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A computer algebra approach to rational general solutions of algebraic ordinary differential equations

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Se i tuoi occhi fossero ciliegie, io non ci troverei niente da dire.
Francesco De Gregori

Contents

Abstract

In this thesis, I approach to algebraic ODEs from Differential Algebra's point of view. I look for rational solutions of AODE, I present an algebrogeometric method to decide the existence of rational solutions of a first-order algebraic ODE and if they exist I give an algorithm to compute them. This method depends heavily on rational parametrizations, in particular for autonomous equations on the parametrization of algebraic curves, and for non-autonomous equations on the parametrization of algebraic surfaces. In the first case I use a proper rational parametrization ϕ of the curve defined by the equation, regarded as algebraic equation, as starting point to find a rational solution of the equation. I look for a rational function f such that $\pi_1(\phi \circ f)$, where π_1 is the projection on the first coordinate, is a rational solution of the equation. I show that such a f exists only in two cases. In the last case, I prove the correspondence between rational solutions of a parametrizable algebraic ODE and rational solutions of a rational first-order linear autonomous differential system of two equations in two variables. I provide an algorithm to compute rational solutions of such system based on its invariant algebraic curves. Rational parametrizations of such curves are good candidates to be rational solution of the system. Since computing rational parametrization of algebraic surfaces and the method presented depends heavily on them. In order to avoid that computation I also study a group of affine transformations which preserves the rational solvability, in order to reduce, when possible, an algebraic ODE to an easier one. Moreover I present the results of the implementation of all these algorithms in two computer algebra system: CoCoA and SINGULAR.

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Introduction

Solving algebraic ordinary differential equations is still a challenge in computer algebra. After the work by J.F. Ritt, *Differential Algebra* and later by E.R. Kolchin, *Differential Algebra and Algebraic Groups,* the theory of differential equations has been rapidly developed from the algebraic point of view. In particular, most of the studies of algebraic ODEs can be seen as a differential counterpart of the one of algebraic equations. Clearly this last ones are much more studied in computer algebra. The first definition of general solution of an algebraic ODE appeared in the paper *The general solution of an ordinary differential equations,* E. Hubert, 1996. Where is given a method to compute a basis of the prime differential ideal defining the general component. While in the papers *Rational general solutions of algebraic ordinary differential equations,* 2004, and *A polynomial time algorithm for finding rational general solutions of first order autonomous ODEs,* 2006, by R. Feng and X-S. Gao the authors give an algorithm for explicitly computing a rational general solution of the autonomous algebraic ODE $F(y, y') = 0$. The method is based on the computation of a proper rational parametrization of the corresponding algebraic curve $F(y, z) = 0$. This last one is the starting point of my work. Following the papers *Rational general solutions of first order non- autonomous parametrizable ODEs*, 2010 and *Rational general solutions of planar rational systems of autonomous ODEs,* 2011 by my advisor F. Winkler, joint with L. X. C. Ngô, I studied how the method ideated by Feng and Gao works in the case of non-autonomous equations. All the work is based on the will of finding an efficient algebrogeometric method to compute rational general solutions of non-linear algebraic ODEs that can be used in the most general cases and, concerning the algorithmic aspect, of seeing how much can be done with the algebraic tools now available in common computer algebra systems, for instance, CoCoA and Singular.

The thesis consists of three main chapters 3,4, and 5. In Chapter 3, we give a criterion of existence for rational general solution of an autonomous first order algebraic ODE based on *Gao's differential polynomial.* Then we present a geometric method to decide the existence of a rational general solution of a autonomous parametrizable algebraic ODE of order 1. In the affirmative case this decision method can be turned into an algorithm for actually computing such a rational general solution. We conclude this chapter with some examples.

In Chapter 4, we present a geometric method to decide the existence of a rational general solution of a non-autonomous parametrizable algebraic ODE of order 1. As in the autonomous case, in the affirmative case this decision method can be turned into an algorithm for actually computing such a rational general solution. More precisely, a proper rational parametrization of the solution surface allows us to reduce the given differential equation to a system of autonomous algebraic ODEs of order 1 and of degree 1 in the derivatives, called the *associated system* with respect to the chosen parametrization. This often turns out to be an advantage because the original differential equation is typically of higher degree in the derivative. One of he main result in this chapter is the one-to-one correspondence between a rational general solution of the associated system and that of the given differential equation. We give a criterion of existence of rational general solution of the associated system under a degree bound for the solution. Then we present a geometric method to compute rational general solutions of the associated system based on the notion of an *invariant algebraic curve* of a planar polynomial system. Since the method presented in this chapter depends heavily on computing rational parametrizations of algebraic surfaces, that computation is hard, so in order to avoid it, in this chapter we define a group of linear transformations. The main idea in the construction of the group is to preserve the rational solvability of the differential equations under the group action. We use the action of this group on the set of parametrizable algebraic ODE, to transform the equations into other equations of a special form, for instance autonomous equations.

In Chapter 5, I present the result of the implementation of the methods presented in chapters 3 and 4, moreover I analyse what can be done so far and what could be the next steps to improve the method.

Almost all the results in this thesis have already been published, in particular the proof of Lemma 4.8, Lemma 4.20, Proposition 4.21 and their consequences are original and the whole chapter 5 is new, moreover it is the first implementation of these methods in SINGULAR and CoCoA.

Chapter 1

Basics of Algebraic Geometry

In this chapter I make an introduction on basic concepts of algebraic geometry, focusing on curves, surfaces and on the properties of their parametrizations. The main results in this chapter are about rationality and properness of a parametrization of an algebraic curve, these properties will be fundamental in next chapters. In the last section I describe a way to decide if a rational parametrization of a rational curve is proper by computing its tracing index. This chapter is based on the contents of [SWPD08].

1.1 Algebraic curves and algebraic surfaces

In this and next chapters we indicate by $\mathbb{K}$ an algebraically closed field of characteristic zero. An *algebraic hypersurface* $\mathcal{S}$ in the n-dimensional affine space $\mathbb{A}^n(\mathbb{K})$ is an algebraic set defined by a non-constant polynomial f in $\mathbb{K}[x_1, \ldots, x_n]$. The squarefree part of f defines the same set $\mathcal{S}$, so we might as well require the defining polynomial to be squarefree. In particular, if $n = 2$ and $n = 3$ we have a *plane algebraic curve* and an *algebraic surface,* respectively.

Definition 1.1. An *algebraic hypersurface* over $\mathbb{K}$ is defined as the set

$$\mathcal{S} = \big\{(a_1, \ldots, a_n) \in \mathbb{A}^n(\mathbb{K}) \mid f(a_1, \ldots, a_n) = 0\big\}$$

for a non constant squarefree polynomial $f(x_1, \ldots, x_n) \in \mathbb{K}[x_1, \ldots, x_n]$.

Definition 1.2. A *parametrization* of a hypersurface $\mathcal{S}$ defined by $f \in \mathbb{K}[x_1, \ldots, x_n]$ is a mapping

$$\phi\colon \mathbb{K}^{n-1} \to \mathbb{A}^n(\mathbb{K}),$$
$$v \mapsto (\phi_1(v), \ldots, \phi_n(v))$$

such that

$$f(\phi_1(v), \ldots, \phi_n(v)) = 0,$$

for each $v \in \mathbb{K}^{n-1}$.

In particular we are interested in rational parametrizations which are invertible.

Definition 1.3. A parametrization ϕ of a hypersurface $\mathcal{S} \subset \mathbb{A}^n(\mathbb{K})$ is called *rational* if ϕ_i is a rational function for each i, i.e. $\phi_i \in \mathbb{K}(v_1, \ldots, v_{n-1})$.

Definition 1.4. A hypersurface $\mathcal{S} \subset \mathbb{A}^n(\mathbb{K})$ is called *rational hypersurface* if it admits a rational parametrization.

In this thesis we consider only non-constant polynomials which defines rational curves or rational surfaces.

Definition 1.5. A parametrization ϕ of a hypersurface $\mathcal{S} \subset \mathbb{A}^n(\mathbb{K})$ is called *proper* if ϕ has an inverse, i.e., there is a rational mapping

$$\eta \colon \mathbb{A}^n(\mathbb{K}) \to \mathbb{K}^{n-1},$$

$$x \mapsto (\eta_1, \ldots, \eta_{n-1})$$

such that

$$(\eta \circ \phi)(v) = v$$

for almost all v, i.e., except for finitely many vectors in $\mathbb{K}^{n-1}$, and

$$\phi \circ \eta(x) = x$$

for almost every $x \in \mathcal{S}$.

We have an equivalent condition of properness of a parametrization.

Theorem 1.1. *Let $\mathcal{S} \subset \mathbb{A}^n(\mathbb{K})$ be a hypersurface and ϕ a rational parametrization of $\mathcal{S}$. Then, $\phi(v_1, \ldots, v_{n-1})$ is proper with inverse map η if and only if*

$$\mathbb{K}(\phi_1(v_1, \ldots, v_{n-1}), \ldots, \phi_n(v_1, \ldots, v_{n-1})) = \mathbb{K}(v_1, \ldots, v_{n-1}). \tag{1.1}$$

Proof. Suppose that the equation (1.1) is true. Then for each $i = 1, \ldots, n-1$ exist $a_{i,J} \in \mathbb{K}$ for $J = (j_1, \ldots, j_n) \in \mathbb{N}^n$, with only finitely many of them different from zero, such that

$$v_i = \sum_J a_{i,J} f_1^{j_1} \cdots f_n^{j_n}.$$

Hence, the map

$$\eta \colon (y_1, \ldots, y_n) \mapsto \big(\cdots, \sum_J a_{i,J} y_1^{j_1} \cdots y_n^{j_n}, \cdots)$$

is clearly the inverse mapping of ϕ.
If ϕ is proper then exists a rational mapping η such that $\eta(\phi(v_1, \ldots, v_n)) = (v_1, \ldots, v_n)$. Since η_i is a rational mapping for each $i = 1, \ldots, n-1$, we have

$$v_i = \eta_i(\phi_1, \ldots, \phi_n) \in \mathbb{K}(\phi_1, \ldots, \phi_n).$$

Then the (1.1) follows. □

1.2 Rational curves

Definition 1.6. A *rational algebraic curve* $\mathcal{C}$ is an algebraic hypersurface of dimension 1. If $\mathcal{C}$ is of codimension 1 it is called *plane rational algebraic curve.*

Definition 1.7. Let $\mathcal{C}$ be a plane curve over $\mathbb{K}$ defined by $f(x, y) \in \mathbb{K}[x, y]$, and let $P = (a, b) \in \mathcal{C}$. We say that P is of multiplicity r on $\mathcal{C}$ if all the derivatives of f up to and including the $(r-1)$th vanish at P but at least one rth derivative does not vanish at P. We denote the multiplicity of P on C by $mult_P(C)$. P is called a *simple* point on $\mathcal{C}$ if $mult_P(\mathcal{C}) = 1$. If $mult_P(\mathcal{C}) = r > 1$, then we say that P is a *multiple* or *singular* point (or *singularity*) of multiplicity r on $\mathcal{C}$ or an r-*fold* point.

Definition 1.8. A singular point P of multiplicity r on a plane algebraic curve $\mathcal{C}$ is called *ordinary* if the r tangents to $\mathcal{C}$ at P are distinct, and *non-ordinary* otherwise.

As we did for the affine case, we can define a curve in the projective plane.

Definition 1.9. A *projective plane algebraic curve* over $\mathbb{K}$ is defined as the set

$$\mathcal{C}_{\mathbb{P}} = \left\{(a : b : c) \in \mathbb{P}^2(\mathbb{K}) \mid F(a, b, c) = 0\right\}$$

for a non-constant squarefree homogeneous polynomial $F(x, y, z) \in \mathbb{K}[x, y, z]$.

Definition 1.10. Let $\mathcal{C}$ be an irreducible curve defined by $f \in \mathbb{K}[x, y]$. The *homogenisation* of $\mathcal{C}$ is the projective curve $\mathcal{C}^*$ defined by $f^* \in \mathbb{K}[x, y, z]$ the homogenisation of f.

We can apply quadratic transformations (blow-ups) for birationally transforming the curve into a curve with only ordinary singularities. For such a curve the *genus* can be readily determined by proper counting the singularities, so we define the genus of an irreducible projective curve with only ordinary singularities as follows

Definition 1.11. Let $\mathcal{C}$ be a projective curve with only ordinary singularities, and let d be the degree of $\mathcal{C}$. Then

$$genus(\mathcal{C}) = \frac{1}{2}\Big[(d-1)(d-2) - mult_P(\mathcal{C})(mult_P(\mathcal{C}) - 1)\Big],$$

for $P \in Sing(\mathcal{C})$, where $Sing(\mathcal{C})$ is the set of the singular points of $\mathcal{C}$.

Theorem 1.2. *Any rational curve is irreducible.*

Proposition 1.3. *Let $\mathcal{C}$ be an irreducible curve and $\mathcal{C}^*$ its corresponding projective curve. Then $\mathcal{C}$ is rational if and only if $\mathcal{C}^*$ is rational. Furthermore, a parametrization of $\mathcal{C}$ can be computed from a parametrization of $\mathcal{C}^*$ and vice versa.*

Theorem 1.4. *An algebraic curve $\mathcal{C}$ is rational if and only if $genus(\mathcal{C}) = 0$.*

Let's now focus on rational parametrizations, first of all we recall now a deep result in field theory due to Lüroth.

Theorem 1.5 (Lüroth's Theorem)**.** *Let $\mathbb{L}$ be a field (not necessarily algebraically closed), t a transcendental element over $\mathbb{L}$. If $\mathbb{K}$ is a subfield of $\mathbb{L}(t)$ strictly containing $\mathbb{L}$, then $\mathbb{K}$ is $\mathbb{L}$-isomorphic to $\mathbb{L}(t)$.*

Thanks to Theorem 1.1 and Lüroth's Theorem we get

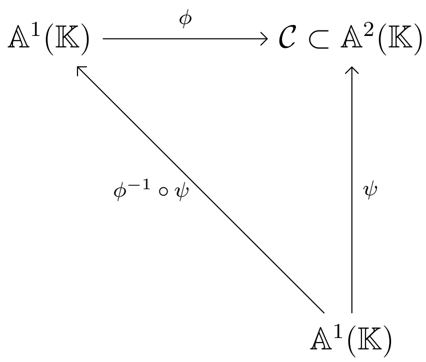
Corollary 1.6. *Every rational curve can be properly parametrized.*

Next two theorems will be fundamental in the method presented in Chapter 2 and Chapter 3.

Theorem 1.7. *Let $\phi(t)$ be a proper parametrization of a rational curve $\mathcal{C}$, and let $\psi(t)$ be any other rational parametrization of $\mathcal{C}$. Then*

1. *There exists a nonconstant rational function $\rho(t) \in \mathbb{K}(t)$ such that $\psi(t) = \phi(\rho(t))$.*
2. *$\psi(t)$ is proper if and only if there exists a linear rational function $\lambda(t) \in \mathbb{K}(t)$, i.e. $\lambda(t) = \frac{at+b}{ct+d}$ with $a, b, c, d \in \mathbb{K}$, such that $\psi(t) = \phi(\lambda(t))$.*

Proof. 1. We consider the following diagram:



Then, since is ϕ is a birational mapping, it is clear that $\rho(t) = \phi^{-1} \circ \psi \in \mathbb{K}(t)$.

2. If ψ is proper, then by the diagram above we see that $\beta = \phi^{-1} \circ \psi$ is a birational mapping from $\mathbb{A}^1(\mathbb{K})$ onto $\mathbb{A}^1(\mathbb{K})$, so we have an isomorphism between the rings of functions, since $\mathbb{A}^1(\mathbb{K})$ is a line its ring of functions is $\mathbb{K}(t)$. Hence, β induces an automorphism $\tilde{\beta}$ of $\mathbb{K}(t)$ defined as:

$$\tilde{\beta} \colon \mathbb{K}(t) \to \mathbb{K}(t),$$
$$t \mapsto \beta(t).$$

Therefore, since $\mathbb{K}$-automorphisms of $\mathbb{K}(t)$ are the invertible rational functions, we see that $\tilde{\beta}$ is our linear rational function.
Conversely, let η be the birational mapping from $\mathbb{A}^1(\mathbb{K})$ onto $\mathbb{A}^1(\mathbb{K})$ defined by the linear rational function $\lambda(t) \in \mathbb{K}(t)$. Then, it is clear that $\psi = \phi \circ \eta \colon \mathbb{A}^1(\mathbb{K}) \to \mathcal{C}$ is a birational mapping, and therefore $\psi(t)$ is proper.

∎

Theorem 1.8. *Let $\mathcal{C}$ be a rational curve defined over $\mathbb{K}$ with defining polynomial $f(x,y) \in \mathbb{K}[x,y]$, and let $\phi(t) = (\phi_1(t), \phi_2(t))$ be a parametrization of $\mathcal{C}$. Then $\phi(t)$ is proper if and only if $deg(\phi(t)) = max\{deg_x(f), deg_y(f)\}$.*
Furthermore, if $\phi(t)$ is proper and $\phi_1(t)$ is nonzero, then $deg(\phi_1(t)) = deg_y(f)$; similarly, if $\phi_2(t)$ is nonzero then $deg(\phi_2(t)) = deg_x(f)$.

If the parametrization $\phi(t)$ is of the form

$$\phi(t) = \Big(\frac{\phi_{11}}{\phi_{12}}, \frac{\phi_{21}}{\phi_{22}}\Big),$$

we define the following polynomials

$$H_1^\phi(t,x) = x\phi_{12}(t) - \phi_{11}(t),$$

$$H_2^\phi(t,y) = y\phi_{22}(t) - \phi_{21}(t).$$

Theorem 1.9. *Let $\phi(t)$ be a proper parametrization in reduced form of a rational affine plane curve $\mathcal{C}$. Then, the defining polynomial of $\mathcal{C}$ is the resultant*

$$res_t(H_1^\phi(t,x), H_2^\phi(t,y)).$$

1.3 Tracing index

Definition 1.12. Let W_1 and W_2 be varieties over $\mathbb{K}$. Let $\phi\colon W_1 \to W_2$ be a rational mapping such that $\phi(W_1) \subset W_2$ is dense. Then ϕ is a *dominant* mapping from W_1 to W_2.

Definition 1.13. The *degree* of the dominant rational mapping ϕ from W_1 to W_2, where $dim(W_1) = dim(W_2)$, is the degree of the finite algebraic field extension $\mathbb{K}(W_1)$ over $\tilde{\phi}(\mathbb{K}(W_2))$, that is

$$degree(\phi) = [\mathbb{K}(W_1) : \tilde{\phi}(\mathbb{K}(W_2))],$$

where $\tilde{\phi}\colon \mathbb{K}(W_2) \to \mathbb{K}(W_1), \rho \mapsto \rho \circ \phi$.

Observe that the notion of degree can be used to characterize the birationality of rational mappings as follows.

Lemma 1.10. *A dominant rational mapping $\phi\colon W_1 \to W_2$ between varieties of the same dimension is birational if and only if $degree(\phi) = 1$.*

Consider now a rational curve $\mathcal{C}$ and a proper rational parametrization $\phi(t)$ of the curve $\mathcal{C}$. We will denote by $\mathcal{F}_\phi(P)$ the fibre of a point $P \in \mathcal{C}$; that is

$$\mathcal{F}_\phi(P) = \phi^{-1}(P) = \big\{t \in \mathbb{K} \mid \phi(t) = P\big\}.$$

Intuitively speaking, the degree of the mapping ϕ measures the number of times the parametrization traces the curve when the parameter takes values in $\mathbb{K}$.

Definition 1.14. Let $\mathcal{C}$ be a rational curve, and let $\phi(t)$ be a rational parametrization of $\mathcal{C}$. Then the tracing index of $\phi(t)$, denoted by index$(\phi(t))$, is the degree of $\phi\colon \mathbb{A}(\mathbb{K}) \to \mathcal{C}, t \mapsto \phi(t)$; i.e., index$(\phi(t))$ is a natural number such that almost all points on $\mathcal{C}$ are generated, via $\phi(t)$, by exactly index$(\phi(t))$ parameter values.

