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**HOLOGRAPHIC COMPUTATION OF  
THE NUCLEON-NUCLEON POTENTIAL**

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# Introduction

Beginning from the experiments of Rutherford, the modern picture of atoms began to form. In this picture, the atom is composed of negatively charged light particles (called electrons) that interact electromagnetically with an heavy, positively charged nucleus. While the orbiting electrons are actually considered elementary particles, the heavy nucleus is an aggregate of interacting heavy particles, called nucleons. Understanding how those baryons interact and what binds those particles together is the main problem of nuclear physics.

Details of those interactions have always been difficult to grasp. Earlier experiments and theoretical models found two types of nucleons: the proton, that has the opposite charge of the electron, and the neutron, that has no net charge. This interaction shows some unfamiliar features: it is strong enough to overcome the repulsive Coulomb interaction between protons, but it heavily relies on the presence of neutrons. In fact, all stable nuclei are composed of a comparable amount of protons and neutrons, and heavier nuclei are found to have more neutrons than protons. This equilibrium between neutrons and protons is delicate, and the addition of one or the other could result in an unstable nucleus.

Further experimental results brought to more questions. Particles with similar characteristics to the nucleons were discovered (like the  $\Delta$  particle), and nucleons and all those similar particles were called baryons. The advent of quantum field theory and its description of forces as interaction between particles and mediators gave the theoretical input for the construction of mesons as mediators of the nuclear force between baryons, and experiments confirmed that point of view. Still, there was a rather large park of baryons and mesons showing similarities that could not be satisfactorily interpreted as coincidences, so those advancements were not considered the final answer.

In his revolutionary work, Murray Gell-Mann used the known similarities between baryons and between mesons, introducing a symmetry transformation that grouped the known baryons and mesons in different irreducible representations of the group  $SU(3)$ . Each baryon (meson) occupied a particular spot in an irreducible representation, and the various baryons (mesons) were distinguished from each other by following the standard rules in the representation theory of symmetry groups. Despite being only an approximate symmetry, this model managed to group all known baryons (mesons), and also predicted the existence of additional particles in the places where there were some blank spots, particles that have been experimentally observed after their theorization. The fundamental representation of  $SU(3)$  was empty, so it was postulated that three particles of half integer spin existed, called quarks, that were the building blocks of hadrons (the collective name for baryons and mesons). The baryons were seen as composed by three quarks, while the mesons were formed by a quark-antiquark pair. Later, a new quantum number was introduced to explain some apparent contradictions (like Fermi-Dirac statistics), called color: each quark was postulated to be in the fundamental representation of another  $SU(3)$  symmetry group, the color group, and this symmetry was taken as local, giving rise to interaction between objects presenting a color charge, mediated by a new kind of particle called gluon. It was postulated that hadrons were always in a color singlet, and that it was not possible to find isolated quarks, that appeared only in hadrons.

This new quantum field theory, called Quantum Chromodynamics (QCD), is our actual theory for describing the strong force, the force that mediates the nuclear interactions. In terms of this force, the interaction of baryons mediated by mesons can be seen in terms of an interaction between quarks and antiquarks mediated by gluons. Baryon interaction, in principle, can be exactly calculated from the QCD Lagrangian, in terms of interaction of the constituent quarks. The standard tool for extracting observables from interacting field theories, perturbation theory, can be applied to QCD (and the theory is fully renormalizable, so in principle the theory makes sense at every perturbative order) but, as the interaction is expected to be strong, we cannot trust the results of standard perturbation theory to give results that agree with experiments. Actually, renormalization group approach shows that the coupling of the quarks to the gluon field is weak at high energies, but becomes very large at low energies, in the energy regime that is of interest in nuclear physics.



QCD is our actual model of the theory of nuclear interactions, that could explain the mechanisms underlying the formation of nuclei. Several attempts at solving the theory (in the sense of computing physical observables from it) have been attempted, ranging from full numerical simulations on a lattice (Lattice QCD), to effective field theories that consider baryons and mesons as fundamental particles, keeping the approximate symmetries of the QCD Lagrangian (as the sigma models). Phenomenological studies that fit enormous banks of data with a small number of parameter have not been abandoned, and they provide precious insights on the qualitative (and, at times, quantitative) features of the nuclear interaction.

Of particular interest is an approximation to QCD, the large  $N$  limit, where the number of colors is brought to infinity. The number of degrees of freedom goes to infinity in this limit, but the theory is actually made simpler: this is because contributions to correlation functions can be given an  $N$  dependence, and the large  $N$  limit consists in just taking the leading terms in the determination of those correlation functions. Corrections can always be made in terms of the parameter  $1/N$ , that is  $1/3$  in real QCD.

This thesis is inserted in this context. Our main goal is to build a model that can be used to describe baryon dynamics, and in particular we look for an interaction potential between the basic nucleons (proton and neutron). To do that, we follow an approach that mixes two ideas: the description of particles as topological solitons in nonlinear field theory and the Anti de Sitter / Conformal Field Theory (AdS/CFT) correspondence.

Topological solitons arise in nonlinear field theories: they are static solutions of the equations of motion that cannot decay in the vacuum. They can be imagined as stable lumps in a field, and a conserved charge (called topological charge) can be associated to those field configurations. The dynamics of those lumps can be approximated as rigid motion of the lumps in space, plus some internal degrees of freedom (like rigid body orientations). Quantizing the coordinates of the lumps, we get a quantum system of finite dimension, that can be studied through ordinary quantum mechanics. This is what is done in the Skyrme model, that we will study in the first chapter.

The action of the Skyrme model is an effective action, and various studies have evidenced the fact that the predictions of the theory varies as one changes the effective action. We want an action that descends uniquely from more general

principles, and for this we will use AdS/CFT correspondence. The main idea is that a string theory in a space is dynamically equivalent to a quantum field theory living on the boundary of that space, in the sense that we can map the observables of one theory in the observables of another. In a limit that we will see to be equivalent to large  $N$  limit of QCD, the string theory can be taken as classical, so the correspondence allows us to compute QFT observables from a classical action.

Using those tools, we study large  $N$  QCD by studying the action of the low energy modes of a string theory. We see that this theory admits topological solitons called instantons, and quantization of the degrees of freedom of an instantonic field of charge one creates a quantum system with states whose transformation properties and quantum numbers are just right to interpret them as states with a definite rest energy, an impulse, a spin and an isospin degrees of freedom. In this picture, we build a charge two field configuration by gluing together two single charge instanton solutions. Due to the non linearity of the theory, this solution is approximative, and we show that it holds in the limit of large 't Hooft coupling  $\lambda$ . In those limits, large  $N$  and  $\Lambda$ , we compute the energy of this field configuration, and interpret the result as a potential of interaction between instantons. This is proposed as a classical potential for baryon interaction, and its structure as infinite sum of Yukawa monopole and dipole interactions is interpreted as the classical analogue of an exchange interaction with a meson mediator. We show how the masses of baryons and mesons can be computed in this model.

After computing the potential, we quantize the coordinates of the two instanton fields, and impose physical constraints to restrict the spectrum of the system. We see that the internal degrees of freedom of the system can be rearranged and interpreted as total spin and isospin of the system, and that they assume only integer values. Among the states that are compatible with our constraints, we find a state with the right angular quantum numbers (spin one and isospin zero) and interpret it as deuteron state. We compute the stability of this state with respect to the splitting in two separated baryons, and make a similar analysis for other low energy states in the spectrum. We comment the large  $N$  and large  $\Lambda$  limit, comparing our results with the qualitative predictions of large  $N$  QCD, and extrapolate the binding energy of the deuteron.

This is our original work, and our results are contained in chapter 4, the final chapter of the thesis. The reader that is just interested in our results is strongly advised to read section (1.4), where a similar construction is made in the Skyrme

model, and then just skip to chapter 4, accepting that in our model baryons can be seen as instantons in the field theory defined by action (4.1.4). The rest of the thesis is dedicated to introducing and explaining the tools that we use to prove that our model is a model for large  $N$ , low energy QCD.

The thesis is organized as follows. In the first chapter, we introduce the concept of topological solitons. We give some basic background in topology, study its application to classical field theory and give various examples of theories admitting topological solitons. We give a more precise notion of collective coordinates and how they can be related to the symmetries of the system. Of particular interest are the Skyrme model and the Yang-Mills instanton, while the Baby Skyrme model is studied as one of the simplest, non trivial examples. We conclude the chapter by reviewing how the Skyrme model is used in nuclear physics to build a quantum picture of baryons, in a way that is similar to ours.

The second chapter is an introduction to the two sides of AdS/CFT correspondence. All the topics are covered from a beginner's point of view, and give the fundamental notions to understand the theories that the AdS/CFT correspondence relates. The chapter is divided in two parts: the first part (or gauge side) is dedicated to introduce conformal field theory and supersymmetry, as the approach taken in AdS/CFT always forces us to deal with supersymmetry (by realizing it in our models or finding a way to explicitly break it to build models appropriate for our energy scales). We will examine the simplifications that conformal field theory imposes on the physical observables, and study  $\mathcal{N} = 4$  supersymmetric Yang Mills theory, that is the theory living at the boundary of the Anti de Sitter space in AdS/CFT. In particular, we are interested in understanding the differences between Super Yang Mills and standard Yang Mills, in terms of additional fields and symmetries. We conclude the first part of the chapter with a brief analysis of the large  $N$  limit in QCD, and collect the main results.

The second part of the second chapter is entirely dedicated to the gravity side of AdS/CFT. We start by building Anti de Sitter space and examining their geometry, to find that one boundary of AdS space can be identified with standard Minkowski space. Then, we introduce string theory. In this introduction, we concentrate on solving the classical equations of motion for free strings, examine the degrees of freedom of the first quantized theory and introduce the concept of supersymmetric string theories, to also have fermionic degrees of freedom. We give a brief qualitative description of interacting string and how they can be related

to the large  $N$  expansion, and then conclude by introducing D-branes, extended objects in string theory that are sources of gravity, and whose low energy dynamics can be described in terms of open strings with their ends attached to it.

In the third chapter, we join those two sides together in the AdS/CFT correspondence. We start with the historical correspondence, that as now is the most well studied and understood theory in the correspondence, that describes supersymmetric Yang Mills theory through the dynamics of strings attached to fixed configurations of branes. By explicitly computing some observables as examples of the correspondence, we examine how this description diverges from standard QCD and then introduce the Sakai-Sugimoto model, a string theory in which supersymmetry is explicitly broken and quark flavors are inserted, showing how the action of this string theory at low energies is the action of standard QCD. We also see how the description of mesons and baryons arise in this particular string theory, and give the limits of application.

The fourth chapter is dedicated to the original work of the thesis. We start by building the charge two field by gluing together two well distanced charge one fields, and interpret the energy of this configuration as an interaction potential. After finding a configuration of the field that minimizes the classical energy, we proceed to quantize the degrees of freedom that correspond to free motion of the instantons without changing the energy (zero modes), and use discrete symmetries of the configuration to impose constraints on the set of states that arise. We also study the motion of instantons that has an energy cost (massive modes), by first trying an harmonic approximation and then quantizing the coordinate of the relative distance separating the two instantons. We then make numerical computations on this model, to study the spectrum of states, examine their binding energy and give conclusions on the large  $N$  and  $\Lambda$  limit of QCD's description of baryons and mesons. We collect our result in the conclusion.

# Chapter 1

## Topological solitons

In this chapter we briefly review the fundamental notions about the applications of topological solitons in classical field theory. In the first section we introduce the basic definitions and concepts, while in the following sections we give some classical examples of topologically non trivial fields, examining in detail the Baby-Skyrme model, the Skyrme model and the  $SU(N)$  (with particular emphasis on  $N = 2$ ) instanton. We conclude building the quantum theory of the Skyrme field using semiclassical quantization and the uses of this theory in modeling atomic nuclei. The main reference for this chapter is [32], an introductory text on topology and solitons.

### 1.1 Introduction to topology

#### 1.1.1 Basic definitions: homotopy groups

Homotopy theory studies the relations between continuous maps on manifolds, giving a rigorous definition of the intuitive concept of continuous deformation of maps. Intuitively, the scope is to classify maps by building an equivalence relation, saying that two maps are equivalent if one can be continuously deformed in the other. This has important applications in physics: as the time evolution of a field is a continuous deformation of the field, we can say that, if the field at a certain

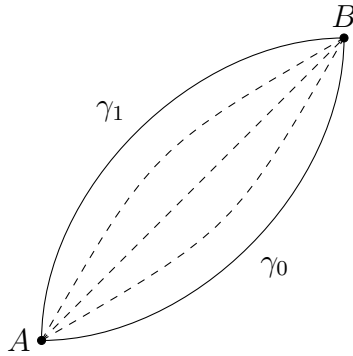


Figure 1.1: A continuous deformation of curves in the plane. Curves are a particular case of the maps that we are considering, with  $X = \mathbb{R}$  or an interval and  $Y = \mathbb{R}^2$ . The curve  $\gamma_0$  gets deformed in the curve  $\gamma_1$ , and we call the deformation function  $H$ , with one argument more than  $\gamma_0$  and  $\gamma_1$ : the intermediate curves (dashed lines) are given by fixing the value of the additional argument of  $H$  to some chosen values between 0 and 1.

time is represented by certain boundary conditions, then the image of the time evolution is contained in the homotopy class of the initial condition. Let us then give the basic definitions.

Let  $X$  and  $Y$  be differentiable manifolds, and let  $x_0 \in X$  and  $y_0 \in Y$  be two certain points on the manifolds: we define the set of based maps  $\mathcal{F}$  as the set of continuous maps between  $X$  and  $Y$  such as, for every  $f \in \mathcal{F}$ ,  $f(x_0) = y_0$ , with  $x_0$  and  $y_0$  called base points. The choice of the base points is arbitrary (assuming connectedness of  $X$ ), and all our results will not depend on the choice of base points.

We now define the *continuous deformation* of maps. Let  $f, g \in \mathcal{F}$  and  $x$  be any point in  $X$ : we define the function (if it exists)

$$H : X \times [0, 1] \rightarrow Y \quad (1.1.1)$$

such as  $H(x, 0) = f(x)$ ,  $H(x, 1) = g(x)$ ,  $H(x_0, t) = y_0$  for all  $t \in [0, 1]$  and with the requirement that  $H$  be continuous in its arguments. Such a function is a continuous deformation between  $f$  and  $g$ , and we say that  $H$  deforms  $f$  into  $g$ .

The existence of such a map is non trivial. As an example, we can take

(1.2), where the dot internal to the left square is removed from the domain. In that case, we see that the figures on the left cannot be deformed into each other without crossing the point, while the figures on the right can.

We now introduce an equivalence relation over  $\mathcal{F}$ . We say that  $f \sim g$  if a continuous deformation of  $f$  into  $g$  exists. This is an equivalence relation, as

- It is trivially reflexive, as for every  $f \in \mathcal{F}$  the function  $H(x, t) = f(x)$  continuously deforms  $f$  into itself.
- It is symmetric, as for every  $f, g \in \mathcal{F}$  such as  $f \sim g$ , calling  $H(x, t)$  the continuous deformation of  $f$  into  $g$ , the function  $H'(x, t) = H(x, 1 - t)$  deforms  $g$  into  $f$ .
- It is transitive, as taking  $f, g, h \in \mathcal{F}$  such as  $H(x, t)$  deforms  $f$  into  $g$  and  $S(x, t)$  deforms  $g$  into  $h$ , then the function

$$L(x, t) = \begin{cases} H(x, 2t) & 0 \leq t \leq \frac{1}{2} \\ S(x, 2t - 1) & \frac{1}{2} \leq t \leq 1. \end{cases} \quad (1.1.2)$$

deforms  $f$  into  $h$ .

The equivalence relation  $\sim$  splits  $\mathcal{F}$  into disjoint equivalence classes, the quotient space  $\mathcal{F}/\sim$ . In the trivial case of figure (1.1) we have that the quotient space is isomorphic to  $\mathbb{Z}_1$ , the group with only one element, as every curve can be continuously deformed into one another. We state without proof that, in the less trivial case of  $\mathbb{R}^2$  with one point removed (figure (1.2)), the quotient space of the based curves modulo the equivalence relation is isomorphic to  $\mathbb{Z}$ , and two curves are equivalent if they wind around the removed point an equal number of times (counting orientation).

We introduce a procedure that is standard when dealing with classical field theory, the compactification of  $\mathbb{R}^n$ . Let us suppose that we have a field theory of a field  $U$  defined on  $\mathbb{R}^n$  with a target space  $Y$ , with the field such as  $\lim_{|x| \rightarrow \infty} U(x) = y_0$ , independent of the direction of the limit (this is necessary for fields with energy density terms of the form  $\partial_i U \partial_i U$ : to have it tend to zero as  $|x| \rightarrow \infty$  we have that the field must tend to a constant field at infinity). Then we can consider the space  $\mathbb{R}^n \cup \{\infty\}$  as the domain of  $U$ , adding the point at infinity. This set is compact

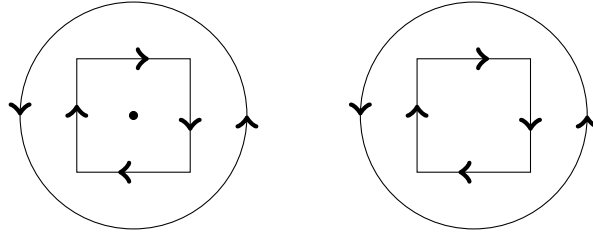


Figure 1.2: Examples of nontrivial homotopy: the dot in the middle of the square is removed from the plane. We see that the left circle winds around the removed dot once, clockwise, while the inner square winds around counterclockwise. The right figures do not wind around the circle. The result is that the right figures are homotopic, while the left are not (and no left figure is homotopic to any right figure).

(as divergent successions can be interpreted as successions converging to the point at infinity), and isomorphic to the  $n$  sphere  $S^n$  (one possible isomorphism is the stereographic projection exemplified for  $S^1$  in figure (1.3), with the pole identified with the point at infinity).

Motivated by this procedure, we specialize to the case where  $X$  is an  $n$ -sphere,  $S^n$ . The set of equivalence classes of the continuous maps is called  $\pi_n(Y)$ , also called the  $n$ -th homotopy group. This set can be endowed with a group structure for  $n \geq 1$ , defining a composition between equivalence classes. We see that in detail for  $n = 1$  and give an intuitive construction of the operation for  $n \geq 1$ . In the case  $n = 1$ , the maps  $f : S^1 \rightarrow Y$  are based loops in  $Y$  at the point  $f(0)$ . Taking two maps  $f(\theta)$  and  $g(\theta)$  with  $f(0) = g(0)$  as base point condition, we can compose them through the function

$$h(\theta) = \begin{cases} f(2\theta) & 0 < \theta < \pi \\ f(2\pi) - f(0) + g(2\theta - 2\pi) & \pi < \theta < 2\pi. \end{cases} \quad (1.1.3)$$

We can see that this operation can be written as an operation between homotopy classes: we say that the composition of the class of  $f$  and the class of  $g$  gives the class of  $h$ . For it to be a good definition, we must prove that the class that is obtained by composing one representative of the class of  $f$  with a representative of the class of  $g$  does not depend on the particular representative. Let  $\tilde{f}$  be a deformation of  $f$  through  $F(\theta, t)$  and  $\tilde{g}$  be a deformation of  $g$  through  $G(\theta, t)$ . We



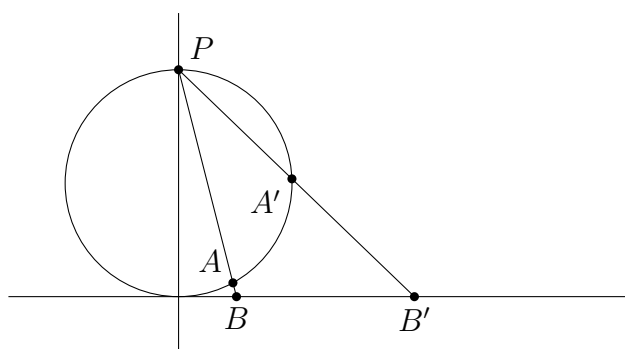


Figure 1.3: Stereographic projection of the line on a circle. The points  $B$  and  $B'$  get mapped to  $A$  and  $A'$ . Getting nearer to  $P$  on the circumference results in a point in a line with arbitrarily high increasing distance from the origin, so  $P$  can be interpreted as the point at infinity on the line.

define  $\tilde{h}$  as the map obtained by composing  $\tilde{f}$  with  $\tilde{g}$ . We can see that the map

$$H(\theta, t) = \begin{cases} F(2\theta, t) & 0 < \theta < \pi \\ F(2\pi, t) - F(0, t) + G(2\theta - 2\pi, t) & \pi < \theta < 2\pi \end{cases} \quad (1.1.4)$$

deforms  $h$  into  $\tilde{h}$ , so if we compose different representatives we obtain a different function, but all functions obtained by composing all possible couples of representatives are homotopic to each other. We can then define this composition operation as an operation on  $\pi_1(Y)$ , and this operation effectively makes  $\pi_1(Y)$  a group, as

- The operation is obviously closed (composition of loops is always a loop, and any loop is contained in an element of  $\pi_1(Y)$ ) and associative, as the composition of  $f, g, h$  is independent of whether we compose  $f, g$  and then compose the result with  $h$  or we compose  $g, h$  and then compose the result with  $f$ .
- The null element of the group is the class of the constant function.
- The inverse element of a class  $[f]$  is given by the class of  $f(2\pi - \theta)$ , the same loop reversed.
- This operation is in general non commutative.

To build the operation for  $\pi_n(Y)$ , a common way of defining it is to modify  $S^n$  back into  $R^n$  and then into a cube, where the boundary of the cube represents the point at infinity. The cube is the tensor product of  $n$  intervals  $[0, 1]$ : we compose a function on its  $n$  coordinates  $x_i$  with another through

$$h(x_1, \dots, x_n) = \begin{cases} f(2x_1, \dots, x_n) & 0 < x_1 < \frac{1}{2} \\ f(1, x_2, \dots, x_n) - f(0, x_2, \dots, x_n) + g(2x_1 - 1, \dots, x_n) & \frac{1}{2} < x_1 < 1. \end{cases} \quad (1.1.5)$$

As before, it can be shown that this defines an operation over  $\pi_n(Y)$ . The only difference from the  $\pi_1(Y)$  case is that the operation is abelian. We quote those results without proof, referring to [33] (chapter 4) for a complete discussion. We conclude analyzing  $\pi_0(Y)$ : as  $S^0$  is the boundary of a segment, it has only two points. One point is used to provide the base condition, so maps are characterized by their values on the other point. If two different based maps map the second point in two different points, those maps are homotopic if and only if the points can be joined with a path, so  $\pi_0(Y)$  counts the number of disconnected pieces composing  $Y$ .

Considering a field theory of a field  $U$  from  $\mathbb{R}^n$  to a target space  $Y$  with the boundary condition  $\lim_{|x| \rightarrow \infty} U(x) = \text{const.}$ , independent of the direction, the relevant homotopy group is  $\pi_n(Y)$ . The computation of the homotopy sets  $\pi_n(Y)$  is highly non trivial, and we usually refer to literature for those results. We conclude this introduction on homotopy groups with a monodimensional example.

We study the homotopy group  $\pi_1(S^1)$ , analyzing all continuous functions of the circle into itself. A function on a circle is written as  $f(\theta)$ , where  $\theta$  is a coordinate on a circle. We take as base points 0 both on domain and target space, imposing  $f(0) = 0$ . By continuity,  $f(2\pi) = f(0) + 2\pi m = 2\pi m$  for some integer  $m$ , called the winding number of the map. Every based continuous map has then an associate integer, its winding number. It is easy to see that two maps with the same  $m$  can be deformed into one another: taking  $f$  and  $g$  with winding number  $m$ , then the function

$$h(\theta, t) = (1 - t)f(\theta) + tg(\theta) \quad (1.1.6)$$

always preserves the base point mapping (as  $h(0, t) = 0$  for all  $t$ ) and is continuous in  $\theta = 2\pi$  for all values of  $t$ , as with fixed  $t$

$$h(2\pi, t) = (1 - t)f(2\pi) + tg(2\pi) = 2\pi m. \quad (1.1.7)$$

$h$  respects the continuity condition on  $S^1$ , so it is an acceptable deformation of  $f$  into  $g$ . Conversely, if  $f$  and  $g$  have two different winding numbers, the same function cannot work (as  $h$  would not be continuous in  $2\pi$ ), and there is no deformation of  $f$  into  $g$ , as any deformation of  $f$  deforms it into a function with the same winding number (as always, this is needed for continuity). We conclude that we can associate an integer to any map by looking at its value on  $2\pi$ , and if (and only if) two maps have that same integer they can be deformed into one another. As there is no limit on the winding number (that can be negative, too), we conclude that  $\pi_1(S^1) = \mathbb{Z}$ . We also study the composition between maps: the composed map  $s(t)$ , obtained through applying composition (1.1.3) to the maps  $f$  and  $g$  has winding number obtained through  $s(2\pi) - s(0) = f(2\pi) + g(2\pi) - f(0) = 2\pi(m + n)$ : the composition of two maps with given winding numbers gives a map with winding number given by the sum of the original winding numbers. The group  $\pi_1(S^1)$  is then abelian and the sum is identical to the sum in  $\mathbb{Z}$ , so  $\pi_1(S^1)$  and  $\mathbb{Z}$  are basically the same group.

### 1.1.2 Topological degree

It is a difficult task to compute homotopy groups, and it is a (less) difficult task to assign functions to elements in the homotopy group, once calculated. There is a tool that greatly helps solving the last problem, converting it in the calculation of an integral of the field. Let us examine this tool.

Let  $X$  and  $Y$  be two oriented manifolds with the same dimension (that we call  $n$ ), and let  $U$  be a representative of continuous based maps from  $X$  to  $Y$ . Topological degree is defined only for maps between manifolds with the same dimension (while homotopy groups are non trivial even if the target space has not the same dimension of the starting sphere), so we are restricting our analysis. Let  $\Omega$  be a volume form on  $Y$  such as

$$\int_Y \Omega = 1. \quad (1.1.8)$$

We define the *topological degree* of  $U$  as the integral on  $X$  of the pull back of  $\Omega$  through  $U$ :

$$\deg U = \int_X U^*(\Omega). \quad (1.1.9)$$

In components, introducing coordinates  $\mathbf{x}$  over  $X$  and  $\mathbf{y}$  over  $Y$ , expressing the action of  $U$  as a law that transforms the coordinates in  $X$  into coordinates in  $Y$  and denoting this map as  $\mathbf{y}(\mathbf{x})$ , decomposing  $\Omega$  as a  $n$ -form over  $Y$

$$\Omega = \Omega(\mathbf{y}) dy^1 \wedge dy^2 \wedge \dots \wedge dy^n, \quad (1.1.10)$$

the pullback of  $\Omega$  is given by

$$U^*(\Omega) = \Omega(\mathbf{y}(\mathbf{x})) \det \left( \frac{\partial y^i(\mathbf{x})}{\partial x^j} \right) dx^1 \wedge dx^2 \wedge \dots \wedge dx^n. \quad (1.1.11)$$

We see that the result of the pullback is to insert the determinant of the Jacobian of the map, that we henceforth call  $\mathcal{J}(\mathbf{x})$ . This is an  $n$ -form over  $X$ , so it can be integrated over the manifold (this is the reason why we choose  $X$  and  $Y$  with the same dimension).

We can show that  $\deg U$  does not depend on the choice of  $\Omega$ , as long as the volume form is normalized. Choosing another volume form  $\tilde{\Omega}$ , we get that the difference of the forms  $\Omega - \tilde{\Omega}$  is another form with null integral. We state without proof (it is a consequence of de Rham's theorem, as explained in chapter 6 of [33]) that this difference is an exact form: as pull back commutes with exterior derivative, then  $U^*(\Omega - \tilde{\Omega})$  is an exact form, integrating over  $X$  to zero. Thus, the degree of the map does not depend on the normalized volume form that is chosen, and this fact can be used to prove that the degree of the map is an integer.

We introduce another method of defining the topological degree, and we show that this definition equals to (1.1.9). Let  $\mathbf{y}$  be a point in the target space such as its counterimage (the set of points in  $X$  mapped to  $\mathbf{y}$ ) is a discrete, finite set, denoting this set as  $\{\mathbf{x}_1, \dots, \mathbf{x}_M\}$  (the set can be empty). We assume that such a point exists, and that at any counterimage of such a point the Jacobian determinant is non null as, if it was, then we would not have a discrete set of counterimages, but a continuous one. We define the modified degree

$$\widetilde{\deg} U = \sum_{i=1}^M \text{sign}(\mathcal{J}(\mathbf{x}_i)). \quad (1.1.12)$$

This procedure is called counting preimages with sign. Apparently, the degree depends on  $\mathbf{y}$  but, when we show that any degree is equal to (1.1.9), then we will have shown  $\mathbf{y}$  independence. This equality is consequence of the  $\Omega$  independence in the definition of (1.1.9). We can choose a volume form that has compact support only in a small neighborhood of  $\mathbf{y}$ : then the pullback of the form will give a

distribution that has compact support, only in the neighborhoods of the points  $\mathbf{x}_i$ , and the integration can be expressed as a sum of integrations on those neighborhoods. Restricting our attention to a neighborhood of one  $\mathbf{x}_i$ , we know that if the neighborhood is small enough (and it can always be made smaller by modifying  $\Omega$ ) then the map  $\mathbf{y}(\mathbf{x})$  is locally invertible, and then we can transform back to the  $y$  coordinates. This means that we gain a factor  $|J|^{-1}(\mathbf{x})$ , exactly compensating the factor  $J(\mathbf{x})$  in (1.1.11) up to a sign. Every contribution from the neighborhoods with this particular volume form gives  $\pm 1$ , so it is equal to counting preimages with signs.

The fact that  $\text{deg}$  is integer also has a nice consequence: as the integral should continuously change for deformations of the field  $U$ , then the degree should be continuous in  $U$ . As it is an integer, the only possibility is that the degree of  $U$  is equal to the degree of any map that is obtained by continuously deforming  $U$ . This means that  $\text{deg}$  can be thought as a function from the equivalence classes of deformable functions to the integers, and allows us to understand to which equivalence class a field belongs.<sup>1</sup>

As before, we finish this part by computing the topological degree of the maps between  $S^1$  and  $S^1$ . We start by giving a volume form on  $S^1$ :

$$\Omega = \frac{1}{2\pi} d\theta. \quad (1.1.13)$$

The  $2\pi$  factor normalizes the form. The pullback through a function  $f$  is given by

$$f^*(\Omega) = \frac{1}{2\pi} \frac{df}{d\theta} d\theta. \quad (1.1.14)$$

The topological charge is given by

$$\text{deg } f = \frac{1}{2\pi} \int_0^{2\pi} \frac{df}{d\theta} d\theta = \frac{1}{2\pi} (f(2\pi) - f(0)) = m. \quad (1.1.15)$$

We see that, in this simple case, the degree of the map is equal to its winding number. The topological degree is then additive under composition: a composition of two fields of charge  $m$  and  $n$  gives a field of charge  $m + n$ . We will see that this is the case in every theory we study, when we find a suitable composition.

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<sup>1</sup>We showed that homotopic fields have the same topological degree, but we did not show that fields that are not homotopic cannot have the same degree. This is non trivial, and is necessary to totally substitute homotopy groups with topological rank, a simpler tool to use, preferable whenever possible. It turns out that this fact is true in the particular theories we will study. For a proof, see chapter 23 of [51]

## 1.2 Classical field theory and topology

### 1.2.1 General overview: solitons and Derrick's theorem

We start exploring the applications of topology to classical field theories. We restrict our attention to class of theories of the form

$$\phi : \mathbb{R}^{d+1} \rightarrow Y, \quad (1.2.1)$$

where  $\phi$  is the field,  $\mathbb{R}^{d+1}$  is the standard Minkowski space with a time dimension and  $d$  space dimensions and  $Y$  is called the target space. For definiteness, let us restrict to fields with no spin: our argumentations will not depend on spin and can be easily generalized.

We write a general action

$$S[\phi] = \int -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - U(\phi, \partial_0 \phi, \partial_i \phi) d^{d+1}x, \quad (1.2.2)$$

with a given potential  $U$  that can also depends from time and space derivatives. We turn to the static theory, neglecting time derivatives. We can write the energy as minus the static action:

$$E[\phi] = \int \frac{1}{2} \partial_i \phi \partial_i \phi + U(\phi, 0, \partial_i \phi) d^d x. \quad (1.2.3)$$

The field  $\phi$  is a static field, sending  $\mathbb{R}^d$  into the target space  $Y$ . We must make some assumptions on the potential  $U$  before continuing. Those assumptions are related to the request that the physical fields have finite energy: this means that the integrand in (1.2.3) must tend to zero as  $|x| \rightarrow \infty$  fast enough for convergence. This means that the field should tend to a constant value, independent of the direction, as with a direction dependent boundary condition the effective potential energy density would not vanish, making the integral diverge. Thus we impose the boundary condition

$$\lim_{|x| \rightarrow \infty} \phi(x) = \phi_0, \quad (1.2.4)$$

with  $\phi_0$  such as  $U(\phi_0, 0, 0) = 0$ , with 0 the minimum of  $U$ . With condition (1.2.4), we can reformulate the theory: as in the previous section, we compactify  $\mathbb{R}^d$  to  $S^d$

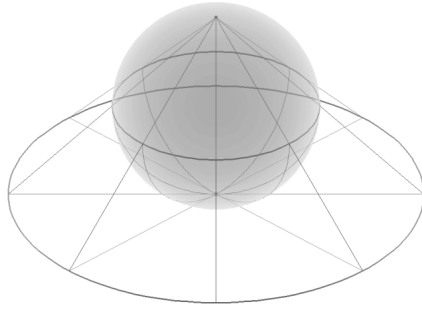


Figure 1.4: Stereographic projection in two dimensions, mapping the plane onto the sphere.

through stereographic projection, and identify the north pole of  $S^d$  as the point at infinity and the south pole as the origin of the axes.

With this compactification, the field theory becomes the theory of a field

$$\phi : S^d \rightarrow Y, \quad \phi(\text{North pole}) = \phi_0, \quad (1.2.5)$$

and the set of based maps  $\mathcal{F}$  (with obvious bases the north pole and the boundary condition) is divided in equivalence classes: this information is contained in the homotopy group  $\pi_d(Y)$ . Let us consider the situation where  $\pi_d(Y) = \mathbb{Z}$  (or a finite subset), that is the situation that we will meet in this thesis: then we label with 0 the equivalence class of the constant field  $\phi(x) = \phi_0$ . We call any field that is not in the equivalence class of the constant field and is a minimum point for the static energy (1.2.3) a *topological soliton* or *soliton*. We can also do more when we take  $Y$  such as  $\dim Y = d$ : in this case we have a natural label for each equivalence class, the topological charge of a map. We interpret the degree of the map  $\phi$  as a conserved charge, called topological charge, and adopt the standard notation to call it  $B[\phi]$  or, simply,  $B$ .  $B$  is a conserved charge in the time evolution of the system as it is homotopy invariant, and time evolution is a continuous deformation of the field.

The presence of solitons divides the configuration space (the infinite dimensional space of possible maps) in different disconnected topological sectors. It is always possible to associate a topological sector to an initial condition: then the motion of the field will always be contained in the topological sector from which it started. To find a stable configuration, the energy has to be minimized in the topological sector, finding a local minimum to the energy functional. A topolog-

ical charge  $B$  always gives a constraint on the energy of the fields, of the form  $E \geq a|B|$ , where  $a$  is just a multiplicative constant that depends on the model and  $|B|$  accounts for the fact that the topological charge can be negative. Such an inequality is called a *Bogomolny bound*. If a field of charge  $B$  with  $E = a|B|$  exists, that field is said to *saturate* the Bogomolny bound: the existence of such a field is not guaranteed, and it does not hold in models such as the Skyrme model.

To have solitons in a theory, it is necessary that the relevant homotopy group be not trivial, and for the field to be static, being a minimizer of (1.2.3): the existence of such a minimum is non trivial. In the study of a theory with solitons, we first check the relevant homotopy group to see if it is non trivial, and then we use Derrick's theorem [14]. The theorem is very simple, as it is a scaling argument: scaling coordinates and fields is a variation of the fields, and the energy must be invariant under the infinitesimal version of the variation. We provide an example with the scalar field  $\phi$  that we introduced: under rescaling  $x \rightarrow \lambda x$  (with  $\lambda > 0$ ), we scale the field as  $\phi(x) \rightarrow \phi(\lambda x)$ . We can note that every energy that is polynomial in the field and its derivatives is a sum of pieces of the form

$$E_n = \int (\partial\phi)^n \phi^m d^d x. \quad (1.2.6)$$

The notation is highly symbolic, and we do not care to specify how the derivative indexes are contracted. The only important thing is the number of derivatives,  $n$ . As an example, an effective potential term with two derivatives is  $E_2$ , while a potential with no derivatives is  $E_0$ . Under scaling, an  $E_n$  piece scales as

$$E_n \rightarrow \lambda^{n-d} E_n. \quad (1.2.7)$$

We can then write a scale dependent energy,  $E[\lambda]$ , that is composed of the sum of various pieces  $E_n$ , multiplied by  $\lambda^{n-d}$ . If there is no minimum to this function, then a static soliton does not exist. If there is such a minimum, then a static soliton can exist, but its existence is not guaranteed.

Derrick's theorem is a no-go theorem: it can suggest us the form of a potential to have stable solitons, and applying it is just a very simple algebraic matter. It must be evaded for static solitons to exist. We now study an example of application, and then propose a way to evade Derrick's theorem that is frequently used in soliton theory. When the energy functional is minimized for a certain  $\lambda$ , that  $\lambda$  becomes a soliton scale.



Let us study a simple example in  $d = 1, 2, 3$  dimensions for a theory of a scalar field with a potential that does not depend on field derivatives. The energy can be written as  $E = E_2 + E_0$ , and the scale-dependent energy is  $E[\lambda] = \lambda^{2-d}E_2 + \lambda^{-d}E_0$ . In the dimensions that we considered

$$E[\lambda] = \begin{cases} \lambda E_2 + \lambda^{-1}E_0 & d = 1, \\ E_2 + \lambda^{-2}E_0 & d = 2, \\ \lambda^{-1}E_2 + \lambda^{-3}E_0 & d = 3. \end{cases} \quad (1.2.8)$$

We see that, in  $d = 1$ , the energy functional is stationary at  $\lambda = \sqrt{E_0/E_2}$ , so static solitons can exist. In  $d = 2$ , the derivative of the energy is  $-2\lambda^{-3}E_0$ , so there is no way to set this derivative to zero (unless  $E_0 = 0$ , but this is the case of the vacuum). So there is no static soliton in a scalar field theory in two spatial dimensions. The same result (for different reasons) holds for  $d = 3$ , as in this case the energy is minimized by  $\lambda = \sqrt{-3E_0/E_2}$ . As the energy terms cannot be negative to have an energy bounded from below, such a  $\lambda$  cannot exist. Repeating the analysis for  $d = 2, 3$  adding a term  $E_4$  to the energy gives us that a minimum can be achieved, so one of the possible ways to evade Derrick's theorem is to add derivative terms to the potential.

Another subtler way to evade the theorem is to modify the theory, by gauging it. A gauge potential will scale as a derivative under rescaling,  $A(x) \rightarrow \lambda A(\lambda x)$ : this way, the covariant derivative term scales as  $D \rightarrow \lambda D$ . The field strength contains derivatives of the gauge field, so it scales as  $F \rightarrow \lambda^2 F$ . The field strength piece then scales as  $E_4$ . This is a very natural way to stabilize solitons with an electromagnetic charge, and it can be seen in use in vortex theory (chapter 7 of [32]). We will see an example of solitons in gauge theories with instantons.

### 1.2.2 Moduli space and soliton dynamics

Solitons are studied as static objects, neglecting all time dependence. The exact problem of the soliton dynamics is difficult to solve, and an approximation to obtain a dynamical model from static solutions is widely used in literature: the moduli space approximation (an example of application is in [19], while a more general introduction is given in [31] and in our main reference, chapter 4).

We review it briefly and then give an elementary example. The moduli space

approximation consists in truncating the infinite dimensional configuration space of the theory to a finite dimensional space, by assuming a rigid motion for the solitons. This is obtained using the symmetries of the theory: as their action on the fields give different fields with the same energy, the symmetries can be used to describe all equivalent configurations of fields. Those symmetries are manifest in the field configuration through the appearance of parameters that can be varied freely, without changing the energy. Those parameters are called *moduli* of the configuration, and the space where they can vary is called a *moduli space*. One easy way to understand the moduli space (at least, its continuous part) is to see it as the orbit of a static soliton under the continuous part of the symmetry group of the theory  $\mathcal{G}$ . This is not enough, as we can almost always find a subgroup of  $\mathcal{G}$  that leaves the field invariant, called the stabilizer  $\mathcal{H}$  of the group:  $\mathcal{H}$  must be quotiented out (ore else we would be overcounting degrees of freedom), so the moduli space is given by  $\mathcal{M} = \mathcal{G}/\mathcal{H}$ . An example of such a calculation is given in [28].

$\mathcal{M}$  is often a differentiable manifold, and there is a way to define a metric on it, starting from the original action. To be concrete, let us introduce a concise notation. Suppose a field configuration has  $N$  continuous moduli of any kind: we denote the field with moduli specified by the  $N$ -vector  $X$  of components  $a_i$  as  $\phi(x, t; X)$ . Moduli space approximation consists in making the field depend on time only through the moduli, that are taken as time dependent:  $\phi(x, t; X) = \phi(x, X(t))$ . Let us suppose the kinetic energy density for the field is of the standard form  $T = \int (1/2) \partial_0 \phi \partial_0 \phi d^d x$ : in the moduli space approximation, this is rewritten making use of chain derivatives:

$$T = \frac{1}{2} \int \partial_0 \phi \partial_0 \phi d^d x = \frac{1}{2} \left( \int \frac{\partial \phi}{\partial X^i} \frac{\partial \phi}{\partial X^j} d^d x \right) \dot{X}^i \dot{X}^j = \frac{1}{2} g_{ij}(X) \dot{X}^i \dot{X}^j, \quad (1.2.9)$$

with an obvious definition for  $g_{ij}(X)$ , that is a rank two, symmetric tensor. We will always check the rank of the metric, that will turn out to be maximal in most problems: this way, the tensor  $g_{ij}$  can be used as a metric on moduli space, and the problem of solving the dynamics of a field is approximated to the problem of finding the geodesic motion of the moduli in the moduli space  $\mathcal{M}$ . This approximation can be used in the case of adiabatic motion, when the kinetic energy is little compared to the energy of the soliton. A formal approach to moduli space approximation and its range of applicability has still to be proven in general field theory, although some studies of particular cases exist, and are cited in [32]. We will ignore those

questions and follow most works in this sector [11, 12, 24, 28] by making use of moduli space approximation.

We will see examples of calculation of moduli space in the next sections, where we examine three particular field theories with topological solitons.

## 1.3 Examples of topology in field theory

### 1.3.1 The Baby Skyrme model

The Baby Skyrme model (studied first with general potential in [36], while a presentation in more modern form can be found in [37]) is a (2+1) dimensional theory of a scalar field

$$\phi : \mathcal{M}^3 \rightarrow S^2 \quad (1.3.1)$$

from Minkowski space to an ordinary 2-sphere. The field is represented as a vector of three fields,  $\phi = (\phi_1, \phi_2, \phi_3)$ , under the constraint  $\phi \cdot \phi = 1$ . Setting  $c = 1$ , the Lagrangian is

$$\mathcal{L} = F \left( -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\kappa}{4} (\partial^\alpha \phi \times \partial^\beta \phi)^2 - \mu^2 (1 - \mathbf{n} \cdot \phi) \right), \quad (1.3.2)$$

where  $\mathbf{n}$  is a constant versor, and we choose coordinates on the target manifold such as  $\mathbf{n} = (0, 0, 1)$ .  $F$  has the dimension of energy, while  $\kappa$  and  $\mu$  have dimension of length: we will choose units such as  $F = 1$  and  $\kappa = 1$ . The static theory energy is given by

$$E = \int \frac{1}{2} \partial_i \phi \partial_i \phi + \frac{\kappa}{4} (\partial_i \phi \times \partial_j \phi)^2 + \mu^2 (1 - \mathbf{n} \cdot \phi) d^2 x. \quad (1.3.3)$$

Motion equations are obtained by varying the field as  $\phi \rightarrow \phi + \epsilon \times \phi$ , automatically satisfying the constraint  $\phi \cdot \phi = 1$  after variation. The motion equations are

$$\partial_i (\phi \times \partial_i \phi + \partial_j \phi \cdot (\partial_j \phi \cdot (\phi \times \partial_i \phi))) = \mu^2 \mathbf{n} \times \phi, \quad (1.3.4)$$

supported by the boundary condition

$$\lim_{|x| \rightarrow \infty} \phi = \mathbf{n}. \quad (1.3.5)$$

With this boundary condition, we can compactify  $\mathbb{R}^2$  to  $S^2$ , and view the static Baby Skyrme field (that we will still call  $\phi$ ) as a map

$$\phi : S^2 \rightarrow S^2. \quad (1.3.6)$$

We now quote a standard result from topology, that can be found in any topology textbook:  $\pi_2(S^2) = \mathbb{Z}$ . This is an hint on existence of solitonic solutions, supported also by the fact that, with four derivatives in the potential, Derrick's theorem does not exclude stable solitons. We can perform a pullback of the standard volume element on  $S^2$  to obtain the topological charge<sup>2</sup>. In order to obtain a topological density expressed as a functional of the constrained coordinates on  $S^2$  ( $\phi_1, \phi_2, \phi_3$ ), we need to express the standard normalized metric on  $S^2$  in terms of the three coordinates. The standard volume element inherited from the standard metric is given by

$$d\Omega = \frac{1}{4\pi} \sin \theta d\theta \wedge d\varphi. \quad (1.3.7)$$

The  $S^2$  angles are related to the constrained coordinates as

$$\theta = \arctan \frac{\sqrt{\phi_1^2 + \phi_2^2}}{\phi_3}, \quad \phi = \arctan \frac{\phi_2}{\phi_1}, \quad (1.3.8)$$

(in particular, note that  $\sin \theta = \sqrt{1 - \phi_3^2}$ ) so the forms relative to the coordinates are

$$d\theta = \frac{\phi_3}{\sqrt{1 - \phi_3^2}} (\phi_1 d\phi_1 + \phi_2 d\phi_2) - \sqrt{1 - \phi_3^2} d\phi_3, \quad (1.3.9a)$$

$$d\varphi = -\frac{\phi_2}{1 - \phi_3^2} d\phi_1 + \frac{\phi_1}{1 - \phi_3^2} d\phi_2. \quad (1.3.9b)$$

Plugging these expressions in the volume form, we get

$$d\Omega = \frac{1}{4\pi} (\phi_1 d\phi_2 \wedge d\phi_3 + \phi_2 d\phi_3 \wedge d\phi_1 + \phi_3 d\phi_1 \wedge d\phi_2). \quad (1.3.10)$$

We see that the volume element is expressed as a sum of three volume elements: in each one, we can choose the coordinates in the wedge product as independent coordinates, and the latter coordinate as dependent from the other two. We can do the pullback term by term. We still use  $(x_1, x_2)$  coordinates on the domain

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<sup>2</sup>As this is the first appearance of a topological degree, we do the computation explicitly. The computation ends at (1.3.13), so one can skip directly to the result and avoid calculations

(with derivatives  $(\partial_1, \partial_2)$ ), so we explicitly write the Jacobian for the pullback of the third term in the sum:

$$\frac{\partial\phi_{1,2}}{\partial x_{1,2}} = \begin{pmatrix} \partial_1\phi_1 & \partial_1\phi_2 \\ \partial_2\phi_1 & \partial_2\phi_2 \end{pmatrix}, \quad \mathcal{J} = \partial_1\phi_1\partial_2\phi_2 - \partial_1\phi_2\partial_2\phi_1, \quad (1.3.11)$$

so the contribution is

$$\frac{1}{4\pi}\phi_3(\partial_1\phi \times \partial_2\phi)_3 dx_1 \wedge dx_2. \quad (1.3.12)$$

Summing every contribution, we obtain the topological charge expression, usually written as

$$B[\phi] = \frac{1}{8\pi} \int \epsilon_{ij}\phi \cdot (\partial_i\phi \times \partial_j\phi) dx_1 dx_2. \quad (1.3.13)$$

We can now find our first example of Bogomolny bound. We start by noting that the energy is a sum of positive terms, so we can concentrate on finding a bound for a single term. Consider the inequality

$$(\partial_i\phi \pm \epsilon_{ij}\phi \times \partial_j\phi)^2 \geq 0. \quad (1.3.14)$$

We can write it as

$$\partial_i\phi\partial_i\phi + (\phi \times \partial_i\phi)^2 \mp \epsilon_{ij}\phi \cdot (\partial_i\phi \times \partial_j\phi) \geq 0. \quad (1.3.15)$$

The second term can be written as  $\partial_i\phi\partial_i\phi$  due to the constraint  $\phi \cdot \phi = 1$  and its consequence  $\phi \cdot \partial_i\phi = 0$ . Integrating the equation, we obtain

$$E_2 \geq \pm 2\pi B. \quad (1.3.16)$$

We can now choose the sign to give the most stringent bound. If the field configuration has positive  $B$  we choose  $+$ , and if it has negative  $B$  we will choose  $-$ . We obtain our Bogomolny bound as

$$E \geq 2\pi|B|. \quad (1.3.17)$$

In every topological sector, the energy is bounded from below by a value that is greater than zero. Equality is attained for those field configurations that solve (1.3.14), without the square and with an  $=$  instead of a  $\geq$ . The existence of fields that satisfy  $E = 2\pi|B|$  is not guaranteed.

