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De Concini and Procesi models of reflection groups and Coxeter groups

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Contents

Introduction

0.1 State of the Art

In 1995, in the fundamental article [8], De Concini and Procesi constructed *wonderful models* for the complement of a subspace arrangement in a given vector space. These models can be described in a combinatorial way and also by an explicit sequence of blow ups; they are smooth varieties in which the complement of these subspaces is unchanged but the family of subspaces is replaced by a divisor with normal crossing.

The interest in these varieties was at first motivated by an approach to Drinfeld's construction [17] of special solutions for the Khniznik - Zamolodchikov equation [39]. Then real and complex De Concini-Procesi models turned out to play a relevant role in several fields of mathematical research: subspace and toric arrangements [10], toric varieties [21], tropical geometry ([12], [22]), box splines and index theory [10], discrete geometry [23], moduli spaces of curves and configuration spaces [25].

The precursors of these models were Fulton and MacPherson's compactifications of classical configurations spaces of smooth algebraic varieties [25]. These compactifications show several properties similar to those characterizing wonderful arrangement models; for instance, the complement of the original configuration space is a normal crossing divisor. Also cardinal notions of building and nested sets, key elements in the combinatorics of arrangement models, emerge initially in the Fulton - MacPherson construction for configuration spaces, thus are inspired by the combinatorics of partition lattice.

Over the years, De Concini- Procesi arrangement models have inspired several more general constructions along the same lines of thought: for instance,

compactifications of conically stratified complex manifolds by MacPherson and Procesi [43], model constructions for mixed real subspace and halfspace arrangements and real stratified manifolds by Gaiffi [27] (that use spherical rather than classical blow ups), compactifications of open varieties by Hu [37] and compactifications of arrangements of subvarieties by Li [41].
From the point of view of homological algebra, *wonderful* models construction has made major contribution on an enduring open question in arrangement theory, since in [8] it was proven that combinatorial data of subspace arrangement determines the cohomology algebra of its complement. Previously, in 1980, Orlik and Solomon already showed an elegant description of the integral cohomology algebra of the hyperplanes arrangement complement in terms of the intersection lattice [44], while Goresky and MacPherson proved that integral cohomology modules of complements of subspace arrangements are determined by the intersection lattice [34].

In general, given a subspace arrangement, there are several De Concini-Procesi models related to it, depending on distinct sets of initial combinatorial data. Among these *building sets* there are always a minimal one and a maximal one with respect to inclusion: as a consequence there are always a minimal and a maximal De Concini-Procesi model.
In this thesis we will deal with minimal De Concini-Procesi models related to classical Coxeter arrangements as well as more general complex reflection groups, as described in the next section.

0.2 Structure of the Thesis

The first chapter is entirely dedicated to recall De Concini and Procesi original combinatorial and geometric constructions arising from subspace arrangements. We describe, given a finite family $\mathcal{K}$ of linear subspaces in a vector space V or more generally in a projective space, an explicit construction of a model for the complement $\mathcal{A}_{\mathcal{K}}$ of the union of the subspaces in the given family. This is a smooth irreducible variety with a proper map to V which is an isomorphism on the preimage of $\mathcal{A}_{\mathcal{K}}$ and such that the complement of this preimage is a divisor with normal crossings.

The general construction consists in forming the closure of the graph of the map

$$\rho : \mathcal{A}_{\mathcal{K}} \to V \times \prod_{A\in\mathcal{K}} \mathbb{P}(V/A)$$

where $\mathcal{K}$ is a family of subspaces of V, $\mathbb{P}(V/A)$ denotes the projective space of lines in V/A and the map from $\mathcal{A}_\mathcal{K}$ to $\mathbb{P}(V/A)$ is the restriction of the canonical projection $V/A \to \mathbb{P}(V/A)$. This map encodes the relative position of each point in the arrangement complement $\mathcal{A}_\mathcal{K}$ with respect to the intersection of subspaces in $\mathcal{K}$. We will call this variety $Y_\mathcal{K}$: it contains $\mathcal{A}_\mathcal{K}$ as open set and for suitable classes of families $\mathcal{K}$ it can be obtained by a sequence of blow up along smooth centers.
The combinatorics is rather complex and it is best described working in the dual space. In the dual let $\mathcal{C}$ be the family of subspaces dual to all the subspaces in $\mathcal{K}$ and their intersections. This family is clearly closed under sum. We say that a family $\mathcal{G} \subset \mathcal{C}$ is a *building set* in $\mathcal{C}$ if every element $X \in \mathcal{C}$ is the direct sum $X = \oplus_i Y_i$ of its maximal elements (i.e. maximal elements Y_i of $\mathcal{G}$ contained in X). Then we say that $X = \oplus_i Y_i$ is the decomposition of X relative to $\mathcal{G}$.
The main theorem states that, if $\mathcal{G}$ is a building set relative to $\mathcal{C}$, the variety $Y_\mathcal{G}$ is smooth and the complement in $Y_\mathcal{G}$ of the open set $\mathcal{A}_\mathcal{G}$ is a divisor with normal crossings. Moreover, one can build a sequence $\mathcal{G}_i$ of building sets (relative to different families $\mathcal{C}_i \subset \mathcal{C}$) so that $Y_{\mathcal{G}_0} = V$ and $Y_{\mathcal{G}_{i+1}}$ is obtained from $Y_{\mathcal{G}_i}$ by blowing up a smooth subvariety.
The last fundamental concept to study these models is the notion of a $\mathcal{G}$ *nested* set, i.e. a family $A_1, A_2, \ldots, A_k$ of elements of $\mathcal{G}$ with the property that, if $B_1, B_2, \ldots, B_h$ are taken out of this family and are pairwise non comparable elements, then they form a direct sum $X = \oplus_{i=1}^h B_i$ and this is the decomposition of X relative to $\mathcal{G}$.

In the second chapter we present main results about integer cohomology rings of De Concini-Procesi models of subspace arrangements.
Firs of all we recall that, in 1997, Yuzvinsky [52] has found bases for the $\mathbb{Z}-$module $H^*(Y_\mathcal{F}, \mathbb{Z})$ when $\mathcal{F}$ is the minimal building set which refines a hyperplane arrangement (i.e. the building set of irreducibles). We summarize a more general construction [26] that provides bases for the $\mathbb{Z}-$modules $H^*(D_\mathcal{S}, \mathbb{Z})$ and $H^*(Y_\mathcal{G}, \mathbb{Z})$, when $D_\mathcal{S}$ are the smooth irreducible divisors indexed by the elements $\mathcal{S} \in \mathcal{G}$ whose union forms the complement of $\mathcal{A}_\mathcal{K}$ in Y_G.
Then we specialize this cohomology basis to the case of classical Coxeter arrangements of type $A_n, B_n(= C_n), D_n$, reducing the computation to summation over trees in a particular combinatorial fashion.

In the third chapter we will consider the real or complexified braid arrange-

ment, i.e. the arrangement given by the hyperplanes defined by the equations $x_i - x_j = 0$, $1 \leq i < j \leq n$ in $\mathbb{R}^n$ or $\mathbb{C}^n$. It turns out that the minimal building set $\mathcal{F}_{A_{n-1}}$ that contains the lines in V^* that are the annihilators of the hyperplanes $x_i - x_j = 0$ is made by all the subspaces in V^* whose annihilators in V are described by equations like $x_{i_1} = x_{i_2} = \cdots = x_{i_k}$ $(k \geq 2)$. This gives a bijective correspondence between the elements of $\mathcal{F}_{A_{n-1}}$ and the subsets of $\{1, 2, \ldots, n\}$ of cardinality at least two. As immediate consequence, a $\mathcal{F}_{A_{n-1}}$-nested set $\mathcal{S}$ is represented by a set (which we still call $\mathcal{S}$) of subsets of $\{1, 2, \ldots, n\}$ with the property that any of its elements has cardinality ≥ 2 and if I and J belong to $\mathcal{S}$ then either $I \cap J = \emptyset$ or one of the two sets is included into the other.
We recall that the minimal projective De Concini-Procesi model of type A_{n-1} is isomorphic to the moduli space $\overline{M}_{0,n+1}$ of $(n+1)$-pointed stable curves of genus 0. This isomorphism carries on the cohomology of the models $Y_{\mathcal{F}_{A_{n-1}}}$ an hidden extended action of the symmetric groups S_{n+1} that has been studied by several authors ([33], [48], [19]). In addition to this natural S_{n+1} action, we present another hidden extended action of the symmetric group on the minimal models of a braid arrangement as described in [3]. Thanks to the combinatorial remark proven in [29], the symmetric group S_{n+k} acts by permutation on the set of $k-$codimensional strata of the minimal model of type A_{n-1}. This happens at a purely combinatorial level, and it does not correspond to a geometric action on the minimal model, nevertheless it give rise to an interesting permutation action on the elements of a basis of the integer cohomology of the complex minimal model. The splitting of these elements into orbits allows to write a generating formula for the Poincaré polynomials of the complex minimal models that is different from the one available in the literature (compare [26], [42] and [52] with [3]).
We follow [3] and we recall briefly here the combinatorial approach that we use: elements of integer cohomology basis can be presented by graphs that are oriented rooted trees on n leaves, with exponents attached to the internal vertices; in [29] a bijection between rooted trees with k internal vertices and the partitions of $\{1, 2, \ldots, n+k-1\}$ into k parts of cardinality ≥ 2 has been described. It turns out that, using this bijection, a new representation of the integer cohomology basis monomials is provided by partitions with exponents.
As an example we present an explicit construction of integer cohomology basis for the minimal complex model $Y_{\mathcal{F}_{A_6}}$ through the presentation of nested sets.

In the fourth and last chapter, we present De Concini-Procesi models as-

sociated with finite irreducible complex reflection groups, giving an account of the results in [31]. We recall that, according to Shephard - Todd classification [50], these are the groups $G(r,p,n)$, with $r,p,n \in \mathbb{Z}^+$ and $p|r$, plus 34 exceptional groups.
Then we extend the building and nested sets language to the more general case of model $Y_{\mathcal{F}_{G(r,p,n)}}$. Here, the combinatorial structure is richer: vertices of two types appear as well as weights attached to the vertices and the leaves. Even if the combinatorial picture is more complicated, also in this more general case exponential formulas for the generating functions of the Betti numbers have been obtained [31].
Finally, we generalize the model example presented in the third chapter in the case $Y_{\mathcal{G}_{(r,1,6)}}$ and we obtain in this exmaple formulas for dimension of non vanishing integer cohomology groups that can be compared with the results obtained before.

Chapter 1

The construction of De Concini-Procesi models

In [8] De Concini and Procesi constructed *wonderful models* for the complement of a subspaces arrangement in a vector space. This first chapter is a short survey on their original construction, therefore all of the proofs are omitted.

1.1 Basic Definitions

Let V be a finite dimensional vector space over an infinite field K and V^* its dual space. Given a finite family $\mathcal{K}$ of subspaces of V^*, for every $A \in \mathcal{K}$ consider its annihilator $A^\perp \subset V$. We also denote by $\mathbb{P}_A$ the projective space of lines in $V/A^\perp$, remarking that a basis of A gives a system of projective coordinates in $\mathbb{P}_A$.
Let $V_\mathcal{K} = \cup_{A\in\mathcal{K}} A^\perp$ be the union of all the subspaces $A^\perp$ and $\mathcal{A}_\mathcal{K}$ be the open set of V complement of $V_\mathcal{K}$. For every $A \in \mathcal{K}$ we have a rational map:

$$\pi_A : V - A^\perp \to V/A^\perp \to \mathbb{P}_A$$

defined outside of $A^\perp$ and thus we have a regular morphism $\mathcal{A}_\mathcal{K} \to \prod_{A\in\mathcal{K}} \mathbb{P}_A$. Finally we have an embedding

$$\rho : \mathcal{A}_\mathcal{K} \to V \times \prod_{A\in\mathcal{K}} \mathbb{P}_A$$

given by the inclusion on the first component and by the maps π_A on the other components.

Definition 1.1. We denote by $Y_\mathcal{K}$ the closure of the image $\rho(\mathcal{A}_\mathcal{K})$.

We first make some general remarks about these *wonderful models*:

1. From the construction of this variety it is clear that it does not depend on the 1-dimensional subspaces A since in this case $\mathbb{P}_A$ is a single point.

2. The projection $p : V \times \prod_{A \in \mathcal{K}} \mathbb{P}_A \to V$, restricted to $Y_{\mathcal{K}}$, is a proper birational map $p_{\mathcal{K}}$ which is an isomorphism on $\mathcal{A}_{\mathcal{K}}$.

3. Given two families $\mathcal{K}_1 \subset \mathcal{K}_2$ we have a canonical projection $p_{\mathcal{K}_1}^{\mathcal{K}_2} : Y_{\mathcal{K}_2} \to Y_{\mathcal{K}_1}$ extending the identity on $\mathcal{A}_{\mathcal{K}_2}$.

4. If f is a linear isomorphism of V, then f extends to as isomorphism $Y_{\mathcal{K}} \to Y_{f(\mathcal{K})}$, where $f(\mathcal{K}) = \{f(A) | A \in \mathcal{K}\}$; in particular the group of linear simmetries of $\mathcal{K}$ acts on $Y_{\mathcal{K}}$.

5. Given A minimal in $\mathcal{K}$ set $\mathcal{K}' := \mathcal{K} - A$ and define the family $\mathcal{K}(A)$ in V^*/A by

$$\mathcal{K}(A) := \{(B + A)/A | B \in \mathcal{K}'\}.$$

By identification we have made $(B+A/A^{\perp}) = B^{\perp} \cap A^{\perp}$, then $V_{\mathcal{K}'} \cap A^{\perp} = V_{\mathcal{K}(A)}$, $\mathcal{A}_{\mathcal{K}'} \cap A^{\perp} = \mathcal{A}_{\mathcal{K}(A)}$, $\mathcal{A}_{\mathcal{K}} = \mathcal{A}_{\mathcal{K}'} - A^{\perp}$. The closure of $\mathcal{A}_{\mathcal{K}(A)}$ in $Y_{\mathcal{K}}$ is isomorphic to $Y_{\mathcal{K}(A)}$ and we have a commutative diagram:

$$\begin{tikzcd}
Y_{\mathcal{K}(A)} \arrow[r, "i"] \arrow[d, "p_{\mathcal{K}}(A)"'] & Y_{\mathcal{K}'} \arrow[d, "p_{\mathcal{K}'}"] \\
A^{\perp} \arrow[r] & V
\end{tikzcd}$$

We start our analysis with a basic case given by the following definition:

Definition 1.2. A set $\mathcal{S}$ of subspaces in V^* is *nested* if, given any $U_1, \ldots, U_k \in \mathcal{S}$ pairwise non comparable, they form a direct sum $U = \oplus_i U_i$ and $U \notin \mathcal{S}$.

A nested set can be recursively constructed as follows. Choose subspaces A_i which form a direct sum. Then in each A_i choose a nested set $\mathcal{S}_i$ containing A_i. Finally set $\mathcal{S}$ to be the union of the $\mathcal{S}_i$.
Let $\mathcal{S}$ be a nested set of subspaces. For every non empty set $A \subset V^*$ the set of subspaces in $\mathcal{S} \cup \{V^*\}$ containing A is linearly ordered and non empty. Let be $p_{\mathcal{S}}(A)$ (or $p(A)$) to be the minimum of them.

Lemma 1.1. The set of $x \in V^* - \{0\}$ with $p_{\mathcal{S}}(x) = U$ equals $U - \cup_i U_i - \{0\}$, where U_i are the maximal proper subspaces of U in $\mathcal{S}$. Given a subspace G there is an $x \in G$ such that $p(x) = p(G)$.

Let $\mathcal{S}$ be a generical nested set of subspaces.

Definition 1.3.

1. A basis b of V^* is *adapted* to $\mathcal{S}$ if, for all $A \in \mathcal{S}$, the set
$$b_A := b \cap A = \{v \in b | p(v) \subset A\}$$
is a basis of A.

2. A marking of a basis b adapted to $\mathcal{S}$ is a choice, for all $A \in \mathcal{S}$, of an element $x_A \in b$ with $p(x_A) = A$.

Lemma 1.2. An adapted basis always exists and for all $A \in \mathcal{S}$, there exists a $x \in b_A$ such that $p(x) = A$.

We now define an order in a marked basis by setting $x \leq y$ if $p(x) \subset p(y)$ and y is marked. Under this ordering each subspace A of the nested set has as basis the set of elements $\{v \in b | v \leq x_A\}$.

As general digression we remark that, given a partial order μ on a finite set E, we can define a map $\rho_\mu : K^E \to K^E$ by the formula
$$a := \textstyle\prod_{b \geq a} u_b$$
where $a \in E$ are chosen as coordinates on the target space while u_a are coordinates on the source of the map. Assuming that the partial ordering is such that the elements greater than any given one a form a linearly ordered set, then the map ρ_μ is a birational morphism.

Consider now a space K^b with coordinates indexed by the basis elements and set $u_a := u_y$ where y is the marked element associated to A. Define $v := u_v \prod_{B \supset A} u_B$ if $A = p(V)$ and v is not marked, or $v := \prod_{B \supset A} u_B$ if $v = x_A$. This is the monomial map associated to the given ordering, and since b is a basis of V^*, we can consider it as map $\rho_\mu : K^b \to V$.