If the parametrization $\phi(t)$ is of the form

$$\phi(t) = \left(\frac{\phi_{11}}{\phi_{12}}, \frac{\phi_{21}}{\phi_{22}}\right),$$

we define the bivariate polynomials

$$G_1^\phi(x, y) = \phi_{11}(x)\phi_{12}(y) - \phi_{12}(x)\phi_{11}(y),$$
$$G_2^\phi(x, y) = \phi_{21}(x)\phi_{22}(y) - \phi_{22}(x)\phi_{21}(y).$$

Theorem 1.11. *Let $\phi(t)$ be a rational parametrization of in reduced form. Then for almost all $a \in \mathbb{K}$ we have*

$$card(\mathcal{F}_\phi(\phi(a))) = deg_y(gcd(G_1^\phi(a, y), G_2^\phi(a, y))).$$

Lemma 1.12. *Let $\phi(t)$ be a rational parametrization in reduced form. Then for almost all values $a \in \mathbb{K}$ of x we have*

$$deg_y(gcd(G_1^\phi(x, y), G_2^\phi(x, y))) = deg_y(gcd(G_1^\phi(a, y), G_2^\phi(a, y))).$$

From the theorem and the lemma above follow

Theorem 1.13. *Let $\phi(t)$ be a rational parametrization in reduced form of the curve $\mathcal{C}$. Then*

$$\text{index}(\phi(t)) = deg_y(gcd(G_1^\phi(x, y), G_2^\phi(x, y)).$$

Theorem 1.14. *A rational parametrization of a curve $\mathcal{C}$ is proper if and only if its tracing index is* 1, *i.e. if and only if $deg_y(gcd(G_1^\phi, G_2^\phi)) = 1$.*

The previous results can be used to derive the following algorithm for computing the tracing index of a given parametrization. This algorithm can also be used for checking the properness of a parametrization.

INPUT: A rational parametrization $\phi(t)$ in reduced form.
OUTPUT: The tracing index of the parametrization $\phi(t)$ given in input, and if $\phi(t)$ is proper or not.

1. Compute the polynomials $G_1^\phi(x, y), G_2^\phi(x, y)$.
2. Determine $G^\phi(x, y) = gcd(G_1^\phi(x, y), G_2^\phi(x, y))$.
3. $l = deg_y(G^\phi(x, y))$.
4. If $l = 1$ then return "$\phi(t)$ is proper and index$(\phi(t)) = 1$" else return "$\phi(t)$ is not proper and index$(\phi(t)) = l$."

We illustrate the algorithm by an example.

Example 1.1. Let $\phi(t)$ be the rational parametrization

$$\phi(t) = \Big(\frac{(t^2-1)t}{t^4-t^2+1}, \frac{(t^2-1)t}{t^6-3t^4+3t^2-1-2t^3}\Big).$$

In step 1 the polynomials

$$G_1^{\phi}(x,y) = x^3y^4 - xy^4 + x^2y^3 - x^4y^3 - y^3 - x^3y^2 + xy^2 + x^4y - x^2y + y + x^3 - x,$$

$$G_2^{\phi}(x,y) = x^4y^6 - x^2y^6 - x^6y^4 + 2x^3y^4 + y^4 - 2x^4y^3 + 2x^2y^3 + x^6y^2 - 2x^3y^2 - y^2 - x^4 + x^2,$$

are generated. Their great common divisor G^{ϕ} is computed in step 2.

$$G^{\phi}(x,y) = xy^2 - x^2y + y - x.$$

In step 3,

$$l = deg_y(G^{\phi}(x,y)) = 2.$$

Thus, index$(\phi(t)) = 2$, and therefore the parametrization is not proper.

Chapter 2

Basics of Differential Algebra

In this chapter I make an easy introduction on first concepts of Differential Algebra essentially based on [Ritt], [Kol73] and [Ngo]. I also make first steps on algebraic ODEs theory from an algebraic point of view defining what an algebraic ODE is and the notion of general solution of an algebraic ODE.

2.1 First definitions

Definition 2.1. Let $\mathcal{R}$ be a commutative ring and $\delta\colon \mathcal{R} \to \mathcal{R}$ be a mapping such that

$$\delta(a+b) = \delta(a) + \delta(b), \text{ and } \delta(ab) = \delta(a)b + a\delta(b).$$

Then $(\mathcal{R}, \delta)$ is called a *differential ring* with the *derivation* δ. If $\mathcal{R}$ is a field then $(\mathcal{R}, \delta)$ is called a *differential field.*

In this thesis we deal with an algebraically closed field of characteristic 0, i.e. it contains $\mathbb{Q}$, named $\mathbb{K}$.

Example 2.1. Let $\mathcal{R}$ be a commutative ring and δ the operator such that

$$\delta(a) = 0, \text{ for every } a \in \mathcal{R}.$$

Clearly δ is a derivation, then $(\mathcal{R}, \delta)$ is a differential ring.

Example 2.2. Let $\mathbb{K}(x)$ be the field of the rational functions over $\mathbb{K}$. Let δ be a derivation on $\mathbb{K}(x)$. Then the conditions

$$\delta(a) = 0 \text{ for all } a \in \mathbb{K}$$

and

$$\delta(x) = 1$$

define uniquely the usual derivation $\frac{d}{dx}$ on $\mathbb{K}(x)$. Therefore, $\left(\mathbb{K}(x), \frac{d}{dx}\right)$ is a differential field.

Note that the concept of differential ring or field can be generalised to an arbitrary finite number of derivations.

Example 2.3. The field of complex meromorphic functions on a given region of the space of m complex variables $x_1, \ldots, x_m$, with the set of operators $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}$ is a differential field.

The differential ring is called *ordinary* if it is only equipped with one derivation. We consider only ordinary differential rings.

Definition 2.2. Let $(\mathcal{R}, \delta)$ be a differential ring. The set $C = \{c \in \mathcal{R} \mid \delta(c) = 0\}$ is called the set of *constants* of $\mathcal{R}$ with respect to the derivation δ. If $\mathcal{R}$ is a field, then C is a subfield of $\mathcal{R}$. In this case, we also call C the *field of constants* of $\mathcal{R}$ with respect to δ.

Example 2.4 (Continuing Example 2.2)**.** The field of constants of $\left(\mathbb{K}(x), \frac{d}{dx}\right)$ is $\mathbb{K}$.

Definition 2.3. Let $(\mathcal{R}, \delta)$ be an ordinary differential ring. An ideal I of $\mathcal{R}$ is called a *differential ideal* if I is closed under the derivation δ, i.e., for all $a \in I$ we have $\delta(a) \in I$.

Definition 2.4. Let S be a set of differential polynomials in $\mathcal{R}$. The differential ideal generated by S, denoted by (S), is the ideal generated by all elements in S and their derivatives. The radical differential ideal generated by S, denoted by $\{S\}$, is the radical of (S).

Consider now the infinite sequence of symbols

$$y, y', y'', \cdots, y^{(n)}, \cdots . \tag{2.1}$$

We call y *differential indeterminate* and $y^{(p)}$ is the pth *derivative* of y. Where $y = y^{(0)}, y' = y^{(1)}$ and $y'' = y^{(2)}$. Furthermore, for every p and for every $q > 0$, $y^{(p+q)}$ is the qth derivative of $y^{(p)}$. It is to be pointed out that in (2.1) only y is a differential indeterminate. We want to use the symbols in (2.1) to build polynomials, clearly only a finite subset of them for each polynomial.

Definition 2.5. Let $(\mathcal{F}, \delta)$ be a differential field. Consider the polynomial ring

$$\mathcal{R} = \mathcal{F}[y, y', y'', \ldots],$$

and the derivation $\tilde{\delta}$ on $\mathcal{R}$ obtained from δ as follows

$$\tilde{\delta}(y^{(i)}) = y^{(i+1)},$$

$$\tilde{\delta}(a) = \delta(a), \text{ for all } a \in \mathcal{F},$$

$$\tilde{\delta}\Big(\sum_{i_1,\cdots,i_k} a_{i_1\cdots i_k} y^{(i_1)} \cdots y^{(i_k)}\Big) =$$

$$= \sum_{i_1,\cdots,i_k} \delta(a_{i_1\cdots i_k}) y^{(i_1)} \cdots y^{(i_k)} + a_{i_1\cdots i_k} y^{(i_1+1)} y^{(i_2)} \cdots y^{(i_k)} +$$

$$+ \cdots + a_{i_1\cdots i_k} y^{(i_1)} \cdots y^{(i_{k-1})} y^{(i_k+1)}.$$

So $(\mathcal{R}, \tilde{\delta})$ defines a differential ring, denoted by $\mathcal{F}\{y\}$. A polynomial in $\mathcal{F}\{y\}$ is called *differential polynomial*.

Example 2.5. Consider the differential field of rational functions $\left(\bar{\mathbb{Q}}(x), \frac{d}{dx}\right)$ then

$$G = xy'^2 + yy' - y^4$$

is an element of $\bar{\mathbb{Q}}(x)\{y\}$. The derivative of $G(x, y, y')$ is

$$\delta(G) = y'^2 + 2xy'y'' + y'^2 + yy'' - 4y^3y',$$

where $\delta = \frac{\tilde{d}}{dx}$ as in the definition above.

We can also define differential polynomial ring in more than one differential indeterminates, for instance consider the differential indeterminates $y_1, \cdots, y_n$, then

$$\mathcal{F}\{y_1, y_2, \cdots, y_n\}$$

is the ring of all poynomial combinations of $y_1, \cdots, y_n$ and of their derivatives with coefficients in $\mathcal{F}$.

Example 2.6. The ring $\mathbb{K}(x)\{y_1, y_2\}$ is a differential polynomial ring.

Definition 2.6. Let $(\mathcal{F}, \delta)$ be a differential field. The *order* of $G \in \mathcal{F}\{y\}$ is the highest $p \in \mathbb{N}$ such that $y^{(p)}$ effectively occurs in G. It is denoted by $\mathrm{ord}(G)$, $\mathrm{ord}_y(G)$ if the ring has more than one differential indeterminate.

Example 2.7. The order of $G = xy'^2 + yy' - y^4$ is $\mathrm{ord}(G) = 1$.

Definition 2.7. A differential polynomial $F \in \mathcal{R} = \mathcal{F}\{y_1, \ldots, y_n\}$ is irreducible if it can not be written as product of two differential polynomials.

2.2 Rankings

Here we give the definitions for ordinary differential rings, for more general settings please check [Kol73].

Definition 2.8. Let $(\mathcal{F}, \delta)$ be a differential field and $\mathcal{R} = (\mathcal{F}\{y_1, \ldots, y_n\}, \tilde{\delta})$ be the differential ring with indeterminates $y_1, \ldots, y_n$. A *ranking* of $(y_1, \ldots, y_n)$ is a total ordering of the set of all derivatives

$$\Delta y := \{\delta^j(y_i) \mid i = 1, \ldots, n; j \in \mathbb{N}\},$$

such that

$$u \leq \delta^j(u) \text{ and } u \leq v \Rightarrow \delta^j(u) \leq \delta^j(v)$$

for all $u, v \in \Delta y$ and $j \in \mathbb{N}$.

We say that $u \in \Delta y$ is *of higher rank than* $v \in \Delta y$ if $v \leq u$, and that u is *of lower rank than* v if $u \leq v$, with respect to a ranking of $(y_1, \ldots, y_n)$.

Definition 2.9. A ranking is said to be:

- *integrated* if for any $\delta^j y_{i_1}, \delta^k y_{i_2}$, there exists $l \in \mathbb{N}$ such that

$$\delta^j y_{i_1} \leq \delta^{k+l} y_{i_2}.$$

- *sequential* if every derivative $\delta^j y_i$ is of higher rank than only a finite number of derivatives.
- *orderly* if

$$j \leq k \Rightarrow \delta^j y_{i_1} \leq \delta^k y_{i_2}$$

for any $i_1, i_2 \in \{1, \ldots, n\}$.

Remark 2.1. An orderly ranking is sequential and integrated.

Example 2.8. Consider the differential polynomial ring $\mathcal{R} = \mathbb{K}(x)\{s, t\}$. Let Δ be the set of the derivatives of $\mathcal{R}$, i.e. $\Delta = \{s^{(i)}, t^{(i)} \mid i \in \mathbb{N}\}$. The *ord-lex ranking* on Δ is the total order defined as follows:

$$\begin{cases} s^{(i)} < s^{(j)}, & \text{if } i < j, \\ t^{(i)} < t^{(j)}, & \text{if } i < j, \\ t^{(i)} < s^{(j)}, & \text{if } i \leq j, \\ s^{(i)} < t^{(j)}, & \text{if } i < j. \end{cases} \tag{2.2}$$

The ord-lex ranking is an orderly ranking.

In this thesis we will consider only the ord-lex ranking.

Definition 2.10. Let $A \in \mathcal{F}\{y_1, \ldots, y_n\}$ such that $A \notin \mathcal{F}$. The highest ranking derivative occurring in A is called the *leader* of A, and it is denoted by u_A. The leading coefficient of A with respect to its leader is called the *initial* of A, and it is denoted by I. The partial derivative of A with respect to its leader is called the *separant* of A, and it is denoted by S_A.

Example 2.9. Consider $G = (s^{(4)})^2 t^{(2)} + x^3 t^{(3)} - 2xs'$ the differential polynomial $G \in \bar{\mathbb{Q}}(x)\{s, t\}$ with the ord-lex ranking defined in Example 2.8. Therefore, The leader of G is $u_G = s^{(4)}$, its initial is $I = t^{(2)}$ and its separant is $S_G = \frac{\partial}{\partial s^{(4)}}(G) = 2s^{(4)} t^{(2)}$.

Remark 2.2. Both the initial and the separant of a differential polynomial G are of lower rank than G.

2.3 Ritt's reduction

Definition 2.11. Let $(\mathcal{F}, \delta)$ be a differential field and $\mathcal{R} = (\mathcal{F}\{y_1, \ldots, y_n\}, \tilde{\delta})$ be the differential ring with indeterminates $y_1, \ldots, y_n$, with a fixed ranking on them. Let $F, G \in \mathcal{R}$ such that $F, G \notin \mathcal{F}$. F is said to be *partially reduced* with respect to G if it is free of the derivatives that occur in u_G.

Definition 2.12. Let $\mathcal{R}, F$ and G be as above then F is said to be *reduced* with respect to G if it is partially reduced with respect to G and if $deg_{u_G}(F) < deg_{u_G}(G)$.

Remark 2.3. With the same notation of the definitions above, if F is of lower rank than G, then $u_F < u_A$, so $deg_{u_G}(F) = 0 < deg_{u_G}(G)$. Fix an orderly ranking, if F is of lower rank than G, then F is also reduced with respect to G, since, in this ranking, F is also lower than all the derivatives of G.

Definition 2.13. $\mathcal{A} \subset \mathcal{R} = \mathcal{F}\{y_1, \ldots, y_n\}$ is called an *autoreduced set* if $\mathcal{A} \cap \mathcal{F} = \emptyset$ and each element in $\mathcal{A}$ is reduced with respect to all the others.

We want now to introduce an important tool in Differential Algebra called *Ritt's reduction*, it is an analogous of the euclidean division for differential polynomials.

Theorem 2.1. *Let $\mathcal{A} \subset \mathcal{R} = \mathcal{F}\{y_1, \ldots, y_n\}$ be an autoreduced set with respect to a ranking and $G \in \mathcal{R}$ a differential polynomial. There exist nonnegative integers $s_i, t_i, i = 1, \ldots, r$, and a differential polynomial $R \in \mathcal{R}$ such that R is reduced with respect to $\mathcal{A}$, its rank is lower than or equal the rank of G and*

$$S_1^{s_1} \cdots S_r^{s_r} I_1^{t_1} \cdots I_r^{t_r} G = \sum_{A \in \mathcal{A}} \sum_j Q_{A,j} \cdot A^{(j)} + R,$$

where S_i an I_i are respectively the separant and the initial of $A_i \in \mathcal{A}$, $i = 1, \ldots, r$ and $Q_A \in \mathcal{R}$.

Proof. If G is reduced with respect to $\mathcal{A}$, then it is enough to take $R = G$, so assume that G is not reduced with respect to $\mathcal{A}$. Let j be the greatest value of i such that G is not reduced with respect to $A_i \in \mathcal{A}$. Let $u_{A_j} = y_p^{(m)}$, and let G be of order h in y_p. We suppose first that $h > m$. If $k_1 = h - m$, then $A_j^{(k_1)}$, the k_1th derivative of A_j, will be of order h in y_p. It will be linear in $y_p^{(h)}$, with S_j for coefficient of $y_p^{(h)}$. Therefore,

$$A_j^{(k_1)} = S_j y_p^{(h)} + D.$$

Since

$$G = \sum_t I_t (y_p^{(h)})^t,$$

we find a nonnegative integer v_1 such that

$$S_j^{v_1} G = C_1 A_j^{(k_1)} + D_1, \tag{2.3}$$

where D_1 is of order less than h in y_p. In order to have a unique procedure, we take v_1 as small as possible.
Suppose that $p < n$, where n is the number of differential indeterminate of $\mathcal{R}$. Let a be an integer such that $p < a \leq n$. We claim that D_1 is not of higher rank than G in y_a. We may limit ourselves to the case in which $D_1 \neq 0$. Also since S_j is free of y_a, we need only treat the case in which y_a is present in G. Let G of order g in y_a. Then the order of D_1 in y_a cannot exceed g because of (2.3). If D_1 were of higher degree than G in $y_a^{(g)}$, C_1 would have to involve $y_a^{(g)}$ to the same degree as D_1 and $C_1 A_j^{(k_1)}$ would contain terms involving $y_a^{(g)}$ and $y_p^{(h)}$ which would be balanced neither by D_1 nor by $S_j^{v_1} G$. This proves our claim.
If D_1 is of order greater than m in y_p, we find a relation

$$S_j^{v_2} D_1 = C_2 A_j^{(k_2)} + D_2 \tag{2.4}$$

with D_2 of lower order than D_1 in y_p and not of higher rank than D_1 (or G) in any y_a with $a > p$. For uniqueness, we take v_2 as small as possible.
We eventually reach a D_u, of order not greater than m in y_p, such that, if

$$s_j = v_1 + \cdots + v_u,$$

we have

$$S_j^{s_j} G = E_1 A_j^{(k_1)} + \cdots + E_u A_j^{(k_u)} + D_u.$$

Furthermore, if $a > p$, D_u is not of higher rank than G in y_a.
If D_u is of order less than m in y_p, D_u is reduced with respect to A_j, as well as to any A_i with $i > j$. If D_u is of order m in y_p, we find, with the algorithm of division, a relation

$$I_j^{t_j} D_u = H A_j + K \tag{2.5}$$

with K reduced with respect to A_j, as well as to $A_{j+1}, \cdots, A_r$. For uniqueness, we take t_j as small as possible. If K is not reduced with respect to $\mathcal{A}$, we treat K as G was treated. For some $l < j$, there are s_l, t_l such that $S_l^{s_l} I_l^{t_l} K$ exceeds, by a linear combination of A_l and its derivatives, a differential polynomial L which is reduced with respect to $A_l, A_{l+1}, \ldots, A_r$. Then

$$S_l^{s_l} S_j^{s_j} I_l^{t_l} I_j^{t_j} G = \sum_h Q_{A_j,h} A_j^{(h)} + \sum_k Q_{A_l,k} A_l^{(k)} + L. \tag{2.6}$$

Continuing, we reach a differential polynomial L as described in the statement. □

Definition 2.14. The differential polynomial R in the Theorem 2.1 is called *pseudo-remainder* of G with respect to $\mathcal{A}$ and it is denoted $\operatorname{prem}(G, \mathcal{A})$.

2.4 Rational general solutions of an algebraic ODE

Let I be a differential ideal in the differential ring $\mathcal{R} = (\mathcal{F}\{y_1, y_2, \ldots, y_n\}, \tilde{\delta})$, where $(\mathcal{F}, \delta)$ is a differential field. Let $\mathcal{L}$ be a differential field extension of $\mathcal{F}$. An element $\xi = (\xi_1, \ldots, \xi_n) \in \mathcal{L}^n$ is called a *zero* of I if for all $F \in I$ we have $F(\xi) = 0$. The *defining differential ideal* of ξ in $\mathcal{R}$ is $D = \{F \in \mathcal{R} \mid F(\xi) = 0\}$.

Definition 2.15. Let P be a prime differential ideal in $\mathcal{R} = (\mathcal{F}\{y_1, y_2, \ldots, y_n\}, \tilde{\delta}),$. where $(\mathcal{F}, \delta)$ is a differential field. A point $\xi \in \mathcal{L}^n$ is called a *generic zero* of P if P is the defining differential ideal of ξ in $\mathcal{R}$.