We can look for solutions of the form

$$\boldsymbol{\phi} = (\sin f(r) \cos \theta, \sin f(r) \sin \theta, \cos f(r)), \quad (1.3.18)$$

where  $r$  and  $\theta$  are polar coordinates on the plane, and write the topological charge in terms of the function  $f$ . It turns out that the boundary conditions that  $f$  must obey to have a charge 1 field are  $f(\infty) = 0$  (for  $\boldsymbol{\phi}(\infty) = \mathbf{n}$ ) and  $f(0) = \pi$  (for  $B = 1$ ). The energy for such a field is a functional of  $f$ :

$$E = \pi \int_0^\infty \left( \frac{1}{2} f'^2 + \frac{(\sin f)^2}{2r^2} (1 + f'^2) + m^2 (1 - \cos f) \right) r dr. \quad (1.3.19)$$

To have a minimum of the energy, we must have

$$\left( 1 + \frac{(\sin f)^2}{r^2} \right) f'' + \left( 1 - \frac{(\sin f)^2}{r^2} \right) \frac{f'}{r} + \frac{\sin 2f}{2r^2} (f'^2 - 1) - m^2 \sin f = 0. \quad (1.3.20)$$

This equation has been solved numerically in [38], obtaining a value of  $E = 1.564 \cdot 4\pi$  for the value  $m = 1$ , so the Bogomolny bound is exceeded.

The moduli space of this solution becomes evident when we note that all fields of the form

$$\boldsymbol{\phi} = (\sin f(|\vec{r} - \vec{R}|) \cos(\theta + \chi), \sin f(|\vec{r} - \vec{R}|) \sin(\theta + \chi), \cos f(|\vec{r} - \vec{R}|)), \quad (1.3.21)$$

where  $\vec{r}$  are the coordinates,  $\vec{R}$  is an  $\mathbb{R}^2$  vector representing the position of the Baby Skyrmion,  $\theta$  is the polar angle from the center  $\vec{R}$  and  $\chi$  is an angle, representing the phase of the object, have the same energy as the original configuration. There is no stabilizer, so the moduli space is  $\mathcal{M} = \mathbb{R}^2 \times S^1$ . We do not enter in detail in the calculation of the metric on the moduli space.

### 1.3.2 The Skyrme model

The Skyrme model, proposed by Skyrme in [43, 44], is a variation of the standard sigma model that admits stable, solitonic solutions. It has been proposed by Skyrme to model baryon and meson dynamics, and it has been used in different ways to give examples of quantum effective theories of nuclear interactions [2, 12, 20, 28].

The model is a theory of a field

$$U : \mathcal{M}^4 \rightarrow SU(N), \quad (1.3.22)$$

with Lagrangian

$$\mathcal{L} = -\frac{F_\pi^2}{16} \text{tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{32e^2} \text{tr}([\partial_\mu U U^\dagger, \partial_\nu U U^\dagger][\partial^\mu U U^\dagger, \partial^\nu U U^\dagger]). \quad (1.3.23)$$

$F_\pi$  and  $e$  are phenomenological constants, and we can scale them away using  $F_\pi/4e$  as unit of energy and  $2/eF_\pi$  as unit of length. The four derivative term provides stability for the soliton, but it is not the unique choice and other stabilizing terms have been studied ([1]). A mass term can be added, of the form  $m^2 \text{tr}(1 - U)$ , but we will neglect it. Motion equations are obtained by performing a left or right variation on  $U$ ,  $\delta U = LU$  (with  $L = -L^\dagger$ ) in the case of left variation, obtaining the motion equation

$$\partial_\mu \left( R^\mu + \frac{1}{4} [R^\nu, [R_\nu, R^\mu]] \right) = 0, \quad (1.3.24)$$

where  $R$  is the right invariant current,  $R_\mu = \partial_\mu U U^\dagger$ . The static energy is

$$E = -\frac{1}{12\pi^2} \int \frac{1}{2} \text{tr}(R_i R_i) + \frac{1}{16} \text{tr}([R_i, R_j]^2) d^3x. \quad (1.3.25)$$

This theory has the usual symmetry  $SU(N)_L \times SU(N)_R$ , but we need to choose a boundary condition to have finite energy: choosing  $U(\infty) = \mathbf{1}$ , we explicitly break the symmetry group to an  $SU(N)$  group, acting as  $SU(N) : U \rightarrow VUV^\dagger$ . Having chosen a boundary condition, we can compactify  $R^3$  to  $S^3$  and use the topological result  $\pi_3(SU(N)) = \mathbb{Z}$ . Derrick's theorem is evaded by the presence of the four derivatives term, so the theory can admit topological solitons, that we call Skyrmions.

Specializing for  $N = 2$ , as  $\dim SU(2) = \dim S^3 = 3$ , we can find a topological charge and a Bogomolny bound. Using the standard, left and right invariant, volume form on  $SU(2)$

$$\Omega = \frac{1}{24\pi^2} \text{tr}[dUU^{-1} \wedge dUU^{-1} \wedge dUU^{-1}], \quad (1.3.26)$$

we can pull back through  $dU = \partial_i U U^\dagger dx^i = R_i dx^i$  and substitute: the integral of the form is the topological charge, that reads (the right normalization factor is the same as in [32])

$$B[U] = -\frac{1}{24\pi^2} \int \text{tr}[R_i R_j R_k] \epsilon_{ijk} d^3x. \quad (1.3.27)$$

Through the topological charge, a Bogomolny bound is found. To find it, we use a geometrical description of the Skyrmion, formulated in [30]. Let us define the stress tensor

$$D_{ij} = -\frac{1}{2} \text{Tr}[R_i R_j]. \quad (1.3.28)$$

This is symmetric and positive definite, so it can be diagonalized and it has three positive eigenvalues,  $\lambda_1^2(x)$ ,  $\lambda_2^2(x)$ ,  $\lambda_3^2(x)$ , depending on the position  $x$  on which the field is evaluated when computing  $D$ . In terms of those objects, the energy and the topological charge are written as

$$E = \frac{1}{12\pi^2} \int (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_3^2 + \lambda_2^2 \lambda_3^2) d^3x \quad B = \frac{1}{2\pi^2} \int \lambda_1 \lambda_2 \lambda_3 d^3x. \quad (1.3.29)$$

From the inequality

$$(\lambda_1 \pm \lambda_2 \lambda_3)^2 + (\lambda_2 \pm \lambda_3 \lambda_1)^2 + (\lambda_3 \pm \lambda_1 \lambda_2)^2 \geq 0, \quad (1.3.30)$$

we can check that

$$E \geq |B| \quad (1.3.31)$$

holds.

We now try to find a  $B = 1$  field and study its properties. We need an ansatz, and we'll use the hedgehog ansatz. To explain it, we note that any  $U$  can be written as

$$U = \exp(i\pi_a(x)\sigma_a), \quad (1.3.32)$$

where  $\sigma_a$  are the standard Pauli matrices, normalized as  $\text{tr}[\sigma_a \sigma_b] = 2\delta_{ab}$ ,  $a$  runs from 1 to 3 and the  $\pi_a$  are scalar fields, the pion fields from familiar effective QCD. The hedgehog ansatz consists in using a radial form for the pion field, of the form<sup>3</sup>

$$\pi_a(x) = f(r) \frac{x_a}{r}. \quad (1.3.33)$$

The radial profile function  $f(r)$  has the boundary condition  $f(\infty) = 0$  (to have  $U(\infty) = \mathbf{1}$ ). In terms of the profile function, the energy and topological charge become

$$E = \frac{1}{3\pi} \int_0^\infty \left( r^2 f'^2 + 2(\sin f)^2 (1 + f'^2) + \frac{(\sin f)^4}{r^2} \right), \quad (1.3.34a)$$

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<sup>3</sup>We are mixing space indexes  $i$  and target indexes  $a$ , allowing  $x$  to have target indexes. Such mixing of indexes often appears when writing explicit solutions of field theories



$$B = -\frac{2}{\pi} \int_0^\infty f'(\sin f)^2 dr = \frac{1}{\pi} f(0). \quad (1.3.34b)$$

We see that the profile function must obey  $f(0) = \pi$  to have  $B = 1$ .  $f$  solves the equation

$$(r^2 + 2(\sin f)^2)f'' + 2rf' + \sin 2f \left( f'^2 - 1 - \frac{(\sin f)^2}{r^2} \right) = 0. \quad (1.3.35)$$

The energy of this Skyrmion is  $E = 1.232$ , so the Bogomolny bound is exceeded. It has been shown in [16] that this field configuration has the minimum energy in this sector.

We now study the moduli space of the Skyrmion hedgehog. We can act on the hedgehog in three ways: we can translate it by a constant 3-vector  $\vec{R}$ , obtaining

$$U(x) \rightarrow U(x - R) = \exp \left( i \frac{f(|\vec{x} - \vec{R}|)}{|\vec{x} - \vec{R}|} (\vec{x} - \vec{R})_a \sigma_a \right). \quad (1.3.36)$$

We can use a global rotation  $M_{ij}$ , transforming the field as

$$U(x) \rightarrow U(M^{-1}x) = \exp \left( i \frac{f(r)}{r} x_b M_{ba} \sigma_a \right). \quad (1.3.37)$$

Or, we can use an isospin rotation from the residual symmetry group after symmetry breaking to transform the field as

$$U(x) \rightarrow A^\dagger U(x) A = \exp \left( i \frac{f(r)}{r} x_a A^\dagger \sigma_a A \right). \quad (1.3.38)$$

The moduli space seems to be obtained through the action of  $\mathbb{R}^3 \times SO(3)_J \times SU(2)_I$ , where  $SU(2)_I$  represents the isospin transformations and  $SO(3)_J$  represents the space rotations. We can now notice that we are overcounting the degrees of freedom. In fact, consider the function  $M : SU(2) \rightarrow SO(3)$ , defined by

$$M(E)_{ab} \sigma_b = E^\dagger \sigma_a E \quad \implies \quad M(E)_{ab} = \frac{1}{2} \text{tr} [\sigma_a E \sigma_b E^\dagger]. \quad (1.3.39)$$

This function is a two-to-one map (mapping  $E$  and  $-E$  in the same object) and can be used to relate the action of an  $SU(2)$  matrix to the action of an  $SO(3)$  matrix (mathematically, it explicitly shows that  $SU(2)$  doubly covers  $SO(3)$ , or  $SO(3) \simeq SU(2)/\mathbb{Z}^2$ ). We can obtain any space rotation from an isospin rotation, so one of those transformations is redundant. We can also note that an isorotation

by the matrix  $A$  is identical to the isorotation by the matrix  $-A$ . There is also an important property of composition:

$$M(UE)_{ab}\sigma_b = (UE)^\dagger\sigma_a UE = M(U)_{ab}E^\dagger\sigma_b E = (M(U)M(E))_{ab}\sigma_b. \quad (1.3.40)$$

This implies  $M(UE) = M(U)M(E)$ , so  $M$  preserves the group structure. In the end, we can say that the moduli space is given by the orbit of the hedgehog solution under the group

$$\mathcal{G} = \mathbb{R}^3 \times SU(2)/\mathbb{Z}^2, \quad (1.3.41)$$

acting on the hedgehog as

$$U(x) \rightarrow \exp\left(i\frac{f(|\vec{x} - \vec{X}|)}{|\vec{x} - \vec{X}|}(\vec{x} - \vec{X})_a A^\dagger \sigma_a A\right). \quad (1.3.42)$$

The coordinates on the moduli space are then a set of 3 positions and an  $SU(2)$  matrix,  $(\vec{R}, A)$ , with the identification  $(\vec{X}, A) \simeq (\vec{X}, -A)$ . This identification will play a fundamental role in the quantization of the moduli space, that we will introduce at the end of the section. The calculation of the metric consists in promoting the moduli to time dependent coordinates and evaluating the kinetical energy. We will refer to [2, 42] for details on the calculation, citing their result: choosing velocities  $v^i = \dot{R}^i$  and  $\omega_a = -i \operatorname{tr}[\dot{A}A^\dagger \sigma^a]$ , we get that the kinetic energy becomes

$$T = \frac{1}{2}M\dot{X}^i\dot{X}^i + \frac{1}{2}\Lambda\omega^a\omega^a, \quad (1.3.43)$$

where  $M$  is the single Skyrmion mass, its energy (1.3.34a), and  $\Lambda$  is the integral

$$\Lambda = \frac{16}{3}\pi \int_0^\infty r^2(\sin f)^2 \left(1 + 4\left(f'^2 + \frac{(\sin f)^2}{r^2}\right)\right) dr. \quad (1.3.44)$$

The kinetical energy is akin to the kinetical energy of a rigid body. The hedgehog dynamic is then the same dynamic of a rigid body, specified by a position in space and a frame centered on the position.

### 1.3.3 The Yang-Mills Instanton

Yang Mills (YM) theory is the theory of interaction of the mediators of an  $SU(N)$  action (when  $N = 3$ , the mediators are called gluons). It is often studied

numerically, performing a Wick rotation on the time coordinate and ending with a theory in euclidean 4-space. The fields are<sup>4</sup>

$$A_\mu : \mathbb{R}^4 \rightarrow su(N), \quad (1.3.45)$$

where  $su(N)$  is the Lie algebra of  $SU(N)$ , the space of antihermitian  $N \times N$  traceless matrices, and the euclidean action is

$$S = -\frac{1}{2g_Y^2} \int \text{tr}[F_{\mu\nu}F_{\mu\nu}]d^4x, \quad (1.3.46)$$

where  $g_Y$  is the Yang-Mills coupling. The equations of motion for this theory are given by

$$D_\mu F_{\mu\nu} = 0. \quad (1.3.47)$$

Defining

$$*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} \quad (1.3.48)$$

and also noting that  $\text{tr}[F_{\mu\nu}F_{\mu\nu}] = \text{tr}[*F_{\mu\nu} * F_{\mu\nu}]$  we can write the action as

$$S = -\frac{1}{4g_Y^2} \int (\text{tr}[(F_{\mu\nu} \mp *F_{\mu\nu})(F_{\mu\nu} \mp *F_{\mu\nu})] \pm 2 \text{Tr}[F_{\mu\nu} * F_{\mu\nu}]d^4x). \quad (1.3.49)$$

As the first term in the sum is always non negative, we can see that the action has a lower bound

$$S \geq 8\pi^2|B|, \quad (1.3.50)$$

where

$$B = -\frac{1}{16\pi^2g_Y^2} \int \text{tr}[F_{\mu\nu} * F_{\mu\nu}]d^4x. \quad (1.3.51)$$

Restricting to the case  $N = 2$ , we can recognize this quantity as the second Chern number associated to the non abelian gauge field, that is an integer (chapter 3 of [32]). We can find a more familiar topological structure by considering the boundary conditions for  $A_\mu$ . To have a finite action,  $A_\mu$  must tend to a field that is gauge equivalent to the vacuum: as the distance from origin  $r$  goes to infinity, we must have

$$A_\mu \rightarrow -\partial_\mu g^\infty (g^\infty)^{-1} \quad (1.3.52)$$

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<sup>4</sup>Check the notation section for our conventions on gauge theories.

for some  $g^\infty \in SU(2)$ , defined on the 3-sphere  $S^3$  at infinity. As  $g^\infty : S^3 \rightarrow SU(2)$ , we are in the same situation as with the Skyrmions: it turns out that the Chern number  $N$  is actually equal to the topological degree of  $g^\infty$ , exactly calculated as in (1.3.27). In this particular field theory, there exist fields such as  $S = 8\pi^2|N|$ , that automatically become local minimums for the action in the  $N$  sector: they are the self-dual and anti-self dual instanton, respecting the equation

$$F_{\mu\nu} = \pm * F_{\mu\nu} \quad (1.3.53)$$

(as can be seen from the decomposition (1.3.49)). Fields with  $+$  have positive charge, fields with  $-$  have negative charge.

There is no need to check if Derrick's theorem prohibits the existence of solitons, as the theory is conformal: this means that the action is scale invariant, and thus the static energy will be scale independent. In particular, we expect not to have a fixed soliton size. We check it anyway: under rescaling,  $A \rightarrow \mu A$  (where  $\mu$  is the scaling parameter) and  $F \rightarrow \mu^2 F$ : the only term in the action is then scale independent, as the kinetic term scales as  $\mu^4$ , exactly compensating the scaling of  $d^4x$ .

We look for a self dual instanton of topological charge  $B = 1$ . Such a solution was found first in [8], and goes under the name of *JNR ansatz*. Let us define the antisymmetric tensor

$$\sigma_{i4} = \sigma_i, \quad \sigma_{ij} = \epsilon_{ijk}\sigma_k \quad (1.3.54)$$

( $\epsilon_{ijk}$  is normalized as  $\epsilon_{123} = 1$ ). This symbol is anti self dual, while the symbol with the definition of  $\sigma_{4i} = \sigma_i$  (and the same definition for  $\sigma_{ij}$ ) is instead a self-dual symbol. As in the case of the Baby Skyrmion and the hedgehog Skyrmion, this symbol is a mixed tensor between the  $su(2)$  indexes (explicit in the fact that  $\sigma$  matrices are present) and the spacetime indexes. We state that a self dual YM field can be written as

$$A_\mu = \frac{1}{2}\sigma_{\mu\nu}\partial_\nu \log \rho, \quad (1.3.55)$$

where  $\rho(x)$  is a scalar function determined by the self duality condition. We compute the field strength:

$$F_{\mu\nu} = \frac{1}{2}(\sigma_{\nu\alpha}\partial_\mu\partial_\alpha \ln \rho - \sigma_{\mu\alpha}\partial_\nu\partial_\alpha \ln \rho) + \frac{1}{4}[\sigma_{\mu\alpha}, \sigma_{\nu\beta}]\partial_\alpha \ln \rho \partial_\beta \ln \rho. \quad (1.3.56)$$

As can be verified by direct computation, this tensor is self dual up to a term proportional to  $\partial_\mu \partial_\mu \rho$ : the self dual tensor equation can then be written as

$$\partial_\mu \partial_\mu \rho = 0. \quad (1.3.57)$$

This is the standard Laplace equation in four dimensions. The  $B = 1$  instanton is then obtained by

$$\rho(x) = 1 + \frac{\lambda^2}{|x - a|^2}, \quad (1.3.58)$$

where  $a$  is an arbitrary four vector indicating the position of the instanton, while  $\lambda$  is an arbitrary, strictly positive real number, interpreted as the instanton size. Those interpretations are supported by the fact that the action density for such a field is peaked on  $a$ , and the action inside the ball  $|x - a| \leq \lambda$  is equal to half the total action,  $4\pi^2$ . The field is non singular everywhere, as the apparent singularity in  $x = a$  can be removed by a gauge transformation, and the action density turns out to be finite there.

For the moduli space computations, we follow [48].  $a$  and  $\lambda$  are 5 moduli for the instanton, but there are also more. Another symmetry on which we can act is a global gauge transformation (a local transformation falling to zero at infinity would not change the physics, so we use a global transformation), which adds 3 other moduli to the space: the total moduli space is then  $\mathcal{M} = \mathbb{R}^4 \times \mathbb{R}^{*,+} \times (SU(2)/\mathbb{Z}^2)$ , where  $\mathbb{R}^{*,+}$  is the positive half of  $\mathbb{R}$  with 0 removed and the quotient of  $SU(2)$  with  $\mathbb{Z}^2$  indicates the fact that an  $SU(2)$  matrix and its opposite give the same action. As  $SU(2) \simeq S^3$ , we can interpret  $\mathbb{R}^* \times SU(2)$  as  $\mathbb{R}^4$  with the origin removed, just by interpreting  $\lambda$  as a radial coordinate and  $SU(2)$  coordinates as angles on a sphere  $S^3$ . The moduli space is then  $\mathcal{M} = \mathbb{R}^4 \times \mathbb{R}^{*,4}/\mathbb{Z}^2$ , where  $*$  indicates removal of the origin. The metric on moduli space must be computed carefully. Let  $\delta_\alpha A_\mu$  be the first order change of  $A_\mu$  when one of the moduli (labeled by the index  $\alpha$ , going from 1 to 8:  $X^\alpha$  will indicate a generic moduli) is varied. Usually, this would be the derivative of  $A_\mu$  with respect to the coordinate  $X^\alpha$ , but in the particular case of a gauge theory one could vary  $A_\mu$  without changing the physical situation, through a gauge transformation. We will be more general and write

$$\delta_\alpha A_\mu = \frac{\partial A_\mu}{\partial X^\alpha} + D_\mu \Omega_\alpha \quad (1.3.59)$$

where  $\Omega_\alpha$  is arbitrary, chosen to enforce

$$\int d^4x \operatorname{tr}[\delta_\alpha A_\mu] D_\mu \eta = 0 \quad \forall \eta \in SU(2) \quad (1.3.60)$$

This enforces the fact that  $\delta_\alpha A_\mu$  must be orthogonal to any gauge transformation, generated by the  $SU(2)$  function  $\eta$ . This is equivalent to say

$$D_\mu \delta_\alpha A_\nu = 0 \quad (1.3.61)$$

where the covariant derivative is calculated using the unvaried background field  $A_\mu$ . The metric on moduli space is given by

$$g_{\mathcal{M},\alpha\beta} = - \int \text{tr}[\delta_\alpha A_\mu \delta_\beta A_\mu] d^4x \quad (1.3.62)$$

The strategy is clear: first, we derive  $A_\mu$  with respect to a modulus, then we fix  $\Omega_\mu$  to have (1.3.61), so we can compute  $\delta_\alpha A_\mu$  avoiding gauge complication. Lastly, we perform the integral. Denoting as  $g_1$  the standard metric on the first  $\mathbb{R}^4$  in the moduli space and as  $g_2$  the standard metric on the second  $\mathbb{R}^4$ , we have that the metric on moduli space is given by

$$g_{\mathcal{M}} = S_{inst}(g_1 + 2g_2) \quad (1.3.63)$$

$S_{inst}$  is the single instanton action,  $S_{inst} = 8\pi^2$ .

We conclude this section by expanding an apparent similarity between instantons and Skyrmions, suggested by the fact that the topological content for a gauge field  $A_\mu$  is contained in its asymptotic form, through the field  $g^\infty$ , that is an  $SU(2)$  matrix. This similitude was investigated in [6]. Let us define

$$U = \mathcal{P} \exp \left( i \int_{-\infty}^{+\infty} A_4(x^i, x^4) dx^4 \right) \quad (1.3.64)$$

that can be explicitly computed by defining the field  $\tilde{U}(x^i, x^4)$  through the boundary condition  $\tilde{U}(x^i, -\infty) = \mathbf{1}$  and the differential equation

$$\frac{\partial \tilde{U}}{\partial x^4} = i A_4 \tilde{U} \quad (1.3.65)$$

We conclude by writing  $U(x^i) = \tilde{U}(x^i, +\infty)$ . This procedure is called *holonomy*. Due to the boundary condition, integrating along the line  $(-\infty, \infty)$  is equivalent to performing an integration on a closed loop on  $S^3$  (as  $-\infty$  and  $+\infty$  are identified). It can be shown that, if  $A$  has a topological charge,  $U$  has the same topological charge. In particular, the charge one instanton field defined through (1.3.55) and (1.3.58) generates an hedgehog Skyrminion, with radial profile function

$$f(r) = \pi \left( 1 - \left( 1 + \frac{\lambda^2}{r^2} \right)^{-\frac{1}{2}} \right) \quad (1.3.66)$$

The moduli of the instanton are inherited by the Skyrmion, as the three moduli describing the instanton position become the three spatial moduli for the Skyrmion (the position along  $x^4$  of the instanton does not matter, as that coordinate is integrated). Further, under a global gauge transformation  $A \rightarrow GAG^\dagger$  the Skyrme field is rotated the same way,  $U \rightarrow GUG^\dagger$ , so the two fields also share the same orientation in  $SU(2)$ . The modulus  $\lambda$  has no analogous counterpart in the Skyrme hedgehog, but it can enter as a parameter in the field (as in (1.3.66)).

## 1.4 Skyrmons and nuclear physics

We conclude this chapter by describing an application of the Skyrme model in nuclear physics. Moduli space approximation defines a set of collective coordinates  $\mathbf{X}$  (the moduli) that can describe, to a certain level of approximation, the field dynamics. This approximate field dynamic is described by the curve  $\mathbf{X}(t)$ , where  $t$  is some parametrization, and in moduli space approximation  $\mathbf{X}(t)$  is a geodesic on the moduli space manifold  $\mathcal{M}$  (equipped with the metric (1.2.9)).

This system can be quantized, and an Hilbert space of quantum states can be defined. In [2] the moduli space of the hedgehog Skyrmion is quantized, and the quantum states of the system are interpreted as proton and neutron states. This Hilbert space can then be used to predict observables. Agreement with experimental data varies greatly if we change the specific model. We review the construction in a slightly different way, that is the standard way to quantize a classical system with finite degrees of freedom.

First, we explain the quantization procedure. The connected part of the Skyrmion moduli space containing identity can be parametrized by the coordinates  $(\mathbf{R}, A)$  where  $\mathbf{R}$  is a three vector and  $A$  an  $SU(2)$  matrix. The ket state in coordinate representation is then defined as  $|\mathbf{X}, F\rangle^5$  and we can define the multiplicative operators  $\hat{\mathbf{R}}$  and  $\hat{A}$  such as  $\hat{\mathbf{R}}|\mathbf{X}, F\rangle = \mathbf{X}|\mathbf{X}, F\rangle$  and  $\hat{A}|\mathbf{X}, F\rangle = F|\mathbf{X}, F\rangle$ . Every component of  $\hat{\mathbf{R}}$  and every matrix element of  $\hat{A}$  commute between each other.

We must find canonical pulses, conjugate to the coordinates, and impose appropriate commutation relations with the coordinates. As the manifold is a

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<sup>5</sup>We change coordinate names in the kets to avoid confusion with operators:  $\mathbf{X}$  is a 3-vector and  $F$  an  $SU(2)$  matrix

tensor product of two submanifolds, we can write  $|\mathbf{X}, F\rangle = |\mathbf{X}\rangle \times |F\rangle$  and work separately on the two parts. The quantization of the  $\mathbb{R}^3$  part is the standard linear momentum quantization, so we will concentrate on quantizing the  $SU(2)$  part.

$SU(2)$  is a group with respect to the standard matrix composition, and any curve  $F(t)$  on  $SU(2)$  can be written as a left translation  $F(t) = L(t)F(0)$  or a right translation  $F(t) = F(0)R(t)$  (with  $L(t)$  and  $R(t)$   $SU(2)$  matrices such as  $L(0) = R(0) = \mathbf{1}$ ). Left and right translation commute, so we will concentrate on left translations for the moment. We can write  $L(t)$  as

$$L(t) = \exp\left(i\frac{\sigma^a}{2}l^a(t)\right) = U(L). \quad (1.4.1)$$

The operator  $U(L)$  acts on the coordinate, and it must be represented on the kets. We denote  $D(U(L))$  as the representation, and define the left angular momenta through  $D(U(L)) = \exp\left(i\hat{J}_L^a l^a(t)\right)$ . Those momenta have the same commutation rules as  $\sigma^a/2$ ,  $[\hat{J}_L^a, \hat{J}_L^b] = i\epsilon^{abc}\hat{J}_L^c$ , so they are standard angular momentums. We define the action of  $\hat{J}_L$  through (we insert dots to indicate standard matrix multiplication to help readability)

$$\hat{A}D(U(L))|F\rangle = L \cdot F \cdot D(U(L))|F\rangle, \quad (1.4.2)$$

that is to say,  $D$  translates a state centered in  $F$  to a state centered in  $LF$ . We then set  $D(U(L))|F\rangle = |LF\rangle^6$ . We calculate the commutator

$$[\hat{A}, D(U(L))]|F\rangle = (L - \mathbf{1}) \cdot F |FL\rangle = (L - \mathbf{1}) \cdot F \cdot D(U(L))|F\rangle. \quad (1.4.3)$$

Expanding  $L$  and  $D$  in powers of  $l$  and keeping only the first order, we get

$$i[\hat{A}, J_L^a]l^a |F\rangle = i\frac{\sigma^a}{2}l^a \cdot \hat{A}|F\rangle \quad (1.4.4)$$

on every state, so we can conclude

$$[\hat{A}, J_L^a] = -i\frac{\sigma^a}{2}\hat{A} = \frac{\sigma^a}{2}\hat{A}. \quad (1.4.5)$$

This is analogous to the fundamental parenthesis,  $[x, p] = i$ . We can repeat the exact same steps for right translations, defining a right momentum  $\hat{J}_R^a$  such as

$$[\hat{A}, \hat{J}_R^a] = -\hat{A}\frac{\sigma^a}{2} \quad [\hat{J}_L^a, \hat{J}_R^b] = 0. \quad (1.4.6)$$

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<sup>6</sup>In general, we could set  $D(U(L))|F\rangle = \exp(i\alpha(L, F))|LF\rangle$ , with a non constant phase. Here we set  $\alpha = 0$ . A case where  $\alpha$  is not zero is studied in chapter 2.7 of [50]



The last commutator derives from the fact that the left and right actions commute, so the vectors in the Lie algebra generating the transformations commute. This is reflected by quantum operators. Lastly, we can note that the operator  $M(\hat{A})_{ba}\hat{J}_L^b$  ( $M$  is defined in (1.3.39)) has the same commutation rules with  $\hat{A}$  as  $-\hat{J}_R^b$ , so we can say that  $\hat{J}_R^a = -M_{ba}(\hat{A})J_L^b$ . As  $M$  is orthogonal, this means  $\hat{J}_R^a\hat{J}_R^a = \hat{J}_L^a\hat{J}_L^a$ .

Our quantization has produced a set of six commuting operators<sup>7</sup>: three momenta  $P^i$ , the 3 projections of the left and right angular momenta  $J_L^3$  and  $J_R^3$  and the common Casimir  $J^2 = J_L^a J_L^a = J_R^a J_R^a$ . We rewrite the ket with the eigenvalues of those operators: a generic state  $|\psi\rangle$  is written as

$$|\psi\rangle = |\mathbf{P}\rangle \times |j, m_l, m_r\rangle, \quad (1.4.7)$$

where we adopted the standard nomenclature for angular momenta:  $j(j+1)$  is the eigenvalue of the Casimir, while  $m_l$  and  $m_r$  are the eigenvalues of the left and right third component of the angular momentum. In Schrödinger's picture, the wavefunction is expressed as

$$\psi(\mathbf{X}, F) = \langle \mathbf{X}, F | \psi \rangle = \exp(i\mathbf{P} \cdot \mathbf{X}) D_{m_l, m_r}^j(F), \quad (1.4.8)$$

where  $D_{m_l, m_r}^j$  is Wigner D-matrix. With a Lagrangian we can identify the operators with physical quantities. Working explicitly with (1.3.43) as Lagrangian (as there is no potential for motion on moduli space) and setting  $\hbar = 1$ , we can verify that

$$P_i = \frac{\partial L}{\partial X^i} = M\dot{X}_i \quad J_{L,i} = \frac{\partial L}{\partial \omega_i} = \Lambda\omega_i \quad (1.4.9)$$

(we could also have identified  $J_R$ ). Then, the Hamiltonian is given by Legendre transforming, and is of the form<sup>8</sup>

$$H = \frac{P_i P_i}{2M} + \frac{J_{L,i} J_{L,i}}{2\Lambda} + M. \quad (1.4.10)$$

Our quantum system is ready, and we have to do some identifications. Neglecting the eigenvalues of the linear momentum (as they just give a momentum to the states, that can be removed with a Galileian boost), we identify each state

<sup>7</sup>We drop the hats on quantum operators, from now on

<sup>8</sup>We add  $M$  to represent the fact that a field with no impulse and no momentum still has a non vanishing energy, given by the Skyrmion mass

with a baryon, of equal total isospin and spin (indicated by  $j$ ) but different projections (indicated respectively by  $m_l$  and  $m_r$ ). Restricting to  $j = \frac{1}{2}$  and eliminating it from the ket, we identify

$$|p, \uparrow\rangle = \left| \frac{1}{2}, \frac{1}{2} \right\rangle \quad |p, \downarrow\rangle = \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \quad |n, \uparrow\rangle = \left| -\frac{1}{2}, \frac{1}{2} \right\rangle \quad |n, \downarrow\rangle = \left| -\frac{1}{2}, -\frac{1}{2} \right\rangle, \quad (1.4.11)$$

where  $p$  and  $n$  are proton and neutron, with up or down spin. Changing the angular momentum, we can also represent higher spin baryons, and we can use standard ladder operators built from  $J_L$  and  $J_R$  to change one state into another.

Those identifications bring to an apparent disaster. From the form of the Hamiltonian (setting the pulse to 0), it is evident that the state  $j = 0$  is energetically favorable with respect to any state  $j = 1/2$ . This would mean that proton and neutron are likely to decay in a spinless baryon, and that is obviously unphysical. We are forgetting that, in no part during the process, we have done something to check if the states represent bosonic or fermionic particles. The statistics is implemented by the stabilizer's discrete part (in this case  $\mathbb{Z}^2$ , symbolizing the equivalence between  $A$  and  $-A$ ). We can solve this problem by introducing Finkelstein-Rubenstein constraints [17]: in this case, the statistic can be implemented with a superselection rule by considering the fact that, if  $A$  and  $-A$  indicate the same state, then any wavefunction must respect  $\psi(-A) = e^{i\alpha}\psi(A)$ , and by iterating twice

$$\psi(-A) = \pm\psi(A). \quad (1.4.12)$$

In this case, the plus sign corresponds to bosonic statistics, while the minus sign corresponds to fermionic statistics. If we want to quantize baryons as fermions, we have to choose the minus sign. Due to the inversion property of the Wigner matrix,  $D_{mp}^j(-A) = (-1)^{2j}D_{mp}^j(A)$ , we have to choose  $j$  half integer, and the states with integer  $j$  are removed from the spectrum of the theory. Thus, proton and neutron are the true ground states, with masses

$$M_{p,n} = M + \frac{3}{8\Lambda}. \quad (1.4.13)$$

We can now express physical quantities by taking their expressions in terms of the Skyrme field with explicit moduli, quantize them and average them on the

state of interest. Various examples of those long but simple calculations are given in [2].

Before going on, we introduce an useful notation, that allows us to skip Wigner matrices and write explicit wavefunctions for the angular parts. As  $A = a_0 \mathbf{1} + ia_i \sigma_i$ , we can promote  $a_0$  and  $a_i$  as constrained coordinate operators (as they always respect  $a_0 a_0 + a_i a_i = 1$ ). From  $\omega_i = -\frac{i}{2} \text{tr} [A^\dagger \dot{A} \sigma_i]$ , we can express the  $\omega_i$  in function of those coordinates and use the fact that  $\omega_i \omega_i = \dot{a}_0 \dot{a}_0 + \dot{a}_i \dot{a}_i$  to make them explicit in the Lagrangian, and calculate canonical impulses  $\pi_0$  and  $\pi_i$  that obey the standard commutation relations, and are represented as standard derivative operators. The left and right invariant angular momenta become

$$J_L^i = \frac{i}{2} \left( a_0 \frac{\partial}{\partial a_i} - a_i \frac{\partial}{\partial a_0} - \epsilon_{ijk} a_j \frac{\partial}{\partial a_k} \right), \quad (1.4.14a)$$

$$J_R^i = \frac{i}{2} \left( -a_0 \frac{\partial}{\partial a_i} + a_i \frac{\partial}{\partial a_0} - \epsilon_{ijk} a_j \frac{\partial}{\partial a_k} \right). \quad (1.4.14b)$$

We can verify that those assignments give the right quantum numbers.

$$\begin{aligned} |p, \uparrow\rangle &= \frac{1}{\pi} (a_1 + ia_2), & |p, \downarrow\rangle &= -\frac{i}{\pi} (a_0 - ia_3), \\ |n, \uparrow\rangle &= \frac{i}{\pi} (a_0 + ia_3), & |n, \downarrow\rangle &= -\frac{1}{\pi} (a_1 - ia_2). \end{aligned} \quad (1.4.15)$$



# Chapter 2

## The two sides of AdS/CFT

In this chapter, we're going to introduce the physical basis for the AdS/CFT duality. AdS/CFT is a duality between two distinct physical theories. On the AdS side, also called the gravity side, we have a theory of superstrings in an Anti de Sitter background, complemented by enough compact dimensions that are needed to have a coherent string theory. On the CFT side, also called the gauge side, we have a Yang Mills supersymmetric QFT, that exhibits no scale, even after quantization. We will also study the large  $N$  limit [53], that will allow us to do computations in the next chapter.

For the sake of brevity, we're going to collect only the results that are needed to understand AdS/CFT, and we'll skip many interesting parts. We refer to [56] for details about string theory, while we refer to [5] and its references for the necessary basis about quantum field theory and supersymmetry.

## 2.1 The gauge side

### 2.1.1 Conformal symmetry

#### Conformal algebra

Conformal symmetry is an extension of the Poincarè group, studying the behavior of fields under rescaling of the coordinates. Roughly speaking, in a conformally invariant theory we can scale our coordinates freely, and the physics does not change: there is no measurable quantity that provides a scale.

One way to understand if a field theory is not conformal is to look at its coupling constants. It is evident that, if the theory has a dimensionful coupling constant (or if the field is massive) then we have a preferred scale. As an example, scalar massless  $\phi^4$  in 4 spacetime dimensions has no dimensionful couplings, so it is a candidate for a conformal theory.

Let us consider  $d$  dimensional Minkowski spacetime. Conformal transformations are transformations that preserve the causal structure of spacetime, meaning that spacelike points remain spacelike *et cetera*. This happens if, under the transformation  $x \rightarrow y(x)$ , the metric changes as  $\Omega^{-2}(y)\eta = e^{2\sigma(y)}\eta$ , with  $\Omega$  nowhere vanishing (so with a definite sign). From this definition, the fact that the conformal transformations form a group is obviously true. We restrict to infinitesimal transformations,  $x^\mu \rightarrow x^\mu + \epsilon^\mu$ : the induced transformation on the metric is

$$\eta_{\mu\nu} \rightarrow \eta_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu. \quad (2.1.1)$$

Expanding in terms of  $\sigma$ , this means

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = 2\sigma(x)\eta_{\mu\nu} \quad (2.1.2)$$

(notice that, for  $\sigma = 0$ , the transformation is a standard Minkowski transformation: Minkowski transformations are part of the conformal transformations, with  $\Omega = 1$ ). The most general solution to this equation is given by

$$(\eta_{\mu\nu}\partial_\rho\partial^\rho + (d-2)\partial_\mu\partial_\nu)\partial \cdot \epsilon = 0. \quad (2.1.3)$$

We make the assumption  $d > 2$  (for  $d = 2$  the discussion would be very different, but we won't need that case) and write the most general solution as

$$\epsilon^\mu(x) = a^\mu + \omega^\mu_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2(b \cdot x)x^\mu. \quad (2.1.4)$$

Every term in the sum contains a particular transformation parameter:  $a$  and  $\omega$  are the standard translation and boost/rotation parameters,  $\lambda$  represents a dilation parameter while  $b$  represents a special conformal transformation. Now we must classify the operators that implement conformal transformations on the states.

Each parameter must be associated to a vector in the Lie algebra of the conformal group, so we define standard momentum  $P^\mu$  associated to  $a^\mu$  and angular momentum  $J^{\mu\nu}$  associated to  $\omega_{\mu\nu}$ . The operator associated to the dilatation parameter  $\lambda$  is denoted as  $D$ , while the operator associated to the special conformal transformation parametrized by  $b^\mu$  is indicated with  $K^\mu$ . The commutators between them are given by

$$\begin{aligned}
[J_{\mu\nu}, J_{\rho\sigma}] &= i(\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\nu\rho}J_{\mu\sigma} - \eta_{\mu\sigma}J_{\nu\rho}) & [P^\mu, P^\nu] &= 0, \\
[J_{\mu\nu}, P_\rho] &= i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) & [J_{\mu\nu}, K_\rho] &= i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu), \\
[D, P_\mu] &= iP_\mu & [D, K_\nu] &= -iK_\nu & [D, J_{\mu\nu}] &= 0, \\
[K_\mu, K_\rho] &= 0 & [K_\mu, P_\nu] &= -2i(\eta_{\mu\nu}D - J_{\mu\nu}).
\end{aligned} \tag{2.1.5}$$

This set of relations forms the conformal algebra.

This algebra has a particular property. The generators  $J$ , commuting between themselves, form the subalgebra of the standard Lorentz group,  $so(d-1, 1)$ . We can define the group  $SO(d, 2)$  as the group of transformations on  $\mathbb{R}^{d+3}$  leaving invariant the Minkowski metric with two time directions, given by the diagonal matrix  $\Lambda$  with elements  $(-1, 1, \dots, 1, -1)$ :  $X^0$  and  $X^{d+1}$  are the time coordinates: this group is generated by the Lie vectors  $J_{AB}$ , with  $A$  running from 0 to  $d+1$ , and the commutation relations are given by the same commutation rules of  $J_{\mu\nu}$ , substituting every  $\eta$  for a  $\Lambda$  and putting the appropriate indexes. We can rearrange the generators of the conformal algebra in this way: introducing  $\mu$  from 0 to  $d-1$ , we see that, defining

$$\bar{J}_{\mu\nu} = J_{\mu\nu}, \quad \bar{J}_{d(d+1)} = -D, \quad \bar{J}_{\mu d} = \frac{1}{2}(K_\mu - P_\mu), \quad \bar{J}_{\mu(d+1)} = \frac{1}{2}(P_\mu + K_\mu). \tag{2.1.6}$$

By direct calculation, it can be verified that  $\bar{J}$  have the same commutation rules that are found in the algebra of  $so(d, 2)$ , so the algebra of the conformal group and the algebra of  $so(d, 2)$  can be identified.

## Field representations

After introducing the symmetry in spacetime, the next step is to classify fields according to the irreducible representations of the conformal group, to build quantum states. We will use the method of induced representations, as in [5] and [50]: we look for the transformation properties of the operator  $\phi(0)$  (where  $\phi$  denotes any kind of second quantized field) and then use a boost to find the transformation properties of  $\phi(x)$ .

We start from the commutation relations with the angular momentum. Those are given by

$$[\phi(0), J_{\mu\nu}] = \mathcal{J}_{\mu\nu}\phi(0), \quad (2.1.7)$$

where  $\mathcal{J}$  is any irreducible representation of the standard Lorentz group. This commutation rule provides us with the necessary quantum numbers to identify the  $SO(3, 1)$  representation. For the dilation generator, we have

$$[\phi(0), D] = i\Delta\phi(0). \quad (2.1.8)$$

$\Delta$  is called the scaling dimension, and is another quantum number used to classify states. Scaling dimension of a field can differ from its physical dimension: as an example, in four dimensions a scalar field (of dimension 1) can have any scaling dimension, independently from its dimensionality. An important result in literature is that the scaling dimension of a field cannot be lower than its physical dimension [21], so the scaling dimension is bounded from below in an unitary representation. From (2.1.5), we have that  $P_\mu$  and  $K_\mu$  are ladder operators:  $P_\mu$  raises the scaling dimension by one, while  $K_\mu$  lowers it by one. There must then be some fields that have commutation rules

$$[\phi(0), K_\mu] = 0. \quad (2.1.9)$$

Those fields are called *primary fields*, and the fields obtained by acting with  $P_\mu$  on the primary fields are called *conformal descendants* of  $\phi$ . Now we can deduce commutation relations in every generic point  $x^\mu$ : calling  $\tau(x) = \exp(-iP_\mu x^\mu)$ , we have that  $\tau(x)\phi(0)\tau^{-1}(x) = \phi(x)$ , so we can obtain the full commutation rules, that read

$$[P_\mu, \phi(x)] = -i\partial_\mu\phi(x), \quad [D, \phi(x)] = -i(\Delta + x^\mu\partial_\mu)\phi(x),$$



$$\begin{aligned}
[J_{\mu\nu}, \phi(x)] &= (-\mathcal{J}_{\mu\nu} + i(x_\mu\partial_\nu - x_\nu\partial_\mu))\phi(x), \\
[K_\mu, \phi(x)] &= (i(-x^2\partial_\mu + 2x_\mu x_\rho\partial^\rho + 2x_\mu\Delta) - 2x^\nu\mathcal{J}_{\mu\nu})\phi(x).
\end{aligned}
\tag{2.1.10}$$

Those rules furnish explicit operator representations of the generators.

Let  $T_{\mu\nu}$  be the symmetric energy-momentum tensor for a conformally symmetric field theory. The conserved classical currents associated to the operators  $D$  and  $K_\mu$  are given by

$$J_{(D)\mu} = x^\nu T_{\mu\nu}, \quad J_{(K)\mu\nu} = x^2 T_{\mu\nu} - 2x_\nu x^\rho T_{\mu\rho}. \tag{2.1.11}$$

Conservation of  $J_{(D)\mu}$  gives  $\partial_\mu J_{(D)}^\mu = 0 = (\partial^\nu x^\rho)T_{\nu\rho} = T_\mu^\mu$  (where we used the symmetry of  $T$  and the fact that  $\partial_\mu T_\nu^\mu = 0$ ): in a conformal field theory, the energy momentum tensor is always traceless. This fact rarely survives quantization: perturbative quantization introduces an energy scale in the theory, and the  $\beta$  functions quantify the theory's scale dependence. If those functions are non zero, then the quantized, renormalized theory has a scale dependence and the trace of  $T_{\mu\nu}$  is corrected by anomalies. Conformal symmetry is then almost always lost in QFT, and few theories keep their conformal invariance when quantized. Supersymmetric YM for 4 supersymmetries is an example of such a theory.

### Correlation functions for CFT

When it survives quantization, conformal symmetry gives stringent bounds on the correlation functions of the theory. In particular, the propagator and the three point functions can be written exactly, up to normalization constants.

The starting point is the non anomalous Ward identity relative to dilations:

$$\sum_{i=1}^N \left( x_i^\mu \frac{\partial}{\partial x_i^\mu} + \Delta_i \right) \langle \phi_1(x_1) \dots \phi_i(x_i) \dots \phi_n(x_n) \rangle = 0, \tag{2.1.12}$$

where  $\Delta_i$  is the scaling dimension of  $\phi_i$ . Let us consider the two point function  $\langle \phi(x)\phi(y) \rangle$  for a scalar field of scaling dimension  $\Delta$ : this function of  $x$  and  $y$  must be a function of  $(x - y)^2$  by Lorentz and Poincarè invariance. By explicit derivation, we can verify that

$$\langle \phi(x)\phi(y) \rangle = \frac{C_\phi}{(x - y)^{2\Delta}}, \tag{2.1.13}$$

where  $C_\phi$  is a normalization constant, that can be set to one after renormalizing  $\phi$ . We can try any power  $(x-y)^a$ , but the only value of  $a$  that solves the Ward identity is  $a = 2\Delta$ . In an analogous way, we can show that, considering two different fields, it is sufficient to substitute  $2\Delta$  with  $\Delta_i + \Delta_j$ , the sum of the scaling dimensions of the fields. In general, the result for a CFT of scalar fields  $\phi_i$  with scaling dimensions  $\Delta_i$  is given by

$$\langle \phi_i(x)\phi_j(y) \rangle = \frac{C_{ij}}{(x-y)^{\Delta_i+\Delta_j}}. \quad (2.1.14)$$

The matrix  $C_{ij}$  is symmetric, so transformations of the fields can make this matrix the identity. For three fields  $\phi_i$  with scaling dimensions  $\Delta_i$ , an analogous result holds:

$$\begin{aligned} \langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3) \rangle &= \\ &= \frac{C_{ijk}}{(x_1-x_2)^{\Delta_1+\Delta_2-\Delta_3}(x_1-x_3)^{\Delta_1+\Delta_3-\Delta_2}(x_2-x_3)^{\Delta_2+\Delta_3-\Delta_1}}. \end{aligned} \quad (2.1.15)$$

We then just need the quantities  $\{\Delta_i, C_{ijk}\}$  to completely fix the two and three point functions. Four point functions are not so constrained, as one could build the adimensional and Lorentz invariant ratio

$$\frac{|x_1-x_2|}{|x_3-x_4|} \quad (2.1.16)$$

or other different similar ratios. Conformal symmetry then imposes stringent bounds on some correlation functions.

## 2.1.2 Supersymmetry

### Supersymmetry algebra

Supersymmetry is an extension of the Poincaré algebra, introducing the concept of *spinor supercharges*. Those charges are generators of transformations that change a bosonic field for a fermionic field, and vice versa. Supersymmetry enriches the spectrum of fundamental particles, by introducing a supersymmetric particle for each standard model particle, of opposite statistic.

First, we introduce the statistic of a field as 0 if the field is bosonic, 1 if the field is fermionic. The product of two fields gets a statistic that is the sum of the

statistics of the single fields, modulo 2: this means that a product of two bosonic fields or two fermionic fields is always bosonic, while the product of a fermionic field and a bosonic field is always fermionic. Fermionic fields are represented by Grassmann variables. We introduce a notation for Weyl spinors: the spinor  $Q_\alpha^a$  represents a left handed spinor, with  $\alpha$  assuming values 1 and 2 and  $a$  assuming values from 1 to  $\mathcal{N}$ , where  $\mathcal{N}$  is the number of supersymmetries and is a parameter of the theory, while  $\bar{Q}_{a\dot{\alpha}} = (Q_\alpha^a)^*$  is a right handed spinor, with  $\dot{\alpha}$  having the same range of  $\alpha$ . We can raise and lower spinor indexes by means of the tensors  $\epsilon_{\alpha\beta}$  and  $\epsilon_{\dot{\alpha}\dot{\beta}}$  with upper or lower indexes, normalized as  $\epsilon_{12} = \epsilon_{\dot{1}\dot{2}} = 1 = -\epsilon^{12} = -\epsilon^{\dot{1}\dot{2}}$  and we denote  $\sigma^\mu = (-1, \sigma^i)$  and  $\bar{\sigma}^\mu = (-1, -\sigma^i)$ . Lastly, we introduce the matrices  $\sigma^{\mu\nu} = i/4(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$  and  $\bar{\sigma}^{\mu\nu} = i/4(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)$ .