Proposition 1.3. The map ρ_μ restricts to an isomorphism between the open set where u_A are all different from 0 and the open set where v_A are all different from 0, and maps the hyperplane defined by $u_A = 0$ in to the subspace $A^\perp$.

To study $Y_{\mathcal{S}}$ choose a basis b adapted to $\mathcal{S}$, give it a marking and consider the map ρ_μ. We notice that the composition of ρ_μ with the rational map $\pi_A : V \to \mathbb{P}_A$ in the projective coordinates for $\mathbb{P}_A$ coming from the basis b_A of A, is given by the preceding formulas. Thus as monomials in the u_x, these coordinates are all divisible by the monomials expressing x_A. We finally deduce that:

Proposition 1.4. The map ρ_μ lifts to an open embedding into $Y_{\mathcal{S}}$.

We denote by $\mathcal{U}^b_{\mathcal{S}}$ the open set in $Y_{\mathcal{S}}$ given by the previous proposition and identify the restriction to $\mathcal{U}^b_{\mathcal{S}}$ of the projection from $Y_{\mathcal{S}}$ to V again with ρ_μ. The open set $\mathcal{U}^b_{\mathcal{S}}$ depends only on the marked elements of the basis and not on the full basis.

Given $\mathcal{S}$, one possible way to select adapted marked bases is the following. Choose for every $B \in \mathcal{S}$ a basis b^B of B of vectors non contained in any $C \in \mathcal{S}$ properly contained in B. For every B choose a vector $x_B \in b^B$. These vectors are linearly independent and thus can be completed to a basis b adapted to $\mathcal{S}$ in which they are marked. As already mentioned, the corresponding open set $\mathcal{U}^b_{\mathcal{S}}$ does not depend on the way in which we complete the basis. Finally we get a finite family $\mathcal{M}$ of open sets.

Proposition 1.5.

1. The variety $Y_{\mathcal{S}}$ is covered by the open sets $\mathcal{U}^b_{\mathcal{S}}$ in the family $\mathcal{M}$.

2. Given a minimal element $A \in \mathcal{S}$ and let $\mathcal{S}' = \mathcal{S} - \{A\}$. Then $Y_{\mathcal{S}}$ it is the blow up of $Y_{\mathcal{S}'}$ along the proper transform Z_A of the subspace $A^\perp$ which is a smooth subvariety.

3. Consider $\mathcal{A} = V - \cup_{A\in\mathcal{S}} A^\perp$ embedded as an open set in $Y_{\mathcal{S}}$. Then $Y_{\mathcal{S}} - \mathcal{A}$ is a divisor with normal crossing with smooth irreducible components $D^{\mathcal{S}}_A$ for $A \in \mathcal{S}$.

4. All intersections of the divisors $D^{\mathcal{S}}_A$ are irreducible.

1.2 Combinatorics of arrangements

Let $\mathcal{C}$ denote a finite set of non zero subspaces of V^* closed under sum.

Definition 1.4. Given a subspace $U \in \mathcal{C}$ a *decomposition* of U is a collection of non zero subspaces $U_1, \ldots, U_k \in \mathcal{C}$ with $U = U_1 \oplus \ldots \oplus U_k$ and, for every subspace $A \subset U$ in $\mathcal{C}$, also $A \cap U_1, \ldots, A \cap U_k$ lie in $\mathcal{C}$ and $A =$

$(A \cap U_1) \oplus \ldots \oplus (A \cap U_k)$.
If a subspace does not admit a decomposition it is called irreducible. The set of all irreducible subspaces of $\mathcal{C}$ is denoted $\mathcal{F}_{\mathcal{C}}$.

Observation 1.1. One can prove that every subspace $U \in \mathcal{C}$ has a unique decomposition $U = \oplus_{i=1}^k U_i$ into irreducible subspaces. In addition, if $A \subset U$ is irreducible, then $A \subset U_i$ for some i.

Observation 1.2. As an example, let us consider a root system Φ in V (a real or complex vector space) and its associated root arrangement: $\mathcal{A}^{\perp}$ is the hyperplane arrangement provided by the hyperplane orthogonal to the roots in Φ. In this case the building set of irreducibles is the set whose elements are the subspaces spanned by the irreducible root subsystems of Φ.

Proposition 1.6. Given a direct sum $U = \oplus_{i=1}^k U_i$, this is a decomposition if and only if, for every irreducible A, $A \subset U$ implies that $A \subset U_i$ for some i.
Given two irreducible subspaces A, B then either $A + B$ is irreducible or $A \oplus B$ is an irreducible decomposition.
If a subspace M is sum of irreducible subspaces M_i, its irreducible components are also sum of subfamilies of the family M_i.

Let $\mathcal{G}$ be a set of subspaces in V^*. Set $\mathcal{C}_{\mathcal{G}}$ the set of subspaces which are sums of subspaces in $\mathcal{G}$ and $\mathcal{F}_{\mathcal{G}} = \mathcal{F}_{\mathcal{C}_{\mathcal{G}}}$ the irreducibles in $\mathcal{C}_{\mathcal{G}}$.

Theorem 1.7. The following three conditions on $\mathcal{G}$ are equivalent:

1. $\mathcal{G}$ satisfies:
   (a) $\mathcal{G} \supset \mathcal{F}_{\mathcal{G}}$.
   (b) If $A, B \in \mathcal{G}$, $A = \oplus_{i=1}^t F_i$ is the irreducible decomposition of A in $\mathcal{F}_{\mathcal{G}}$ and $B \supset F_i$ for some i, then $A + B \in \mathcal{G}$.
2. Every element $C \in \mathcal{C}_{\mathcal{G}}$ is the direct sum $C = G_1 \oplus \ldots \oplus G_k$ of the maximal elements $G_i \in \mathcal{G}$ contained in C.
3. $\mathcal{G}$ satisfies
   (a) $\mathcal{G} \supset \mathcal{F}_{\mathcal{G}}$
   (b) If $A, B \in \mathcal{G}$ and $A + B$ is not a decomposition, then $A + B \in \mathcal{G}$.

Definition 1.5. A set $\mathcal{G}$ satisfying the previous three equivalent conditions will be called a *building* set.

The following proposition gives us some fundamental examples of building sets.

Proposition 1.8. Let $\mathcal{C}$ be a set of non zero subspaces closed under sum. Then

1. $\mathcal{C}$ is a building set.

2. $\mathcal{F}_{\mathcal{C}}$ is a building set.

3. Let $\mathcal{G} \subset \mathcal{F}_{\mathcal{C}}$ be a subset with the property that, if $X \in \mathcal{C}_{\mathcal{G}}$, then all its irreducible components in $\mathcal{F}_{\mathcal{C}}$ lie in $\mathcal{G}$. Then $\mathcal{G}$ is a building set.

4. Let $A \in \mathcal{F}_{\mathcal{C}}$ be a minimal element. Denote with $\phi : V^* \to V^*/A$ the quotient morphism, and consider the set $\overline{\mathcal{G}} = \{\phi(B) | B \in \mathcal{F}_{\mathcal{C}} - \{A\}\}$. Then

   (a) the set $\mathcal{H} = \mathcal{F}_{\mathcal{C}} - \{A\}$ is a building set;

   (b) if $\phi(G) \in \mathcal{F}_{\mathcal{G}}$ then either G is irreducible in $\mathcal{C}$ or $G \supset A$ and there is a irreducible $G_0 \subset G$ such that $G = G_0 \oplus A$ is the irreducible decomposition of G.

   (c) $\overline{\mathcal{G}}$ is a building set in V^*/A.

In general, given a set of subspaces $\mathcal{G}$, there are different building sets $\mathcal{B}$ of subspaces of V^* such that $\mathcal{C}_{\mathcal{B}} = \mathcal{C}_{\mathcal{G}}$; if we order by inclusion the collection of such sets, it turns out that the minimal element is $\mathcal{F}_{\mathcal{G}}$ and the maximal one is $\mathcal{C}_{\mathcal{G}}$.

Definition 1.6. Let $\mathcal{G}$ be a building set. A subset $\mathcal{S} \subset \mathcal{G}$ will be called *nested relative to* $\mathcal{G}$ or $\mathcal{G}$-nested if

1. $\mathcal{S}$ is nested.

2. Given a subset $\{A_1, \ldots, A_h\}$ of pairwise non comparable elements in $\mathcal{S}$, then $C = \oplus_{i=1}^{h} A_h$ is the decomposition of C in $\mathcal{G}$.

We remark that given a nested set $\mathcal{S}$, the minimal set of subspaces $\mathcal{C}$ closed under sum and containing $\mathcal{S}$ is formed by the direct sums of the families of non comparable elements of $\mathcal{S}$. Furthermore $\mathcal{S}$ coincides with the set of irreducible elements of $\mathcal{C}$.

1.3 The Model $Y_{\mathcal{G}}$

Let us now take a building set $\mathcal{G}$ and let $\mathcal{A}_{\mathcal{G}} = \mathcal{A}$ be the usual complement. As in the previous sections we take the variety $Y_{\mathcal{G}}$ as the closure of the embedding of $\mathcal{A}$ in $V \times \prod_{G \in \mathcal{G}} \mathbb{P}_G$. Take a $\mathcal{G}$-nested set $\mathcal{S}$ and a marked basis b adapted to it. Let $p(X) = p_{\mathcal{S}}(X)$ for any non zero subset $X \subset V$, and $\rho_m : K^b \to V$ the morphism associated to $\mathcal{S}$ and B.

Lemma 1.9.
1. Given any $x \in V^* - \{0\}$, suppose $A = p(x) \in \mathcal{S}$. Then $x = x_A P_x(u_v)$, where P_x is a polynomial depending only on the variables u_v, $v < x_A$.
2. If $\mathcal{S}$ is a maximal nested and $G \in \mathcal{G}$ then $p(G) = A \in \mathcal{S}$ and there is an $x \in G$ with $A = p(x)$ such that, writing $x = x_A P_x(u_v)$, P_x does not vanish in 0.

From the previous lemma, for a $G \in \mathcal{G}$ we shall define polynomials $P_x^G(u)$ by the formula $x = x_A P_x^G(u)$. Let Z_G be the subvariety in K^b defined by the vanishing of these polynomials. It is defined in such a way that the map $K^b \to V \to V/G^\perp$ can be composed in $K^b - Z_G$ with the rational map $V/G^\perp \to \mathbb{P}(V/G^\perp) = \mathbb{P}_G$ giving a regular morphism.

Definition 1.7. Given a $\mathcal{G}$ nested set $\mathcal{S}$ define the open set $\mathcal{U}_{\mathcal{S}}^b$ as the complement in K^b of the union of all the varieties $Z_G, G \in \mathcal{G}$.

On the set $\mathcal{U}_{\mathcal{S}}^b$ are defined all the rational morphism to $\mathbb{P}_G$ and so we get an embedding $j_{\mathcal{S}}^b$ of $\mathcal{U}_{\mathcal{S}}^b$ in $Y_{\mathcal{G}}$. One can now prove

Theorem 1.10.
1. The map $j_{\mathcal{S}}^b$ is an open embedding.
2. $Y_{\mathcal{G}} = \cup_{\mathcal{S}} j_{\mathcal{S}}^b(U_{\mathcal{S}}^b)$. In particular $Y_{\mathcal{G}}$ is smooth.
3. Set $D_{\mathcal{S}}^b$ equal to the divisor defined by $\prod_{A \in \mathcal{S}} u_A = 0$. Set $D = \cup_{\mathcal{S}} j_{\mathcal{S}}^b(D_{\mathcal{S}}^b)$. Then $A = Y_{\mathcal{G}} - D$ and D is a divisor with normal crossing.

The next more precise statement will be useful in cohomology computations.

Theorem 1.11.
1. The complement D of $\mathcal{A}$ in $Y_{\mathcal{G}}$ is the union of smooth irreducible divisors D_G indexed by elements $G \in \mathcal{G}$, where D_G is the unique irreducible component in D such that $\delta(D_G) = G^\perp$ with $\delta : Y_{\mathcal{G}} \to V$ is the projection map.
2. The divisors $D_{A_1}, \dots, D_{A_n}$ have nonempty intersection if and only if the set $S = \{A_1, \dots, A_n\}$ is $\mathcal{G}$-nested. In this case the intersection is transversal and we obtain a smooth irreducible variety $D_S = \cap_{i=1}^n D_{A_i}$.
3. Let G a minimal element in $\mathcal{G}$ and let $\mathcal{G}' = \mathcal{G} - \{G\}$. Then $Y_{\mathcal{G}}$ is obtained from $Y_{\mathcal{G}'}$ by blowing up the proper transform T_G of $G^\perp$ and T_G is isomorphic to the variety $Y_{\overline{\mathcal{G}}}$ where $\overline{\mathcal{G}}$ is induced in V^*/G by $\mathcal{F}_{\mathcal{G}}$ as in Proposition 1.8.

1.3.1 Projective arrangements

Associated to a nonempty family $\mathcal{K}$ of non zero subspaces in V^* one can also consider the configuration of linear subspaces $\mathbb{P}(A^\perp)$ in $\mathbb{P}(V)$. As in the first section we set $\overline{\mathcal{V}} := \cup_{A\in\mathcal{K}}\mathbb{P}(A^\perp)$ and $\overline{\mathcal{A}}_{\mathcal{K}}$ the open set of $\mathbb{P}(V)$ complement of $\overline{\mathcal{V}}$.
Using the same notation, the multiplicative group K^* acts on $\mathcal{A}_{\mathcal{K}}$ and $\overline{\mathcal{A}}_{\mathcal{K}} = \mathcal{A}_{\mathcal{K}}/K^*$. The regular morphism $\mathcal{A}_{\mathcal{K}} \to \prod_{a\in\mathcal{K}}\mathbb{P}_A$ is constant on K^* orbits and we get an induced morphism $\overline{\mathcal{A}}_{\mathcal{K}} \to \prod_{A\in\mathcal{K}}\mathbb{P}_A$, its graph is a closed subset of $\overline{\mathcal{A}}_{\mathcal{K}} \times \prod_{A\in\mathcal{K}}\mathbb{P}_A$ which embeds as open set into $\mathbb{P}(V) \times \prod_{A\in\mathcal{K}}\mathbb{P}_A$. Finally we have an embedding

$$\rho : \overline{\mathcal{A}}_{\mathcal{K}} \to \mathbb{P}(V) \times \textstyle\prod_{A\in\mathcal{K}}\mathbb{P}_A.$$

Definition 1.8. We let $\overline{Y}_{\mathcal{K}}$ to be the closure of the image of $\overline{\mathcal{A}}_{\mathcal{K}}$ under ρ.

Consider the projection $p : Y_{\mathcal{K}} \to V$ and let $Y^0_{\mathcal{K}} = Y_{\mathcal{K}} - p^{-1}(0)$ be an open set in $Y_{\mathcal{K}}$; it is K^* stable and it equals the closure of $\mathcal{A}_{\mathcal{K}}$ in the embedding $\mathcal{A}_{\mathcal{K}} \to V - \{0\} \times \prod_{A\in\mathcal{K}}\mathbb{P}(A)$. Furthermore, since $V - \{0\} \times \prod_{A\in\mathcal{K}}\mathbb{P}_A/K^* = \mathbb{P}(V) \times \prod_{A\in\mathcal{K}}\mathbb{P}_A$ it is possible to show that $\overline{Y}_{\mathcal{K}} = \overline{Y}^0_{\mathcal{K}}/K^*$.

We finally apply all this to the case of a building set $\mathcal{G}$ containing V^* and deduce immediately

Theorem 1.12. 1. $\overline{Y}_{\mathcal{G}}$ is a smooth projective and irreducible variety.

2. $Y_{\mathcal{G}}$ is the total space of a line bundle on $p^{-1}(0) = D_{V^*}$ and $\overline{Y}_{\mathcal{G}}$ is isomorphic to $p^{-1}(0) = D_{V^*}$.

3. The morphism $p : \overline{Y}_{\mathcal{G}} \to \mathbb{P}(V)$ is surjective and restricts to an isomorphism on $\overline{\mathcal{A}}_{\mathcal{G}}$.

4. $\overline{D} = \overline{Y}_{\mathcal{G}} - \overline{\mathcal{A}}_{\mathcal{G}}$ is a divisor with normal crossing. The irreducible components of $\overline{D}$ are smooth and in one to one correspondence with the elements $F \in \mathcal{G} - \{V^*\}$.

5. Given a subset $\mathcal{S} \subset \mathcal{G} - \{V^*\}$ the corresponding divisors have nonempty intersection if and only if $\mathcal{S}$ is nested.

An explicit description in local coordinates is the following.

Take $\mathcal{S}$ nested and consider a marked basis b and the associated open set $U^b_{\mathcal{S}}$. By the explicit description of the coordinates u_v, we immediately see that $U^b_{\mathcal{S}}$ is K^* stable: the coordinates u_v for v not maximal are fixed, while the maximal coordinate x_{V^*} is multiplied by the scalars in K^*. If the nested set is maximal contained in a building set $\mathcal{G}$ we can consider the open set $\mathcal{U}_{\mathcal{S}}^b$ defined as the complement of the union of the varieties Z_G of equations P_x, $x \in G$.