By *algebraic ODE* of order n we mean a polynomial a relation given by

$$F(x, y, y', ..., y^{(n)}) = 0,$$

where F is a differential polynomial in $\mathbb{K}(x)\{y\}$ where $\mathbb{K}$ is a differential field and the derivation $'$ is the usual $\frac{d}{dx}$. A solution of $F(x, y, y', ..., y^{(n)}) = 0$, is a function $y = f(x)$ such that $F(x, f(x), f'(x), ..., f^{(n)}(x)) = 0$. We can look at F as a differential polynomial in the differential ring $(\mathbb{K}(x)\{y\}, \delta)$, where y is the differential indeterminate and δ is the extension of $\frac{d}{dx}$. The proof of next theorem can be found in [Ritt] (Chapter II,7).

Theorem 2.2 (Differential Nullstellensatz)**.** *Let*

$$F_1, \ldots, F_p$$

be any finite system of differential polynomials in $(\mathbb{K}(x)\{y\}, \delta)$ *and let* G *be such that every common zero of* $F_1, \ldots, F_p$ *is a zero of* G*. There exists* $m \in \mathbb{N}$ *such that*

$$G^m = \sum_{i=1}^{p} \sum_{j} A_{i,j} F_i^{(j)},$$

where $A_{i,j}$ *are differential polynomial for all* i *and* j*, that is exactly*

$$G \in \{F_1, \ldots, F_p\}.$$

In particular if $F_1, \ldots, F_p$ *has no zeros,*

$$\{F_1, \ldots, F_p\} = 1.$$

Note now that if $(\mathcal{K}, \delta)$ is a differential extension of $(\mathbb{K}(x), \frac{d}{dx})$. A solution of $F(x, y, y', ..., y^{(n)}) = 0$, in $\mathcal{K}$ is an element $\eta \in \mathcal{K}$ such that $F(x, \eta, \delta\eta, ..., \delta^n\eta) = 0$. Therefore, if η is a solution of $F = 0$ then it is a solution of $\delta^m F = 0$ for every $m \in \mathbb{N} \setminus \{0\}$. Hence, η is a zero of the differential ideal (F) generated by F. Thanks to the Differential Nullstellensatz the set of differential polynomials in $\mathbb{K}(x)\{y\}$ vanishing on the zeros of F is the radical differential ideal $\{F\}$.

Lemma 2.3. *If* u *and* v *are indeterminates and if* j *is a nonnegative integer then*

$$u^{j+1}v^{(j)} \equiv 0 \quad (uv, (uv)', \ldots, (uv)^{(j)}).$$

Proof. The statement is clearly true for $j = 0$. We make an induction to $j = r$ with $r > 0$, assuming that the lemma is true for $j \leq r - 1$. By induction we have

$$u^r v^{(r-1)} \equiv 0 \quad (uv, (uv)', \ldots, (uv)^{(r-1)}). \tag{2.7}$$

Then

$$u^r v^{(r)} + r u^{r-1} u' v^{r-1} \equiv 0 \quad (uv, (uv)', \ldots, (uv)^{(r)}).$$

Multiplying by u and using (2.7) we get the thesis. □

Remark 2.4. Consider a perfect ideal $I \in \mathcal{F}\{y_1, \dots, y_n\}$, i.e. an ideal such that if $G^n \in I$ for some $n \in \mathbb{N}$ then $G \in I$, and let $FG \in I$. By the lemma with $j = 1$, $F^2G' \in I$. So $FG' \in I$. In general, if I is a perfect differential ideal and if $FG \in I$ then $F^{(i)}G^{(j)}$ for every i and j.

Theorem 2.4. *Consider $F \in \mathbb{K}(x)\{y\}$ then we can decompose $\{F\}$ as*

$$\{F\} = (\{F\} : S) \cap \{F, S\},$$

where S is the separant of F and $\{F\} : S = \{G \in \mathbb{K}(x)\{y\} \mid SG \in \{F\}\}$.

Proof. Since $\{F\}$ is contained in each ideal in the second member it is enough to show that the second member is in $\{F\}$. Let $A \in \{F, S\}$. There exists $m \in \mathbb{N}$ such that

$$A^m = B + C$$

with $B \in (F)$ and $C \in (S)$. Now, let A also belong to $(\{F\} : S)$. Then $SA \in \{F\}$, since a radical ideal is perfect, according to Remark 2.4, $S^{(j)}A \in \{F\}$. Then $AC \in \{F\}$. So that $A^{m+1} \in \{F\}$. □

Proposition 2.5. *Let F be an irreducible differential polynomial in $\mathbb{K}(x)\{y_1 \dots, y_n\}$. Then $\{F\} : S$, where S is the separant of F, is a prime differential ideal.*

Proof. Let $AB \in (\{F\} : S)$. Assume $\text{ord}_{y_p}(F) = m$. The process of reduction used for forming remainders shows the existence of relations

$$S^a A \equiv R, \quad S^b B \equiv T, \quad (F), \tag{2.8}$$

with R and T of order at most m in y_p. We shall prove that at least one of R and T is divisible by F. From (2.8) we have

$$SRT \equiv S^{a+b+1}AB, \quad (F). \tag{2.9}$$

Since the second member in (2.9) is divisible by F we have

$$SRT \equiv 0, \quad (F).$$

Let then

$$(SRT)^c = MF + M_1F' + \cdots + M_qF^{(q)}.$$

We have

$$F^{(q)} = Sy_p^{(m+q)} + U,$$

where U is of order less than $m + q$ in y_p. We replace y_p^{m+q} in $F(q)$ and in M_i for every i by $-U/S$. Clearing fractions, we find the relation

$$S^d(RT)^c = NF + N_1F' + \cdots + N_{q-1}F^{(q-1)}.$$

Continuing, we find that some $S^e(RT)^c$ is divisible by F. As F is algebraically irreducible, and it is not a factor of S, F must be a factor of at least one of R and T.
Suppose that R is divisible by F. By (2.8) $SA \in \{F\}$ so that $A \in (\{F\} : S)$. Thus $(\{F\} : S)$ is prime. □

Consider again $\mathbb{K}(x)\{y\}$. It is a radical Noetherian ring, i.e., a ring in which every radical differential ideal is finitely generated. So, in this case, $\{F, S\}$ can be decomposed as the intersection of finite number of prime differential ideals. From that decomposition eliminating the ideals which are divisors of $\{F\} : S$ we can obtain a unique minimal decomposition of $\{F\}$ into an intersection of prime differential ideals. Furthermore, $\{F\} : S$ is one of this minimal component and moreover it is the unique component of $\{F\}$ that does not contain the separant S, in fact if $\{F\} : S$ contains S, then $S^2 \in \{F\}$. Hence, $S \in \{F\}$, a contradiction with $deg_{y^{(n)}}(S) < deg_{y^{(n)}}(F)$.

Definition 2.16. A generic zero of the prime differential ideal $\{F\} : S$ is called a *general solution* of $F(y) = 0$. Where generic zero η of $\{F\} : S$ means that for any $G \in \mathbb{K}(x)\{y\}$ $G(\eta) = 0$ if and only if $G \in (\{F\} : S)$.

Definition 2.17. A zero of $\{F, S\}$ is called a *singular solution* of $F(y) = 0$.

It is important to solve the ideal membership problem for $\{F\} : S$, that can be done using Ritt's reduction introduced in the previous section. We remember that let $F, G \in \mathbb{K}(x)\{y\}$ the pseudo-remainder of G with respect to F is the differential polynomial R such that

$$S^s I^t G = \sum_j Q_j F^{(j)} + R,$$

where s and t are nonnegative integers, S is the separant of F and I is the initial of F, Q_j are differential polynomials. In particular R is reduced with respect to F and $\mathrm{ord}_y(R) \leq \mathrm{ord}_y(F)$, and if $\mathrm{ord}_y(F) = n$ then $deg_{y^{(n)}}(R) < deg_{y^{(n)}}(F)$.

Theorem 2.6. *For every* $G \in \mathbb{K}(x)\{y\}$, $G \in (\{F\} : S)$ *if and only if* $\mathrm{prem}(G, F) = 0$.

Proof. Suppose $G \in (\{F\} : S)$. We have a relation

$$S^s G \equiv H, \quad (F), \tag{2.10}$$

with $ord_y(H) \leq ord_y(F)$. Now SB is in $\{F\}$ so that, as in the proof of Proposition 2.5, H is divisible by F. Conversely, if $\mathrm{prem}(G, F) = 0$, we have again (2.10), but with H is divisible by F. Therefore $G \in (\{F\} : S)$. □

Corollary 2.7. *Suppose that* η *is a general solution of* $F(y) = 0$. *Then for every* $G \in \mathbb{K}(x)\{y\}$, $G(\eta) = 0$ *if and only if* $\mathrm{prem}(G, F) = 0$.

Chapter 3

Rational general solutions of autonomous algebraic ordinary differential equations

In this chapter I present a criterion to decide if an autonomous algebraic differential equation of the first order has a rational general solution. If it exists, I give an algorithm to compute it. The algorithm is based on the rational parametrization of the curve defined by the equation regarded as algebraic equation, which means that the differential variables are considered as independent algebraic variables.

3.1 A criterion for existence of rational general solutions

The contents of this and next section can be found in [FG04] and in [FG06].

Definition 3.1. An algebraic ordinary differential equation of the first order is called *autonomous* if it does not depend directly on the independent variable x, i.e. it is of the form

$$F(y, y') = 0,$$

where $F \in \mathbb{K}(x)\{y\}$ is a differential polynomial over a field $\mathbb{K}$.

In this chapter F will always be a first order non-zero differential polynomial with coefficients in $\mathbb{K}$.

The aim of this chapter is to find a general solution y of $F = 0$ of the form

$$y = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0}{x^m + b_{m-1} x^{m-1} + \cdots + b_0},$$

where a_i, b_j are constants in some extension of $\mathbb{K}$. We put $deg_x(y) = \max\{m, n\}$ We call a solution y of $F = 0$ *nontrivial* if $deg_x(y) > 0$. The differential polynomial F can be seen as an algebraic polynomial, i.e. as an element of the polynomial ring $K[y, z]$. In this case the zeros of F defines a curve in the plane.

Definition 3.2. Let $F = 0$ be a first order autonomous algebraic ODE, the curve defined by $F = 0$ as element of $\mathbb{K}[y, z]$ is called *solution curve* of the differential equation $F = 0$.

Let F be a differential polynomial in $\mathbb{K}(x)\{y\}$ and S its separant with respect to the ord-lex ranking. In this chapter we indicate by Σ_F the differential ideal $\{F\} : S$. In the previous chapter we have defined a *rational general solution* of the equation $F = 0$ as a generic zero of Σ_F.

We want to obtain a criterion to say if an autonomous algebraic ODE admits a rational general solution, the first step in this direction is the definition of a really special polynomial named *Gao's differential polynomial.*

Definition 3.3. Let $\{y, y^{(1)}, y^{(2)}, \dots\}$ be a set of differential variables, the differential polynomial $\mathcal{D}_{n,m}$ defined by

$$\mathcal{D}_{n,m} = \begin{vmatrix} \binom{n+1}{0}y^{(n+1)} & \binom{n+1}{1}y^{(n)} & \cdots & \binom{n+1}{m}y^{(n+1-m)} \\ \binom{n+2}{0}y^{(n+2)} & \binom{n+2}{1}y^{(n+1)} & \cdots & \binom{n+2}{m}y^{(n+2-m)} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n+m+1}{0}y^{(n+m+1)} & \binom{n+m+1}{1}y^{(n+m)} & \cdots & \binom{n+m+1}{m}y^{(n+1)} \end{vmatrix}$$

is called *Gao's differential polynomial.*

The main property of $\mathcal{D}_{n,m}$ is contained in the following lemma

Lemma 3.1. *The solutions $\bar{y}$ of the equation $\mathcal{D}_{n,m} = 0$ have the following form*

$$\bar{y}(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{x^m + b_{m-1}x^{m-1} + \cdots + b_0},$$

where a_i, b_j are constants in some extension of $\mathbb{K}$.

Proof. Let's prove it by induction on m. Suppose $m = 0$, in this case $\mathcal{D}_{n,m} = y^{(n+1)}$. The solutions of $y^{(n+1)} = 0$ are the polynomials $c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$, where the c_j are arbitrary constants. Suppose now that the theorem is true for $m < k + 1$ and let's prove it for $m = k + 1$. If $\bar{y}$ is a zero for $\mathcal{D}_{n,m}$, then $\bar{y}$ is rational by inductive hypothesis. Now we suppose that $\mathcal{D}_{n,k}(\bar{y}) \neq 0$. Since $\mathcal{D}_{n,k+1}(\bar{y}) = 0$, there exists $Q_0, Q_1, \dots, Q_{k+1} \in \mathbb{K}$, not all zero, such that

$$\begin{pmatrix} \binom{n+1}{0}\bar{y}^{(n+1)} & \binom{n+1}{1}\bar{y}^{(n)} & \cdots & \binom{n+1}{k+1}\bar{y}^{(n-k)} \\ \binom{n+2}{0}\bar{y}^{(n+2)} & \binom{n+2}{1}\bar{y}^{(n+1)} & \cdots & \binom{n+2}{k+1}\bar{y}^{(n+1-k)} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n+k+2}{0}\bar{y}^{(n+k+2)} & \binom{n+k+2}{1}\bar{y}^{(n+k+1)} & \cdots & \binom{n+k+2}{k+1}\bar{y}^{(n+1)} \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \\ \vdots \\ Q_{k+1} \end{pmatrix} = 0. \tag{3.1}$$

Without loss of generality, we can assume that $Q_{k+1} = 0$ or 1. From (3.1) we get $\sum_{i=0}^{k+1}\binom{j}{i}\bar{y}_{j-i}Q_i = 0$ for $j = n+1,\ldots,n+k+2$. Differentiating $\sum_{i=0}^{k+1}\binom{j}{i}\bar{y}_{j-i}Q_i = 0$, we obtain

$$(\sum_{i=0}^{k+1}\binom{j}{i}\bar{y}_{j-i}Q_i)' = \sum_{i=0}^{k+1}\binom{j}{i}\bar{y}_{j-i+1}Q_i + \sum_{i=0}^{k+1}\binom{j}{i}\bar{y}_{j-1}Q_i' = 0.$$

Thanks to the property of binomial coefficients $\binom{j+1}{i} = \binom{j}{i} + \binom{j}{i-1}$, we have

$$\sum_{i=0}^{k+1}\binom{j+1}{i}\bar{y}_{j-i+1}Q_i = \sum_{i=0}^{k+1}\binom{j}{i}\bar{y}_{j-i+1}Q_i + \sum_{i=0}^{k}\binom{j}{i}\bar{y}_{j-1}Q_{i+1} = 0,$$

then the above equations imply that

$$\sum_{i=0}^{k}\binom{j}{i}\bar{y}_{j-1}(Q_i' - Q_{i+1}) + \binom{j}{k+1}\bar{y}_{j-k-1}Q_{k+1}' = 0$$

for $j = n+1,\ldots,n+k+1$.
Since $Q_{k+1}' = 0$, we have $\sum_{i=0}^{k}\binom{j}{i}\bar{y}_{j-i}(Q_i' - Q_{i+1}) = 0$ for $j = n+1,\ldots,n+k+1$ which can be written in the matrix form:

$$\begin{pmatrix} \binom{n+1}{0}\bar{y}^{(n+1)} & \binom{n+1}{1}\bar{y}^{(n)} & \cdots & \binom{n+1}{k+1}\bar{y}^{(n-k)} \\ \binom{n+2}{0}\bar{y}^{(n+2)} & \binom{n+2}{1}\bar{y}^{(n+1)} & \cdots & \binom{n+2}{k+1}\bar{y}^{(n+1-k)} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n+k+2}{0}\bar{y}^{(n+k+2)} & \binom{n+k+2}{1}\bar{y}^{(n+k+1)} & \cdots & \binom{n+k+2}{k+1}\bar{y}^{(n+1)} \end{pmatrix} \begin{pmatrix} Q_0' - Q_1 \\ Q_1' - Q_2 \\ \vdots \\ Q_k' - Q_{k+1} \end{pmatrix} = 0.$$

Since $\mathcal{D}_{n,k}(\bar{y}) \neq 0$, we get $Q_{i+1} = Q_i'$ for $i = 0,1,\ldots,k$. According to this Q_0 must be a polynomial of degree k or $k+1$ if $Q_{k+1} = 0$ or 1, respectively. In the last case the leading coefficient of Q_0 will be $\frac{1}{(k+1)!}$. Then

$$\sum_{i=0}^{k+1}\binom{n+1}{i}\bar{y}_{n+1-i}Q_i = \sum_{i=0}^{n+1}\binom{n+1}{i}\bar{y}_{n+1-i}Q_i = 0,$$

putting $Q_j = 0$ for $j = k+2, k+3,\ldots,n+1$. Now the equation becomes

$$(\bar{y}Q_0)^{n+1} = 0,$$

and follows that

$$\bar{y} = \frac{a_n x^n + \cdots + a_0}{Q_0},$$

where a_i are arbitrary constants. □

This lemma implies the following criterion of existence for rational general solutions of an algebraic ODE.

Theorem 3.2. *Let F be an irreducible differential polynomial. Then the differential equation $F = 0$ has a rational general solution $\bar{y}$ if and only if there exist non-negative integers n and m such that* $\text{Prem}(\mathcal{D}_{n,m}, F) = 0$, *where* Prem *is the pseudo-remainder for differential polynomials defined in the first chapter.*

Proof. Let $\bar{y} = \frac{P(x)}{Q(x)}$ be a rational general solution of $F = 0$. We choose non-negative integers n, m such that $n \geq deg(P(x))$ and $m \geq deg(Q(x))$. Thanks to Lemma 3.1 we have

$$\mathcal{D}_{n,m}(\bar{y}) = 0,$$

since $\bar{y}$ in a generic zero of the differential ideal Σ_F generated by F we have

$$\mathcal{D}_{n,m} \in \Sigma_F,$$

which is equivalent to

$$\text{Prem}(\mathcal{D}_{n,m}, F) = 0.$$

On the other hand $\text{Prem}(\mathcal{D}_{n,m}, F) = 0$ implies that $\mathcal{D}_{n,m} \in \Sigma_F$. Let m be the least integer such that $\mathcal{D}_{n,m} \in \Sigma_F$. Since every zero y of Σ_F must be a zero of $\mathcal{D}_{n,m}$, it must must be of the form

$$y = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0},$$

with $b_m \neq 0$, otherwise $m - 1$ is such that $\mathcal{D}_{n,m-1} \in \Sigma_F$. Then the generic zero of Σ_F has the following form

$$\bar{y} = \frac{\bar{a}_n x^n + \bar{a}_{n-1}x^{n-1} + \cdots + \bar{a}_0}{\bar{b}_m x^m + \bar{b}_{m-1}x^{m-1} + \cdots + \bar{b}_0}.$$

Remember that Σ_F is prime and that implies the existence of a generic zero. So $F = 0$ has a rational general solution. □

3.2 Computing rational general solutions

In previous sections we worked in an arbitrary field $\mathbb{K}$, now we choose $\mathbb{K} = \bar{\mathbb{Q}}$, so $F(y, y')$ will be an autonomous first order differential polynomial with coefficient in $\mathbb{Q}$ and irreducible over $\bar{\mathbb{Q}}$, with $F(y, y_1)$ we indicate the same polynomial when it is considered as an algebraic polynomial.

It is well know that the solution set of an algebraic ODE with constant coefficients is invariant of the independent variable x, which means that if $\bar{y}(x)$ is a solution, then $\bar{y}(x + k)$ is a solution for every arbitrary constant k. We can use this fact to reduce the problem of finding a rational general solution to the problem of finding a nontrivial rational solution.

Lemma 3.3. *Let* $\bar{y} = \frac{\bar{a}_n x^n + \bar{a}_{n-1}x^{n-1} + \cdots + \bar{a}_0}{x^m + \bar{b}_{m-1}x^{m-1} + \cdots + \bar{b}_0}$ *be a non trivial solution of the equation* $F(y, y') = 0$, *where* $\bar{a}_i, \bar{b}_j \in \mathbb{Q}$, *and* $\bar{a}_n \neq 0$. *Then*

$$\hat{y} = \frac{\bar{a}_n (x + c)^n + \bar{a}_{n-1}(x + c)^{n-1} + \cdots + \bar{a}_0}{(x + c)^m + \bar{b}_{m-1}(x + c)^{m-1} + \cdots + \bar{b}_0}$$

is a rational general solution of the equation $F(y, y') = 0$, *where* c *is an arbitrary transcendental constant over* $\mathbb{Q}$.