With those notations, we can define the supersymmetry algebra. Introducing the supercharges  $\mathcal{Q}_\alpha^a$  and  $\bar{\mathcal{Q}}_{\dot{\alpha}}^a$  and recalling the standard commutation relations of  $J_{\mu\nu}$  and  $P_\mu$  from (2.1.5) (without the conformal group extension), we can impose the following commutation relations:

$$\begin{aligned} [\mathcal{Q}_\alpha^a, J^{\mu\nu}] &= (\sigma^{\mu\nu})_\alpha^\beta \mathcal{Q}_\beta^a, & [\bar{\mathcal{Q}}_{\dot{\alpha}}^a, J^{\mu\nu}] &= \epsilon_{\dot{\alpha}\dot{\beta}} (\bar{\sigma}^{\mu\nu})^{\dot{\beta}\dot{\gamma}} \bar{\mathcal{Q}}^{\dot{\gamma}a}, \\ [\mathcal{Q}_\alpha^a, P^\mu] &= 0, & [\bar{\mathcal{Q}}_{\dot{\alpha}}^a, P^\mu] &= 0, \\ \{\mathcal{Q}_\alpha^a, \mathcal{Q}_{b\dot{\beta}}\} &= 2\sigma_{\alpha\dot{\beta}}^\mu P_\mu \delta_b^a, & \{\mathcal{Q}_\alpha^a, \mathcal{Q}_\beta^b\} &= \epsilon_{\alpha\beta} Z^{ab}, & \{\bar{\mathcal{Q}}_{a\dot{\alpha}}, \bar{\mathcal{Q}}_{b\dot{\beta}}\} &= \epsilon_{\dot{\alpha}\dot{\beta}} \bar{Z}_{ab}. \end{aligned} \quad (2.1.17)$$

In the first and second line, we have written the commutation rules of  $\mathcal{Q}$  and its conjugate with the rest of the Poincaré algebra, stating that spinor charges transform as spinors and are scalars under translations. In the last line (where  $\{\cdot, \cdot\}$  is the anticommutator) we have written the commutation rules of the spinor charges between themselves, introducing the antisymmetric *central charges*  $Z^{ab}$  and  $\bar{Z}_{ab} = (Z^\dagger)^{ab}$ , that commute with all other generators due to the Jacobi identity. The commutation relations are invariant under the transformation (called *R-symmetry*)

$$\mathcal{Q}_\alpha^a \rightarrow R_b^a \mathcal{Q}_\alpha^b, \quad \bar{\mathcal{Q}}_{a\dot{\alpha}} \rightarrow \bar{\mathcal{Q}}_{b\dot{\alpha}} (R^\dagger)_a^b. \quad (2.1.18)$$

where the matrix  $R$  can be taken as an  $U(\mathcal{N})$  matrix in four dimensions. Denoting as  $T^j$  matrices in the Lie algebra of  $U(\mathcal{N})$ , we complete the algebra by writing the commutation relations

$$[\mathcal{Q}_\alpha^a, T^j] = B_b^{ja} \mathcal{Q}_\alpha^b, \quad [\bar{\mathcal{Q}}_{a\dot{\alpha}}, T^j] = -B_{ab}^j \bar{\mathcal{Q}}_{\dot{\alpha}}^b, \quad [T_i, T_j] = if_{ijk} T^k. \quad (2.1.19)$$

In the Lie algebra, we raise and lower indices freely. We now study field representations of the extended algebra.

### Field representations

In the standard Lorentz algebra, we define the Pauli-Lubanski vector as

$$W_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma. \quad (2.1.20)$$

A set of commuting operators is then given by the 3-impulse,  $P^i$ , the squared mass,  $-P^\mu P_\mu$ , the modulus of the Pauli-Lubanski vector  $W^\mu W_\mu$  (that is proportional to the spin of the field) and the component  $W_3$  (that is proportional to the helicity). We can also write  $W^2 = C_{\mu\nu} C^{\mu\nu}$ , where  $C_{\mu\nu} = W_\mu P_\nu - W_\nu P_\mu$ . When we extend the algebra through supersymmetry,  $-P^\mu P_\mu$  is still a Casimir, while  $W^2$  is not. We can find a new Casimir through

$$\tilde{W}_\mu = W_\mu - \frac{1}{4} \bar{Q}_{a\dot{\alpha}} \bar{\sigma}_\mu^{\dot{\alpha}\alpha} Q_\alpha \quad \tilde{C}_{\mu\nu} = \tilde{W}_\mu P_\nu - \tilde{W}_\nu P_\mu, \quad (2.1.21)$$

summing over the supersymmetry indices. With those modifications, we find that  $P^i$ ,  $-P^\mu P_\mu$ ,  $\tilde{W}^2 = \tilde{C}_{\mu\nu} \tilde{C}^{\mu\nu}$  and  $\tilde{W}_3$  (with the same interpretations as before) commute.

The discussion of the supersymmetric spectrum is not of particular interest to us, so we cite the results. Particles are divided in massless and massive, and all particles in the same multiplet share the same mass (as  $-P_\mu P^\mu$  is a Casimir). The remarkable feature of supersymmetry is that there is the same number of bosonic and fermionic particles in each multiplet. Supercharges are represented on the multiplets either trivially or as ladder operators: in particular, they take a state of helicity  $\lambda$  to a state of helicity  $\lambda \pm 1/2$ , changing the statistic of the particle (in accord with the fact that they are represented by Grassmann-valued operators). As an example, for  $\mathcal{N} = 1$  supersymmetry and massless particles, we have that one component of the supercharge spinor has to be realized trivially, while the other raises the helicity by  $1/2$ . Starting with a definite helicity and momentum  $p$ , we have the states

$$|p, \pm\lambda\rangle \quad \left| p, \pm \left( \lambda + \frac{1}{2} \right) \right\rangle, \quad (2.1.22)$$

where we added opposite helicities to have a multiplet of  $CPT$ : if  $\lambda = 1$ , the first state can be used to represent a classical gauge boson while the second state represents its supersymmetric partner, the gaugino.

In the case of massless representations, we can build  $2^{\mathcal{N}}$  states starting from a definite helicity. In the case of massive representations with vanishing central charges, we represent all spinorial components of the supercharges as ladder operators, so we get a total of  $2^{2\mathcal{N}}$  states from a given helicity. If central charges are nonvanishing, we get a shortened representation, with less states available.

### $\mathcal{N} = 1$ Superspace

To incorporate supersymmetry in a natural way, we can extend the range of physical coordinates, admitting the existence of Grassmann-valued coordinates  $\theta$  and  $\bar{\theta}$ . The coordinates are compactly written as

$$z^A = (x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}). \quad (2.1.23)$$

Fields taking values on  $z^A$  are denoted as *superfields*. The  $\theta$  and  $\bar{\theta}$  dependence can be easily solved, from the fact that with Grassmann variables Taylor series have to be finite. Denoting, from now on,  $\theta \cdot \chi = \epsilon^{\alpha\beta} \theta_\alpha \chi_\beta$ ,  $\bar{\theta} \cdot \bar{\chi} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\theta}_{\dot{\alpha}} \bar{\chi}_{\dot{\beta}}$ ,  $\theta^2 = \theta \cdot \theta$  and  $\bar{\theta}^2 = \bar{\theta} \cdot \bar{\theta}$  and  $\theta \sigma^\mu \bar{\chi} = \theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}}$  we can write any superfield  $\mathbf{F}(z)$  (that can carry any type of superspace index, but we'll take it as scalar for now) as

$$\begin{aligned} \mathbf{F}(z) = & f^{(1)}(x) + \theta \cdot f^{(2)}(x) + \bar{\theta} \cdot f^{(3)}(x) + \theta^2 f^{(4)}(x) + \bar{\theta}^{(2)} f^{(5)}(x) + \\ & + \theta \sigma^\mu \bar{\theta} f_\mu^{(6)}(x) + \theta^2 \bar{\theta} \cdot \bar{f}^{(7)}(x) + \bar{\theta}^2 \theta \cdot f^{(8)}(x) + \bar{\theta}^2 \theta^2 f^{(9)}(x), \end{aligned} \quad (2.1.24)$$

where the spin and vector indices of  $f$  agree with the fact that we're taking  $\mathbf{F}$  as a scalar. The idea is that, writing an action for  $\mathbf{F}$  in superspace, truncations of this action can give supersymmetric actions. Supersymmetry transformations are implemented as

$$\delta_\epsilon \mathbf{F} = (\epsilon \cdot \mathcal{Q} + \bar{\epsilon} \cdot \bar{\mathcal{Q}}) \mathbf{F}, \quad (2.1.25)$$

where  $\mathcal{Q}$  and  $\bar{\mathcal{Q}}$  are promoted as operators, represented as

$$\mathcal{Q}_\alpha = \frac{\partial}{\partial \theta^\alpha} - i \sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}} \partial_\mu, \quad \bar{\mathcal{Q}}_{\dot{\alpha}} = \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i \theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu. \quad (2.1.26)$$

In their non infinitesimal version, we represent supersymmetry transformations as  $G(x, \theta, \bar{\theta}) = \exp(-ix^\mu P_\mu + i\theta \cdot \mathcal{Q} + i\bar{\theta} \cdot \bar{\mathcal{Q}})$ , and define  $G(x, \theta + \xi, \bar{\theta} + \bar{\xi}) = G(0, \xi, \bar{\xi}) G(x, \theta, \bar{\theta})$  when  $\chi$  is infinitesimal. Equivalently, we can introduce another representation by reversing the order of  $G(0, \xi, \bar{\xi})$  and  $G(x, \theta, \bar{\theta})$  when defining

infinitesimal composition. It is customary to call those generators  $\mathcal{D}$ , and the transformation is generated by  $(\epsilon \cdot \mathcal{D} + \epsilon \cdot \bar{\mathcal{D}})$ . Their operator version are

$$\mathcal{D}_\alpha = \frac{\partial}{\partial \theta^\alpha} + i\sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}} \partial_\mu, \quad \bar{\mathcal{D}}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha \sigma_{\alpha\dot{\beta}}^\mu \epsilon^{\dot{\beta}\dot{\alpha}} \partial_\mu. \quad (2.1.27)$$

The field (2.1.25) has too many degrees of freedom to have them all in the same supersymmetric multiplet, so we have to impose constraints on the superfield to describe a single multiplet.

The fields that we will need are the chiral superfield  $\Phi$  and the vector superfield  $V$ . Both of them are scalar in superspace. The chiral superfield is constrained by the equation

$$\bar{\mathcal{D}}_{\dot{\alpha}} \Phi = 0. \quad (2.1.28)$$

The most general superfield form respecting this constraint is

$$\begin{aligned} \Phi = & \phi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) + \frac{1}{4}\theta^2\bar{\theta}^2\partial_\rho\partial^\rho\phi(x) + \\ & + \sqrt{2}\theta \cdot \psi(x) - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \theta^2 F(x). \end{aligned} \quad (2.1.29)$$

The degrees of freedom are a scalar complex field  $\phi(x)$  and a Weyl spinor  $\psi(x)$  without its complex conjugate (this justifies the name of the field). In addition, a complex scalar  $F(x)$  is added, and in our applications it plays the role of an auxiliary field and not a dynamical one. A vector field  $V$  (still scalar in superspace) is defined by the reality condition

$$V = V^*. \quad (2.1.30)$$

The most general superfield is then written as

$$\begin{aligned} V = & C(x) + i\theta \cdot \chi(x) - i\bar{\theta} \cdot \bar{\chi}(x) + \frac{i}{2}\theta^2(M(x) + iN(x)) - \theta\sigma^\mu\bar{\theta}A_\mu(x) - \\ & - \frac{i}{2}\bar{\theta}^2(M(x) - iN(x)) + i\theta^2\bar{\theta} \cdot \left( \bar{\lambda}(x) + \frac{i}{2}\sigma^\mu\partial_\mu\chi(x) \right) - \\ & - i\bar{\theta}^2\theta \cdot \left( \lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}(x) \right) + \frac{1}{2}\theta^2\bar{\theta}^2 \left( D(x) + \frac{1}{2}\partial_\rho\partial^\rho C(x) \right). \end{aligned} \quad (2.1.31)$$

There is a large field content: four scalars ( $C(x), D(x), M(x), N(x)$ ), four components of a vector ( $A_\mu(x)$ ) and eight fermions ( $\chi(x), \bar{\chi}(x), \lambda(x), \bar{\lambda}(x)$ ). There is a

particularity: if  $V$  is a vector field, then  $V + \Phi + \Phi^*$  obviously also is, for any choice of  $\Phi$ . Then, we can use a particular  $\Phi$  to set a gauge for the vector superfield. In the Wess-Zumino gauge, the vector superfield is

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu(x) + i\theta^2\bar{\theta}\cdot\bar{\lambda}(x) - i\bar{\theta}^2\theta\cdot\lambda(x) + \frac{1}{2}\theta^2\bar{\theta}^2D(x). \quad (2.1.32)$$

The components  $A_\mu$  represent gauge bosons, the components  $\lambda$  and  $\bar{\lambda}$  represent the gaugino fields and  $D$  can be considered as an auxiliary field. We can also define two field strength from the vector field:

$$W_\alpha = -\frac{1}{4}\bar{\mathcal{D}}^2\mathcal{D}_\alpha V \quad W_{\dot{\alpha}} = -\frac{1}{4}\mathcal{D}^2\bar{\mathcal{D}}_{\dot{\alpha}}V. \quad (2.1.33)$$

Those fields are invariant under the translation by  $\Phi + \Phi^*$  and can be used in the definition of an action.

Superfield Lagrangians can be made by writing any combinations of fields that are invariant under supersymmetry transformations and using Grassmann integration to project out field degrees of freedom. In our notation,  $d\theta^2 = 1/2d\theta_1d\theta_2$  and similar for  $d\bar{\theta}$ . The most general Lagrangian that can be written using only chiral superfields is

$$\mathcal{L} = \int K(\Phi, \Phi^*)d\theta^2d\bar{\theta}^2 + \int W(\Phi)d\theta^2 + \int W^*(\Phi^*)d\bar{\theta}^2, \quad (2.1.34)$$

with  $K$  a real function of  $\Phi$  and  $\Phi^\dagger$  and  $W(\Phi)$  holomorphic. When we consider a set of fields  $\Phi_a$ , we can introduce a non abelian symmetry: by defining  $\Phi = \Phi_a T_a$  where  $T_a$  generate an irreducible representation of some gauge group and choosing

$$K(\Phi, \Phi^\dagger) = \Phi_a^\dagger \Phi_a \quad W = 0, \quad (2.1.35)$$

we can convert the Lagrangian in a gauge invariant Lagrangian with respect to the transformation

$$\Phi \rightarrow \exp(i\Omega(x))\Phi, \quad (2.1.36)$$

by postulating the existence of a vector superfield  $V$  transforming as

$$e^V \rightarrow e^{i\Omega^\dagger(x)}e^V e^{-i\Omega(x)}, \quad (2.1.37)$$

by modifying  $K$  as  $K(\Phi, \Phi^\dagger) = \text{tr}(\Phi^\dagger e^V \Phi e^{-V})$  and adding a field strength piece, that reads

$$\frac{1}{4g_Y^2} \left( \int \text{tr}(W^\alpha W_\alpha)d\theta^2 + \int \text{tr}(\bar{W}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}})d\bar{\theta}^2 \right) \quad (2.1.38)$$

or, by introducing a constant

$$\tau = \frac{\Theta}{2\pi} + i\frac{4\pi}{g_Y^2}, \quad (2.1.39)$$

we can also use the term

$$\frac{1}{8\pi^2} \text{Im} \int \text{tr} (\tau W_\alpha W^\alpha) d\theta^2 \quad (2.1.40)$$

to provide a CP violating  $\Theta$  term.

### $\mathcal{N} = 4$ super Yang Mills

In 4 spacetime dimensions, we want to create a theory of massless particles by including the maximum number possible of supersymmetries. Every independent supercharge can be applied on a field: to avoid having in our representations fields with spin greater than 1, we can have a maximum of  $\mathcal{N} = 4$  supersymmetries (as four steps of one half divide  $-1$  from  $1$ ). In superspace formalism, this is done by introducing three chiral superfields  $\Phi^i$ , with  $i = 1, \dots, 3$  in the  $N \times N$  matrix field  $\Phi = \Phi^i \sigma^i$ , and a vector field. We write the action as

$$\begin{aligned} \mathcal{S} = \int d^4x \text{tr} \left( \int d\theta^2 d\bar{\theta}^2 \text{tr} (\Phi^\dagger e^V \Phi e^{-V}) + \frac{1}{8\pi} \text{Im} \left( \tau \int d\theta^2 W_\alpha W^\alpha \right) + \right. \\ \left. + \left( i g_Y \frac{\sqrt{2}}{3!} \int d\theta^2 \epsilon_{ijk} \Phi^i [\Phi^j, \Phi^k] + c.c. \right) \right). \end{aligned} \quad (2.1.41)$$

By writing explicit field components and performing Grassmann integrations, we have the supersymmetric Lagrangian

$$\begin{aligned} \mathcal{L} = \text{tr} \left( -\frac{1}{2g_Y^2} F_{\mu\nu} F^{\mu\nu} + \frac{\Theta}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} - i\bar{\lambda}_a \bar{\sigma}^\mu D_\mu \lambda_a - D_\mu \phi_i D^\mu \phi_i + \right. \\ \left. + g_Y C_{abi} \lambda_a [\phi_i, \lambda_b] + g_Y \bar{C}_{iab} \bar{\lambda}_a [\phi_i, \bar{\lambda}_b] + \frac{g_Y^2}{2} [\phi_i, \phi_j]^2 \right), \end{aligned} \quad (2.1.42)$$

where the fields  $\lambda_a$  are four spinor fields (called the gaugino) and  $\phi_i$  are six real scalars, and both of those fields are in representation of the  $R$ -symmetry group  $SU(4)$ : to be more precise, the four fermionic fields  $\lambda$  and  $\bar{\lambda}$  transform under the fundamental representation of  $SU(4)$  (that has dimension 4), while the six scalar fields  $\phi^i$  transform under the antisymmetric representation of  $SU(4)$  (that



has dimension 6).  $C$  and  $\bar{C}$  are Clebsch-Gordan coefficients related to the group  $SU(4)$  and the covariant derivative is with respect to the dynamical field  $A_\mu$ . This is the action of super Yang Mills, for gauge group  $SU(N)$ . This theory has important properties, as the classical theory is conformally invariant, and conformal symmetry survives even after quantization. This happens because it can be shown by writing Feynmann rules and computing loop diagrams the divergences cancel without renormalization, so there is no need to introduce a dynamical scale and scale invariance is preserved.

### 2.1.3 The large $N$ limit of QCD

We conclude the discussion of the gauge side by introducing an approximation to QCD, introduced by 't Hooft in [47] and reviewed and expanded by Witten in [53], that is a perturbative expansion of QCD taking as expansion parameter the number of colors in the theory. Graphs for an arbitrary gauge theory  $SU(N)$  contain the parameter  $N$  explicitly, so a development in powers of  $\frac{1}{N}$  can be defined. Although the expansion parameter can be objected to be not so small in real QCD, the simplifications that happen in the limit  $N \rightarrow \infty$  make the theory much simpler to study, and this expansion is closely related to the genus expansion in perturbative string theory, that we'll introduce in the next section.

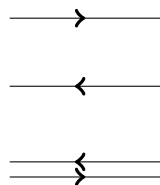


Figure 2.1: Propagators in double line notation. From top to bottom: quark, antiquark and gluon.

The  $N$  dependence of the Feynmann diagrams is studied by introducing the double line notation, as in figure (2.1). The quark and antiquark are represented by an arrow to the left or to the right, while the gluon is represented as a double arrow, one to the left and one to the right. In this notation, the vertices of the theory are as in figure (2.2). We also redefine the coupling constant  $g_Y = \sqrt{\lambda/N}$  with  $\lambda$  called the 't Hooft coupling, independent of  $N$ . In the large  $N$  limit, the coupling constant goes to zero, but the number of fields (parametrized by the

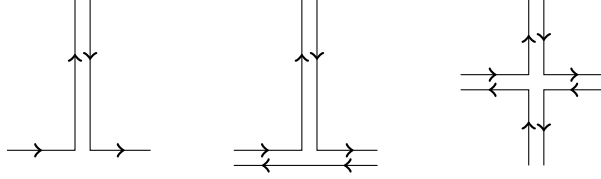


Figure 2.2: Double line vertices. From left to right: gluon-quark-antiquark, three gluons and four gluons. Each vertex gets a factor  $N$ , while each propagator gets a factor  $N^{-1}$ .

index  $a$ , running from 1 to  $N$ ) goes to infinity so a nontrivial limit can exist. To understand the limit, we write a set of rules:

- The gluon matrix  $A_\mu$  has  $N^2 - 1$  independent components (one less for the condition  $\text{tr } A_\mu = 0$ ). For  $N \rightarrow \infty$ , we can neglect the 1.
- A quark vector in the fundamental representation has  $N$  independent components.
- As the coupling constant appears just in front of the Lagrangian when using matrix fields, a propagator will give a contribution of  $g^2 = \lambda/N$ , while a vertex will give a contribution of  $g^{-2} = N/\lambda$ . We can remove the  $\lambda$  dependence and say that, in a graph, every distinct propagator gets a factor of  $N^{-1}$ , while every vertex gets a factor of  $N$ .
- Any closed line (counting the gluon lines as separate) is a sum over all colors, so it brings a combinatorial factor of  $N$ .

We now study the  $N$  dependence of a graph with  $V$  vertices,  $E$  propagators and  $F$  loops. Using our simple rules, a diagram  $(V, E, F)$  gets an  $N$  overall dependence equal to

$$N^{V+F-E} = N^\chi \quad (2.1.43)$$

where  $\chi$  is called the *Euler characteristic* of the graph. There is also another interpretation of the Euler characteristic: it is related to the genus  $g$ , that in this case represents the number of lines that cross each other without forming a vertex. Diagrams with  $g = 0$  are called *planar diagrams*, and they are the only surviving diagrams in the large  $N$  limit. Thus, in the large  $N$  limit planar diagrams give the leading order, while adding intersections will give subleading contributions.



Figure 2.3: To the left, a planar graph, with 4 color loops, 4 three gluon vertices and 1 four gluon vertex. To the right, a non planar graph similar to the planar one: there is no four gluon vertex, but there is the same amount of three gluons vertices and just one closed color loop. The left graph has an overall dependence  $N^2$  and genus  $g = 0$ , while the right graph has an overall dependence  $N^0$  and genus  $g = 1$ .

In this framework, 't Hooft and Witten studied the phenomenology of mesons and hadrons and their interactions. We cite their relevant results:

- States like glueballs are suppressed, and they decouple from baryons and mesons.
- Mesons are composed by a quark-antiquark couple and have finite masses in the large  $N$  limit, and their mutual interactions are suppressed. Meson states are stable when  $N \rightarrow \infty$ .
- Baryons are composed by  $N$  quarks or  $N$  antiquarks to have a color singlet, so their mass go to infinity when  $N \rightarrow \infty$ . Principal interactions are baryon-baryon and baryon-antibaryon interaction, as the mesons are too light to sensibly modify the state of a baryon, but the main interactions can be described through exchange of virtual mesons.

## 2.2 The gravity side

### 2.2.1 Anti de Sitter space

Anti de Sitter spacetime is a maximally symmetric space with a metric obeying Einstein's equations, provided with a negative cosmological constant. It is the background for the string theory that we will define, so we have to study its properties. We follow [7] for the presentation.

We first introduce the concept of maximally symmetric spacetime. In General Relativity, a symmetry of a spacetime  $\mathbb{S}$  is expressed through a diffeomorphism from  $\mathbb{S}$  to itself, depending on a continuous parameter. Formally, the parameter dependent diffeomorphism  $\phi_t$  goes from  $\mathbb{S} \times \mathbb{R}$  to  $\mathbb{S}$ , and at a fixed time it can be used to make a pullback of the metric on  $\mathbb{S}$  (that we denote as  $g_{\mathbb{S}}$ ). If

$$(\phi_t)^* g_{\mathbb{S}} = g_{\mathbb{S}}, \quad (2.2.1)$$

then  $\phi_t$  is called an isometry and its tangent vector field  $\xi^a$  is called a Killing vector. It can be shown that  $\phi_t$  is an isometry if and only if the tangent vector  $\xi^a$  obeys

$$\nabla^b \xi^a + \nabla^a \xi^b = 0 \quad (2.2.2)$$

(with  $\nabla$  the derivative operator associated to  $g_{\mathbb{S}}$ ). Thus, the search of isometries is substituted by the task of searching solutions to (2.2.2). There is a maximum of  $d(d+1)/2$  independent Killing vector fields on a spacetime of dimension  $d$ . As an example,  $\mathbb{R}^3$  with the standard Euclidean metric admits six independent Killing vector fields, namely

$$\xi_1 = \partial_x, \quad \xi_2 = \partial_y, \quad \xi_3 = \partial_z, \quad \xi_4 = x\partial_y - y\partial_x, \quad \xi_5 = x\partial_z - z\partial_x, \quad \xi_6 = y\partial_z - z\partial_y. \quad (2.2.3)$$

The first three vectors generate translations, while the last three generate rotations. For a maximally symmetric,  $d$  dimensional spacetime, it can be proven that the Riemann tensor can be simply expressed in terms of the metric as (taking  $\mathcal{R}$  as a constant, the spacetime curvature)

$$R_{abcd} = \frac{1}{d(d-1)} \mathcal{R} (g_{ac}g_{bd} - g_{ad}g_{bc}). \quad (2.2.4)$$

The converse is also true: if the Riemann tensor of a spacetime can be related to the metric through (2.2.4), then the spacetime is maximally symmetric. It can be proven directly by an easy contraction that the Ricci scalar associated to this Riemann tensor is given by the constant  $R$  (hence the naming), and it is constant all over the space. As through a rescaling of the coordinate one can always modify the magnitude of the scalar curvature (but never the sign), we divide maximally symmetric spaces in three classes: those with positive curvature, those with zero curvature and those with negative curvature. We're interested in Anti de Sitter spaces, which are maximally symmetric spacetimes with negative curvature.

The space  $AdS_{d+1}$  can be explicitly built through an embedding in a Minkowski space with two times: taking  $\mathbb{R}^{d+2}$  as the manifold  $\mathbb{R}^{d+2}$  with the coordinate system  $(X^0, X^1, \dots, X^d, X^{d+1})$  (coordinates spanning from  $-\infty$  to  $\infty$ ) and the metric

$$\bar{\eta} = -(dX^0)^2 + \sum_{i=1}^d dX^i dX^i - (dX^{d+1})^2. \quad (2.2.5)$$

The Anti de Sitter space  $AdS_{d+1}$  is defined as the set of points with coordinates respecting

$$-(X^0)^2 + \sum_{i=1}^d X^i X^i - (X^{d+1})^2 = -L^2, \quad (2.2.6)$$

with  $L$  a fixed real constant. This definition suggests the coordinate change

$$\begin{aligned} X^0 &= R \cos \tau \sec \rho, \\ X^i &= R \tan \rho \Omega_i \quad (i = 1, \dots, d), \\ X^{d+1} &= R \sin \tau \sec \rho. \end{aligned} \quad (2.2.7)$$

$\Omega_i$  are angular coordinates on  $S^d$ , respecting  $\sum_{i=1}^d \Omega_i^2 = 1$ , while the ranges of the other coordinates are given by  $R \in (0, \infty)$ ,  $\tau \in (-\infty, \infty)$ ,  $\rho \in [0, \frac{\pi}{2})$ . In those coordinates, (2.2.6) becomes  $R = L$ . Thus a coordinate system for  $AdS_{d+1}$  is provided by (2.2.7) with  $R = L$ . The induced metric  $g$  is given by

$$g = \frac{L^2}{(\cos \rho)^2} (-d\tau^2 + d\rho^2 + (\sin \rho)^2 dS^d), \quad (2.2.8)$$

where  $dS^d$  is the metric on  $S^d$ . This is a standard choice of the coordinate patch, but there is another, more useful patch. Letting  $\bar{x}^2 = \sum_{i=1}^{d-1} (x^i)^2$ , we can define coordinates  $(z, x^i, t)$  such as

$$\begin{aligned} X^0 &= \frac{z}{2} (1 + z^{-2}(L^2 + \bar{x}^2 - t^2)), \\ X^{d+1} &= \frac{Lt}{z}, \\ X^i &= \frac{Lx^i}{z} \quad (i = 1, \dots, d-1), \\ X^d &= \frac{z}{2} (1 + z^{-2}(-L^2 + \bar{x}^2 - t^2)). \end{aligned} \quad (2.2.9)$$

We can explicitly check that, by square summing those definitions, we obtain that the constraint (2.2.6) are defined for all coordinates. In those coordinates (all ranging from  $-\infty$  to  $\infty$ , excluding  $z = 0$ ), the metric becomes

$$g = \frac{L^2}{z^2}(dz^2 + (d\bar{x})^2 - dt^2). \quad (2.2.10)$$

From this metric, we can explicitly calculate the Ricci curvature,  $\mathcal{R} = -20/L$ . The coordinate patch is singular in  $z = 0$ , dividing the space in two disconnected pieces: one with  $z > 0$  and one with  $z < 0$ . From now on, when we refer to *AdS* space, we intend the part that is parametrized by  $z > 0$ .  $z = 0$  is a boundary, but the prefactor of the metric can be scaled away, giving a definite metric in  $z = 0$ . The metric on the boundary is conformally equivalent to

$$g_{\text{bound}} = d\bar{x}^2 - dt^2. \quad (2.2.11)$$

We can see that the boundary  $z = 0$  is conformally equivalent to  $d$  dimensional Minkowski space  $\mathbb{R}^{d-1|1}$ . This is the main fact that motivates the holographic principle, stating that for a theory of quantum gravity the information stored in the whole space on which the theory is defined is entirely contained in its boundary conditions, that define a field theory living on the boundary. We have seen that the (conformal) boundary of  $AdS_5$  is  $R^{3|1}$ , so we can relate a field theory on  $R^{3|1}$  to a string theory (that is a theory of quantum gravity) in  $AdS_5$  (with the addition of compact dimensions to make the theory well defined, as we'll see when discussing strings).

We conclude our review of *AdS* spaces by talking about its symmetries. Standard Lorentz symmetry on  $\mathbb{R}^{d|2}$  (the set of transformations leaving metric (2.2.5) invariant) leaves the definition of *AdS* space (2.2.6) invariant. *AdS* space inherits then the whole symmetry group  $SO(d, 2)$  of the original space. This can be related to the conformal group in  $d$  dimensions with just one time direction, as we have seen through identification (2.1.11), so we can say that the symmetry group of  $AdS_{d+1}$  space is the conformal group in  $d$  spatial dimensions. Counting the generators of the conformal algebra, we have that  $AdS_{d+1}$  space has  $(d+1)(d+2)/2$  independent symmetries, so we can find  $(d+1)(d+2)/2$  independent Killing vectors: as  $AdS_{d+1}$  is  $d+1$  dimensional, we have a confirmation of the fact that *AdS* is a maximally symmetric space.

## 2.2.2 String theory

### Classical bosonic string

String theory studies the dynamics of extended objects. The path of a particle in spacetime defines a *world line*, a one parameter curve in spacetime, and is formally represented by a function from  $\mathbb{R}$  to the target manifold in which the object moves, called  $\mathcal{M}$ . This function is said to provide an embedding of the curve in spacetime. We follow [5] and [56] for the presentation.

In the case of strings, we want to describe extended objects, that must be specified by two parameters instead of one. A string is a function

$$X(\tau, \sigma) : \mathbb{R} \times [0, \sigma_0] \rightarrow \mathcal{M}. \quad (2.2.12)$$

The range of the second parameter is taken to be limited, with  $\sigma_0$  a positive real: this choice reflects the fact that we want to describe finite length strings. The sets of all points that are in the image of  $X$  are called the *world sheet* of the string. The first parameter,  $\tau$ , can be interpreted as a time parameter, while the second one is a string parameter. It has no sense to distinguish points on the strings, with the exception of the points parametrized by the functions  $X(\tau, 0)$  and  $X(\tau, \pi)$ , called the *endpoints* of the string. The string equations of motion are provided by a variational principle: an action can be defined by pulling the metric of  $\mathcal{M}$  back to the parameter space. This way we obtain the *Nambu-Goto action*, that reads

$$S[X] = -T_0 \int \sqrt{|\gamma|} d\tau d\sigma, \quad (2.2.13)$$

where  $T_0$  is called the *string tension* and is the only dimensionful parameter in the theory,  $\gamma$  is the pullback of the metric on  $\mathcal{M}$ , that we call  $g$ :

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta} g_{\mu\nu}. \quad (2.2.14)$$

$\mu$  and  $\nu$  are indices on  $\mathcal{M}$ , while  $\alpha$  and  $\beta$  are indices on the parameter space, assuming values 0 and 1: the parameters are  $\xi^0 = \tau$  and  $\xi^1 = \sigma$ . We specialize in the case  $\mathcal{M} = \mathbb{R}^{d,1}$ , replacing  $g$  by the standard Minkowski metric  $\eta$ : in this case, denoting  $\dot{X} = \partial X / \partial \tau$  and  $X' = \partial X / \partial \sigma$ , we can explicitly write the pullback as

$$S[X] = -T_0 \int \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2} d\tau d\sigma. \quad (2.2.15)$$

This action is really difficult to use, as it involves a square root and is not polynomial in  $X$  and its derivatives. A classically equivalent action is given by the *Polyakov action*:

$$S[X] = -\frac{T_0}{2} \int \sqrt{|h|} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} d\tau d\sigma. \quad (2.2.16)$$

Here  $h_{\alpha\beta}$  is an auxiliary symmetric non degenerate matrix field, living on the parameter space, that is not dynamic (as the action does not contain derivatives of  $h$ ), while  $h^{\alpha\beta}$  is its inverse:  $h$  plays the role of a non dynamical metric in the parameter space. From now on we will use a different constant instead of  $T_0$ : we introduce the parameter  $\alpha'$ , frequent in literature, related to  $T_0$  through  $T_0 = 1/2\pi\alpha'$ . The quantity  $\sqrt{\alpha'}$ , due to its dimensions, is called *string length*. The equation of motion for  $h$  imposes constraints:

$$T_{\alpha\beta} = -\frac{4\pi\alpha'}{\sqrt{|h|}} \frac{\delta S}{\delta h^{\alpha\beta}} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X^\nu \eta_{\mu\nu} = 0. \quad (2.2.17)$$

Those constraints are called *Virasoro constraints*, and they are necessary to have an unitary quantum theory of strings. Polyakov action has many symmetries, that we can use to make the string description pretty convenient:

- In Minkowski space there is the standard Poincarè symmetry, acting on the embedding function as  $X^\mu = \Lambda^\mu_\nu X^\nu + a^\mu$ , with  $\Lambda$  a Lorentz matrix and  $a^\mu$  a constant vector.  $h$  is unchanged under this transformation.
- We can reparametrize the string, introducing functions  $\tau'(\tau, \sigma)$  and  $\sigma'(\tau, \sigma)$ : in that case, the embedding function just changes arguments,  $X(\tau, \sigma) \rightarrow X(\tau', \sigma')$ , while  $h$  changes as a proper tensor,

$$h_{\alpha\beta}(\tau, \sigma) = \frac{\partial \xi'^\gamma}{\partial \xi^\alpha} \frac{\partial \xi'^\delta}{\partial \xi^\beta} h_{\gamma\delta}(\tau', \sigma'). \quad (2.2.18)$$

- Lastly, we can do a Weyl transformation, an internal symmetry on the parameter space that does not change the coordinates: this is given by  $h_{\alpha\beta}(\tau, \sigma) \rightarrow e^{2\omega(\tau, \sigma)} h_{\alpha\beta}$  and the embedding function is totally unchanged.

Using reparametrization invariance and Weyl transformations, we can choose a comfortable gauge to do calculations. We will use Weyl gauge:

$$h_{\alpha\beta}(\tau, \sigma) = e^{2\omega(\tau, \sigma)} \eta_{\alpha\beta}, \quad \eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (2.2.19)$$



In this gauge, the function  $\omega$  gets canceled in the motion equations, and the action reads

$$S[X] = \frac{1}{4\pi\alpha'} \int ((\dot{X})^2 - (X')^2) d\tau d\sigma, \quad (2.2.20)$$

so the motion equations are

$$\begin{cases} (\partial_\tau^2 - \partial_\sigma^2)X(\tau, \sigma) = 0, \\ X'(\tau, \sigma_0) \cdot \delta X(\tau, \sigma_0) - X'(\tau, 0) \cdot \delta X(\tau, 0) = 0 \quad \forall \tau, \\ (\dot{X} \pm X')^2 = 0. \end{cases} \quad (2.2.21)$$

The first vector equation is the motion equation, that is equal to the wave equation in 1 + 1 dimensions. The second equation is imposed to have variation of  $S$  at the string boundaries vanish, while the two equations in the third row are the Virasoro constraint, expressed in Weyl gauge.

Before imposing boundary conditions, we review the way string solutions are found. We introduce *light cone coordinates* on the parameter space:  $\sigma_\pm = \tau \pm \sigma$ , and introduce the respective derivatives through  $\partial_\tau = \partial_+ + \partial_-$  and  $\partial_\sigma = \partial_+ - \partial_-$ . This way, the first equation of (2.2.21) becomes

$$\partial_+ \partial_- X(\tau, \sigma) = 0. \quad (2.2.22)$$

A solution to this equation is obtained by decomposing  $X$  in the sum of a left moving part, depending only on  $\sigma_+$ , and a right moving part, depending only on  $\sigma_-$ :

$$X(\tau, \sigma) = X_L(\sigma_+) + X_R(\sigma_-). \quad (2.2.23)$$

In light cone coordinates, Virasoro constraints read

$$\partial_+ X \cdot \partial_+ X = 0, \quad \partial_- X \cdot \partial_- X = 0. \quad (2.2.24)$$

We can decompose in Fourier sum both pieces, obtaining

$$X_L^\mu(\sigma_+) = \frac{\tilde{x}_0^\mu}{2} + \frac{\alpha'}{2} \tilde{p}^\mu \sigma_+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\tilde{\alpha}_n^\mu}{n} e^{-in\sigma_+}, \quad (2.2.25a)$$

$$X_R^\mu(\sigma_-) = \frac{x_0^\mu}{2} + \frac{\alpha'}{2} p^\mu \sigma_- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} e^{-in\sigma_-}. \quad (2.2.25b)$$

Here,  $x_0, p, \alpha_n, \tilde{x}_0, \tilde{p}, \tilde{\alpha}_n$  are constants, and reality of the solutions imposes  $\tilde{\alpha}_n^\mu = (\tilde{\alpha}_{-n}^\mu)^*$  and  $\alpha_n^\mu = (\alpha_{-n}^\mu)^*$  (the sum runs over both positive and negative integers). The constants  $x_0$  and  $\tilde{x}_0$  can be related to string position,  $p$  and  $\tilde{p}$  describe the pulse of the string while  $\alpha_n$  and  $\tilde{\alpha}_n$  are coefficients that allow the string to vibrate transversally. We still have to impose Virasoro constraints: to do that, it is convenient to define  $\alpha_0^\mu = \sqrt{\alpha'/2}p^\mu$  and  $\tilde{\alpha}_0^\mu = \sqrt{\alpha'/2}\tilde{p}^\mu$ . Explicit differentiation gives

$$\partial_+ X \cdot \partial_+ X = \alpha' \sum_m \tilde{L}_m e^{-im\sigma_+} = 0 \quad \partial_- X \cdot \partial_- X = \alpha' \sum_m L_m e^{-im\sigma_-} = 0, \quad (2.2.26)$$

(now 0 is included in the sum) where we defined

$$\tilde{L}_m = \frac{1}{2} \sum_n \tilde{\alpha}_n \cdot \tilde{\alpha}_{m-n} \quad L_m = \frac{1}{2} \sum_n \alpha_n \cdot \alpha_{m-n}. \quad (2.2.27)$$

To have (2.2.26), we must impose  $L_m = 0$  and  $\tilde{L}_m = 0$  for each value of  $m$ .

To complete the discussion of the classical solution, we have to impose boundary conditions. There are various choices that we can make to impose them. We can impose a periodicity condition, namely  $X(\tau, \sigma_0) = X(\tau, 0)$  and  $X'(\tau, \sigma_0) = X'(\tau, 0)$  for all  $\tau$ . A kind of string with this boundary condition is called a *closed string*, due to the fact that its world sheet will have the topology of a cylinder. For closed strings, we set  $\sigma_0 = 2\pi$ : then we can see that we just have to impose  $p^\mu = \tilde{p}^\mu$ , and we can freely set  $x_0^\mu = \tilde{x}_0^\mu$ . This way, the boundary conditions are automatically satisfied, and the string solution reads

$$X^\mu(\tau, \sigma) = x_0^\mu + 2\alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{e^{-in\tau}}{n} (\tilde{\alpha}_n^\mu e^{-in\sigma} + \alpha_n^\mu e^{in\sigma}). \quad (2.2.28)$$

Periodicity is manifest, as the only  $\sigma$  dependence is in the complex exponentials, and  $\sigma$  is multiplied by an integer. This justifies choosing  $\sigma_0 = 2\pi$ .

If we do not impose a periodicity condition, then we have an *open string*, and the topology of the world sheet will be that of a strip. The string has two distinct, unidentified endpoints, and the most general choice that we can make is to have both terms in the second line of (2.2.21) vanish. Let us choose  $\sigma_0 = \pi$  and call  $\sigma^*$  a generic endpoint, 0 or  $\pi$ : then we can impose a *Neumann boundary condition*, letting  $X'^\mu(\tau, \sigma^*) = 0$ , or we can impose a *Dirichlet boundary condition*, letting the

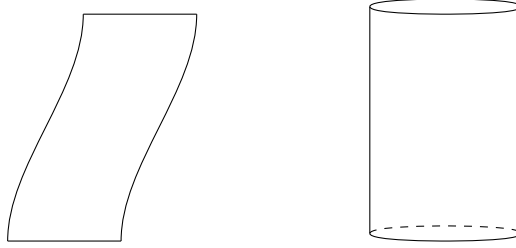


Figure 2.4: Example in 2+1 dimensions: to the left, a worldsheet with the topology of a strip, on which an open string propagates. To the right, a worldsheet with the topology of a cylinder, on which a closed string propagates.

endpoint be fixed,  $\delta X^\mu(\tau, \sigma^*) = 0$ . We can impose those conditions independently on each endpoint and on each component of  $X^\mu$ , so we'll have  $DD$  boundaries if we impose Dirichlet conditions on each endpoint,  $NN$  boundaries,  $ND$  or  $DN$  boundaries depending on the particular condition imposed at 0 and  $\pi$ . The only physical constraint that we have is that we must have  $NN$  conditions in time direction, or else we would have a string that is fixed in time. Dirichlet boundary conditions are best discussed in the context of D-branes, that we'll introduce later. For now, we impose  $NN$  conditions on every coordinate. To do that, we must have  $p^\mu = \tilde{p}^\mu$ , we can choose  $x_0^\mu = \tilde{x}_0^\mu$  and, lastly, we have to impose  $\alpha_n^\mu = \tilde{\alpha}_n^\mu$  for each  $n$ . The solution then reads

$$X^\mu(\tau, \sigma) = x_0^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{e^{-in\tau}}{n} \alpha_n^\mu \cos(n\sigma). \quad (2.2.29)$$

The derivative with respect to  $\sigma$  is a sum of terms proportional to sines of  $n\sigma$ , so it vanishes when  $\sigma$  equals to 0 or  $\pi$ . In the case of closed strings, we have two sets of independent oscillations,  $\alpha$  and  $\tilde{\alpha}$ , while in the case of open strings we just have one set.

### Quantization of the bosonic string

We can try a naive open string quantization by imposing canonical commutation relations. We can identify the canonical impulse by varying the Polyakov action with respect to  $\dot{X}$ : we can then identify the canonical pulse as

$$\Pi^\mu(\tau, \sigma) = \frac{\dot{X}^\mu(\tau, \sigma)}{2\pi\alpha'}. \quad (2.2.30)$$

Then, we impose standard equal time commutation relations:

$$[X^\mu(\tau, \sigma), \Pi^\nu(\tau, \sigma')] = i\eta^{\mu\nu} \delta(\sigma - \sigma'). \quad (2.2.31)$$

From those relations, the following relations must hold:

$$[x_0^\mu, p^\nu] = i\eta^{\mu\nu} \quad [\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu} \delta_{m,-n}. \quad (2.2.32)$$

We can define creation and annihilation operators through (here we take  $m > 0$ )

$$a_m^\mu = \frac{1}{\sqrt{m}} \alpha_m^\mu \quad a_m^{\dagger\mu} = \frac{1}{\sqrt{m}} \alpha_{-m}^\mu, \quad (2.2.33)$$

respecting the commutation relation  $[a_m^\mu, a_n^{\dagger\nu}] = \eta^{\mu\nu} \delta_{mn}$  of a canonical oscillator. All other commutation relations are 0. A ket is then described by a  $d + 1$  vector, representing the overall string momentum, and an infinite set of natural numbers, indicating the eigenvalues of the infinite series of oscillations that the string can have.

All seems to go straightforwardly, but there is a problem:  $[a_m^0, a_m^{\dagger 0}] = -1$  means that states with oscillations in the 0 direction give rise to negative norm states, spoiling the unitarity of the theory and the probabilistic interpretation of wavefunctions. Such a quantum theory is not acceptable. This happens because we did not impose Virasoro constraints: we can hope that, promoting the constants  $L_m$  to *Virasoro operators*, the physical condition for a state  $|\psi\rangle$ , expressed as  $L_m |\psi\rangle = 0$ , suffices to totally decouple negative norm states from positive norm states and to recover unitarity.

To do that, we first use *light cone gauge* for quantization: defining  $X^\pm = X^0 \pm X^1$  and using latin indices  $i = 2, \dots, d$  to indicate the rest of the coordinates, we can reparametrize the string imposing  $X^+(\tau, \sigma) = x_0^+ + 2\alpha' p^+ \tau$  with  $p^+$  a constant, and dropping  $X^-$  as a degree of freedom, that is completely determined by  $X^+$  and the other coordinates  $X^i$  through the Virasoro constraints, up to a constant  $x_0^-$ . Proceeding this way, we're spoiling explicit Lorentz invariance, but there are ways to quantize the string keeping explicit Lorentz invariance, as BRST quantization (see [39] for a reference). The degrees of freedom are then  $x_0^-, p^+, a_m^i, a_n^{\dagger i}$ , with nontrivial commutation relations

$$[x_0^-, p^+] = i \quad [x_0^i, p^j] = i\delta^{ij} \quad [a_m^i, a_n^{\dagger j}] = \delta^{ij}. \quad (2.2.34)$$

We can now form physical states: they are specified by an overall pulse  $k$  and by the oscillation numbers of the independent oscillators, collectively denoted as  $N_{in}$ : a generic state is written as

$$|N, k\rangle = \left[ \prod_{i=2}^d \prod_{n=1}^{\infty} \frac{(a_n^{\dagger i})^{N_{in}}}{\sqrt{N_{in}!}} \right] |0, k\rangle. \quad (2.2.35)$$

We have to impose quantum Virasoro constraints, as we have imposed only the classical version. This is done easily for  $L_m$  and  $m > 0$ : it is sufficient to recall the definition of  $\alpha_0^\mu$  in terms of  $p^\mu$  and substitute the appropriate operators in  $L_m$ , then we say that physical states  $|\psi\rangle$  are the ones in the kernel of  $L_m$ ,  $L_m |\psi\rangle = 0$ . As  $L_m$  is an infinite sum of products of commuting operators, no normal ordering ambiguities arise. In the case of  $L_0$ , we have products of non commuting operators, so we have to follow a prescription for normal ordering. This is done by redefining  $L_0$  in terms of quantum operators, as

$$L_0 = \alpha' p \cdot p + \sum_{i=2}^d \sum_{n=1}^{\infty} n a_n^{\dagger i} a_n^i. \quad (2.2.36)$$

We then have to impose  $(L_0 - a) |\psi\rangle = 0$ , with  $a$  a constant that arises from normal ordering. The constant  $a$  has a well defined value to have a coherent theory, expressed in terms of the number of spacetime dimensions,  $d + 1$ : we just cite the result, referring to [49] for details. We have to set

$$a = -\frac{d-1}{24}. \quad (2.2.37)$$

We can now find the masses of open string states. To do that, we can identify the squared mass of a state as the result of the operator  $-p \cdot p$  on a state: from the Virasoro constraint and the explicit value of  $a$ , we get that the squared mass of a state is given by the application of the operator

$$M^2 = \frac{1}{\alpha'} \left( \sum_{i=2}^d \sum_{n=1}^{\infty} n a_n^{\dagger i} a_n^i - \frac{d-1}{24} \right). \quad (2.2.38)$$

We see that the mass contribution of oscillators created by  $a_n^{\dagger i}$  grows with  $n$ , so  $a_1^{\dagger i}$  create the lightest states. Let us now consider the lightest excited state,  $a_1^{\dagger i} |0, k\rangle$ . This state transforms as an object in the fundamental representation of  $SO(d-1)$ , as the index runs from 2 to  $d+1$ . As we're considering Minkowski space with  $d+1$  dimensions, we know from the representation of symmetry groups through the

method of the little group (chapter 2 of [50]) that a state transforming as an object in a representation of  $SO(d)$  must be massive, while a state transforming as an object in a representation of  $SO(d-1)$  must be massless<sup>1</sup>. To have a coherent theory we're forced to impose that  $a_1^{\dagger i} |0, k\rangle$  be massless: this constraints the number of space dimensions to  $d = 25$ , called the *critical dimension*. This is a feature of quantum string theory, that is coherent only with a fixed number of dimensions: different theories give different dimension, as we'll see in the case of the fermionic string.

States of the form  $a_1^{\dagger i} |0, k\rangle$  are interpreted as one particle states of standard quantum field theory. By their index type, they represent massless vectors. String states are then identified with particle states of standard quantum field theory, so the lightest states are candidates for gauge bosons. The vacuum state has  $M^2 = -\frac{1}{\alpha'}$ , negative mass square, and this is what is called a *tachyon state*. Its presence in a quantum theory does not spoil unitarity, but causality is lost. The tachyon state is obviously not decoupled from the rest of the theory, as it is the ground state from which all states are built, so the presence of tachyons is a defect of bosonic string theory, that we cannot remedy. We'll still continue our analysis of this theory, as it is perfect to understand what happens when quantizing string theories.