We end this section by remarking that if $\mathcal{G}$ is a building set in V^*, then the set $\mathcal{G}'$ of subspaces in $V^* \oplus K$ formed by the subspaces $H \oplus \{0\}$, $h \in \mathcal{G}$, and the line $L = \{0\} \oplus K$ is again a building set and the complement of the union of the corresponding linear spaces in $\mathbb{P}(V \oplus K)$ equals the open set $\mathcal{A}_{\mathcal{G}}$. Thus applying the above results, we obtain a compactification of $\mathcal{A}_{\mathcal{G}}$ in which the complement of $\mathcal{A}_{\mathcal{G}}$ is a divisor with normal crossing whose irreducible components are in bijection with the subset $\mathcal{S} \subset \mathcal{G}'$ which are either $\mathcal{G}$-nested subsets or are such that $\mathcal{S} - \{L\}$ is $\mathcal{G}$-nested.

1.4 Examples on Root Systems

First of all we need to recall some basic terminology, in particular as it concerns the combinatorial data of an arrangement.
The combinatorial data associated with an arrangement $\mathcal{K} = \{U_1, U_2, \ldots, U_n\}$ is recorded in a partially order set, the *intersection lattice* $\mathcal{L} = \mathcal{L}(\mathcal{K})$: the set of intersections of subspaces in $\mathcal{K}$ ordered by reversed inclusion. We adopt terminology from the theory of partially order sets and denote the unique minimum in $\mathcal{L}(\mathcal{K})$ (the vector space V corresponding to the empty intersection) by $\hat{0}$ and the unique maximum element (the overall intersection of subspaces in $\mathcal{K}$) by $\hat{1}$. In the general theory the elements of the intersection lattice are labeled by the codimension of the corresponding intersection. For arrangements of hyperplanes, this information is registered in the rank function of the lattice: the codimension of an intersection X corresponds to the number of elements in a maximal chain in the interval $(\hat{0}, X]$ in $\mathcal{L}(\mathcal{K})$.
Furthermore, we can consider the *order complex* $\Delta(\mathring{\mathcal{L}})$ of the interior part $\mathring{\mathcal{L}} = \mathcal{L} \setminus \{\hat{0}, \hat{1}\}$ of the intersection lattice; it is the abstract simplicial complex formed by the linearly ordered subsets in $\mathring{\mathcal{L}}$. Besides $\Delta(\mathring{\mathcal{L}})$, we will refer to the cone over $\Delta(\mathring{\mathcal{L}})$ obtained by extending the linearly ordered sets in $\mathring{\mathcal{L}}$ by the maximal element $\hat{1}$ in $\mathcal{L}$: we will denote this complex by $\Delta(\mathcal{L} \setminus \{\hat{0}\})$.

Example 1.1. The arrangement A_{n-1}, given by the hyperplanes

$$H_{ij} = x_i = x_j, \qquad \text{for } 1 \leq i < j \leq n,$$

in a real n-dimensional vector space is called the (real) $(n-1)$-*rank braid arrangement.* There is a natural complex version of this braid arrangement: it consists of hyperplanes H_{ij} in $\mathbb{C}^n$ given by the same linear equations. At

times, we will use the notation $A_{n-1}^{\mathbb{R}}$, $A_{n-1}^{\mathbb{C}}$ if we want to mark the real or complex setting.
First of all, we observe that the diagonal $\Delta = \{x \in \mathbb{K}^n | x_1 = \ldots = x_n\}$ is the intersection of all hyperplanes in A_{n-1}. One can consider A_{n-1} as an arrangement in real or complex $(n-1)$-dimensional space $V = \mathbb{K}^n/\Delta$ without changing the overall setting of the complement.
As the intersection lattice of the braid arrangement A_{n-1} we recognize the *partition lattice* Π_n, the set of partitions of $\{1, 2, \ldots, n\}$ ordered by reversed refinement. The correspondence to intersections in the braid arrangement can be described as follows: the parts of a partition correspond to sets of coordinates with identical entries, thus to the set of points in the corresponding intersection of hyperplanes.
In Figure 1.1 we sketch the real braid arrangement A_2 in $V = \mathbb{R}^3/\Delta$, its intersection lattice Π_3, and the order complex $\Delta(\Pi_3 \setminus \{\hat{0}\})$, denoting partitions in Π_3 by their non-trivial parts. The depicted complex is a cone over $\Delta(\overline{\Pi_3})$.

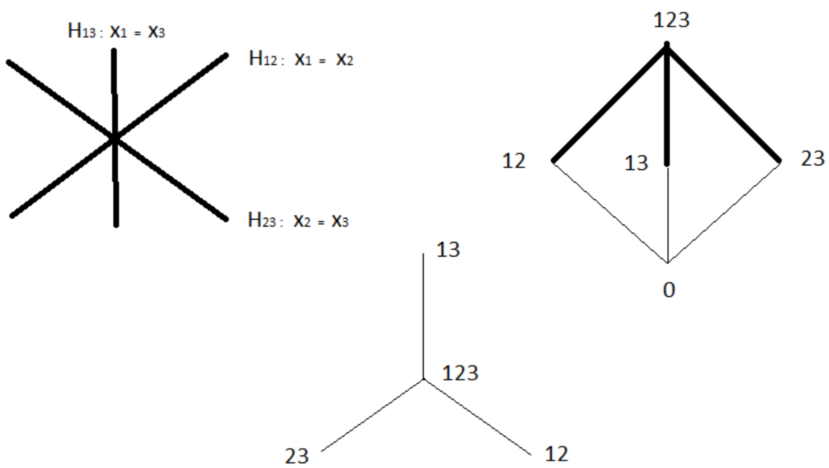

Figure 1.1

Example 1.2. Choosing the maximal building set in the partition lattice Π_n, we obtain the order complex $\Delta(\Pi_n \setminus \{\hat{0}\})$ as the associated complex of nested sets.
Instead, the minimal building set $\mathcal{G}_{\min}$ in Π_n is given by partitions with exactly one part of size larger or equal to 2. We can identify these partitions with subsets of $\{1, 2, \ldots, n\}$ of size larger or equal to 2. A collection of such subsets is nested if and only if for any pair of subsets they are either disjoint or one is contained in the other.

For the rank 3 partition lattice Π_3, it is easy to observe that maximal and minimal building sets coincide, $\mathcal{G} = \Pi_3 \setminus \{\hat{0}\}$. The nested set complex $\mathcal{N}(\Pi_3, \mathcal{G})$ is the order complex $\Delta(\Pi_3 \setminus \{\hat{0}\})$ represented in Figure 1.1.
For the rank 4 partition lattice Π_4, nested set complexes for the minimal $\mathcal{N}(\Pi_4, \mathcal{G}_{\min})$ and maximal $\mathcal{N}(\Pi_4, \mathcal{G}_{\max})$ building set are depicted in Figures 1.2 and 1.3: recall that the complexes are 2-dimensional cones, both with maximum 1234 over $\Delta(\mathring{\Pi}_4)$, and we only draw their bases for graphics reasons.

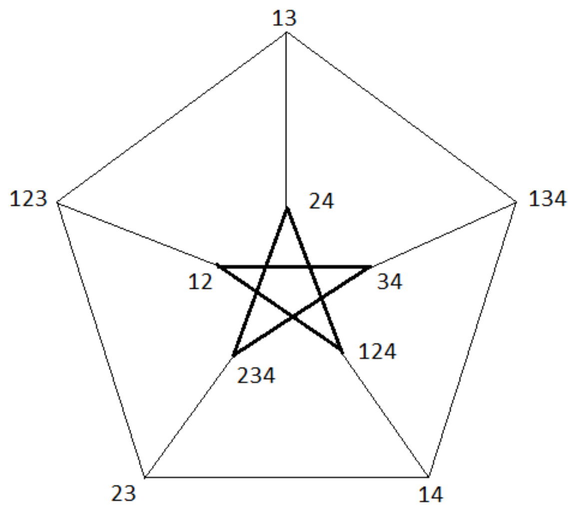

Figure 1.2: The nested set complex $\mathcal{N}(\Pi_4, \mathcal{G}_{\min})$

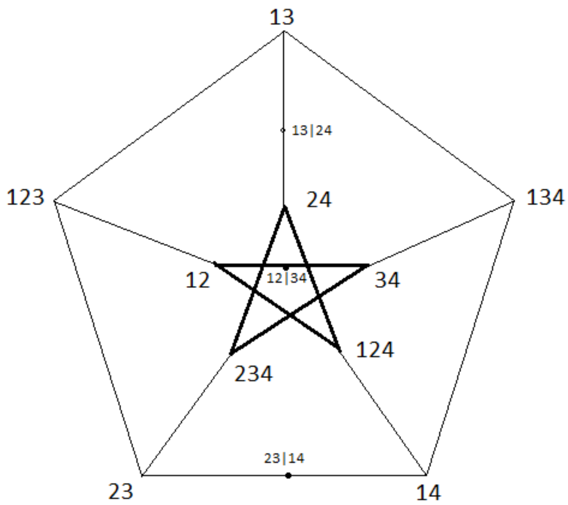

Figure 1.3: The nested set complex $\mathcal{N}(\Pi_4, \mathcal{G}_{\max})$

Chapter 2

The cohomology

We are now interested in the integer cohomology rings of $Y_{\mathcal{G}}$ and of the subvarieties D_S. They have been studied in [8], where presentations as quotients of polynomial algebras have been obtained.

2.1 Cohomology rings

First of all suppose to fix a $\mathcal{G}$-nested set $S \subset \mathcal{G}$. Let us take a subset $\mathcal{H} \subset \mathcal{G}$ such that there is an element $B \in \mathcal{G}$ satisfying $A \subsetneq B$ for all $A \in \mathcal{H}$. Set $S_B = \{A \in S : A \subsetneq B\}$. As in [8], we define the nonnegative integer $d^S_{\mathcal{H},B}$.

Definition 2.1. Define

$$d^S_{\mathcal{H},B} = \dim B \text{ - } \dim \ (\sum_{A \in \mathcal{H} \cup S_B} A).$$

Notice that $d^S_{\mathcal{H},B} \geq 0$ and, if $S' \supset S$ are two $\mathcal{G}$-nested sets $d^S_{\mathcal{H},B} \geq d^{S'}_{\mathcal{H},B}$. To these data we associate the polynomial in $\mathbb{Z}[c_A]_{A \in \mathcal{G}}$ given by

$$P^S_{\mathcal{H},B} = (\prod_{A \in \mathcal{H}} c_A)(\sum_{C \supset B} c_C)^{d^S_{\mathcal{H},B}}.$$

We let I_S be the ideal in $\mathbb{Z}[c_A]$ generated by these polynomials, for fixed S as $\mathcal{H}, B$ vary. We notice again that, if $S' \supset S$ sre two $\mathcal{G}$-nested set, the polynomial $P^{S'}_{\mathcal{H},B}$ divides $P^S_{\mathcal{H},B}$, so $I'_S \supset I_S$ and also we can assume that $\mathcal{H} \cap S_B = \emptyset$.

In [8] the cohomoolgy of the varieties D_S is computed as follows.

Theorem 2.1. The natural map

$$\phi_S : \mathbb{Z}[c_A] \to H^*(D_S, \mathbb{Z})$$

defined by sending c_A to the cohomology class $[D_A]$ associated to the divisor D_A restricted to D_S, induces an isomorphism between $H^*(D_S,\mathbb{Z})$ and $\mathbb{Z}[c_A]/I_S$. In particular, keeping the above definitions also in the case $S=\emptyset$, we obtain

$$H^*(Y_{\mathcal{G}},\mathbb{Z}) \simeq \mathbb{Z}[c_A]/I_{\emptyset}.$$

Finally, let us recall the following lemma, since it points out some relevant polynomials in the ideals I_S.

Lemma 2.2. Let $\mathcal{H} \subset \mathcal{G}$ be such that $\mathcal{H} \cup \mathcal{S}$ is not $\mathcal{G}$-nested. Then $\prod_{A\in\mathcal{H}} c_A \in I_S$.

Observation 2.1. Notice that, if $\overline{\mathcal{H}}$ denotes the set of maximal elements of $\mathcal{H}$, we have that $P^S_{\overline{\mathcal{H}},B}$ divides $P^S_{\mathcal{H},B}$. Therefore I_S is generated by the polynomials $\prod_{A\in\mathcal{H}} c_A$ for $\mathcal{H}\cup S$ not $\mathcal{G}$-nested, and $P^S_{\mathcal{H},B}$ for $\mathcal{H}\cup S$, $\mathcal{G}$-nested.

Example 2.1. Keeping in mind the description of braid arrangements given in Examples 1.1 and 1.2, one can read integer cohomology rings as follows. For Π_3 and its only building set $\mathcal{G}_{\min}=\mathcal{G}_{\max}$ we have the following expression:

$$H^*(Y_{\mathcal{G}_{\min}},\mathbb{Z}) \simeq \mathbb{Z}[c_A]/I_{\emptyset} = \mathbb{Z}[c_{12},c_{13},c_{23},c_{123}]/\mathcal{I}_3 \text{ with}$$
$$\mathcal{I}_3 =< c_{12}c_{13}, c_{12}c_{23}, c_{13}c_{23}, c_{12}+c_{123}, c_{13}+c_{123}, c_{23}+c_{123}, c_{123}^2 >$$

In fact, from Lemma 2.2 and Observation 2.1 we are interested only in monomials of the form $\prod_{A\in\mathcal{H}} c_A$ for $\mathcal{H}\cup S$ not $\mathcal{G}$-nested, and $P^S_{\mathcal{H},B}$ for $\mathcal{H}\cup S$, $\mathcal{G}$-nested: the former are $c_{12}c_{13}, c_{12}c_{23}, c_{13}c_{23}$, while the latter are $c_{12}+c_{123}, c_{13}+c_{123}, c_{23}+c_{123}, c_{123}^2$.
Finally, noting that in the quotient ring the following relations hold $c_{12}=c_{23}=c_{13}=-c_{123}$ and $c_{12}c_{13}=c_{12}c_{23}=c_{13}c_{23}=c_{123}^2$, one can write the cohomology ring in the simpler form $H^*(Y_{\mathcal{G}_{\min}},\mathbb{Z}) \simeq \mathbb{Z}[c_{123}]/[c_{123}^2]$.

2.2 Bases for cohomology rings

Let $\mathcal{G}$ be a building set and let $s\subset\mathcal{G}$ be a $\mathcal{G}$-nested set. Set $S_A=\{E\in A \mid E\subsetneq A\}$. As described in [26] we want to give a $\mathbb{Z}$-basis for $H^*(D_S,\mathbb{Z})$. As already mentioned, when $S=\emptyset$, we will get a basis for $H^*(Y_{\mathcal{G}},\mathbb{Z})$.

Definition 2.2. A function $f:\mathcal{G}\to\mathbb{N}$ is called $\mathcal{G},S$-admissible if it is $f=0$ or if $f\neq 0$, supp $f\cup S$ is $\mathcal{G}$-nested and, for every $A\in$ supp f,

$$f(A) < d^S_{(suppf)_A,A} = \dim A - \dim\, (\sum_{B \in (suppf)_A \cup S_A} B).$$

Since supp$f \cup S$ is $\mathcal{G}$-nested, we observe that $d^S_{(suppf)_A,A}$ for every $A \in$ supp f.
Now, given a $\mathcal{G}, S$-admissible function f, we will call $\mathcal{G}, S$-admissible the monomial $m_f = \prod_{A \in \mathcal{G}} (c_A)^{f(A)}$. Thanks to Lemma 2.2 all such monomials lie in $H^*(D_S, \mathbb{Z}) \simeq \mathbb{Z}[c_A]/I_S$.

Theorem 2.3. Given $\mathcal{G}$ and S as above, the set $\mathcal{B}_{\mathcal{G},S}$ of $\mathcal{G}, S$-admissible monomials is a $\mathbb{Z}$ basis for $H^*(D_S, \mathbb{Z})$.

