infty$.*

If the network is a tree and T is finite, the curvature is uniformly bounded and only the first two situations listed above can happen. If instead $T=+\infty$, for every sequence of times $t_i\to+\infty$, there exists a subsequence (not relabeled) such that the evolving networks $\mathbb{S}_{t_i}$ converge in $C^{1,\alpha}\cap W^{2,2}$, for every $\alpha\in(0,1/2)$, to a possibly degenerate (see Definition 7.1) regular network with zero curvature (hence, "stationary" for the length functional), as $i\to\infty$.

PROOF. Let $\mathbb{S}_t$ be a smooth evolution by curvature of a network with two triple junctions and (possibly) fixed end–points on $\partial\Omega$, with Ω regular, open and strictly convex subset of $\mathbb{R}^2$, in a maximal time interval $[0,T)$.

If a loop is present, by Proposition 8.35, the maximal time of smooth existence T is finite. If such time T is smaller than the "natural" time that the loop shrinks (depending on the number of curves composing the loop, as in Proposition 8.35), the network is locally a tree, uniformly for $t\in[0,T)$. Hence, every blow–up limit at any point $x_0\in\overline{\Omega}$ cannot contain loops, then Proposition 10.7 shows that it must have zero curvature, thus, by Theorem 8.8 and Proposition 8.31 the curvature of $\mathbb{S}_t$ is uniformly bounded along the flow and (see Proposition 8.21) converges, as $t\to T$, to a degenerate regular network $\mathbb{S}_T$ with vertices that are either a regular triple junction, an end–point, or

- a 4–point where the four concurring curves have opposite unit tangents in pairs and form angles of $120/60$ degrees between them (collapse of the curve joining the two triple junctions of $\mathbb{S}_t$);
- a 2–point at an end–point of the network $\mathbb{S}_t$ where the two concurring curves form an angle of 120 degrees among them (collapse of the curve joining a triple junction to such end–point of $\mathbb{S}_t$).

The same conclusion clearly holds if $\mathbb{S}_0$ is a tree and T is finite.

If instead the time T coincides with the vanishing time of a loop of the network, by Proposition 8.35, the curvature is unbounded and there must exists a reachable point for the flow $x_0\in\Omega$ and a sequence of times $t_j\to T$ such that, the associate sequence of rescaled networks $\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_j}$, as in Proposition 7.22, converges in $C^{1,\alpha}_{\mathrm{loc}}\cap W^{2,2}_{\mathrm{loc}}$, for any $\alpha\in(0,1/2)$, to a limit degenerate regular shrinker $\widetilde{\mathbb{S}}_\infty$ which is either a Brakke spoon, or a standard lens or a fish.

If $T=+\infty$, hence $\mathbb{S}_0$ is a tree, then $\mathbb{S}_t$ converges, as $t\to+\infty$, to a regular network with zero curvature (a stationary point for the length functional). Indeed, as the total length of the network decreases, we have the estimate

$$\int_0^{+\infty}\int_{\mathbb{S}_t} k^2\,ds\,dt\leq L(0)<+\infty\,, \tag{10.3}$$

by the first equation in Proposition 11.1. Then, suppose by contradiction that for a sequence of times $t_j \nearrow +\infty$ we have $\int_{\mathbb{S}_{t_j}} k^2\,ds \geq \delta$ for some $\delta > 0$. By the following estimate, which is inequality 10.4 in Lemma 10.17 of [**70**],

$$\frac{d}{dt}\int_{\mathbb{S}_t} k^2\,ds \leq C\Big(1+\Big(\int_{\mathbb{S}_t} k^2\Big)\Big)^3,$$

holding (in the case of fixed end–points) with a uniform constant C independent of time, we would have $\int_{\mathbb{S}_{\widetilde{t}}} k^2\,ds \geq \frac{\delta}{2}$, for every $\widetilde{t}$ in a uniform neighborhood of every t_j. This is clearly in contradiction with the estimate (10.3). Hence, $\lim_{t\to+\infty}\int_{\mathbb{S}_t} k^2\,ds = 0$ and, consequently, for every sequence of times $t_i \to +\infty$, there exists a subsequence (not relabeled) such that the evolving networks $\mathbb{S}_{t_i}$ converge in $C^{1,\alpha}\cap W^{2,2}$, for every $\alpha \in (0, 1/2)$, to a possibly degenerate regular network with zero curvature (hence, “stationary” for the length functional), as $i \to \infty$. □

PROPOSITION 10.10. *Let $\mathbb{S}_0$ be a network with two triple junctions and without end–points on $\partial\Omega$ and $\mathbb{S}_t$ an evolution by curvature in $[0, T)$, with $T < +\infty$. Then, as $t \to T$, the total length of the network $L(t)$ cannot go to zero.*

PROOF. The network $\mathbb{S}_0$ can only be a Θ–shaped or an eyeglasses–shaped network, as in the following figure, indeed, if some end–points are present, clearly the total length cannot go to zero.

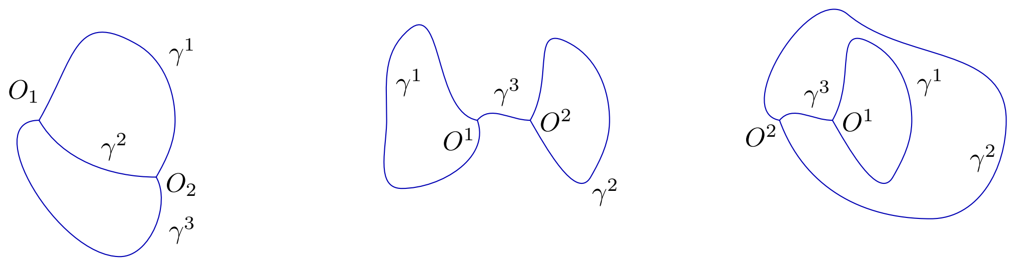


FIGURE 21. A Θ–shaped network and two different embeddings in $\mathbb{R}^2$ of eyeglasses–shaped networks (type A and type B).