We now turn to closed strings. There are two sets of oscillators,  $\alpha_n$  and  $\tilde{\alpha}_n$ . As in the case of open strings, we quantize in light cone gauge, as the string parametrization  $X^+(\tau, \sigma) = x_0^+ + 2\alpha' p^+ \tau$  is compatible with the periodicity condition for closed strings. We can define creation and destruction operators  $a_n^i, a_n^{\dagger i}, \tilde{a}_n^i$  and  $\tilde{a}_n^{\dagger i}$  as before, and use the Virasoro constraints to determine  $X^-$ . In the end, we've got the following operators and nontrivial commutation relations

$$[x_0^-, p^+] = i \quad [x_0^i, p^j] = i\delta^{ij} \quad [a_m^i, a_n^{\dagger j}] = \delta^{ij} \quad [\tilde{a}_m^i, \tilde{a}_n^{\dagger j}] = \delta^{ij}. \quad (2.2.39)$$

We have two sets of Virasoro operators,  $L_m$  and  $\tilde{L}_m$ , with obvious definitions. Only  $L_0$  and  $\tilde{L}_0$  have normal ordering ambiguities. To solve them, they are redefined as in (2.2.36) and the constants  $a$  and  $\tilde{a}$  are introduced to account for normal ordering. The quantum constraints on physical states  $|\psi\rangle$  are ( $m > 0$ )

$$L_m |\psi\rangle = \tilde{L}_m |\psi\rangle = 0 \quad (L_0 - a) |\psi\rangle = (\tilde{L}_0 - \tilde{a}) |\psi\rangle = 0, \quad (2.2.40)$$

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<sup>1</sup>As an example in standard four dimensional field theory, a massive vector field is described by a three vector,  $v^i$ , while  $v^0$  is constrained: the three vector transforms as a vector in  $SO(3)$ . The electromagnetic field has two degrees of freedom, transforming under a representation of  $SO(2)$

and  $a$  and  $\tilde{a}$  are expressed in terms of spacetime dimensions as

$$a = \tilde{a} = -\frac{d-1}{12}. \quad (2.2.41)$$

Ket states are specified by an overall pulse and the occupation numbers  $N_{in}$  and  $\tilde{N}_{in}$ , and a generic state can be expressed as

$$|N, \tilde{N}, k\rangle = \left[ \prod_{i=2}^d \prod_{n=1}^{\infty} \frac{(a_n^{\dagger i})^{N_{in}} (\tilde{a}_n^{\dagger i})^{\tilde{N}_{in}}}{\sqrt{N_{in}!} \sqrt{\tilde{N}_{in}!}} \right] |0, 0, k\rangle. \quad (2.2.42)$$

There is an important level matching condition: from Virasoro constraints

$$(L_0 - a - \tilde{L}_0 + \tilde{a}) |N, \tilde{N}, k\rangle, \quad (2.2.43)$$

the explicit forms (2.2.36) and the fact that  $a = \tilde{a}$ , we must have on physical states

$$\sum_{i=2}^d \sum_{n=1}^{\infty} n a_n^{\dagger i} a_n^i = \sum_{i=2}^d \sum_{n=1}^{\infty} n \tilde{a}_n^{\dagger i} \tilde{a}_n^i. \quad (2.2.44)$$

As an example, states of the form  $a_1^{\dagger i} |0, 0, k\rangle$  are not in the physical spectrum of closed strings. The mass is obtained as before, and reads

$$M^2 = \frac{2}{\alpha'} \left( \sum_{i=2}^d \sum_{n=1}^{\infty} n a_n^{\dagger i} a_n^i + \sum_{i=2}^d \sum_{n=1}^{\infty} n \tilde{a}_n^{\dagger i} \tilde{a}_n^i + \frac{1-d}{12} \right). \quad (2.2.45)$$

The lightest excited physical state is

$$a_1^{\dagger i} a_1^{\dagger j} |0, 0, k\rangle. \quad (2.2.46)$$

This state transforms in a mixed representation of  $SO(d-1)$ , as there is no symmetry condition on the indices. As the little group is  $SO(d-1)$ , this state must be massless, so the critical dimension in closed strings is still  $d = 25$ , and the ground state is still a tachyon with the same negative square mass.

Open string states contain massless vector bosons, and closed string states are interpreted in the same way. A two indices tensor can be decomposed in the sum of a symmetric traceless tensor, an antisymmetric tensor and a scalar tensor. The symmetric part is interpreted as the graviton, while the antisymmetric part and the trace part are respectively called *Kalb-Ramond field* and *dilaton*.

Those interpretations are justified in the following way: by calling  $g_{\mu\nu}(X)$  the wavefunction associated to the symmetric part of (2.2.46),  $B_{\mu\nu}(X)$  the antisymmetric part of the wavefunction (Kalb-Ramond field) and by  $\phi(X)$  the scalar part, the most general Polyakov action with the same symmetries of the original one in the presence of a closed string background is given by

$$S_P = -\frac{1}{4\pi\alpha'} \int \sqrt{|h|} (h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu} + \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu} + \alpha' R_h \phi) d\tau d\sigma, \quad (2.2.47)$$

where  $R_h$  is the Ricci scalar associated to the background metric  $h$ . The vanishing of the variation with respect to  $h$  gives equations of motions for the fields  $g$ ,  $B$  and  $\phi$ . Varying the action with respect to  $h$  and taking the trace of the result, we have

$$T_\alpha^\alpha = -\frac{1}{2\alpha'} \beta_{\mu\nu}^g h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2\alpha'} \beta_{\mu\nu}^B \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2} \beta^\phi R_h = 0, \quad (2.2.48)$$

where the  $\beta$  functions are, to order  $\alpha'$

$$\beta_{\mu\nu}^g = -\alpha' \left( R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi - \frac{1}{4} H_{\mu\rho\sigma} H_\nu^{\rho\sigma} \right), \quad (2.2.49a)$$

$$\beta_{\mu\nu}^B = \alpha' \left( -\frac{1}{2} \nabla^\rho H_{\rho\mu\nu} + \nabla^\rho \phi H_{\rho\mu\nu} \right), \quad (2.2.49b)$$

$$\beta^\phi = \alpha' \left( \frac{d-25}{6\alpha'} - \frac{1}{2} \nabla^2 \phi + \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{24} H_{\mu\nu\rho} H^{\mu\nu\rho} \right), \quad (2.2.49c)$$

where the covariant derivative  $\nabla$  and the Ricci tensor  $R_{\mu\nu}$  are calculated from the tensor  $g$  and  $H$  is the exterior derivative of  $B$ , in components  $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ . The physical condition is that the  $\beta$  functions must vanish separately. In the case of constant dilaton field and vanishing Kalb-Ramond field, only the first  $\beta$  function has to vanish, giving the Einstein equations in vacuum  $R_{\mu\nu} = 0$ . This way quantum gravity is included in string theory. Lastly, we remark that from the following action

$$S = \frac{1}{2\tilde{\kappa}} \int \sqrt{|g|} e^{-2\phi} \left( R + 4\nabla_\mu \phi \nabla^\mu \phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{2(d-25)}{3\alpha'} \right) d^{d+1}x, \quad (2.2.50)$$

the equations of motion for the  $\beta$  functions descend, so it can be used as an effective action, and the normalization constant  $\tilde{\kappa}$  will be specified in a moment. Assuming  $\phi_0$  as the boundary condition for  $\phi$ , we can rewrite the action by defining

$$\tilde{\phi} = \phi - \phi_0 \quad \kappa = \tilde{\kappa} e^{\phi_0} = \sqrt{8\pi G_{26}} \quad \tilde{g}_{\mu\nu} = e^{\frac{4}{1-d}\tilde{\phi}} g_{\mu\nu}, \quad (2.2.51)$$



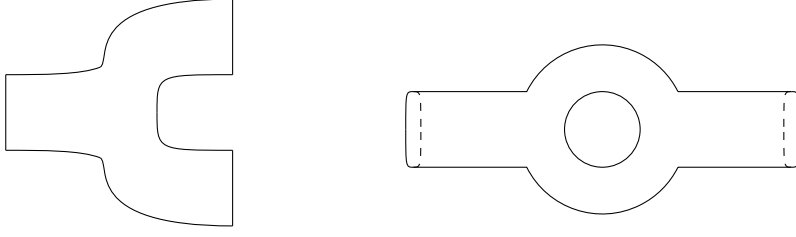


Figure 2.5: Two typical examples of interacting string graphs. To the left, an open string separates in two open strings. There is no hole in the middle of the graphs, so the left diagram has  $g = 0$ . To the right, a closed string splitting in two closed strings, then rejoining in a single one. With an hole in the middle, the right diagram has  $g = 1$ .

(where  $G_{26}$  is the 26-dimensional gravitational constant) obtaining the action

$$S = \frac{1}{2\kappa^2} \int \sqrt{|\tilde{g}|} \left( \tilde{R} - \frac{4}{d-1} \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{12} e^{-\frac{8}{d-1}\tilde{\phi}} H_{\mu\nu\rho} H^{\mu\nu\rho} - 2 \frac{d-25}{3\alpha'} e^{\frac{4}{d-1}\tilde{\phi}} \right) d^{d+1}x. \quad (2.2.52)$$

### String interactions

Here we give a short picture of interaction of strings. The interaction of strings is a vast topic, and explaining the procedure that brings us to calculable results is out of the scope of this thesis, so we limit ourselves to the general ideas that can be used to visualize string interactions and to understand the genus expansion, that is related to the  $1/N$  expansion of QCD. We refer to [39] for details. String interaction happens when the metric of spacetime is promoted as a propagating field, so a string will self-interact through gravity. Free string worldsheets have the topology of a strip for open strings, and a cylinder for closed strings: we can picture interactions through worldsheets with more involved topologies, as strips separating or cylinders separating and rejoining, as illustrated in figure (2.5). The basic idea for calculations is to use Feynmann path integral, using as functional variables the string  $X$  and the background metric  $h$ , imposing constraints that derive from Weyl symmetries. Thus, the basic object is of the form

$$Z = \int_{\Sigma} [dX][dh] e^{-S_P}, \quad (2.2.53)$$

where  $\Sigma$  denotes integration over all possible worldsheets (akin to integration over all paths),  $[dX]$  indicates functional integration,  $S_P$  is the Polyakov action, Wick rotation has been performed and the sources are suppressed. We can add to  $S_P$  a term: defining

$$S'_P = S_P - \frac{\lambda}{4\pi} \int_{\Sigma} \sqrt{|h|} R_h d\tau d\sigma = S_P - \lambda\chi, \quad (2.2.54)$$

we can verify that this addition does not change the equations of motion. Turns out that  $\chi$  is a topological term, and it is entirely determined by the topology of the worldsheet. For closed string worldsheets, labeling as  $g$  the genus of the sheet, or the number of holes, we have that  $\chi = 2 - 2g$ . Then, we can write an expansion in  $g$ :

$$Z = \sum_g \int_{\Sigma_g} [dX][dh] e^{-S_P - \lambda(2-2g)} = \sum_g e^{-\lambda(2-2g)} \int_{\Sigma_g} [dX][dh] e^{-S_P}, \quad (2.2.55)$$

where  $\Sigma_g$  denotes all worldsheets with genus  $g$ . We see that sheet contributions are suppressed if  $g$  is non zero, so the genus number can be used to define an order of expansion. Closed string coupling is identified with  $g_{cs} = e^\lambda$ , so the genus expansion in powers  $g_{cs}^{2n}$ , with  $n$  assuming values from  $-1$  to  $\infty$ : if the coupling constant is small (or  $\lambda$  is small) then higher genus worldsheets decay. An important fact is that  $\lambda$  is not an external constant, but it is related to the dilaton field  $\phi$ , or its expectation value, thus the interaction is inserted in a natural way. We do not prove this, but we note that the motivation can be intuitively understood by (2.2.50), where the dilaton field multiplies the whole action of the effective fields, that can be thought as the dilaton field multiplying the interaction pieces. In the case of open strings, all proceeds in the same way, with a fundamental difference: as the open string worldsheet has more boundaries than a closed string worldsheet with the same genus, the contribution of  $\chi$  is varied: to account for that, we identify  $g_{os} = e^{\frac{\lambda}{2}}$ . Then we have an expansion in powers of  $g_{os}^{2n}$ , with  $n$  from  $-1$  to  $\infty$ , as before.

### Fermionic string

The previous quantization has a fundamental limit (besides the tachyon state, that disrupts causality): all states obtained by applying the creation operators are of bosonic statistics. To introduce fermionic degrees of freedom in the spectrum,

we must introduce the fermionic string. This is done by introducing the embedding  $\Psi^\mu$  and its conjugate  $\bar{\Psi}^\mu$ , that are Grassmann fields, vectors in the target space and spinors in the parameter space. Introducing  $\gamma$  matrices on the parameter space, we write the Polyakov action in conformal gauge and flat space as

$$S_P = -\frac{1}{4\pi\alpha'} \int \eta^{\alpha\beta} (i\bar{\Psi}^\mu \gamma_\alpha \partial_\beta \Psi^\nu + \partial_\alpha X^\mu \partial_\beta X^\nu) \eta_{\mu\nu} d\tau d\sigma. \quad (2.2.56)$$

We have added the bosonic string to have the most general action possible, but from now on we concentrate on the fermionic part. Writing in component  $\Psi = (\psi_-, \psi_+)$  and introducing light cone coordinates on the parameter space, we have

$$S_P = \frac{i}{2\pi\alpha'} \int (\psi_-^\mu \partial_+ \psi_{-\mu} + \psi_+^\mu \partial_- \psi_{+\mu}) d\tau d\sigma. \quad (2.2.57)$$

The equations of motion with boundary conditions are then

$$\begin{cases} \partial_+ \psi_- = 0, & \partial_- \psi_+ = 0, \\ (\psi_- \cdot \delta\psi_- - \psi_+ \cdot \delta\psi_+) |_0^{\sigma_0} = 0. \end{cases} \quad (2.2.58)$$

The last row gives the boundary conditions.<sup>2</sup>

As in the bosonic string, the boundary conditions can be solved by imposing open or closed topology. We start by considering open strings: then we have to make the differences in (2.2.58) vanish separately at  $\sigma_0$  (that we take as  $\pi$  as in open bosonic strings) and at 0. As overall signs are irrelevant in fermionic strings, we solve those constraints by setting  $\psi_-(\tau, 0) = \psi_+(\tau, 0)$ . We have two different choices for the endpoint  $\pi$ : the condition  $\psi_-(\tau, \pi) = \psi_+(\tau, \pi)$  gives rise to states that are in the *Ramond* sector, or R sector, while the opposite choice  $\psi_-(\tau, \pi) = -\psi_+(\tau, \pi)$  gives the second sector, called *Neveu-Schwartz* or NS sector. Functions solving those constraints are expanded as

$$\text{R sector: } \psi_\pm^\mu = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} d_n^\mu e^{-in\sigma_\pm}, \quad (2.2.59)$$

$$\text{NS sector: } \psi_\pm^\mu = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z} - \frac{1}{2}} b_n^\mu e^{-in\sigma_\pm}. \quad (2.2.60)$$

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<sup>2</sup>Constraints as Virasoro constraints should be imposed, but one should also consider additional constraints derived by the fact that there is a supersymmetry transformation rotating the bosonic string in the fermionic one and vice versa. We will omit discussion of those constraints, and take the resulting mass formulas from [39]

In the NS sector the sum runs over half integers. Quantization of the theory is done in light cone gauge, in a way similar to the bosonic string. Reintroducing the coordinates  $i = 2\dots d$  we must have

$$\{b_m^i, b_{-n}^j\} = \delta^{ij} \delta_{mn}, \quad \{d_m^i, d_{-n}^j\} = \delta^{ij} \delta_{mn}. \quad (2.2.61)$$

We then interpret  $b_n$  and  $d_n$  with  $n > 0$  as annihilation operators, those with  $n < 0$  as creation operators and a vacuum state  $|0\rangle$  (with an underscript  $NS$  or  $R$  to indicate which operators annihilate the state). The spectrum of states can then be obtained by acting with creation operators on the vacuum state.

There are two problems to address before going further. In the  $NS$  sector, the state  $b_{-1/2}^i |0_{NS}\rangle$  has mass

$$M^2 = \frac{1}{\alpha'} \left( \frac{1}{2} - \frac{d-1}{16} \right), \quad (2.2.62)$$

and must be massless for the same reason as in the bosonic string: this implies  $d = 9$ , so with fermionic strings the critical dimension turns out to be lower than in the bosonic string. This implies that the vacuum state is tachyonic. In the  $R$  sector, there is an operator  $b_0^i$  that is not a creation or destruction operator, but it anticommutes with all other operators. Acting with  $b_0^i$  on  $|0_R\rangle$ , we obtain another state with the same energy from which particle states can be built, so the ground state is degenerate.

Both problems are solved by introducing the *GSO projection*, a superselection rule on states: introducing the operator  $(-)^F$  that acts as identity on  $|0_{NS}\rangle$  and anticommutes with all creation operators, we have that  $(-)^F$  effectively counts how many times a fermionic operator is applied to a state modulo 2, so  $(-)^F$  has only eigenvalues 1 and  $-1$ . In the  $R$  sector, we choose a ground state and proceed in the same way: the other degenerate ground state will have eigenvalue  $-1$ . GSO projection consists in choosing as physical states only those in a fixed subspace of  $(-)^F$ : in the  $NS$  sector we only choose those with eigenvalue  $-1$ , so the true vacuum is  $b_{-1/2}^i |0_{NS}\rangle$ , belonging to the vector representation of  $SO(8)$ . In the  $R$  sector we can choose the eigenvalue and the ground states have different chirality: they are in the spinor representations of  $SO(8)$  with definite chirality. We introduce a notation to indicate irreducible representations of  $SO(8)$ , including spinorial representations:  $\mathbf{1}$  is the scalar representation,  $\mathbf{35}_\pm$  are symmetric representations that are either dual or self dual,  $\mathbf{8}$  and  $\mathbf{8}'$  are the two (chiral) fundamental spinorial representations,  $\mathbf{8}_V$  is the vector representation. We will also

Sector	State	$(-)^F$	Representation	$m^2$
R+	$ 0_R\rangle$	+1	<b>8</b>	0
R-	$ 0'_R\rangle$	-1	<b>8'</b>	0
NS-	$ 0_{NS}\rangle$	-1	<b>1</b>	$-1/2\alpha'$
NS+	$b_{-1/2}^i  0_{NS}\rangle$	+1	<b>8<sub>V</sub></b>	0

Table 2.1: Field content of the ground states in open superstring quantum theories.

need other higher spin representations: **28** denotes a representation of differential forms with two indices, **56<sub>T</sub>** is a tensorial representation of differential forms with three indices, **35<sub>±</sub>** denote differential forms with four indices, whose exterior derivatives satisfy a self duality (or anti self duality) condition, **35** is a representation of symmetric, traceless, two indices tensors, while **56** and **56'** denote spinor vector representations. The representations of the ground states are collected in table (2.1) Here we have conventionally named  $|0_R\rangle$  as the ground state of R sector with eigenvalue +1 for the fermion number operator and as  $|0'_R\rangle$  the state obtained through application of  $d_0^i$ .

We now analyze the closed string spectrum. As for bosonic string, we obtain that the closed string can be seen as two copies of the open string, and the quantum states obey a level matching condition. Each closed string is defined by the choice of two open string sectors: as an example, we can have the (R+,R-) string, whose quantum states are built by applying creation operators on the vacuum  $|0_R, 0'_R, k\rangle$  (where  $k$  is an overall string momentum) and physical states are subject to the level matching condition (2.2.44) with obvious substitutions. We can choose freely the sectors, but we always exclude the NS- sector, as it contains a tachyon. Using results from representation theory of  $SO(8)$ , we write the ground states representations in the sectors that we will use in table (2.1).

Two common quantum superstring theories that are used in AdS/CFT correspondence are *Type IIA* and *Type IIB* supergravity. They are obtained by studying the effective action of closed strings in the sectors

$$\text{Type IIA : } (\text{NS+}, \text{NS+}), (\text{R+}, \text{NS+}), (\text{NS+}, \text{R-}), (\text{R+}, \text{R-}), \quad (2.2.63a)$$

$$\text{Type IIB : } (\text{NS+}, \text{NS+}), (\text{R+}, \text{NS+}), (\text{NS+}, \text{R+}), (\text{R+}, \text{R+}). \quad (2.2.63b)$$

The field content of those theories is given by summing the appropriate entries of

Sector	Representation
(NS+,NS+)	$\mathbf{8}_V \otimes \mathbf{8}_V = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}$
(R+,NS+)	$\mathbf{8} \otimes \mathbf{8}_V = \mathbf{8}' \oplus \mathbf{56}$
(R+,R+)	$\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_+$
(NS+,R-)	$\mathbf{8}_V \otimes \mathbf{8}' = \mathbf{8} \oplus \mathbf{56}'$
(R+,R-)	$\mathbf{8} \otimes \mathbf{8}' = \mathbf{8}_V \oplus \mathbf{56}_T$

Table 2.2: Field content of the ground state in some sectors of closed superstring quantum theories. Masses are omitted, as we choose only sectors with massless ground states. Sectors that differ only by permutation of open string sectors (like (R+,R-) and (R-,R+)) share the same field content.

Field	Representation	Field	Representation
$\phi$	$\mathbf{1}$	$\phi, C_0$	$\mathbf{1}^2$
$\psi, \bar{\psi}$	$\mathbf{8}, \mathbf{8}'$	$\psi_I, I = 1, 2$	$\mathbf{8}'^2$
$C_1$	$\mathbf{8}_V$	$C_2, B_2$	$\mathbf{28}^2$
$B_2$	$\mathbf{28}$	$g_{ij}$	$\mathbf{35}$
$g_{\mu\nu}$	$\mathbf{35}$	$C_4$	$\mathbf{35}_+$
$\lambda, \bar{\lambda}$	$\mathbf{56}, \mathbf{56}'$	$\lambda_I, I = 1, 2$	$\mathbf{56}^2$
$C_3$	$\mathbf{56}_T$		

Table 2.3: Physical fields and their  $SO(8)$  representation in Type II A (left) and Type II B (right) supergravity.  $B_n$  and  $C_n$  are differential forms with  $n$  indexes.  $B_2$  is interpreted as the Kalb-Ramond field,  $g$  as the metric and  $\phi$  as the dilaton.

table (2.2):

$$\text{Type IIA : } \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8}' \oplus \mathbf{8}_V \oplus \mathbf{28} \oplus \mathbf{35} \oplus \mathbf{56} \oplus \mathbf{56}' \oplus \mathbf{56}_T, \quad (2.2.64a)$$

$$\text{Type IIB : } \mathbf{1}^2 \oplus \mathbf{8}'^2 \oplus \mathbf{28}^2 \oplus \mathbf{35} \oplus \mathbf{35}_+ \oplus \mathbf{56}^2. \quad (2.2.64b)$$

We can see that Type IIA supergravity is nonchiral, as for each representation of definite chirality the representation of opposite chirality is also contained. On the other and, Type IIB supergravity is chiral, and some representations appear twice. We give names to the fields in the representations as in table (2.3).

We can write an effective action for supergravity, by combining the fields of table (2.3) in an action. We start with Type IIB, concentrating only on the bosonic

part. We write the field strengths

$$\mathcal{F}_p = dC_{p-1} \quad H_3 = dB_2 \quad \tilde{\mathcal{F}}_3 = \mathcal{F}_3 - C_0 H_3 \quad \tilde{\mathcal{F}}_5 = \mathcal{F}_5 - \frac{1}{2}C_2 H_3 + \frac{1}{2}B_2 \mathcal{F}_3, \quad (2.2.65)$$

where wedge product between forms is implicit. Self duality of  $\tilde{\mathcal{F}}_5$  has to be imposed:  $\tilde{\mathcal{F}}_5 = *\tilde{\mathcal{F}}_5$ , where  $*$  denotes Hodge dual. We also define  $\bar{\mathcal{F}}$  as the complex conjugate of  $\mathcal{F}$  and define

$$\int \sqrt{|g|} |\mathcal{F}_p|^2 d^{10}X = \int \sqrt{|g|} g_{\mu_1 \nu_1} \dots g_{\mu_p \nu_p} \bar{\mathcal{F}}^{\mu_1 \dots \mu_p} \mathcal{F}^{\nu_1 \dots \nu_p} d^{10}x. \quad (2.2.66)$$

In those terms, the action is

$$S_{IIB} = \frac{1}{2\tilde{\kappa}_{10}^2} \int \left( \sqrt{|g|} \left( e^{-2\phi} \left( R + 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}|H_3|^2 \right) - \frac{1}{2} \left( |\mathcal{F}_1|^2 + |\tilde{\mathcal{F}}_3|^2 + |\tilde{\mathcal{F}}_5|^2 \right) \right) - \frac{1}{2} C_4 H_3 \mathcal{F}_3 \right) d^{10}x. \quad (2.2.67)$$

$R$  is calculated from  $g$ , and the last term is a topological term, and is not multiplied by the metric. The constant  $\tilde{\kappa}_{10}$  can be related to Newton's constant in ten dimensions by defining  $\kappa = \tilde{\kappa} e^{\phi_0} = \sqrt{8\pi G_{10}}$ , where  $\phi_0$  is the boundary condition for the dilaton field,  $\phi$ .

For Type IIA supergravity we use the same conventions, with the addition of

$$\tilde{\mathcal{F}}_4 = dC_3 - C_1 \mathcal{F}_3. \quad (2.2.68)$$

The action is then

$$S_{IIA} = \frac{1}{2\tilde{\kappa}_{10}^2} \int \left( \sqrt{|g|} \left( e^{-2\phi} \left( R + 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}|H_3|^2 \right) - \frac{1}{2} \left( |\mathcal{F}_2|^2 + |\tilde{\mathcal{F}}_4|^2 \right) \right) - \frac{1}{2} B \mathcal{F}_4 \mathcal{F}_4 \right) d^{10}x. \quad (2.2.69)$$

## D-branes

We conclude this introduction on string theory by discussing *D-branes*. When imposing boundary condition on the open strings, we discarded Dirichlet boundary

Brane/Coordinate	0	1	2	3	4	5	6	7	8	9
D4-brane	○	—	○	○	—	—	○	—	○	—

Table 2.4: A typical scheme used to define D-branes. We call the brane Dp-brane, and indicate with a dot tangential coordinates and with a line normal coordinates.

conditions. Those are easily visualized with the use of D-branes. Dirichlet boundary condition assumes that some string endpoints can be fixed to hypersurfaces, breaking impulse conservation (by breaking translational invariance): impulse conservation can be recovered with the use of D-branes, that become dynamical objects of the theory, with an action, gravitational field and charge.

We start by studying a single D-brane, following the presentation of [56]. A D-brane is an hypersurface in spacetime. Let us assume the D-brane of dimension  $p + 1$ : it is customary to study it by introducing  $p + 1$  tangential coordinates and their completion,  $d - p$  normal coordinates. We write the coordinates as

$$\underbrace{X^0, X^1, \dots, X^p}_{p+1 \text{ tangential coordinates}}, \quad \underbrace{X^{p+1}, \dots, X^{d+1}}_{d-p \text{ normal coordinates}}. \quad (2.2.70)$$

We choose the normal coordinates such as the hypersurface representing the D-brane is at coordinates  $\bar{x}^a$ , where  $a$  runs from  $p + 1$  to  $d + 1$ . A physical condition is that the time coordinate (here expressed by  $X^0$ ) must always be tangential to the D-brane, as we want string endpoints attached to the brane to always move forward in time. Each normal coordinate is a constraint on string endpoint motion. The case of bosonic string in 25 space dimensions with N boundary conditions on all coordinates and endpoints can be thought here as a D-brane with  $p = 25$  filling the whole space. A D-brane is defined by indicating which coordinates are tangential and which coordinates are normal, and it is customary to talk about Dp-branes, embedding their dimensionality in their name.

When we define open strings in a spacetime with a filling D-brane, we must have the string endpoints on the D-brane at all times. This amounts to imposing Dirichlet boundary conditions on the normal coordinates, and Neumann conditions on the tangential coordinates. We can expand the string embedding  $X$  in those coordinates as

$$X^a(\tau, \sigma) = \bar{x}^a + \sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^a e^{-in\tau} \sin n\sigma, \quad (2.2.71)$$



while we make the N expansion for coordinates  $\alpha = 0, \dots, p$ . There is no zero mode in this expansion. Quantizing such a string is done by rotating into light cone coordinates in the tangential subspace, so we divide the tangential coordinates in light cone coordinates  $X_{\pm} = X^0 \pm X^1$  and use the index  $i$  to run from 2 to  $p+1$ . We get that states are obtained by acting on the vacuum with creation operators  $a_n^{\dagger i}$  and  $a_n^{\dagger a}$ , the critical dimension is always  $d = 25$  and the mass of those states is given by

$$m^2 = \frac{1}{\alpha'} \left( -1 + \sum_{n=1}^{\infty} \sum_{i=2}^{p+1} n a_n^{\dagger i} a_n^i + \sum_{n=1}^{\infty} \sum_{a=p+2}^{d+1} n a_n^{\dagger a} a_n^a \right). \quad (2.2.72)$$

The ground state is tachyonic, while the first excited states generated by  $a_1^{\dagger i}$  and  $a_1^{\dagger a}$  are massless. Furthermore,  $a_1^{\dagger i}$  has  $p-1$  indices, the same number of indices that a massless vector boson on the brane would have, so the state generated by  $a_1^{\dagger i}$  can be thought as a Maxwell field on the brane. We can assign a classical EM charge to string endpoints, and they will interact with such a Maxwell field.

We now consider the situation where  $N$  parallel Dp-branes are present, specified by the normal coordinates  $\bar{x}_i^a$  (the lower latin index on the symbols  $\bar{x}^a$  is not a Lorentz index, but a counting label from 1 to  $N$ ). We still impose NN conditions on tangential coordinates. String endpoints can begin on the  $i$ -th brane and end on the  $j$ -th brane, and their mode expansion is

$$X^a(\tau, \sigma) = \bar{x}_i^a + (\bar{x}_j^a - \bar{x}_i^a) \frac{\sigma}{\pi} + \sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^a e^{-in\tau} \sin n\sigma. \quad (2.2.73)$$

Strings starting from the  $i$ -th brane and ending on the  $j$ -th brane are denoted by the labels  $[ij]$ , called Chern-Paton factors. Each different choice of  $[ij]$  gives rise to a different sector in quantum theory: oscillations from the string [12] contribute to the [12] sector, and so on: the vacuum is the tensor product of all vacuums in the single  $[ij]$  sectors. The mass for the modes of a string in the  $[ij]$  sector is given by

$$m_{[ij]}^2 = \frac{1}{\alpha'} \left( -1 + \sum_{n=1}^{\infty} \sum_{i=2}^{p+1} n a_n^{\dagger i} a_n^i + \sum_{n=1}^{\infty} \sum_{a=p+2}^{d+1} n a_n^{\dagger a} a_n^a \right) + \sum_{a=p+2}^{d+1} \left( \frac{\bar{x}_j^a - \bar{x}_i^a}{2\pi\alpha'} \right)^2. \quad (2.2.74)$$

We see that stretched strings get a mass contribution. In the case of coincident branes, we can still use Chern-Paton factors to distinguish the sectors, and all

states  $a_1^{\dagger i} |0_{[ij]}\rangle$  are massless vectors in the  $[ij]$  sectors. As there are  $N^2$  possible sectors, we have  $N^2$  gauge bosons, that interact when the string endpoints that generate the bosons in the sector touch on the same brane. The numbers and index structure are just right to identify the  $N^2$  vector fields as belonging to the adjoint representation of  $U(N)$ . Thus, as a single brane has a Maxwell field living on it,  $N$  coincident branes have a non abelian  $U(N)$  gauge field living on them.

In the case of a Dp- and a Dq-brane (with  $q < p$ ), we assume that we can divide the coordinates as

$$\underbrace{X^0, \dots, X^q}_{\text{Common tangential}}, \quad \underbrace{X^{q+1}, \dots, X^p}_{\text{Mixed coordinates}}, \quad \underbrace{X^{p+1}, \dots, X^{d+1}}_{\text{Common normal}}. \quad (2.2.75)$$

On the mixed coordinates, we have to use DN (or ND, depending on string orientation) boundary conditions. Calling  $\bar{x}^r$  the Dp-brane coordinates (with  $r$  running from  $q+2$  to  $p+1$ ), we can expand the mixed coordinates as

$$X^r(\tau, \sigma) = \bar{x}^r + i\sqrt{2\alpha'} \sum_{n \in \mathbb{Z}_{\text{odd}}} \frac{2}{n} \alpha_{n/2}^r e^{-i\frac{n}{2}\tau} \cos\left(\frac{n\sigma}{2}\right). \quad (2.2.76)$$

Quantizing in light cone gauge, we discover that the mass of the modes of a string stretching from one brane to the another is given by

$$m^2 = \sum_{a=p+2}^{d+1} \left( \frac{\bar{x}_j^a - \bar{x}_i^a}{2\pi\alpha'} \right)^2 + \frac{1}{\alpha'} \left( N_m - 1 + \frac{1}{16}(p-q) \right), \quad (2.2.77)$$

where the weighted number of modes  $N_m$  is given by

$$N_m = \sum_{n=1}^{\infty} \sum_{i=2}^{q+1} n a_n^{\dagger i} a_n^i + \sum_{n=1}^{\infty} \sum_{a=p+2}^{d+1} n a_n^{\dagger a} a_n^a + \sum_{n \in \mathbb{Z}_{\text{odd}}}^{\infty} \sum_{r=q+2}^{p+1} \frac{n}{2} a_{n/2}^{\dagger r} a_{n/2}^r. \quad (2.2.78)$$

We still have to introduce brane dynamics. This is done by introducing an action, the *Dirac-Born-Infeld action* or DBI action, that is the most general possible action for a D-brane in curved spacetime. The natural tensors that can be written on a D-brane are the pullback of the metric  $g_{\mu\nu}$ , the pullback of the Kalb-Ramond field  $B_{\mu\nu}$  and, additionally, on a single string a gauge field can be present. Denoting as  $r, s, t, u, \dots$  indices on the brane (from 0 to  $p+1$ ) and as  $\xi^r$  coordinates on the D-brane parameter space, we have that the DBI action for the brane is

$$S_{DBI} = -\frac{1}{(2\pi)^p \alpha'^{(p+1)/2}} \int e^{-\phi} \sqrt{|P[g]_{rs} + P[B]_{rs} + 2\pi\alpha' F_{rs}|} d^{p+1}\xi, \quad (2.2.79)$$

where  $F$  is the Maxwell field strength associated to the field living on the brane and the bars denote absolute value of the determinant. In supergravity, there are  $n$  forms that can couple to the brane. Namely, a  $p$  brane couples naturally to a form  $C_{p+1}$  through the pullback

$$S_p = \frac{1}{g_S (2\pi)^p \alpha'^{(p+1)/2}} \int_{\Sigma_{p+1}} P[C_{p+1}], \quad (2.2.80)$$

where  $g_S = e^{\phi_0}$  and  $\Sigma_{p+1}$  is the brane worldsheet, and we call the coupling constant  $\mu_P$ . Lastly, we can generalize the previous piece in a Chern-Simons type interaction,

$$S_{CS} = \mu_P \int \sum_q P[C_{p+1}] \wedge e^{P[B] + 2\pi\alpha' F}, \quad (2.2.81)$$

where we only take from the exponential the terms with the right indices to get a  $p+1$ -form to integrate.

We now study a supergravity solution, where the D-brane plays the role of a soliton-like object. A Dp-brane solution in 10 spacetime dimensions reads (where  $i$  still goes from 0 to  $p$  and  $a$  from  $p+1$  to  $d+1$ , as in previous discussion of a single brane spectrum, and define  $z^2 = x^i x^i$ )<sup>3</sup>

$$g = H_p(z)^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + H_p(z)^{1/2} dx^i dx^j \delta_{ij}, \quad (2.2.82a)$$

$$e^\phi = g_S H_p(z)^{(3-p)/4} \quad B = 0, \quad (2.2.82b)$$

$$C_{p+1} = (H_p(z)^{-1} - 1) dx^0 \wedge dx^1 \wedge \dots \wedge dx^p. \quad (2.2.82c)$$

When plugged into supergravity equations, we get

$$\partial_i \partial_i H_p(z) = 0 \quad (2.2.83)$$

everywhere but in the origin: this is solved by

$$H_p = 1 + \left( \frac{L_p}{z} \right)^{7-p}, \quad (2.2.84)$$

giving the boundary condition of Minkowski space for  $r \rightarrow \infty$ , with vanishing  $C_{p+1}$  form at infinity (necessary for finiteness of energy) and sending the dilaton to its

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<sup>3</sup>We are using a different index convention from usual, that is explained in the notation, section Holography.

constant VEV.  $L_p$  is an integration constant. It can be recovered by integrating  $dC_{p+1} = \mathcal{F}_{p+2}$  on an  $8 - p$  dimensional sphere at infinity, that gives the charge to the brane:

$$\mathcal{Q} = \frac{1}{2\kappa_{10}^2} \int_{S^{8-p}} *\mathcal{F}_{p+2}. \quad (2.2.85)$$

In this ansatz, it turns out that  $\mathcal{Q} = N\mu_P$ , so the charge is quantized in units of  $\mu_P$  and we can choose  $N$  to have a particular solution. This charge is conserved in time evolution, and is related to  $L_p$  through

$$L_p^{7-p} = (4\pi)^{(5-p)/2} \Gamma\left(\frac{7-p}{2}\right) g_S N \alpha'^{(7-p)/2}. \quad (2.2.86)$$

Using this ansatz in Type IIA or Type IIB restricts the brane dimensionality, as the coupling fields  $C_p$  have to be found among the physical fields of the theory, listed in table (2.3). A Dp-brane has dimension  $p + 1$ , so it naturally couples to a  $C_{p+1}$  form that can be integrated on its worldsheet. In Type IIA we only have forms with odd number of indices (we can obtain higher indices forms through Hodge dual: as an example, we can define  $C_7 = *C_3$ ), so this solution only works for  $p$  even. In Type IIB we can only have  $p$  odd for the same reason.

# Chapter 3

## The AdS/CFT correspondence

In this chapter, we'll use the tools that we have developed in the previous chapters to introduce the AdS/CFT correspondence. This correspondence has first been proposed by Maldacena in 1998 [29] (and reviewed by Maldacena himself et al. in [3]). The proposal arises from the similarity between Witten's large N limit of QCD and string genus expansion, and relates superstring theory in  $AdS_5 \times S^5$  (where  $S^5$  is introduced to have 10 dimensions and define a superstring theory, and is a compact space) with a conformal theory defined on the boundary of  $AdS_5$ . The correspondence has not yet proved in its most general form, but it has been widely used to get quantitative predictions for QCD. In particular, we'll concentrate on the Sakai-Sugimoto model [40, 41], a model that describes low energy QCD in this framework, obtaining an effective description for nuclear interaction.

### 3.1 Basics of the correspondence

The AdS/CFT correspondence, relating string theories and gauge theories, is stated in three forms. In its strongest form, the conjecture is expressed in table (3.1).

Gravity side and gauge side are dynamically equivalent in AdS/CFT, in the sense that they only differ mathematically, but describe the same physics. Besides the *strongest form* of the conjecture expressed in (3.1), there are two limits that

<p><math>\mathcal{N} = 4</math> Super Yang Mills Theory with gauge group <math>SU(N)</math> and Yang Mills coupling <math>g_Y</math></p> <p>is <i>dynamically equivalent</i> to</p> <p>type IIB string theory, with string constant <math>\alpha'</math> and coupling <math>g_S</math> on <math>AdS_5 \times S^5</math> of radius <math>L</math> and <math>N</math> units of flux of <math>\mathcal{F}_5</math> on <math>S^5</math></p> <p>The map between constants is <math>g_Y^2 = 2\pi g_S</math> and <math>2g_Y^2 N = L^4/\alpha'^2</math></p>
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Table 3.1: Strongest form of the AdS/CFT conjecture.

make computations possible.

As those limits are related to the large  $N$  limit of the gauge theory, we reintroduce the 't Hooft coupling,  $\lambda = g_Y^2 N$ , that is fixed as  $N \rightarrow \infty$ . In the *strong form* of the conjecture, we let  $N$  on the gauge side go to infinity, suppressing all non-planar diagrams, while  $\lambda$  remains arbitrary. On the other side, this corresponds to the approximation of free strings,  $g_S \rightarrow 0$ : we can see that in this framework planar diagrams correspond to free strings, and this could be an hint of a deeper relationship. In the *weak form*, we let  $N$  go to infinity as before (so  $g_S \rightarrow 0$ ), and we also take  $\lambda$  to go to infinity: this corresponds, on the gravity side, to the limit  $L^4/\alpha'^2 \rightarrow \infty$ , so the radius of the  $AdS$  space becomes large with respect to string length  $\sqrt{\alpha'}$ . With the curvature radius going to infinity, the Ricci scalar goes to zero: we can then state that we are mapping a strongly coupled gauge theory (as  $\lambda \rightarrow \infty$  implies  $g_Y \rightarrow \infty$ ) to a free string theory in a weakly curved space.

### 3.1.1 Symmetry matching

A first step in understanding the duality is to confront the symmetries of the two theories.  $AdS_5$  space is maximally symmetric, and its symmetry group is isomorphic to the conformal group in four spacetime dimensions. Part of this symmetry can be interpreted in an interesting way. Let us consider  $AdS$  metric (2.2.10) and perform the coordinate change  $z \rightarrow 1/u$ . In those coordinates, the

metric reads

$$g = L^2 \left( \eta_{\mu\nu} dx^\mu dx^\nu + \frac{du^2}{u^2} \right). \quad (3.1.1)$$

Dilation symmetry  $(x, z) \rightarrow (\lambda x, \lambda z)$ , leaving the metric invariant, is realized in those coordinates as  $(x, u) \rightarrow (\lambda x, u/\lambda)$ . We can take a different approach: a scaling of the form  $(x, u) \rightarrow (x, u/\lambda) = (x, u')$  can be reabsorbed by successively scaling  $x$ , as  $(x, u') \rightarrow (\lambda x, u')$ . Performing successive scalings with  $\lambda > 1$  brings us to points in the spacetime that have a coordinate  $u$  approaching 0 and coordinates  $x$  approaching infinity, while doing the scalings with  $\lambda < 1$  brings us to the opposite region. It is natural to identify  $u$  as an energy scale in a QFT. The fact that a scaling of the energy can be reabsorbed through a scaling of the coordinates matches the conformal symmetry of the QFT living at the boundary, the Super Yang Mills theory: as the theory is conformal the scale dependence is trivial, and in the ten dimensional space it can be reabsorbed through a coordinate transformation.

$S^5$  can be embedded in  $\mathbb{R}^6$ : the sphere is invariant under  $SO(6)$ , that is isomorphic to  $SU(4)$ .  $SU(4)$  acts on the QFT at the boundary by leaving the gauge field invariant (as it is in the scalar representation), rotating the four fermions  $\lambda$  (that form a multiplet in the fundamental representation of  $SU(4)$ ) and the six scalars  $\phi$  (that form a multiplet in the antisymmetric representation of  $SU(4)$ ). The symmetry group of the compact space is then implemented in the QFT as an internal symmetry.

## 3.2 Details of the duality

### 3.2.1 D3-branes in high and low energy regimes

The AdS/CFT correspondence consists in interpreting the two sides of the correspondence as two equivalent descriptions of the low energy limit of a string theory. In particular, the low string coupling limit of that string theory is given by a field theory, where the fields are the massless degrees of freedom of open strings with their endpoint on some Dp-branes, and this limit would correspond to the gauge side of the correspondence. In contrast, the large string coupling limit of

Brane/Coordinate	0	1	2	3	4	5	6	7	8	9
D3-brane	○	○	○	○	—	—	—	—	—	—

Table 3.2: D3-branes embedding: the dot represents a tangential direction, the line a normal direction. We can choose our coordinates and branes locations to have the branes localized at  $x^4 = \dots = x^9 = 0$ .

the low energy string theory is given by an explicit solution of Dp-branes, that becomes a source of gravitational interaction (that is represented as interaction with closed strings, that contain the graviton). We now expand on those limits. Let us consider a stack of  $N$  coincident D3-branes (to have a coupling with  $C_4$ , thus a non null flux of  $\mathcal{F}_5$ ), that extend as in table (3.2), and consider the low string coupling limit,  $g_S N \ll 1$  (where we have to include an  $N$  to account for the fact that strings can interact with  $N$  different branes, to each with coupling  $g_S$ ). In this limit the branes can be considered as an immobile background, and the theory is a theory of closed strings and open strings ending on the brane, with interactions between closed and open strings. We can write the action as

$$S = S_{\text{open}} + S_{\text{closed}} + S_{\text{int}}, \quad (3.2.1)$$

with obvious definitions. Those actions have been already discussed in the previous chapter. For  $S_{\text{closed}}$ , we can take (2.2.68), with just the field  $C_4$ . Low string coupling is obtained by expressing  $\tilde{\kappa}_{10} = g_S \kappa_{10}$ , with  $\kappa_{10}$  independent of  $g_S$  (we remember that  $\kappa_{10} = \sqrt{8\pi G_{10}}$ ) and taking a constant dilaton to make the coupling constant cancels the  $e^{-2\phi}$ , and appear as multiplying the rest of the action, that will be neglected in this limit. The only surviving terms in the low coupling regime are then (renaming  $\kappa_{10}$  as  $\kappa$  for brevity)

$$S_{\text{closed}} = \frac{1}{2\kappa^2} \int \sqrt{|g|} (g_S^2 e^{-2\phi} (R + 4\partial_\mu \phi \partial^\mu \phi)) d^{10}x. \quad (3.2.2)$$

Taking the limit of constant dilaton and expanding the metric as  $g = \eta + \kappa h$ , we have

$$S_{\text{closed}} = \frac{1}{2} \int \partial h \cdot \partial h d^{10}x + o(\kappa) \quad (3.2.3)$$

(the notation is symbolic, the integrand is the kinetic term for the graviton). Low coupling closed string dynamic is then given by graviton dynamic, represented by the metric perturbation  $h$ . The rest of the interaction can be written from the DBI



action (2.2.79) in the low coupling regime, with vanishing Kalb-Ramond field. We recall the action for a single brane

$$S_{DBI} = -\frac{1}{(2\pi)^p \alpha'^{(p+1)/2}} \int e^{-\phi} \sqrt{|P[g]_{rs} + 2\pi\alpha' F_{rs}|} d^4\xi. \quad (3.2.4)$$

When performing the pullback of the metric of the 10 dimensional space in a 4 dimensional space, the extra coordinates  $x^i$  can be taken as fields, expressing fluctuations in the brane positions: we define  $2\pi\alpha'\phi^{i-3} = x^i$ , where  $i$  goes from 4 to 9. Low coupling limit is obtained expanding  $g$  as before, and keeping the dilaton. The result can be identified as the sum of a self propagation for the open strings and the interaction term between open and closed strings, up to additional powers of  $\alpha'$ :

$$S_{\text{open}} = -\frac{1}{2\pi g_S} \int \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \eta_{\mu\nu} \partial^\mu \phi^i \partial^\nu \phi^i + o(\alpha') \right) d^4x, \quad (3.2.5a)$$

$$S_{\text{int}} = -\frac{1}{8\pi g_S} \int (\phi F^{\mu\nu} F_{\mu\nu} + o(\alpha')) d^4x. \quad (3.2.5b)$$

In the first line, we have a  $U(1)$  theory (as we're using a single brane) in flat space and 6 scalar fields (with an internal rotation symmetry, as in Super Yang Mills), while in the second line we see that the gauge field interacts with the dilaton of the closed string sector. The case for  $N$  branes is straightforward: it amounts in redefining the transverse fields  $\phi^i$  as matrix fields, valued in  $U(N)$ , do the same for the gauge field and add traces. We have an additional element coming from the  $U(N)$  theory, a self interaction between the transverse fields:

$$V = \frac{1}{2\pi g_S} \sum_{i,j} \int \text{tr}[\phi_i, \phi_j]^2 d^4x. \quad (3.2.6)$$

In the low energy limit, that in string theory is given by  $\alpha' \rightarrow 0$  (that implies  $\kappa \rightarrow 0$ ), the open string part becomes the bosonic action of  $\mathcal{N} = 4$  supersymmetric Yang-Mills, the interaction part is proportional to the dilaton, so performing a similar rescaling as in (2.2.51) we can see that the whole interaction term vanishes. Low energy and coupling limit of this theory is then given by an  $U(N)$  Yang-Mills theory and free supergravity.

In the high coupling regime, we have a theory of closed strings interacting in a brane background, where the branes source the various fields of Type IIB supergravity with which the closed strings interact. The branes provide a background

field, of the form

$$g = H_3(z)^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + H_3(z)^{1/2} dx^i dx^j \delta_{ij}, \quad (3.2.7a)$$

$$e^\phi = g_S, \quad (3.2.7b)$$

$$C_4 = (H_3(z)^{-1} - 1) dx^0 \wedge dx^1 \wedge \dots \wedge dx^3 + \dots, \quad (3.2.7c)$$

$$H_3(r) = 1 + \left(\frac{L}{z}\right)^4. \quad (3.2.7d)$$

with  $z^2 = x_4^2 + \dots + x_9^2$ , and  $L^4 = 4\pi g_S N \alpha'^2$ , as from (2.2.86), the dots denote additions to  $C_4$  that make  $\mathcal{F}_5$  self dual and the dilaton field is taken as constant. In the limit of  $z \gg L$ , we have that  $H_3 = 1$  and we recover ten dimensional flat Minkowski space, with vanishing  $C_4$ . In the opposite case, we have the *near-horizon limit* or *throat limit*: we can neglect the 1 in  $H_3$ , and obtain, passing in spherical coordinates on the  $i$  coordinates:

$$g = \frac{z^2}{L^2} (\eta_{\mu\nu} dx^\mu dx^\nu + dz^2) + L^2 d\Omega_5, \quad (3.2.8a)$$

$$C_4 = \frac{z^4}{L^4} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 + \dots \quad (3.2.8b)$$

$d\Omega_5$  indicates spherical coordinates on  $S^5$ . We see that, in the near horizon limit, the metric of the space coincides with the metric of a five dimensional Anti de Sitter space (given in (2.2.10)) and the metric of  $S^5$ . We can say that, in this limit, the 10-dimensional space becomes  $AdS_5 \times S^5$ . We study the behavior of two types of closed strings, dominant in the low energy limit: we can have closed strings in the flat background, localized at  $z \gg L$ , or we can have strings localized in the throat,  $z \ll L$ . Measuring energies from the flat background, we have that we must correct the energy  $E$  of a string excitation in the closed throat to compensate for the redshift: as this is a stationary spacetime, redshift factor is given by  $\sqrt{-g_{00}}$ , so the energy of a mode in the throat  $E$ , when observed by a spectator at infinity, gets redshifted to  $H^{-\frac{1}{4}} E$ . Thus, low energy modes of closed strings with  $z \gg L$  are obtained from low frequency vibrations, while the frequency of the vibration must grow in the throat zone, to account for redshift. Those two kinds of strings, vibrating at very different frequencies, can be taken as noninteracting. The high coupling regime is then given by a theory of two kinds of noninteracting closed strings, one kind localized on an  $AdS \times S^5$  spacetime, while the other is localized in  $\mathbb{R}^{9|1}$ .