Proof. First we prove that the elements in $\mathcal{B}_{\mathcal{G},S}$ span $H^*(D_S, \mathbb{Z})$.
For a certain function $g : \mathcal{G} \to \mathbb{N}$ let $m_g = \prod_{A \in \mathcal{G}} (c_A)^{g(A)}$ be a monomial in $H^*(D_S, \mathbb{Z})$: because of Lemma 2.2 supp g must be $\mathcal{G}$-nested. Let us suppose that g is not $\mathcal{G}, S$-admissible, therefore there is an $A \in$ supp g such that $g(A) > d^S_{(suppf)_A,A}$. We call such an A a *bad component* for the monomial if it is minimal with this property. We will prove the claim by reverse induction on the rank of bad components.
We notice that if a bad component A of m_g is a maximal element in $\mathcal{G}$ then the polynomial $P^S_{(suppg)_A,A}$ divides m_g, so $m_g = 0$. Otherwise, given a bad component A, we note that the polynomial

$$(\prod_{B \in (suppg)_A} c_B)(c_A)^{d^S_{(suppg)_A,A}}$$

divides m_g so, using the expression of $P^S_{(suppg)_A,A}$, we can express m_g as sum of monomials that are in $\mathcal{B}_\mathcal{G}$ or have bad components strictly greater than the ones of m_g. We can conclude by using inductive hypothesis.
It remains to prove the linear independence of monomials in $\mathcal{B}_{\mathcal{G},S}$; we will do it first in the simple case $S = \emptyset$.
Let $\chi(Y_\mathcal{G})$ denote the Euler - Poincar characteristic of $Y_\mathcal{G}$. Now, given a minimal element $G \in \mathcal{G}$ and keeping the same notation as in Theorem 1.11, we already know that $Y_\mathcal{G}$ can be obtained by blowing up $Y_{\mathcal{G}'}$ along a subvariety isomorphic to $Y_{\overline{\mathcal{G}}}$, so we deduce

$$\chi(Y_\mathcal{G}) = \chi(Y_{\mathcal{G}'}) + (\dim G - 1)\chi(Y_{\overline{\mathcal{G}}}).$$

Since the odd degree components of $H^*(Y_\mathcal{G}, \mathbb{Z})$ are zero, it suffices to show that $|\mathcal{B}_\mathcal{G}| = \chi(Y_\mathcal{G})$.
We will proceed by induction on the cardinality of $\mathcal{G}$, the case $|\mathcal{G}| = 1$ being obvious. Given $\mathcal{G}$ and G as before we can divide admissible functions in two

sets: $Z_1 = \{f(G) = 0\}$ and $Z_2 = \{f(G) > 0\}$. We observe that there is a bijective correspondence between M_{Z_1}, the set of monomials associated to admissible functions in Z_1, and $\mathcal{B}_{\mathcal{G}'}$, so $|Z_1| = |\mathcal{B}_{\mathcal{G}'}|$. Moreover, if $f \in Z_2$ satisfies $f(B) > 0$ for some $B \neq G$, we have that either $B \cap G = \{0\}$ or $G \subset B$. So the function $\overline{f} : \overline{\mathcal{G}} \to \mathbb{N}$ constructed by putting $\overline{f}(\overline{D}) = f(D)$ if $f(D) > 0$ and 0 otherwise, is $\overline{\mathcal{G}}$-admissible.
We next observe that the established correspondence between Z_2 and the set of $\overline{\mathcal{G}}$-admissible functions is surjective and $\dim G - 1$ to 1, so we have

$$|Z_2| = |\mathcal{B}_{\overline{\mathcal{G}}}|(\dim G - 1).$$

We have then proved that $|\mathcal{B}_{\mathcal{G}}|$ satisfies the same recurrence relation as $\chi(Y_{\mathcal{G}})$:

$$|\mathcal{B}_{\mathcal{G}}| = |Z_1| + |Z_2| = |\mathcal{B}_{\mathcal{G}'}| + |\mathcal{B}_{\overline{\mathcal{G}}}|(\dim G - 1).$$

Thus the first claim follows by induction. Let now suppose $S \neq \emptyset$ to be a $\mathcal{G}$-nested set. The proof will be given in various steps.

Step 1. $S \cup \{G\}$ is not $\mathcal{G}$-nested.

In this case S is $\mathcal{G}'$-nested and the restrictions to D_S of the natural projection $p : Y_{\mathcal{G}} \to Y_{\mathcal{G}'}$ is an isomorphism onto its image, i.e. the variety D'_S associated to S in $Y_{\mathcal{G}'}$.
The theorem is true, by induction, for $H^*(D'_S), \mathbb{Z}$, and since a function $f : \mathcal{G} \to \mathbb{N}$ is S-admissible if and only if $\mathrm{supp} f \subset \mathcal{G}'$ and $f_{|\mathcal{G}'}$ is S-admissible, it is also true for $H^*(D_S, \mathbb{Z})$.

Step 2. $S \cup \{G\}$ is $\mathcal{G}$-nested but $G \notin S$.

In this case S is still $\mathcal{G}'$-nested. In addition, we can consider the set $\overline{S} = \{\overline{A} : A \in S\} \subset \overline{\mathcal{G}}$ which turns out to be $\overline{\mathcal{G}}$-nested.
From Theorem 1.11 we know that if D'_S is the subvariety associated to S in $Y_{\mathcal{G}'}$, then D_S can be obtained by blowing up D'_S along a subvariety isomorphic to $D_{\overline{S}}$ in $Y_{\overline{\mathcal{G}}}$. So the proof is analogous to the one of the case $S = \emptyset$ and we omit it.

Step 3. $G \in S$.

In this final case, let $\tilde{S} = S - G$ and $\overline{\tilde{S}} \subset \overline{\mathcal{G}}$ be the projection of $\tilde{S}$ in $\overline{\mathcal{G}}$: it turns out to be $\overline{\mathcal{G}}$-nested. Now D_S is the exceptional divisor in $D_{\tilde{S}}$, i.e. it is the preimage of $D_{\overline{\tilde{S}}}$ in $Y_{\mathcal{G}}$. Then it is a $\mathbf{P}^{\dim G - 1}$ bundle over $D_{\overline{\tilde{S}}}$, so

$$\dim_{\mathbb{Z}} H^*(D_S, \mathbb{Z}) = (\dim_{\mathbb{C}} G)(\dim_{\mathbb{Z}} H^*(D_{\overline{\tilde{S}}}, \mathbb{Z})).$$

But now, given an S-admissible function $f : \mathcal{G} \to \mathbb{N}$ we can define an $\tilde{S}$-admissible function $\tilde{f} : \overline{\mathcal{G}} \to \mathbb{N}$ as follows: $\tilde{f}(\overline{D}) = f(D)$ for every $D \in \mathcal{G}$ and $D \neq G$.
This map turns out to be surjective and $(\dim G)$ to 1, so we finally obtain

$$|\mathcal{B}_{\mathcal{G},S}| = (\dim G)|\mathcal{B}_{\overline{\mathcal{G}},\overline{\tilde{S}}}|,$$

and the theorem is proved by induction.

□

Remark 2.1. It follows from Theorem 5.2 of [8] that the degree of generators c_A of $R = H^*(Y_{\mathcal{G}}, \mathbb{Z})$ is 2. Grading R by one half of the degree, from now on we assume that the degree of these generators is 1, i.e. the grading of $R = \oplus_{q \geq 0} R_q$ is generated by the standard grading of the polynomial ring $\mathbb{Z}[c_A]/I_{\emptyset}$.
We denote by Δ_q, $q > 0$, the set of monomials in R that corresponds to $\mathcal{G}, \emptyset$-admissible functions with $\sum_{A \in \mathcal{G}} f(A) = q$ and we also put $\Delta_0 = \{1\}$.
It is now obvious that, for every q, the set Δ_q is a basis of R_q over $\mathbb{Z}$.

2.3 Models of root systems

In this section we specify the monomial basis for the reflection arrangements of classical types A_n, B_n $(= C_n)$ and D_n.

Type A_n We have already introduced minimal wonderful model of the braid arrangement of type A_{n-1}. We saw that $\mathcal{F}_{A_{n-1}}$ can be identified with the poset of all subsets of $\{1, 2, \ldots n\}$ of cardinality at least two ordered by inclusion. A collection $\mathcal{S}$ of these subsets is nested if and only if for every $I, J \in \mathcal{S}$ either $I \cap J = \emptyset$, or one of these sets lies in to the other. We note that for every $\mathcal{S} \in \mathcal{F}_{A_{n-1}}$ we have rank $\mathcal{S} = |\mathcal{S}| - 1$.
Nested collections in $\mathcal{F}_{A_{n-1}}$ admit another interpretation in term of trees. Suppose that $\mathcal{S}$ is a nested collection, fix a maximal $A \in \mathcal{S}$ and consider $S_A = \{B \in \mathcal{S} | B \subsetneq A\}$. Then we can define a rooted tree $\mathcal{F}(A)$ corresponding to $\mathcal{S}_A$. For the set of vertices of $\mathcal{F}(A)$ we take $\{A\} \cup \mathcal{S}_A \cup A$, with A itself serving as the root. Two elements of $\mathcal{S}_A$ are connected by an edge if one of them is a maximal subset of the other. An element $p \in A$ is connected by an edge to $B \in \mathcal{S}_A$ if B is the minimal set of $\mathcal{S}_A$ containing p. Since $\mathcal{S}_A$ is nested $\mathcal{F}(A)$ is indeed a tree. The set of leaves of $\mathcal{F}(A)$ coincides with A. Then for any nested collection $\mathcal{S}$ we obtain a forest $\mathcal{F}(\mathcal{S})$ with at most n leaves

whose connected components are trees $\mathcal{F}(A)$ where A is running through the maximal elements of $\mathcal{S}$ (see Figure 2.1).

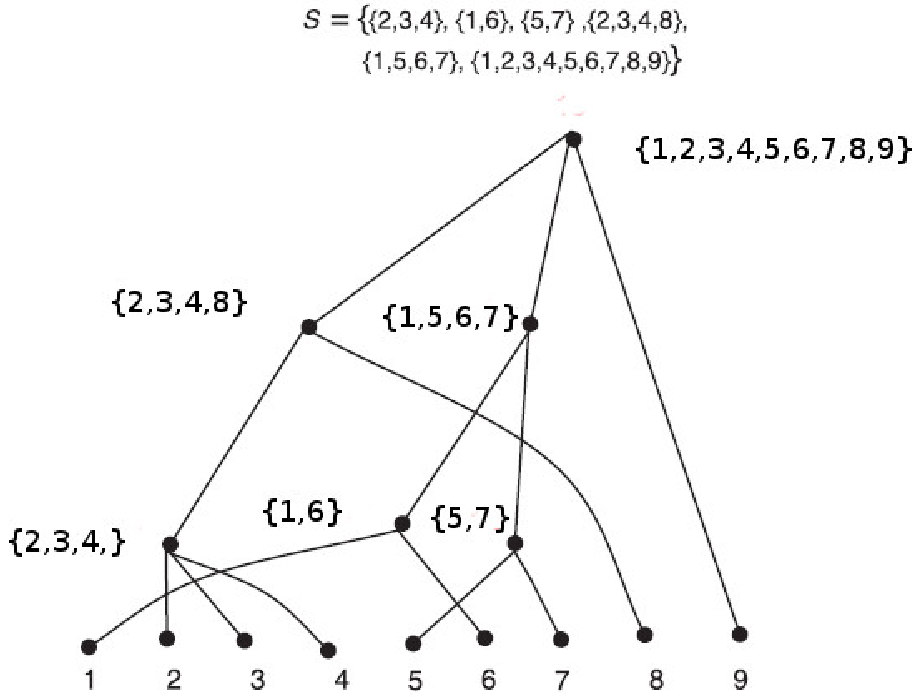

Figure 2.1: On top of the picture there is a nested set S with 6 elements in $\mathcal{F}_{A_8}$. In the middle there is its representation by an oriented labeled rooted tree.

We will always consider rooted trees provided with the standard direction of growth from the root to the leaves. Then it is not hard to see that for every nested collections $\mathcal{S}$ and for every internal vertex v of $\mathcal{F}(\mathcal{S})$ that is not a leaf, we have $d^{\emptyset}_{\mathcal{S},v} = |out(v)| - 1$, where $out(v)$ is the set of all outgoing edges from v. Thus the basis Δ_q of R_q is in one to one correspondence with the forests whose internal vertices are marked by positive integers $f(v)$ such that $f(v) < |out(v)| - 1$ and $\sum_v f(v) = q$. Notice that the necessary and sufficient condition for the existence of such marking is $|out(v)| \geq 3$ for every internal vertex v.

Type B_n The arrangement of type B_n can be defined in a n-dimensional space by the equations $x_i = 0$, $x_i = \pm x_j, 1 \leq i < j \leq n$. Each element of the intersection lattice L is defined by a system of several of these equations. In order to describe $\mathcal{F}_{B_n}$ we fix an element $X \in L$ and consider two cases.

1. There are equations $x_i = 0$ among equations for X. Denote by $\mathcal{S}$ the set of all such subscripts i. Then the arrangement $\mathcal{A}_{\mathcal{S}}$ of hyperplanes $x_i = 0$, $i \in \mathcal{S}$, and $x_i = \pm x_j$, $i, j \in \mathcal{S}$, is an irreducible component of $\mathcal{A}_X$. Thus if $X \in \mathcal{F}$ then $\mathcal{A}_{\mathcal{S}} = \mathcal{A}_X$ and we will identify X with $\mathcal{S}$ and call $\mathcal{S}$ *strong*. Notice that $\mathcal{S}$ can be any nonempty subset of $\{1, 2, \ldots, n\}$.

2. There is no equation $x_i = 0$ among equations for X. Thus from any pair of equations $x_i = x_j$ and $x_i = -x_j$ only one can be among equations for X. As in the case of type A_n the equations $x_i = x_j$ define a partition of $\mathcal{A}_X$. The equations $x_i = -x_j$ connect some elements of this partition tying them in pairs. Again if $X \in \mathcal{F}_{B_{n-1}}$ then the described diagram is connected. Thus in this case X can be identified with a subset of $\{1, 2, \ldots, n\}$ of cardinality at least 2 provided with an unordered partition $\{S_1, S_2\}$ ($\mathcal{S} = S_1 \cup S_2$). The element X is defined then by the system $\{x_i = x_j, x_i = -x_k\}$, where i and j belong both to one of the sets S_r ($r = 1, 2$) and i and k belong to the different sets. We call these subsets $\mathcal{S}$ *weak*.

The order on $\mathcal{F}_{B_n}$ can be now interpreted as follows. All subsets of $\{1, 2, \ldots, n\}$ larger than a strong subset are strong and they are ordered by inclusion. A weak subset $\mathcal{S}$ is smaller than a strong subset $\mathcal{T}$ if $\mathcal{S} \subset \mathcal{T}$. A weak subset $\mathcal{S} = S_1 \cup S_2$ is smaller than a weak subset $\mathcal{T} = T_1 \cup T_2$ if first $\mathcal{S} \subset \mathcal{T}$ and second either $S_i \subset T_i$ ($i = 1, 2$) or $S_1 \subset T_2$ and $S_2 \subset T_1$.
One can now easily computes ranks of elements of $\mathcal{F}_{B_n}$ after the identification of them with strong and weak sets above. If $\mathcal{S}$ is a strong set then $\mathrm{rank}\mathcal{S} = |\mathcal{S}|$, if it is weak then $\mathrm{rank}\mathcal{S} = |\mathcal{S}| - 1$. A collection of elements is nested if and only if the following two conditions hold. First, every two sets in this collection are either disjoint or one is embedded into the other. Second, no two strong sets in this collection are incomparable that is they are totally ordered by inclusion. Notice that a set $\mathcal{S}$ can be represented twice in the collection: as a strong set and as a weak set with partition.

It is again convenient to associate forests of rooted trees with the nested collections. This can be done similarly to type A_{n-1} except now the vertices of each tree are divided in two classes: weak and strong. It is useful to regard the leaves (that do not correspond to elements of $\mathcal{F}_{B_n}$) as weak vertices; then in each tree, for any strong vertex v all vertices closer to the root than v are strong. Furthermore, a forest can have at most one connected component with strong vertices. Let us call the forests satisfying all these conditions *admissible*.

Of course, after passing to admissible forests we loose the partitions on weak sets; this will amount to counting the same forest as many times as there are nested collections corresponding to this forest. In particular the following lemma shows that exponents of admissible monomial m_f depend only on the forest structure.

Lemma 2.4. Let $\mathcal{S}$ be a nested set and v an internal vertex of the corresponding forest. Then $d^{\emptyset}_{\mathcal{S},v} = d_{\mathcal{S},v} = |out(v) - 1|$, if v is weak and $d_{\mathcal{S},v} = |wout(v)|$ if v is strong, where $wout(v)$ is the set of all edges outgoing from v to weak vertices (including leaves).

Proof. By definition, we have $d_{\mathcal{S},v}$ = rank v - $\sum_{w\in out(v)}$ rank w, where we put rank$w = 0$ if w is a leaf. By the expressions for the rank above we see that rank$w$ is the number of leaves connected by a directed path to w if w is strong, and the same number decreased by 1 if w is weak. The result follows. □

Now we want to compute the number of nested collections corresponding to the same forest.

Lemma 2.5. Let $\mathcal{S}$ be a nested collection and $\mathcal{F}(\mathcal{S})$ the corresponding forest. Let $\pi = \pi(\mathcal{F}(\mathcal{S}))$ be the number of the nested collections $\mathcal{T}$ such that $\mathcal{F}(\mathcal{T}) = \mathcal{F}(\mathcal{S})$. Then $\log_2 \pi = \sum$ rank v, where the sum is taken with respect to all closest to the roots weak vertices of $\mathcal{F}(\mathcal{S})$.