Consider first the case of a Θ–shaped network. For both regions the equation of the evolution of the area is

$$A'(t) = -\frac{4\pi}{3},$$

as shown in equation (7.3). If $A^1(t) \neq A^2(t)$, then a loop shrinks before the other and $\lim_{t\to T} L(t) \neq 0$. Hence, $A^1(t) = A^2(t) = 4\pi(T-t)/3$, for every $t \in [0, T)$. Taking a blow–up limit $\widetilde{\mathbb{S}}_\infty$ at a hypothetical vanishing point $x_0 \in \Omega$, such limit also must contain two loops with equal finite area, since every rescaled network of the sequence $\widetilde{\mathbb{S}}_{x_0,t}$, converging to $\widetilde{\mathbb{S}}_\infty$, contains two regions with area equal to $2\pi/3$ (the rescaling factor is $1/\sqrt{2(T-t)}$, see Section 6.2) and the two loops cannot vanish, going to infinity (neither collapsing to a core by the constant area), because they are contiguous and at least one is present in the possible limit shrinker (Brakke spoon, lens or fish). Then, $\widetilde{\mathbb{S}}_\infty$ cannot be a Brakke spoon, a standard lens or a fish, but the curvature must be unbounded, by Proposition 8.35, hence, this situation is not possible.

We now analyse an eyeglasses–shaped network of “type B” (see Figure 21). We call A^1 the area enclosed in the curve γ^1, A^2 the area between γ^1 and γ^2 and A^3 the sum of A^1 and A^2. Arguing, as before, by means of equation (7.3) and Gauss–Bonnet theorem, we get that it must be

$$A^1(t) = 5\pi(T-t)/3, \qquad A^2(t) = 2\pi(T-t)/3, \qquad A^3(t) = 7\pi(T-t)/3.$$

Again, the two loops cannot vanish in the rescaling procedure by the same argument of the previous case and we exclude also this situation by the lack of a shrinker with two regions.

Arguing as before, in the case of a eyeglasses–shaped network of "type A" as in Figure 21, the evolution equation for the area of the regions is

$$A'(t) = -\frac{5\pi}{3},$$

but, in this situation, we cannot exclude a priori that one of the two loops goes to infinity along the converging sequence of rescaled networks, getting a Brakke spoon as blow–up limit (lens and fish are clearly not possible because of the eyeglasses topology). Anyway, following the proof of Proposition 8.31, when the blow–up limit around a point $x_0 \in \Omega$ is a Brakke spoon, there exists a small annulus around x_0 and a time $t_0 \in [0,T)$ such that, for every $t \in (t_0,T)$, the network $\mathbb{S}_t$ in such annulus is a graph over a (piece of a) halfline through the point x_0. In particular, $\mathbb{S}_t$ does not collapse to the point x_0, and we have a contradiction. Hence, also this case is impossible. $\square$

To conclude the analysis of the long time behavior of the flows of network with two triple junctions it is more convenient to consider each topological type of network separately.

10.5. The theta. We call A^1 the area enclosed by the curves γ^1 and γ^2 and, respectively, A^2 the area enclosed by γ^2 and γ^3, as one can see in Figure 22.

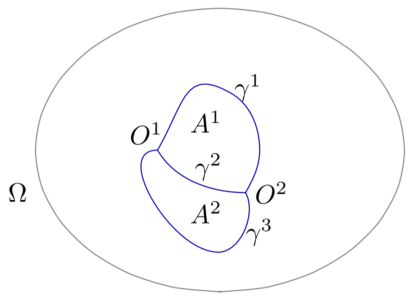


FIGURE 22. Theta.

Let x_0 be a reachable point of the flow, from Proposition 7.22 we know that the sequence of rescaled networks $\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_j}$ converges in $C^{1,\alpha}_{\mathrm{loc}} \cap W^{2,2}_{\mathrm{loc}}$, for any $\alpha \in (0,1/2)$, to a limit $\widetilde{\mathbb{S}}_\infty$. The possible $\widetilde{\mathbb{S}}_\infty$ are:

- a straight line through the origin;
- a standard triod;
- two lines that cross each other forming angles of 120 and 60 degrees;
- a standard lens;
- a fish.

We know from Proposition 8.35 that the maximal time T of existence of a smooth flow is finite and bounded by $\frac{3}{4\pi}\min\{A^1(0), A^2(0)\}$ Moreover from Equation (7.3) we know that the areas in the two loops are linearly in time decreasing.

If $T < \frac{3}{4\pi}\min\{A^1(0), A^2(0)\}$, then $\widetilde{\mathbb{S}}_\infty$ is the degenerate regular network composed by four halflines forming alternate angles of 60 and 120 degrees, both areas are not gone to zero, the length of only one curve has gone to zero and the two triple junctions have collapsed in a 4–point and the curvature stays bounded during all the evolution (Proposition 8.31).

If $T = \frac{3}{4\pi}\min\{A^1(0), A^2(0)\}$, then $\widetilde{\mathbb{S}}_\infty$ is a standard lens or a fish, $\lim_{t\to T} A^i(t) \to 0$ (where A^i has the smallest area between the two at $t=0$), if the length of the curves bordering A^i goes to zero, then the L^2–norm of the curvature is unbounded (Proposition 8.35).

We notice that no other phenomena (all the lengths go to zero as $t \to T$) are possible because of the lack of "suitable" blow up limit (Proposition 10.10). This fact says also to us that in a special symmetry situation ($A^1(0) = A^2(0)$) the only possible first singularity onset is a 4–point

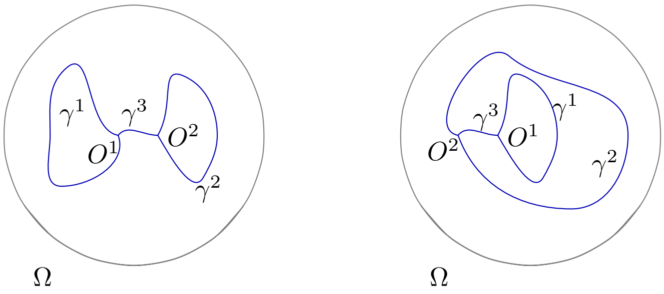


FIGURE 23. Eyeglasses: "type A" and "type B".

formation: the limit of the length of a curve that connects the two 3–points goes to zero as $t \to T$, and the curvature remains bounded.