In synthesis, we have that the same theory behaves differently for different values of the coupling limit, when taking the low energy limit:

- In the low coupling limit, we have two non interacting theories: a  $\mathcal{N}=4$  Super Yang Mills and a theory of gravity (hence, closed strings) on  $\mathbb{R}^{9|1}$ .
- In the high coupling limit, we also have two uncoupled theories: closed strings on  $\mathbb{R}^{9|1}$  and closed strings in the throat, with geometry  $AdS_5 \times S^5$ .

It is assumed that, as those two limits are the limits of the same theory, the different theories should be dynamically equivalent. Gravitation on  $\mathbb{R}^{9|1}$  is included in both limits: we can then state the Maldacena conjecture as the equivalence of the two remaining pieces of the theories, supergravity on  $AdS_5 \times S^5$  and Super Yang Mills on flat space.

### 3.2.2 Dimensional reduction

To realize the duality is to build an invertible map that gives us correlation functions of the CFT side in terms of objects of the supergravity description. To do that, we must find out what to do with the 5 extra coordinates on the gravity side: we now introduce the concept of *Kaluza-Klein dimensional reduction*, that consents us to discard the compact dimensions of a spacetime. Let us consider the situation where we want to reduce from  $\mathbb{M} \times S^n$  to  $\mathbb{M}$ , with  $\mathbb{M}$  an arbitrary manifold. To do that, we first recall some basic facts about spherical harmonics on an arbitrary n-sphere,  $S^n$ .

Spherical harmonics are defined by embedding the sphere in  $\mathbb{R}^{n+1}$ , and using standard Cartesian coordinates on  $\mathbb{R}^{n+1}$ . Each spherical harmonic  $Y_{(l)}^A$  of rank  $l$  is defined through a totally traceless tensor  $C_{i_1, \dots, i_l}^A$ , where indexes are raised and lowered with the standard Euclidean metric and they run from 1 to  $n+1$ . To fix normalizations, those tensors are taken to be orthonormal,  $C_{i_1, \dots, i_l}^A C^{B, i_1, \dots, i_l} = \delta^{AB}$ . Calling  $x^i$  the Cartesian coordinates on  $\mathbb{R}^{n+1}$ , a spherical harmonic is given by

$$Y_{(l)}^A = C_{i_1, \dots, i_l}^A x^{i_1} \dots x^{i_l}. \quad (3.2.9)$$

Passing to polar coordinates, we obtain (at fixed radius) the spherical harmonics in terms of coordinates on  $S^n$ . The Laplacian on the sphere  $\square_{S^n}$  has  $Y_{(l)}^A$  as eigenvectors, with eigenvalues depending on their rank  $l$ . Denoting as  $L$  the sphere

radius

$$\square_{S^n} Y_{(l)}^A = -\frac{1}{L^2} l(l+n-1) Y_{(l)}^A. \quad (3.2.10)$$

We can expand any function with coordinates on an arbitrary manifold containing coordinates in  $S^n$  (that we collectively denote with  $\Omega$ , while we collectively denote with  $x$  the rest of the coordinates) through

$$\varphi(x, \Omega) = \sum_{A, l=0}^{\infty} Y_{(l)}^A(\Omega) \varphi_{(l)}^A(x). \quad (3.2.11)$$

The sum runs over the spherical harmonic rank and over the index  $A$ , that labels different spherical harmonics with the same rank.

We make an explicit example of dimensional reduction in the simplest case. Let us consider  $\mathbb{R}^{n+1}$ : we take the last coordinate and compactify it on a circle of radius  $R$ . The resulting space is then  $R^n \times S^1$ , and we call  $x$  the collective coordinate on  $R^n$  and  $\theta$  the coordinate on  $S^1$ . Fields are distinguished by their (anti)periodicity on the coordinate  $\theta$ , that ranges from 0 to  $2\pi R$ : we can impose the boundary condition  $\phi(x, \theta + 2\pi R) = \pm \phi(x, \theta)$ . We choose  $\pm 1$  as only possible phases to have the condition  $\phi(x, \theta + 4\pi R) = \phi(x, \theta)$ : winding around the circle twice gives us the same value<sup>1</sup>. We first expand a periodic field  $\phi$ , using standard Fourier decomposition:

$$\phi(x, \theta) = \sum_{j \in \mathbb{Z}} \phi^{(j)}(x) e^{ij \frac{\theta}{R}}. \quad (3.2.12)$$

If the original field obeys the massless propagation equation  $(\square_n + \partial_\theta^2) \phi(x, \theta) = 0$ , with  $\square_n$  the standard Laplacian on  $\mathbb{R}^n$ , the component fields  $\phi^{(j)}(x)$  will obey a massive propagation equation on  $\mathbb{R}^n$ :

$$\left( \square_n - \left( \frac{j}{R} \right)^2 \right) \phi^{(j)}(x) = 0. \quad (3.2.13)$$

Each field  $\phi^{(j)}$  obeys a massive propagation equation, with the exception of the  $j = 0$  field, that is massless. If we impose antiperiodic boundary condition, we expand the field  $\psi$  as

$$\psi(x, \theta) = \sum_{j \in \mathbb{Z}} \psi^{(j)}(x) e^{i(j+\frac{1}{2}) \frac{\theta}{R}}. \quad (3.2.14)$$

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<sup>1</sup>This closely resembles statistical mechanics at a finite temperature  $\beta$ , that plays the role of the compactified dimension: bosonic fields are periodic, fermionic fields are antiperiodic.

The massless equation becomes, for the component field

$$\left( \square_n - \left( \frac{j + \frac{1}{2}}{R} \right)^2 \right) = 0. \quad (3.2.15)$$

Here, every field  $\psi^{(j)}$  obeys a massive propagation equation.

Dimensional reduction allows us to trade the  $S^5$  in the geometry for a theory on a smaller space,  $AdS_5$ , but with more degrees of freedom. This is crucial in holography, and it is another step towards the Minkowski space in the conformal boundary of  $AdS_5$ , where the gauge theory that we wish to study lives. Calculations of Kaluza-Klein reductions of the theories that we're interested in are usually very cumbersome and we won't give other examples. We refer to [25] for the detailed reduction of Type IIB theory from  $AdS_5 \times S^5$  to  $AdS_5$ .

Dimensional reduction on  $S^1$  is particularly important. By imposing antiperiodic boundary conditions for a field, we can make every field coming from the reduction massive. As we have seen, using superstring theories forces us to deal with fields that are peculiar to supersymmetry, as gauginos. Dimensional reduction can be used to give mass to some fields, by imposing antiperiodic boundary conditions for them, effectively removing them from the low energy theory (at least, until the energy scale becomes comparable to  $1/R$ ). We will use this technique in the Sakai-Sugimoto model, where we will use an additional compact direction to give mass to supersymmetric partners, and make them negligible in the low energy theory.

### 3.2.3 Holography

We now make the correspondence explicit, giving a list of steps to perform in order to obtain correlation functions for the *CFT* side from physical results on the *AdS* side. Let us call  $\varphi$  all fields in the supersymmetric Yang Mills living on the *AdS* boundary and let us consider composite operators  $O$ , products of the fields  $\varphi$  that are gauge invariant and have a fixed scaling dimension,  $\Delta$ : the objective is to calculate the correlation functions  $\langle O(x_1) \dots O(x_n) \rangle$ . This is usually done in QFT by introducing source fields for the operators  $O$  (called  $J(x)$ ) and then

building the partition function

$$Z[J] = \int [d\phi] \exp \left( iS_{CFT} + i \int J(x)O(x)d^4x \right) = \exp(iW[J]). \quad (3.2.16)$$

Connected contributions to correlation functions are then obtained as

$$\langle O(x_1)\dots O(x_n) \rangle_C = (-i)^n \frac{\delta W[\phi]}{\delta J(x_1)\dots\delta J(x_n)} \Big|_{J=0}. \quad (3.2.17)$$

On the *AdS* side, in the large  $N$  and  $\lambda$  limits the theory of strings goes to its noninteracting classical limit. The important object is the Kaluza-Klein reduced supergravity action  $S_{SUGRA}(\phi)$ , written in terms of some fields  $\phi(x, z)$  that solve the appropriate motion equations, coming from Type IIB equations. The map is

$$Z_{CFT}[J] = \exp(iS_{SUGRA}[\phi]) \Big|_{\lim_{z \rightarrow 0} \phi(x,z) = \mathcal{J}(x)}, \quad (3.2.18)$$

or, in terms of the generator  $W$

$$W[J] = S_{SUGRA}[\phi] \Big|_{\lim_{z \rightarrow 0} \phi(x,z) = \mathcal{J}(x)}. \quad (3.2.19)$$

The boundary condition  $\mathcal{J}(x)$  is a classical field living entirely on the Minkowski boundary of *AdS*, and in principle it can be determined from the sources  $J(x)$ . The difficult part of the computation consists in finding the duals, a mapping  $O \leftrightarrow \phi$  and  $J \leftrightarrow \mathcal{J}$ , and this depends on the details of the problem.

We schematically synthesize the procedure.

- We start with an operator (composite or elementary) on the gauge side  $O$  and introduce its source  $J$ . The objective is to calculate  $W[J]$ .
- We then find a gravity dual field to the operator, a field  $\phi(x, z)$  among the fields created from Kaluza-Klein reduction solving the equations of motion of Type IIB superstring, with an arbitrary boundary condition  $\mathcal{J}(x)$  on the boundary. We will say, from now on, that the field  $\phi$  lives in the *bulk* of the theory. We must also find a relation between  $\mathcal{J}(x)$  and  $J(x)$ .
- With the field  $\phi(x, z)$ , we calculate the action  $S_{SUGRA}$ . This action will be a functional of the boundary condition  $\mathcal{J}$ , that is functions of  $J(x)$ . By making the  $J(x)$  dependence explicit, we obtain the functional  $W[J]$ .

- Once we have  $W[J]$  we can use it as a standard generating functional to obtain correlation functions.

The supergravity equations of motion for the Kaluza-Klein fields are often complicated, but we can expand  $W[J]$  in terms of  $J$  in the standard way. If we need an  $n$ -point correlation function, we must find a term in  $W[J]$  that contain  $n$  times  $J$ . To find such a term, we can also develop the supergravity Lagrangian in terms of the field  $\phi$  that we need, and keep all orders up to  $n$ . This will be sufficient to find the dependence of  $W[J]$  up to  $J^n$  terms.

This duality approach has its advantages. First, it is a nonperturbative approach, so it does not rely on smallness of coupling constants. On the contrary, it can give us insights on the high coupling strength regime of QCD, once we find out a way to make the CFT more similar to QCD (this will be studied in the Sakai-Sugimoto model). Second, instead of numerically calculating generating functionals, once we establish a duality the problem consists in solving PDEs with arbitrary boundary conditions. This is still an hard task, but it can be done through numerical analysis with much more ease.

Before going on to detailed examples, we make a quick one: if the *CFT* has a global symmetry and a conserved current  $J^\mu$ , it is natural to couple it to a gauge vector field in the bulk,  $A_\mu$ . Let us restrict, for simplicity, to an abelian bulk gauge field: then the coupling  $J^\mu A_\mu$  is always gauge invariant, as

$$\int A_\mu J^\mu d^d x \rightarrow \int A_\mu J^\mu d^d x - \int \partial_\mu \Lambda J^\mu d^d x = \int A_\mu J^\mu d^d x + \int \Lambda \partial_\mu J^\mu d^d x. \quad (3.2.20)$$

The last term vanishes due to current conservation,  $\partial_\mu J^\mu = 0$ , when  $\Lambda$  is well behaved. This reasoning can also be used in reverse: a gauge field in the bulk must be coupled to a conserved current for the coupling to be gauge invariant. This is an example of a general fact in AdS/CFT: global symmetries on the boundary become gauge symmetries in the bulk.

As stated, finding a gravity dual is the hardest part. We illustrate a typical example in the next part.

### 3.2.4 Bulk scalar field

To do this example, we will think in reverse: we take a scalar field  $\phi(x, z)$  in the bulk among the Kaluza-Klein fields and find which boundary operators can have this field as source. We consider the action

$$S = -\frac{1}{2\kappa^2} \int \sqrt{|g|} (\partial^\Sigma \phi \partial_\Sigma \phi + m^2 \phi^2) d^5x, \quad (3.2.21)$$

where  $g$  is the metric of  $AdS_5$ . The motion equation is expressed in terms of the Laplacian on  $AdS_5$ , that is

$$\square|_g = \frac{1}{L^2} (z^2 \partial_z^2 - 3z \partial_z + z^2 \eta^{\mu\nu} \partial_\mu \partial_\nu). \quad (3.2.22)$$

The motion equation and the boundary conditions are

$$(\square|_g - m^2)\phi(x, z) = 0 \quad \lim_{z \rightarrow 0} \phi(x, z) = \mathcal{J}(x), \quad (3.2.23)$$

where  $\mathcal{J}(x)$  is arbitrary.

To find a solution, we perform the Wick rotation. This is possible in  $AdS$  because the metric is diagonal, so  $x^0$  still plays the role of a time coordinate, and applying to  $x^0$  the same procedures as in the notation section gives a correctly Wick rotated theory. We assume plane wave form for the  $x$  dependence.

$$\phi(x, z) = \exp(ip_\mu x_\mu) \phi_p(z). \quad (3.2.24)$$

We also define  $\phi_p(z) = (pz)^2 \varphi(pz)$ : with this definition,  $\varphi(pz)$  must solve the equation

$$(pz)^2 \varphi'' + (pz) \varphi' - ((pz)^2 + \nu^2) \varphi = 0, \quad (3.2.25)$$

where the derivatives are with respect to  $pz$  and  $\nu^2 = 4 + m^2 L^2$ . This is a modified Bessel equation, of solution

$$\varphi(pz) = C(p) K_\nu(pz) + D(p) I_\nu(pz). \quad (3.2.26)$$

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<sup>2</sup> $\nu$  is well defined for any positive mass squared, but it is also well defined when  $0 \geq m^2 L^2 \geq -4$ . Fields with negative mass squared in Minkowski space can exist, but the action has a maximum in correspondence of those fields and so they are unstable. In the case of  $AdS_{d+1}$  space, as shown in [13], the contribution from the spacetime curvature actually allows us to have a slightly negative mass: solutions with  $m^2 L^2 \geq -d^2/4$  are found to be stable. The mass is chosen at the end of the calculation as it is identified with physical quantities: this allows us to have a greater mass range to choose from.



$I$  and  $K$  are the modified Bessel functions. Of those,  $I_\nu$  goes to infinity as its argument goes to infinity, while  $K$  decays exponentially. For this reason, to have finite action we must set  $D(p) = 0$ , so the solution is

$$\phi_p(z) = z^2 C(p) K_\nu(pz). \quad (3.2.27)$$

We can extract the behavior of the function at the boundary  $z = 0$  by using the expansion

$$K_\nu(x) \simeq \frac{1}{2} \left( \left( \frac{x}{2} \right)^\nu \Gamma(-\nu) + \left( \frac{x}{2} \right)^{-\nu} \Gamma(\nu) \right), \quad (3.2.28)$$

valid for small  $x$ . Near the boundary, we have

$$\phi_p(z) \simeq \frac{1}{2} C(p) \left( \left( \frac{(pz)^{2+\nu}}{2^\nu} \right) \Gamma(-\nu) + \left( \frac{(pz)^{2-\nu}}{2^{-\nu}} \right) \Gamma(\nu) \right). \quad (3.2.29)$$

As  $\nu > 2$ , the first term in the sum vanishes when  $z \rightarrow 0$ , while the second surely diverges. We can now find the correspondence of  $J$  with the boundary condition of  $\phi_p$ : to have a non divergent  $J(x)$ , we impose the identification on the Fourier transform

$$J(p) = \frac{2^{\nu-1}}{\Gamma(\nu)} \lim_{z \rightarrow 0} z^{\nu-2} \phi_p(z) = C(p) p^{2-\nu}. \quad (3.2.30)$$

$J(x)$  is found by transforming back. We have added some constants for later convenience.

With the solution at hand, we can calculate the action. We first write its Euclidean version

$$S = \frac{1}{2\kappa^2} \int \sqrt{|g|} (\partial_\Sigma \phi \partial^\Sigma \phi + m^2 \phi^2) d^5 x, \quad (3.2.31)$$

where  $g$  is now given by

$$g = \frac{L^2}{z^2} (\delta_{\mu\nu} dx^\mu dx^\nu + dz^2). \quad (3.2.32)$$

We perform an integration by parts

$$S = \frac{1}{2\kappa^2} \int \nabla_\Sigma \sqrt{|g|} (\phi \partial^\Sigma \phi) d^5 x + \frac{1}{2\kappa^2} \int \phi (-\square + m^2) \phi d^5 x \quad (3.2.33)$$

( $\nabla$  is the covariant derivative with respect to the metric). The second term is zero, as  $\phi$  solves the equations of motion, so we just have to evaluate the boundary term. We have two distinct boundaries: the one at  $z = \infty$  and the other at  $z = 0$ . At  $z = \infty$  the Bessel function decays exponentially, so there is no border term from  $\infty$ . To avoid divergences, we impose an  $UV$  cutoff by placing the boundary at  $z = \epsilon$ . Using Stokes, we have

$$S = -\frac{1}{2\kappa^2} \int g^{zz} \sqrt{|g|} (\phi(x, \epsilon) \partial_\epsilon \phi(x, \epsilon)) d^4x. \quad (3.2.34)$$

<sup>3</sup> We can now insert the metric coefficients and perform the Fourier transform, obtaining

$$S = -\frac{1}{2\kappa^2} \frac{L^3}{(2\pi)^4} \int \frac{\phi_p(\epsilon) \partial_\epsilon \phi_q(\epsilon)}{\epsilon^3} d^4p d^4q. \quad (3.2.35)$$

We write the integrand, using (3.2.27) and (3.2.28)

$$\frac{C(p)C(q)}{4} p^2 q^2 \left( \left(\frac{\epsilon}{2}\right)^{2\nu} p^\nu q^\nu (\Gamma(-\nu))^2 (2 + \nu) + \left(\frac{\epsilon}{2}\right)^{-2\nu} p^{-\nu} q^{-\nu} (\Gamma(\nu))^2 (2 - \nu) + \right. \\ \left. \Gamma(\nu)\Gamma(-\nu)((2 + \nu)p^{-\nu} q^\nu + (2 - \nu)p^\nu q^{-\nu}) \right). \quad (3.2.36)$$

The first term of the sum vanishes for  $\epsilon \rightarrow 0$ , while the second diverges and the third is constant. To get rid of the divergent term, we use *holographic renormalization* [9]: we add to the action a boundary term, that does not change motion equations in the bulk. Calling  $\gamma$  the restriction of the metric to the boundary

$$S \rightarrow S + \frac{FL^3}{2\kappa^2(2\pi)^4} \int \sqrt{|\gamma|} \phi_p(\epsilon) \phi_q(\epsilon) d^4p d^4q, \quad (3.2.37)$$

where  $F$  is chosen to remove the divergence. Substituting  $\phi_p(\epsilon)$  and power developing, after some algebra we choose  $F = (2 - \nu)/L^4$  to remove the divergence. The action then becomes

$$S = -\frac{1}{4\kappa^2} \frac{L^3}{(2\pi)^4} \int d^4p d^4q C(p)C(q) \Gamma(\nu)\Gamma(-\nu) \nu p^{2-\nu} q^{2+\nu}. \quad (3.2.38)$$

We use (3.2.30): the result is identified as the Wick rotated generating functional  $W$  for the  $CFT$ .

$$W[J] = -\frac{L^3}{4\kappa^2} \Gamma(\nu)\Gamma(-\nu) \nu \int J(p)J(-p) p^{2\nu} \frac{d^4p}{(2\pi)^4}. \quad (3.2.39)$$

---

<sup>3</sup>A minus sign has to be inserted to account for the orientation of the boundary.

From this identification, we have

$$\langle O(p)O(-p) \rangle = -\frac{L^3}{2\kappa^2}\Gamma(\nu)\Gamma(-\nu)\nu p^{2\nu}. \quad (3.2.40)$$

The position space propagator is then proportional to

$$\langle O(x)O(y) \rangle \propto \frac{1}{|x-y|^{2\nu+4}}, \quad (3.2.41)$$

so we can identify the scaling dimension of  $O$  through  $\Delta = \nu + 2$ . Inverting, we get the relation  $m^2 L^2 = \Delta(\Delta - 4)$ , that tells us what value of  $m^2 L^2$  we must choose to have a field in the bulk that is dual to an operator of scaling dimension  $\Delta$ . Not any  $\Delta$  can be represented, though: as  $\Delta = \nu + 2$  and  $\nu > 0$ , we will surely have  $\Delta \geq 2$ .

### 3.2.5 Wilson loops and static quark-antiquark potential

Even without explicitly inserting quark flavors in the model, we can calculate the static quark-antiquark potential, in the limit where the particles are taken as infinitely massive and at a distance  $R$ . This is done through calculating the Wilson loop on a rectangle in the  $(x^0, x^1)$  plane, of lengths  $T$  and  $R$ , in Euclidean space and in the limit  $T \gg R$  [52], through

$$\langle W(C) \rangle \propto \exp(-V(R)T) \quad (3.2.42)$$

The Wilson loop in Euclidean space around a loop  $C$  is formally expressed as

$$W(C) = \text{Tr} \left( \mathcal{P} \exp \left( i \oint_C A_\mu(x) dx^\mu(s) \right) \right), \quad (3.2.43)$$

where  $\mathcal{P}$  denotes path ordering, and  $x^\mu(s)$  parametrizes the loop. In *AdS/CFT* framework, the proposed gauge dual is a two dimensional surface  $\Sigma$ , that extends from the boundary  $z = 0$  (where it becomes the rectangle around which we're

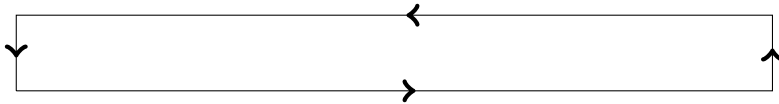


Figure 3.1: Rectangle on which the Wilson loop is calculated, with orientation. We assume  $T \gg R$ : the short side has length  $R$ , the long side has length  $T$ .

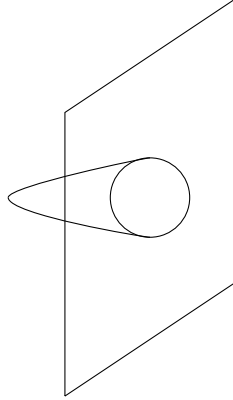


Figure 3.2: Holographic dual of a circular Wilson loop.

calculating the Wilson loop) in the  $AdS$  space, and its surface minimizes the Nambu-Goto action (2.2.13). The correct identification in the weak form of the correspondence is

$$\langle W(C) \rangle \simeq \exp(-S_{NG}(\Sigma)), \quad \partial\Sigma = C. \quad (3.2.44)$$

We perform the calculation explicitly. Assuming that the loop lies in the  $(\tau, x^1 = x)$  plane ( $\tau$  is the Euclidean time), the surface  $\Sigma$  will lie in the  $(\tau, x, z)$  hypersurface. Using the Euclidean metric for  $AdS$  (3.2.32), we write the Nambu-Goto action as

$$S_{NG}(\Sigma) = \frac{1}{2\pi\alpha'} \int_{\Sigma} \left( \det \left( \sqrt{g_{MN} \frac{\partial \Sigma^M(\sigma_1, \sigma_2)}{\partial \sigma_\alpha} \frac{\partial \Sigma^N(\sigma_1, \sigma_2)}{\partial \sigma_\beta}} \right) \right) d\sigma_1 d\sigma_2, \quad (3.2.45)$$

where  $\Sigma^M$  parametrizes the surface. The only components of  $\Sigma$  that are not vanishing are along the  $\tau$ ,  $x$  and  $z$  directions. We parametrize it by choosing  $\Sigma^\tau = \sigma_1$ ,  $\Sigma^x = \sigma_2$  and  $\Sigma^z = \Sigma^z(\sigma_2)$ : in this parametrization, the surface is entirely specified by a function  $z(x) = \Sigma^z(\Sigma^x)$ . As always, we must impose an UV cutoff to go on the boundary: we impose the boundary condition  $z(-R/2) = z(R/2) = \epsilon$ . We explicitly calculate the determinant and integrate over  $\sigma_1 = \tau$  (the integrand

is constant), obtaining

$$S_{NG}(\Sigma) = \frac{T}{2\pi\alpha'} \int_{-R/2}^{R/2} \sqrt{g_{\tau\tau}g_{xx} + z'(x)^2 g_{\tau\tau}g_{zz}} dx = \frac{TL^2}{2\pi\alpha'} \int_{-R/2}^{R/2} \frac{\sqrt{1 + z'(x)^2}}{z(x)^2} dx. \quad (3.2.46)$$

We can adimensionalize the integral, by changing variables as  $x = r/R$  and  $z = l/R$ . We also use the *CFT* variables using the duality map.

$$S_{NG}(\Sigma) = \frac{T}{\pi} \sqrt{\frac{\lambda}{2}} \frac{1}{R} \int_{-1/2}^{1/2} \frac{\sqrt{1 + l'(r)^2}}{l(r)^2} dr. \quad (3.2.47)$$

We must now solve a variational problem to determine  $l$ , requiring that the integral value must be minimized under the boundary conditions  $l(-1/2) = l(1/2) = \epsilon/R$ . We interpret the integrand as a Lagrangian: as there is no explicit  $r$  dependence, the Hamiltonian (without substituting the canonical pulse) is conserved: we obtain

$$l^2 \sqrt{1 + l'^2} = l_*^2 \implies l' = \sqrt{\left(\frac{l_*}{l}\right)^4 - 1}, \quad (3.2.48)$$

where  $l_*$  is a constant. As we expect  $z$  to be symmetric for  $x \rightarrow -x$ , we can impose  $l_* = l(0)$  to implement  $l'(0) = 0$ . We integrate and change variables to obtain

$$r = l_* \int_{l(r)/l_*}^1 \frac{s^2}{\sqrt{1 - s^4}} ds. \quad (3.2.49)$$

We can obtain  $l^*$  by imposing  $l(R/2) = \epsilon/R$ , obtaining

$$l_* = \frac{\Gamma(1/4)}{2\sqrt{\pi}\Gamma(3/4)}. \quad (3.2.50)$$

Inserting  $l'$  in (3.2.47) and performing the integration, we obtain

$$S_{NG}(\Sigma) = \frac{T}{\pi} \frac{\sqrt{2\lambda}}{l_*} \frac{1}{R} \int_{\epsilon/R}^1 \frac{1}{\sqrt{1 - s^4}} \frac{ds}{s^2}. \quad (3.2.51)$$

The integral can be solved exactly, obtaining

$$S_{NG}(\Sigma) = \frac{T}{\pi} \frac{\sqrt{2\lambda}}{l_*} \frac{1}{R} \left( \frac{R}{\epsilon} - \frac{\sqrt{\pi}\Gamma(3/4)}{\Gamma(1/4)} \right). \quad (3.2.52)$$

In the limit  $\epsilon \rightarrow 0$  the first term diverges, but it is also  $R$  independent: we can safely interpret it as a self energy. We can then safely neglect it and interpret the resulting potential as the interaction potential. Using the holographic identification (3.2.44) and (3.2.42), we can write the interaction potential between static quark and antiquark as<sup>4</sup>

$$V(R) = -\frac{4\pi^2}{\Gamma(1/4)^2} \frac{\sqrt{2\lambda}}{R}. \quad (3.2.53)$$

We see that the potential in Super Yang Mills goes as  $1/R$  and not  $R$ , as it would be expected from a confining theory. We conclude that Super Yang Mills is *not* confining (as we would expect from a conformal theory), so we will have to modify the *AdS/CFT* correspondence to use it for *QCD*. The main idea is intact: QCD in four dimensions is equivalent to some string theory on  $AdS_5 \times S^5$ . We'll have to modify our theory in the bulk to have QCD in the boundary: this is done in the Sakai-Sugimoto model, that is the argument of the next section.

### 3.3 The Sakai-Sugimoto model

The Sakai-Sugimoto model [40, 41] goes towards an holographic dual of QCD. Its main characteristics are the inclusion of quark flavors through the use of probe branes, the complete breaking of supersymmetry, the presence of confinement and the pattern of spontaneous breaking of the axial non abelian symmetry, when more than one flavor is inserted. We will see how this model is related to the low energy limit of QCD, and how an holographic description of baryons in terms of quantum instantons can be obtained [24]. To agree with conventions used in the Sakai-Sugimoto model, we will make use of the string length parameter, that we recall to be defined as  $l_S = \sqrt{\alpha'}$ .

#### 3.3.1 The brane background: D4 background

We first introduce the brane background in the 10 dimensional space where the string theory is defined. This has been first studied in [54], and expanded

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<sup>4</sup>Equation (3.2.42) comes with a  $\propto$  symbol: proportionality constants can be interpreted as counterterms, used to remove terms that are diverging constants, as in (3.2.52).

Brane/Coordinate	0	1	2	3	(4)	5	6	7	8	9
D4-brane	○	○	○	○	○	—	—	—	—	—

Table 3.3: D4 brane configuration. The parentheses around the fourth dimension indicate that the dimension is compactified.

in [27], and is often referred as *Witten background*. The main difference with the *AdS/CFT* classical model is the use of Dp-branes with  $p$  even: we are then using Type IIA superstring theory. We will study a D4-brane background, where a stack of  $N$  D4-branes are oriented as in table (3.3), with an important modification: we compactify one of the dimensions tangent to the brane on a circle. We will specify the radius of that circle by demanding regularity of the solution. In a certain limit that we will specify, the solution becomes the solitonic solution of Type IIA supergravity. There are some slight differences from the solution we have obtained studying D-branes. The solution reads (we use conventions as in [40])

$$g = \left(\frac{U}{R}\right)^{\frac{3}{2}} (\eta_{\mu\nu} dx^\mu dx^\nu + f(U) d\tau^2) + \left(\frac{R}{U}\right)^{\frac{3}{2}} \frac{dU^2}{f(U)} + R^{\frac{3}{2}} U^{\frac{1}{2}} d\Omega_4, \quad (3.3.1)$$

$$e^\phi = g_S \left(\frac{U}{R}\right)^{\frac{3}{4}}, \quad \mathcal{F}_4 = dC_3 = \frac{2\pi N}{V_4} \epsilon_{S^4}, \quad f(U) = 1 - \frac{U_{KK}^3}{U^3} \quad R^3 = \pi g_S N l_S^3.$$

Some comments are in order. We are defining a scale  $U_{KK}$ , with dimensions of length. This scale indirectly sets the radius of the  $S^1$  coordinate  $\tau$ , as we can think as the coordinates  $(U, \tau)$  being the radial and angular coordinates in  $\mathbb{R}^2$ : regularity at the origin of  $\mathbb{R}^2$  demands that the coordinate  $\tau$  must be periodic with period

$$\delta\tau = \frac{2\pi}{M_{KK}}, \quad M_{KK} = \frac{3}{2} \frac{U_{KK}^{\frac{1}{2}}}{R^{\frac{3}{2}}}. \quad (3.3.2)$$

To have the right signature of spacetime, we must impose  $f(U) > 0$ : this implies that the coordinate  $U$  is bounded from below by the scale  $U_{KK}$ . Minkowski spacetime is located on the boundary  $U \rightarrow \infty$ . The radius of the circle in the  $\tau$  direction shrinks as  $u \rightarrow U_{KK}$ : if we set the period as in (3.3.2), the radius of the circle will shrink to zero smoothly, and the manifold presents no singularities. The form  $F_4$  is normalized through the total surface of  $S^4$ ,  $V_4 = 8\pi^2/3$ , and is proportional to the standard volume element of  $S^4$   $\epsilon_{S^4}$ , giving the quantized flux

$$\int_{S^4} \mathcal{F}_4 = 2\pi N. \quad (3.3.3)$$

We now propose the duality: the field theory at the boundary  $U \rightarrow \infty$  is specified by the number of colors  $N$ , the 't Hooft coupling  $\lambda$  and the scale  $M_{KK}$  (that can be considered an energy cutoff: our approximation will break down for energies larger than  $M_{KK}$ ), while on the string side we have the  $AdS$  radius  $L$ , the string coupling  $g_S$  and the scale  $U_{KK}$ . We relate those quantities through the map

$$\frac{R^3}{l_S^2} = \frac{1}{2} \frac{\lambda}{M_{KK}}, \quad \frac{U_{KK}}{l_S^2} = \frac{2}{9} \lambda M_{KK}, \quad \frac{g_S}{l_S} = \frac{1}{2\pi} \frac{\lambda}{NM_{KK}}. \quad (3.3.4)$$

As in (3.3.1),  $l_S$  is related to the other parameters through  $R^3 = \pi g_S N l_S^3$ .

Supergravity approach is applicable when the curvature of spacetime is everywhere small, when compared to the fundamental string length. We attain maximum curvature for  $U = U_{KK}$ : in this case, the curvature turns out to be of order  $(U_{KK} R^3)^{-\frac{1}{2}}$ . We must require

$$\frac{U_{KK}^{\frac{1}{2}} R^{\frac{3}{2}}}{l_S^2} \gg 1. \quad (3.3.5)$$

Using our dictionary, we see that the quantity on the LHS is actually proportional to  $\lambda$ : then the supergravity approach is reliable only when the 't Hooft coupling for the theory at the boundary is large,  $\lambda \gg 1$  [27]. Using this background one can calculate the expectation values of Wilson loops, to check if the theory shows confinement: the calculation is done in [54], showing that this theory does indeed include quark confinement.

### 3.3.2 Inserting quark flavors: D8- $\overline{D8}$ probe branes

With  $N$  coincident  $D4$  branes in the space, we have a rather poor number of possible particles: by taking only the massless ones, strings beginning and ending on the  $D4$  brane give rise to a gauge field  $A_\mu$ , the gauginos  $\lambda$  and the scalar transverse oscillations  $\phi$ . We are still missing quarks in this picture, and we now review the way they are inserted in the Sakai-Sugimoto model.

We insert quark flavors in the theory by inserting  $N_f$  D8-branes and  $\overline{D8}$ -branes in the spacetime as in (3.4) (with the bar over the  $\overline{D8}$  branes indicating that those branes have opposite charge with respect to the D8 branes) treating them as *probe branes*: we suppose  $N_f \ll N$  (that is certainly true in the large  $N$  limit), so the



Brane/Coordinate	0	1	2	3	(4)	5	6	7	8	9
D8- $\overline{\text{D8}}$ -brane	o	o	o	o	—	o	o	o	o	o

Table 3.4: D8- $\overline{\text{D8}}$  brane configuration.

contribution of the probe branes on the metric of spacetime (called *backreaction*) can be considered negligible. For now, we concentrate on the embedding of a single D8 brane, that is entirely specified by a function  $U(\tau)$ . This function can be used to minimize the area of the branes, inserted in the background (3.3.1). Pulling the background metric on the brane, we have

$$g|_8 = \left(\frac{U}{R}\right)^{\frac{3}{2}} \eta_{\mu\nu} dx^\mu dx^\nu + \left( \left(\frac{U}{R}\right)^{\frac{3}{2}} f(U) + \left(\frac{R}{U}\right)^{\frac{3}{2}} \frac{U'}{f(U)} \right) d\tau^2 + R^{\frac{3}{2}} U^{\frac{1}{2}} d\Omega_4, \quad (3.3.6)$$

where  $U'$  is the derivative of  $U$  with respect to  $\tau$ . The DBI action with just the metric pullback is given by the determinant of  $g|_8$  through (integrating all compact degrees of freedom but  $\tau$ )

$$S_{DBI} \propto \int e^{-\phi} \sqrt{|g|_8} d^4 x d\tau \propto \int U^4 \sqrt{f(U) + \left(\frac{R}{U}\right)^3 \frac{U'^2}{f(U)}} d^4 x d\tau. \quad (3.3.7)$$

We can determine  $U(\tau)$  by minimizing  $S_{DBI}$ . The motion equation is easily found, as the coordinate  $\tau$  does not explicitly appear in the integrand. Assuming boundary conditions  $U(0) = U_0$  and  $U'(0) = 0$ , we can implicitly write the solution as

$$\tau(U) = \pm U_0^4 f(U_0)^{\frac{1}{2}} \int_{U_0}^U \frac{dx}{\left(\frac{x}{R}\right)^{\frac{3}{2}} f(x) \sqrt{x^8 f(x) - U_0^8 f(U_0)}}. \quad (3.3.8)$$

Let us analyze the solution. By taking its derivative, we have

$$\tau'(U) = \pm \left(\frac{U}{R}\right)^{-\frac{3}{2}} f(U)^{-1} \frac{U_0^4 f(U_0)^{\frac{1}{2}}}{\sqrt{U^8 f(U) - U_0^8 f(U_0)}}. \quad (3.3.9)$$

Let us analyze  $U_0 = U_{KK}$ . In this case, the derivative  $\tau'(U)$  is null for every  $U$ , so the embedding solution  $\tau(U)$  must be constant. The integral  $\tau(U)$  is ill defined when  $U_0 = U_{KK}$  (as the factor multiplying the integral goes to zero, but the integral diverges), so we change variables as

$$y(x) = \frac{f(x)}{f(U_0)}, \quad x(y) = \frac{U_{KK}}{(1 - f(U_0)y)^{\frac{1}{3}}}. \quad (3.3.10)$$

In those coordinates, the integral is written as

$$\tau(U) = \pm \frac{R^{\frac{3}{2}}}{3U_{KK}} \int_1^{\frac{f(U)}{f(U_0)}} \frac{x^{\frac{5}{2}}(y)}{y \sqrt{\left(\frac{x(y)}{U_0}\right)^8 y - 1}} dy. \quad (3.3.11)$$

Now the integral is well defined for each value of  $U_0$ , including  $U_{KK}$ : in that simple case, we have  $f(U_0) \rightarrow 0$ , so the integral goes from 1 to  $\infty$  and the function  $x(y)$  acquires constant value  $U_{KK}$ . We can, in this case, explicitly compute the integral, obtaining

$$\tau(U) = \pm \frac{\pi R^{\frac{3}{2}}}{3U_{KK}^{\frac{1}{2}}} = \pm \frac{\delta\tau}{4}. \quad (3.3.12)$$

To have a stable embedding, we must assign constant value to the angular coordinate  $\tau$  of the flavor branes. We choose to embed the D8 branes to have  $\tau = \delta\tau/4$  and, for the  $\overline{\text{D8}}$  branes,  $\tau = -\delta\tau/4$ . In this configuration, the branes are antipodal, joining at  $U = U_{KK}$ .

We should check the stability of this configuration with respect to small oscillations, but we just refer to the calculation in [40].

### 3.3.3 Open strings on the branes

We now study the possible excitations of the branes that we have placed, starting with the excitations of the color brane. Strings starting and ending on the color brane give rise to an  $U(N)$  gauge field,  $A$ , its gauginos  $\lambda$  and  $\bar{\lambda}$  transforming in a spinorial representation of  $SO(5)$  and a set of five scalar fields,  $\phi^i$ , transforming in the fundamental representation of  $SO(5)$ . The gauge field  $A$  is furthermore divided in its Minkowski part  $A_\mu$ , transforming as a vector in  $SO(3,1)$ , and an additional  $A_4$ , scalar under  $SO(3,1)$ , and also scalar under the gauge group when we consider gauge transformations independent of the compact coordinate.

Here the role of the compact  $S^1$  becomes clear. By imposing antiperiodic boundary conditions for the fermionic fields on the coordinate  $S^1$  and Fourier expanding as in section (3.2.2), we have that every component arising from the reduction of the gaugino gets a mass of order  $M_{KK}$ . As the supersymmetric partners of the gauge fields become massive, the boundary conditions on  $S^1$  completely

break supersymmetry, and for energies  $E < M_{KK}$  the contribution from the gauginos is negligible. The scalar fields  $\phi^i$  can get massive from loop corrections, while the potential  $A$  is protected by gauge symmetry from becoming massive in the renormalization process (with the exception of the extra component  $A_4$ ). Thus, considering only the color branes, the resulting theory (at low energy) is a pure  $U(N)$  Yang Mills theory.

Quark flavors are added as strings with an end on the color branes and the other on one of the flavor branes. As analyzed in [45], D4-D8 branes give rise to two different fermions with different chirality, a fermion index (representing the flavor brane on which one side of the string ends) and a color index (representing the freedom of choosing a color brane for the other side of the string), while D4- $\overline{\text{D8}}$  strings give rise to two other fields that can be interpreted as the antifermions. Imposing the GSO projection, we are left with a left-handed quark field from D4-D8 strings,  $q_L^f$ , and a right-handed quark field from D4- $\overline{\text{D8}}$  strings,  $\bar{q}_R^f$ . Both quark fields are in the scalar representation of  $SO(5)$ .

As argued in [40], D8- $\overline{\text{D8}}$  strings are massive, due to the brane separation at any  $U > U_{KK}$ , so we can ignore them in the low energy limit. We're left with the excitations of the flavor branes: restricting only to bosonic excitations, we get a gauge field  $A$  on each brane, for gauge group  $U(N_f)$ , and the quarks that arise from color-flavor strings are charged in the fundamental representations of the appropriate  $U(N_f)$ . Those two independent symmetries will be studied in detail next section, where we will see how they give rise to the  $U(N_f)_L \times U(N_f)_R$  global symmetry.

### 3.3.4 Gauge fields on flavor branes

We now turn our attention to the excitations of the flavor branes. First, we describe a coordinate change that will be useful in this section.

Let us restrict to the  $(U, \tau)$  subspace. We can see those coordinates as coordinates on a plane, where  $U$  represents distance from origin and  $\tau$  the polar angle. There is a peculiarity, though:  $U$  has a lower limit  $U_{KK} > 0$ . We then make a new radial coordinate  $r$  and rescale the angle  $\tau$  to have a period of  $2\pi$  through

$$U^3 - U_{KK}^3 = U_{KK} r^2, \quad \theta = \frac{2\pi}{\delta\tau} \tau. \quad (3.3.13)$$

Field	$U(N)$	$SO(5)$	$SO(3,1)$	$U(N_f)_L$	$U(N_f)_R$
$A_\mu$	<b>adj</b>	<b>1</b>	$(\frac{1}{2}, \frac{1}{2})$	<b>1</b>	<b>1</b>
$q_L^f$	<b>fund.</b>	<b>1</b>	$(\frac{1}{2}, \mathbf{0})$	<b>fund.</b>	<b>1</b>
$\bar{q}_R^f$	<b>fund.*</b>	<b>1</b>	$(\mathbf{0}, \frac{1}{2})$	<b>1</b>	<b>fund.*</b>

Table 3.5: Fields of the model and their charge under the transformations. We indicate as  $U(N_f)_L$  and  $U(N_f)_R$  the gauge groups that arise from the symmetry of the D8 and  $\overline{\text{D8}}$  branes, respectively.  $SO(5)$  parametrizes transformations that are transverse to the D4 brane, while  $U(N)$  is the gauge group for color brane symmetry. A \* symbol indicates conjugate representation. We do not indicate gauge fields on the flavor branes, as we'll study them in detail in the next section. Note that, as  $SO(5)$  is not a QCD symmetry, we want all physical fields to be scalars under the group.

Then, we can introduce Cartesian coordinates  $y$  and  $z$  through

$$y = r \cos \theta, \quad z = r \sin \theta. \quad (3.3.14)$$

The metric of the  $(U, \tau)$  plane becomes

$$g|_{\mathfrak{s},(U,\tau)} = \frac{4}{9} \left( \frac{R}{U} \right)^{\frac{3}{2}} \left( \frac{U_{KK}}{U} dr^2 + r^2 d\theta^2 \right) = \quad (3.3.15)$$

$$\frac{4}{9} \left( \frac{R}{U} \right)^{\frac{3}{2}} \left( (1 - h(r)z^2) dz^2 + (1 - h(r)y^2) dy^2 - 2h(r)zy dz dy \right)$$

with

$$h(r) = \frac{1}{r^2} \left( 1 - \frac{U_{KK}}{U} \right). \quad (3.3.16)$$

The embedding of the D8 and  $\overline{\text{D8}}$  branes translates, in those coordinates, to  $y = 0$ , as  $\tau = \delta\tau/4$  corresponds to  $\theta = \pi/2$ . This makes the embedding simpler, and the metric on the  $z$  line is obtained by restraining the previous metric to  $y = 0$ , so  $z = r$ . Lastly, we define  $U_z(z) = (U_{KK}^3 + U_{KK}z^2)^{\frac{1}{3}}$ . We note that the coordinate  $z$  spans from  $-\infty$  to  $\infty$  and the point  $z = 0$ , corresponding to  $U = U_{KK}$ , is the point at which the branes join: in the upper space we embed the D8-brane, while in the lower space we embed the  $\overline{\text{D8}}$  brane (coherent with the fact that the first brane is at angle  $\theta = \pi/2$ , the second one is at angle  $\theta = -\pi/2$ ).

We now write the *DBI* action for the D8 brane, making use of the new coordinates. We define  $T = 1/((2\pi)^8 l_s^9)$  and  $\tilde{T} = (2/3)R^{3/2}U_{KK}^{1/2}TV_{4gS}^{-1}$ . As we're

working with a non abelian gauge theory (with gauge group  $U(N_f)$ ), we have to generalize (2.2.79) in the case where many coincident strings are placed in the spacetime. This is naturally done by introducing the symmetrized trace, defined as

$$\text{S Tr}(A_1 \dots A_N) = \frac{1}{N!} \text{Tr}(A_1 \dots A_N + \text{all permutations}). \quad (3.3.17)$$

Using the symmetrized trace, we rewrite the *DBI* action as (the integral on  $z$  runs over all positive values, up to  $\infty$ )

$$S_{D8,DBI} = -T \int e^{-\phi} \text{S Tr} \left( \sqrt{|\det(g|_8 + 2\pi l_S^2 F)|} \right) d^4 x dz d\Omega_4. \quad (3.3.18)$$

We rewrite the metric on the D8 brane, with the  $z$  coordinate:

$$g|_8 = \left( \frac{U}{R} \right)^{\frac{3}{2}} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{4}{9} \left( \frac{R}{U} \right)^{\frac{3}{2}} (1 - h(z)z^2) dz^2 + R^{\frac{3}{2}} U^{\frac{1}{2}} d\Omega_4. \quad (3.3.19)$$

We can expand in terms of the gauge field  $A$ . This field has 9 components, as it lives on the D8 brane: the Minkowski part  $A_\mu$ , the holographic component  $A_z$  and the components on  $S^4$ ,  $A_\alpha$ , with  $\alpha$  spanning the remaining indices. We want all our fields to be  $SO(5)$  singlets after Kaluza-Klein reduction, so we impose  $A_\alpha = 0$  and require that all of our fields depend only on the coordinates  $(x^\mu, z)$ . We can expand the action and perform the sphere integral, obtaining

$$S_{D8,DBI} = -2\pi^2 \tilde{T} l_S^4 \int \frac{R^3}{U_z} \text{tr}(F^{\mu\nu} F_{\mu\nu}) + \frac{9}{2} \frac{U_z^3}{U_{KK}} \text{tr}(F_z^\mu F_{\mu z}) d^4 x dz. \quad (3.3.20)$$

We see that the DBI part is equivalent to a Yang-Mills theory in warped space. We now use our dictionary (3.3.4) to write constants in terms of Yang-Mills constants. We also remove our dimensionful units. As explained in [41], we can set

$$\frac{2}{9} M_{KK}^2 l_S^2 = \lambda^{-1} \quad (3.3.21)$$

and then set  $M_{KK} = 1$ : this determines the other constants, through

$$U_{KK} = 1, \quad R^3 = \frac{9}{4}, \quad \frac{1}{g_S l_S^3} = \frac{4\pi}{9} N. \quad (3.3.22)$$

In those simple units, we can write the action as (defining  $H(z) = (1 + z^2)^{\frac{2}{3}}$ )

$$S_{D8,DBI} = -\frac{\lambda N}{216\pi^3} \int \frac{1}{2} H(z)^{-\frac{1}{2}} \text{tr}(F_{\mu\nu} F^{\mu\nu}) + H(z)^{\frac{3}{2}} \text{tr}(F_{\mu z} F^\mu_z) d^4 x dz. \quad (3.3.23)$$

We are also interested to Chern-Simons terms that can arise from the interaction with the form  $C_3$ . From the Chern Simons term (2.2.81), the coupling (generalized to  $N$  coincident strings) is<sup>5</sup>

$$S_{D8,CS} = \frac{1}{48\pi^3} \int_{D8} C_3 \text{tr}(F^3). \quad (3.3.24)$$

This integral can be computed by defining  $\omega_5(A)$  such as  $d\omega_5(A) = \text{Tr}(F^3)$ . By explicit computation,  $\omega_5(A)$  can be written as

$$\omega_5(A) = \text{tr} \left( F^2 A - \frac{1}{2} F A^3 + \frac{1}{10} A^5 \right). \quad (3.3.25)$$

We can then use Stokes to write the integral as<sup>6</sup>

$$S_{D8,CS} = \frac{N}{24\pi^2} \int \omega_5(A). \quad (3.3.26)$$

We have performed the integral on the angular variables through (3.3.1) and the  $F$  independence from the angular coordinates, so the integral is on the Minkowski spacetime and the coordinate  $z$ . A problem arises when noting that  $\omega_5(A)$  is not gauge independent: we postulate that  $\omega_5(A)$  takes this form when we choose a gauge where  $A$  falls off to 0 when  $z \rightarrow \pm\infty$ .