Proof. To compute π we need to compute the number of ways of defining one or two element partitions on the weak vertices of $\mathcal{F}(\mathcal{S})$ so that they are included in each other in the opposite direction to the order on $\mathcal{F}(\mathcal{S})$. Notice that the closest to the roots weak vertices correspond to the maximal weak sets in $\mathcal{S}$. Thus any partitions on these sets will uniquely define partitions on all the weak sets of $\mathcal{S}$. Choose $\mathcal{T}$ one of these sets and put $k = |\mathcal{T}|$. Then the number of unordered partitions on $\mathcal{T}$ with at most two parts is 2^{k-1} and recall that $k-1 =$ rank $\mathcal{T}$. Multiplying with respect to all such sets we get the result. □

Summing up we see that the basis Δ_q, for $q > 0$, is mapped onto the set of admissible forests of rooted trees having not more than n leaves whose internal vertices are marked by positive integers $f(v)$ such that $\sum_v f(v) = q$, and $f(v) < |out(v)-1|$ if v is weak and $f(v) < |wout(v)|$ if v is strong. Notice the following necessary and sufficient condition for the existence of such a marking for some $q > 0$: for any weak vertex v we should have $|out(v)| \geq 3$

and any strong vertex should be connected by edges to at least 2 weak vertices.

Type D_n The arrangement of this type can be defined in a n-dimensional space by the equations $x_i = \pm x_j$, $1 \leq i < j \leq n$. The poset $\mathcal{F}_{D_n}$ can be described similarly to type B_n with the main difference that every strong subset of $\{1, 2, \ldots, n\}$ should have at least 3 elements (e.g. for the element $X \in L$ given by the system $\{x_1 = x_2, x_1 = -x_2\}$ the arrangement $\mathcal{A}_X$ is not irreducible). These arrangements will be described in more detail in 4.1.

Chapter 3

The Braid Case

In this section we will consider the real or complexified braid arrangement, i.e. the arrangement given by the hyperplanes defined by the equations $x_i - x_j = 0$, $1 \leq i < j \leq n$ in $\mathbb{R}^n$ or $\mathbb{C}^n$. As mentioned before, $\mathcal{F}_{A_{n-1}}$ is the minimal building set that contains the lines in V^* that are the annihilators of the hyperplanes $x_i - x_j = 0$: it is made by all the subspaces in V^* whose annihilators in V are described by equations like $x_{i_1} = x_{i_2} = \cdots = x_{i_k}$ ($k \geq 2$). We already observed that there is a bijective correspondence between the elements of $\mathcal{F}_{A_{n-1}}$ and the subsets of $\{1, 2, \ldots, n\}$ of cardinality at least two. As immediate consequence, a $\mathcal{F}_{A_{n-1}}$-nested set $\mathcal{S}$ is represented by a set (which we still call $\mathcal{S}$) of subsets of $\{1, 2, \ldots, n\}$ with the property that any of its elements has cardinality ≥ 2 and if I and J belong to $\mathcal{S}$ then either $I \cap J = \emptyset$ or one of the two sets is included into the other.

3.1 Nested sets and set partitions

First of all we are interested to point out a purely combinatorial action on the poset $\mathcal{B}(n-1)$ that indexes the strata of the model $Y_{\mathcal{F}_{A_{n-1}}}$. This does not correspond to an action on the variety $Y_{\mathcal{F}_{A_{n-1}}}$, but it gives rise, as we will see later, to an useful permutation action on the monomials of the Yuzvinski basis of $H^*(Y_{\mathcal{F}_{A_{n-1}}}, \mathbb{Z})$.
Let us denote by $F^k(\mathcal{B}(n-1))$ the subset of $\mathcal{B}(n-1)$ made by the elements of cardinality $k+1$ (these elements indicize the k-codimensional strata of $Y_{\mathcal{F}_{A_{n-1}}}$). In [29] it has been described an explicit bijection between $F^k(\mathcal{B}(n-1))$ and the set of unordered partitions of $\{1, 2, \ldots, n+k\}$ into $k+1$ parts of cardinality greater than or equal to 2. To recall this bijection, we identify the elements of $Y_{\mathcal{F}_{A_{n-1}}}$ with subsets of $\{1, 2, \ldots, n\}$ as before.

Definition 3.1. We fix the following strict partial ordering on $Y_{\mathcal{F}_{A_{n-1}}}$: given I and J in $Y_{\mathcal{F}_{A_{n-1}}}$, we put $I < J$ if the minimal number in I is less than the minimal number in J.

Let us consider a nested set $\mathcal{S}$ that belongs to $F^k(\mathcal{B}(n-1))$. It can be represented by an oriented rooted tree on n leaves, namely the sets $\{1\}$, $\{2\}$, $\cdots$, $\{n\}$. Now we put labels on the vertices of this tree, starting by labeling the vertices $\{1\}$, $\{2\}$, $\cdots$, $\{n\}$ respectively by the labels $1, 2, \ldots, n$.
Then we can partition the set of vertices of the tree into levels: level 0 is made by the leaves, and in general, level j is made by vertices v such that the maximal lenght of an oriented path that connects v to a leaf is j.
Now we can label the internal vertices of the tree in the following way. Let us suppose that there are q vertices at level 1. These vertices correspond to pairwise disjoint elements of $\mathcal{F}_{A_{n-1}}$, therefore we can totally order them using the ordering in Definition 3.1, and we label them with the numbers from $n+1$ (the minimum) to $n+q$ (the maximum).
At the same way, if there are t vertices in level 2, we can label them with the numbers from $n+q+1$ to $n+q+t$, and so on. At the end of the process, the root is labeled with the number $n+k+1$. We can now associate to this tree an unordered partition of $\{1, 2, \ldots, n+k\}$ into $k+1$ parts by assigning to every internal vertex v the set of the labels of the vertices covered by v.

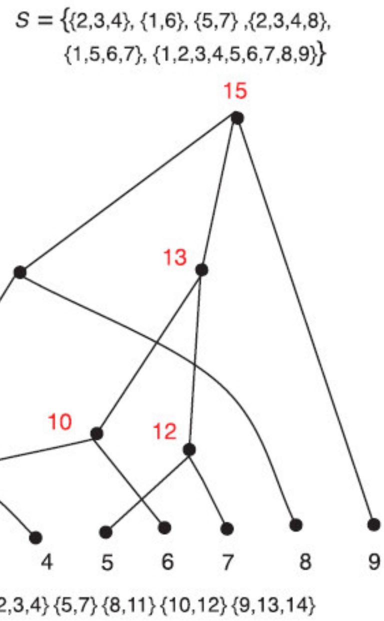

Figure 3.1: The same nested set S with 6 elements in $\mathcal{F}_{A_8}$ in Figure 2.1. At the bottom one can read the resulting partition of $\{1, 2, \ldots, 14\}$ into 6 parts.

This bijection allows us to consider new actions of the symmetric group on $\mathcal{B}(n-1)$: every subset $F^k(\mathcal{B}(n-1))$ is equipped with an action of S_{n+k}.

Remark 3.1. We observe that when $k > 2$, if we first embed S_n into S_{n+k} in the standard way and then restrict the S_{n+k} action to S_n we do not obtain the natural S_n action on $\mathcal{B}(n-1)$. For example, let us consider $n = 6$ and $k = 4$, and the following nested set $\mathcal{S} \in \mathcal{B}(5)$:

$$\mathcal{S} = \{\{1,2\},\{3,4\},\{3,4,5\},\{3,4,5,6\},\{1,2,3,4,5,6\}\}.$$

On one hand, the natural action of the transposition $(1,3)$ sends $\mathcal{S}$ to

$$\mathcal{S}' = \{\{2,3\},\{1,4\},\{1,4,5\},\{1,4,5,6\},\{1,2,3,4,5,6\}\}.$$

On the other hand, the partition of $\{1,2,\ldots,10\}$ associated with $\mathcal{S}$ is

$$\{1,2\},\{3,4\},\{5,8\},\{6,9\},\{7,10\},$$

that is sent by the transposition $(1,3)$ to $\{2,3\},\{1,4\},\{5,8\},\{6,9\},\{7,10\}$. This last partition corresponds to the nested set

$$\mathcal{S}'' = \{\{1,4\},\{2,3\},\{2,3,5\},\{2,3,5,6\},\{1,2,3,4,5,6\}\}$$

and we notice that $\mathcal{S}' \neq \mathcal{S}''$.

Moreover, we observe that the natural S_6 action on $F^4(\mathcal{B}(5))$ and the S_6 action restricted from S_{10} differ in the number of orbits, therefore when we consider the associated permutation representations they differ in the multiplicity of the trivial representations.

3.2 The S_{n+k} action on the Yuzvisnki basis

As we observed in the preceding section, the combinatorial action of S_{n+k} on $F^k(\mathcal{B}(n-1))$ can be read as an action on the k-codimensional strata of $Y_{\mathcal{F}_{A_{n-1}}}$. Moreover we notice that this action can in turn be extended to the Yuzvinski basis of $H^*(Y_{\mathcal{F}_{A_{n-1}}},\mathbb{Z})$. In fact we can represent this basis by labeled partitions in the way described by the following example.

Example 3.1. Let $n = 9$ and let us consider the monomial $c^3_{A_1}c^2_{A_2}$ in the Yuzvinski basis of $H^{10}(Y_{\mathcal{F}_{A_8}},\mathbb{Z})$, where $\{A_1, A_2\}$ is the nested set given by the subspaces $A_1 = \{1,3,5,7,9\}$, $A_2 = \{2,4,6,8\}$. Since V does not belong to this nested set, we write this monomial as $c^3_{A_1}c^2_{A_2}c^0_V$. Now we can associate to the nested set $\{A_1, A_2, V\}$ the following partition of the set $\{1,2,\ldots,11\}$:

$$\{1,3,5,7,9\}\{2,4,6,8\}\{10,11\}$$

where, A_1 corresponds to $\{1,3,5,7,9\}$, A_2 corresponds to $\{2,4,6,8\}$ and V corresponds to $\{10,11\}$. Finally we associate to $c_{A_1}^3 c_{A_2}^2$ the following labeled partition of $\{1,2,\dots,11\}$:

$$\{1,3,5,7,9\}^3\{2,4,6,8\}^2\{10,11\}^0.$$

We notice that this representation provides us an easy way to find the bounds for the exponents in the Yuzvinski basis. More in detail, the bounds $d^{\{V\}}_{(suppf)_A,A}$ can be translated in this language in the following way. Let I be a part of a labeled partition of $\{1,2,\dots n+k\}$ that represents a monomial in the Yuzvinski basis: then the exponent α_I of I satisfies $0 \leq \alpha_I \leq |I|-2$. Moreover, it may be equal to 0 only if I contains the number $n+k$, i.e. when I represents V, and in the monomial the variable c_V does not appear. In particular all the sets in the partition have cardinality ≥ 3 except eventually for the set containing $n+k$, that may have cardinality equal to 2.
Now we observe that S_{n+k} acts on the labeled partitions of $\{1,2,\dots,n+k\}$ into $k+1$ parts, and this provides us with a permutation action on the monomials of Yuzvinski basis of $H^*(Y_{\mathcal{F}_{A_{n-1}}},\mathbb{Z})$:

1. S_{n+k} acts on the set of all the monomials that are represented by a labeled partition of $\{1,2,\dots,n+k\}$ into $k+1$ parts with all positive labels;

2. S_{n+k-1} acts on the set of all the monomials that are represented by a labeled partition of $\{1,2,\dots,n+k\}$ into $k+1$ parts with one of the labels equal to 0. If there is a part labeled by 0, it must contain the number $n+k$ and S_{n+k-1} is embedded into S_{n+k} as the subgroup that keeps $n+k$ fixed.

This representation, once restricted in the standard way to S_n, is not isomorphic to the natural S_n representation.

Example 3.2. Let us consider, for instance, the multiplicity of the trivial representation in $H^6(Y_{\mathcal{F}_{A_7}},\mathbb{Z})$. The key point is provided by the monomials of type $c^1_{A_1}c^1_{A_2}c^1_{A_3}c^0_V$ that span an invariant subspace W for both the natural S_8 action and the extended S_{10} action. By an argument similar to that of Remark 2, one can check that on W the natural S_8 representation and the S_8 representation restricted from S_{10} differ in the multiplicity of the trivial representation (that is respectively 3 and 4).

The orbits of this action can be used to write a generating formula for the Poincaré polynomials of the models $Y_{\mathcal{F}_{A_{n-1}}}$ (see [?] that is different from the recursive formula for the Poincaré series discovered in [26], [42] and [52].

Let us denote by $\psi(q,t,z)$ the following exponential generating series:

$$\psi(q,t,z) = 1 + \sum_{n\geq 2, \mathcal{S}\in\mathcal{N}(\mathcal{F}_{A_{n-1}})} P(\mathcal{S}) z^{|\mathcal{S}|} \frac{t^{n+|\mathcal{S}|-1}}{(n+|\mathcal{S}|-1)!}$$

where, for every $n \geq 2$,

- $\mathcal{S}$ range over all the nested sets of the building set $\mathcal{F}_{A_{n-1}}$;
- $P(\mathcal{S})$ is the polynomial, in the variable q, that expresses the contribution to $H^*(Y_{\mathcal{F}_{A_{n-1}}}, \mathbb{Z})$ provided by all the monomials m_f in the Yuzvinski basis such that $suppf = \mathcal{S}$. For instance, with reference to the last example, if $\mathcal{S}$ is the nested set $\{A_1, A_2\}$, then $P(\mathcal{S}) = (q+q^2+q^3)(q+q^2)$ since we have to take into account all the possible ways to label the partition
$$\{1,3,5,7,9\}\{2,4,6,8\}\{10,11\},$$
while if $\mathcal{S}$ is $\{A_2, V\}$ then $P(\mathcal{S}) = (q+q^2+q^3+q^4)(q+q^2)$ since we are dealing with the possible labelings of the partitions
$$\{2,4,6,8\}\{1,3,5,7,9,10\}.$$

We observe that the series $\psi(q,t,z)$ encodes the same information of the Poincaré series: for a fixed n, the Poincaré polynomial of the model $Y_{\mathcal{F}_{A_{n-1}}}$ can be read from the coefficients of the monomials whose z,t components is $t^k z^s$ with $k-s = n-1$.

Theorem 3.1 (see [3], Theorem 10.1)**.** We have the following formula for the series $\psi(q,t,z)$
$$\psi(q,t,z) = e^t \prod_{i\geq 3} e^{zq[i-2]_q \frac{t^i}{i!}}$$
where $[j]_q$ denotes the q-analog of j: $[j]_q = 1 + q + \ldots + q^{j-1}$.

Proof. It is again useful to think of the monomials of the Yuzvinski basis as labeled partitions. Then we can single out the contribution given to ψ by all the parts represented by subsets with cardinality ≥ 3 and with non trivial label. If in a partition there is only one such part, its contribution is $z(q+q^2+\ldots+q^{i-2})\frac{t^i}{i!}$; if there are j such parts their contributions is $z^j(q+q^2+\ldots+q^{i-2})^j \frac{(\frac{t^i}{i!})^j}{j!}$. The total contribution of all the parts represented by subsets with cardinality ≥ 3 and with non trivial label is provided by

$e^{z(q+q^2+\ldots+q^{i-2})\frac{t^i}{i!}} - 1$.
Let us now calculate the contribution to ψ that comes from the parts with cardinality ≥ 2 and with label equal to 0. For every monomial in the basis there is at most one such part and its contribution is $\frac{t^{i-1}}{(i-1)!}$. The exponent is now decreased to $i-1$ to take into account that such part does not contribute to the cardinality $|\mathcal{S}|$. The total contribution of the elements with label equal to 0 is therefore $\sum_{i\geq 2} \frac{t^{i-1}}{(i-1)!}$. Summing up all contributions, we observe that the expression

$$e^t \prod_{i\geq 3} e^{zq[i-2]_q \frac{t^i}{i!}}$$

allows us to take into account the contribution to ψ of all the possible monomials in the Yuzvinski basis. □

3.2.1 A model example: $Y_{\mathcal{F}_{A_6}}$

Let us consider the model $Y_{\mathcal{F}_{A_6}}$. If one wants to compute the Poincaré polynomial of this model, one can single out all the monomials in ψ whose z, t component is $t^k z^s$, with $k - s = 6$. Therefore, starting from the product of exponential functions that appears in the formula

$$\psi(q,t,z) = e^t \prod_{i\geq 3} e^{zq[i-2]_q \frac{t^i}{i!}}$$

we are only interested in the following truncated series expansions:

$$\begin{aligned}
\psi(q,t,z) =&(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!})\cdot \\
&(1 + \frac{zt^3}{3!}q + \frac{z^2t^6}{2!3!3!}q^2 + \frac{z^3t^9}{3!3!3!3!}q^3)\cdot \\
&(1 + \frac{zt^4}{4!}(q+q^2) + \frac{z^2t^8}{2!4!4!}(q+q^2)^2)\cdot \\
&(1 + \frac{zt^5}{5!}(q+q^2+q^3))\cdot \\
&(1 + \frac{zt^6}{6!}(q+q^2+q^3+q^4))\cdot \\
&(1 + \frac{zt^7}{7!}(q+q^2+q^3+q^4+q^5)) \cdots \text{other terms}
\end{aligned}$$

We can now single out contributions to the Poincaré polynomial taking into account every pair of exponents (k, s) with $k - s = 6$ for the component $t^k z^s$.