PROPOSITION 10.11. *Let $\mathbb{S}_t$ be a theta-shaped network evolving by curvature. Then, $T < +\infty$ and as $t \to T$ the following behavior are possible:*

- *If $T = \frac{3}{4\pi}\min\{A^1(0), A^2(0)\}$ an area shrinks and if the length of two curves that bound the area goes to zero we have $\limsup_{t\to T}\int_{\mathbb{S}_t} k^2\, ds = +\infty$.*
- *If $T < \frac{3}{4\pi}\min\{A^1(0), A^2(0)\}$ the length of only one curve goes to zero, the curvature remains bounded.*

If $A^1(0) = A^2(0)$ the first case is not possible.

10.6. The eyeglasses. We analyze the two different embedding in $\mathbb{R}^2$ of this network (see Figure 23). We remind that for an eyeglasses "type B", calling A^1 the area of the internal loop, A^2 the area between the inner and the outer closed curve and A^3 the sum of the two areas we have the following evolution equations:

$$A^1(t) = 5\pi(T-t)/3, \qquad A^2(t) = 2\pi(T-t)/3, \qquad A^3(t) = 7\pi(T-t)/3.$$

Moreover $A^3(t) > A^1(t)$ for every $t \in [0, T)$.

Consider a reachable point for the flow $x_0 \in \mathbb{R}^2$, then the possible limits $\widetilde{\mathbb{S}}_\infty$ that arise as $t \to T$ in the sequence of rescaled networks $\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_j}$ (we know that the limit exists and that the convergence is C^1 from Proposition 7.22) are:

- a straight line through the origin;
- a standard triod;
- four halflines concurring in the origin that form angles of 120 and 60 degrees;
- a Brakke spoon.

We notice that there is no other possible candidates for $\widetilde{\mathbb{S}}_\infty$, indeed the shrinkers with two triple junctions (standard lens and fish) are not topological compatible with the eyeglasses shape. If one tries to construct a shrinker that is topological compatible with the eyeglasses shape simply attaching together pieces of Abresch-Langer curves and satisfying the 120 degrees condition at the junctions, one violates the embeddedness of the network or the convexity of the closed region of the shrinker (see **[1]** and **[15]** for a deeper study on the properties of the Abresch-Langer curve and of the shrinkers).

In the case of the eyeglasses some loop are present, hence T is finite.

We first analyze the behavior of a "type A" eyeglasses network: the bound for the maximal time of existence T is:

$$T \leq \frac{3}{5\pi}\min\{A^1(0), A^2(0)\}. \tag{10.4}$$

If $T < \frac{3}{5\pi}\min\{A^1(0), A^2(0)\}$, then the sequence of the rescaled $\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_j}$ converges as $j \to \infty$ to four halflines concurring in the origin that form angles of 120 and 60 degrees, and by Proposition 8.31 the curvature is bounded, only the length of the curve γ^3 goes to zero, the two triple

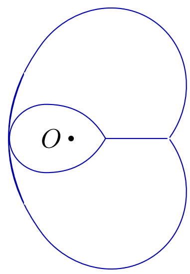


FIGURE 24. An eyeglasses shape shrinker does not exist.

junctions collapse in a 4-point.

If $T = \frac{3}{5\pi}\min\{A^1(0), A^2(0)\}$, then the limit $\widetilde{\mathbb{S}}_\infty$ is a Brakke spoon, $\lim_{t\to T} A^1(t) = 0$ or $\lim_{t\to T} A^2(t) = 0$ (also both) and if $\liminf_{t\to T} L^1(t) = 0$ or $\liminf_{t\to T} L^2(t) = 0$, then $\limsup_{t\to T}\int_{\mathbb{S}_t} k^2\,ds = \infty$ and $T = \frac{3}{5\pi}\min\{A^1(0), A^2(0)\}$. We underline that, differently from the case of the theta, also both areas could shrink down to a point as $t \to T$, provided that the length of γ^3 stay positive during all the evolution, if this is the case then $A^1(0) = A^2(0)$.

We can avoid the case in which the lengths of all the curves go to zero because of lack of "good" shrinkers (Proposition 10.10). Reasoning as in Proposition 10.10 one also gets that the case in which $L^3 \to 0$, but also another length goes to zero is not possible.

For the "type B", the situation is more complicated since the two areas have different evolution equations. The maximal time of existence in bounded by $\frac{3A^1(0)}{5\pi}$, and could be attain only if $\frac{A^1(0)}{A^2(0)} < \frac{5}{2}$.

Indeed if $T < \frac{3A^1(0)}{5\pi}$, then $\widetilde{\mathbb{S}}_\infty$ is composed by four halflines concurring in the origin that form angles of 120 and 60 degrees and the curvature is bounded.

If $T = \frac{3A^1(0)}{5\pi}$ (hence the initial areas satisfy $\frac{A^1(0)}{A^2(0)} < \frac{5}{2}$, otherwise the maximal time of existence should be $\frac{3A^1(0)}{2\pi} < \frac{3A^1(0)}{5\pi}$, the time in which A^2 should vanish) then the limit $\widetilde{\mathbb{S}}_\infty$ is a Brakke spoon and if $\lim_{t\to T} A^1(t) = 0$, $\liminf_{t\to T} L^1(t) = 0$, the curvature is not bounded and $T = \frac{3A^1(0)}{5\pi}$.

As for "type A" we can can avoid the case in which the lengths of all the curves go to zero and the case in which $L^3 \to 0$, but also another length goes to zero (Proposition 10.10). This implies that if $\frac{A^1(0)}{A^2(0)} \geq \frac{5}{2}$ the only possible singularity is a 4–point formation.

PROPOSITION 10.12. *Let $\mathbb{S}_0$ be an initial eyeglasses network and let $\mathbb{S}_t$ be its smooth evolution by curvature, in a maximal time interval $[0, T)$. Then $T < +\infty$ and there are these possibilities:*

- *only the length of the curve that do not form the loop goes to zero and the curvature stays bounded (both for "type A" and "type B");*
- *for "type A" at least one area goes to zero, $T = \frac{3}{5\pi}\min\{A^1(0), A^2(0)\}$ and if the length of the curves composing the loops (one or both) goes to zero the curvature is unbounded;*
- *for "type B" the internal area goes to zero, $T = \frac{3A^1(0)}{5\pi}$ and the length of the curve enclosing this area goes to zero the curvature is unbounded.*

10.7. The lens. The main difference with respect to the theta and the eyeglasses in that the lens presents two boundary points P^1 and P^2, that increase the list of the possible limits $\widetilde{\mathbb{S}}_\infty$, that could be:

If $x_0 \in \Omega$:

- a straight line through the origin;
- a standard triod;
- four halflines from the origin that form angles of $120/60$ degrees;

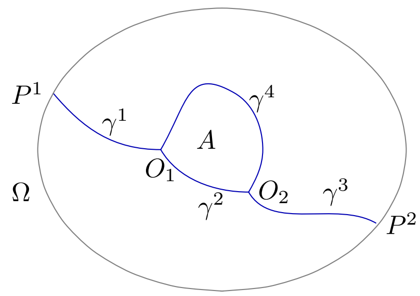


FIGURE 25. Lens.