This is the action that we will need in this work. In principle, other forms could couple through CS terms, but we will neglect them. The action for the gauge field on the flavor branes acquires the form

$$S_{D8} = - \frac{N}{24\pi^2} \left( \frac{\lambda}{9\pi} \int \frac{1}{2} H(z)^{-\frac{1}{2}} \text{tr}(F_{\mu\nu} F^{\mu\nu}) + H(z)^{\frac{3}{2}} \text{tr}(F_{\mu z} F^\mu_z) - \omega_5(A) d^4 x dz \right). \quad (3.3.27)$$

It is interesting to note that we can also write the action more compactly, by defining the metric on the five dimensional space  $(x, z)$

$$g = H(z) \eta_{\mu\nu} dx^\mu dx^\nu + \frac{1}{H(z)} dz^2 \quad (3.3.28)$$

---

<sup>5</sup>We are here normalizing our CS term differently with respect to (2.2.81). We won't pause on this normalization, that is explained in Appendix A of [40].

<sup>6</sup>Actually, there are problems of definition for (3.3.24) when  $\mathcal{F}_4$  has a nontrivial flux. The integral after Stokes (3.3.26) is always well defined, so we should really take it as definition of the Chern Simons term.

In this five-dimensional space, we revert to our standard notations with capital Greek indices: the action is

$$S = -\frac{N\lambda}{216\pi^3} \int \sqrt{|g|} \frac{1}{2} \text{tr}(F_{\Gamma\Delta} F^{\Gamma\Delta}) d^4 x dz + \frac{N}{24\pi^2} \int \omega_5(A) d^4 x dz. \quad (3.3.29)$$

This is the action of a pure Yang Mills (gauge group  $U(N_f)$ ) in warped space with metric (3.3.28), plus a topological Chern Simons term. We have analyzed the excitation of the D8 gauge field, with  $z > 0$ , and we can repeat the analysis for the  $\overline{\text{D8}}$  field, with  $z < 0$ : the complete action is then (3.3.29), with the  $z$  integration running from  $-\infty$  to  $\infty$ . The two fields join at the branes joining point  $z = 0$ .

### 3.3.5 The pion in the Sakai-Sugimoto model

We now illustrate how the Sakai-Sugimoto model can be used to build an effective theory of baryons and mesons, making a phenomenological model of low energy QCD. We will see that this model is closely related to the Skyrme model, and phenomenological parameters like the pion decay constant can be related to the gravity side constants.

We start by performing a mode expansion of the potential. We start simple, from the  $N_f = 1$  case: in this case, the gauge field is Abelian. We expand in terms of complete sets functions  $\psi_n(z)$  and  $\phi_n(z)$ , whose properties will be defined later.

$$A_\mu(x, z) = \sum_{n=1}^{\infty} B_{\mu,n}(x) \psi_n(z), \quad A_z(x, z) = \sum_{n=1}^{\infty} \varphi_n(x) \phi_n(z). \quad (3.3.30)$$

Let us define  $F_{\Sigma\Xi, B, n} = \partial_\Sigma B_{\Xi, n} - \partial_\Xi B_{\Sigma, n}$  and consider just the DBI contribution to the D8- $\overline{\text{D8}}$  configuration, neglecting the CS term. Let us consider, for now, the terms containing only the  $B_{\mu, n}$  in the action. They read

$$S = -\frac{\kappa}{2} \int \left( \frac{1}{2} H(z)^{-\frac{1}{2}} \sum_{m, n=1}^{\infty} F_{\mu\nu, B, n} F_{B, m}^{\mu\nu} \psi_n \psi_m + H(z)^{\frac{3}{2}} \sum_{m, n=1}^{\infty} B_{\mu, n} B^{\mu, m} \psi'_n \psi'_m \right) d^4 x dz. \quad (3.3.31)$$

where  $\kappa$  is the prefactor to the DBI part in (2.2.79), a convention that we will adopt in this part of this work only to offer better confrontation with our main

references and  $\psi'_n$  is just the derivative of  $\psi_n$ . We define the scalar products

$$(f, g) = \int_{-\infty}^{+\infty} \frac{f(z)g(z)}{H^{\frac{1}{2}}} dz, \quad \langle f, g \rangle = \int_{-\infty}^{+\infty} H^{\frac{3}{2}} f(z)g(z) dz. \quad (3.3.32)$$

so we can rewrite the action as

$$S = -\frac{\kappa}{2} \int \left( \sum_{m,n=1}^{\infty} \frac{1}{2} F_{\mu\nu, B, n} F_{B, m}^{\mu\nu}(\psi_n, \psi_m) + \sum_{m,n=1}^{\infty} B_{\mu, n} B^{\mu, m} \langle \psi'_n, \psi'_m \rangle \right) d^4x. \quad (3.3.33)$$

We now choose our  $\psi_n$  functions to have  $\kappa(\psi_n, \psi_m) = \delta_{nm}$ . We also impose that the  $\psi_n$  solve the equation

$$H^{\frac{1}{2}} \partial_z (H^{\frac{3}{2}} \partial_z \psi_n(z)) + k_n^2 \psi_n(z) = 0 \quad (3.3.34)$$

for some numbers  $k_n$ . This way, by partial integration, we can see that

$$\delta_{nm} = \kappa(\psi_n, \psi_m) = \kappa \frac{\langle \psi'_n, \psi'_m \rangle}{k_n^2}. \quad (3.3.35)$$

Substituting those scalar products, we obtain a Proca action for the fields  $B_\mu$ . We now turn on the contribution of the fields  $\varphi_n$ . We impose  $\kappa \langle \phi_n, \phi_m \rangle = k_n^2 \delta_{nm}$ : with this choice, we can just set  $\phi_n = \psi'_n$ . The  $\phi_n$  obey the differential equation

$$\partial_z (H^{\frac{1}{2}} \partial_z (H^{\frac{3}{2}} \phi_n(z))) + k_n^2 \phi_n(z) = 0. \quad (3.3.36)$$

There is a subtlety: if we set  $k_n = 0$ , we see that  $\phi_n \propto H^{-\frac{3}{2}}$  is a solution. We also have that  $\langle \phi_0, \phi_0 \rangle$  converges, and it has value  $\pi$ , so we can include  $\phi_0$  in the expansion of  $A_z$ . The right normalization to have  $\kappa \langle \phi_0, \phi_0 \rangle = 1$  is

$$\phi_0 = \frac{1}{\sqrt{\kappa\pi}} H(z)^{-\frac{3}{2}}. \quad (3.3.37)$$

We note that the primitive of  $\phi_0$ , that would be  $\psi_0 = \frac{2}{\pi} \arctan z$ , still solves (3.3.34), but does not fall off to infinity and is not normalizable under the scalar product  $(\psi_0, \psi_0)$ , so we cannot include it in the mode expansion.

We can numerically calculate the functions  $\psi_n$  in the following way. Let  $f(k)$  be the asymptotic value for  $z \rightarrow +\infty$  of an even solution to (3.3.34) with  $k$  in place of  $k_n$ , and let  $g(k)$  be the same for odd functions. Searching for normalizable solutions of (3.3.34) then amounts in finding the zeroes of  $f(k)$  and  $g(k)$ . We plot



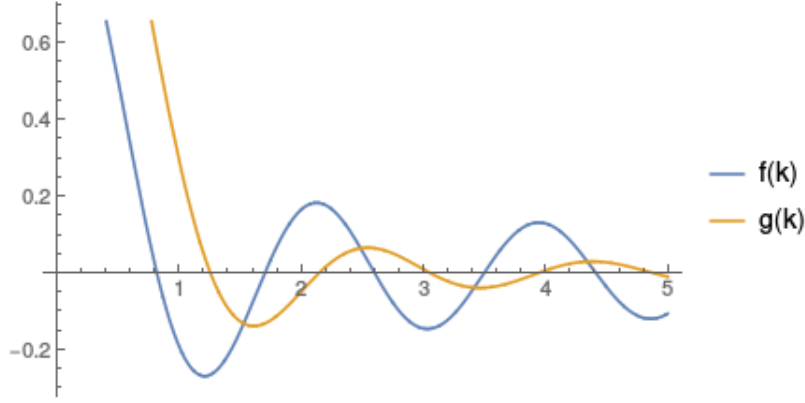


Figure 3.3: Asymptotic functions  $f(k)$  and  $g(k)$ , plotted against  $k$ . Their zeroes indicate a meson mass. We can see the alternation of the zeroes.

those functions in (3.3). The zeroes of  $f(k)$  are the entries of  $k_n$  with  $n$  odd, while the zeroes of  $g(k)$  are the entries of  $k_n$  with  $n$  even. In addition, we have to insert  $k_0 = 0$ . Equation (3.3.34) becomes then an ordinary PDE, to solve numerically.

To insert the  $\varphi_n$  fields, we must explicitly compute  $F_{\mu z, B, n}$ , that reads

$$F_{\mu z, B, n} = \partial_\mu \varphi_0 \phi_0 + \sum_{n=1}^{\infty} (\partial_\mu \varphi_n - B_{\mu, n}) \phi_n. \quad (3.3.38)$$

We can use the original gauge symmetry to transform  $B_{\mu, n} \rightarrow B_{\mu, n} + \partial_\mu \varphi_n$ , effectively reabsorbing every  $\varphi_n$  but  $\varphi_0$ , for which no analogous  $B_{\mu, 0}$  exists. The final action reads

$$S = - \int \left( \frac{1}{2} \partial_\mu \varphi_0 \partial^\mu \varphi_0 + \sum_{n=1}^{\infty} \left( \frac{1}{4} F_{\mu\nu, B, n} F_{B, n}^{\mu\nu} + \frac{1}{2} k_n^2 B_{\mu, n} B_n^\mu \right) \right). \quad (3.3.39)$$

We can recognize the dynamics of a massless pseudoscalar (due to its parity for  $z \rightarrow -z$ ) field, that can be identified with the pion field, and a tower of heavier vector meson states.

We are working in a gauge where  $A_\pm$  fall to zero as  $z \rightarrow \pm\infty$ . This condition is preserved by a group of gauge transformations with matrices  $g(x, z) \in U(N_f)$  satisfying

$$\lim_{z \rightarrow \pm\infty} g(x, z) = g_\pm(x), \quad \lim_{z \rightarrow \infty} \partial_\pm g(x, z) = 0. \quad (3.3.40)$$

The  $U(N_f)$  matrices  $g_+$  and  $g_-$  have a clear interpretation in terms of the holonomy

$$U(x) = \mathcal{P} \exp \left( i \int_{-\infty}^{+\infty} A_z(x, z) dz \right). \quad (3.3.41)$$

In fact it can be noted that, if we take  $g_{\pm}$  independent of  $x$  and, under a transformation with the matrix function  $g$ , the action on the holonomy is

$$U(x) \rightarrow g_+ U(x) g_-^{-1}. \quad (3.3.42)$$

This is exactly the  $U(N_f) \times U(N_f)$  transformation in the sigma models of the pion matrix  $U$ . The fact that we generically have  $g_{\pm}$  dependent on  $x$  corresponds to the fact that in the bulk theory global symmetries are represented by gauge symmetries, and the value at the boundary of the corresponding gauge fields can be used as an external current for the global conserved currents. Now, we apply the gauge transformation

$$h(x, z) = \mathcal{P} \exp \left( -i \int_0^z A_z(x, z') dz' \right). \quad (3.3.43)$$

It can be verified immediately that, after gauge transformation, we have  $A_z = 0$ . Defining  $\xi_{\pm} = \lim_{z \rightarrow \pm\infty} h(x, z)$ , we also have that, in the limit  $z \rightarrow \pm\infty$ ,  $A_{\mu} \rightarrow -i \xi_{\pm} \partial_{\mu} \xi_{\pm}^{-1}$ . Now we expand  $A_{\mu}$  in terms of the eigenfunctions  $\psi_n(z)$ : as we are dropping the requirement for  $A_{\mu}$  to go to zero at  $z \rightarrow \pm\infty$ , we can insert  $\psi_0 = \frac{2}{\pi} \arctan z$  in the development. For simplicity, we drop the terms proportional to  $\psi_n$  with  $n \neq 0$ . The expansion, in terms of  $\psi_{\pm}(z) = \frac{i}{2}(1 \pm \psi_0(z))$ , reads

$$A_{\mu}(x) = A_{\mu,L}^{\xi_+}(x) \psi_+(z) + A_{\mu,R}^{\xi_-}(x) \psi_-(z), \quad (3.3.44)$$

where  $\xi_{\pm}$  are the limits of  $h$  and the coefficients in the expansion are written in the form

$$A_{\mu,L}^{\xi_+} = \xi_+ A_{\mu,L} \xi_+^{-1} + i \xi_+ \partial_{\mu} \xi_+^{-1}, \quad (3.3.45a)$$

$$A_{\mu,R}^{\xi_-} = \xi_- A_{\mu,R} \xi_-^{-1} + i \xi_- \partial_{\mu} \xi_-^{-1}. \quad (3.3.45b)$$

From the asymptotics of  $\psi_{\pm}$ , we can see that  $A_L^{\xi_+}$  is the limit of the gauge field for  $z \rightarrow +\infty$ , while  $A_R^{\xi_-}$  is the limit for  $z \rightarrow -\infty$ . There is a nice interpretation of those fields in terms of the  $U(N_f)_L \times U(N_f)_R$  symmetry of the chiral model. We can perform a transformation on the whole space given by the  $z$  independent matrix  $l(x)$ , and successively two separate transformations, one for  $z > 0$  and one

for  $z < 0$ , also independent of  $z$  and denoted as  $s_{\pm}(x)$ . The fields are transformed as

$$A_{\mu,L} \rightarrow g_+ A_{\mu,L} g_+^{-1} + i g_+ \partial_{\mu} g_+^{-1}, \quad (3.3.46a)$$

$$A_{\mu,R} \rightarrow g_- A_{\mu,L} g_-^{-1} + i g_- \partial_{\mu} g_-^{-1}, \quad (3.3.46b)$$

$$\xi_{\pm} \rightarrow h \xi_{\pm} g_{\pm}^{-1}. \quad (3.3.46c)$$

The interpretation is now clear:  $A_L$  and  $A_R$  are the gauge fields that correspond to the gauge symmetry  $U(N_f)_L \times U(N_f)_R$ , that is the dual of the global symmetry of the chiral models. Those fields are the holographic duals of the flavor currents  $J_L$  and  $J_R$ .

We now set  $U(x) = \xi_+^{-1}(x) \xi_-(x)$  and fix residual gauge transformations by requiring  $\xi_- = 1$ : then

$$A_{\mu}(x, z) = U^{-1}(x) \partial_{\mu} U(x) \psi_+. \quad (3.3.47)$$

With this form of the gauge field, we explicitly compute the action in terms of  $U$ . After long but simple calculations, we have

$$S = -\kappa \int \text{tr} (A(U^{-1} \partial_{\mu} U)^2 + B[U^{-1} \partial_{\mu} U, U^{-1} \partial_{\nu} U]^2) d^4 x \quad (3.3.48)$$

with

$$A = \frac{9}{4\pi}, \quad B = \frac{9}{8\pi^4} \left( \left( \arctan z + \frac{\pi}{2} \right)^2, \left( \arctan z - \frac{\pi}{2} \right)^2 \right) = \frac{9}{8\pi^4} b \quad (3.3.49)$$

(the numerical integral  $b$  can be approximated by 15.25...). We can make the identifications

$$f_{\pi}^2 = \frac{4\kappa}{\pi}, \quad e^2 = \frac{1}{32\kappa b} \quad (3.3.50)$$

to recover the Skyrme model, and introduce a relation between the pion decay constant  $f_{\pi}$  and the parameters  $N$  and  $\lambda$  from the large  $N$  theory. We have then proved that the decomposition of the gauge field on the flavor branes give rise to meson dynamics, in particular to the pion dynamics. We did not introduce the rest of the tower, formed by the analogues of  $B_{\mu}$  in the  $U(N_f)$  case, but we just refer to [40] for that insertion.

### 3.3.6 The baryons in the Sakai-Sugimoto model

We conclude this introduction on the Sakai-Sugimoto model by introducing a concept that will be studied in more detail in the next chapter. In the language of holographic QCD, baryons are represented by wrapped D4 branes on the sphere  $S^4$  [55]. This picture involves complicate branes configurations, but there is an alternative description of baryons: from the fact that the action of the flavor branes can be written as the Skyrme action, and as baryons can be seen as solitons in the Skyrme model, it is tempting to identify instantonic configurations of the gauge fields on the flavor branes with the holographic duals of baryons.

A wrapped D4 brane has a charge, indicating the wrapping number around the sphere. Let this number be  $n$ , and  $\mathcal{R}^4$  the subspace  $(x^1, x^2, x^3, z)$ : as argued in [15], this number is related to the instantonic charge of the field on the flavor branes as

$$n = \frac{1}{8\pi^2} \int_{\mathcal{R}^4} \text{tr}(F_{IJ}F_{KL})\epsilon_{IJKL} = B[A]. \quad (3.3.51)$$

As  $A$ , we take the gauge field of the unbroken  $U(N_f)_V$  vector symmetry. By dividing  $A = A_{inst} + a\mathbf{1}$ , where  $A_{inst}$  is an instantonic configuration of charge  $n$  and  $a$  is an  $U(1)$  perturbation, we get from the Chern Simons term in the D8 action

$$S = nN \int_{\mathbb{R}} a. \quad (3.3.52)$$

This action has an immediate interpretation. The form of the action is similar to the electromagnetic coupling to a point like particle, where the integral is taken on the world line of the particle: under the  $U(1)_V$  group, our configuration has charge  $nN$ . A single quark has charge 1, and it takes  $N$  quarks to form a baryon: the interpretation of the instanton of charge  $n$  as representing an  $n$  baryons system is then supported by this calculation. Furthermore, as  $\text{tr}(F_{IJ} * F_{IJ}) = d\omega_3(A)$ , with

$$\omega_3(A) = \text{tr} \left( AF - \frac{1}{3}A^3 \right) \quad (3.3.53)$$

and using the limits from the previous section

$$\lim_{z \rightarrow +\infty} A_\mu(x, z) = U^{-1} \partial_\mu U, \quad \lim_{z \rightarrow -\infty} A_\mu(x, z) = 0, \quad (3.3.54)$$

we can express the charge as

$$n = \frac{1}{8\pi^2} \int_{\partial\mathcal{R}^4} \omega_3(A)|_{z=+\infty} = -\frac{1}{24\pi^2} \int_{\mathbb{R}^3} \text{tr}(U^{-1}dU)^3, \quad (3.3.55)$$

as the topological charge (and, consequently, the baryon number) is expressed in the Skyrme model.

Studying baryons in the Sakai-Sugimoto model corresponds then to the problem of identifying the topological sectors of the gauge theory on the flavor branes, and then finding explicit, classical solutions. Unfortunately, due to the geometry of the space, this is a difficult task, and only approximate solutions are known even for the sector of charge 1, as we cannot even use the standard technique of radial symmetry [10,23,24]. In the next chapter, we will present the modern calculations on the charge 1 sector, and then we will begin the study of the charge 2 sector, that should contain interesting physical phenomena, as the binding of proton and neutron in the formation of the deuteron.



# Chapter 4

## Sakai-Sugimoto solitons

In the previous chapter, we have found a possible candidate for an holographic dual of QCD, the Sakai-Sugimoto model. We have also exhibited a low energy action for the gauge field on the flavor branes ((3.3.29)) and we have also found that we can identify the dual of baryons in QCD as instantonic configurations of that gauge theory. In this chapter, we explicitly study the low energy action (3.3.29), concentrating on solitonic solutions of the theory. We state that states of the approximate quantum mechanical system obtained from moduli space approximation can be identified with physical baryons, as explained in (1.4). The chapter is organized in a brief review of the works in [10, 24] where the sector with topological charge 1 is studied both classically and quantum mechanically. Those results will be used in the second part, where we begin the study of the sector of charge 2, by gluing together two solutions at large spatial distances and find an interaction potential between the two objects. This potential is proposed as an interaction potential between nucleons, and the system is quantized to show that in the spectrum there is a state with the same quantum numbers of the deuteron that has minimal energy, with a quantization scheme that is parallel to the scheme used in [11, 12]. We use the quantum model to confront our potential with the one obtained in [26] through a different approach. Lastly, we use our quantum model to give physical predictions about the binding energy of the deuteron in our model, comparing with experimental results, and we give some indications about how the analysis can be generalized for topological sectors of arbitrarily high charge, provided that the single instantons are located at the appropriate distances.

## 4.1 Topological sector of charge one: the baryon

### 4.1.1 Classical solution

We start by following [10, 24], studying the field theory of action (3.3.29) at a classical level and the moduli space of the instantonic solution. We temporarily adopt the convention to call the gauge field  $\tilde{A}$  and its associated strength tensor as  $\tilde{F}$ . In those terms, we are studying a field theory

$$\tilde{A} : \mathcal{R} \rightarrow su(N_f), \quad (4.1.1)$$

where  $\mathcal{R}$  has the topology of  $\mathbb{R}^{4|1}$ , and metric

$$g = H(z)\eta_{\mu\nu}dx^\mu dx^\nu + \frac{1}{H(z)}dz^2, \quad (4.1.2)$$

where  $H(z)$  is

$$H(z) = \left(1 + \frac{z^2}{U_{KK}^2}\right)^{\frac{2}{3}}. \quad (4.1.3)$$

As in [41], we work in units of  $M_{KK}$  (that does not explicitly appears in the gauge theory): with this choice of units, we have  $U_{KK} = 1$ . From now on, we adopt those units. The action is given by

$$S = -\frac{N\lambda}{216\pi^3} \int \sqrt{|g|} \frac{1}{2} \text{tr}(\tilde{F}_{\Gamma\Delta}\tilde{F}^{\Gamma\Delta}) d^4x dz + \frac{N}{24\pi^2} \int \omega_5(\tilde{A}) d^4x dz. \quad (4.1.4)$$

It is interesting to note that  $N$  is an overall multiplicative constant: the classical theory is thus completely independent of  $N$ , and quantum corrections are always negligible when we take  $N$  to infinity.

We want to find static solutions of this theory. To do so, we divide the field in two components: an abelian component  $\hat{A}$  and the non abelian part  $A$  and doing the same for the field strength, as

$$\tilde{A}_\Gamma = A_\Gamma + \frac{1}{N_f}\hat{A}_\Gamma, \quad \tilde{F}_{\Gamma\Xi} = F_{\Gamma\Xi} + \frac{1}{N_f}\hat{F}_{\Gamma\Xi}. \quad (4.1.5)$$

We rescale the action as

$$\mathcal{S} = \frac{216\pi^3}{N\lambda} S \quad (4.1.6)$$



and define the coupling

$$\Lambda = \frac{8\lambda}{27\pi}. \quad (4.1.7)$$

Furthermore, we restrict to  $N_f = 2$  for simplicity. The rescaled action reads

$$\begin{aligned} \mathcal{S} = & \int \left( -\frac{1}{4H^{\frac{1}{2}}}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} - \frac{H^{\frac{3}{2}}}{2}\hat{F}_{\mu z}\hat{F}^{\mu z} - \frac{1}{2H^{\frac{1}{2}}}\text{tr}(F_{\mu\nu}F^{\mu\nu}) - H^{\frac{3}{2}}\text{tr}(F_{\mu z}F^{\mu z}) \right) d^4x dz + \\ & + \frac{1}{\Lambda} \int \left( \hat{A}_\Gamma \text{tr}(F_{\Delta\Sigma}F_{\Xi\Phi}) + \frac{1}{6}\hat{A}_\Gamma\hat{F}_{\Delta\Sigma}\hat{F}_{\Xi\Phi} \right) \epsilon^{\Gamma\Delta\Sigma\Xi\Phi} d^4x dz. \end{aligned} \quad (4.1.8)$$

The term in the second line is the Chern-Simons term, up to total derivatives. We now perform the static ansatz: we choose

$$A_I = A_I(x_J), \quad A_0 = 0, \quad \hat{A}_I = 0, \quad \hat{A}_0 = \hat{A}_0(x_I). \quad (4.1.9)$$

that is, we remove all dependence from time coordinates, the time part of the non abelian gauge potential and the space part of the abelian gauge potential. In this ansatz, we also suppose that  $\hat{A}_0$  is not a propagating field, but a constrained field fixed by the equations of motion. With this ansatz, the action reads

$$\begin{aligned} \mathcal{S} = & \int \left( \frac{1}{2H^{\frac{1}{2}}}(\partial_i\hat{A}_0)^2 + \frac{H^{\frac{3}{2}}}{2}(\partial_z\hat{A}_0)^2 - \frac{1}{2H^{\frac{1}{2}}}\text{tr}(F_{ij}^2) - H^{\frac{3}{2}}\text{tr}(F_{iz}^2) \right) d^4x dz + \\ & + \frac{1}{\Lambda} \int \hat{A}_0 \text{tr}(F_{IJ}F_{KL})\epsilon_{IJKL} d^4x dz. \end{aligned} \quad (4.1.10)$$

and the motion equations are (where  $D$  is the covariant derivative with respect to the field  $A$ )

$$\frac{1}{H^{\frac{1}{2}}}D_j F_{ji} + D_z(H^{\frac{3}{2}}F_{zi}) = \frac{1}{\Lambda}\epsilon_{iJKL}F_{KL}\partial_j\hat{A}_0, \quad (4.1.11a)$$

$$H^{\frac{3}{2}}D_j F_{jz} = \frac{1}{\Lambda}\epsilon_{ijk}F_{jk}\partial_i\hat{A}_0, \quad (4.1.11b)$$

$$\frac{1}{H^{\frac{1}{2}}}\partial_i\partial_i\hat{A}_0 + \partial_z(H^{\frac{3}{2}}\partial_z\hat{A}_0) = \frac{1}{\Lambda}\text{tr}(F_{IJ}F_{KL})\epsilon_{IJKL}. \quad (4.1.11c)$$

The last equation defines  $\hat{A}_0$  as the non homogeneous solution of the equation, obtained through convoluting the Green function of the LHS operator with the RHS. We will come to this point explicitly.

The presence of a gauge field used to stabilize a soliton is not a peculiarity of this model, and it has been amply studied as an alternative term used to stabilize

the Skyrmion (where the interaction is added ad hoc). See [1, 46] for examples about that problem, and [18] for an example of a Baby Skyrmion stabilized by a similar interaction.

To have a finite action solution, the non abelian gauge field must approach pure gauge configuration on the sphere at infinity,  $S_\infty^3$ :

$$A_I(x^I)|_{S_\infty^3} = g\partial_I g, \quad g : S_\infty^3 \rightarrow SU(2). \quad (4.1.12)$$

As  $\pi_3(SU(2)) = \mathbb{Z}$ , we have a discrete (but infinite) number of topological sectors, labeled by the topological charge

$$B = \int B^0(x, z)d^3xdz = -\frac{1}{32\pi^2} \int \text{tr}(F_{IJ}F_{KL})\epsilon_{IJKL}d^3xdz \quad (4.1.13)$$

that assumes integer values. We have an additional constrained field,  $\hat{A}_0$ , that can be interpreted as an electrostatic potential for the electric field  $\hat{F}_{0I} = -\partial_I \hat{A}_0$ , sourced by the topological charge.

We want to find a solution for the  $B = 1$  sector. We assume a central ansatz  $A_I = A_I(\rho)$ , with  $\rho = \sqrt{x^I x^I}$ , even if the curvature along the  $z$  direction explicitly breaks invariance for translations along  $z$ . We make the 't Hooft ansatz: defining

$$\sigma_{ij} = \epsilon_{ijk}\sigma_k, \quad \sigma_{zi} = \sigma_i, \quad \sigma_{IJ} = -\sigma_{JI}, \quad (4.1.14)$$

we can try the ansatz

$$\hat{A}_0 = a(\rho), \quad A_I = -\sigma_{IJ}\partial_J b(\rho). \quad (4.1.15)$$

We now have to explicitly plug ansatz (4.1.15) in the topological charge (4.1.13) to impose the constraint  $B = 1$  (that will give boundary conditions for  $b$ ), and then we can minimize the action with fields of this form. The full solution can be found in [10], here we just cite the results. The appropriate boundary conditions are

$$\lim_{\rho \rightarrow \infty} \rho^2 b(\rho) = 1, \quad b'(0) = 0. \quad (4.1.16)$$

$a(\rho)$  is obtained from  $b(\rho)$  through equation (4.1.11c). Inserting the ansatz in the action and developing in order of  $1/\Lambda$ , we see that, at order  $\Lambda^0$  in the scaled action and neglecting warp factors, we have the same action of the instanton. We can then find  $b(\rho)$ :

$$b(\rho) = \frac{1}{\Lambda(\rho^2 + \mu^2)}, \quad (4.1.17)$$

where  $\mu^2$  represents the instanton size. Up to  $\Lambda^{-2}$ , the rescaled energy  $\mathcal{E} = -\mathcal{S}$  is given by

$$\mathcal{E} = 2\pi^2 \left( 4 + \frac{2}{3}\mu^2 + \frac{256}{5\Lambda^2\mu^2} \right) + o(\Lambda^{-2}). \quad (4.1.18)$$

Now we note the first difference from the classical instanton: the instanton size was a modulus of the solution, and the energy was independent from it. In this case size matters: the metric corresponds to a gravitational field, for which the equilibrium position is  $z = 0$ . The energy of the instanton then grows with its size, and with the gravitational effect alone the instanton becomes pointlike and placed at  $z = 0$ . The second term in the energy represents the influence of gravity. The instanton would shrink to zero size, would it not be for the Chern-Simons term: the abelian field  $\hat{A}_0$  acts as an effective electric potential, and as the topological charge density is positive everywhere the net effect of the electric field is to expand the graviton, giving the third contribution to the energy. Those two effects combine to give an instanton of definite size, with

$$\mu = \frac{4}{\sqrt{\Lambda}} \left( \frac{3}{10} \right)^{\frac{1}{4}}. \quad (4.1.19)$$

$\mu$  ceases to be a modulus.  $a$  is given by

$$a(\rho) = \frac{8}{\Lambda} \frac{\rho^2 + 2\mu^2}{(\rho^2 + \mu^2)^2}. \quad (4.1.20)$$

In normal units, the soliton has energy (that we interpret as rest mass)

$$E = M_0 = \frac{N\Lambda}{8} + \sqrt{\frac{2}{15}}N. \quad (4.1.21)$$

We now turn our attention to the moduli space. We have explicit translational invariance along the  $x^i$  coordinates, so we have three moduli  $X^i$ , indicating the position of the instanton in physical space. Translations along the holographic direction  $z$  are not a symmetry, so the position  $Z$  is not a modulus, neither is the instanton size  $\mu$ , fixed to a specific value. We also have global gauge transformations, that do not fall off to zero at infinity. We get as moduli space

$$\mathcal{M} = \mathbb{R}^3 \times SU(2) \quad (4.1.22)$$

The calculation of the metric on the moduli space is similar to the standard calculation for the standard YM instanton. It reads

$$g|_{\mathcal{M}} = dx^i dx^i + 2\mu^2 d\Omega_{SU(2)}, \quad (4.1.23)$$

where  $d\Omega_{SU(2)}$  is the standard  $SU(2)$  metric. Actually, in [24] the instanton size  $\mu$  and the coordinate along the  $z$  direction are considered as approximate moduli, and the appropriate potentials obtained from small modifications of  $\mu$  and  $z$  from their equilibrium values are calculated (as an example, the potential for  $\mu$  is given by the second two terms in (4.1.18)). In the same reference, it is argued that the masses of those modes are expected to be of order  $\Lambda^0$ , so their contribution can become important when extrapolating  $\Lambda$  to physical values, so a quantization of those modes is necessary to confront physical values. We will not pursue this road, as their quantization is way more difficult in the topological sector of charge 2. We will leave this issue for a future analysis.

### 4.1.2 The linear approximation

We now perform an expansion in  $1/\Lambda$  of the gauge potential that we have just obtained. The objective is to find an expression for the fields and the equations of motion (4.1.11a), (4.1.11b) and (4.1.11c) and identify the *linear region* of the soliton, the region of space where we can approximate the gauge potential with its first term in the  $1/\Lambda$  expansion. As before, we take the results from [10], giving just a review of the facts that we will need to compute the interaction potential between two instantons.

We define the  $1/\Lambda$  approximation through

$$A_I = A_I^{(1)} + A_I^{(2)} + \dots \quad (4.1.24)$$

where each term  $A_I^{(n)}$  is of order  $1/\Lambda^n$ . We are interested in the motion equations for the field  $A_I^{(1)}$ . In the linear zone (that is given by  $\rho > 1/\sqrt{\Lambda}$  as argued in [10]<sup>1</sup>) we can take only the  $A_I^{(1)}$  contributions to the action and the equations of motion,

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<sup>1</sup>Actually, in the cited article the authors note that the linear approximation is valid up to  $\rho < \ln \Lambda$ : in the region  $\rho > \ln \Lambda$  the contributions  $A_I^{(n)}$  with  $n > 1$  become more important than  $A_I^{(1)}$ , so the linear approximation breaks down in that region. We will consider the situation where  $\Lambda \gg 1$  and neglect that zone.

effectively linearizing the system. The RHS sides of the motion equations are 0 in the linear region, but their contribution is not trivial: as an example, the RHS of the third equation is proportional to the topological charge density, and its integral on the whole space is not vanishing. Before proceeding, we divide the field  $A_i = A_i^+ + A_i^-$ , where the superscript indicates parity with respect to  $z \rightarrow -z$ : the  $A_z$  part is an even function, in the gauge where the core potential has been obtained, so  $A_z = A_z^+$ . Restricting to the order  $1/\Lambda$  terms in the motion equations (and dropping the (1) superscript), we have

$$\frac{\partial_i \partial_i}{H^{\frac{1}{2}}} \hat{A}_0 + \partial_z (H^{\frac{3}{2}} \partial_z \hat{A}_0) = \text{source1}, \quad (4.1.25a)$$

$$\frac{\partial_j \partial_j}{H^{\frac{1}{2}}} A_i^+ + \partial_z (H^{\frac{3}{2}} \partial_z A_i^+) = \text{source2}, \quad (4.1.25b)$$

$$H^{\frac{3}{2}} (\partial_i \partial_i A_z^+ - \partial_i \partial_z A_i^-) = \text{source3}, \quad (4.1.25c)$$

$$\frac{\partial_j \partial_j A_i^- - \partial_j \partial_i A_j^-}{H^{\frac{1}{2}}} - \partial_z (H^{\frac{3}{2}} (\partial_i A_z^+ - \partial_z A_i^-)) = \text{source4}. \quad (4.1.25d)$$

where the source terms are delta functions or derivatives, centered in  $(x, z) = (0, 0)$ . By developing the core solution to first order in  $1/\Lambda$ , we obtain explicit expressions for  $\hat{A}$ ,  $A_I$  and use them to calculate the source terms.

To do that, we use the functions defined in (3.3.34), (3.3.36). We change the normalization of the functions in order to have a normalization that is independent of  $N$  and  $\Lambda$ , by imposing  $\psi_n(0) = 1$  for  $n$  odd or  $\psi'_n(0) = 1$  for  $n$  even, where the prime is the derivative with respect to  $z$ . Obviously, the derivatives  $\phi_n(z) = \partial_z \psi_n(z)$  follow opposite boundary conditions. Their normalization now reads

$$(\psi_n, \psi_m) = c_n \delta_{nm}, \quad \langle \phi_n, \phi_m \rangle = d_n \delta_{nm}. \quad (4.1.26)$$

where  $c_n$  and  $d_n$  have to be determined numerically. As  $k_n^2 (\psi_n, \psi_m) = \langle \psi'_n, \psi'_m \rangle$  we have  $k_n^2 c_n = d_n$ . The only particular value is the norm of  $\phi_0(z) = H^{-\frac{3}{2}}(z)$ : we have  $d_0 = \pi$ , while  $c_0$  is divergent. With this normalization, the completeness relations are

$$\sum_{n=1}^{\infty} \frac{\psi_n(z) \psi_n(z')}{H^{\frac{1}{2}}(z) c_n} = \delta(z - z'), \quad \sum_{n=1}^{\infty} H^{\frac{3}{2}}(z) \frac{\phi_n(z) \phi_n(z')}{d_n} = \delta(z - z'). \quad (4.1.27)$$

We define, following [23]

$$G(x, z, x', z') = -\frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{\psi_n(z) \psi_n(z')}{c_n} \frac{e^{-k_n |x-x'|}}{|x-x'|}, \quad (4.1.28a)$$

$$L(x, z, x', z') = -\frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{\phi_n(z)\phi_n(z')}{d_n} \frac{e^{-k_n|x-x'|}}{|x-x'|}. \quad (4.1.28b)$$

Those functions can be used as Green functions for the operators in the left hand sides of the motion equations, as they obey (with the functions evaluated in  $(x, z, x', z')$ )

$$\frac{\partial_i \partial_i}{H^{\frac{1}{2}}(z)} G + \partial_z (H^{\frac{3}{2}}(z) \partial_z G) = \delta^3(x-x') \delta(z-z'), \quad (4.1.29a)$$

$$\partial_i \partial_i L - \partial_z \partial'_z G = \delta^3(x-x') \delta(z-z'), \quad (4.1.29b)$$

$$\partial_z (H^{\frac{3}{2}}(z) L) + H^{-\frac{1}{2}}(z) \partial_{z'} G = 0. \quad (4.1.29c)$$

We now take the linear approximation to the core solution from [10]. In terms of the functions  $G$  and  $L$ , they can be written as

$$\hat{A}_0(x, z) = -\frac{32\pi^2}{\Lambda} G(x, z, 0, 0), \quad A_i^+(x, z) = -2\pi\mu^2 \epsilon_{ijk} \sigma_k \partial_j G(x, z, 0, 0), \quad (4.1.30a)$$

$$A_i^-(x, z) = -2\pi\mu^2 \sigma_i \partial_{z'} G(x, z, 0, z')|_{z'=0}, \quad A_z^+(x, z) = -2\pi\mu^2 \sigma_i \partial_i L(x, z, 0, 0). \quad (4.1.30b)$$

We now apply the operators of the linear motion equations, obtaining the form of the source terms:

$$\frac{\partial_i \partial_i}{H^{\frac{1}{2}}} \hat{A}_0 + \partial_z (H^{\frac{3}{2}} \partial_z \hat{A}_0) = -\frac{32\pi^2}{\Lambda} \delta^3(x) \delta(z), \quad (4.1.31)$$

$$\frac{\partial_j \partial_j}{H^{\frac{1}{2}}} A_i^+ + \partial_z (H^{\frac{3}{2}} \partial_z A_i^+) = -2\pi\mu^2 \epsilon_{ijk} \sigma_k \partial_j \delta^3(x) \delta(z), \quad (4.1.32)$$

$$H^{\frac{3}{2}} (\partial_i \partial_i A_z^+ - \partial_i \partial_z A_i^-) = -2\pi\mu^2 \sigma_i \partial_i \delta^3(x) \delta(z), \quad (4.1.33)$$

$$\frac{\partial_j \partial_j A_i^- - \partial_j \partial_i A_j^-}{H^{\frac{1}{2}}} - \partial_z (H^{\frac{3}{2}} (\partial_i A_z^+ - \partial_z A_i^-)) = 2\pi\mu^2 \sigma_i \delta^3(x) \partial_z \delta(z). \quad (4.1.34)$$

We can generalize the linear form with an arbitrary  $SU(2)$  phase  $G$  and an arbitrary  $\mathbb{R}^3$  position,  $X$ : this is done by substituting  $G(x, z, 0, 0)$  with  $G(x, z, X, 0)$  (and analogous for  $L$ ), and every occurrence of the Pauli matrices  $\sigma_i$  with  $G\sigma_i G^\dagger$ .

We will use the linear form of the fields and the values of the motion equations when calculating the interaction potential.

### 4.1.3 Quantum model

We now write an effective zero mode lagrangian for the Sakai-Sugimoto instanton, that will give us a description of the baryon in our model. As stated, the moduli space is

$$\mathcal{M} = \mathbb{R}^3 \times SU(2), \quad (4.1.35)$$

with metric

$$g|_{\mathcal{M}} = dX^i dX^i + 2\mu^2 d\Omega_{SU(2)}. \quad (4.1.36)$$

The zero mode lagrangian is then given by (in non scaled units)

$$L = \frac{1}{2} M_0 \dot{X}^i \dot{X}^i + M_0 \mu^2 \omega_i \omega_i - M_0, \quad (4.1.37)$$

where  $\omega_i$  represent the angular velocity on  $SU(2)$ ,  $\omega_i = -i \operatorname{tr} [G^\dagger \dot{G} \sigma_i]$ . We have the same Lagrangian and moduli space as in section (1.4), so the quantization scheme is the same. We define canonical momenta

$$P^i = M_0 \dot{X}^i \quad J_i = 2M_0 \mu^2 \omega_i, \quad (4.1.38)$$

and write the Hamiltonian as

$$H = \frac{P^i P^i}{2M_0} + \frac{J_i J_i}{4M_0 \mu^2} + M_0. \quad (4.1.39)$$

We impose canonical commutation relations

$$[X^i, P^j] = i\delta^{ij} \quad [G, J_i] = -i \frac{i\sigma_i}{2} G, \quad (4.1.40)$$

with all other commutator vanishing: we then write a generic ket state as

$$|\psi\rangle = |p^i, j, m_l, m_r\rangle, \quad (4.1.41)$$

with  $p^i$  impulse,  $j$  assuming only semi-integer values to quantize the instanton as a soliton and  $m_l$  and  $m_r$  being interpreted, respectively, as isospin and spin of the particle. In the rest frame of the instanton,  $p^i = 0$ , the energy eigenvalues are

$$E_1 = \frac{j(j+1)}{4M_0 \mu^2} + M_0. \quad (4.1.42)$$

We identify proton and neutron states as the lowest energy states, with  $j = 1/2$ : their mass is given by

$$E_{1,pn} = M_0 + \frac{3}{16M_0\mu^2}. \quad (4.1.43)$$

The proton is identified as the particle with isospin up, while the neutron has isospin down. States with an higher value of  $j$  (always being semi integer) give heavier baryons: as an example, we identify states with  $j = 3/2$  with the  $\Delta$  states.

(4.1.42) can be taken as a baryon mass formula, giving the mass of the baryonic states in our model. Some comments are in order. As  $M_0 \propto N$ , we get the right dependence from the number of colors, and the baryon is infinitively massive in the approximation  $N \rightarrow \infty$ . Quantum corrections due to the spinning are subleading, of order  $N^{-1}$ , and they correctly become neglectable when  $N \rightarrow \infty$ , as expected from the fact that  $N$  multiplies the action.  $M_0$  is also proportional to  $\Lambda$ : when  $\Lambda \rightarrow \infty$ , the interaction of the gauge field  $\hat{A}_0$  with the topological charge becomes weak (as the relative term is multiplied by  $\Lambda^{-1}$ ). In this case, the soliton size shrinks to zero (as the electric field does not contrast the shrinking anymore) and we get an infinitively massive point-like instanton.

## 4.2 Topological sector of charge two

### 4.2.1 The interaction potential

We now perform the main calculation of this thesis, the holographic potential between nucleons. To do so, we place the instantons with their cores distanced, with a distance  $R$  greater than  $1/\Lambda$ , but we set both holographic coordinates for the two instantons to zero to minimize the energy. We write the single instanton fields by writing the first one as in (4.1.30b),(4.1.30a) and writing the second one by translating it to  $(R, 0, 0)$  and assigning an arbitrary phase matrix  $G$ .

In figure (4.1), we picture the situation. We call  $\tilde{A}^p$  the gauge field centered in the origin,  $(0, 0, 0)$ , and  $\tilde{A}^q$  the gauge field centered in  $(R, 0, 0)$ . Due to the distance of the fields, we can take the gauge field in the whole space to be  $\tilde{A}^p + \tilde{A}^q$ : in the  $P$  region,  $\tilde{A}^q$  is small and can be considered as a small perturbation, while the opposite situation happens in  $Q$ . There is a zone contained within  $Q$  (that we



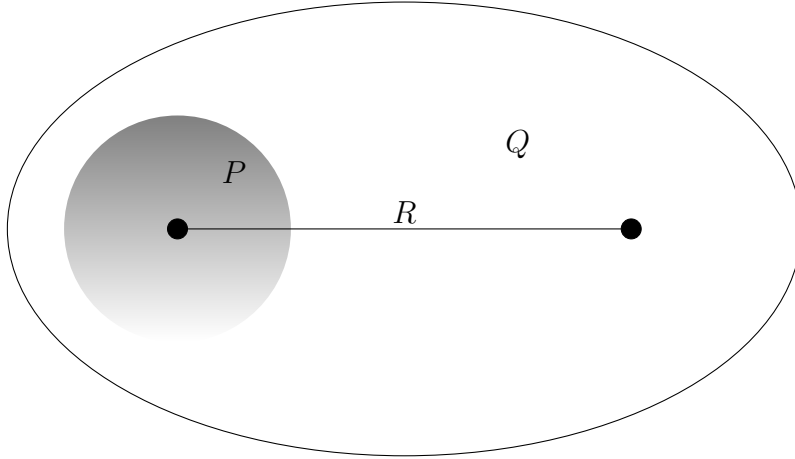


Figure 4.1: Soliton configuration for the charge two sector: the soliton location is indicated by the dots. The shaded region, called  $P$ , is where we can approximate the field to the core field, studied previously. The radius of  $P$  is greater than  $1/\Lambda$  so the border of the region lies in the zone where we can approximate the field with its linear approximation. It is also assumed that the distance is greater than  $2/\Lambda$ : this way, we can consider the contribution of the field centered in  $(R, 0, 0)$  as a linear perturbation to the field sourced by the instanton in the origin. On the border of  $P$ , the linear approximation holds for both instantons.

do not need to specify) where both fields are weak, and can be approximated by their linear form.

The energy of the configuration can be found by using the fact that the  $B = 2$  field can be approximated by the sum of two  $B = 1$  fields, and one of the coefficients of the sum can always be taken as a linear perturbation. We start by writing the scaled energy, through an integration by parts:

$$\mathcal{E} = \int \left( \frac{1}{2H^{\frac{1}{2}}} \text{tr}(F_{ij}^2) + H^{\frac{3}{2}} \text{tr}(F_{iz}^2) - \frac{1}{2} \hat{A}_0 \left( \frac{\partial_i \partial_i}{H^{\frac{1}{2}}} + \partial_z (H^{\frac{3}{2}} \partial_z) \right) \hat{A}_0 \right) d^3 x dz. \quad (4.2.1)$$

In the integration by parts, we have used the fact that the functions  $\hat{A}_0$  are supposed to vanish at the boundaries fast enough for the energy to be finite. We split this integral in two: we will see that the first two terms (called  $\mathcal{E}_1$ ) give the dipole interaction contribution, while the last one (called  $\mathcal{E}_2$ ) gives a monopole interaction.

Let us start with the evaluation of the monopole term

$$\mathcal{E}_2 = - \int \frac{1}{2} \hat{A}_0 \square \hat{A}_0 d^3 x dz, \quad (4.2.2)$$

where  $\square$  is the Laplace-Bertrami operator, that is the operator between the  $\hat{A}_0$ 's in (4.2.1). In our approximation, we can divide the topological charge density as  $B^0 \simeq B^{0,p} + B^{0,q}$ , so we also divide the gauge field in  $\hat{A}^p + \hat{A}^q$ , such as  $\square \hat{A}^p = -(32\pi^2 \Lambda^{-1}) B^{0,p}$  and similar for  $\hat{A}^q$ . (4.2.2) becomes

$$\mathcal{E}_2 = -\frac{1}{2} \int (\hat{A}_0^p \square \hat{A}_0^p + \hat{A}_0^q \square \hat{A}_0^q + \hat{A}_0^p \square \hat{A}_0^q + \hat{A}_0^q \square \hat{A}_0^p) d^3 x dz. \quad (4.2.3)$$

The terms  $\hat{A}_0^p \square \hat{A}_0^p + \hat{A}_0^q \square \hat{A}_0^q$  contribute to the self energies of the instanton, and we are really interested in the cross terms (as we will subtract the self energies to extract the potential). Let us then take  $\hat{A}_0^p \square \hat{A}_0^q = -32\pi^2 \Lambda^{-1} \hat{A}^p B^{0,q}$ .  $B^{0,q}$  is peaked in the  $q$  zone, where  $\hat{A}^p$  must be taken as its linear approximation. We can then suppose  $B^{0,q}$  to be strongly localized in the  $(R, 0, 0)$  point, as a delta function:  $B^{0,q} \simeq \delta^3(x - R)\delta(z)$ . In this approximation the topological charge of the soliton  $\tilde{A}_q$  is still one. Any contribution that tends to enlarge the soliton comes from the electrostatic field, and is then multiplied by some negative power of  $\Lambda$ : as we're keeping the linear order in  $\Lambda$  we can neglect those contributions. With the  $\delta$  functions, the integral is easily performed, and we can do that with the other term, too. Summing everything and removing self energies, we get to the monopole part of the potential. Using the linear forms of the field, we have

$$\mathcal{V}_{mp} = \frac{16\pi^2}{\Lambda} (\hat{A}_0^p(R, 0) + \hat{A}_0^q(0, 0)) = \frac{256\pi^3}{\Lambda^2} \sum_{n=1}^{\infty} \frac{1}{c_{2n-1}} \frac{e^{-k_{2n-1}R}}{R}. \quad (4.2.4)$$

This is the monopole potential, where only the contribution of  $k_n$  with odd  $n$  matters. This monopole interaction can be interpreted as a classical analogous of the exchange potential between the instantons, that interact by exchanging mesons  $\omega_{2n-1}$  with masses  $k_{2n-1}$ .