- $(k,s)=(6,0)$: the only contribution is given by the trivial term $\frac{t^6}{6!}[1]$;
- $(k,s)=(7,1)$: now the contributions have to be found among multiplications of terms of the form $\frac{zt^i}{i!}(q+q^2+\ldots,q^{i-2})$ by $\frac{t^{7-i}}{(7-i)!}$ for $i=3,4,5,6,7$. Summing up all such terms we found $\frac{zt^7}{7!}[\frac{7!}{3!4!}q+\frac{7!}{4!3!}(q+q^2)+\frac{7!}{5!2!}(q+q^2+q^3)+\frac{7!}{6!1!}(q+q^2+q^3+q^4)+\frac{7!}{7!}(q+q^2+q^3+q^4+q^5)]$.
- $(k,s)=(8,2)$: in this case there are two different ways to build the expression t^8z^2. We can consider multiplication of terms z^2t^j by $\frac{t^{8-j}}{(8-j)!}$ for $j=6,8$, or multiplications of terms zt^k by zt^{8-k} for $k=4,5$. The total amount of the contribution is $\frac{z^2t^8}{8!}[\frac{8!}{3!3!2!2!}q^2+\frac{8!}{4!4!2!}(q+q^2)^2+\frac{8!}{4!3!}q(q+q^2)+\frac{8!}{5!3!}q(q+q^2+q^3)]$.
- $(k,s)=(9,3)$: the last possible contribution comes from the single term $\frac{z^3t^9}{9!}[\frac{9!}{3!3!3!}q^3]$.

Finally, taking into account all the possible contributions, we can now express the Poincaré polynomial of the model $Y_{\mathcal{F}_{A_6}}$:

$$1+99q+715q^2+715q^3+99q^4+q^5.$$

Remark 3.2. We notice that it is possible in small examples to verify the correctness of the formula for the series $\psi(q,t,z)$ directly from the expression for the admissible monomials in the Yuzvinski basis. For instance, referring to our example $Y_{\mathcal{F}_{A_6}}$, one can try to list all the possible admissible monomials m_f for every generating set Δ_q $(q>0)$. We recall from Definition 2.2 that not every $\mathcal{F}_{A_6}$-nested set $\mathcal{S}$ brings with itself an admissible function f, i.e. an admissible set of exponents for the elements of $\mathcal{S}$, as shown in the following example.

Example 3.3. Let us consider the $\mathcal{F}_{A_6}$-nested sets

$$\mathcal{S}_1=\{\{1,2,3,4,5,6,7\},\{1,2,3,4,5,6\}\},\ \ \mathcal{S}_2=\{\{1,2,3,4,5,6,7\},\{1,2,3,4,5\}\},$$

$$\mathcal{S}_3=\{\{1,2,3,4,5,6,7\}\},$$

in our usual partitions language, within an admissible function f. From the very definition one can evaluate f on the nested set $\mathcal{S}_i$, $i=1,2,3$ to find every possible monomial m_f referred to these sets.

- First consider $\mathcal{S}_1 = \{\{1,2,3,4,5,6,7\},\{1,2,3,4,5,6\}\}$. We have

$$f(A) < d^{\emptyset}_{(\mathrm{supp} f)_A, A} = \dim A - \dim\Big(\sum_{B \in (\mathrm{supp} f)_A} B\Big)$$

  for every $A \in \mathrm{supp} f$. In particular $f(\{1,2,3,4,5,6,7\}) < 1$ and we conclude that there are no admissible monomials indexed with $\mathcal{S}_1$ because the only possible exponent for the element $\{1,2,3,4,5,6,7\} \in \mathcal{S}_1$ is 0.

- We consider now the slightly different case $\mathcal{S}_2$. This time an admissible function f satisfies $f(\{1,2,3,4,5,6,7\}) < 2$ and $f(\{1,2,3,4,5\}) < 4$. Therefore all the admissible monomial are $c_{\{1,2,3,4,5,6,7\}} \cdot c^j_{\{1,2,3,4,5\}}$, for $j = 1,2,3$.

- In the latter case $\mathcal{S}_3$ our admissible functions satisfy $f(\{1,2,3,4,5,6,7\}) < 6$, therefore we find admissible monomials of the form $c^j_{\{1,2,3,4,5,6,7\}}$ for $j = 1,2,3,4,5$.

Finally, one can recover the exact expression of Poincaré series collecting all possible *admissible nested sets*, i.e. nested sets that generate admissible monomials, and assigning to every such monomial its contribution to the Poincaré polynomial, as summarized in the following table.

Nested set structure	Monomial	Tot. nested sets	Contribution
$\mathcal{S}$	m_f		$P(\mathcal{S})$
$\{123\}$	$c_{\{123\}}$	$\binom{7}{3} = 35$	$35q$
$\{123\}, \{456\}$	$c_{\{123\}} \cdot c_{\{456\}}$	$\frac{\binom{7}{3}\cdot\binom{4}{3}}{2} = 70$	$70q^2$
$\{1234\}$	$c_{\{1234\}}$	$\binom{7}{4} = 35$	$35q$
$\{1234\}$	$c^2_{\{1234\}}$	$\binom{7}{4} = 35$	$35q^2$
$\{1234\}, \{567\}$	$c_{\{1234\}} \cdot c_{\{567\}}$	$\binom{7}{4} = 35$	$35q$
$\{1234\}, \{567\}$	$c^2_{\{1234\}} \cdot c_{\{567\}}$	$\binom{7}{4} = 35$	$35q^2$
$\{12345\}$	$c_{\{12345\}}$	$\binom{7}{5} = 21$	$21q$
$\{12345\}$	$c^2_{\{12345\}}$	$\binom{7}{5} = 21$	$21q^2$
$\{12345\}$	$c^3_{\{12345\}}$	$\binom{7}{5} = 21$	$21q^3$
$\{12345\}, \{123\}$	$c_{\{12345\}} \cdot c_{\{123\}}$	$\binom{7}{5} \cdot \binom{5}{3} = 210$	$210q^2$
$\{123456\}$	$c_{\{123456\}}$	$\binom{7}{6} = 7$	$7q$
$\{123456\}$	$c^2_{\{123456\}}$	$\binom{7}{6} = 7$	$7q^2$
$\{123456\}$	$c^3_{\{123456\}}$	$\binom{7}{6} = 7$	$7q^3$
$\{123456\}$	$c^4_{\{123456\}}$	$\binom{7}{6} = 7$	$7q^4$
$\{123456\}, \{1234\}$	$c_{\{123456\}} \cdot c_{\{1234\}}$	$\binom{7}{6} \cdot \binom{6}{4} = 105$	$105q^2$
$\{123456\}, \{1234\}$	$c_{\{123456\}} \cdot c^2_{\{1234\}}$	$\binom{7}{6} \cdot \binom{6}{4} = 105$	$105q^3$
$\{123456\}, \{123\}$	$c_{\{123456\}} \cdot c_{\{123\}}$	$\binom{7}{6} \cdot \binom{6}{3} = 140$	$140q^2$
$\{123456\}, \{123\}$	$c^2_{\{123456\}} \cdot c_{\{123\}}$	$\binom{7}{6} \cdot \binom{6}{3} = 140$	$140q^3$
$\{1234567\}$	$c_{\{1234567\}}$	1	q
$\{1234567\}$	$c^2_{\{1234567\}}$	1	q^2
$\{1234567\}$	$c^3_{\{1234567\}}$	1	q^3
$\{1234567\}$	$c^4_{\{1234567\}}$	1	q^4
$\{1234567\}$	$c^5_{\{1234567\}}$	1	q^5
$\{1234567\}, \{12345\}$	$c_{\{1234567\}} \cdot c_{\{1,2,3,4,5\}}$	$\binom{7}{5} = 21$	$21q$
$\{1234567\}, \{12345\}$	$c^2_{\{1234567\}} \cdot c_{\{12345\}}$	$\binom{7}{5} = 21$	$21q^2$
$\{1234567\}, \{12345\}$	$c^3_{\{1234567\}} \cdot c_{\{12345\}}$	$\binom{7}{5} = 21$	$21q^3$
$\{1234567\}, \{12345\}, \{123\}$	$c_{\{1234567\}} \cdot c_{\{12345\}} \cdot c_{\{123\}}$	$\binom{7}{5} \cdot \binom{5}{3} = 210$	$210q^3$
$\{1234567\}, \{123\}$	$c_{\{1234567\}} \cdot c_{\{123\}}$	$\binom{7}{3} = 35$	$35q^2$
$\{1234567\}, \{123\}$	$c^2_{\{1234567\}} \cdot c_{\{123\}}$	$\binom{7}{3} = 35$	$35q^3$
$\{1234567\}, \{123\}$	$c^3_{\{1234567\}} \cdot c_{\{123\}}$	$\binom{7}{3} = 35$	$35q^4$
$\{1234567\}, \{123\}, \{456\}$	$c_{\{1234567\}} \cdot c_{\{123\}} \cdot c_{\{456\}}$	$\frac{\binom{7}{3}\cdot\binom{4}{3}}{2} = 70$	$70q^3$
$\{1234567\}, \{1234\}$	$c_{\{1234567\}} \cdot c_{\{1234\}}$	$\binom{7}{4} = 35$	$35q^2$
$\{1234567\}, \{1234\}$	$c^2_{\{1234567\}} \cdot c_{\{1234\}}$	$\binom{7}{4} = 35$	$35q^3$
$\{1234567\}, \{1234\}$	$c_{\{1234567\}} \cdot c^2_{\{1234\}}$	$\binom{7}{4} = 35$	$35q^3$
$\{1234567\}, \{1234\}$	$c^2_{\{1234567\}} \cdot c^2_{\{1234\}}$	$\binom{7}{4} = 35$	$35q^4$

Therefore the Poincaré polynomial of $Y_{\mathcal{F}_{A_6}}$ is $1 + 99q + 715q^2 + 715q^3 + 99q^4 + q^5$, as we expected from the computation of the generating series $\psi(q, t, z)$.

Observation 3.1. We notice that, starting from either the exponential formula for $\psi(q, t, z)$ or the equivalent language of combinatorial nested set over partition of $\{1, 2, \ldots, n\}$, it is possible to compute expressions for the dimension of the cohomology groups $H^2(Y_{\mathcal{F}_{A_{n-1}}}, \mathbb{Z})$ and $H^4(Y_{\mathcal{F}_{A_{n-1}}}, \mathbb{Z})$.

For instance, we have the following expression for the dimension of the first not vanishing cohomology group

$$\dim\, H^2(Y_{\mathcal{F}_{A_{n-1}}}, \mathbb{Z}) = \; 2^n - \frac{n(n+1)}{2} - 1.$$

In fact one can easily observe that we are interested in counting $\mathcal{F}_{A_{n-1}}$-nested set associated with admissible monomials on Δ_1; these are in one to one correspondence with subsets of $\{1, 2, \ldots, n\}$ with cardinality ≥ 3. Removing from the total number of subsets 2^n every singletons and pairs of the form $\{i\}$, $i = 0, \ldots, n$, and $\{i, j\}$, $1 \leq i \lneq j \leq n$, and also the empy set $\emptyset$, the desired formula follows.

Increasing the size of the cohomology groups, the formulas for their dimension become more elaborate. For example, our language of combinatorial nested set can help us to find expression for the dimension of the second non vanishing cohomology group $H^4(Y_{\mathcal{F}_{A_{n-1}}}, \mathbb{Z})$, as follows.
We are interested in counting admissible monomials of three distinct forms: c_A^2, $c_A \cdot c_B$ with $B \subsetneq A$, and $c_A \cdot c_B$ with $A \cap B = \emptyset$, for every subset $A, B \subset \{1, 2, \ldots, n\}$ of cardinality at least 3.

Monomials of the form c_A^2 arise every time $n - 2 \geq 2$, i.e. $n \geq 4$ and they are in one to one correspondence with the subsets of $\{1, 2, \ldots, n\}$ of cardinality at least 4. Thus they are $\sum_{k=4}^{n} \binom{n}{k}$.

Then we observe that monomials of the second type, $c_A \cdot c_B$ with $B \subsetneq A$, are admissible if and only if $|A| - |B| \geq 2$ and $|B| \geq 3$ (see Example 3.3). Thus they arise only when $n \geq 5$ and their total number is given by $\sum_{i=5}^{n}\sum_{k=3}^{i-2}\binom{n}{i}\cdot\binom{i}{k}$

Monomials of the third type arise only if $n \geq |A| + |B| \geq 6$. These kind of monomials are in one to one correspondence with different admissible monomials in the set $\Delta_1^{n-1-k} \times \Delta_1^{k-1}$ for k that varies from 3 to $n-3$. Taking care of all possible collisions (i.e. every monomial of the form $c_A \cdot c_B$ with $|A| = |B|$ appears twice in our expression), we find the following contribution $\sum_{i=3}^{\lfloor \frac{n}{2} \rfloor}\sum_{k=i}^{n-i}\alpha_{ik}\binom{n}{i}\cdot\binom{n-i}{k}$, where $\alpha_{ik} = 1$ if $i \neq k$ or $\frac{1}{2}$ if $i = k$.

Finally, we can sum up all possible contributions given by admissible monomials of all three species to find the following formula:

$$\begin{aligned}\dim H^4(Y_{\mathcal{F}_{A_{n-1}}}, \mathbb{Z}) = &\sum_{k=4}^{n}\binom{n}{k} + \sum_{i=5}^{n}\sum_{k=3}^{i-2}\binom{n}{i}\cdot\binom{i}{k} \\ &+ \sum_{i=3}^{\lfloor \frac{n}{2} \rfloor}\sum_{k=i}^{n-i}\alpha_{ik}\binom{n}{i}\cdot\binom{n-i}{k},\end{aligned}$$

where $\alpha_{ik} = 1$ if $i \neq k$ or $\frac{1}{2}$ if $i = k$.

Chapter 4

Models of complex reflection groups

In this section we are interested in studying De Concini - Procesi models associated with finite irreducible complex reflection groups. We recall that, according to Shephard - Todd classification [50], these are the groups $G(r,p,n)$, with $r,p,n \in \mathbb{Z}^+$ and $p|r$, plus 34 exceptional groups.
If we call $C(r)$ the cyclic group of order r generated by a primitive r-th root of unity ζ, then the group $G(r,1,n)$, the full monomial group, is the wreath product of $C(r)$ and the symmetric group S_n. For our purposes it is useful to view $G(r,1,n)$ as the group generated by all the complex reflections in $GL(\mathbb{C}^n)$ whose reflecting hyperplanes are described by equations $x_i = \zeta^\alpha x_j$, whith $\alpha = 0, 1, \ldots, r-1$, and $x_i = 0$.
Its elements are all the linear transformations $g(\sigma, \epsilon) : \mathbb{C}^n \to \mathbb{C}^n$ defined on the standard basis by $g(\sigma,\epsilon) = \epsilon(i)e_{\sigma(i)}$, where $\sigma \in S_n$ and ϵ ranges among the functions from $\{1, 2, \ldots, n\}$ to $C(r)$.
The group $G(r,p,n)$ is the subgroup of $G(r,1,n)$ formed by all the $g(\sigma,\epsilon)$ such that the product $\epsilon(1)\epsilon(2)\cdots\epsilon(n)$ is a power of ζ^p. If $p < r$ the sets of reflecting hyperplanes of $G(r,p,n,)$ and $G(r,1,n)$ coincide. The set of reflecting hyperplanes of $G(r,r,n)$ is obtained from that of $G(r,1,n)$ by deleting the coordinate hyperplanes.
We observe that $G(1,1,n) = S_n$ is the well known Weyl group of type A_{n-1}, while $G(2,1,n)$ and $G(2,2,n)$ are respectively the Weyl groups of type B_n (=C_n) and D_n.

4.1 Extension to $G(r,1,n)$, $G(r,p,n)$ and $G(r,r,n)$

It is natural to extend the building and nested sets language to the more general case of model $Y_{\mathcal{F}_{G(r,p,n)}}$. The building sets of irreducibles $\mathcal{F}_{G(r,1,n)}$ (that is equal to $\mathcal{F}_{G(r,p,n)}$ when $p < r$) consists of two families of subspaces. The *strong* subspaces $\overline{H}_{i_1,\ldots,i_k}$ that are the annihilators of the subspaces in $\mathbb{C}^n$ described by the equations $x_{i_1} = \cdots = x_{i_k} = 0$, with $1 \leq k \leq n$. We can represent them by associating to $\overline{H}_{i_1,\ldots,i_k}$ the subset $\{0, i_1, \ldots, i_k\} \subset \{0, 1, \ldots, n\}$ (the word *strong* comes from the analyis of B_n in Section 2.3). The second family is made by the *weak* subspaces that are the annihilators $H_{i_1,i_2,\ldots,i_k}(\alpha_2, \ldots, \alpha_k)$ of the subspaces in $\mathbb{C}^n$ described by the equations: $x_{i_1} = \zeta^{\alpha_2} x_{i_2} = \cdots = \zeta^{\alpha_k} x_{i_k}$, with $2 \leq k \leq n$ and $0 \leq \alpha_s \leq r-1$. For every choice of $i_1 < \cdots < i_k$, we can represent these weak subspaces by associating to $H_{i_1,\ldots,i_k}(\alpha_2, \ldots, \alpha_k)$ the weighted subset $\{i_1, i_2^{\alpha_2}, \ldots, i_k^{\alpha_k}\}$ of $\{1, \ldots, n\}$. The weights are considered modulo r and if a weight is 0 we will omit it.