- a standard lens;
- a fish.

If $x_0 \in \partial\Omega$

- a halfline from the origin;
- two halflines from the origin that form an angle of 120 degrees.

One obtains this list, having in mind the topology of the lens network, reasoning as for the previous cases.

Because of the presence of a loop in the structure of the network, we know that the maximal time of existence T is finite and is less or equal than $\overline{T} = \frac{3A(0)}{4\pi}$.

If $T = \frac{3A(0)}{4\pi}$ the sequence $\widetilde{\mathbb{S}}_{x_0,\mathfrak{t}_j}$ converges to a standard lens or to a fish, then the area enclosed in the loop disappears. Moreover we know from Proposition 8.35 that if $\lim_{t\to T} L^2(t) = 0$ and $\lim_{t\to T} L^4(t) = 0$, then the L^2–norm of curvature in not bounded.

If instead $T < \frac{3A(0)}{4\pi}$ the degenerate regular network $\widetilde{\mathbb{S}}_\infty$ is composed by four halflines, the curvature stays bounded (Proposition 8.31), and only one length between L^2 and L^4 goes to zero. The blow up limit is centered in P^1 or in P^2 and is composed by two halflines forming an angle of 120 degrees, then by Proposition 8.31 the curvature do not go to infinity, $\lim_{t\to T} L^1(t) = 0$ or $\lim_{t\to T} L^3(t) = 0$ (or even both) .

The contemporary collapse of the two triple junctions onto an end–point on $\partial\Omega$ is avoided by Lemma 9.8.

PROPOSITION 10.13. *Let $\Omega \subset \mathbb{R}^2$ be an open,regular and strictly convex set. Let $\mathbb{S}_0$ an initial lens-shaped network with fixed end–points on $\partial\Omega$ and $\mathbb{S}_t$, with $t \in [0, T)$ a smooth evolution by curvature. Then the maximal time of existence T is bounded by $\frac{3A(0)}{4\pi}$ and as $t \to T$ we can have the following cases:*

- *the area enclosed in the loop goes to zero, the lengths of the two curves bounding the loop go to zero, the curvature is unbounded and $T = \frac{3A(0)}{4\pi}$;*
- *the length of only one curve bounding the loop goes to zero, there is a 4–point formation and the curvature remains bounded;*
- *the length of at least one of the curves connecting the triple junctions with the end–points on the boundary of Ω goes to zero, there is a 2–point formation and the curvature remains bounded.*

10.8. The island. As for all the previous networks, that present one loop, the finite maximal time of existence T of a smooth flow in bounded by

$$T = \frac{3A(0)}{5\pi}\,.$$

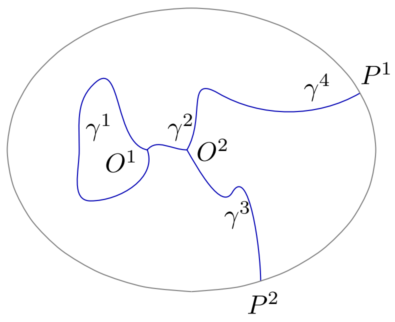


FIGURE 26. Island.

If we apply Proposition 5.7 in the case of the island-shaped network, we can have the following degenerate regular networks $\widetilde{\mathbb{S}}_\infty$:
If $x_0 \in \Omega$

- a straight line through the origin;
- a standard triod;
- a Brakke spoon;
- four halflines from the origin forming angles of $120/60$ degrees.

If $x_0 \in \partial\Omega$

- a halfline from the origin;
- two halflines forming an angle of 120 degrees.

If T is $\frac{3A(0)}{5\pi}$, then $\widetilde{\mathbb{S}}_\infty$ is a Brakke spoon, the area enclosed in the loop goes to zero, if $\lim_{t\to T} L^1(t) = 0$, the curvature unbounded (Proposition 8.35 applies).

If $T < \frac{3A(0)}{5\pi}$, the degenerate regular limit network $\widetilde{\mathbb{S}}_\infty$ is composed by four halflines from the origin forming angles in pair of $120/60$ degrees, there is the collapse of O^1 and O^2 into a 4–point, from Proposition 8.31 we know that the curvature stays bounded during the evolution.

Another possibility is that we have $\lim_{t\to T} L^3(t) = 0$ (or respectively $\lim_{t\to T} L^4(t) = 0$, both together is not possible as $P^1 \neq P^2$) and the triple junction O^2 hit the boundary in P^1 (or respectively P^2), forming a 2–point. In this case $\widetilde{\mathbb{S}}_\infty$ is being composed by two halflines with an angle of 120 degrees between them. Also in this case from Proposition 8.31 we know that the curvature locally stays bounded during the evolution.

Instead Lemma 9.8 excluded that, as $t \to T$, both O^1 and O^2 collapses onto a boundary point. We can reason similarly to the case of the eyeglasses to exclude that both $\liminf_{t\to T} L^1(t) = 0$ and $\liminf_{t\to T} L^2(t) = 0$, so it cannot be that the area disappear and also there is a multi–point formation.

We notice that instead can happen that $\liminf_{t\to T} L^1(t) = 0$ and $\liminf_{t\to T} L^3(t) = 0$ (or respectively $\liminf_{t\to T} L^4(t) = 0$): concurrently the area shrinks and a 3–point collapses to an end–point.