The contribution of the dipole part can be calculated through a trick, similar to the one used in [38]. Dividing the space in the  $P$  and  $Q$  region, we split the integral as

$$\int_{\mathcal{R}} = \int_P + \int_Q. \quad (4.2.5)$$

In the  $P$  region, we can take as a first approximation the whole gauge field to be coincident with  $A_I^p$ . Then, we relax this approximation admitting variations of the form  $\delta A_I^p = A_I^q$ , always taking the first order in  $A_I^q$ . The integral over the  $P$  region of the unperturbed field is a contribution to its self energy, while the variation of this energy accounts for the interaction between the instantons, so it is the only piece that we need. We write the variations that we need:

$$\delta F_{IJ}^p = D_I^p A_J^q - D_J^p A_I^q \quad \delta \int \text{tr}(F_{IJ}^p F_{IJ}^p) d^3 x dz = 4 \int \text{tr}(F_{IJ}^p D_I^p A_J^q) d^3 x dz. \quad (4.2.6)$$

Notation is obvious:  $F^p$  and  $D^p$  are the field strength and the covariante derivative built from  $A_I^p$ . We can do the same in the  $Q$  region, interchanging the roles of the fields. Noting

$$P_{ij}^{(p,q)} = \frac{2F_{ij}^{(p,q)}}{H^{\frac{1}{2}}} \quad P_{iz}^{(p,q)} = 2H^{\frac{3}{2}} F_{iz}^{(p,q)}, \quad (4.2.7)$$

we can write

$$\mathcal{V}_{dp} = \int_P \text{tr}(P_{IJ}^p D_I^p A_J^q) d^3 x dz + \int_Q \text{tr}(P_{IJ}^q D_I^q A_J^p) d^3 x dz. \quad (4.2.8)$$

As the gauge field in the core region goes as  $1/\Lambda$  for great  $\Lambda$  and so does the linear approximation, we can approximate the covariant derivative with the usual one. We can then use Stokes, using the fact that  $\partial P = -\partial Q$ : we get

$$\mathcal{V}_{dp} = \int_{\partial P} (P_{IJ}^p A_J^q - P_{IJ}^q A_J^p) d\Gamma_I, \quad (4.2.9)$$

where  $d\Gamma_I$  is a normal vector field to  $\partial P$ , pointing outwards (remember that  $P$  is a ball in four dimensions). In the region  $\partial P$ , both fields take their linear form, so we can linearize the field strength tensors (neglecting the commutator) and approximate every  $A^{(p,q)}$  with their linear approximations. We use Stokes again to return inside the  $P$  region. Derivatives act only on the field strength, as when they act on the gauge field the first term cancels the second one. Using the linear motion equations, we have that  $\partial_I P_{IJ}^q = 0$ , as we're integrating in the  $P$  region and the core of  $A^q$  is outside it. Performing the division in parity components, we have the integral

$$\mathcal{V}_{dp} = 2 \int_P \text{tr} \left( A_i^{+,q} \left( \frac{\partial_j \partial_j A_i^{+,p}}{H^{\frac{1}{2}}} + \partial_z (H^{\frac{3}{2}} \partial_z A_i^{+,p}) \right) \right) d^3 x dz + \quad (4.2.10)$$

$$\begin{aligned}
& +2 \int_P H^{\frac{3}{2}} \operatorname{tr} (A_z^{+,q} (\partial_i \partial_i A_z^{+,p} - \partial_i \partial_z A_i^{-,p})) d^3 x dz + \\
& +2 \int_P \operatorname{tr} \left( A_i^{-,q} \left( \frac{\partial_j \partial_j A_i^{-,p} - \partial_j \partial_i A_j^{-,p}}{H^{\frac{1}{2}}} - \partial_z (H^{\frac{3}{2}} (\partial_i A_z^{+,p} - \partial_z A_i^{-,p})) \right) \right) d^3 x dz.
\end{aligned}$$

Using motion equations (4.1.32), (4.1.33) and (4.1.34), we get that the operators in the parentheses applied to the  $A^p$  fields give terms proportional to a Dirac delta, so the integrals are simply done by evaluating  $A^q$  at the origin and adding the necessary constants and derivatives. The first line of the potential reads

$$\mathcal{V}_{dp,1} = \frac{256\pi^3}{\Lambda^2} \frac{6}{5} \sum_{n=1}^{\infty} \frac{1}{c_{2n-1}} \left( M_{ii} k_{2n-1}^2 R^2 \frac{e^{-k_{2n-1}R}}{R^3} - M_{ij} \partial_i \partial_j \frac{e^{-k_{2n-1}R}}{R} \right). \quad (4.2.11)$$

Here we have used the explicit form of  $\mu^2$  (4.1.19) to obtain the  $\Lambda^{-2}$  dependence, just as with the monopole term. The matrix  $M_{ij} = M_{ij}(G)$  is equal to

$$M_{ij}(G) = \frac{1}{2} \operatorname{tr} (\sigma_i G \sigma_j G^\dagger). \quad (4.2.12)$$

This term can be interpreted as a sum of Yukawa dipole interactions between the two objects, mediated by the infinite tower of mesons  $\rho_{2n-1}$  that have the same masses as the  $\omega_{2n-1}$  mesons. While the monopole interaction is always repulsive, the dipole interaction depends on the phase matrix  $G$ , that is interpreted as the isorotation that we must perform on the first object to obtain the same isorientation of the second object. We will give an easy way to visualize the matrix with the complete potential.

The last part of the potential comes from the last two lines of (4.2.10). They combine in the term

$$\mathcal{V}_{dp,2} = -\frac{256\pi^3}{\Lambda^2} \frac{6}{5} \sum_{n=0}^{\infty} \frac{1}{d_{2n}} \left( M_{ii} k_{2n}^2 R^2 \frac{e^{-k_{2n}R}}{R^3} - M_{ij} \partial_i \partial_j \frac{e^{-k_{2n}R}}{R} \right). \quad (4.2.13)$$

There are some fundamental differences between  $\mathcal{V}_{dp,1}$  and  $\mathcal{V}_{dp,2}$ . The first one is the overall sign, and the very important difference comes from the fact that we are also summing a  $k_0$  contribution: as  $k_0 = 0$ ,  $\mathcal{V}_{dp,2}$  contains a massless, long range interaction. The particle that we classically take as the mediator of this long range interaction is the pion, that is massless in our model. The other mesons, of mass  $k_{2n}$ , are interpreted as a tower of  $a_{2n}$  mesons.

Now that we have a final result for the interaction potential, we scale back to physical units and perform some changes, to have a more general result that we

will use in the following sections. We denote the coordinates of the first instanton as  $(X_1, B)$  and the coordinates of the second instanton as  $(X_2, C)$ , where  $X$  are 3-vectors. We make the change of variables  $X_{1,i} - X_{2,i} = r_i$  and  $R_i = (X_{1,i} + X_{2,i})/2$ , as usual in two bodies problem: the potential will only depend on the relative distance  $r_i$ . It is also easy to be convinced that  $G$  has to be substituted simply by  $B^\dagger C$ , indicating the relative orientation of the two objects. We define

$$P_{ij}(r, k) = \delta_{ij}((rk)^2 + rk + 1) - \frac{r_i r_j}{r^2}((rk)^2 + 3rk + 3), \quad (4.2.14)$$

with  $r$  on the RHS indicating the modulus of the position vector, and express the potential as

$$V(r, B^\dagger C) = \frac{4\pi N}{\Lambda} \left( \sum_{n=1}^{\infty} \left( \frac{1}{c_{2n-1}} \frac{e^{-k_{2n-1}r}}{r} + \frac{6}{5} \frac{1}{c_{2n-1}} M_{ij}(B^\dagger C) P_{ij}(r, k_{2n-1}) \frac{e^{-k_{2n-1}r}}{r^3} - \frac{6}{5} \frac{1}{d_{2n}} \frac{e^{-k_{2n}r}}{r^3} M_{ij}(B^\dagger C) P_{ij}(r, k_{2n}) \right) - \frac{6}{5\pi} \frac{1}{r^3} M_{ij}(B^\dagger C) P_{ij}(r, 0) \right). \quad (4.2.15)$$

We have separated the pion contribution from the rest of the  $a$  meson tower and explicitly calculated  $d_0 = \pi$ .

### 4.2.2 Looking for a bound state: the classical deuteron

We can obtain a classical description of the deuteron by looking for a minimum configuration, where we choose the coordinates of our instantons to minimize (4.2.15).

The first thing to do is to calculate the coefficients  $d_n$  and  $c_n$ . By taking  $k_n$  as in section (3.3.5), we can solve the PDE (3.3.34) and (3.3.36) to find the functions  $\psi_n$  and  $\phi_n$ . Then, we set  $c_n = (\psi_n, \psi_n)$  and  $d_n = \langle \phi_n, \phi_n \rangle$  (as a check, we could verify that  $k_n^2 c_n = d_n$ ).

We now have to choose the relative orientation of the instantons. To do that, it is useful to use axis-angle notation to write the matrix  $M_{ij}$ . As  $M_{ij}$  is an  $SO(3)$  matrix, it can be specified by giving two components of a versor, the axis of rotation  $n$  (where the third component is decided from the normalization of the vector, up to sign) and an angle  $\alpha$ , indicating the rotation around the axis (counterclockwise). We can then express any  $M$  through

$$M_{ij} = \cos \alpha \delta_{ij} + (1 - \cos \alpha) n_i n_j + \epsilon_{ijk} n_k \sin \alpha. \quad (4.2.16)$$

The orientation dependent part is then given by

$$M_{ij}P_{ij}(r, k) = \left(1 + \cos \alpha - (1 - \cos \alpha) \frac{(n \cdot r)^2}{r^2}\right) (rk)^2 - \quad (4.2.17)$$

$$- (1 - \cos \alpha) \left(3 \frac{(n \cdot r)^2}{r^2} - 1\right) (rk + 1).$$

We need a negative contribution from the dipole part to contrast the monopole part. Our best bet is to choose  $r$  and  $\alpha, n$  to get a positive contribution from  $M_{ij}P_{ij}$ , as that would mean that the long range force mediated by the pion is attracting the two objects, contrasting the potential. We then choose the configuration of *phase opposition*, where we choose  $r$  and  $n$  to be orthogonal and  $\alpha$  to indicate an half rotation: we can choose  $r_i = (R, 0, 0)$ ,  $n_i = (0, 0, 1)$  and  $\alpha = \pi$ , corresponding to  $M_{ij}P_{ij} = 2rk + 2$ . This corresponds to  $B^\dagger C = \pm i\sigma_3$ : we will choose  $B = \mathbf{1}$  and  $C = i\sigma_3$  as phase opposition configuration. Later numerical analysis with Mathematica confirmed that the global minimum is attained in phase opposition. The potential in the phase opposition is plotted against the distance  $R$  in figure (4.2).

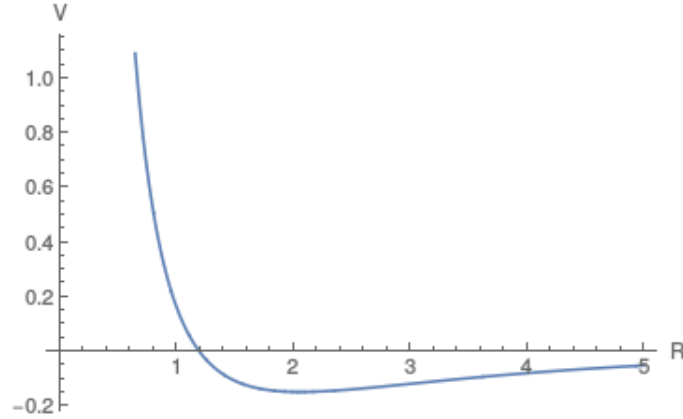


Figure 4.2: Attractive channel potential, with  $N/\Lambda = 1$ . We can note the existence of a local minimum around  $x = 2$ : that minimum is also a global minimum.

We will further analyze the potential during the later sections, obtaining physical results.

We confront our potential with the potential obtained in [26] through considering an effective QFT of fermions (representing baryons) exchanging bosons (the mesons) obtained from the Sakai-Sugimoto model, by calculating scatterings

between the instantons mediated by intermediate mesons at tree level. Note that there is yet another normalization difference for the functions  $\psi_n, \phi_n$ : in the cited article  $(\psi_n, \psi_m) = \delta_{nm} = \langle \phi_n, \phi_m \rangle$ . The correct identifications to make are then (LHS normalized as in this chapter, RHS normalized as in the cited article)

$$\frac{1}{c_n} = \psi_n(0)^2 \quad \frac{1}{d_n} = \psi'_n(0)^2. \quad (4.2.18)$$

We get that the potentials look the same, apart from a numerical coefficient of three in front of the dipole part: our dipoles are three times as strong as in the cited article. The reason of this difference will be clarified in the next parts.

### 4.2.3 Moduli space: the zero mode manifold

We want to identify the manifold of zero modes, the subspace in the twelve dimensional space  $\mathcal{M}_2$ , parametrized by the coordinates  $(X_1, B, X_2, C)$ , on which the potential assumes a constant value.  $\mathcal{M}_2$  can be defined by introducing an alternative notation: we indicate with  $A(x - X)$  a B=1 instanton field centered in  $X$ , that is a 3-vector, and with standard isoorientation. In this notation, an arbitrary field of topological charge 2 in the linear approximation can be written as

$$BA_I(x - X_1)B^\dagger + CA_I(x - X_2)C^\dagger. \quad (4.2.19)$$

The space  $\mathcal{M}_2$  is defined as the set of field configurations of the form (4.2.19).

This manifold can be defined by using the symmetries of the system. The symmetry group of the action is

$$\mathcal{L} = \mathbb{R}^3 \times SU(2)_I \times SU(2)_J \times \mathcal{P}, \quad (4.2.20)$$

where  $\mathbb{R}^3$  is the group of space translation,  $SU(2)_I$  is the global part of the gauge group,  $SU(2)_J$  is the double covering of the rotation group  $SO(3)$  and  $\mathcal{P}$  is the parity operation, that sends  $x \rightarrow -x$  while keeping the holographic coordinate invariant. Let  $\tilde{A}$  be any static gauge field: the continuous part of  $\mathcal{L}$  acts on  $\tilde{A}$  according to

$$\tilde{A}(x, z) \rightarrow U[M(E)^* \tilde{A}(M(E)^{-1}x, z)]U^\dagger, \quad (4.2.21)$$

where  $U \in SU(2)_I$ ,  $E \in SU(2)_J$ ,  $M$  is the usual transformation from  $SU(2)$  to  $SO(3)$ ,  $M^*$  is the pullback on the vector field (rotating the fields  $A_i$  and leaving the field  $A_z$  invariant). The parity operation acts on the fields as

$$\tilde{A}_i(x, z) \rightarrow \tilde{A}_i(-x, z) \quad \tilde{A}_z(x, z) \rightarrow -\tilde{A}_z(-x, z). \quad (4.2.22)$$

We want to explicitly apply the transformation to the configuration  $A^p + A^q$  that we used to calculate the potential. As the transformation properties of the core solution and the linear approximation are the same, we can just use the linear approximation fields: all calculations can be repeated in the core regions.

We start from a minimum energy configuration

$$\tilde{A}_I(x) = A_I \left( x - \frac{R}{2} \right) + \sigma_3 A_I \left( x + \frac{R}{2} \right) \sigma_3, \quad (4.2.23)$$

where we define  $R = (R_0, 0, 0)$  and  $R_0$  as the position of the minimum of the potential in the attractive channel. From the linear approximation, we study the action of  $\mathcal{L}$  on the field  $A_I(x-X)$ . The  $\mathbb{R}^3$  part is trivial: taking  $a$  as translation, it is sufficient to translate  $x$  as  $A_I(x-X) \rightarrow A_I(x-a-X)$ . An  $SU(2)_I$  transformation acts in the usual way:

$$A_I(x-X) \rightarrow U A_I(x-X) U^\dagger, \quad (4.2.24)$$

while an  $SU(2)_J$  transformation acts as

$$A_i(x-X) \rightarrow M(E)_{ij} A_j(M(E)^{-1}x-X) \quad A_z(x-X) \rightarrow A_z(M(E)^{-1}x-X). \quad (4.2.25)$$

We can do some manipulations to the  $SU(2)_J$  transformation. When we studied the skyrmion moduli space in (1.3.2), we saw that an isorotation and a rotation are equivalent. In that case, we were rotating around the place where the skyrmion was centered. In this case, our instantons (whose moduli space resembles the skyrmion moduli space) are not located in the center, so there are some differences. In the following,  $M = M(E)$ .

$$A_i(x-X, z) \rightarrow -2\pi^2 \mu^2 M_{ij} (\epsilon_{jml} \sigma_l \partial_m^M + \sigma_j \partial_{z'}) G(M^{-1}x, z, M^{-1}MX, z')|_{z'=0}. \quad (4.2.26)$$

After transforming, the derivative  $\partial_m^M$  is with respect to  $M^{-1}x$ . Note that we have multiplied  $X$  by identity: by using the fact that  $G$  only depends on  $|x-x'|$ , we



can remove  $M^{-1}$ . We must then transform the derivative according to

$$\partial_m^M = \frac{\partial}{\partial(M^{-1}x)^m} = \frac{\partial M_{ka}(M^{-1}x)^a}{\partial(M^{-1}x)^m} \frac{\partial}{\partial x^k} = M_{km} \frac{\partial}{\partial x^k} = M_{km} \partial_k. \quad (4.2.27)$$

Then, we substitute in the expression for  $A_i$ , obtaining

$$A_i(x - R, z) \rightarrow -2\pi^2 \mu^2 (M_{ij} M_{km} \epsilon_{jml} \sigma_l \partial_k + M_{ij} \sigma_j \partial_{z'}) G(x, z, MX, z')|_{z'=0}. \quad (4.2.28)$$

We can use the fact that  $\epsilon$  is an invariant tensor,  $\epsilon_{ijk} M_{ai} M_{bj} M_{ck} = \epsilon_{abc}$ , by substituting  $\epsilon_{jml} M_{ij} M_{km} \sigma_l \partial_m = \epsilon_{ijk} M_{kl} \sigma_l \partial_j$ . Then, we use  $M_{ij} \sigma_j = E^\dagger \sigma_i E$  to obtain

$$A_i(x - X, z) \rightarrow E^\dagger A_i(x - M(E)X, z) E. \quad (4.2.29)$$

The action on  $A_z$  is the same:

$$A_z(x - X, z) \rightarrow -2\pi^2 \mu^2 \sigma_i \partial_i^M L(M^{-1}x, z, 0, 0). \quad (4.2.30)$$

Working as before, we get

$$A_z(x - X, z) \rightarrow E^\dagger A_z(x - M(E)X, z) E. \quad (4.2.31)$$

Regarding parity, it is trivial to verify that (remembering that  $\epsilon$  takes a minus sign for the parity operation)

$$A_i(x - X, z) \rightarrow A_i(x + X, z) \quad A_z(x - X, z) \rightarrow A_z(x + X, z). \quad (4.2.32)$$

The action of the continuous part of  $G$  on the fields is then

$$\begin{aligned} \tilde{A}_I(x, z) \rightarrow & U E^\dagger A \left( x - M(E) \frac{R}{2}, z \right) (U E^\dagger)^\dagger + \\ & + U \sigma_3 E^\dagger A \left( x + M(E) \frac{R}{2}, z \right) (U \sigma_3 E^\dagger)^\dagger. \end{aligned} \quad (4.2.33)$$

Eventually, parity can be used to change the sign of  $\frac{R}{2}$ . A discrete subgroup of  $\mathcal{L}$  forms the isotropy group of  $\tilde{A}$ . To describe it, we use the notation in [28] (where a similar analysis in the Skyrme model is given):  $O_{ai}$  represents an isorotation of  $\pi$  around the  $a$ -th isoaxis and the  $i$ -th spatial axis, while  $\mathcal{P}_{ai}$  represents the composition  $\mathcal{P}O_{ai}$ .  $a$  and  $i$  take values 1, 2, 3, and they can eventually take value 0 to indicate no rotation around that axis (e.g.,  $O_{02}$  is a pure spatial rotation around the 2 axis, represented by the matrix  $E = i\sigma_2$  while  $U$  takes value  $\mathbf{1}$ ). It is easily

verified that  $\mathcal{P}_{21}$ ,  $\mathcal{P}_{22}$  and  $\mathcal{P}_{33}$  leave  $\tilde{A}_I$  invariant. Those 3, together to the identity 1, generate the stabilizer, that is composed of 8 distinct elements:

$$H = \{1, O_{11}, O_{12}, O_{03}, \mathcal{P}_{30}, \mathcal{P}_{21}, \mathcal{P}_{22}, \mathcal{P}_{33}\}. \quad (4.2.34)$$

We have to quotient this stabilizer out. The zero mode manifold is then given by the orbit of (4.2.23) under the group  $\mathcal{L}/H$ . As we can obtain the second four transformations from the first four and parity, we can rewrite this group in a simpler way: calling  $V = \{1, O_{11}, O_{12}, O_{03}\}$  we have that the previous group is diffeomorphic to

$$\mathcal{G} = \mathbb{R}^3 \times SU(2)_I \times SU(2)_J/V. \quad (4.2.35)$$

We can then finally say that the zero mode manifold  $\mathcal{Z}$ , that is a subset of  $\mathcal{M}_2$ , is given by the orbit of (4.2.23) under (4.2.35). Coordinates on  $\mathcal{Z}$  can be read from (4.2.34), and they are the two  $SU(2)$  matrices  $(U, E)$ .

We want to build a Lagrangian on this manifold. For each instanton, we derive its kinetic energy through metric (4.1.23). In our usual coordinates  $(X_1, B, X_2, C)$  and defining angular velocities  $\omega_{B,i} = -i \operatorname{tr}(B^\dagger \dot{B} \sigma_i)$  and analogous for  $\omega_{C,i}$  we take the result from the  $B = 1$  sector to write the metric as

$$g|_{\mathcal{M}} = dX_1^i dX_1^i + 2\mu^2 d\Omega_{SU(2),B} + dX_2^i dX_2^i + 2\mu^2 d\Omega_{SU(2),C}. \quad (4.2.36)$$

The kinetic energy on  $\mathcal{M}_2$  is then

$$T = \frac{1}{2} M_0 \left( \dot{X}_1^i \dot{X}_1^i + \dot{X}_2^i \dot{X}_2^i + 2\mu^2 \omega_{B,i} \omega_{B,i} + 2\mu^2 \omega_{C,i} \omega_{C,i} \right). \quad (4.2.37)$$

We modify the spatial coordinates as usual, defining a mass center coordinate  $r_i$  and a global translation  $X_i$ . From now on, we will neglect global translations, redefining  $\mathcal{M}_2$  through the coordinates  $(r, B, C)$ , that specify a field configuration through

$$BA_I \left( x - \frac{r}{2} \right) B^\dagger + CA_I \left( x + \frac{r}{2} \right) C^\dagger. \quad (4.2.38)$$

The kinetic energy becomes

$$T = \frac{1}{2} M_0 \left( \frac{1}{2} \dot{r}^i \dot{r}^i + 2\mu^2 \omega_{B,i} \omega_{B,i} + 2\mu^2 \omega_{C,i} \omega_{C,i} \right). \quad (4.2.39)$$

We must embed  $\mathcal{Z}$  into  $\mathcal{M}_2$ , finding a law that allows us to find the coordinates on  $\mathcal{M}_2$  through the coordinates of  $\mathcal{Z}$ . The embedding law is obtained by confronting (4.2.34) with (4.2.38):

$$\begin{cases} r_i = M(E)_{ij}R_j, \\ B = UE^\dagger, \\ C = Ui\sigma_3E^\dagger. \end{cases} \quad (4.2.40)$$

To transform the kinetic energy in the zero mode manifold, we need to transform the velocities. We define angular velocities  $\omega_i$  relative to the matrix  $E$  and  $\Omega_i$  relative to the matrix  $U$ . First, we compute the derivative  $\dot{M}_{ij}(E)$ . Inverting the relation defining  $\omega_i$ , we get

$$E^\dagger \dot{E} = \frac{i}{2} \omega_i \sigma_i. \quad (4.2.41)$$

This can be used to compute

$$\begin{aligned} \dot{M}_{ij}\sigma_j &= \dot{E}^\dagger \sigma_i E + E^\dagger \sigma_i \dot{E} = \\ &E^\dagger \sigma_i E E^\dagger \dot{E} - E^\dagger \dot{E} E^\dagger \sigma_i E = \frac{i}{2} M_{ij} \omega_k [\sigma_j, \sigma_k] = M_{ij} \epsilon_{kjl} \omega_k \sigma_l. \end{aligned} \quad (4.2.42)$$

This implies

$$\dot{M}_{ij} = \epsilon_{lkj} M_{il} \omega_k. \quad (4.2.43)$$

In the following, we denote  $M'_{ij} = \frac{1}{2} \text{Tr}(\sigma_i E \sigma_3 \sigma_j \sigma_3 E^\dagger)$ , while  $M$  is the usual  $SO(3)$  matrix associated to  $E$ .

$$\begin{cases} B^\dagger \dot{B} = E(U^\dagger \dot{U} - E^\dagger \dot{E})E^\dagger \implies \omega_{B,i} = M_{ij}(\Omega_j - \omega_j), \\ C^\dagger \dot{C} = E(\sigma_3 U^\dagger \dot{U} \sigma_3 - E^\dagger \dot{E})E^\dagger \implies \omega_{C,i} = M'_{ij} \Omega_j - M_{ij} \omega_j, \\ \dot{r}_i = \dot{M}_{ij} R_j = \epsilon_{jkl} M_{ik} \omega_l R_j. \end{cases} \quad (4.2.44)$$

We obtain

$$\begin{aligned} \omega_{B,i} \omega_{B,i} + \omega_{C,i} \omega_{C,i} &= 2\omega_i \omega_i + 2\Omega_i \Omega_i - 2(\delta_{kl} + M_{ik} M'_{il}) \omega_k \Omega_l, \\ \dot{r}_i \dot{r}_i &= \epsilon_{jkl} M_{ik} \omega_l R_j \epsilon_{acb} M_{ia} \omega_c R_b = \omega^2 R^2 - (\omega \cdot R)^2 = (\omega_2^2 + \omega_3^2) R_0^2. \end{aligned} \quad (4.2.45)$$

The matrix  $\delta_{kl} + M_{ik} M'_{il}$  has only a non null element, the element 33 that has value 2 (this because  $M_{ik} M'_{il}$  represents a rotation of  $\pi$  around axis 3). The kinetic energy in the zero mode manifold  $\mathcal{M}$  then becomes

$$T|_{\mathcal{Z}} = \frac{1}{2} M_0 \left( 4\mu^2 \omega_1^2 + \left( 4\mu^2 + \frac{R_0^2}{2} \right) \omega_2^2 + \frac{R_0^2}{2} \omega_3^2 + 4\mu^2 (\Omega_1^2 + \Omega_2^2 + (\Omega_3 - \omega_3)^2) \right). \quad (4.2.46)$$

In the zero mode manifold, the potential attains its minimum value, that we call  $V_{min}$ . We also have to add  $2M_0$  to the potential function as the rest mass of the two objects. The Lagrangian is then given by

$$L|_{\mathcal{Z}} = T|_{\mathcal{Z}} - V_{min} - 2M_0. \quad (4.2.47)$$

#### 4.2.4 The quantum deuteron: quantizing the zero mode manifold

We quantize the zero mode manifold  $\mathcal{Z}$  by calculating the conjugate momenta from  $L_{\mathcal{Z}}$ : calling  $L_i$  the momenta obtained by deriving with respect to  $\omega_i$ , while  $I_i$  are obtained by deriving with respect to  $\Omega_i$ , we have

$$\begin{aligned} J_1 &= 4M_0\mu^2\omega_1, & J_2 &= M_0 \left( 4\mu^2 + \frac{R_0^2}{2} \right) \omega_2, & J_3 &= M_0 \left( \left( 4\mu^2 + \frac{R_0^2}{2} \right) \omega_3 - 4\mu^2\Omega_3 \right), \\ I_1 &= 4M_0\mu^2\Omega_1, & I_2 &= 4M_0\mu^2\Omega_2, & I_3 &= 4M_0\mu^2(\Omega_3 - \omega_3). \end{aligned} \quad (4.2.48)$$

The Hamiltonian is

$$H|_{\mathcal{Z}} = \frac{1}{2} \left( \frac{J_1^2}{4M_0\mu^2} + \frac{J_2^2}{M_0 \left( 4\mu^2 + \frac{R_0^2}{2} \right)} + \frac{(J_3 + I_3)^2}{\frac{1}{2}M_0R_0^2} + \frac{I_1^2 + I_2^2 + I_3^2}{4M_0\mu^2} \right) + 2M_0 + V_{min}. \quad (4.2.49)$$

Quantization proceeds as usual. We impose canonical commutation relations

$$[U, I_i] = -i\frac{i\sigma_i}{2}U \quad [E, J_i] = -i\frac{i\sigma_i}{2}E. \quad (4.2.50)$$

Let us call  $j, j_{3,L}, j_{3,R}$  the eigenvalues relative to the momenta  $J_i$  and  $i, i_{3,L}, i_{3,R}$  the eigenvalues relative to the momenta  $I_i$ . We identify  $J_i$  with the deuteron spin, and  $I_i$  with the deuteron isospin (as the first one arises from spatial rotation, the second one from isospacial rotations). Actually, as we already have distinct spin and isospin operators (totally unrelated), we can neglect the projections  $i_{3,R}$  and  $j_{3,R}$ , fixing their values. Renaming the other projections, the ket is specified by

$$|\psi\rangle = |i, i_3, j, j_3\rangle. \quad (4.2.51)$$

We have to add Finkelstein-Rubenstein constraints. Actually, we need a more generic form of those constraints. We previously stated that a wavefunction for a

baryon described by the matrix  $B$  must be odd under the sign inversion  $B \rightarrow -B$ . This is a specific case of the more general Finkelstein-Rubenstein constraints. In the moduli space of a single instanton, we consider the curve  $B(t)$ , starting at identity at  $t = 0$  and finishing at minus identity at  $t = 1$ . In  $SU(2)$  this is an open curve, but due to the invariance under  $B \rightarrow -B$  of the classical configuration this path gives a closed loop in the moduli space. As  $\pi_4(SU(2)) = \mathbb{Z}_2$ , loops are divided in two topological sectors. When quantizing the system, we must assign a phase to each loop: we choose to assign a phase of  $-1$  to loops homotopic to the one that we have just considered, and a phase of  $+1$  to loops that are not homotopic. In our case, this procedure must be repeated for each (non-identity) element in the stabilizer group  $V$ : we have to assign a phase to loops ending at  $O_{11}$ ,  $O_{12}$  and  $O_{03}$ , and we can only assign two phases, as  $\pi_4(SU(2) \times SU(2)) = \pi_4(SU(2)) = \mathbb{Z}_2$ . The situation is the same as in [28], and we recover the same results.

We first consider a rotation of a single instanton. We are looking for a path  $(U(t), E(t))$  that is implemented as a closed loop in  $\mathcal{Z}$ , that can be interpreted as a rotation of one instanton while keeping the other one fixed. We are then requiring  $U(t)E^\dagger(t) = \mathbf{1} \forall t$  and  $U(1)i\sigma_3E^\dagger(1) = -\sigma_3$ . Such a path is given by

$$\begin{cases} U(t) = \exp(i\sigma_1 \frac{\pi}{2}t), \\ E(t) = \exp(i\sigma_1 \frac{\pi}{2}t). \end{cases} \quad (4.2.52)$$

Incidentally,  $U(1) = i\sigma_1$  and  $E(1) = i\sigma_1$ : a rotation of a single instanton by  $2\pi$  is equivalent to a contemporaneous rotation and isorotation of the whole two-instanton system by  $\pi$ . This means that the path associated to  $O_{11}$  is noncontractible. A path ending at  $O_{21}$  is homotopic to the previous, through the homotopy

$$\begin{cases} U(s, t) = \exp(i\sigma_1 \frac{\pi}{2}t), \\ E(s, t) = \exp(i\sigma(s) \frac{\pi}{2}t), \\ \sigma(s) = \sigma_1 \cos \frac{\pi}{2}s + \sigma_2 \sin \frac{\pi}{2}s, \end{cases} \quad (4.2.53)$$

with  $s$  going from 0 to 1, as  $t$ . This means that  $O_{21}$  and  $O_{11}$  are endpoints of noncontractible loops, while  $O_{03}$ , that can be obtained by composing the two previous transformations, is the endpoint of a contractible loop. Thus the phase  $-1$  must be associated to the first and second loop, the phase  $+1$  must be associated to the third.

Those constraints give restrictions to the possible physical kets of the system.

As  $J_i$  generates rotations and  $I_i$  generates isorotations, we must have the following constraints on physical states  $|ph\rangle$ :

$$\begin{aligned} \exp(i\pi(\pm J_1 \pm I_1)) |ph\rangle &= -|ph\rangle, & \exp(i\pi(\pm J_2 \pm I_1)) |ph\rangle &= -|ph\rangle, & (4.2.54) \\ \exp(\pm i\pi J_3) |ph\rangle &= |ph\rangle. \end{aligned}$$

The  $\pm$  factors indicate the fact that, as example, the transformation  $O_{03}$  can be represented by two different couples of matrices,  $(i\sigma_3, \mathbf{1})$  and  $(-i\sigma_3, \mathbf{1})$ , while the other symmetries can be realized by four couples of matrices.

Those constraints remove some states from the physical spectrum. The states with  $j = i = 0$  are obviously unphysical, as rotation operators act as the identity and paths with assigned phase  $-1$  cannot be implemented. We then look for the ground state among the physical states with the lowest angular momentum. The only states that are compatible with constraints (4.2.55) are

$$|D\rangle = |0, 0, 1, 0\rangle, \quad |I_0\rangle = |1, 0, 0, 0\rangle, \quad |I_1\rangle = \frac{1}{\sqrt{2}}(|1, 1, 0, 0\rangle + |1, -1, 0, 0\rangle). \quad (4.2.55)$$

We see that  $|D\rangle$  has the right quantum numbers to be identified as the deuteron state (isospin singlet and spin triplet). We want our deuteron to be the state of minimum energy, so we compute  $H|_Z$  on the states that we have found:

$$\begin{aligned} H|_Z |D\rangle &= \left( \frac{1}{8\mu^2 M_0} \left( 1 + \frac{1}{1 + \frac{R_0^2}{8\mu^2}} \right) + 2M_0 + V_{min} \right) |D\rangle, \\ H|_Z |I_0\rangle &= \left( \frac{1}{4\mu^2 M_0} + 2M_0 + V_{min} \right) |I_0\rangle, \\ H|_Z |I_1\rangle &= \left( \frac{1}{4M_0\mu^2} \left( 1 + \frac{4\mu^2}{R_0^2} \right) + 2M_0 + V_{min} \right) |I_1\rangle. \end{aligned} \quad (4.2.56)$$

For every value of  $R_0$  and  $\mu$ , the deuteron state turns out to be the lowest energy state, with the lowest rotational energy contribution.

### 4.2.5 The expectation value of the potential

We are now in the position to understand the origin of the factor of 3 that differs between our potential and the one in [26]. For this, we compute the expectation value of the potential on the deuteron state. We can use a method from [23]

to compute matrix elements between one baryon states, then adapt our results to the deuteron.

In the  $B = 1$  topological sector, the  $SU(2)$  part of the quantized wavefunction can be written using four real coordinates  $a^I$  under the constraint  $a_I a_I = 1$ . There are two kinds of angular momenta that can be written through those  $a_I$ , left and right invariant: we will use them both. They are given by

$$I_i = \frac{i}{2} \left( a_0 \frac{\partial}{\partial a_i} - a_i \frac{\partial}{\partial a_0} - \epsilon_{ijk} a_j \frac{\partial}{\partial a_k} \right), \quad (4.2.57)$$

$$J_i = \frac{i}{2} \left( -a_0 \frac{\partial}{\partial a_i} + a_i \frac{\partial}{\partial a_0} - \epsilon_{ijk} a_j \frac{\partial}{\partial a_k} \right). \quad (4.2.58)$$

From now on, we will work by fixing the eigenvalues of  $I$  and  $J$  to  $1/2$ , and always omit that number from the ket.

Eigenvalues of the angular momenta are built in a similar way to the spherical harmonics: they are given by a linear combination of the four coordinates, with the condition that the sum of the squares of the coefficients must be 0. We can nicely organize the states in a matrix: calling  $\mathbf{a}$  the  $SU(2)$  matrix  $\mathbf{a} = a_0 \mathbf{I} + ia_i \sigma_i$ , we can verify through direct calculation that

$$\frac{1}{\pi} \sigma_2 \mathbf{a} = \frac{1}{\pi} \begin{pmatrix} a_1 + ia_2 & -i(a_0 - ia_3) \\ i(a_0 + ia_3) & -(a_1 - ia_2) \end{pmatrix} := \begin{pmatrix} \langle a_I | p, \uparrow \rangle & \langle a_I | p, \downarrow \rangle \\ \langle a_I | n, \uparrow \rangle & \langle a_I | n, \downarrow \rangle \end{pmatrix}. \quad (4.2.59)$$

The ket is organized as follows: the first letter stands for the  $I_3$  eigenvalue. Proton ( $p$ ) corresponds to isospin  $\frac{1}{2}$ , while neutron ( $n$ ) corresponds to isospin  $-\frac{1}{2}$ . Arrows describe the  $J_3$  eigenvalue in the obvious way:  $\uparrow$  for  $\frac{1}{2}$ ,  $\downarrow$  for  $-\frac{1}{2}$ . One can check that the assignments of the quantum numbers are correct by explicitly applying  $I_3$  and  $J_3$  to the single states.

The useful information here is given by the peculiar form of matrix elements of an operator. Let  $O$  be any operator depending on the  $SU(2)$  degrees of freedom: matrix elements are of the form

$$\langle I_3, J_3 | O(a_I) | I'_3, J'_3 \rangle = \frac{1}{\pi^2} \int d\Omega_a (\sigma_2 \mathbf{a})_{I_3, J_3} O(a_I) (\sigma_2 \mathbf{a})_{I'_3, J'_3} \quad (4.2.60)$$

( $d\Omega_a$  is the Haar invariant measure on  $SU(2)$ ). In interesting cases,  $O(a_I)$  depends on the coordinates through the matrix  $\mathbf{a}$ . The following integrals from [23] are useful:

$$\int d\Omega_a \mathbf{a}_{ij} \mathbf{a}_{kl}^{-1} = \pi^2 \delta_{il} \delta_{jk} \quad (4.2.61)$$

$$\int d\Omega_a \mathbf{a}_{ij} \mathbf{a}_{kl}^{-1} \mathbf{a}_{mn} \mathbf{a}_{pq}^{-1} = \frac{\pi^2}{3} (2(\delta_{il} \delta_{mq} \delta_{jk} \delta_{np} + \delta_{iq} \delta_{ml} \delta_{jp} \delta_{nk}) - \delta_{il} \delta_{mq} \delta_{jp} \delta_{nk} - \delta_{jk} \delta_{np} \delta_{iq} \delta_{ml}). \quad (4.2.62)$$

Two particle states (described by two unit vectors,  $a_I$  and  $b_I$ ) are direct product of two one particle kets, appropriately changing the coordinates. As an example

$$\begin{aligned} \langle a_I, b_I | p, \uparrow, n, \uparrow \rangle &:= \langle a_I | p, \uparrow \rangle \langle b_I | n, \uparrow \rangle = \\ \frac{i}{\pi^2} (a_1 + ia_2)(b_0 + ib_3) &= \frac{1}{\pi^2} (\sigma_2 \mathbf{a})_{\frac{1}{2}, \frac{1}{2}} (\sigma_2 \mathbf{b})_{-\frac{1}{2}, \frac{1}{2}}. \end{aligned} \quad (4.2.63)$$

The rules to find the eigenvalues of the total spin and isospin are similar to the rules for the three dimensional angular momentum, in particular, two nucleons can have total spin 1 or 0 and total isospin 1 or 0, but they do not need in principle to be equal. As before, when we compose two particles we will have two projections of the (iso)spin, a left and a right invariant operator, and in principle we would need to give both eigenvalues. We will neglect this additional eigenvalue, and we will write the value of the projections of (iso)spin as sums of the projections on single particle states (as example, the state that we just wrote has  $I_3 = 0$  and  $J_3 = 1$ ).

The state of deuteron has  $J = 1, J_3 = 0, I = 0, I_3 = 0$ . Denoting it as  $|D\rangle$ , we can verify that the only combination of states with the right quantum numbers (and eigenvalue  $-1$  under exchange of particles, that in this model consists in switching the first two eigenvalues with the last two) is given by

$$|D\rangle = \frac{1}{2} (|p, \uparrow, n, \downarrow\rangle + |p, \downarrow, n, \uparrow\rangle - |n, \downarrow, p, \uparrow\rangle - |n, \uparrow, p, \downarrow\rangle). \quad (4.2.64)$$

Using those techniques, we can compute the expectation value of the potential that we've written. In particular the spatial variables are intact after averages. We take states that are localized around the position of equilibrium, so we do not quantize spatial coordinates. The only average to perform is the average of the phase matrix  $M_{ij}$ . We remember that this phase matrix is related to the isospin orientation matrices  $B$  and  $C$  as  $M_{ij} = M(B^\dagger C) = M(B^\dagger)M(C)$ , and every  $M$  factor contains two coordinate matrices. We can then use formula (4.2.63) to compute the matrix elements in the 16 dimensional basis formed by the combinations of single particle states, and then evaluate the resulting matrix on the deuteron vector. The result is

$$\langle D | M | D \rangle = \frac{1}{3} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (4.2.65)$$



This matrix is proportional to the phase opposition matrix that we found classically, with an extra factor of  $1/3$ . The expectation value of our potential is exactly equal to the potential found in [26].

The lack of the factor  $1/3$  in the classical potential is given by the fact that we are choosing the phases to be locked in the attractive channel. The relative phase among the instantons is not considered a coordinate. Actually, in a full quantum approach one should consider the relative phase as a coordinate, and find the contribution to the energy eigenvalues of those coordinates. Once we write a wavefunction for the relative coordinate, the average of the dipole part of the potential should get a factor in between  $1/3$  and  $1$ .

The next two sections are dedicated to preparing the analysis of the massive modes, the distance between the objects and the relative phase. We will propose a Lagrangian for those coordinates, but solving the full quantum system requires a separate study, that we postpone for future analysis.

### 4.2.6 The massive modes: harmonic approximation

We now extend our quantization scheme to massive modes, such as relative translations and changes of relative phase. Before quantizing the exact dynamics on  $\mathcal{M}_\epsilon$ , we first approximate the massive modes through the harmonic approximation. To do so, it is convenient to switch back to  $(r, B, C)$  coordinates.

To perform this approximation, we calculate the second derivatives of the potential with respect to the coordinates. The derivatives with respect to the spatial coordinates  $r$  are standard derivatives, but we need a coordinate representation of the matrices  $(B, C)$  to identify numerical coordinates for the derivation. We choose coordinates through the exponential map

$$B = \exp\left(iB_i \frac{\sigma_i}{2}\right), \quad C = \exp\left(iC_i \frac{\sigma_i}{2}\right). \quad (4.2.66)$$

$B_i$  and  $C_i$  are real, unconstrained numerical coordinates. They have a finite range but, as we are interested in small changes of  $B_i$  and  $C_i$ , we do not need to specify the range. In those coordinates, the velocities are

$$\omega_{B,i} = -i \operatorname{tr}\left(B^\dagger \dot{B} \sigma_i\right) = \dot{B}_i, \quad \omega_{C,i} = -i \operatorname{tr}\left(C^\dagger \dot{C} \sigma_i\right) = \dot{C}_i, \quad (4.2.67)$$

$$V = \begin{pmatrix} 0.142 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.662 & 0 & 0.681 & 0 & 0 & 0.681 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.681 & 0 & 0.701 & 0 & 0 & 0.701 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.542 & 0 & 0 & -0.542 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.681 & 0 & 0.701 & 0 & 0 & 0.701 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.542 & 0 & 0 & 0.542 \end{pmatrix} \frac{N}{\Lambda}$$

Table 4.1: Potential matrix at the equilibrium position.

and, after canonical quantization of the matrix coordinates  $B$  and  $C$ , we recover the quantum commutation relations

$$[B_i, J_{B,j}] = i\delta_{ij} \quad [C_i, J_{C,j}] = i\delta_{ij}, \quad (4.2.68)$$

with  $J_{B,i} = 2M_0\mu^2\omega_{B,i}$  and analogous for  $J_{C,i}$ . Those coordinates can be used as canonical coordinates, and we can perform the little oscillations approximation in the standard way. Returning back to the Lagrangian, we perform the derivatives and set the coordinates to their equilibrium values,  $r = (R_0, 0, 0)$ ,  $B_i = (0, 0, 0)$ ,  $C_i = (0, 0, \pi)$ . Calling  $\eta_a$  the displacement from equilibrium coordinates (with  $a = 1, \dots, 9$ ), the approximated Lagrangian can be written as

$$L|_{\mathcal{M}_2} = \frac{1}{2}M_{ab}\dot{\eta}^a\dot{\eta}^b - \frac{1}{2}V_{ab}\eta^a\eta^b - V_{min} - 2M_0, \quad (4.2.69)$$

where the mass matrix  $M_{ab}$  is the diagonal matrix of eigenvalues

$$\left[ \frac{1}{2}M_0, \frac{1}{2}M_0, \frac{1}{2}M_0, 2M_0\mu^2, 2M_0\mu^2, 2M_0\mu^2, 2M_0\mu^2, 2M_0\mu^2, 2M_0\mu^2 \right], \quad (4.2.70)$$

and  $V_{ab}$  has been computed numerically and written in table (4.1) Solving the secular equation  $\det(\omega^2 M_{ab} - V_{ab})$  we obtain three non null frequencies, as expected.

$$\begin{aligned} \omega_1 &= \frac{1.509}{\Lambda} & \omega_2 &= \frac{0.995}{\sqrt{\Lambda}} + o\left(\frac{1}{\sqrt{\Lambda^5}}\right), \\ \omega_3 &= \frac{1.131}{\sqrt{\Lambda}} + \frac{4.678}{\sqrt{\Lambda^3}} + o\left(\frac{1}{\sqrt{\Lambda^5}}\right). \end{aligned} \quad (4.2.71)$$

We can identify  $\omega_1$  with the radial oscillation, that allows the constituents of the deuteron to vibrate along the axis joining them: this interpretation is suggested

by its  $\Lambda$  dependence, as the translational mode inertia is proportional to  $\Lambda$  and all entries in the  $V$  matrix are multiplied by  $\Lambda^{-1}$ , giving an overall  $\Lambda^{-2}$  dependence of the squared frequency. The other two frequencies are relative to small, non global isorotations of the two objects, that cost energy. The dependence  $\Lambda^{-\frac{1}{2}}$  of the leading order comes from the fact that the moment of inertia has leading order proportional to  $M_0\mu^2$ , that is proportional to  $\Lambda^0$ , giving an overall  $\Lambda^{-1}$  dependence of the squared frequencies.

The quantum hamiltonian is readily written. We also include the contribution from the zero modes.

$$H|_{\mathcal{M}_2} = \sum_{i=1,2,3} \omega_i \left( a_i^\dagger a_i + \frac{1}{2} \right) + H|_{\mathcal{Z}} + V_{min} + 2M_0. \quad (4.2.72)$$

The ground states of the oscillators then give a contribution to the energy of the deuteron. The ground state of this Hamiltonian is given by

$$E_{0,approx} = \frac{\omega_1 + \omega_2 + \omega_3}{2} + V_{min} + 2M_0 + \frac{1}{8\mu^2 M_0} \left( 1 + \frac{1}{1 + \frac{R_0^2}{8\mu^2}} \right). \quad (4.2.73)$$

We will investigate this result numerically in another section.

### 4.2.7 The massive modes: full Lagrangian

After using the harmonic approximation, we write the Lagrangian on  $\mathcal{M}_2$  to set up the quantum system. We want to have as many cyclic coordinates as possible, as their contribution to energy eigenvalues is trivial. We then introduce the coordinates  $(\eta, U, F, E)$ :  $\eta$  is a number, representing the absolute value distance between the instantons,  $U$  and  $E$  are global rotation and isorotation matrices, while  $F$  is an  $SU(2)$  matrix describing the relative orientation. We have a total of 10 coordinates, while  $\mathcal{M}_2$  is 9 dimensional: the reason of this apparent discrepancy will be clarified.

The coordinates  $(\eta, U, F, E)$  are related to the coordinates  $(r, B, C)$  through

$$\begin{cases} r_i = M(E)_{i1}\eta, \\ B = UE^\dagger, \\ C = UFE^\dagger. \end{cases} \quad (4.2.74)$$

We must introduce velocities relative to the matrix  $F$ , and we call them  $\xi_i$ . We note that, if we perform the transformation

$$\begin{cases} U \rightarrow UN(\alpha), \\ E \rightarrow EN(\alpha), \\ F \rightarrow N^\dagger(\alpha)FN(\alpha), \end{cases} \quad (4.2.75)$$

with

$$N(\alpha) = \cos \frac{\alpha}{2} \mathbf{1} + i \sin \frac{\alpha}{2} \sigma_1, \quad (4.2.76)$$

we have that the coordinates  $r_i, B, C$  do not change. This is the reason of the additional coordinate: the new set of coordinates has the advantage of isolating the zero modes from the massive modes, but an unphysical mode appears, and it must be taken care of. The fact that this mode is unphysical is confirmed by computing the new kinetic energy: through the velocities transformation

$$\begin{cases} \omega_{B,i} = M(E)_{ij}(\Omega_j - \omega_j), \\ \omega_{C,i} = M(E)_{ij}(M(F)_{kj}\Omega_k + \xi_j - \omega_j), \\ \dot{r}_i = \dot{M}(E)_{i1}\eta + M(E)_{i1}\dot{\eta}. \end{cases} \quad (4.2.77)$$

Calculations are performed as usual, and the kinetic energy in the new coordinates read

$$\begin{aligned} T|_{\mathcal{M}_2} = M_0 & \left( \frac{1}{4}\dot{\eta}^2 + (2\mu^2 + \frac{1}{4}\eta^2)\omega_i\omega_i - \frac{1}{4}\eta^2\omega_1^2 + \right. \\ & \left. + \mu^2(2\Omega_i\Omega_i + \xi_i\xi_i - 2\xi_i\omega_i) - 2\mu^2 M(F)_{ij}\omega_i\Omega_j - 2\mu^2(\delta_{ij} + M(F)_{ij})\omega_i\Omega_j \right). \end{aligned} \quad (4.2.78)$$

By writing the kinetic energy in scalar product form, we can explicitly verify that the determinant of the associated matrix is 0, its rank being 9: one of the motion equations is really a constraint. The potential in those coordinates assumes the form<sup>2</sup>

$$\begin{aligned} V(\eta, F) = \frac{4\pi N}{\Lambda} & \left( \sum_{n=1}^{\infty} \left( \frac{1}{c_{2n-1}} \frac{e^{-k_{2n-1}\eta}}{\eta} + \frac{6}{5} \frac{1}{c_{2n-1}} M_{ij}(i\sigma_3) P_{ij}(\eta, k_{2n-1}) \frac{e^{-k_{2n-1}\eta}}{\eta^3} \right. \right. \\ & \left. \left. - \frac{6}{5} \frac{1}{d_{2n}} \frac{e^{-k_{2n}\eta}}{\eta^3} M_{ij}(i\sigma_3) P_{ij}(\eta, k_{2n}) \right) - \frac{6}{5\pi} \frac{1}{\eta^3} M_{ij}(i\sigma_3) P_{ij}(\eta, 0) \right). \end{aligned} \quad (4.2.79)$$

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<sup>2</sup>In  $P_{ij}$ , the first argument should be a vector: it is sufficient to insert  $\eta$  as the vector  $(\eta, 0, 0)$ .