Proof. Clearly $\hat{y}$ is still a zero of Σ_F. For any $G \in \mathbb{Q}\{y\}$ which satisfies $G(\hat{y}) = 0$, let $R = \text{Prem}(G, F(y, y'))$. Then $R(\hat{y}) = 0$. Suppose that $R \neq 0$, since F is irreducible and $deg(R, y') < deg(F, y')$ by the definition of Prem, there exist two differential polynomials $P, Q \in \mathbb{Q}\{y\}$ such that $PF+QR \in \mathbb{Q}[y]$ and $PF+QR \neq 0$. Thus we have $(PF+QR)(\hat{y}) = 0$ since both F and R has $\hat{y}$ as zero. Because c is a transcendental constant over $\mathbb{Q}$ we have $PF + QR = 0$, a contradiction. Hence $R = 0$ which means $G \in \Sigma_F$. Since this is true for every G then $\hat{y}$ is a generic zero of Σ_F, i.e. a rational general solution. □

If $\bar{y} = r(x)$ is a non trivial solution of the differential equation $F(y, y') = 0$, then we can regard $(r(x), r'(x))$ as a parametrization of the curve defined by the algebraic equation $F(y, y_1) = 0$. Now we want to show that $(r(x), r'(x))$ is a proper parametrization of $F(y, y_1) = 0$, this will allow us to look for rational solutions of $F(y, y') = 0$ among the proper parametrizations of the curve $F(y, y_1) = 0$.

Proposition 3.4. *Let* $f(x) = \frac{p(x)}{q(x)} \notin \bar{\mathbb{Q}}$ *be a rational function in* x *such that* $gcd(p(x), q(x)) = 1$. *Then* $\bar{\mathbb{Q}}(f(x)) \neq \bar{\mathbb{Q}}(f'(x))$.

Proof. Clearly if $f'(x)$ is a constant the proposition is true, so assume that $deg(f'(x)) > 0$. Suppose that $\bar{\mathbb{Q}}(f(x)) = \bar{\mathbb{Q}}(f'(x))$. Since $f(x), f'(x)$ are transcendental over $\bar{\mathbb{Q}}$, thanks to the Theorem in Section 63 of [VDW1] we have

$$f(x) = \frac{af'(x) + b}{cf'(x) + d},$$

where $a, b, c, d \in \bar{\mathbb{Q}}$. Differentiating we have

$$f'(x) = \frac{p'(x)q(x) - p(x)q'(x)}{q(x)^2},$$

then by substitution

$$\frac{p(x)}{q(x)} = \frac{a(p'(x)q(x) - p(x)q'(x)) + bq(x)^2}{b(p'(x)q(x) - p(x)q'(x)) + dq(x)^2},$$

which implies that $q(x)|cp(x)q'(x)$, which means $q(x)|cq'(x)$ because $gcd(q(x), p(x)) = 1$. So $c = 0$ or $q'(x) = 0$, we get respectively

$$f(x) = \frac{a}{d}f'(x) + \frac{b}{d},$$

or

$$p(x) = c_1 p'(x) + c_2,$$

where $c_1, c_2 \in \bar{\mathbb{Q}}$. This is impossible, because $f(x)$ is a rational function and $p(x)$ is a constant polynomial if $q(x) \in \bar{\mathbb{Q}}$. □

Theorem 3.5. *Let $f(x) = \frac{p(x)}{q(x)} \notin \bar{\mathbb{Q}}$ be a rational function in x such that $gcd(p(x), q(x)) = 1$. Then $\bar{\mathbb{Q}}(f(x), f'(x)) = \bar{\mathbb{Q}}(x)$.*

Proof. From Lüroth's Theorem, there exists $g(x) = \frac{u(x)}{v(x)}$ such that $\bar{\mathbb{Q}}(f(x), f'(x)) = \bar{\mathbb{Q}}(g(x))$, where $u(x), v(x) \in \bar{\mathbb{Q}}[x]$, and $gcd(u(x), v(x)) = 1$. We can assume that $deg(u(x)) > deg(v(x))$. Otherwise, we have $\frac{u}{v} = c + \frac{w}{v}$ with $c \in \bar{\mathbb{Q}}$ and $deg(w(x)) > deg(v(x))$, and $\frac{w}{v}$ is also a generator of $\bar{\mathbb{Q}}(g(x))$. Then we have

$$f(x) = \frac{p_1(g(x))}{p_2(g(x))},$$

$$f'(x) = \frac{p_2(g(x))}{q_2(g(x))} = \frac{g'(x)(p_1'(g(x))q_1(g(x)) - p_1(g(x))q_1'(g(x)))}{q_1^2(g(x))},$$

for some polynomials $p_i(x), q_i(x)$ for $i = 1, 2$. The two equations above imply that $g'(x) \in \bar{\mathbb{Q}}(g(x))$. Indeed, if $g'(x) \notin \bar{\mathbb{Q}}$, we have

$$[\bar{\mathbb{Q}}(x) : \bar{\mathbb{Q}}(g'(x))] = [\bar{\mathbb{Q}}(x) : \bar{\mathbb{Q}}(g(x))][\bar{\mathbb{Q}}(g(x)) : \bar{\mathbb{Q}}(g'(x))]. \tag{3.2}$$

However, we have $[\bar{\mathbb{Q}}(x) : \bar{\mathbb{Q}}(g(x))] = deg(g(x)) = deg(u(x))$ and $[\bar{\mathbb{Q}}(x) : \bar{\mathbb{Q}}(g'(x))] \leq 2deg(u(x)) - 1$. So the equation (3.2) tells us that $[\bar{\mathbb{Q}}(g(x)) : \bar{\mathbb{Q}}(g'(x))] < 2$, which implies

$$\bar{\mathbb{Q}}(g(x)) = \bar{\mathbb{Q}}(g'(x)),$$

a contradiction by Proposition (3.4). Hence $g'(x) \in \bar{\mathbb{Q}}$, which means that $g(x) = ax + b$, for some constants a, b. □

The theorem proven just above tells us that if $\bar{y}$ is a nontrivial rational solution of the equation $F(y, y') = 0$ the parametrization $(\bar{y}(x), \bar{y}'(x))$ of the curve $F(y, y_1) = 0$ is proper. Thanks to the following lemma we can deduce a degree bound for rational general solutions of autonomous algebraic ODE.

Lemma 3.6. *Let $f(x) = \frac{p(x)}{q(x)} \notin \bar{\mathbb{Q}}$ be a rational function in x such that $gcd(p(x), q(x)) = 1$. Then $deg(f(x)) - 1 \leq deg(f(x)) \leq 2deg(f(x))$.*

Proof. The inequality $deg(f(x)) \leq 2deg(f(x))$ is a direct consequence of the definitions. If $q(x) \in \bar{\mathbb{Q}}$, then $deg((\frac{p(x)}{q(x)})') = deg((\frac{p(x)}{q(x)})) - 1$. Hence the result is true. Assume that $q(x) \notin \bar{\mathbb{Q}}$. We are in a algebraically closed field, so we can assume that

$$q(x) = (x - a_1)^{\alpha_1}(x - a_2)^{\alpha_2} \cdots (x - a_r)^{\alpha_r}.$$

Then $(\frac{p(x)}{q(x)})' = \frac{u(x)}{v(x)}$ where

$$u(x) = p'(x)\prod_{i=1}^{r}(x - a_i) - p(x)\big(\sum_{i=1}^{r}\prod_{j \neq i}\alpha_i(a - a_j)\big),$$

$$v(x) = (x - a_1)^{\alpha_1+1}(x - a_2)^{\alpha_2+1} \cdots (x - a_r)^{\alpha_r+1}.$$

Since $gcd(u(x), v(x)) = 1$ we have

$$deg\Big(\Big(\frac{p(x)}{q(x)}\Big)'\Big) = max\{deg(p(x)) + r - 1, deg(q(x)) + r\},$$

which is greater than $deg((\frac{p(x)}{q(x)})) - 1$. This completes the proof. □

As announced we got the following

Theorem 3.7. *If the autonomous algebraic ODE $F(y, y') = 0$ has a rational general solution $\bar{y}$, then we have*

$$\begin{cases} deg(\bar{y}) = deg(F, y'), \\ deg(F, y') - 1 \leq deg(F, y) \leq 2deg(F, y'). \end{cases}$$

Proof. Assume $\bar{y} = r(x)$ for some rational function $r(x)$. Then $(r(x), r'(x))$ is a proper parametrization of the curve $F(y, y') = 0$, thanks to the Theorem 1.8 we have that $deg(r(x)) = deg(F, y')$. And Lemma 3.6 applied to $r(x)$ give us the required inequalities. □

We can use the Theorem 3.7 to get an upgraded version of the criterion of existence of rational general solution of autonomous algebraic ODE presented in the previous section.

Theorem 3.8. *Let $F(y, y') = 0$ be an autonomous algebraic ODE and $d = deg(F, y')$. $F(y, y') = 0$ has rational general solutions if and only if* $\text{Prem}(\mathcal{D}_{d,d}, F) = 0$.

Proof. It follows immediately from Theorem 3.2 and Theorem 3.7. □

What we have said so far about the rational general solutions of algebraic autonomous ODE suggests us that we can use the results on rational algebraic curves to obtain an algorithm to compute these solutions. Indeed, we can construct a nontrivial rational solution of $F(y, y') = 0$ starting from a proper rational parametrization of the curve $F(y, z) = 0$.

Theorem 3.9. *Let $(r(x), s(x))$ be a proper rational parametrization of the curve $F(y, z) = 0$, where $r(x), s(x) \in \bar{\mathbb{Q}}(x)$. Then $F(y, y') = 0$ has a nontrivial rational solution if and only if we have one of the following relations*

$$\frac{s(x)}{r'(x)} = \bar{a},$$

or

$$\frac{s(x)}{r'(x)} = \bar{a}(x - \bar{b})^2,$$

where $\bar{a}, \bar{b} \in \bar{\mathbb{Q}}$ and $\bar{a} \neq 0$. if the first relation is true then the rational solution of $F(y, y') = 0$ is given by $\bar{y} = r(\bar{a}x)$, if the second one is the rational solution is given by $\bar{y} = r(\bar{b} - \frac{1}{\bar{a}x})$.

Proof. Let $\bar{y} = p(x)$ a nontrivial rational solution of the equation $F(y, y') = 0$. Then we know that $(p(x), p'(x))$ is a proper rational parametrization of the curve $F(y, z) = 0$. Then there exists a linear rational function $f(x) = \frac{ax+b}{cx+d}$ where $ad - bc \neq 0$, i.e. the map f is invertible, such that

$$p(x) = r(f(x)),$$
$$p'(x) = s(f(x)) = (r(f(x)))' = f'(x)r'(f(x)). \tag{3.3}$$

If $c = 0$, then $f(x) = \frac{a}{d}x + \frac{b}{d}$, so $f'(x) = \frac{a}{d}$. From the equation (3.3) we have $s(f(x)) = \bar{a}r'(f(x))$, where $f(x) = \bar{a}x + \bar{b}$, with $\bar{a} = \frac{a}{d}$ and $\bar{b} = \frac{c}{d}$. Suppose now that $c \neq 0$, we can write $f(x) = \frac{ax+b}{cx+d} = \frac{a}{c} + \frac{bc-ad}{c(cx-d)}$. Then

$$f'(x) = \frac{(ad - bc)}{(cx + d)^2} = \frac{c^2(f(x) - \frac{a}{c})^2}{ad - bd}.$$

Again from (3.3), we have

$$\bar{a}(x - \bar{b})^2 r'(x) = s(x),$$

where $\bar{a} = \frac{c^2}{ad-bc}$ and $\bar{b} = \frac{a}{c}$. In both situations we get a non trivial rational solution $p(x) = r(f(x))$ of $F(y, y') = 0$. For the other direction put $p(x) = r(f(x))$. Equation (3.3) tells us that $p'(x) = f'(x)r'(f(x))$, by hypothesis we have $\bar{f}'(x)r'(f(x)) = s(f(x))$, so we have

$$p'(x) = s(f(x)).$$

That implies

$$F(p(x), p'(x)) = F(r(f(x)), s(f(x))) = 0,$$

because $(r(x), s(x))$ is a parametrization of $F(y, z) = 0$. So $\bar{y} = p(x)$ is a nontrivial rational solution of $F(y, y') = 0$. □

Clearly once we obtain a non trivial rational solution of the equation $F(y, y') = 0$, applying Lemma 3.3 we get a rational general solution.

In previous chapters we have seen that an irreducible algebraic curve defined by a polynomial over $\mathbb{Q}$ can be parametrized over an extension field of $\mathbb{Q}$ of degree at most two. Next result tell us that algebraic ODEs which admit a rational general solution define an algebraic curve which can be parametrized over $\mathbb{Q}$.

Theorem 3.10. *If the algebraic ODE $F(y, y') = 0$ has a rational general solution, then the coefficients of the rational general solution can be chosen in $\mathbb{Q}$.*

Proof. It is enough to prove that the coefficients of a nontrivial rational solution are in $\mathbb{Q}$. Thanks to Theorem 3.1 in the paper [SW97] and Theorem 3.9, we get a nontrivial rational solution $r(x)$ of $F(y, y') = 0$ whose coefficients belong to $\mathbb{Q}(\alpha)$ where $\alpha^2 \in \mathbb{Q}$. There exist $p_j(x), q_j(x) \in \mathbb{Q}[x]$ for $j = 1, 2$ such that

$$r(x) = \frac{\alpha p_1(x) + p_2(x)}{x^m + \alpha q_1(x) + q_2(x)},$$

and we can assume that $gcd_{\mathbb{Q}(\alpha)[x]}(\alpha p_1(x) + p_2(x), x^m + \alpha q_1(x) + q_2(x)) = 1$. Moreover, we assume that $deg(q_j(x)) \leq m-2$, which may be achieved by a proper linear transformation, this implies that if $m = 0$, i.e. we have a polynomial solution, then $q_j(x) = 0$. Clearly if $\alpha \in \mathbb{Q}$ there is nothing to prove. Assume $\alpha \notin \mathbb{Q}$. Using field extension automorphisms it is easy to check that

$$\bar{r}(x) = \frac{-\alpha p_1(x) + p_2(x)}{x^m - \alpha q_1(x) + q_2(x)},$$

is also a nontrivial rational solution of $F(y, y') = 0$. Each nontrivial rational solution of $F(y, y') = 0$ defines a proper parametrization of the curve $F(y, z) = 0$, then it exists a rational linear transformation $f(x)$ such that $r(x) = \bar{r}(f(x))$ and $r'(x) = \bar{r}'(f(x))$. The last equation implies that $f(x) = x + c$ for some $c \in \mathbb{Q}(\alpha)$, because $r'(x) = f'(x)\bar{r}'(f(x))$. So

$$\frac{\alpha p_1(x) + p_2(x)}{x^m + \alpha q_1(x) + q_2(x)} = \frac{-\alpha p_1(x+c) + p_2(x+c)}{x^m - \alpha q_1(x+c) + q_2(x+c)}.$$

Since both rational function are reduced we have

$$x^m + \alpha q_1(x) + q_2(x) = x^m - \alpha q_1(x+c) + q_2(x+c).$$

If $m > 0$, we have $c = 0$ because $deg(q_j(x)) \leq m-2$, the equality becomes

$$x^m + \alpha q_1(x) + q_2(x) = x^m - \alpha q_1(x) + q_2(x),$$

$$2\alpha q_1(x) = 0,$$

and as well

$$2\alpha p_1(x) = 0,$$

which imply $q_1(x) = p_1(x) = 0$ and the thesis in this case. If $m = 0$, then $r(x)$ is a polynomial. We can assume $r(x) = (a_n\alpha + \tilde{a}_n)x^n + \alpha p_1(x) + p_2(x)$ where $p_i(x) \in \mathbb{Q}(x)$, $deg(p_i(x)) \leq n-2$ and $a_n, \tilde{a}_n \in \mathbb{Q}$, and at least one of a_n and $\tilde{a}_n$ is not 0. As above we have

$$(a_n\alpha + \tilde{a}_n)x^n + \alpha p_1(x) + p_2(x) = (-a_n\alpha + \tilde{a}_n)(x+c)^n - \alpha p_1(x+c) + p_2(x+c),$$

we obtain

$$2a_n\alpha = 0,$$

$$2\alpha p_1(x) = 0.$$

These imply $a_n = p_1(x) = 0$ and the thesis. □

The results presented in this section lead to an algorithm which decides if an autonomous algebraic ODE has a rational general solution, and if yes it computes it.

INPUT: A first order differential polynomial $F(y, y')$ with coefficients in $\mathbb{Q}$, which is irreducible over $\bar{\mathbb{Q}}$.
OUTPUT:A rational general solution of $F(y, y') = 0$ if it exists.

1. Verify the degree bound in Theorem 3.7. If $deg(F, y) < deg(F, y') - 1$ or $deg(F, y) > 2deg(F, y')$ then the algorithm terminates and $F(y, y') = 0$ has no rational general solution.
2. Compute a proper rational parametrization $(r(x), s(x))$ of the curve $F(y, y_1) = 0$.
3. Let $A = \frac{s(x)}{r'(x)}$
   (a) If $A = a \in \mathbb{Q}$ we get a rational general solution $\bar{y} = r(a(x + c))$.
   (b) If $A = a(x-b)^2$ for $a, b \in \mathbb{Q}$ we get a rational general solution $\bar{y} = r(\frac{ab(x+c)-1}{a(x+c)})$.
   (c) Otherwise F has no rational solution.

Clearly the algorithm is correct and it terminates, there are not infinite processes. The complexity of the algorithm depends on the algorithm chosen to compute the parametrization.

Example 3.1. Consider the differential equation

$$F(y, y') = y'^3 + 4y'^2 + (27y^2 + 4)y' + 27y^4 + 4y^2$$

1. $d = deg(F, y')$ and $e = deg(F, y)$. $d = 3$, $e = 4$. We have $d - 1 < e < 2d$.
2. A proper parametrization of $F(y, y_1) = 0$ is given by
$$(r(x), s(x)) = (216x^3 + 6x, -3888x^4 - 36x^2).$$
3. $r'(x) = 648x^2 + 6$, we have
$$A = \frac{s(x)}{r(x)} = -6x^2.$$
   So $a = -6$, and $b = 0$.
4. $$\bar{y} = \left(r\left(b - \frac{1}{a(x + c)}\right)\right) = \frac{(x + c)^2 + 1}{(x + c)^3}$$
   is a rational general solution of $F(y, y') = 0$.

3.3 Examples

We implemented the algorithm above in two computer algebra systems: SINGULAR by Kaiserslautern University and CoCoA by Genova University. To compute the parametrizations we used the function in SINGULAR. For more details about the implementation see Chapter 5.