PROPOSITION 10.14. *Let $\Omega \subset \mathbb{R}^2$ be a regular, open and strictly convex set and let $\mathbb{S}_0$ be a network with two triple junctions, with the shape of an island and with two end–points P^1 and P^2 fixed on $\partial\Omega$. Then, if $\mathbb{S}_t$ is the smooth evolution by curvature of $\mathbb{S}_0$ in a maximal time interval $[0, T)$, T is finite and, as $t \to T$:*

- *the area A shrinks down to a point, if the length of the curve that bound the area goes to zero the curvature is unbounded and $T = \frac{3A(0)}{5\pi}$;*
- *the triple junction O^2 collapses onto an end–point on $\partial\Omega$, the curvature remains bounded and $T \leq \frac{3A(0)}{5\pi}$;*

- *the lengths of the curve γ^2 goes to zero, the curvature remains bounded and $T < \frac{3A(0)}{5\pi}$.*

10.9. The tree. We conclude with the analysis of the long time behavior of the only network with two triple junctions which does not present loops. Consequently it is the only case where we could expect global existence.

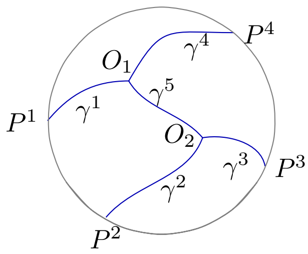


FIGURE 27. Tree.

Consider the sequence of rescaled networks $\widetilde{\mathbb{S}}_{x_0,t_j}$ and its C^1_{loc}–limit $\widetilde{\mathbb{S}}_\infty$ that can be only:
If $x_0 \in \Omega$

- a straight line through the origin;
- a standard triod;
- four halflines from the origin forming angles in pair of $120/60$ degrees.

If $x_0 \in \partial\Omega$

- a halfline;
- two halflines from the origin forming an angle of 120 degrees.

If T is finite, then either $\widetilde{\mathbb{S}}_\infty$ is composed by four halflines from the origin forming angles in pair of $120/60$ degrees ($L(\gamma^5)$ goes to zero as $t \to T$, the two triple junctions collapse but the curvature remains bounded) or $\widetilde{\mathbb{S}}_\infty$ is composed by two halflines from the origin forming an angle of 120 degrees (the length of one or two curves between γ^1 , γ^2 , γ^3 and γ^4 goes to zero as $t \to T$ a triple junction touches the boundary and the curvature remains bounded).

The case in which the length of both one between the four curves with fixed end–points on $\partial\Omega$ and $L(\gamma^5)$ goes to zero is excluded by Proposition 9.8.

Otherwise if $T = \infty$ no lengths go to zero during all the evolution, and the network tends to the configuration of minimal length, that is the Steiner configuration (possibly degenerate) that connects the four fixed end–points.
We underline that in all the possible scenarios the curvature remains bounded.

PROPOSITION 10.15. *Given a smooth embedded tree network $\mathbb{S}_0$ with two triple junctions evolving by curvature in a strictly convex, open regular domain $\Omega \subset \mathbb{R}^2$ with fixed end–point on $\partial\Omega$, then either the inferior limit of the length of a curve goes to zero as $t \to T$ (either two triple junctions collapse producing a 4–point where the four concurring curves have opposite unit tangents in pairs and form angles of $120/60$ degrees between them or a triple junctions collapses onto a boundary point and the unit tangents of two curves concurring at such end–point form angle of 120 degrees), or $T = +\infty$ and $\mathbb{S}_t$ tends, as $t \to \infty$ to the generalized Steiner configuration connecting the fixed end–points. In all cases the curvature stays bounded during all the smooth evolution.*

REMARK 10.16. In the case of the triod we can show an easy example that explain why we cannot exclude the situation in which the triple junction collapses on a boundary point. If the end–points P^1, P^2 and P^3 are fixed in such a way that the triangle with that points for vertices does not admit the Steiner configuration, then one of the three lengths goes to zero. This happens if an angle between the three points P^i is greater than 120 degrees. Something similar could also happen for the tree. For instance if Ω is a circle and the four points P^1, P^2 , P^3 and P^4 are all in the same quarter the generalized Steiner configuration that connect the P^i and the O^i (keeping them distinct) does not exist.

11. Restarting the flow after a singularity

We resume in the following propositions the behavior of the evolving regular network at a singular time, assuming the multiplicity–one conjecture 8.1 and the uniqueness assumption 7.19.

PROPOSITION 11.1. *If* **M1** *is true and the evolving regular network is a tree (no loops) or no regions are collapsing, then the only "singularities" are given by either the collapse of a curve with the two triple junctions at its end–points (only) going to coincide, producing a* 4*–point where the four concurring curves have opposite unit tangents in pairs and form angles of* $120/60$ *degrees between them, or the collapse of a curve getting to an end–point of the network letting two curves concurring at such end–point forming an angle of* 120 *degrees between them.*

PROPOSITION 11.2. *If* **M1** *is true and the uniqueness assumption* **U** *holds, then the only "singularities" are given by the situations described above in the case of a tree or*

- *the collapse of a region (loop) with more than one and less than six boundary curves, creating a multi–point (that can coincide with an end–point of the evolving network if also the curve getting to such end–point collapses), hence a non–regular network,*
- *the collapse of a region bounded by a single curve which if it happens at a point of* Ω *produce a curve with an end–point which is a 1–point of the limit network, or if it happens at an end–point of the network, such region and the curve connecting it to the end–point both collapse to such end–point.*

The next step, after this description, is to understand how the flow can continue after a singular time. There are clear situations where the flow simply ends, for instance if all the network collapses to a single point (inside Ω since this cannot happen to an end–point on the boundary, which means that the network $\mathbb{S}_t$ is actually without end–points at all), like a circle shrinks down to a point in the evolution of a closed embedded single curve, see, for instance, the following example.

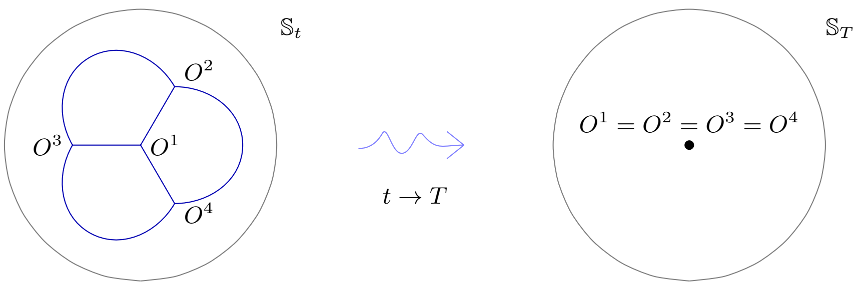


FIGURE 28. A clover shrinker collapsing to a single point.