Observation 4.1. We observe that the building sets of irreducibles $\mathcal{F}_{G(r,r,n)}$ can be obtained from $\mathcal{F}_{G(r,1,n)}$ by removing some strong subspaces, i.e. the hyperplanes $\overline{H}_i$ for $i = 1, 2, \ldots, n$. Moreover, in the case $r = 2$ one needs to remove also the two dimensional subspaces $\overline{H}_{i,j}$: in fact we notice that the subspaces $\overline{H}_{i,j}$ are irreducible only if $r \geq 3$, since the r lines $H_{i,j}(\alpha)$ belong to $\mathcal{F}_{G(r,r,n)}$, and when $r \geq 3$ it is not true that the two dimensional subspace $\overline{H}_{i,j}$ is the direct sum of the maximal subspaces of $\mathcal{F}_{G(r,r,n)}$ contained in it.

Observation 4.2. This last observation explains that, when $n \geq 3$ and $r \geq 3$, the varieties $Y_{\mathcal{F}_{G(r,r,n)}}$ and $Y_{\mathcal{F}_{G(r,1,n)}}$ are isomorphic: the building set $\mathcal{F}_{G(r,1,n)}$ can be obtained from $\mathcal{F}_{G(r,r,n)}$ by adding some lines.

4.1.1 Nested set for $\mathcal{F}_{G(r,1,n)}$ and $\mathcal{F}_{G(r,r,n)}$

According to our representation of the irreducibles subspaces, a nested set for $\mathcal{F}_{G(r,1,n)}$ is represented by a set $\{A_1', \ldots, A_m'\}$ of possibly weighted subsets of $\{0, 1, \ldots, n\}$ with the following properties:

- the subsets that contain 0 are not weighted and they are linearly ordered by inclusion;
- the subsets that do not contain 0 are weighted;
- for any pair of subsets A_i', A_j', we have that, beside their weights, they are one included into the other or disjoint: if both represent weak subspaces one included into the other, then their weights must be compatible. Since we adopt for A_i', A_j' the notation with weights reduced

modulo r, this means that, up to multiplication of all weights of A_i' by the same power of ζ, the weights associated to the same numbers must be equal.

We can represent a $\mathcal{F}_{G(r,1,n)}$-nested set in a way similar to the one described in the braid case $\mathcal{F}_{G(1,1,n)}$. Here, a nested set is represented by an oriented *weighted forest*. Every internal vertex v represents the subset made by the leaves that belong to the subtree stemming from v. In particular, if v is a weak vertex then it represents a weighted subset in the following way: if v is the root of a tree, we put its weight to be equal to 0; then, given any weak vertex w, the weight that one appends to the leaf i, in its associated weighted subset, is given by the sum of the weights that one finds in the oriented path from w to i. According to the above rules, there is a unique way to put the weights in the weighted forest respecting our notations.
A description of the same type holds for the nested sets of $\mathcal{F}_{G(r,r,n)}$ when $n \geq 3$ and $r \geq 3$ (the only difference is that the substes $\{0, j\}$ do not appear). As for the nested sets of $\mathcal{F}_{G(2,2,n)}$ (≥ 2), also the subsets $\{0, j, k\}$ do not appear. Furthermore, there is an exception to the rule that for any pair of subsets A_i', A_j', forgetting their weights, they are one included into the other or disjoint: in a nested set there may be one (and only one) pair $\{i, j\}$, in this case any other element B of the nested set satisfies (forgetting its weights) $B \cap \{i, j\} = \emptyset$ or $\{0, i, j\} \subsetneq B$.

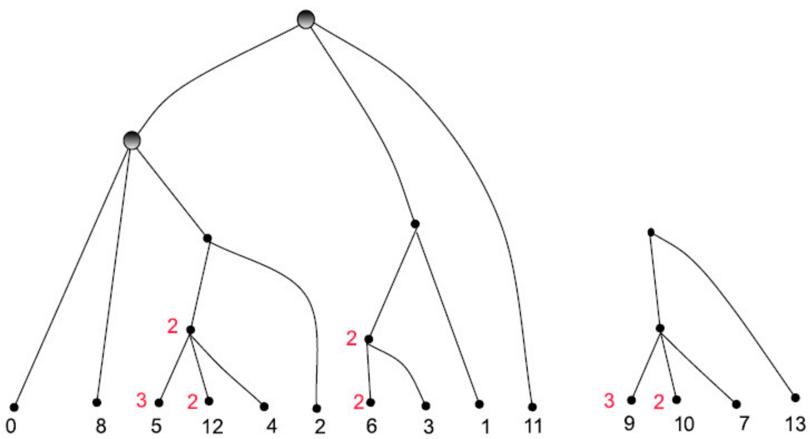

Figure 4.1: A nested set in the case $G(4, 1, 13)$. The *big* internal vertices represent the strong subspaces and red numbers are the weights. The nested set represented in the picture is therefore made by the strong subspaces: $\{0, 2, 4, 5, 8, 12\}, \{0, 1, 2, 3, 4, 5, 6, 8, 11, 12\}$ and by the weak subspaces $\{4, \overset{3}{5}, \overset{2}{12}\}, \{2, \overset{2}{4}, \overset{1}{5}, 12\}, \{3, \overset{2}{6}\}, \{1, \overset{2}{3}, 6\}\{7, \overset{3}{9}, \overset{2}{10}\}, \{7, \overset{3}{9}, \overset{2}{10}, 13\}$.

Observation 4.3. A monomial from the basis of $Y_{\mathcal{F}_{G(r,a,n)}}$ (with $a = 1$ or r) can be represented by a weighted partition with exponents in the following way. Suppose at first that the support of the monomial is given by a nested set formed only by weak subspaces: for instance the support of the monomial

$$c_{\{4,\overset{3}{5},\overset{2}{12}\}}\, c^2_{\{2,\overset{2}{4},\overset{1}{5},8,11,12\}}\, c_{\{7,\overset{3}{9},\overset{2}{10}\}}\, c^3_{\{1,3,6,7,\overset{3}{9},\overset{2}{10},13\}}$$

in $H^*(Y_{\mathcal{F}_{G(4,1,13)}})$, is described by the weighted forest in the Figure 4.2.

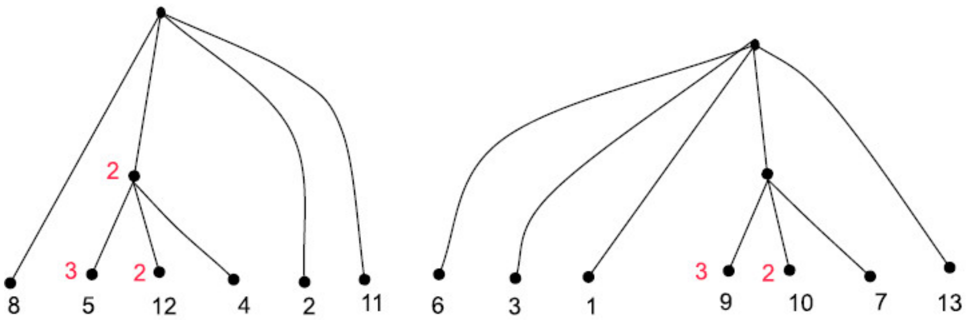

Figure 4.2

Then, following [29], we can label the internal vertices as in the next figure: we put labels level by level, and the label of a vertex v is less than the label of a vertex w if, and only if, the subtree that stems from v contains a leaf whose label is smaller than the labels of all the leaves in the subtree that stems from w. If the forest has more than one connected component, we add an extra vertex on top with the maximum label.

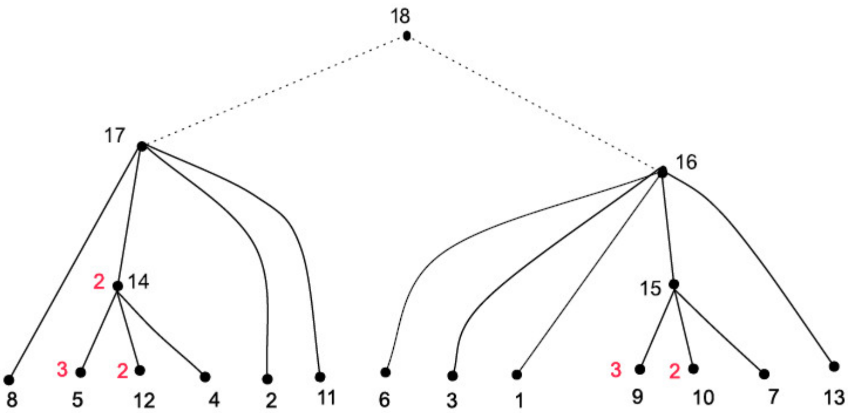

Figure 4.3: We can label internal vertices of the forest in Figure 4.2 with the numbers 14,15,16,17 and we add a vertex on top with maximum label 18.

We can now associate to the support of the monomial m_f a weighted partition by looking at the internal vertices of the labeled forest and taking into account, for each internal vertex, the labels and weights of the vertices covered by it. For instance, considering the weighted forest represented in the previous figure, we associate to it the weighted partition:

$$\{4,\overset{3}{5},\overset{2}{12}\}\{2,8,11,\overset{2}{14}\}\{7,\overset{3}{9},\overset{2}{10}\}\{1,3,6,13,15\}\{16,17\}.$$

Finally, we can associate to the monomial

$$c_{\{4,\overset{3}{5},\overset{2}{12}\}}\, c^{2}_{\{2,\overset{2}{4},\overset{1}{5},8,11,12\}}\, c_{\{7,\overset{3}{9},\overset{2}{10}\}}\, c^{3}_{\{1,3,6,7,\overset{3}{9},\overset{2}{10},13\}}$$

the following weighted partition of $\{1,2,\ldots,17\}$ with exponents attached in order to keep into account the exponents in the monomial:

$$\{4,\overset{3}{5},\overset{2}{12}\}\{2,8,11,\overset{2}{14}\}^2\{7,\overset{3}{9},\overset{2}{10}\}\{1,3,6,13,15\}^3\{16,17\}^0$$

Remark 4.1. This construction, if applied to a forest with more then one connected component, adds an extra vertex on top of the figure. In this case we obtain a part with exponent 0, as $\{16,17\}^0$ in the previous example. We notice that this part necessarily contains the maximum of $\{1,2,\ldots,17\}$, i.e. 17.

Let us suppose now that the support of the monomial contains some strong subspaces: then this support can be represented by a forest that has a shape like the one suggested in the Figure 4.4.

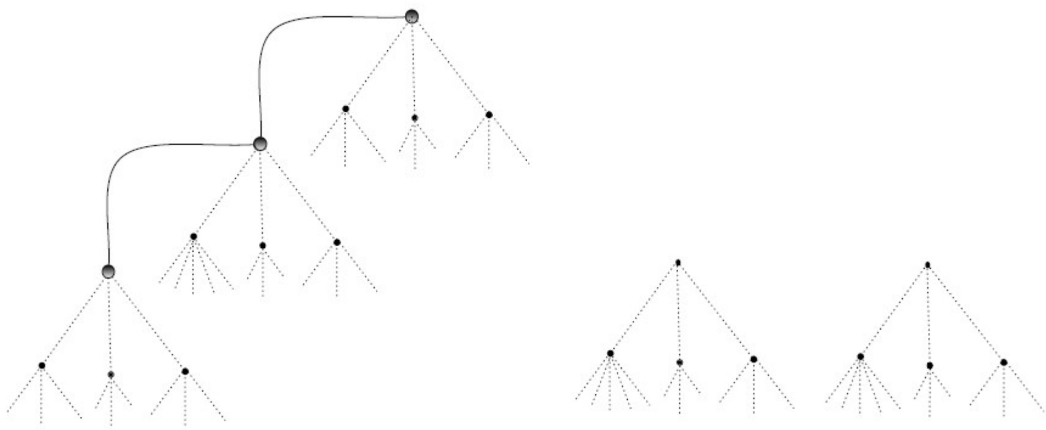

Figure 4.4

There may be at most one connected component with strong vertices. Then the strong vertices form a chain, and below each of them there is a subgraph made by weak vertices. There may be other connected components that are made only by weak vertices.

4.2 Poincaré series

We now describe some theorems from [31] that extend the results of Chapter 3 to the computation of Betti numers of $Y_{\mathcal{F}_{G(r,1,n)}}$ ($= Y_{\mathcal{F}_{G(r,p,n)}}$ when $p < r$) and $Y_{\mathcal{F}_{G(r,r,n)}}$. We will give a non recursive formula for the Poincaré series

$$\Phi_{G(r,a)}(q,t) = 1 + \sum_{n\geq 2} \mathrm{Poin}(Y_{\mathcal{F}_{G(r,a,n)}})(q)\frac{t^n}{n!}$$

where $\mathrm{Poin}(Y_{\mathcal{F}_{G(r,a,n)}})(q)$ is the Poincaré polynomial of the model $Y_{\mathcal{F}_{G(r,a,n)}}$.

We start by extracting the contribution to $\Phi_{G(r,a)}(q,t)$ given by monomials of the Yuzvinski basis whose support does not contain strong subspaces.

Definition 4.1. Let us denote by $K_{G(r,a)}(q,t,z)$ the following exponential generating series:

$$K_{G(r,a)}(q,t,z) = 1 + \sum_{n\geq 2,\mathcal{S}} P(\mathcal{S}) z^{|\mathcal{S}|} \frac{t^{n+|\mathcal{S}|-1}}{(n+|\mathcal{S}|-1)!}$$

where, for every $n \neq 2$ (while $a = 1$ or $a = r$, and r remain fixed)

- $\mathcal{S}$ ranges over all the $\mathcal{F}_{G(r,a,n)}$-nested set that do not contain strong subspaces;
- $P(\mathcal{S})$ is the polynomial that expresses the contribution to $H^*(Y_{\mathcal{F}_{G(r,a,n)}}, \mathbb{Z})$ given by all the monomials m_f in the Yuzvinski basis such that $suppf = \mathcal{S}$.

We observe that the series $K_{G(r,a)}(q,t,z)$ does not change in the two cases $G(r,1,n)$ or $G(r,r,n)$, since only computations over weak subspaces are involved. We present now an explicit formula that is essentially a variant of the one found in the braid case, i.e. the model $Y_{\mathcal{F}_{G(1,1,n)}}$.

Theorem 4.1 (see [31], Theorem 5.1)**.** We have the following formula for the series $K_{G(r,a)}(q,t,z)$ (when $a = 1$ or $a = r$):

$$K_{G(r.a)}(q,t,z) = e^t \prod_{i\geq 3} e^{\frac{z}{r} q[i-2]_q \frac{(rt)^i}{i!}}$$

where $[j]_q$ denotes the q-analog of j: $[j]_q = 1 + q + \ldots + q^{j-1}$.

Proof. Let us denote by $\mathcal{B}(r,1)_{\text{weak}}$ the set of all the Yuzvinski basis monomials of all the models $Y_{\mathcal{F}_{G(r,1,n)}}$ ($n \geq 3$) whose support is made by weak subspaces. As we observed before, we can represent these monomials as weighted partitions with exponents attached to the parts. Let us denote by $\mathcal{P}(r,1)$ the union, for every $j \geq 3$, of the set of weighted partitions of $\{1, 2, \ldots j\}$ with exponents, such that:

- at most one of the parts has exponents equal to 0 (and cardinality ≥ 2). If this part exists, it contains the maximum number j;
- the other parts I have cardinality ≥ 3 and their exponent α_I satisfies $1 \leq \alpha_I \leq |I| - 2$.