Example 3.2. Equation:

$$F(y, y') = -y'^2 + y - 1 = 0$$

SINGULAR:

```
(c2+2cx+x2+4)/4
```

CoCoA:

```
1/4x^2 + 1/2xc + 1/4c^2 + 1
```

Example 3.3. Equation:

$$F(y, y') = 9y^4 - \frac{1}{3}y'^3 - 9y'y^2 + \frac{4}{3}y'^2 + \frac{4}{3}y^2 - \frac{4}{3}y'$$

SINGULAR:

```
(-c2-2cx-x2-1)/(c3+3c2x+3cx2+x3)
```

CoCoA:

```
(-x^2 - 2xc - c^2 - 1)/(x^3 + 3x^2c + 3xc^2 + c^3)
```

Example 3.4. Equation:

$$F(y, y') = \frac{2}{27}y'^3 - 2y^2 - \frac{2}{27}y'^2 + \frac{8}{27}y = 0$$

SINGULAR:

```
x^3 + 3x^2c + 3xc^2 - 1/3x + c^3 - 1/3c + 2/27
```

CoCoA:

```
(27c^3+81c^2x+81cx^2-9c+27x^3-9x+2)/27
```

Example 3.5. Equation:

$$F(y, y') = \frac{8}{189}y^6 + \frac{2}{7}y^4 - \frac{2}{7}y^2y'^2 - \frac{2}{7}y'^3 = 0$$

SINGULAR:

```
-27/(c3+3c2x+3cx2+9c+x3+9x)
```

CoCoA:

```
-27/(x^3 + 3x^2c + 3xc^2 + 9x + c^3 + 9c)
```

Example 3.6. Equation:

$$F(y,y') = \frac{2}{3}y^4y'^3 - 18y^6 - \frac{2}{3}y^4y'^2 - \frac{16}{3}y^3y'^3 - \frac{2}{3}y^2y'^4 + \frac{532}{3}y^5 + \frac{10}{9}y^2y'^3 + \frac{2}{3}y'^5$$
$$-50y^4 + \frac{500}{9}y^2y'^2 + \frac{400}{9}yy'^3 + \frac{50}{9}y'^4 - \frac{20500}{9}y^3 + \frac{500}{3}yy'^2$$
$$+\frac{500}{9}y'^3 - \frac{12500}{3}y'^2 + \frac{1250}{9}y'^2 - \frac{25000}{9}x - \frac{6250}{9} = 0$$

SINGULAR:

```
(92389579776c5+461947898880c4x-2377847197440c4+923895797760c3x2
-9511388789760c3x +24479632045440c3+923895797760c2x3-14267083184640c2x2
+73438896136320c2x-126007336746720c2+461947898880cx4-9511388789760cx3
+73438896136320cx2-252014673493440cx +324307344255180+92389579776x5
-2377847197440x4+24479632045440x3-126007336746720x2+324307344255180x
-333962642704019)/(92389579776c2+184779159552cx-858749299200+92389579776x2
-858749299200x+2064783344832)
```

CoCoA:

```
(x^5+5x^4c-4015/156x^4+10x^3c^2-4015/39x^3c+3224045/12168x^3+10x^2c^3
-4015/26x^2c^2+3224045/4056x^2c -2588908135/1898208x^2+5xc^4-4015/39xc^3
+3224045/4056xc^2-2588908135/949104xc+2078893232405/592240896x+c^5
-4015/156c^4+3224045/12168c^3-2588908135/1898208c^2+2078893232405/592240896c
-333962642704019/92389579776)/(x^2+2xc-725/78x+c^2-725/78c+543877/24336)
```

Chapter 4

Rational general solutions of non-autonomous algebraic ordinary differential equations

As I did in the previous chapter for autonomous equations, here I want to present a way to decide if a rational general solution of first order non-autonomous algebraic ordinary differential equation exists, and if yes to compute it. This time the algorithm is based on the parametrization of rational surfaces. Parametrizing rational surfaces is a hard step, in the last part of the chapter I present a way to reduce our equation to an easier case trough affine transformations.

4.1 Associated System

The contents of this and next section are treated in [CNW10].

We have the obvious definition

Definition 4.1. An algebraic ODE $F = 0$, where F is a differential polynomial in $\mathbb{K}(x)\{y\}$ is said to be *non-autonomous* if it is not autonomous, i.e. x occurs in F.

Example 4.1. Consider the differential polynomial $F(x, y, y') = y'^3 - 4xyy' + 8y'^2$ in $\bar{\mathbb{Q}}(x)\{y\}$ then

$$F(x, y, y') \equiv y'^3 - 4xyy' + 8y'^2 = 0$$

is a non-autonomous algebraic ODE.

As in the autonomous case we can consider F as an algebraic polynomial, $F \in \mathbb{K}[x, y, y_1]$. In this sense F defines a zero set.

Definition 4.2. Let $F = 0$ be a first order non-autonomous algebraic ODE, the surface defined by $F = 0$ as element of $\mathbb{K}[x, y, z]$ is called *solution surface* of the differential equation $F = 0$, where $\mathbb{K}$ is algebraically closed and of characteristic zero.

Example 4.2 (Continuing Example 4.1)**.** The solution surface of the equation in the Example 4.1 is the algebraic surface defined by

$$F(x, y, z) \equiv z^3 - 4xyz + 8z^2 = 0.$$

We introduce now the set of all first order algebraic ODEs

$$\mathcal{AODE} = \{F(x, y, y') = 0 \mid F(x, y, z) \in \mathbb{K}[x, y, z]\},$$

and its subset

$$\mathcal{PODE} = \{F \in \mathcal{AODE} \mid \text{ the surface } F = 0 \text{ has a proper rational parametrization } \}.$$

In this section we consider a first order ODE $F(x, y, y') = 0$, with $F \in \mathbb{K}(x)\{y\}$, where $\mathbb{K}$ is an algebraically closed field of characteristic zero. We can suppose the differential polynomial F to be irreducible, otherwise we could consider one of its factors. Moreover, we consider only polynomial such that its solution surface is a rational solution.

Consider a rational solution $\bar{y} = f(x)$ of $F(x, y, y') = 0$, it is an element of $\mathbb{K}(x)$ such that

$$F(x, f(x), f'(x)) = 0.$$

So a rational solution of $F(x, y, y') = 0$ defines a parametric curve in the space which lies on the solution surface $F(x, y, z) = 0$.

Definition 4.3. Let $\bar{y} = f(x)$ be a rational solution of $F(x, y, y') = 0$. The parametric space curve

$$\sigma : \mathbb{K} \longrightarrow \mathbb{K}^3$$

$$x \mapsto (x, f(x), f'(x)),$$

is called the *solution curve* of $\bar{y} = f(x)$.

Let consider a proper rational parametrization of the solution surface $F(x, y, y') = 0$

$$\phi : \mathbb{K}^2 \longrightarrow \mathbb{K}^3$$

$$(s, t) \mapsto (\chi_1(s, t), \chi_2(s, t), \chi_3(s, t)),$$

where $\chi_i(s, t) \in \mathbb{K}(s, t)$ for $i = 1, 2, 3$.

Definition 4.4. Let $\bar{y} = f(x)$ be a rational solution of $F(x, y, y') = 0$. Let ϕ be a proper rational parametrization of the solution surface $F(x, y, z) = 0$. Let σ be the solution curve of $\bar{y} = f(x)$. The solution curve σ is *parametrizable* by ϕ if and only if σ is almost contained in $Im(\phi) \cap Dom(\phi^{-1})$, i.e. except for finitely many points.

Proposition 4.1. *Let $F(x, y, z) = 0$ be the solution surface of $F(x, y, y') = 0$ with rational proper parametetrization*

$$\phi(s, t) = (\chi_1(s, t), \chi_2(s, t), \chi_3(s, t)).$$

The differential equation $F(x,y,y')=0$ has a rational solution $\bar{y}=f(x)$ such that its solution curve is parametrizable by ϕ if and only if the system

$$\begin{cases}\chi_1(s(x),t(x))=x\\ \chi_2(s(x),t(x))'=\chi_3(s(x),t(x))\end{cases} \tag{4.1}$$

has a rational solution $(s(x),t(x))$. In this case $\bar{y}=\chi_2(s(x),t(x))$ is a rational solution of the equation $F(x,y,y')=0$.

Proof. Assume that $\bar{y}=f(x)$ is a rational solution of $F(x,y,y')=0$, which is parametrizable by ϕ. Consider the function

$$(s(x),t(x))=\phi^{-1}(x,f(x),f'(x)),$$

it is rational because it is composition of rational functions, indeed f and ϕ^{-1} are both rational. We have

$$\phi(s(x),t(x))=\phi(\phi^{-1}(x,f(x),f'(x)))=(x,f(x),f'(x)).$$

This means that $(s(x),t(x))$ is a rational solution of the system

$$\begin{cases}\chi_1(s(x),t(x))=x\\ \chi_2(s(x),t(x))=f(x)\\ \chi_3(s(x),t(x))=f'(x),\end{cases}$$

which is clearly equivalent to the system

$$\begin{cases}\chi_1(s(x),t(x))=x\\ \chi_2(s(x),t(x))'=\chi_3(s(x),t(x)).\end{cases}$$

Conversely, assume that $(s(x),t(x))$ is a rational solution of the system (4.1) and consider the function $\bar{y}=\chi_2(s(x),t(x))$. We get

$$\begin{aligned}F(x,\bar{y},\bar{y}')&=F(x,\chi_2(s(x),t(x)),(\chi_2(s(x),t(x)))')\\&=F(\chi_1(s(x),t(x)),\chi_2(s(x),t(x)),\chi_3(s(x),t(x)))\\&=F(\phi(s(x),t(x)))=0,\end{aligned}$$

where the last equality is because ϕ is a parametrization of the solution surface $F(x,y,z)=0$. So $\bar{y}=\chi_2(s(x),t(x))$ is a rational solution of the equation $F(x,y,y')=0$ and clearly it is parametrizable by ϕ. □

Remark 4.1. Observe that the rational parametrizations of the solution surface $F(x,y,z)=0$ are not unique. So if ϕ_1 and ϕ_2 are two different rational proper parametrization of the solution surface $F(x,y,z)=0$ it may exist a solution curve associated to a solution $\bar{y}=f(x)$ of $F(x,y,y')=0$ is not parametrizable by both parametrizations ϕ_1 and ϕ_2. This happens when the solution curve of f is not almost contained in $Im(\phi_1)\cap Dom(\phi_1)\cap Im(\phi_2)\cap Dom(\phi_2)$.

Differentianting the first equation of the system (4.1) we obtain the following system

$$\begin{cases} (\chi_1(s(x), t(x)))' = 1 \\ (\chi_2(s(x), t(x)))' = \chi_3(s(x), t(x)) \end{cases}$$

that can be expanded to

$$\begin{cases} \frac{\partial \chi_1(s(x),t(x))}{\partial s} \cdot s'(x) + \frac{\partial \chi_1(s(x),t(x))}{\partial t} \cdot t'(x) = 1 \\ \\ \frac{\partial \chi_2(s(x),t(x))}{\partial s} \cdot s'(x) + \frac{\partial \chi_2(s(x),t(x))}{\partial t} \cdot t'(x) = \chi_3(s(x), t(x)). \end{cases} \tag{4.2}$$

The system (4.2) can be seen as a linear system of two equation in two variables $s'(x), t'(x)$. Suppose now that the coefficients matrix is invertible, i.e.

$$det \begin{pmatrix} \frac{\partial \chi_1(s(x),t(x))}{\partial s} & \frac{\partial \chi_1(s(x),t(x))}{\partial t} \\ \\ \frac{\partial \chi_2(s(x),t(x))}{\partial s} & \frac{\partial \chi_2(s(x),t(x))}{\partial t} \end{pmatrix} \not\equiv 0, \tag{4.3}$$

applying Cramer's rule to the system (4.2) we get that $(s(x), t(x))$ is a rational solution of the autonomous system of differential equations

$$\begin{cases} s' = \frac{f_1(s,t)}{g(s,t)} \\ \\ t' = \frac{f_2(s,t)}{g(s,t)}, \end{cases} \tag{4.4}$$

where $f_1(s,t), f_2(s,t), g(s,t) \in \mathbb{K}(s,t)$ are defined by

$$\begin{aligned} f_1(s,t) &= \frac{\partial \chi_2(s,t)}{\partial t} - \chi_3(s,t) \cdot \frac{\partial \chi_1(s,t)}{\partial t}, \\ f_2(s,t) &= \chi_3(s,t) \cdot \frac{\partial \chi_1(s,t)}{\partial s} - \frac{\partial \chi_1(s,t)}{\partial s}, \\ g(s,t) &= \frac{\partial \chi_1(s,t)}{\partial s} \cdot \frac{\partial \chi_2(s,t)}{\partial t} - \frac{\partial \chi_1(s,t)}{\partial t} \cdot \frac{\partial \chi_2(s,t)}{\partial s}. \end{aligned}$$

If $g(s,t) \equiv 0$, which means that the determinant (4.3) is zero, then $(s(x), t(x))$ is a solution of the system

$$\begin{cases} \bar{g}(s,t) = 0 \\ \bar{f}_1(s,t) = 0, \end{cases} \tag{4.5}$$

where $\bar{g}(s,t)$ and $\bar{f}_1(s,t)$ are the numerators of $g(s,t)$ and $f_1(s,t)$, respectively. Observe that if $gcd(\bar{g}(s,t), \bar{f}_1(s,t))$ is constant, i.e. it does not depend on s and t, the solution $(s(x), t(x))$ is an intersection point of two algebraic curves $\bar{g}(s,t) = 0$ and $\bar{f}_1(s,t) = 0$, so it does not define a solution for the equation $F(x, y, y') = 0$ because the condition (4.1) is not satisfied. Then we have that $(s(x), t(x))$ defines a curve if and only if $gcd(\bar{g}(s,t), \bar{f}_1(s,t))$ is a non-constant polynomial in s and t.

Definition 4.5. If the determinant in (4.3) is not zero the autonomous system (4.4) is called the *associated system* of the algebraic ODE $F(x, y, y') = 0$ with respect to the parametrization ϕ of the solution surface $F(x, y, z) = 0$.

Example 4.3 (Continuing Example 4.2)**.** The solution surface in the Example 4.2 is parametrized by

$$\phi(s,t) = (t, -4s^2 \cdot (2s - t), -4s(2s - t)),$$

and its inverse mapping is

$$\phi^{-1}(x, y, z) = (\frac{y}{z}, x).$$

We compute the rational mappings $f_1(s, t), f_2(s, t)$ and $g(s, t)$ as above

$$f_1(s,t) = 4s \cdot (3s - t), \quad f_2(s,t) = 8s \cdot (3s - t),$$

$$g = 8s(3s - t).$$

Then the associated system of $F = 0$ with respect to the parametrization ϕ is

$$\begin{cases} s' = \frac{1}{2} \\ t' = 1. \end{cases}$$

Now we want to define what a rational general solution of a system is and then to prove that there is a one-to-one relation between the rational general solutions of the equation $F(x, y, y') = 0$ and the rational general solutions of its associated system with respect to a proper rational parametrization of the solution surface $F(x, y, y_1) = 0$. So far it seems legitimate to ask why solving a system of ODEs should be easier than solving a single equation. The answer to this question lays in the special form of an associated system, indeed the degree of derivatives is one and the equation of the system are autonomous ones.

In order to define what a rational general solution for a system is let's consider the system

$$\begin{cases} s' = \frac{N_1(s,t)}{M_1(s,t)} \\ t' = \frac{N_2(s,t)}{M_2(s,t)}, \end{cases} \tag{4.6}$$

where $M_1, N_1, M_2, N_2 \in \mathbb{K}[s, t]$ and $M_1 \neq 0$, $M_2 \neq 0$. Let's now define two differential polynomials $F_1, F_2 \in \mathbb{K}\{s, t\}$ as

$$F_1 = M_1 s' - N_1,$$

$$F_2 = M_2 t' - N_2.$$

Remark 4.2. We are considering $\mathbb{K}\{s, t\}$ with the ord-lex ranking introduced in the Example 2.8. Note that with this ranking the initial and the separant of F_i are the same for $i = 1, 2$. Moreover with respect to that order, F_1 is reduced with respect to F_2 and F_1 is reduced with respect to F_1, so the set $\{F_1, F_2\}$ is autoreduced.

Remark 4.3. The differential pseudo-remainder $R = \text{Prem}(G, F_1, F_2)$ is for every $G \in \mathbb{K}(x)\{s, t\}$ is a polynomial in $\mathbb{K}(x)[s, t]$ because F_1 and F_2 are of order 1 and degree 1.

Lemma 4.2. *Let*

$$I = \{G \in \mathbb{K}(x)\{s, t\} | \text{Prem}(G, F_1, F_2) = 0\}.$$

Then I *is a prime differential ideal in* $\mathbb{K}(x)\{s, t\}$.

Proof. Denote (F_1, F_2) the differential ideal generated by F_1 and F_2. Consider M_1 and M_2 as above and denote $H^\infty = \{M_1^{m_1} M_2^{m_2} | m_1, m_2 \in \mathbb{N}\}$. Then

$$(F_1, F_2) : H^\infty = \{G \in \mathbb{K}(x)\{s, t\} | \exists J \in H^\infty, JG \in (F_1, F_2)\},$$

is a prime differential ideal. In fact, suppose that $(F_1, F_2) : H^\infty$ contains a differential polynomial PQ but neither P nor Q. Let R_P and R_Q be, respectively, the remainders of P and Q with respect to F_1, F_2. Then $(F_1, F_2) : H^\infty$ contains $R_P R_Q$ but neither R_P nor R_Q. In what follows, every L_i is a power product of the separants and of the initials of F_1, F_2, i.e. n element of H^∞. Some $L_1 R_P R_Q$ has an expression linear in F_1 and F_2 and their derivatives. Let $F_1^{(k)}$ be the highest derivative of F_1 or F_2 in this expressions, the other case is analogous. Suppose that $k > 0$. Then

$$F_1^{(k)} = M_1 s^{(k+1)} + U,$$

where U is of lower order than $F_1^{(k)}$ in s. We replace $s^{(k+1)}$ by $-\frac{U}{M_1}$ in $L_1 R_P R_Q$. Clearing fractions, we have an expression of the form $L_2 R_P R_Q$ which is free of $F_1^{(k)}$. Continuing we find a $L_t R_P R_Q$ which is linear in F_1, F_2.
In R_P and R_Q may occur derivatives of s and t of order higher than 1, we indicate these as follows

$$v_{1k} = s^{(k)},$$
$$v_{2l} = t^{(l)},$$

for $k, l > 1$, moreover we indicate by v_1 and v_2, respectively, s and t. We consider the polynomial prime ideal $I_0 = (M_1(v_1, v_2)z_1 - N_1(v_1, v_2), M_2(v_1, v_2)z_2 - N_2(v_1, v_2))$ in $\mathbb{K}[z_1, z_2, v_1, v_2, v_{12}, \ldots, v_{1r_1}, v_{22}, \ldots, v_{2r_2}]$ with an ordering which respects the ord-lex ranking. Observe that J_t, R_P and R_Q, regarded as algebraic polynomials do not belong to I_0, neither is their product. Then the differential polynomial $J_t R_P R_Q$ cannot be linear in F_1 and F_2, contradiction. Thus $(F_1, F_2) : H^\infty$ is prime. If I is defined as in the statement we prove that

$$I = (F_1, F_2) : H^\infty.$$

Since M_1 and M_2 are the separant and initial of F_1 respectively of F_2 it is clear that $I \subseteq (F_1, F_2) : H^\infty$. Let $G \in (F_1, F_2) : H^\infty$. Then there exists $J \in H^\infty$ such that $JG \in (F_1, F_2)$. If $R = \text{Prem}(G, F_1, F_2)$ then we have

$$J_1 G - R \in (F_1, F_2),$$

for some $J_1 \in H^\infty$. It follows that $JJ_1G - JR \in (F_1, F_2)$, but since $JG \in (F_1, F_2)$ it implies that $JR \in (F_1, F_2)$. Note that $R, J \in \mathbb{K}(x)[s,t]$, this is possible if and only if $JR = 0$ because F_1 and F_2 are both of order 1. We must have $R = 0$ because J is not the zero polynomial. Then $G \in I$ and we have the equality. □

Definition 4.6. Let $M_1, N_1, M_2, N_2 \in \mathbb{K}[s,t]$, with $M_1 \neq 0$ and $M_2 \neq 0$. A rational solution $(s(x), t(x))$ of the system

$$\begin{cases} s' = \frac{N_1(s,t)}{M_1(s,t)} \\ \\ t' = \frac{N_2(s,t)}{M_2(s,t)} \end{cases}$$

is called a *rational general solution* if for any $G \in \mathbb{K}(x)\{s,t\}$

$$G(s(x), t(x)) = 0$$

if and only if

$$\text{Prem}(G, M_1 s' - N_1, M_2 t' - N_2) = 0.$$

As in the definition of rational general solution for an algebraic ODE $F(x, y, y') = 0$ a rational general solution of the system (4.6) can be seen as a generic zero of the ideal

$$I = \{G \in \mathbb{K}(x)\{s,t\} | \text{Prem}(G, F_1, F_2) = 0\}.$$

The Lemma 4.2 tells us that I is a prime ideal so it has a generic zero in a differential extension of $\mathbb{K}(x)$. We can consider the smallest differential extension such that contains both generic zeros of $F(x, y, y') = 0$ and of the system (4.6) to be the field where our solutions are. Now we want to give a necessary and sufficient condition for a rational solution being general, in few words we are going to prove that a rational solution is general if and only if it depends on a transcendental constant over the field of coefficients.