In other situations, how the flow should continue is easy to guess or define, for instance if a part of the network collapses forming a 2–point, that can be also seen simply as an interior corner point of a single curve (see the following figure).

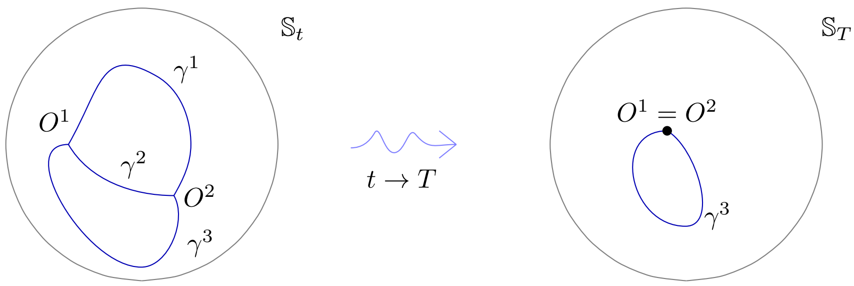


FIGURE 29. Collapse of both the curves γ^1, γ^2 and the region they enclose point $O^1 = O^2$ leaving a closed curve γ^3 with a corner at $O^1 = O^2$ of 120 degrees.

Here, we can restart the network, by means of the work of Angenent [**10, 11, 12**] where the evolution of curves with corners is also treated (see Remark 1.2). In general, one would need an analogue of the small time existence Theorem 3.7 or 3.18, for networks with 2–points or with curves with corners.

Instead, a situation that really needs a "decision" about whether and how the flow should continue after the singularity, is depicted in the following figures.

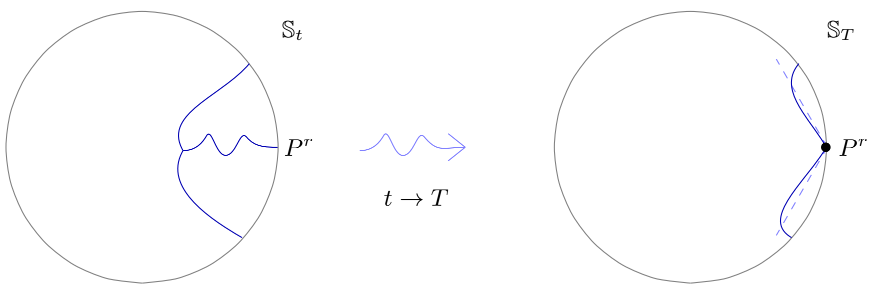


FIGURE 30. A limit network with two curves arriving at the same end–point on $\partial\Omega$.

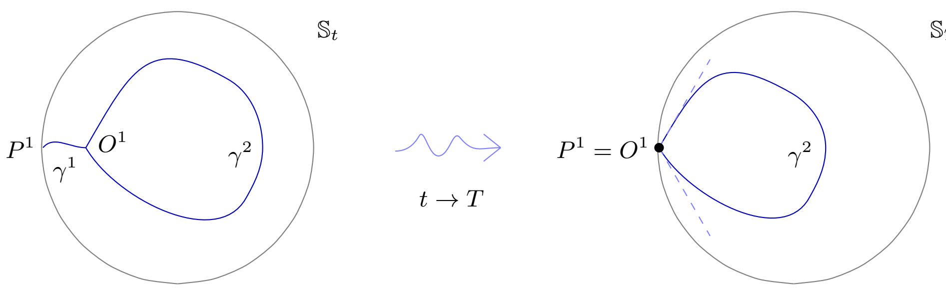


FIGURE 31. Collapse of the curve γ^1 leaving a closed curve γ^2 with an angle of 120 degrees at an end–point.

One can decide that the flow stops at $t = T$ or that the curves become extremal curves of a new network that must have, for every $t > T$, a fixed end in the end–point P^r (this would require some analogues of the small time existence Theorems 3.7 and 3.18 for this class of non–regular networks, which are actually possible to be worked out). Anyway, the subsequent analysis becomes more troublesome because of such concurrency at the same end–point, indeed, it should be allowed that, at some time $t > T$, a new curve and a new 3–point "emerges" from such end–point.

Another situation that also needs a decision, but in this case easier, is described in the following figures.

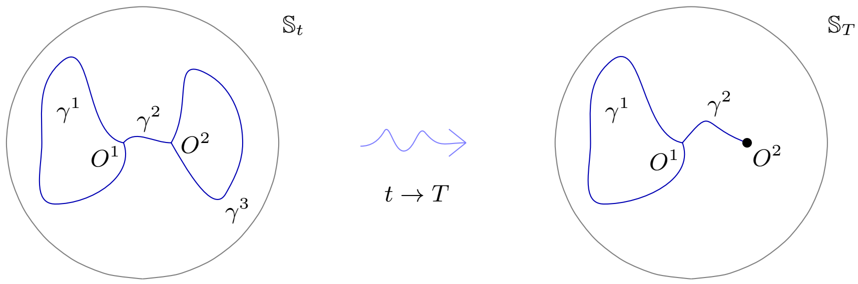


FIGURE 32. Collapse of the curves γ^3 and the region enclosed to the point O^3 leaving a curve γ^2 with a 1–point as an end–point.

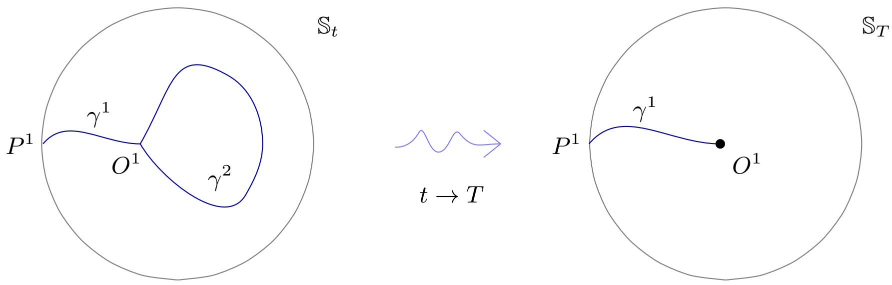


FIGURE 33. Collapse of the curves γ^2 and the region enclosed to the point O^1 leaving a curve γ^1 with a 1–point as an end–point.