As a consequence of Theorem 2.1 in [29], we know that the map from $\mathcal{B}(r,1)_{\text{weak}}$ to $\mathcal{P}(r,1)$ is bijective.
Then we can first single out the contribution given to $K_{G(r,a)}(q,t,z)$ by all the parts represented by subsets with cardinality $i \geq 3$ and with nonzero exponent. If in a weighted partition there is only one such part its contribution is $\frac{z}{r}(q + q^2 + \ldots + q^{i-2})\frac{(rt)^i}{i!}$. If there are j such parts the associated contribution is $(\frac{z}{r})^j(q + q^2 + \ldots + q^{i-2})^j \frac{(\frac{(rt)^i}{i!})^j}{j!}$. The total contribution of all the parts represented by subsets with cardinality $\imath \geq 3$ and with non zero exponent is provided by

$$e^{\frac{z}{r}(q+q^2+\ldots+q^{i-2})\frac{(rt)^i}{i!}} - 1.$$

Let us now focus on the contribution that comes from the part with cardinality $i \geq 2$ and with exponent equal to 0. For every monomial in the basis there is at most one such part, and its contribution is $\frac{t^{i-1}}{(i-1)!}$. We note that, by construction, all the weights of the numbers that belong to this part are equal to 0, since it does not represent a subspace in the support of the monomial.
The total contribution of the elements with exponent equal to 0 is therefore $\sum_{i\geq 2} \frac{t^{i-1}}{(i-1)!}$. Summing up, we observe that the expression

$$e^t \prod_{i\geq 3} e^{\frac{z}{r}q[i-2]_q\frac{(rt)^i}{i!}}$$

takes into account the contribution to $K_{G(r.a)}(q,t,z)$ of all the elements in $\mathcal{P}(r,1)$. $\square$

Let us introduce the following series that will play a key role in computing the contribution of a weak subtree that stems from a strong vertex:

$$\gamma_{G(r,1)}(q,t,z) = (\sum_{i\geq 2} \frac{t^{i-1}}{(i-1)!} q[i-1]_q) \prod_{i\geq 3} e^{\frac{z}{r} q[i-2]_q \frac{(rt)^i}{i!}}.$$

We can evaluate the series $\gamma_{G(r,1)}(q,t,z)$ in $z = \frac{\partial}{\partial t}$ and then integrate formally with respect to the variable t (with constant equal to 0). As a result we get a new series in the variables q, t which we denote by $\Gamma_{G(r,1)}(q,t)$:

$$\Gamma_{G(r,1)}(q,t) = \int \gamma_{G(r,1)}(q,t,\frac{\partial}{\partial t}).$$

Remark 4.2. We remark that every time we say that we evaluate a series in $z = \frac{\partial}{\partial t}$, we mean that first we compute the series expansion and only then we put $z = \frac{\partial}{\partial t}$ in every monomial in the final expression.

With the same process we define the new series

$$\mathcal{K}_{G(r,1)}(q,t) = 1 + \int K_{G(r,1)}(q,t,\frac{\partial}{\partial t}).$$

We are now ready to find a formula for the Poincaré series $\Phi_{G(r,1)}(q,t)$ for the models $Y_{\mathcal{F}_{G(r,1,n)}}$.

Theorem 4.2 (see [31], Theorem 5.2)**.** We have the following formula for the Poincaré series of the models $Y_{\mathcal{F}_{G(r,1,n)}}$:

$$\Phi_{G(r,1)}(q,t) = \frac{1}{1-\Gamma_{G(r,1)}(q,t)} \mathcal{K}_{G(r,1)}(q,t).$$

Proof. First of all we observe that

$$\mathcal{K}_{G(r,1)}(q,t) = 1 + \int K_{G(r,1)}(q,t,\frac{\partial}{\partial t}) = 1 + t + \sum_{n\geq 2, \mathcal{S}\text{weak}} P(\mathcal{S})\frac{t^n}{n!}$$
$$= 1 + t + \sum_{n\geq 2} \mathrm{Poin}^w(Y_{\mathcal{F}_{G(r,1)}})(q)\frac{t^n}{n!}$$

where we are considering only *weak* nested set $\mathcal{S}$ made by weak subspaces, while $\text{Poin}^w(Y_{\mathcal{F}_{G(r,1)}})(q)$ is the contribution to the Poincaré polynomial given by the basis monomials whose support is a *weak* nested set.
At this point we observe that the difference between the series $\gamma_{G(r,1)}(q,t,z)$ and $K_{G(r,1)}(q,t,z)$ consists only in the first exponential factor ($\sum_{i\geq 2} \frac{t^{i-1}}{(i-1)!} q[i-1]_q$): the q-polynomial $q[i-1]_q$ counts the contribution to the Poincaré series given by a strong vertex which covers i weak vertices in the related graph. Then the evaluation $z = \frac{\partial}{\partial t}$ and the subsequent integral transform, gives us the correct contribution to the Poincaré series of a strong vertex and of the weak subgraph stemming from it.
Since the strong vertex are linearly ordered, if there are m strong vertices their contribution is given by $\Gamma_{G(r,1)}(q,t)^m$, so the total contribution of strong vertices is

$$\Gamma_{G(r,1)}(q,t) + \Gamma_{G(r,1)}(q,t)^2 + \Gamma_{G(r,1)}(q,t)^3 + \dots$$

Finally, multiplying by $\mathcal{K}_{G(r,1)}(q,t)$ we take into account the contribution of the components of the forest that do not have strong vertices, proving our formula.

□

Remark 4.3. We recall that, for $r \geq 3$, the series $\Phi_{G(r,r)}(q,t)$ coincides with $\Phi_{G(r,1)}(q,t)$, but when $r = 2$, i.e. the D_n case, we need a slight modification as explained in the next theorem.

Theorem 4.3 (see [31], Theorem 5.3)**.** We have the following formula for the Poincaré series of the model $Y_{\mathcal{F}_{G(2,2)}}$:

$$\Phi_{G(2,2)}(q,t) = \frac{(1 - q^{\frac{t^2}{2}})}{1 - \Gamma_{G(2,1)}(q,t)} \mathcal{K}_{G(2,1)}(q,t).$$

Proof. When $r = 2$, the only modification we have with respect to the computation of the previous theorem is that among the forests that represent the supports of Yuzvinski monomial, we do not have forests whose lower strong vertex corresponds to a two dimensional subspace $\overline{H}_{i,j}$. The contribution to $\Phi_{G(2,1)}(q,t)$ of the associated monomials is computed from the series

$$\frac{q^{\frac{t^2}{2}}}{1 - \Gamma_{G(2,1)}(q,t)} \mathcal{K}_{G(2,1)}(q,t),$$

so we have to subtract it from $\Phi_{G(2,1)}(q,t)$, proving the formula. □

4.2.1 A model example: $Y_{\mathcal{F}_{G(r,1,6)}}$

With arguments similar to the ones used in Section 3.2.1, we can recover the Poincaré polynomial of the more general model $Y_{\mathcal{F}_{G(r,1,n)}}$, for instance the model $Y_{\mathcal{F}_{G(r,1,6)}}$.

Starting from the generating series

$$\Phi_{G(r,1)}(q,t) = \frac{1}{1-\Gamma_{G(r,1)}(q,t)}\mathcal{K}_{G(r,1)}(q,t).$$

we are interested in single out the coefficient (in the variables q, r) of the component $\frac{t^6}{6!}$. Some easy calculations suggest us to consider only the following series expansions:

$$\begin{aligned}
\mathcal{K}_{G(r,1)}(q,t) = & 1 + \int \{(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\frac{t^5}{5!})(1+\frac{zt^3qr^2}{3!}+\frac{z^2t^6q^2r^4}{2\cdot 3!3!}) \\
& (1+\frac{zt^4(q+q^2)r^3}{4!})(1+\frac{zt^5(q+q^2+q^3)r^4}{5!}) \\
& (1+\frac{zt^6(q+q^2+q^3+q^4)r^5}{6!}) + \text{other terms}\}_{z=\frac{\partial}{\partial t}} \\
= & 1+t+\frac{t^2}{2}+\frac{t^3}{3}+\frac{t^4}{4!}+\frac{t^5}{5!}+\frac{t^6}{6!}+\frac{t^3qr^2}{3!}+\frac{t^4(q+q^2)r^3}{4!}+ \\
& \frac{t^5(q+q^2+q^3)r^4}{5!}+\frac{t^6(q+q^2+q^3+q^4)r^5}{6!}+\frac{t^4qr^2}{3!}+ \\
& \frac{t^5(q+q^2)r^3}{4!}+\frac{t^6(q+q^2+q^3)r^4}{5!}+\frac{t^5qr^2}{2\cdot 3!}+\frac{t^6(q+q^2)r^3}{2\cdot 4!}+ \\
& \frac{t^6qr^2}{3!3!}+\frac{t^5q^2r^4}{2\cdot 3!}+\frac{7t^6(q^2+q^3)r^5}{3!4!} + \text{other terms.}
\end{aligned}$$

and

$$
\begin{aligned}
\Gamma_{G(r,1)}(q,t) = &\int \{(tq + \frac{t^2(q+q^2)}{2!} + \frac{t^3(q+q^2+q^3)}{3!} + \frac{t^4(q+q^2+q^3+q^4}{4!} + \\
&+ \frac{t^5(q+q^2+q^3+q^4+q^5)}{5!})(1+\frac{zt^3qr^2}{3!} + \frac{z^2t^6q^2r^4}{2\cdot 3!3!}) \\
&(1+\frac{zt^4(q+q^2)r^3}{4!})(1+\frac{zt^5(q+q^2+q^3)r^4}{5!}) + \text{other terms}\}_{z=\frac{\partial}{\partial t}} \\
= &\frac{t^2q}{2} + \frac{t^3(q+^2)}{3!} + \frac{t^4(q+q^2+q^3)}{4!} + \frac{t^5(q+q^2+q^3+q^4)}{5!} + \\
&\frac{t^6(q+q^2+q^3+q^4+q^5)}{6!} + \frac{t^4q^2r^2}{3!} + \frac{t^5(q^2+q^3)r^3}{4!} + \\
&\frac{t^6(q^2+q^3+q^4)r^4}{5!} + \frac{t^5(q^2+q^3)r^2}{2\cdot 3!} + \frac{t^6(q^2+2q^3+q^4)r^3}{2\cdot 4!} + \\
&\frac{t^6(q^2+q^3+q^4)r^2}{3!3!} + \frac{7t^6q^3r^4}{2\cdot 3!3!} + \text{other terms}.
\end{aligned}
$$

Then it is sufficient to observe that

$$
\Phi_{G(r,1)}(q,t) = (1+\Gamma_{G(r,1)}(q,t) + \Gamma_{G(r,1)}(q,t)^2 + \Gamma_{G(r,1)}(q,t)^3 + \ldots)\mathcal{K}_{G(r,1)}(q,t)
$$

to compute the correct expression of the Poincaré polynomial of the model $Y_{\mathcal{F}_{G(r,1,n)}}$:

$$
\begin{aligned}
\Phi_{G(r,1)}(q,t)_{[t^6]} = &1 + q(57 + 20r^2 + 15r^3 + 6r^4 + r^5) \\
&+ q^2(302 + 220r^2 + 75r^3 + 82r^4 + 36r^5) \\
&+ q^3(302 + 220r^2 + 75r^3 + 82r^4 + 36r^5) \\
&+ q^4(57 + 20r^2 + 15r^3 + 6r^4 + r^5) + q^5.
\end{aligned}
$$

Observation 4.4. We observe that one can evaluate at $r = 1$ the expression of the Poincaré polynomial in the general case $G(r,1,n)$ to recover formulas for the braid case analyzed in Section 3.2.1.
For instance, it easy to check the following equality:

$$
P(q)_{Y_{\mathcal{F}_{G(r,1,6)}}}|_{r=1} = 1 + 99q + 715q^2 + 715q^3 + 99q + 1,
$$

the same polynomial found in the braid case $Y_{\mathcal{F}_{A_6}}$.

Remark 4.4. In Section 3.2.1 we proposed a combinatorial verification to the formula for the Poincaré series, enumerating all possible admissible Yuzvinski monomials along with their corrispective contribution to the Poincaré

polynomial. This enumeration allowed us to find formulas for the first cohomology groups.
A similar process can be done in this more general case of complex reflection groups, taking care of the more complex structure of underlying combinatorial data.

For instance, in the next table we collect all possible *admissible* $Y_{\mathcal{F}_{G(r,1,6)}}$-nested set, i.e. nested sets that generate admissible monomials, along with their related contribution to the Poincaré polynomial. We remark that in this more general case the structure of the nested sets is slightly different. In particular we can have three different forms of nested set:

- nested sets that purely refer to some *strong* subspaces, for example $\{\{0,1,2,3,4,5\},\{0,1,2\}\}$ in our usual (weighted) partition disguise;
- (weighted) nested sets that purely refer to some *weak* (weighted) subspaces, for example $\{\{1,2,\overset{1}{3},4,\overset{2}{5},6\},\{1,2,\overset{1}{3},4\}\}$;
- *mixed* nested sets that contain some strong and some weak (weighted) subspaces, for example $\{\{0,1,2,3,4,5\},\{0,1,2\},\{3,\overset{2}{4},5\}\}$.

Summing up all the possible contributions collected in the table, we find, as expected, the same expressione as before for the Poincaré polynomial of $Y_{\mathcal{F}_{G(r,1,n)}}$:

$$\begin{aligned} P_{G(r,1)}(q,r) = &1 + q(57 + 20r^2 + 15r^3 + 6r^4 + r^5) \\ &+ q^2(302 + 220r^2 + 75r^3 + 82r^4 + 36r^5) \\ &+ q^3(302 + 220r^2 + 75r^3 + 82r^4 + 36r^5) \\ &+ q^4(57 + 20r^2 + 15r^3 + 6r^4 + r^5) + q^5. \end{aligned}$$

Nested set structure	Total nested sets	Contribution
$\mathcal{S}$		$P(\mathcal{S})$
Strong		
$\{012\}$	15	$15q$
$\{0123\}$	20	$20(q+q^2)$
$\{01234\}, \{012\}$	90	$90q^2$
$\{01234\}$	15	$15(q+q^2+q^3)$
$\{012345\}$	6	$6(q+q^2+q^3+q^4)$
$\{012345\}, \{012\}$	60	$60q(q+q^2)$
$\{012345\}, \{0123\}$	60	$60q(q+q^2)$
$\{0123456\}$	1	$q+q^2+q^3+q^4+q^5$
$\{0123456\}, \{01234\}$	15	$15q(q+q^2+q^3)$
$\{0123456\}, \{01234\}, \{012\}$	90	$90q^3$
$\{0123456\}, \{0123\}$	20	$20(q+q^2)^2$
$\{0123456\}, \{012\}$	15	$15q(q+q^2+q^3)$
Weak		
$\{123\}$	20	$20qr^2$
$\{1234\}$	15	$15(q+q^2)r^3$
$\{12345\}$	6	$6(q+q^2+q^3)r^4$
$\{12345\}, \{123\}$	60	$60q^2r^4$
$\{123456\}$	1	$(q+q^2+q^3+q^4)r^5$
$\{123456\}, \{123\}$	20	$20q(q+q^2)r^5$
$\{123456\}, \{1234\}$	15	$15q(q+q^2)r^5$
$\{123\}, \{456\}$	10	10^2r^4
Mixed		
$\{012\}, \{345\}$	60	$60q^2r^2$
$\{012\}, \{3456\}$	15	$15q(q+q^2)r^3$
$\{0123\}, \{456\}$	20	$20q(q+q^2)r^2$
$\{01234\}, \{123\}$	60	$60q^2r^2$
$\{012345\}, \{123\}$	60	$60q(q+q^2)r^2$
$\{012345\}, \{1234\}$	30	$30q(q+q^2)r^3$
$\{0123456\}, \{123\}$	20	$20q(q+q^2+q^3)r^2$
$\{0123456\}, \{1234\}$	15	$15(q+q^2)^2r^3$
$\{0123456\}, \{12345\}$	6	$6q(q+q^2+q^3)r^4$
$\{0123456\}, \{12345\}, \{123\}$	60	$60q^3r^4$
$\{0123456\}, \{123\}, \{456\}$	10	$10q^3r^4$
$\{0123456\}, \{01234\}, \{123\}$	60	$60q^3r^2$
$\{0123456\}, \{012\}, \{345\}$	60	$60q^3r^2$

Observation 4.5. Also in this more general case we notice that, starting from either the formula for $\Phi_{G(r,1)}(q,t,z)$ or the equivalent language of combinatorial nested set over (weighted) partition of $\{0,1,2,\ldots,n\}$, it is possible to compute expressions for the dimension of the cohomology groups $H^{2m}(Y_{\mathcal{F}_{G(r,1,n-1)}},\mathbb{Z})$.
The first non vanishing cohomolgy group has dimension given by the following formula

$$\dim\, H^2(Y_{\mathcal{F}_{G(r,1,n-1)}},\mathbb{Z}) = \; 2^{n-1} - 2n + 1 - \frac{(n-1)(n-2)}{2}r + \frac{(1+r)^{n-1}-1}{r}.$$

First of all we observe that evaluating the previous expression for $r=1$ we found the same result for $H^2(Y_{\mathcal{F}_{A_{n-1}}},\mathbb{Z})$, as expected. Then recall that we are interested in counting only admissible monomials c_A with A either *strong* or *weak* subspace (Mixed nested sets give their contribution only in higher dimension).
The total ammount of *strong* subspace is in one to one correspondence with subspace of $\{0,1,\ldots,n-1\}$ with cardinality at least 3 that necessarily contain the first element 0: they are $\sum_{i=2}^{n-1}\binom{n-1}{i} = 2^{n-1}-n$. On the other hand, *weak* subspaces are in one to one correspondence with weighted subspace of $\{1,2,\ldots,n-1\}$ with at least 3 elements. Since we can fix the first exponent, the correct number is given by the following summation

$$\sum_{i=3}^{n-1}\binom{n-1}{i}r^{i-1} = \frac{(1+r)^{n-1}-1}{r} - (n-1) - \frac{(n-1)(n-2)}{2}r.$$

In addition to the number of *strong* subspaces previously calculated, we find the desired expression

$$\dim\, H^2(Y_{\mathcal{F}_{G(r,1,n-1)}},\mathbb{Z}) = \; 2^{n-1} - 2n + 1 - \frac{(n-1)(n-2)}{2}r + \frac{(1+r)^{n-1}-1}{r}.$$

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