Lemma 4.3. *Let $(s(x), t(x))$ be a rational general solution of the system (4.6). Let G be a bivariate polynomial in $\mathbb{K}(x)[s,t]$. If $G(s(x), t(x)) = 0$, then $G = 0$.*

Proof. G is an algebraic polynomial in s and t, and since F_1 and F_2 defined as above are both of order one we have

$$\text{Prem}(G, M_1 s' - N_1, M_2 t' - N_2) = G.$$

Since G is a rational general solution $G(s(x), t(x)) = 0$ implies

$$\text{Prem}(G, M_1 s' - N_1, M_2 t' - N_2) = 0.$$

Then $G = 0$. □

Proposition 4.4. *Let $\mathbb{L}$ be a field big enough to contain the constants of a rational solution of the system (4.6). Let*

$$s(x) = \frac{a_k x^k + a_{k-1} x^{k-1} + \cdots + a_0}{b_l x^l + b_{l-1} x^{l-1} + \cdots + b_0}$$

and

$$t(x) = \frac{c_n x^n + c_{n-1} x^{n-1} + \cdots + c_0}{d_m x^m + d_{m-1} x^{m-1} + \cdots + d_0}$$

be a non-triavial rational solution where $a_i, b_i, c_i, d_i \in \mathbb{K}$ and $b_l, d_m \neq 0$. If $(s(x), t(x))$ is a non-trivial rational general solution of the system (4.6), then there exist a constant among the coefficients of $s(x)$ and $t(x)$, which is transcendental over $\mathbb{K}$, where $\mathbb{K}$ is the field which contains the coefficients of the equations of the system.

Proof. Define

$$S = (b_l x^l + b_{l-1} x^{l-1} + \cdots + b_0)s - (a_k x^k + a_{k-1} x^{k-1}) + \cdots + a_0$$

and

$$T = (d_m x^m + d_{m-1} x^{m-1} + \cdots + d_0)t - (c_n x^n + c_{n-1} x^{n-1} + \cdots + c_0).$$

Let $G = \text{res}_x(S, T)$ be the resultant of S and T with respect to x. G is a polynomial in s and t by definition of resultant, its coefficients depend on a_i, b_i, c_i, d_i so $G \in \mathbb{K}[s, t]$. It is easy to check that $G(s(x), t(x)) = 0$ so thanks to Lemma 4.3 we have $G = 0$. On the other hand thanks to the Theorem 1.9 we know that G is the rational curve parametrized by $(s(x), t(x))$ then $G \neq 0$, a contradiction. Therefore, there is a coefficient of $s(x)$ or $t(x)$ that does not belong to $\mathbb{K}$, but $\mathbb{K}$ is algebraically closed so the coefficient which is not in $\mathbb{K}$ must be transcendental over $\mathbb{K}$. □

Proposition 4.5. *Let $(s(x), t(x))$ be a rational solution of the system (4.6). Let $G(s, t)$ be a polynomial such that $G = 0$ is the rational algebraic curve defined by $(s(x), t(x))$. If there is an arbitrary transcendental constant in the set of coefficients of $G(s, t)$ then $(s(x), t(x))$ is a rational general solution of the system (4.6).*

Proof. Consider $H \in \mathbb{K}(x)\{s, t\}$ be a differential polynomial such that $H(s(x), t(x)) = 0$. Put

$$R = \text{Prem}(P, s' M_1(s, t) - N_1(s, t), t' M_2(s, t) - N_2(s, t)),$$

where N_1, N_2 and M_1, M_2 denominators and numerators of the right hand side of system (4.6). Then $R \in \mathbb{K}[s, t]$ and $R(s(x), t(x)) = 0$. Among the coefficients of $G(s, t)$ there is an arbitrary constant, so we can look $G(s, t)$ as a family of polynomial in $\mathbb{K}[s, t]$ by substituting the transcendental constant with a number in $\mathbb{K}$. The Theorem 1.2 tells us that is a family of irreducible polynomials since $G(s, t) = 0$ is a family of rational algebraic curves. So R must be a multiple of each element of the family $G(s, t)$, but this is possible if and only if $R = 0$. It follows that $(s(x), t(x))$ is a rational general solution of the system (4.6). □

The last two propositions give us a necessary and sufficient condition for a non-trivial rational solution of the system (4.6) $(s(x), t(x))$ to be a rational general solution. It requires the rational general solution to contain at least a transcendental constant over $\mathbb{K}$, usually this transcendental constant occurs as an arbitrary constant.

Theorem 4.6. *There is a one-to-one correspondence between rational general solutions of the algebraic ODE $F(x, y, y') = 0$ and the rational general solutions of the associated system of $F = 0$ with respect to the proper rational parametrization $\phi(s,t) = (\chi_1(s,t), \chi_2(s,t), \chi_3(s,t))$ of the surface $F(x, y, z) = 0$*

$$\begin{cases} s' = \frac{\chi_{2t} - \chi_3 \cdot \chi_{1t}}{\chi_{1s} \cdot \chi_{2t} - \chi_{1t} \cdot \chi_{2s}}, \\ t' = \frac{\chi_3 \cdot \chi_{1s} - \chi_{2s}}{\chi_{1s} \cdot \chi_{2t} - \chi_{1t} \cdot \chi_{2s}}, \end{cases} \tag{4.7}$$

with $\chi_{1s} \cdot \chi_{2t} - \chi_{1t} \cdot \chi_{2s} \neq 0$.

(i) *Let $y = f(x)$ be a rational general solution of $F(x, y, y') = 0$. Suppose the solution curve of f is parametrizable by $\phi(s,t) = (\chi_1(s,t), \chi_2(s,t), \chi_3(s,t))$. Let*

$$(s(x), t(x)) = \phi^{-1}(x, f(x), f'(x)),$$

and

$$g(s,t) = \frac{\partial \chi_1(s,t)}{\partial s} \cdot \frac{\partial \chi_2(s,t)}{\partial t} - \frac{\partial \chi_1(s,t)}{\partial t} \cdot \frac{\partial \chi_2(s,t)}{\partial s}.$$

if $g((s(x), t(x))) \neq 0$, then $(s(x), t(x))$ is a rational general solution of the associated system of $F = 0$ with respect to ϕ.

(ii) *Let $(s(x), t(x))$ be a rational general solution of the system (4.7) and let $c = \chi_1(s(x), t(x)) - x$. Then*

$$y = \chi_2(s(x - c), t(x - c))$$

is a rational general solution of the equation $F(x, y, y') = 0$.

Proof. Since $g(s(x), t(x)) \neq 0$, by the Proposition 4.1 $(s(x), t(x))$ is a solution of (4.7). Consider a differential polynomial $P \in \mathbb{K}(x)\{y\}$ such that $P(s(x), t(x)) = 0$. Let N_1, N_2 and M_1, M_2 be the numerators and the denominators of the righthand side of the system (4.7), let

$$R = \mathrm{Prem}(P, M_1 s' - n_1, M_2 t' - N_2).$$

Since we are computing a pseudo remainder with respect to linear differential polynomials we have $R \in \mathbb{K}(x)[s,t]$. We want to prove that $R = 0$. We know that

$$R(s(x), t(x)) = R(\phi^{-1}(x, f(x), f'(x))) = 0.$$

Let us consider the rational function $R(\phi^{-1}(x, y, z)) = \frac{U(x,y,z)}{V(x,y,z)}$, clearly U and V are polynomials. Then $U(x, y, y')$ is a differential polynomial such that

$$U(x, f(x), f'(x)) = 0.$$

Since $U(x,y,y')$ vanishes on a generic zero of the differential ideal generated by F the differential pseudo remainder of U with respect to F must be zero. But both $U(x,y,y')$ and $F(x,y,y')$ are differential polynomials of order 1 we have the reduction

$$I^m U(x,y,y') = Q_0 F,$$

where I is the initial of F, $m \in \mathbb{N}$ and Q_0 is a differential polynomial of order 1 in $\mathbb{K}(x)\{y\}$. Therefore,

$$R(s,t) = R(\phi^{-1}(\phi(s,t))) = \frac{U(\phi(s,t))}{V(\phi(s,t))} = \frac{Q_0(\phi(s,t))F(\phi(s,t))}{I^m(\phi(s,t))V(\phi(s,t))} = 0$$

because $F(\phi(s,t)) = 0$ and $I(\phi(s,t)) \neq 0$. Thus $(s(x),t(x))$ is a rational general solution of (4.7). Let' s prove now the second statement. Let $(s(x),t(x))$ be a rational general solution of the system (4.7). In the associated system we have the relation

$$(\chi_1(s,t))' = 1,$$

substituting $s(x)$ and $t(x)$ in $\chi_1(s,t)$ we have

$$(\chi_1(s(x),t(x)))' = 1,$$

from which we get

$$\chi_1(s(x),t(x)) = x + c,$$

for some constant c. So

$$\chi_1(s(x-c),t(x-c)) = x$$

and

$$y = \chi_2(s(x-c),t(x-c))$$

is a rational solution of the differential equation $F(x,y,y') = 0$. Consider a differential polynomial $G \in \mathbb{K}(x)\{y\}$ such that $G(y) = 0$. Let

$$R = \mathrm{Prem}(G,F)$$

be the differential pseudo remainder of G with respect to F. It follows that $R(y) = 0$. We have to prove that $R = 0$. We want to prove that $R = 0$, this will imply that y is a generic zero. Assume that $R \neq 0$. Then

$$R(\chi_1(s,t),\chi_2(s,t),\chi_3(s,t)) = \frac{W(s,t)}{Z(s,t)},$$

where $W(s,t), Z(s,t) \in \mathbb{K}[s,t]$. On the other hand,

$$R(\chi_1(s(x),t(x)),\chi_2(s(x),t(x)),\chi_3(s(x),t(x))) = 0.$$

Hence,

$$W(s(x),t(x)) = 0.$$

By Lemma 4.3 we must have $W(s,t) = 0$. Thus

$$R(\chi_1(s,t), \chi_2(s,t), \chi_3(s,t)) = 0.$$

Since F is irreducible and $deg_{y'}R < deg_{y'}F$, we have $R = 0 \in \mathbb{K}[x,y,z]$. Therefore, y is a rational general solution of $F(x,y,y') = 0$. □

What we said above leads to the following algorithm

INPUT: A non-autonomous algebraic ODE $F(x,y,y') = 0$.
OUTPUT: A rational general solution of $F(x,y,y') = 0$, if it exists.

1. Compute a rational proper parametrization $\phi = (\phi_1, \phi_2, \phi_3)$ of the solution surface $F(x,y,z) = 0$.
2. Compute the associated system of $F(x,y,y') = 0$ with respect to ϕ.
3. Compute a rational general solution $(s(x), t(x))$ of the associated system.
4. Compute $c = \phi_1(s(x), t(x)) - x$.
5. Return $y = \phi_2(s(x-c), t(x-c))$.

Example 4.4 (Continuing Example 4.3)**.** A rational general solution of the system in the Example 4.3 is given by

$$(s(x), t(x)) = (\frac{1}{2}x + C_1, x + C_2),$$

where C_1, C_2 are arbitrary constants. We compute $c = \chi_1(s(x), t(x)) - x$ and we get

$$c = C_2.$$

From the theorem above we obtain that a rational general solution of the equation $F(x,y,y') = 0$ is

$$y = -4(s(x - C_2))^2 \cdot (2s(x - C_2) - t(x - C_2)) = -C(x + C)^2,$$

where $C = 2C_1 - C_2$.

4.2 A criterion for existence of rational general solutions

In the previous section we proved that finding a rational general solution of an algebraic ODE $F(x,y,y') = 0$ is equivalent to finding a rational general solution of its associated system with respect to a rational proper parametrization of its solution surface. Since both equations in the associated system are autonomous we can derive a criterion of existence of rational general solution of non-autonomous equations from the Gao's differential polynomial in an similar way as we have done for the autonomous case.

Theorem 4.7. *Let* $M_1, N_2, M_2, N_2 \in \mathbb{K}[s,t]$, *with* $M_1, M_2 \neq 0$. *The autonomous system*

$$\begin{cases} s' = \frac{N_1(s,t)}{M_1(s,t)}, \\ \\ t' = \frac{N_2(s,t)}{M_2(s,t)}, \end{cases} \tag{4.8}$$

has a general rational solution $(s(x), t(x))$ *with degree* $deg(s(x)) \leq n$ *and* $deg(t(x)) \leq m$ *if and only if*

$$\begin{cases} \text{Prem}(D_{n,n}(s), M_1 s' - N_1, M_2 t' - N_2) = 0, \\ \text{Prem}(D_{m,m}(t), M_1 s' - N_1, M_2 t' - N_2) = 0, \end{cases} \tag{4.9}$$

where $D_{n,m}(x)$ *is the Gao's differential polynomial defined by*

$$\mathcal{D}_{n,m}(s) = \begin{vmatrix} \binom{n+1}{0} s^{(n+1)} & \binom{n+1}{1} s^{(n)} & \cdots & \binom{n+1}{m} s^{(n+1-m)} \\ \binom{n+2}{0} s^{(n+2)} & \binom{n+2}{1} s^{(n+1)} & \cdots & \binom{n+2}{m} s^{(n+2-m)} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n+m+1}{0} s^{(n+m+1)} & \binom{n+m+1}{1} s^{(n+m)} & \cdots & \binom{n+m+1}{m} s^{(n+1)} \end{vmatrix}$$

Proof. Suppose that the system (4.8) has a rational general solution $(s(x), t(x))$ with degree $deg(s(x)) \leq n$ and $deg(t(x)) \leq m$. Thanks to the Lemma 3.1 $(s(x), t(x))$ is a solution of $D_{n,n}(s)$ and $D_{m,m}(t)$. Then by definition of rational general solutions of the system (4.8) we have

$$\begin{cases} \text{Prem}(D_{n,n}(s), M_1 s' - N_1, M_2 t' - N_2) = 0, \\ \text{Prem}(D_{m,m}(t), M_1 s' - N_1, M_2 t' - N_2) = 0. \end{cases}$$

On the other hand, if the conditions in (4.9) are fulfilled, then $D_{n,n}(s)$ and $D_{m,m}(t)$ belong to the ideal I defined by

$$I = \{G \in \mathbb{K}(x)\{s,t\} | \text{prem}(G, M_1 s' - N_1, M_2 t' - N_2) = 0\}.$$

We have seen that I is a prime differential ideal so it has a generic zero η. η is a zero of $D_{n,n}(s)$ and $D_{m,m}(t)$, then it is a rational function for Lemma 3.1. □

Remark 4.4. If we have a degree bound of the rational solutions of the system (4.8), then the previous theorem gives us a criterion of existence of a rational general solution of the system (4.8).

4.3 Invariant algebraic curves of a differential system

The proof of Lemma 4.8 is original, all the other results are treated in [CNW11]. First of all we consider a polynomial differential system of the form

$$\begin{cases} s' = P(s,t), \\ t' = Q(s,t), \end{cases} \tag{4.10}$$

where P and Q are polynomials in $\mathbb{K}[s,t]$ with $\mathbb{K}$ is algebraically closed and of characteristic zero, as so far has been.

Definition 4.7. An *invariant algebraic curve* of the polynomial differential system (4.10) is an algebraic curve $G(s,t) = 0$ such that

$$\frac{\partial G}{\partial s} \cdot P + \frac{\partial G}{\partial t} \cdot Q = GK, \tag{4.11}$$

for some polynomial $K \in \mathbb{K}[s,t]$, K is called *cofactor* of the invariant algebraic curve $G(s,t) = 0$.

The invariant algebraic curves of a polynomial system are well studied in literature, they can be computed by setting up an upper bound for the degree of the polynomial $G(s,t) = 0$ and deriving a system of a algebraic equations with variables the coefficient of $G(s,t)$ by undetermined coefficients method.

Observe that the polynomial K in (4.11) is uniquely defined by the quotient of the division of the lefthand side of (4.11) by G, moreover we have an upper bound for the degree of K given by $max\{degP, degQ\} - 1$. Thanks to the following lemma we can reduce the problem to finding only irreducible invariant algebraic curves.

Lemma 4.8. *Let $G(s,t) = 0$ be an algebraic curve and $G(s,t) = \prod_{i=1}^{m} G_i^{n_i}(s,t)$ be the decomposition of $G(s,t) = 0$ into irreducible factors. $G(s,t) = 0$ is an invariant algebraic curves of the system*

$$\begin{cases} s' = P(s,t), \\ t' = Q(s,t), \end{cases} \tag{4.12}$$

if and only if the irreducible curves $G_i(s,t) = 0$ are invariant algebraic curves of the same system (4.12) with cofactors K_i and $K = \sum_{i=1}^{m} n_i K_i$.

Proof. Assume that each $G_i(s,t) = 0$ is an invariant algebraic curve of the system (4.12) with cofactor K_i.

$$\begin{aligned} P\frac{\partial G(s,t)}{\partial s} + Q\frac{\partial G(s,t)}{\partial t} &= P\frac{\partial}{\partial s}\Big(\prod_{i=1}^{m} G_i^{n_i}(s,t)\Big) + \frac{\partial}{\partial t}\Big(\prod_{i=1}^{m} G_i^{n_i}(s,t)\Big) \\ &= P\Big(\sum_{i=1}^{m}\prod_{j\neq i} n_i G_i^{n_i-1}\frac{\partial G_i(s,t)}{\partial s} G_j^{n_j}\Big) + Q\Big(\sum_{i=1}^{m}\prod_{j\neq i} n_i G_i^{n_i-1}\frac{\partial G_i(s,t)}{\partial t} G_j^{n_j}\Big) \\ &= \sum_{i=1}^{m}\prod_{j\neq i} n_i G_i^{n_i-1} G_j^{n_j}\Big(P\frac{\partial G_i(s,t)}{\partial s} + Q\frac{\partial G_i(s,t)}{\partial t}\Big) \\ &= \sum_{i=1}^{m}\prod_{j\neq i} n_i G_i^{n_i-1} G_j^{n_j} K_i G_i \\ &= \sum_{i=1}^{m}\prod_{j\neq i} n_i G_i^{n_i} G_j^{n_j} K_i \\ &= \sum_{i=1}^{m} n_i K_i G(s,t) \\ &= KG \end{aligned}$$

where $K = \sum_{i=1}^{m} n_i K_i$. Therefore, $G(s,t) = 0$ is an invariant algebraic curve of the system (4.12).
Conversely, if $G(s,t) = 0$ is an invariant algebraic curve of the system (4.12), first of all we consider a special case. If $G(s,t) = G'(s,t)^k$ with $k \in \mathbb{N}$ and $G'(s,t)$ irreducible we have

$$KG(s,t) = P\frac{\partial}{\partial s}G + Q\frac{\partial}{\partial t}G = PkG'(s,t)^{k-1}\frac{\partial}{\partial s}G'(s,t) + QkG'(s,t)^{k-1}\frac{\partial}{\partial t}G'(s,t).$$

Dividing both sides by $G'(s,t)^{k-1}$ we have

$$\frac{1}{k}KG'(s,t) = P\frac{\partial}{\partial s}G'(s,t) + Q\frac{\partial}{\partial t}G'(s,t).$$

Suppose now that $G(s,t) = A(s,t) \cdot B(s,t)$ with $gcd(A,B) = 1$. Then we have

$$KAB = KG = P\frac{\partial}{\partial s}G + Q\frac{\partial}{\partial t}G = P(\frac{\partial}{\partial s}A \cdot B + A \cdot \frac{\partial}{\partial s}B) + Q(\frac{\partial}{\partial t}A \cdot B + A \cdot \frac{\partial}{\partial t}B),$$

$$KAB = (P\frac{\partial}{\partial s}A + Q\frac{\partial}{\partial t}A) \cdot B + (P\frac{\partial}{\partial s}B + Q\frac{\partial}{\partial t}B) \cdot A,$$

$$(KB - P\frac{\partial}{\partial s}B - Q\frac{\partial}{\partial t}B)A = (P\frac{\partial}{\partial s}A + Q\frac{\partial}{\partial t}A)B.$$

Note that A must divide the righthand side of last equality, since $gcd(A,B) = 1$ we must have $A|P\frac{\partial}{\partial s}A + Q\frac{\partial}{\partial t}A$, that means

$$K_A A = P\frac{\partial}{\partial s}A + Q\frac{\partial}{\partial t}A,$$

for some polynomial K_A. The same thing can be done for $B(s,t)$. The special case and this last result give us the thesis easily. □

Remark 4.5. Let $H = gcd(P,Q)$ and $P = P_1 H, Q = Q_1 H$. Then every invariant algebraic curve $G'(s,t) = 0$ of the system

$$\begin{cases} s' = P_1(s,t), \\ t' = Q_1(s,t) \end{cases}$$

is an invariant algebraic curve of (4.10), indeed

$$\frac{\partial}{\partial s}G' \cdot P_1 + \frac{\partial}{\partial t}G' \cdot Q_1 = G' \cdot K,$$

for some polynomial K, then it is enough to multiply both sides by H. If $G(s,t) = 0$ is and invariant algebraic curve of (4.10), then

$$(\frac{\partial}{\partial s}G \cdot P_1 + \frac{\partial}{\partial t}G \cdot Q_1) \cdot H = G \cdot K,$$

for some polynomial K. Since $G(s,t)$ is irreducible we have either $G|H$ or $G|(\frac{\partial}{\partial s}G \cdot P_1 + \frac{\partial}{\partial t}G \cdot Q_1)$. In the second case, $G(s,t) = 0$ is an invariant algebraic curve for the system just above. In the first case, $G(s,t)$ is an irreducible factor of $H(s,t)$. If $(s(x), t(x))$ is a parametrization of $G(s,t) = 0$ then

$$P(s(x), t(x)) = Q(s(x), t(x)) = 0.$$

In this case, $(s(x), t(x))$ is a solution of the system (4.10) if and only if $s(x)$ and $t(x)$ are both constants.