If the limit network $\mathbb{S}_T$ contains a curve (or curves) which ends in a 1–point, it is actually natural to impose that such curve vanishes for every future time, so considering only the evolution of the network of the rest of the network $\mathbb{S}_T$ according to the above discussion (cutting away such a curve will produce a 2–point or the empty set, in the figures above, for instance).

We state a special case of a theorem by Ilmanen, Neves and Schulze [**53**, Theorem 1.1], regarding the short time existence of a motion by curvature starting from a non–regular network, allowing us to continue the flow after the collision of the two triple junctions.

THEOREM 11.3. *Let $\mathbb{S}_T$ be a non–regular, connected embedded, C^1 network with bounded curvature having a single 4–point with the four concurring curves having unit tangent vectors forming angles of 120 and 60 degrees. Then, there exists $\widetilde{T} > T$ and a smooth flow of connected regular networks $\mathbb{S}_t$, locally tree–like, for $t \in (T, \widetilde{T})$, unique in this class, such that $\mathbb{S}_t$ is a regular Brakke flow for $t \in [T, \widetilde{T})$. Moreover away from the 4–point of $\mathbb{S}_T$, the convergence is in C^2_{loc} (or as smooth as $\mathbb{S}_0$).*
Furthermore, there exists a constant $C > 0$ such that $\sup_{\mathbb{S}_t} |k| \leq C/\sqrt{t-T}$ and the length of the shortest curve of $\mathbb{S}_t$ is bounded from below by $\sqrt{t-T}/C$, for all $t \in (T, \widetilde{T})$.

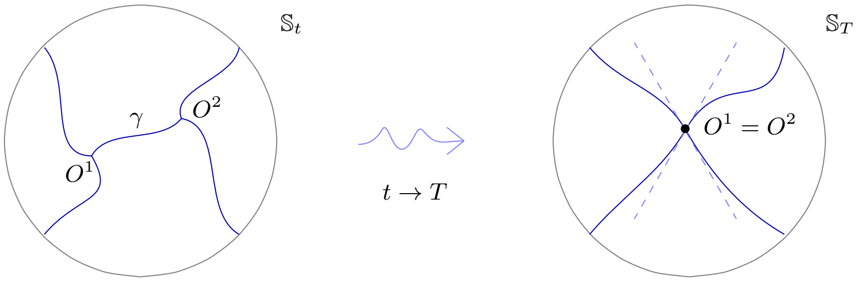


FIGURE 34. A limit "nice" collapse of a single curve γ producing a non–regular network $\mathbb{S}_T$.

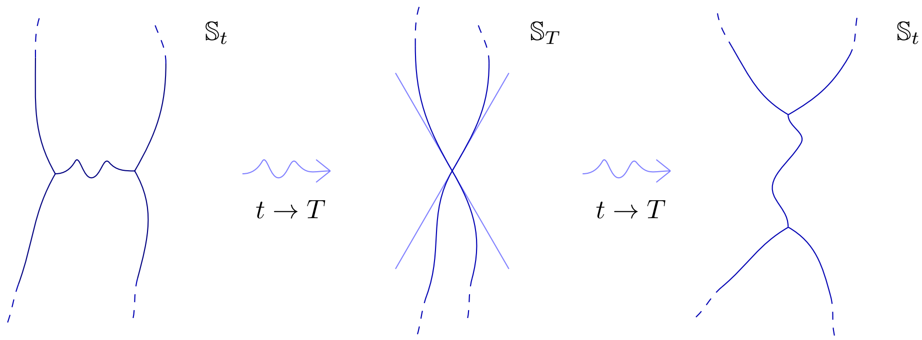


FIGURE 35. The local description of a "standard" transition.

REMARK 11.4. Notice that the transition, passing by $\mathbb{S}_T$, is not symmetric: when $\mathbb{S}_t \to \mathbb{S}_T$, as $t \to T^-$, the unit tangents, hence the four angles between the curves, are continuous, while when $\mathbb{S}_t \to \mathbb{S}_T$, as $t \to T^+$, there is a "jump" in such angles, precisely, there is a "switch" between the angles of 60 degrees and the angles of 120 degrees.

REMARK 11.5. A regular C^2 network $\mathbb{S} = \bigcup_{i=1}^n \sigma^i(I_i)$ is called a *self–expander* if at every point $x \in \mathbb{S}$ there holds
$$\underline{k} - x^\perp = 0\,.$$
Let x_0 be the 4–point of $\mathbb{S}_T$ and consider the rescalings
$$S_{x_0,\mathfrak{t}} = \frac{\mathbb{S}_t - x_0}{\sqrt{2(t-T)}}\,,$$
with $\mathfrak{t}(t) = -\frac{1}{2}\log(t-T)$. Then as $\mathfrak{t} \to +\infty$ the rescaled networks $\widetilde{S}_{x_0,\mathfrak{t}}$ tend to unique connected self–expander $\widetilde{S}_\infty$ which "arises" from the network given by the union of the halflines from the origin generated by the unit tangent vectors of the four concurring curves at x_0 (see [**76**]).

REMARK 11.6. For a general network, a flow of an initial non–regular network given by the general version of the above theorem, is not unique, even if $\mathbb{S}_T$ is composed only by halflines from the origin and we search a solution between the tree–like self–expanding networks. In particular, in our situation there exist two non–connected self–expanding solutions (see the explicit solutions for initial data composed by an even number of halflines in [**76**, Section 3]). Considering only the *locally connected* network flows has a clear "physical" meaning: such a "choice" ensures that initially separated regions remain separate during the flow.

Finally, if we are in the situation of a non–regular limit network $\mathbb{S}_T$ described by Theorem 11.2, after the collapse of a region of $\mathbb{S}_t$, as $t \to T$ (see for instance the following figures), one will need the extension of Theorem 11.3, in order to restart the flow.