The coordinates  $U$  and  $E$  do not appear in the potential, so their contribution to the energy eigenvalues remains the same as in the zero mode quantization. The Lagrangian should be supplemented with a term implementing some constraint to account for the zero mode given by the symmetry under (4.2.75). This is a complicated problem to solve, that we postpone to a future study. For now, we set  $F = i\sigma_3$ ,  $\xi_i$  to zero and we keep the radial coordinate quantized. Locking the relative phase in the attractive channel, we choose to neglect the factor in front of the dipole part that could arise from quantization of this coordinate, as discussed in (4.2.5). Defining the momentum  $\Pi_\eta = M_0\dot{\eta}/2$ , imposing canonically  $[\eta, \Pi_\eta] = i$  and noting  $V(\eta) = V(\eta, i\sigma_3)$ , we get the Hamiltonian

$$H|_{\mathcal{M}_2} = \frac{\Pi_\eta^2}{M_0} + H|_Z + V(\eta) + 2M_0. \quad (4.2.80)$$

We note that the term  $V_{min}$  is absent. We are not taking instantons fully localized in their equilibrium positions, but we allow their distance to be a quantum variable.

### 4.2.8 Insertion of quark mass: massive pion

We conclude the theoretical study of this model by inserting a term that can be used to give mass to the quarks. As massive quarks explicitly break the axial  $SU(N_f)$  symmetry, this result in a massive Goldstone boson. In this model, giving mass to quarks explicitly makes the pion massive.

The insertion of a mass quark term in the gravity dual is a complicated issue, studied in [4, 22]. We just cite the result: insertion of pion mass in the  $N_f = 2$  Sakai-Sugimoto low energy action is done through the term

$$S_m = \frac{\Lambda^{\frac{3}{2}}}{16\sqrt{2}\pi^{\frac{3}{2}}} \int P[M \exp\left(-i \int_{-\infty}^{+\infty} A_z dz\right) + \exp\left(i \int_{-\infty}^{+\infty} A_z dz\right) M - 2\mathbf{1}] d^3x dz. \quad (4.2.81)$$

$P$  denotes path ordering and  $M$  is the mass matrix, a diagonal  $2 \times 2$  matrix containing as entries the mass of the up and down quark. We approximate them to be the same: in this case  $M = m\mathbf{1}$ , with  $m$  adimensional. This term turns out to give a mass  $mM_{KK}$  to quarks in the boundary theory. It is interesting to note the fact that this term is similar to the Skyrme mass term, (see as example the Skyrme chapter in [32]), when we identify the exponential as the Skyrme field  $U$ .

We study this nonlocal term by approximating the exponential to the first nontrivial order, giving a crude estimate of the pion mass. We obtain the approximated action

$$S_{mass} = -\frac{\Lambda^{\frac{3}{2}}}{16\sqrt{2}\pi^{\frac{3}{2}}} \int d^3x dz dz' \text{tr}[A_z(x, z)A_z(x, z')]. \quad (4.2.82)$$

We study the effect of this term in the nonlinear region. The only changed motion equation is

$$H^{\frac{3}{2}}(z)(\partial_i \partial_i A_z^+ - \partial_i \partial_z A_i^-) - \frac{\sqrt{32\pi^3 \Lambda} m}{N} \int_{-\infty}^{+\infty} A_z(x, z') dz' = -2\pi^2 \mu^2 \sigma_i \partial_i \delta^3(x) \delta(z). \quad (4.2.83)$$

We want to study the effect of the additional term in the first term of the tower defining  $A_z$ , that is the pion term. To this end, it is sufficient to make the ansatz

$$A_i^- = 0 \quad A_z^+ = -2\pi \mu^2 H(z)^{-\frac{3}{2}} \partial_i \frac{e^{-k_0|x|}}{|x|}. \quad (4.2.84)$$

Neglecting the  $\delta(z)$  on the RHS (that is accounted for when writing the whole tower), we have to set

$$k_0^2 = \frac{\sqrt{(2\pi)^5 \Lambda}}{N} m. \quad (4.2.85)$$

$k_0$  is the pion mass, in units of  $M_{KK}$ . The potential is modified by just modifying the last part: as the functions  $\psi(z)$  and  $\phi(z)$  do not change in this crude approximation, everything remains the same until the last part. All the non numerical results in the previous sections hold, provided that we use as potential

$$V(\eta, F) = \frac{4\pi N}{\Lambda} \left( \sum_{n=1}^{\infty} \left( \frac{1}{c_{2n-1}} \frac{e^{-k_{2n-1}\eta}}{\eta} + \frac{6}{5} \frac{1}{c_{2n-1}} M_{ij}(F) P_{ij}(\eta, k_{2n-1}) \frac{e^{-k_{2n-1}\eta}}{\eta^3} - \frac{6}{5} \frac{1}{d_{2n}} \frac{e^{-k_{2n}\eta}}{\eta^3} M_{ij}(F) P_{ij}(\eta, k_{2n}) \right) - \frac{6}{5\pi} \frac{e^{-k_0\eta}}{\eta^3} M_{ij}(F) P_{ij}(\eta, k_0) \right). \quad (4.2.86)$$

### 4.3 Calculations and results

We conclude our study of the B=2 sector by giving some numerical calculations in this model. We are particularly interested in the binding energies, and

in the confrontation between the classical and quantum model. We also want to determine the quantum spectrum of the system: this should be composed of only one bound state, as there is no known bound state between proton and neutron that is different from the deuteron state. We organize this last section as follows: we start by computing the meson and baryon masses, then we give some numerical classical result for the deuteron and we use our potential to give a rather crude estimate of the binding energies of nuclei with  $B$  between 2 and 6. Then, we switch to the quantum picture, determining the spectrum of states of the system and studying how the binding energy of the deuteron is modified when taking into account various levels of quantum corrections. We compare the results with the experimental results.

We focus our attention on the  $N$  and  $\Lambda$  dependence of the physical quantities. As we are working in the  $\Lambda \rightarrow \infty$  and  $N \rightarrow \infty$ , we will comment those limits before trying to extrapolate to physical values. As in real physics we have  $N = 3$  and  $\Lambda$  is often chosen to fit the pion decay constant (3.3.50), giving a value of  $\Lambda = \Lambda_{SS} \simeq 1.569$ , we do not expect quantitatively correct results, but we will make qualitative considerations to motivate further study of the model.

For completeness, we recall the dependence of  $f_\pi$  and  $e$  from the physical parameters. In our units,  $f_\pi$  has the dimension of a mass, while  $e$  is adimensional.

$$f_\pi = \frac{\sqrt{N\Lambda}}{4\pi^{\frac{3}{2}}}, \quad e \simeq \frac{15.88}{\sqrt{N\Lambda}}. \quad (4.3.1)$$

### 4.3.1 Meson and baryon masses

We confront the meson masses in our model with the ones present in [35]. We take the masses of the first mesons  $\rho$ ,  $a$  and  $\omega$  and we compare them with the masses  $k_1, k_2$ . The comparison is reported in tables (4.2).

The baryon mass formula

$$M_j = M_0 + \frac{j(j+1)}{4M_0\mu^2}. \quad (4.3.2)$$

can be used to compute the baryon masses. Using the explicit values, we get

$$M_j = \frac{N\Lambda}{8} + \sqrt{\frac{2}{15}}N + \frac{1}{4N}\sqrt{\frac{5}{6}}j(j+1) + o(N^{-2}, \Lambda^{-1}). \quad (4.3.3)$$

Meson	Predicted mass ( $M_{KK}$ )	Observed mass ( $MeV$ )
$\rho$	0.819	775.3
$\omega$	0.819	782.7
$a$	1.257	980.0
Meson mass ratio	Predicted ratio	Observed ratio
$\rho/\omega$	1	0.9905
$\rho/a$	0.652	0.791
$\omega/a$	0.652	0.799

Table 4.2: Comparison of the experimental values and the predicted values of the masses. In the second table, we give the adimensional ratios, finding good accord with the experimental values. All those ratios are independent of the parameters of the theory.

The pieces proportional to  $N$  are the rest mass of the instanton: as the whole action is proportional to  $N$ , they are of order  $N^1$ . Those are the leading terms in the mass: we reproduce the large  $N$  result that the baryon mass should go as  $N$  when  $N \rightarrow \infty$ . The third part is the quantum correction due to the fact that the baryons always have non null spin (being  $j$  semi integer). The states with  $j = 1/2$  are interpreted as proton and neutron states, while the states with  $j = 3/2$  are interpreted as  $\Delta$  states.

The masses of the nucleons with  $j = 1/2$  and  $j = 3/2$  are

$$M_{\frac{1}{2}} = M_0 + \frac{3}{16M_0\mu^2}, \quad M_{\frac{3}{2}} = M_0 + \frac{15}{16M_0\mu^2}. \quad (4.3.4)$$

Evaluating at  $\Lambda_{SS}$ , we get  $M_{\frac{1}{2}} = 1.704$  and  $M_{\frac{3}{2}} = 1.784$  (in units of  $M_{KK}$ ). Their mass difference is

$$\Delta M_n = M_{\frac{3}{2}} - M_{\frac{1}{2}} = \frac{3}{4M_0\mu^2}, \quad (4.3.5)$$

that, evaluated at  $\Lambda_{SS}$ , gives a difference of 0.08. We choose to fit  $M_{KK}$  by fitting the mass of the  $\rho$  meson, obtaining  $M_{KK} = 949MeV$ . With this value of  $M_{KK}$ , our baryons are quite heavy ( $M_{\frac{1}{2}} \simeq 1617MeV$  and  $M_{\frac{3}{2}} \simeq 1693MeV$ ) and their mass difference is too small (circa  $80MeV$ ) when comparing those values with the physical values (respectively,  $938MeV$ ,  $1240MeV$  and circa  $300MeV$ ).



### 4.3.2 Analysis of the classical potential

From now on, we take the pion as massless, unless stated otherwise. In the attractive channel, the potential (sketched in figure (4.2)) assumes a minimum in  $R_0 = 2.059$ , of value  $V_{min} = -0.152N/\Lambda$ . The classical energy in the B=2 sector is then given by

$$E_{2,c} = 2M_0 - 0.152\frac{N}{\Lambda}. \quad (4.3.6)$$

Everything is of order  $N$ , and in the large  $N$  limit the second term (that can be interpreted as the classical energy) goes to infinity: this parallels what happens in large  $N$  QCD. If  $\Lambda \rightarrow \infty$ , we get non interacting baryons (of infinite mass and zero size).

We confront the value of  $E_{2,c}$  with the classical energy of the B=1 sector,  $E_{1,c} = M_0$ , calculating the classical binding ratio. This turns out to be independent of  $N$ . We have

$$B_{2,c} = \frac{E_{2,c} - 2E_{1,c}}{2E_{1,c}} = -\frac{1.216}{2.921\Lambda + \Lambda^2}. \quad (4.3.7)$$

As this quantity is always negative, for every value of  $\Lambda$  and for every value of  $N$  the classical deuteron turns out to be bound.

The experimental value for comparison is calculated from the data of [35]: with  $m_d$  deuteron mass,  $1875.613MeV$ ,  $m_p$  proton mass,  $938.272MeV$  and  $m_n$  neutron mass,  $939.565MeV$ , we have

$$B_{2,exp} = \frac{m_d - (m_p + m_n)}{m_p + m_n} = -1.2 \cdot 10^{-3}. \quad (4.3.8)$$

For comparison, we choose  $\Lambda$  as in [40], to fit the experimental value of the pion decay constant:  $\Lambda = \Lambda_{SS} = 1.569$ . With this value of  $\Lambda$  it is evident that our approximations are not so solid. With  $\Lambda_{SS}$  we get  $B_{2,c} = -0.173$ , two orders of magnitude greater than the experimental value.

We now use the potential to give some predictions about equilibrium configurations for sectors with  $2 \leq B \leq 6$ , provided that the instantons are far away (each instanton core is localized in the linear zone of all other instantons). Calculations are done as follows: for B instantons, we define the potential  $V_B$  as the sum of

single potentials (4.2.15) between all instantons, then we find a minimum energy configuration. To find it, we use axis-angle notation (4.2.16) for the matrix  $M_{ij}$  and we put one instanton in the center, with spatial and angular coordinates put to zero. We allow the phase coordinates of the remaining  $B - 1$  instantons to assume all possible values, but we restrict the spatial coordinates to impose particularly symmetric configurations. This is done because the number of free coordinates in principle should be  $6(B - 1)$  in each  $B$  sector, so the number of coordinates to fix grows rather quickly. We report the results of our analysis in table (4.3), where we list the binding energies in different sectors, trying different shapes for the solution. Here,  $n$  denotes the topological sector, and  $V_{min,n}$  are to be multiplied by  $N/\Lambda$ : consequently, the binding ratios  $B_n = V_{min}/nE_1$  is to be multiplied by  $\Lambda^{-2}$ , keeping only the  $\Lambda^1$  contribution in  $E_1$ . We compare our results to the experimental results in figure (4.3). We note that the minimum energy configuration among those that we have tried in the 3, 4 and 5 sector consist of instantons placed on the edges of a face centered cubic lattice. This result resembles the results of [20], so an immediate future direction would be to find global minimums in those sectors and see if the global minimums reproduce the results in the article.

$n$	Shape	Details	$V_{min,n}$	$B_n$
2	Line	Distance=2.059	-0.152	-0.608
3	Straight line	Neighbor distance=5.000	-0.249	-0.644
	Equilateral triangle	Side=4.878	-0.386	-0.824
4	Rectangle	Sides=2.138,4.737	-0.386	-0.772
	Tetrahedron	Side=5.456	-0.507	-1.014
5	Pentagon	Side=2.844	-0.438	-0.701
	Rectangular pyramid	Base=2.884,2.196 Height=3.193	-0.451	-0.722
	Tetrahedron + particle*	Center distance=3.679	-0.644	-1.030
6	Icosahedron	Base=3.034,4.195 Height=1.766	-0.851	-1.135

Table 4.3: Particular many-bodies configurations and relative binding ratios. Various shapes are considered, and the details on the shape of minimum energy are found. In the 5 sector, we have a particular configuration: as there is no known stable nucleus composed of five baryons, we tried a configuration with a  $n = 4$  tetrahedron plus a free instanton. This turned out to give the minimum energy configuration.

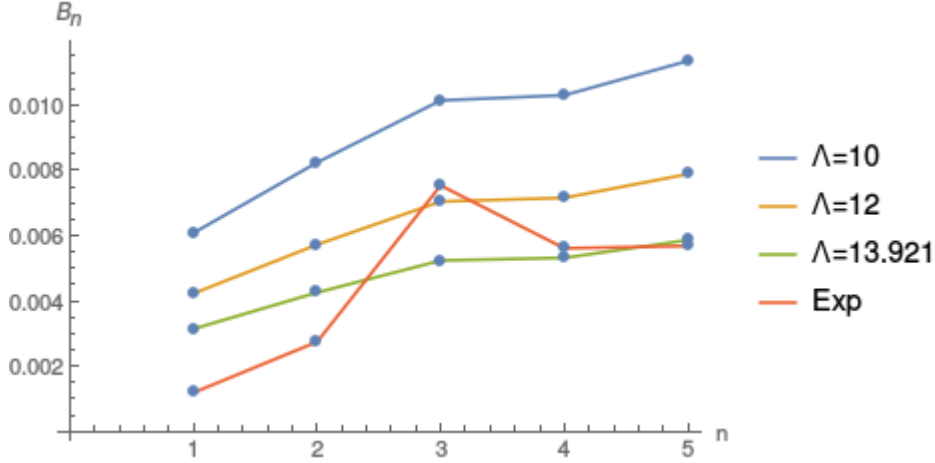


Figure 4.3: Binding ratios for various values of  $\Lambda$ . The binding ratios at  $\Lambda = \Lambda_{SS}$  are large and are not reported: this is expected, as quantization is necessary to lower the binding energy. We report three values of  $\Lambda$ , where the last one has been determined by minimizing the squared distances to the experimental values. We do not reproduce the bump at  $n = 4$ , but we see that the difference between  $n = 4$  and  $n = 3$  is greater than the differences between  $n = 5$  or  $n = 6$  and  $n = 4$ . All binding ratios are calculated for stable nuclei by dividing their binding energy per nucleon with the average proton-neutron mass. The exception is for  $n = 5$ , where the mass of the unstable  ${}^5\text{Li}$  is taken. Nuclear data are taken from [34].

### 4.3.3 The quantum spectrum

We now use the quantum model that we have developed to study the contribution of quantum corrections to the energy. We will start from the contribution that arises from the quantization of zero modes, then we will add the contribution of the massive modes, first trying the harmonic approximation and then numerically solving the Schrodinger equation for the radial coordinate. We will see how the results vary in those approaches. We study the dependence of the first terms in the  $1/\Lambda$  approximation, for  $\Lambda \rightarrow \infty$ , with a particular attention to the physical region, where  $\Lambda$  is comparable to  $\Lambda_{SS}$  to fit the pion decay constant. We will see if, extrapolating those results to physical  $\Lambda$ , the quantum model gives qualitatively correct answers, or if we need to further study the model, inserting higher order contributions.

From the quantization of the zero modes, the wavefunction for the system can

be indicated as

$$|\psi\rangle = |i, i_3, j, j_3\rangle. \quad (4.3.9)$$

We have seen that the Finkelstein-Rubenstein constraints allow only three states:

$$|D\rangle = |0, 0, 1, 0\rangle, \quad |I_0\rangle = |1, 0, 0, 0\rangle, \quad |I_1\rangle = \frac{1}{\sqrt{2}}(|1, 1, 0, 0\rangle + |1, -1, 0, 0\rangle). \quad (4.3.10)$$

The first state has spin one and isospin zero, and we identify it with the deuteron. Bound states of spin zero and isospin one have not been observed, so we have to check which state is the ground state and which state has less energy than the state containing two nucleons (accounting for their rotational energy).

The quantum Hamiltonian is given by

$$H = \frac{1}{2} \left( \frac{J_1^2}{4M_0\mu^2} + \frac{J_2^2}{M_0 \left(4\mu^2 + \frac{R_0^2}{2}\right)} + \frac{(J_3 + I_3)^2}{\frac{1}{2}M_0R^2} + \frac{I_1^2 + I_2^2 + I_3^2}{4M_0\mu^2} \right) + 2M_0 + V_{min}. \quad (4.3.11)$$

In the base given by the three previous states, the Hamiltonian is diagonal, and the eigenvalues have been written in (4.2.56). We write them at leading orders, up to  $N^{-1}$  and  $\Lambda^{-1}$ .

$$E_D = 2M_0 - \frac{0.152N}{\Lambda} + \frac{1}{N} \left( 0.114 + \frac{1.236}{\Lambda} \right) + o(\Lambda^{-2}), \quad (4.3.12a)$$

$$E_{I_0} = 2M_0 - \frac{0.152N}{\Lambda} + \frac{1}{N} \left( 0.228 - \frac{0.667}{\Lambda} \right) + o(\Lambda^{-2}), \quad (4.3.12b)$$

$$E_{I_1} = 2M_0 - \frac{0.152N}{\Lambda} + \frac{1}{N} \left( 0.228 + \frac{0.902}{\Lambda} \right) + o(\Lambda^{-2}). \quad (4.3.12c)$$

We see that the rotational corrections to the masses are of order  $1/N$ , as in large  $N$  QCD. We also see that, at leading order in  $\Lambda$ , the deuteron state turns out to be the lowest energy state, while the  $|I_0\rangle$  and  $|I_1\rangle$  states have the same energy. The coefficients of the  $1/\Lambda$  terms indicate that we must have a pretty high  $\Lambda$  for this ordering to be true ( $\Lambda > 20$ ), but from the analytical forms (4.2.56) we can analytically see that this ordering holds for generic  $\Lambda$ . The large  $\Lambda$  limit suggests  $E_{I_1} > E_{I_0}$  everywhere, but the analytical forms give the opposite result, so we cannot extrapolate those values to physical  $\Lambda$ , but we need the full energies.

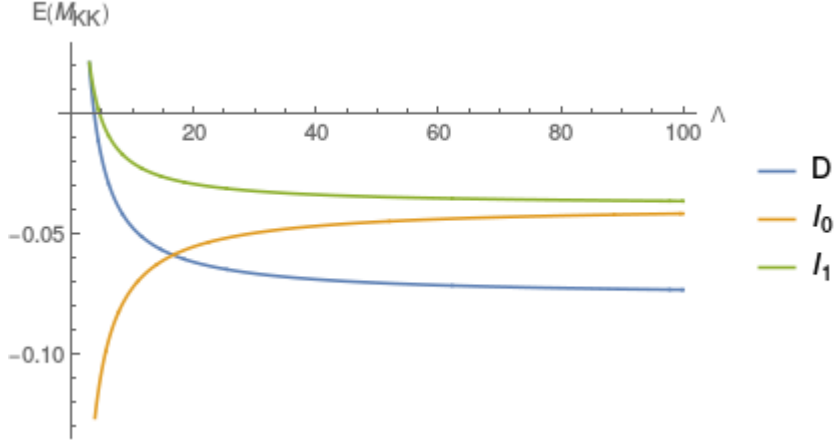


Figure 4.4: Binding energies with respect to  $\Lambda$  at order  $1/\Lambda$ . We see that the physical ordering  $E_D < E_{I_0} < E_{I_1}$  is inverted for  $\Lambda < 20$ . The separation between the asymptotic values is of order  $N^{-1}$ . At large  $\Lambda$ , all three states are bound.

We now calculate the binding energies. To this scope, we need the baryon mass formula to calculate  $E_1$  for the proton and neutron states, from (4.1.43). We approximate the rotational term to  $o(\Lambda^{-2})$  and keep the mass  $M_0$  as it is, as it will cancel in the binding energy.

$$E_1 = M_0 + \sqrt{\frac{15}{2}} \frac{1}{16N} - \frac{1}{2N\Lambda} + o(\Lambda^{-2}). \quad (4.3.13)$$

To leading orders, we get the differences

$$\Delta E_D = E_D - 2E_1 = -\sqrt{\frac{5}{6}} \frac{1}{4N} + \frac{2.236}{N\Lambda} - \frac{0.152N}{\Lambda} + o(\Lambda^{-2}), \quad (4.3.14a)$$

$$\Delta E_{I_0} = E_{I_0} - 2E_1 = -\sqrt{\frac{5}{6}} \frac{1}{8N} + \frac{0.333}{N\Lambda} - \frac{0.152N}{\Lambda} + o(\Lambda^{-2}), \quad (4.3.14b)$$

$$\Delta E_{I_1} = E_{I_1} - 2E_1 = -\sqrt{\frac{5}{6}} \frac{1}{8N} + \frac{1.902}{N\Lambda} - \frac{0.152N}{\Lambda} + o(\Lambda^{-2}). \quad (4.3.14c)$$

At leading order in  $\Lambda$ , every state is bound, but the deuteron state's binding energy turns out to be twice the binding energy of the other two states. We plot the binding energies in (4.4). For completeness, we plot in (4.5) the binding energies calculated from the full form of the zero modes contribution, that is exact.

We now consider the massive modes. At first, we consider the harmonic approximation, with frequencies given by (4.2.72). The contribution from those

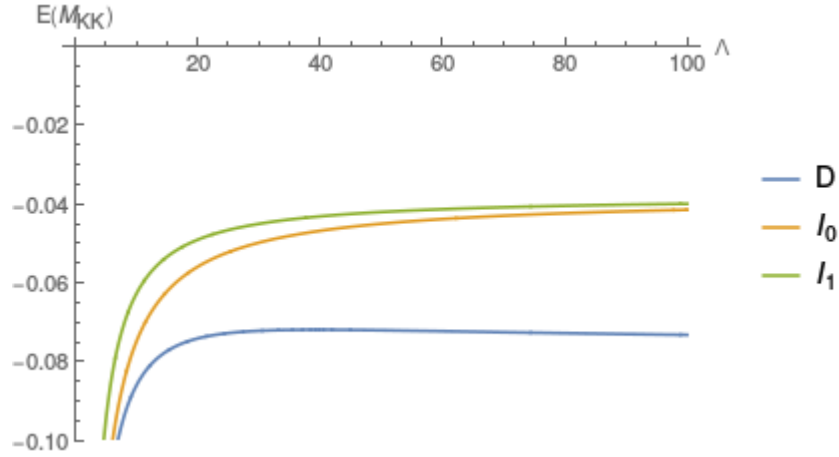


Figure 4.5: Binding energies with respect to  $\Lambda$ , with full form of the zero modes rotational energies. The physical ordering is always obeyed, and the states are always bound.

frequencies is given by

$$E_{mass} = \frac{\omega_1 + \omega_2 + \omega_3}{2} = \frac{1.063}{\sqrt{\Lambda}} + \frac{0.755}{\Lambda} + o\left(\Lambda^{\frac{3}{2}}\right). \quad (4.3.15)$$

We add this energy to the state energies and plot the result in (4.6). This quantum contribution goes to zero as  $\Lambda \rightarrow \infty$ , so it does not affect the asymptotic behavior. The contribution is great with respect to the other contributions: the result is that all three states are unbound for  $\Lambda \lesssim 200$ , while in the range  $200 \lesssim \Lambda \lesssim 800$  only the deuteron state is bound, while the other two states decay in two nucleons.

We now quantize the radial coordinate and neglect the other two massive coordinates. A generic state in the system can then be written as

$$|\psi\rangle = |f\rangle \times |i, i_3, j, j_3\rangle. \quad (4.3.16)$$

The radial dependence factorizes from the zero mode dependence, and we call  $f(\eta)$  the wavefunction relative to the radial coordinate. To factorize the  $\Lambda$  dependence, we take the mass  $M_0$  at leading order in  $\Lambda$ , discarding the  $\Lambda^0$  factor: this will produce a binding energy proportional to  $1/\Lambda$ , substituting  $V_{min}$ . To numerically solve the problem, we choose  $N = 3$ . We numerically solve with Mathematica the equation

$$\left(-\frac{1}{M_0}\partial_\eta^2 + V(\eta) - E\right) f(\eta) = 0, \quad f(0) = 0, \quad f'(0) = 1 \quad (4.3.17)$$

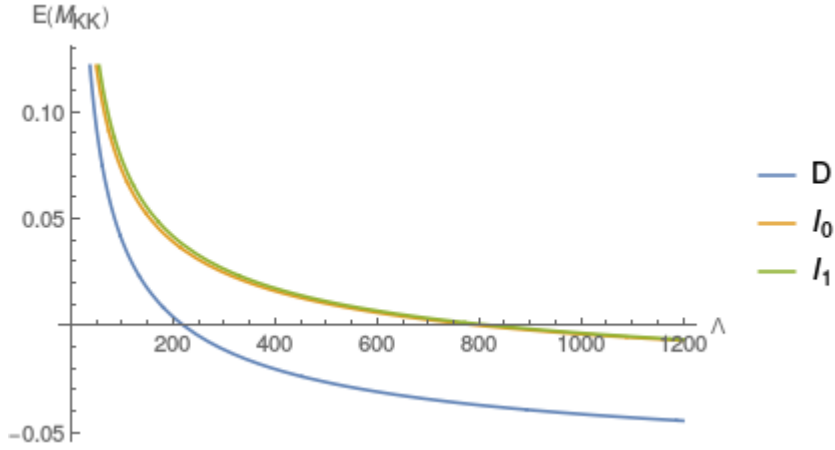


Figure 4.6: Addition of the massive modes. For small  $\Lambda$ , the result is inverted. In this graph the energies of  $I_0$  and  $I_1$  are approximatively equal.

(the value of  $f'(0)$  is chosen arbitrarily,  $f$  will need to be normalized) and then plot in (4.7) the asymptotic value of  $f$  (at a large  $\eta$ , say  $\eta = 1000$ ). If  $f$  represents a bound state solution, then it will have value approximatively zero at that value.

We plot the wavefunction in (4.8). We obtain a binding energy of

$$V_{quan} = -\frac{0.022}{\Lambda} + o(\Lambda^{-2}), \quad (4.3.18)$$

much smaller than the classical binding energy. We substitute  $V_{quan}$  to  $V_{min}$  in the binding energy formulas, with exact rotational energy contributions. We plot those results in (4.9).

We repeat the calculations adding the pion mass. In terms of the quark mass  $m$ , this is given by

$$m_\pi^2 = \frac{\sqrt{(2\pi)^5 \Lambda}}{N} m. \quad (4.3.19)$$

We choose to use  $m_\pi$  as input. As  $M_{KK} = 949 MeV$  and the experimental value is  $m_\pi = 135 MeV$ , we must have  $m_\pi = 0.142$ . The pion mass is then given by

$$m = 2 \cdot 10^{-4} \frac{N}{\sqrt{\Lambda}}. \quad (4.3.20)$$

The minimum separation changes to  $R_0 = 2.228$  and

$$V_{min} = -0.115 \frac{N}{\Lambda}. \quad (4.3.21)$$

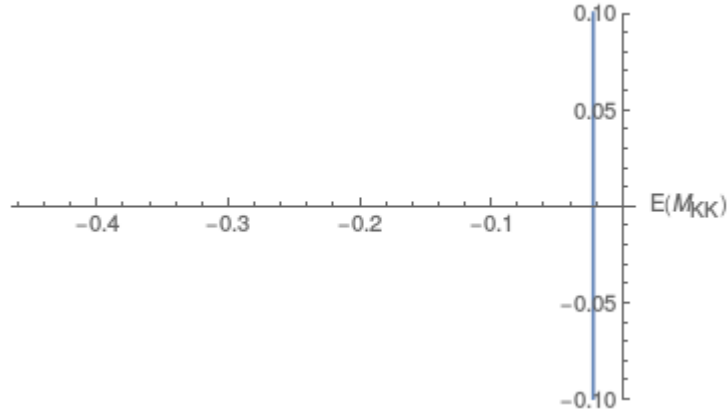


Figure 4.7: Asymptotic value of  $f(\eta)$  against the energy parameter  $E$ . Every crossing of the blue line indicates a bound state, and the energy of the bound state can be read from the intersection of the blue line with the  $x$  axis, that represents the energy. There is only one bound state, so no radial excited states are found.

As expected, with a massive pion the bound becomes weaker and the classical nucleon-nucleon distance becomes greater. The rotational energies are affected by the  $R_0$  change, but their form is the same. Quantizing the radial coordinate, we get a single radial bound state, of energy

$$V_{quan} = -\frac{0.003}{\Lambda} + o(\Lambda^{-2}). \quad (4.3.22)$$

We plot the same quantities as before: the binding energies of the states  $|D\rangle, |I_0\rangle, |I_1\rangle$ , in figure (4.10). The picture is similar to the final picture with no pion mass: the large  $\Lambda$  limit is the same, but at low  $\Lambda$  we get that the  $|I_1\rangle$  state is unbound, while the  $|I_0\rangle$  state and the  $|D\rangle$  state have similar energy.

Computing the deuteron energy with  $M_{KK} = 949 MeV$  at  $\Lambda_{SS}$  gives a binding energy of approximately  $15.72 MeV$ , against an experimental binding energy of  $2.26 MeV$ . The picture of this sector in the large  $\Lambda$  limit is complete: we have three states, and among those we have a state with the quantum numbers of the deuteron. Of all those states, the deuteron is the state at lowest energy, but all three are bound states (stable with respect to the decay in two separated nucleons). At  $\Lambda_{SS}$  one of the states is unbound when taking into account the quantization of the radial coordinate, but we still have two bound states. Our model predicts at  $\Lambda_{SS}$  the existence of an excited state of the deuteron, that is unphysical. As our



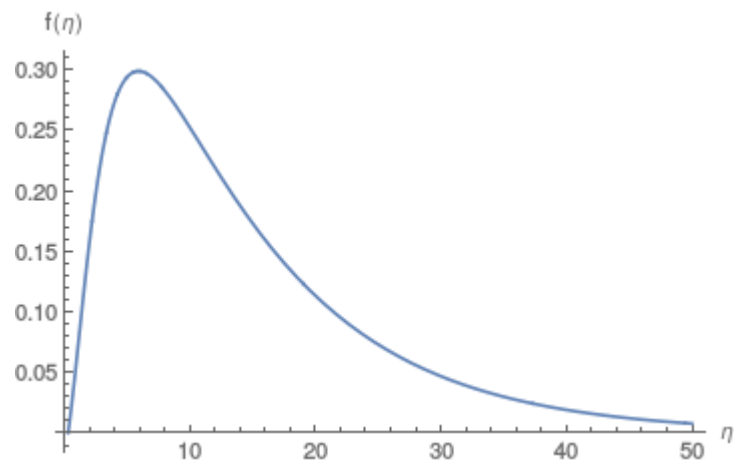


Figure 4.8: Wavefunction of the ground state of the quantum system (4.3.17), normalized. It has no nodes, confirming the fact that it is the ground state wavefunction.

results are valid in the large  $\Lambda$  limit, we need in principle to add corrections to our results. We wish for those corrections to remove the unphysical excited state from the bound states.

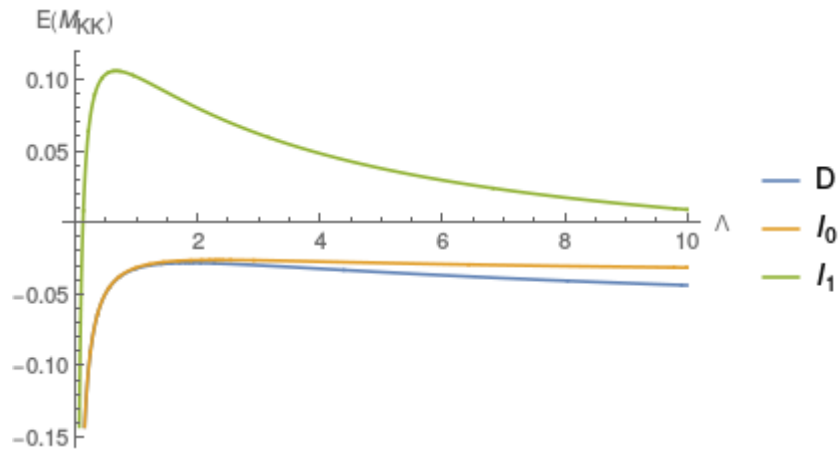


Figure 4.9: Binding energies with radial dependence quantized. The value of  $\Lambda$  that is used to fit the pion decay constant with  $M_{KK}$  to fit the  $\rho$  mass is  $\Lambda = 1.569$ : we see that in this zone the state  $|I_1\rangle$  is not bound.

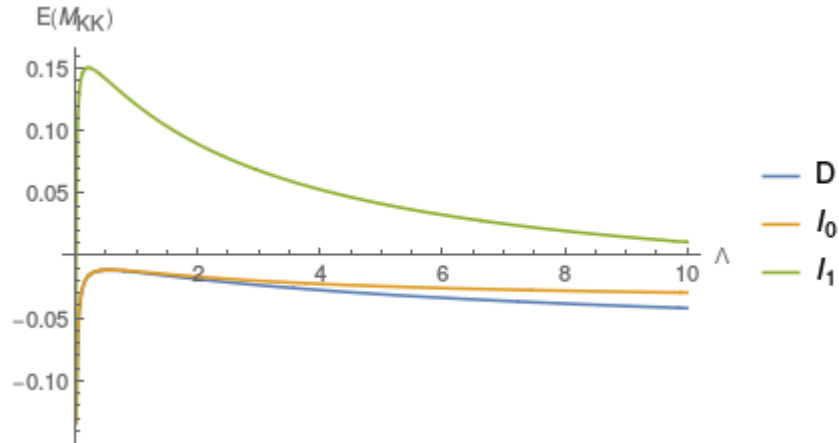


Figure 4.10: Binding energies with radial coordinate quantized and pion mass inserted. As before, we see great differences from the low  $\Lambda$  and high  $\Lambda$  zones. Here, we zoomed in the low  $\Lambda$  region: the results in the high  $\Lambda$  region are the same as before.

# Conclusions

In this thesis, we examined how the Sakai-Sugimoto model is built from the principles of holographic QCD, and what properties does it share with real QCD. We have stated (through proving or referencing) that Sakai-Sugimoto model contains both  $U(N) \simeq SU(N)$  color gauge fields (in the  $N \rightarrow \infty$  limit) and massless quark flavors in the fundamental representation of the gauge group, and has properties like quark confinement and the  $U(N_f)_L \times U(N_f)_R$  symmetry. From the action on the flavor branes, we have obtained a Yang-Mills  $U(N_f)$  theory in curved space, whose solitonic solutions can be interpreted as baryons.

We have summarized how in previous works [10, 24] this picture has already been used to obtain baryon properties from instantons of topological charge 1, and we have extended that description to instantons of topological charge 2. The action of the theory depends principally on two parameters:  $N$  (the number of colors) and  $\Lambda$  (the 't Hooft coupling): while  $N$  multiplies the whole effective action and has no effect on the classical theory (it is just a scaling factor),  $\Lambda^{-1}$  weights the Chern Simons term against the standard curved space Yang Mills action: in the static theory, the presence of the Chern Simons term generates a static field that is akin to the classical electrostatic field, while the topological charge acts as an electric charge: point particles with topological charge of the same sign repel, and the combined action of gravity and Chern-Simons term generate an (approximatively) spherical instanton of dimension  $\mu \propto \Lambda^{-\frac{1}{2}}$ . The classical solution has moduli space  $\mathbb{R}^3 \times SU(2)$ : those moduli can be interpreted as position and spin-isospin of the instanton. The classical picture of the baryon is then that of a spherical body (in three dimensions), with an orthonormal frame attached to it that represents its orientation, exactly as a rigid body. In the quantum picture, the body has an intrinsic, half integer spin, so states of proton and neutron can be interpreted as spinning spheres with  $j = 1/2$ .

We have extended this picture to the charge 2 sector, to build a quantum model of the deuteron. Working in the limit  $\Lambda \rightarrow \infty$ , we can place the instantons at a great spatial distance with respect to their sizes  $2\mu$ , and compute the static energy of the theory at order  $\Lambda^{-1}$ . We interpret the difference between this energy and the energy of two separated instantons as an interaction potential between the instantons and, using the arguments of Sakai and Sugimoto, we have interpreted this interaction potential as a classical potential for nucleon-nucleon interaction. We have shown that this potential depends on the relative distance and the relative orientations of the single instantons, so the potential splits in a central part and a non central part. We have identified a maximally attractive channel by fixing the relative orientation and shown the existence of a classical bound state, computing the separation between the two objects and the binding energy. The resulting picture of the two instantons system is analogous to a rigid rotator, composed of two masses attached at a fixed distance, with a rotational degree of freedom that is interpreted as the classical spin and an internal degree of freedom that is interpreted as an additional angular momentum, the isospin.

Quantization forces this rotor to rotate. We have found three rotational states (with spin zero and isospin one or viceversa) that are compatible with the requirement that the charge 1 instanton must be quantized as a fermion, and one of them has the quantum numbers of the deuteron: it has spatial spin one and isospin zero. We computed the rotational energies of the three states and found that they are of order  $1/N$ , while for  $\Lambda \rightarrow \infty$  the two other additional states become degenerate, with a rotational energy that is double the energy of the deuteron. Quantizing the distance between the instantons, we have found one radial bound state at  $N = 3$ . Confronting the energy of those three low lying states with the energy of infinitely separated instantons, we obtained that, in the large  $\Lambda$  limit, all of those three states are bound states.

Large  $N$  and large  $\Lambda$  limits are pretty different. We summarize them.

- In the large  $N$  limit, the picture is entirely classical: more and more states arise from quantizing the radial coordinate, until they form a continuum. The rotational energies of the states tend to zero in this limit, so the three states become degenerate. Mass and interaction of the baryons go to infinity as  $N$ , so the interaction term is always present. This picture is in agreement with large  $N$  QCD.

- In the large  $\Lambda$  limit, the baryons shrink to zero size, but acquire infinite mass. The interaction potential goes to zero, but at any  $\Lambda$  we can calculate  $\Lambda$  independent rotational energies, in the sector of charge 2 and in the sector of charge 1. The binding energy from the quantization of the radial coordinate goes to zero, but at any  $\Lambda$  at least one state is present. The energy of the non deuteron state tend to the same value, twice than the energy of the deuteron state. All states are stable and do not decay in two separated instantons.

Extrapolating to physical values can be challenging. Sakai-Sugimoto model requires us to take the limit  $N \rightarrow \infty$ , and the linear approximation that we have used to calculate the potential is equivalent to keeping only the dominant terms in the  $1/\Lambda$  expansion. As the physical value of  $\Lambda$  that is used extensively in literature (to fit the pion decay constant) is even smaller than the color number,  $\Lambda_{SS} = 1.569$ , we need in principle higher  $1/\Lambda$  corrections to the interaction potential to extrapolate numerical values that can be considered correct in the model and can be used to confront physical data. The picture in the large  $\Lambda$  limit is complete, but unrealistic, as it predicts two possible excited states of the deuteron that are still bound, but to  $\Lambda_{SS}$  we have that at least one of those two extra states become unbound, and can be considered a scattering state.

Quantization of the lowest energy states is not complete: we have neglected approximated moduli with light mass (instanton sizes and holographic direction position), that have been used in [24] to identify the extra states with nucleon resonances. The hope for the large  $\Lambda$  picture is that the contribution from those moduli effectively separates the deuteron state from the extra state at  $\Lambda_{SS}$ , giving a picture that is at least qualitatively correct. We also have to take into account the relative rotation between the two instantons.

This will be our concern for the near future, completing the picture with particular attention to the  $\Lambda_{SS}$  region. If we obtain satisfactory results, we would pursue a full simulation of the model, that is needed to get finite  $\Lambda$  reliable results. The model of Sakai and Sugimoto is very promising in getting a picture of non abelian gauge theories (at least, in the large  $N$  limit and low energy limit), because it is a top down approach that has very few parameters to fit, so we believe that it deserves a more in depth analysis.



# Notations and symbols

## Differential geometry

When describing spacetimes, we will always use a "mostly pluses" signature. This means that, in  $d + 1$  dimensions, we take the Minkowski metric to be

$$\eta = \text{diag}(-1, \underbrace{1, \dots, 1}_d)$$

Otherwise, for a generic metric tensor  $g$ , we always have  $\text{signature}(g) = (d, 1, 0)$ . We will always set  $c = 1$ . With this signature, for a  $d + 1$ -vector  $x^\mu$   $x^0$  is the "time" component, while  $x^i$  is the "space" vector. We always use Greek indexes to denote indexes from 0 to  $d$ , while we use Latin indexes for indexes from 1 to  $d$ , unless differently noted. On target manifolds, we will often use Latin indexes from the beginning of the alphabet ( $a, b, c, \dots$ ), but we won't be too pedantic and, when two indexes sets have the same range, we can freely exchange them. We denote the scalar product between two vectors with  $x \cdot y = x^\mu y_\mu$ , and use  $(x)^2$  as shorthand for  $x \cdot x$ .

The Levi Civita symbol  $\epsilon$  is normalized as  $\epsilon^{0123\dots d} = 1$ , this implies that  $\epsilon_{0123\dots d} = -1$ .

Wick rotation is defined as  $x_E^{d+1} = ix^0$  and  $x^i = x_E^i$ , replacing the 0 coordinate with a  $d + 1$  coordinate. After this rotation, the metric becomes the identity, with all plus signs. We always suppress the labels  $_E$  after Wick rotating. See the last section for indications about the ranges of the various kinds of indices. The  $\epsilon$  tensor is changed according to  $\epsilon_{123\dots d+1,E} = \epsilon_{0123\dots d}$ . The euclidean action is defined as  $-S_E = iS$ , after expressing  $S$  in terms of the Euclidean fields.

When indexes are repeated, sum convention is always intended, unless otherwise specified.

## Pauli matrices

We indicate the Pauli matrices as  $\sigma^a$  or  $\sigma^i$ , neglecting distinction between upper and lower indices, and use normalization  $\text{tr}[\sigma_a\sigma_b] = 2$  and  $[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c$ . An explicit representation is given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\sigma$  matrices obey  $\sigma_a\sigma_b = \delta_{ab}\mathbf{1} + i\epsilon_{abc}\sigma_c$ , where  $\mathbf{1}$  is taken as symbol for identity in all dimensions (in this case, it is the two dimensional identity).

## Fourier transformations

We define the Fourier transform of a function in  $\mathbb{R}^n$  as

$$\phi(p) = \int e^{-ip \cdot x} \phi(x) d^n x$$

and its inverse as

$$\phi(x) = \int e^{ip \cdot x} \phi(p) \frac{d^n p}{(2\pi)^n}$$

We always omit tildes on functions: the transformation is denoted by the argument  $p$ . The product  $p \cdot x$  is done with the metric of the space on which  $\phi$  is defined.

## Gauge theories

We define a gauge transformation on an arbitrary multiplet of fields  $\phi$  as

$$\phi \rightarrow e^{i\Lambda(x)}\phi = U(x)\phi$$



where  $\Lambda(x)$  is an arbitrary real function for  $U(1)$  symmetry, an  $N \times N$  matrix of the form  $\Lambda(x) = g_Y T^a \Lambda^a(x)$  with  $T^a$  the Hermitian generators of the representation of the group to which the multiplet  $\Phi$  belongs,  $\Lambda^a$  arbitrary real functions and  $g_Y$  is a coupling constant.  $\phi$  can be expressed as  $\phi = g_Y \phi^a T^a$ . The index  $a$  runs from 1 to the dimension of the Lie algebra of the group. In the case of the fundamental representation, the generators are denoted as  $\tau^a$  and normalized as  $\text{tr}(\tau^a \tau^b) = \frac{1}{2} \delta^{ab}$ . This means that, in  $SU(2)$ , the generators are given by  $\tau^a = \frac{\sigma_a}{2}$ . Any  $T^a$  obey the commutation relation  $[T^a, T^b] = i f^{abc} T^c$ , with the totally antisymmetric tensor  $f$  the same for all representations.

The gauge field is hermitian,  $A_\mu = A_\mu^\dagger$ , and for a fixed representation is decomposed in components through  $A_\mu = g_Y A_\mu^a T^a$ .  $A_\mu^a$  are real fields. The covariant derivative of fields in the adjoint representation (transforming as  $\phi \rightarrow U \phi U^\dagger$ ) is given by

$$D_\mu \phi = \partial_\mu \phi + i[A_\mu, \phi] = g_Y \partial_\mu \phi^a T^a + i g_Y^2 x A_\mu^a \phi^b [T^a, T^b]$$

The transformation rule for the gauge field is then

$$A_\mu \rightarrow U A_\mu U^\dagger + i \partial_\mu U U^\dagger$$

The field strength tensor is  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ . Pure gauge Yang Mills in curved space (metric  $g$ ) is then given by

$$S_{YM} = -\frac{1}{2g_Y^2} \int \sqrt{|g|} \text{tr}[F_{\mu\nu} F^{\mu\nu}] d^{d+1}x$$

$g$  is the metric determinant, always negative in Minkowski space, and the prefactor has been chosen to assure canonical normalization for the vector fields  $A_\mu^a$ . We perform Wick rotation on the gauge field by  $A_{0,E} = -iA_0$  and  $A_{i,E} = A_i$ : this means that  $F_{0i,E} = iF_{0i}$  and  $F_{ij,E} = F_{ij}$ .

## Indices range

During this thesis, we will have to do with manifolds of different dimensionality, and we will need to use many types of indices. When dealing with Minkowski space, we make the usual choice of labeling indices with greek letters, like  $\mu, \nu$ , going from 0 to 3. This convention persists when we Wick rotate, but after Wick

rotation the index range is from 1 to 4. The spatial part of Minkowski space is described through latin letters, like  $i, j$ , going from 1 to 3. In string theory, we have to deal with  $d$  dimensional spaces: in this case, we still use greek letters to cover the whole manifold (from 0 to  $d-1$ ), but latin letters indicate the use of light cone coordinates, and they run from 2 to  $d-1$ . While dealing with the AdS/CFT correspondence, we will use manifolds of the form  $\mathcal{M} \times S^n$ : in this case, we use greek letters to cover  $\mathcal{M}$  and latin letters to cover  $S^n$ . After Kaluza-Klein reduction, we always end up with a space with the topology of AdS: this can be seen as the standard space with an additional spatial dimension. In this case, we indicate the extra dimension with  $z$ , use capital greek letters  $\Xi, \Psi$  to cover the whole AdS space (from 0 to 4) and lowercase letters to exclude the non holographic part (from 0 to 3). If we want to exclude time, we use latin letters: uppercase  $I, J$  to include the holographic part (from 1 to 4), lowercase otherwise (from 1 to 3).

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