Notice that if $G(s,t) = 0$ is an irreducible invariant algebraic curve of (4.10) the polynomial $P\frac{\partial}{\partial s}G + Q\frac{\partial}{\partial t}G$ is in the ideal generated by G. Then finding invariant algebraic curves of a polynomial system is a membership problem. It follows that the algorithm to compute the invariant algebraic curves of a system is

INPUT: A polynomial differential system, as (4.10).
OUTPUT: The set of irreducible invariant algebraic curves of the system given in input.

1. Consider a monic generic polynomial of degree k in s and t.
2. Compute the remainder R of the division of $P\frac{\partial}{\partial s}G + Q\frac{\partial}{\partial t}G$ by G
3. Put $R = 0$, and derive the algebraic system Σ given by vanishing of coefficients of R.
4. Solving the system Σ.
5. Return the set of irreducible algebraic invariant curves of the system

Clearly to make this algorithm terminate we need an upper bound for the degree of the irreducible invariant algebraic curves. Such bound is given in the paper [Car94] in terms of foliations for a special case, when there are not *dicritical singularities.* This case results to be the general one. More details on this facts can be found in [Car94] and [Ngo]. Concerning the practical point of view we choose a reasonable upper bound.

Example 4.5. Consider the differential polynomial system

$$\begin{cases} s' = st, \\ t' = s + t^2. \end{cases}$$

Notice that we use the degree lexicographic order with $s > t$. We start looking for the invariant algebraic curve of the system of degree 1. Then we consider the two monic irreducible generic polynomials of degree 1

$$G(s,t) = t + c, \quad G(s,t) = s + bt + c.$$

If $G(s,t) = t + c$ we have $G_t = 0$, $G_t = 1$, and

$$G_sP + G_tQ = s + t^2. \tag{4.13}$$

Since the righthand side of (4.13) is not divisible by $G(s,t)$, so $G(s,t) = 0$ can not be an invariant algebraic curve. If $G(s,t) = s + bt + c$ we have $G_t = b$ and $G_s = 1$.

$$G_s P + G_t Q = st + b(s + t^2). \tag{4.14}$$

The remainder of the division of the righthand side of (4.14) by $G(s,t)$ is

$$R(s,t) = (-c - b^2)t - bc.$$

It follows that $G(s,t) = s + bt + c$ defines an invariant algebraic curve if and only if $b = c = 0$. therefore, $G(s,t) = s$ is an invariant algebraic curve. Similarly, we ask for the invariant algebraic curves of degree 2. Consider the generic polynomials

$$G(s,t) = t^2 + ds + et + f, \quad G(s,t) = st + ct^2 + ds + et + f,$$

$$G(s,t) = s^2 + bst + ct^2 + ds + et + f.$$

If $G(s,t) = t^2 + ds + et + f$, then the remainder of the division of $G_s P + QG_t$ by G is

$$R(s,t) = ef + (e^2 - 2f)t + (de + e)s + (2 - d)st.$$

So we need $d = 2$ and $e = f = 0$. Then $G(s,t) = t^2 + 2s$ is an invariant algebraic curve for the system. Again with the same computations if $G(s,t) = st + ct^2 + ds + et + f$ we get that $G(s,t)$ is not an invariant algebraic curve for any choice of its coefficients. instead if $G(s,t) = s^2 + bst + ct^2 + ds + et + f$, $G(s,t) = 0$ is an invariant algebraic curve if and only if $b = e = f = 0$ and $d = 2c$, i.e. $G(s,t) = s^2 + ct^2 + 2cs$, in this case $G(s,t) = 0$ depends on an arbitrary constant and it defines a whole family of invariant algebraic curves of the system.

Let's now move on a more general case, consider again a system of the form

$$\begin{cases} s' = \frac{N_1(s,t)}{M_1(s,t)} \\ \\ t' = \frac{N_2(s,t)}{M_2(s,t)}, \end{cases} \tag{4.15}$$

where N_1, N_2, M_1, M_2 are polynomials with $M_1 \neq 0$ and $M_2 \neq 0$.

Definition 4.8. An algebraic curve $G(s,t) = 0$ is called an *invariant algebraic curve* of the rational system (4.15) if

$$\frac{\partial}{\partial s} G(s,t) M_2(s,t) N_1(s,t) + \frac{\partial}{\partial t} G(s,t) M_1(s,t) N_2(s,t) = G(s,t) K(s,t)$$

for some polynomial $K(s,t)$. An invariant algebraic curve $G(s,t) = 0$ of the rational system (4.15) is called a *rational invariant algebraic curves* if $G(s,t) = 0$ is a rational curve.

This means that the invariant algebraic curves of the rational system (4.15) are the invariant algebraic curves of the polynomial system

$$\begin{cases} s' = M_2 N_1, \\ t' = M_1 N_2. \end{cases} \tag{4.16}$$

Remark 4.6. A non-constant common factor of M_1, M_2 defines an invariant algebraic curve of the system (4.15). However, it willl not generate any solution to the system (4.15). See also Remark 4.5.

4.4 Rational solutions of the rational differential systems

The contents of this section can be found in [CNW11].

Our goal is computing rational solutions of a differential system like (4.15), each solution defines a rational invariant algebraic curve of the system, we want to use rational invariant algebraic curves as candidates to be a rational solution of the system, in fact we will see that not all invariant algebraic curves define a rational solution for the system.

Proposition 4.9. *Let $(s(x), t(x))$ be a non-trivial rational solution of the system (4.15). Let $G(s, t) = 0$ be the implicit form of the irreducible curve parametrized by $(s(x), t(x))$. Then*

$$\frac{\partial}{\partial s}G(s,t)M_2(s,t)N_1(s,t) + \frac{\partial}{\partial t}G(s,t)M_1(s,t)N_2(s,t) = G(s,t)K(s,t)$$

for some polynomial $K(s, t)$.

Proof. Clearly by definition of $G(s, t) = 0$ we have

$$G(s(x), t(x)) = 0.$$

Differentiating this equation with respect to x we obtain

$$\frac{\partial}{\partial x}G(s(x),t(x)) = \frac{\partial}{\partial s}G(s(x),t(x)) \cdot s'(x) + \frac{\partial}{\partial t}G(s(x),t(x)) \cdot t'(x) = 0.$$

Since $(s(x), t(x))$ is a solution of the system we have

$$\frac{\partial}{\partial s}G(s(x),t(x)) \cdot \frac{N_1(s(x),t(x))}{M_1(s(x),t(x))} + \frac{\partial}{\partial t}G(s(x),t(x)) \cdot \frac{N_2(s(x),t(x))}{M_2(s(x),t(x))} = 0.$$

Hence, the polynomial $\frac{\partial}{\partial s}G \cdot M_2N_1 + \frac{\partial}{\partial t}G \cdot M_1N_2$ is in the ideal of the curve generated by $G(s, t)$. In other words

$$\frac{\partial}{\partial s}G \cdot M_2N_1 + \frac{\partial}{\partial t}G \cdot M_1N_2 = G \cdot K,$$

for some polynomial K. □

Lemma 4.10. *Let $G(s, t) = 0$ be an irreducible rational invariant algebraic curve of the system (4.15). Let $(s(x), t(x))$ be a rational proper parametrization of $G(s, t) = 0$. Then we have*

$$s'(x) \cdot M_1(s(x),t(x))N_2(s(x),t(x)) = t'(x) \cdot M_2(s(x),t(x))N_1(s(x),t(x)).$$

Moreover, if $G \nmid M_1$ and $G \nmid M_2$, then

$$s'(x) \cdot \frac{N_2(s(x),t(x))}{M_2(s(x),t(x))} = t'(x) \cdot \frac{N_1(s(x),t(x))}{M_1(s(x),t(x))}.$$

Proof. Since $G(s(x),t(x)) = 0$, differentiating we have

$$\frac{\partial}{\partial s}G(s(x),t(x)) \cdot s'(x) + \frac{\partial}{\partial t}G(s(x),t(x)) \cdot t'(x) = 0.$$

But $G(s,t) = 0$ is an invariant algebraic curve too, so we have

$$\frac{\partial}{\partial s}G(s(x),t(x))M_2(s(x),t(x))N_1(s(x),t(x)) + \frac{\partial}{\partial t}G(s(x),t(x))M_1(s(x),t(x))N_2(s(x),t(x)) = 0.$$

Note that $\frac{\partial}{\partial s}G(s(x),t(x)) \neq 0$ and $\frac{\partial}{\partial t}G(s(x),t(x)) \neq 0$ because $G(s,t) = 0$ is irreducible. Therefore,

$$\begin{vmatrix} s'(x) & t'(x) \\ M_2(s(x),t(x))N_1(s(x),t(x)) & M_1(s(x),t(x))N_2(s(x),t(x)) \end{vmatrix} = 0.$$

Moreover, if $G \nmid M_1$ and $G \nmid M_2$, then $M_1(s(x),t(x)) \neq 0$ and $M_2(s(x),t(x)) \neq 0$, hence

$$s'(x) \cdot \frac{N_2(s(x),t(x))}{M_2(s(x),t(x))} = t'(x) \cdot \frac{N_1(s(x),t(x))}{M_1(s(x),t(x))}.$$

□

As announced the lemma tells us that not every rational parametrization of a rational invariant algebraic curve will provide a rational solution of the system. But they are good candidates for being rational solutions of the system.

Definition 4.9. A rational invariant algebraic curve of the system (4.15) is called a *rational solution curve* if it admits a rational parametrization which is a solution of the system.

Now we need a sufficient and necessary condition to say when a rational invariant algebraic curve is a solution curve, next theorem provides this condition.

Theorem 4.11. *Let $G(s,t) = 0$ be a rational invariant algebraic curve of the system*

$$\begin{cases} s' = \frac{N_1(s,t)}{M_1(s,t)} \\ \\ t' = \frac{N_2(s,t)}{M_2(s,t)}, \end{cases} \tag{4.17}$$

such that $G(s,t) \nmid M_1(s,t)$ and $G(s,t) \nmid M_2(s,t)$. Let $(s(x),t(x))$ be a rational proper parametrization of $G(s,t) = 0$. Then $G(s,t) = 0$ is a rational solution curve of the system (4.17) if and only if one of the following differential equations has a rational solution $T(x)$:

1. *When $s'(x) \neq 0$:*

$$T' = \frac{1}{s'(T)} \cdot \frac{N_1(s(T),t(T))}{M_1(s(T),t(T))}.$$

2. When $t'(x) \neq 0$:

$$T' = \frac{1}{t'(T)} \cdot \frac{N_2(s(T), t(T))}{M_2(s(T), t(T))}.$$

If there is such a rational solution $T(x)$, then the rational solution of the system (4.17) corresponding to $G(s,t) = 0$ is given by

$$(s(T(x)), t(T(x))).$$

Proof. Assume that $(\bar{s}(x), \bar{t}(x))$ is a rational solution of the system (4.17) corresponding to $G(s,t) = 0$. Then $(\bar{s}(x), \bar{t}(x))$ is a rational proper parametrization of $G(s,t) = 0$. Since $(s(x), t(x))$ is a proper parametrization of $G(s,t) = 0$, there exists a rational function $T(x)$ such that

$$\bar{s}(x) = s(T(x)), \quad \bar{t}(x) = t(T(x)). \tag{4.18}$$

Since $(\bar{s}(x), \bar{t}(x))$ is a rational solution of the system (4.17) we have

$$\begin{cases} \bar{s}'(x) = \frac{N_1(\bar{s}(x), \bar{t}(x))}{M_1(\bar{s}(x), \bar{t}(x))} \\ \\ \bar{t}(x)' = \frac{N_2(\bar{s}(x), \bar{t}(x))}{M_2(\bar{s}(x), \bar{t}(x))}. \end{cases}$$

On the other hand, from (4.18) we obtain

$$\begin{cases} \bar{s}'(x) = s'(T(x)) \cdot T'(x), \\ \bar{t}'(x) = t'(T(x)) \cdot T'(x). \end{cases}$$

Therefore,

$$T'(x) \cdot s'(T(x)) = \frac{N_1(\bar{s}(x), \bar{t}(x))}{M_1(\bar{s}(x), \bar{t}(x))},$$

$$T'(x) \cdot t'(T(x)) = \frac{N_2(\bar{s}(x), \bar{t}(x))}{M_2(\bar{s}(x), \bar{t}(x))}.$$

If $s'(x) \neq 0$ or $t'(x) \neq 0$, we have

$$T' = \frac{1}{s'(T)} \cdot \frac{N_1(s(T), t(T))}{M_1(s(T), t(T))}$$

or

$$T' = \frac{1}{t'(T)} \cdot \frac{N_2(s(T), t(T))}{M_2(s(T), t(T))},$$

respectively. Conversely, we can assume without loss of generality that $s(x)$ is non-constant and $T(x)$ is a rational solution of the fist differential equation. Thanks to Lemma 4.10 we have

$$s'(T) \cdot \frac{N_2(s(T), t(T))}{M_2(s(T), t(T))} = t'(T) \cdot \frac{N_1(s(T), t(T)}{M_1(s(T), t(T)))}.$$

If $t'(T) \equiv 0$, then $\frac{N_2(s(T),t(T))}{M_2(s(T),t(T))} = 0$ and $t(x) = c$ for some constant c. Clearly $(s(T(x)), c)$ is a rational solution of the system (4.17), so $G(s,t) = 0$ is a rational solution curve of the system. If $t'(T)$ is not 0, then

$$\frac{1}{t'(T)} \cdot \frac{N_2(s(T),t(T))}{M_2(s(T),t(T))} = \frac{1}{s'(T)} \cdot \frac{N_1(s(T),t(T))}{M_1(s(T),t(T))}.$$

Therefore, $T(x)$ is also a rational solution of the second differential equation.

$$\frac{\partial}{\partial x} s(T(x)) = s'(T(x)) \cdot T'(x) = s'(T(x)) \cdot \frac{1}{s'(T)} \cdot \frac{N_1(s(T),t(T))}{M_1(s(T),t(T))} = \frac{N_1(s(T),t(T))}{M_1(s(T),t(T))},$$

and

$$\frac{\partial}{\partial x} t(T(x)) = t'(T(x)) \cdot T'(x) = t'(T(x)) \cdot \frac{1}{t'(T)} \cdot \frac{N_2(s(T),t(T))}{M_2(s(T),t(T))} = \frac{N_2(s(T),t(T))}{M_2(s(T),t(T))},$$

hence $(s(T(x)), t(T(x)))$ is a rational solution of the system (4.17). So $G(s,t) = 0$ is a rational solution curve. □

Last Theorem tells us that if we are in the case $s'(x)$ not zero, we need to find a rational solution of the autonomous differential equation

$$T' = \frac{1}{s'(T)} \cdot \frac{N_1(s(T),t(T))}{M_1(s(T),t(T))}$$

to compute a rational solution of the system (4.17).

We want now to prove that the solvability of the autonomous differential equations in Theorem 4.11 do not depend on the choice of the proper rational parametrization of the rational invariant algebraic curve $G(s,t) = 0$.

Theorem 4.12. *Let $G(s,t) = 0$ be a rational invariant algebraic curve of the system (4.17) such that $G(s,t) \nmid M_1(s,t)$ and $G(s,t) \nmid M_2(s,t)$. Let $\phi_1(x) = (s_1(x), t_1(x))$ and $\phi_2(x) = (s_2(x), t_2(x))$ be two proper rational parametrizations of the curve $G(s,t) = 0$ such that $s_1'(x) \neq 0$ and $s_2'(x) \neq 0$. Then the two autonomous differential equations*

$$T_1' = \frac{1}{s_1'(T_1)} \cdot \frac{N_1(s_1(T_1), t_1(T_1))}{M_1(s_1(T_1), t_1(T_1))} \tag{4.19}$$

and

$$T_2' = \frac{1}{s_2'(T_2)} \cdot \frac{N_1(s_2(T_2), t_2(T_2))}{M_1(s_2(T_2), t_2(T_2))} \tag{4.20}$$

are such that one of them have a rational solution if and only if the other one has. Moreover T_1 and T_2 can be chosen such that

$$\phi_1(T_1) = \phi_2(T_2).$$

Proof. Suppose that (4.19) has a rational solution $T_1(x)$. Then the rational general solution of (4.17) corresponding to $G(s,t)=0$ is $(s_1(T_1(x)), t_1(T_1(x)))$. Since $(s_1(T_1(x)), t_1(T_1(x)))$ and $(s_2(x), t_2(x))$ are both a proper rational parametrization of the same curve $G(s,t)=0$, there exists a rational function $T_2(x)$ such that

$$s_2(T_2(x)) = s_1(T_1(x)), \quad t_2(T_2(x)) = t_1(T_1(x)).$$

Hence,

$$s_2'(T_2(x))T_2'(x) = s_1'(T_1(x))T_1'(x) = \frac{N_1(s_1(T_1), t_1(T_1))}{M_1(s_1(T_1), t_1(T_1))} = \frac{N_1(s_2(T_2), t_2(T_2))}{M_1(s_2(T_2), t_2(T_2))}.$$

This means that $T_2(x)$ is a rational solution of (4.20). Exactly the same thing can be done starting from (4.20). □

Theorem 4.13. *Suppose that $s(x)$ is a non constant rational function and $N_1(s(x), t(x)) \neq 0$. Then every rational solution of*

$$T' = \frac{1}{s'(T)} \cdot \frac{N_1(s(T), t(T))}{M_1(s(T), t(T))}$$

is of the form

$$T(x) = \frac{ax+b}{cx+d},$$

where a, b, c and d are constants.

Proof. Assume that $T(x)$ is a rational solution of the above differential equation. Then, by Proposition 3.5 $(T(x), T'(x))$ forms a proper parametrization of the algebraic curve $H(T,U)=0$ defined by the numerator of

$$s'(T) \cdot M_1(s(T), t(T)) \cdot U - N_1(s(T), t(T)).$$

Since the degree of $H(T,U)$ with respect to U is 1, the degree of $T(x)$ is also 1 because of the degree bound of proper parametrizations (Chapter 2). □

Corollary 4.14. *Every non-trivial rational solution of the system (4.17) is proper, i.e. It defines a rational proper parametrization of the rational solution curve.*

Proof. By Theorem 4.11, a non-trivial rational solution of the system (4.17) is a composition of a proper parametrization with an invertible linear rational function, hence, it is a proper parametrization. □

Observe that if $(s(x), t(x))$ is a rational solution of the system (4.17), then, because the equations in (4.17) are autonomous ones,

$$(s(x+c), t(x+c))$$

is also a rational solution of the system for every constant c. Next Theorem tells us that this the only way to obtain rational general solution from the same rational solution curve.

Theorem 4.15. *Let $(s_1(x), t_1(x))$ and $(s_2(x), t_2(x))$ be non-trivial rational solutions of the differential system (4.17) corresponding to the same rational invariant algebraic curve. Then there exists a constant c such that*

$$(s_1(x+c), t_1(x+c)) = s_2((x), t_2(x)).$$

Proof. By Corollary 4.14, we have proven that this solutions are proper. Since $(s_1(x), t_1(x))$ and $(s_2(x), t_2(x))$ are rational parametrizations of the same invariant algebraic curve, there exists a linear rational function $T(x)$ such that

$$(s_2(x), t_2(x)) = (s_1(T(x)), t_1(T(x))).$$

So

$$\begin{cases} s_1'(T(x))T'(x) = s_2'(x) = \frac{N_1(s_2(x),t_2(x))}{M_1(s_2(x),t_2(x))} = \frac{N_1(s_1(x),t_1(x))}{M_1(s_1(x),t_1(x))} \\ t_1'(T(x))T'(x) = t_2'(x) = \frac{N_2(s_2(x),t_2(x))}{M_2(s_2(x),t_2(x))} = \frac{N_2(s_1(x),t_1(x))}{M_2(s_1(x),t_1(x))}. \end{cases} \tag{4.21}$$

It follows that

$$T'(x) = 1.$$

Therefore, $T(x) = x + c$ for some constant c. □

What we said above leads to the following algorithm to solve a linear rational differential system as (4.17)

INPUT: A rational differential system as (4.17).
OUTPUT: The corresponding rational solution of (4.17), if it exists.