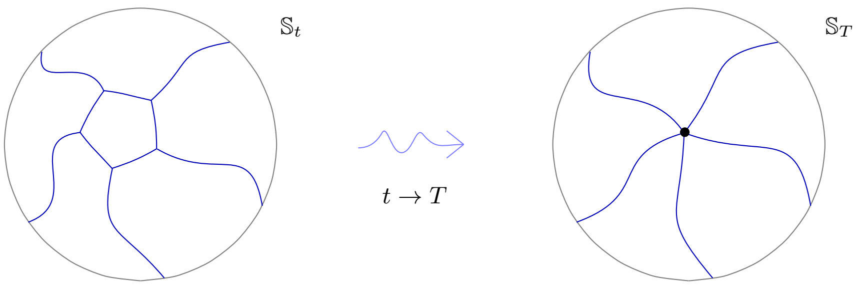


FIGURE 36. Less "nice" examples of collapse and convergence to non–regular networks $\mathbb{S}_T$.

REMARK 11.7. We are able to establish a restarting theorem after the onset of the first singularity only in the case in which the multiplicity–one conjecture holds, the curvature remains bounded, no regions vanish and a triple junction does not collapse to an end–point on the boundary of Ω. At the moment, the restarting of the flow after all the other types of singularities is only conjectural.

REMARK 11.8. Notice that Theorem 11.3 gives only a short time existence result, indeed, it is not possible to say in general if and when another singularity could appear. In particular, we are not able to exclude that the singular times may accumulate.

We conclude this section with a "speculative" conjecture about the singularity formation.

CONJECTURE 11.9. The "generic" singularity is (locally) the collapse of a single curve (only two 3–points colliding)? That is, for a dense set of initial regular network at the maximal time of smooth existence, the situation is the one described in Proposition 11.1?

11.1. Standard transitions for network with two triple junctions. If the previous Conjecture 11.9 is true, the study of the evolution of networks with only *two* triple junctions is quite interesting since they locally describe what happens at a "typical" singular time. Moreover, as we have seen in Corollary 9.11, that the multiplicity–one conjecture holds for such networks.

At this point is natural to list the "standard transition" from one topological type of network with two triple junctions to another (see the classification in Figure 10.1) passing by a 4–point formation.

Consider a tree–shaped network, if at the maximal time T no "boundary" curve collapses, then $\mathbb{S}_t$ converges to a limit network $\mathbb{S}_T$ with bounded curvature and, restarting the flow by means of Theorem 11.3, we get another regular tree which is the only "other" possible connected, regular tree, joining the four fixed end–points P^1, P^2, P^3 and P^4. That is, the "standard" transition at time T, as in Figure 35, transforms one in the other and vice versa. The natural question, if during the flow there could appear infinite singular times (and also whether they could "accumulate") producing an "oscillation" phenomenon between the two structures, has no answer at the moment.

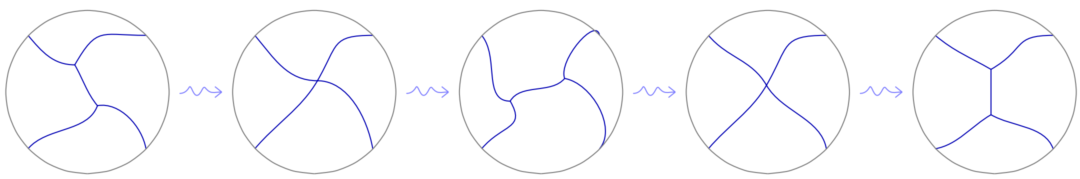


FIGURE 37. The "standard" transition for a tree–shaped network.

The lens and the island shapes are in a sense "dual", with the meaning that a "standard" transition, as in Figure 35, transforms one in the other and vice versa. As before, we do not know if during the flow this kind of "oscillation" phenomenon can happen infinite (possibly accumulating) times.

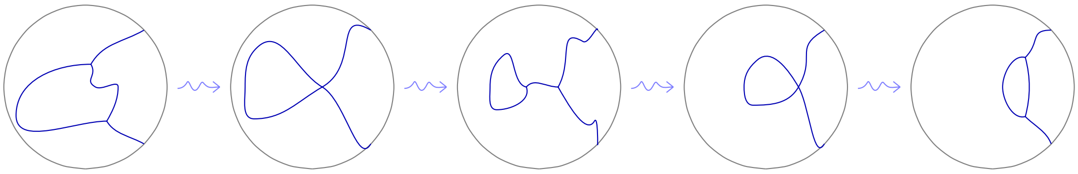


FIGURE 38. A "standard" transition trough a 4–point transforms a lens in a island and vice versa.

As before, there is a sort of "duality" between the theta and eyeglasses shapes: a "standard" transition transforms one in the others and viceversa. Again, we do not know if this kind of "oscillation" can happen infinite times.

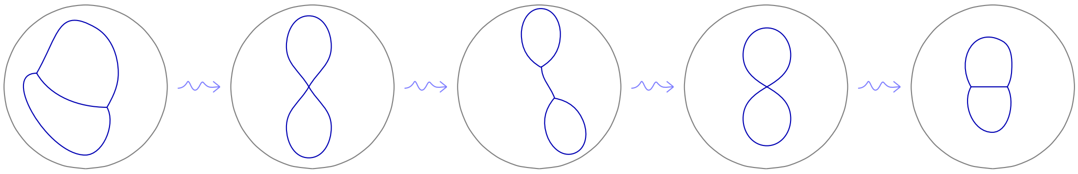


FIGURE 39. An example of evolution from a theta-shaped network to an Eyeglasses "type A" passing by a 4–point formation.

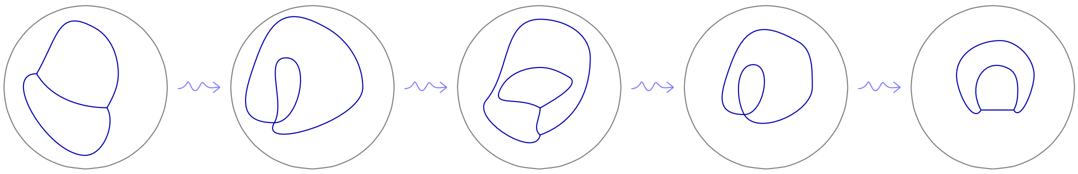


FIGURE 40. An example of evolution from a theta–shaped network to an Eyeglasses "type B" passing by a 4–point formation.

REMARK 11.10. As we said in Remark 11.4, all these transitions of the networks between these "dual" topological shapes are not reversible in time. The angles between the curves are continuous as $t \to T^-$, discontinuous as $t \to T^+$.

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