1. Compute the set of irreducible invariant algebraic curves of the system.
2. Find a rational invariant algebraic curve $G(s,t) = 0$ such that $G \nmid M_1$ and $G \nmid M_2$.
3. Compute a rational proper parametrization η of the curve $G(s,t) = 0$.
4. If it exists, compute a rational solution of one of the following equations:

   If $s'(x) \equiv 0$
   $$T' = \frac{1}{t'(T)} \cdot \frac{N_2(s(T), t(T))}{M_2(s(T), t(T))},$$
   else
   $$T' = \frac{1}{s'(T)} \cdot \frac{N_1(s(T), t(T))}{M_1(s(T), t(T))}.$$
5. If $T(x)$ exists, return
   $$(s(T(x)), t(T(x))).$$

Example 4.6. Consider the rational differential system

$$\begin{cases} s' = \frac{-2(-(t-1)^2+s^2)(t-1)^2}{((t-1)^2+s^2)^2} \\ \\ t' = \frac{-4(t-1)^3 s}{((t-1)^2+s^2)^2}. \end{cases} \tag{4.22}$$

First we compute the set of invariant algebraic curves of the system (4.22),

$$\{t-1=0, s \pm \sqrt{-1}(t-1)=0, s^2+t^2+(-1-c)t+c=0\},$$

where c is an arbitrary constant. Note that one can prove that the system has no irreducible invariant curves of degree higher than 2. Observe that we discard the curves $s \pm \sqrt{-1}(t-1)=0$ because they divide the denominators of the system. The invariant algebraic curve $t-1=0$ can be parametrized by $(x,1)$. $s'(x)=1$, so the corresponding equation is

$$T' = 0.$$

Hence, $T(x)=C$ for some constant C. So $(s(x),t(x))=(C,1)$ is a rational solution corrensponding to the rational solution curve $t-1=0$. Consider now the invariant algebraic curves

$$s^2+t^2+(-1-c)t+c=0.$$

This is a family of conic curves depending on the constant parameter c. Let $c=-1$. Then we obtain the curve

$$s^2+t^2-1=0,$$

which is the unit circle. Consider the parametrization given by the stereographic projection

$$\phi(x) = \big(\frac{-2x}{x^2+1}, \frac{-x^2+1}{x^2+1}\big).$$

Hence, the corresponding equation is

$$T' = T^2.$$

This implies that

$$T(x) = -\frac{1}{x}.$$

A rational solution of the system is

$$(s(x),t(x)) = \big(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1}\big).$$

The Proposition 4.5 gives us a necessary and sufficient condition for a rational solution of a rational differential system, as (4.17), to be a rational general solution of the system. According to this proposition the solution must have at least a coefficient which depends on an arbitrary constant. From the algorithmic point of view we can obtain a rational general solution for the system starting from a rational parametrization of a whole family of invariant algebraic curves as in the following example

Example 4.7 (Continuing Example 4.6)**.** We now consider the whole family of invariant algebraic curves of the system (4.22) depending on the constant parameter c

$$G(s,t) = s^2 + t^2 + (-1-c)t + c = 0.$$

This determines a rational curve in $\mathbb{A}^2(\overline{\mathbb{K}(c)})$, which has the proper rational parametrization

$$\phi(x) = \big(\frac{(c-1)x}{1+x^2}, \frac{cx^2+1}{1+x^2}\big).$$

The corresponding autonomous differential equation is

$$T' = -\frac{2T^2}{c-1}.$$

Hence

$$T(x) = \frac{c-1}{2x}.$$

Now we substitute $T(x)$ into $\phi(x)$ to obtain a rational general solution of the system (4.17)

$$(s(x), t(x)) = \big(\frac{2(c-1)^2 x}{4x^2 + (c-1)^2}, \frac{c(c-1)^2 + 4x^2}{4x^2 + (c-1)^2}\big).$$

4.5 Some examples

Now we are able to compute rational general solutions of an autonomous algebraic ODE $F(x,y,y') = 0$ given a rational proper parametrization of the solution surface $F(x,y,z) = 0$. As in the autonomous case we implemented the algorithms of the previous sections in two different computer algebra systems: Singular and CoCoA. In next examples we consider parametrizations of the solution surface easy computable or known in literature.

Example 4.8. Equation:

$$F(x,y,y') = xy'^2 + yy' - y^4 = 0$$

Solution surface:

$$F(x,y,z) = xz^2 + yz - y^4 = 0$$

Solution surface's parametrization:

$$\phi(s,t) = \big(s, \frac{t}{t^2 - s}, \frac{-t^3}{s(t^2-s)^2}\big)$$

Singular[1]:

[1]My algorithm gives in output two elements (1) and (2) respectively numerator and denominator of the rational solution.

(1) -1

(2) x-1

CoCoA:

```
-1/(x-1)
```

Example 4.9. Equation:

$$F(x, y, y') = y'^2 + 3y' - 2y - 3x = 0$$

Solution surface:

$$F(x, y, z) = z^2 + 3z - 2y - 3x = 0$$

Solution surface's parametrization:

$$\phi(s,t) = \big(\frac{2s + st + t^2}{s^2}, \frac{-t^2 - 3s}{s^2}, \frac{t}{s}\big)$$

Singular:

(1) x2

(2) 2

CoCoA:

```
1/2x^2
```

Example 4.10. Equation:

$$F(x, y, y') = x^2y'^2 - 4x(y + 2)y' + 4y(y + 2) = 0$$

Solution surface:

$$F(x, y, z) = x^2z^2 - 4x(y + 2)z + 4y(y + 2) = 0$$

Solution surface's parametrization:

$$\phi(s,t) = \big(s, \frac{-2st(st - 4)}{(st - 2)^2}, \frac{8t}{(st - 2)^2}\big)$$

Singular:

(1) 2x2+4x

(2) 1

CoCoA:

```
2x^2+4x
```

Example 4.11. Equation:

$$F(x, y, y') = xy'^2 + yy' + y^4 = 0$$

Solution surface:

$$F(x, y, z)xz^2 + yz + y^4 = 0$$

Solution surface's parametrization:

$$\phi(s, t) = \big(s, \frac{t^3}{-s^6 + s^3t^2}, \frac{-t^5}{s^{10} - 2s^7t^2 + s^4t^4}\big)$$

SINGULAR:

(1) -1

(2) x-1

CoCoA:

```
-1/(x-1)
```

4.6 Affine transformations and rational general solutions

The main results in this section are from [CNSW12], but the Lemma 4.20 and the proposition 4.21 are original. In the algorithm to compute a rational general solution of a non-autonomous differential equation $F(x, y, y') = 0$ given in Section 4.1 the hardest step is computing a proper rational parametrization of the solution surface $F(x, y, z) = 0$. To avoid this step we want to investigate on affine transformations which send rational general solutions in general solutions, thanks to such transformations we can transform an equation into an easier one to solve.

Definition 4.10. A parametric curve of the form $C(x) = (x, f(x), f'(x))$, where $f(x)$ is a rational function, is called *integral curve of the space.*

We define a group of affine linear transformations on $\mathbb{K}(x)^3$ mapping an integral curve of the space to another one. This group can act on the set of all algebraic ODEs of order 1 and it is compatible with the solution curves of the corresponding differential equations. Let $L : \mathbb{K}(x)^3 \to \mathbb{K}(x)^3$ be an affine linear transformation defined by

$$L(v) = Av + B,$$

where A is an invertible 3×3 matrix over $\mathbb{K}$, B is a column vector over $\mathbb{K}$ and v is a column vector over $\mathbb{K}(x)$. We want to determine A and B such that for any $f(x) \in \mathbb{K}(x)$, there exists $g \in \mathbb{K}(x)$ with

$$L\begin{pmatrix} x \\ f(x) \\ f'(x) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} x \\ f(x) \\ f'(x) \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} x \\ g(x) \\ g'(x) \end{pmatrix}. \tag{4.23}$$

From (4.23) we have the following equalities

$$\begin{cases} x = a_{11}x + a_{12}f(x) + a_{13}f'(x) + b_1 \\ g(x) = a_{21}x + a_{22}f(x) + a_{23}f'(x) + b_2 \\ g'(x) = a_{31}x + a_{32}f(x) + a_{33}f'(x) + b_3. \end{cases} \tag{4.24}$$

Differentiating the second equation of (4.24) we obtain

$$\begin{cases} x = a_{11}x + a_{12}f(x) + a_{13}f'(x) + b_1 \\ a_{21} + a_{22}f'(x) + a_{23}f''(x) = a_{31}x + a_{32}f(x) + a_{33}f'(x) + b_3. \end{cases} \tag{4.25}$$

Since (4.25) must be true for every $f(x) \in \mathbb{K}(x)$ we have

$$a_{11} = 1, \qquad b_1 = a_{12} = a_{13} = a_{23} = a_{32} = a_{31} = 0$$

$$a_{21} = b_3 = b, \quad a_{22} = a_{33} = a, \quad b_2 = c,$$

with $a, b, c \in \mathbb{K}$. Therefore,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ b & a & 0 \\ 0 & 0 & a \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ c \\ b \end{pmatrix}.$$

Since we want A to be invertible we ask $a \neq 0$. We call $\mathcal{G}$ the set of all such affine transformations. We represent the elements of $\mathcal{G}$ by a pair of matrices $\begin{bmatrix} A & B \end{bmatrix}$.

Proposition 4.16. *The set $\mathcal{G}$ with the usual composition of functions is a group.*

Proof. Consider $L_1, L_2 \in G$,

$$L_i = \left[\begin{pmatrix} 1 & 0 & 0 \\ b_i & a_i & 0 \\ 0 & 0 & a_i \end{pmatrix} \quad \begin{pmatrix} 0 \\ c_i \\ b_i \end{pmatrix} \right], \qquad for \quad i = 1, 2.$$

We must verify that $(L_1 \circ L_2) \in \mathcal{G}$ and the existence of the inverse for every $L \in \mathcal{G}$ and of the unit element. Clearly, the associative property is verified since the composition of functions is associative in general. As unit element we take the element $\begin{bmatrix} Id_3 & 0 \end{bmatrix}$, where Id_3 is the identity matrix 3×3.

$$
\begin{aligned}
(L_1 \circ L_2)\begin{pmatrix} x \\ f(x) \\ f'(x) \end{pmatrix} &= L_1\left(\begin{pmatrix} 1 & 0 & 0 \\ b_2 & a_2 & 0 \\ 0 & 0 & a_2 \end{pmatrix}\begin{pmatrix} x \\ f(x) \\ f'(x) \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 \\ b_2 \end{pmatrix}\right) \\
&= L_1\begin{pmatrix} x \\ b_2x + a_2f(x) + c_2 \\ a_2f'(x) + b_2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ b_1 & a_1 & 0 \\ 0 & 0 & a_1 \end{pmatrix}\begin{pmatrix} x \\ b_2x + a_2f(x) + c_2 \\ a_2f'(x) + b_2 \end{pmatrix} + \begin{pmatrix} 0 \\ c_1 \\ b_1 \end{pmatrix} \\
&= \begin{pmatrix} x \\ (b_1 + a_1b_2)x + a_1a_2f(x) + a_1c_2 + c_1 \\ a_1a_2f'(x) + b_2a_1 + b_1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ b_1 + a_1b_2 & a_1a_2 & 0 \\ 0 & 0 & a_1a_2 \end{pmatrix}\begin{pmatrix} x \\ f(x) \\ f'(x) \end{pmatrix} + \begin{pmatrix} 0 \\ a_1c_2 + c_1 \\ b_2a_1 + b_1 \end{pmatrix}.
\end{aligned}
$$

Hence, $L_1 \circ L_2 \in \mathcal{G}$.
Consider now the element H of $\mathcal{G}$

$$
H = \left[\begin{pmatrix} 1 & 0 & 0 \\ -\frac{b_1}{a_1} & \frac{1}{a_1} & 0 \\ 0 & 0 & \frac{1}{a_1} \end{pmatrix} \quad \begin{pmatrix} 0 \\ -\frac{c_1}{a_1} \\ -\frac{b_1}{a_1} \end{pmatrix}\right],
$$

thanks to the above computations we have

$$
H \circ L_1 = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right],
$$

and

$$
L_1 \circ H = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right].
$$

So $H = L_1^{-1}$ and $\mathcal{G}$ is a group as wanted. □

Now we want to define how the group $\mathcal{G}$ just defined acts on the set $\mathcal{AODE}$.

$$
\varphi \colon \mathcal{G} \times \mathcal{AODE} \to \mathcal{AODE}
$$

$$
(L, F) \mapsto L \cdot F = (F \circ L^{-1})(x, y, y') = F\big(x, -\frac{b}{a}x + \frac{1}{a}y - \frac{c}{a}, -\frac{b}{s} + \frac{1}{a}y'\big),
$$

where

$$L = \left[\begin{pmatrix}1 & 0 & 0\\ b & a & 0\\ 0 & 0 & a\end{pmatrix} \quad \begin{pmatrix}0\\ c\\ b\end{pmatrix}\right].$$

Lemma 4.17. *The map $\varphi\colon \mathcal{G} \times \mathcal{AODE} \to \mathcal{AODE}$ defined above is a group action of $\mathcal{G}$ on $\mathcal{AODE}$.*

Proof. We have

$$\begin{aligned}(L_1 \circ L_2) \cdot F &= F \circ (L_1 \circ L_2)^{-1}\\ &= F \circ (L_2^{-1} \circ L_1^{-1})\\ &= (F \circ L_2^{-1}) \circ L_1^{-1}\\ &= L_1 \cdot (L_2 \cdot F),\end{aligned}$$

and $I \cdot F = F$. Therefore, this is an action of the group $\mathcal{G}$ on $\mathcal{AODE}$. □

Remark 4.7. Let $F \in \mathcal{PODE} \subset \mathcal{AODE}$ and $\psi(s,t)$ be a proper rational parametrization of the solution surface of F, then $L \circ \psi(s,t)$ is a proper rational parametrization of the solution surface of $(L \cdot F)$, because

$$(L \cdot F)((L \circ \psi(s,t))) = F(L^{-1}((L \circ \psi(s,t)))) = F(\psi(s,t)) = 0.$$

Therefore, $(L \cdot F) \in \mathcal{PODE}$ and φ is a group action on the set $\mathcal{PODE}$ too.

Theorem 4.18. *Let $F \in \mathcal{PODE}$ and $L \in \mathcal{G}$. For every proper rational parametrization $\psi(s,t)$ of the surface $F(x,y,z) = 0$ the associated system of $F(x,y,y') = 0$ with respect to ψ and the associated system of $(L \cdot F)(x,y,y') = 0$ with respect to $L \circ \psi(s,t)$ are equal.*

Proof. Let $\psi(s,t) = (\chi_1(s,t), \chi_2(s,t), \chi_3(s,t))$ be a proper rational parametrization of $F(x,y,y') = 0$. Then $L \cdot F$ can be parametrized by $(L \circ \psi)(s,t)$. The associated system of $F(x,y,y') = 0$ with respect to $\psi(s,t)$ is $\left\{s' = \frac{f_1}{g}, t' = \frac{f_2}{g}\right\}$ where

$$f_1 = \begin{vmatrix}1 & \frac{\partial}{\partial t}\chi_1\\ \chi_3 & \frac{\partial}{\partial t}\chi_2\end{vmatrix}, \quad f_2 = \begin{vmatrix}\frac{\partial}{\partial s}\chi_1 & 1\\ \frac{\partial}{\partial s}\chi_2 & \chi_3\end{vmatrix}, \text{ and } g = \begin{vmatrix}\frac{\partial}{\partial s}\chi_1 & \frac{\partial}{\partial t}\chi_1\\ \frac{\partial}{\partial s}\chi_2 & \frac{\partial}{\partial t}\chi_2\end{vmatrix}.$$

We have

$$(L \circ \psi)(s,t) = (\chi_1, b\chi_1 + a\chi_2 + c, b + a\chi_3),$$

where a, b, c are constants and $a \neq 0$. So the associated system of $(L \cdot F)(x,y,y') = 0$ with respect to $(L \circ \psi)$ is $\left\{s' = \frac{\tilde{f}_1}{\tilde{g}}, t' = \frac{\tilde{f}_2}{\tilde{g}}\right\}$ where

$$\tilde{f}_1 = \begin{vmatrix}1 & \frac{\partial}{\partial t}\chi_1\\ b + a\chi_3 & b\frac{\partial}{\partial t}\chi_1 + \frac{\partial}{\partial t}a\chi_2\end{vmatrix} = af_1, \quad \tilde{f}_2 = \begin{vmatrix}\frac{\partial}{\partial s}\chi_1 & 1\\ b\frac{\partial}{\partial s}\chi_1 + a\frac{\partial}{\partial s}\chi_2 & b + a\chi_3\end{vmatrix} = af_2,$$

and

$$\tilde{g} = \begin{vmatrix} \frac{\partial}{\partial s}\chi_1 & \frac{\partial}{\partial t}\chi_1 \\ b\frac{\partial}{\partial s}\chi_1 + a\frac{\partial}{\partial s}\chi_2 & b\frac{\partial}{\partial t}\chi_1 + a\frac{\partial}{\partial t}\chi_2 \end{vmatrix} = ag.$$

Therefore, the associated system of $F(x, y, y') = 0$ with respect to ψ and the associated system of $(L \cdot F)(x, y, y') = 0$ with respect to $(L \circ \psi)(s, t)$ are equal. □

In Chapter 3 we have seen the algorithm to compute rational general solutions of autonomous algebraic ODEs, if they exist. Since the autonomous case is much more easier than the non-autonomous one it can be interesting to see when a non-autonomous equation can be transformed into an autonomous one. Next corollary gives us necessary condition for that.

Corollary 4.19. *Let $F \in \mathcal{PODE}$ and $L \in \mathcal{G}$ such that $(L \cdot F)(x, y, y') = 0$ is an autonomous algebraic ODE. There exist a proper rational parametrization $\psi(s, t)$ of $F(x, y, z) = 0$ such that its associated system is of the form*

$$\left\{s' = 1, t' = \frac{M(t)}{N(t)}\right\}. \tag{4.26}$$

Proof. Since $(L \cdot F)(x, y, y') = 0$ is an autonomous parametrizable ODE, the plane algebraic curve $(L \cdot F)(y, z) = 0$ is rational, and for every proper rational parametrization $(f(t), g(t))$ of $(L \cdot F)(y, z) = 0$ the associated system of $(L \cdot F)(x, y, y') = 0$ with respect to $\psi(s, t) = (s, f(t), g(t))$ is of the form

$$\left\{s' = 1, t' = \frac{g(t)}{f'(t)}\right\}.$$

□

Remark 4.8. The inverse of the corollary is not true. Consider the equation

$$F(x, y, y') = y - y'^2 - y' - y'x = 0.$$

It belongs to $\mathcal{PODE}$ and

$$\psi_1(s, t) = (s, t^2 + t + s, t).$$

is a proper rational parametrization of $F(x, y, z) = 0$. The associated system with respect to ψ_1 is

$$\{s' = 1, t' = 0\},$$

which is of the form (4.26). Consider L the generic element of $\mathcal{G}$, we have

$$(L \cdot F)(x, y, y') = -\frac{1}{a^2}y'^2 - \frac{1}{a}y' + 2\frac{b}{a^2}y' - \frac{1}{a}xy' + \frac{1}{a}y + \frac{b}{a} - \frac{b^2}{a^2} - \frac{c}{a},$$

to make this autonomous we must have

$$\frac{1}{a} = 0,$$

but this is impossible.

We want to discover when an equation $F \in \mathcal{PODE}$ can be transformed into an easier one by the action of $\mathcal{G}$. Algorithmically we work as in Remark 4.8, i.e. we apply the generic element L of $\mathcal{G}$ to the F then we try to obtain a new equation of a special form easy to solve $(L \cdot F)$ by vanishing some coefficients. We obtain an algebraic system Σ in the unknowns a, b, c. By solving the system we obtain the transformation L which is used to obtain a rational general solution of F from a rational general solution of $(L \cdot F)$. We said we would like to avoid computing a rational proper parametrization of the solution surface, so we try to transform non-autonomous equation into autonomous ones, or into equation of a particular form which allow us to obtain easily a proper rational parametrization of the solution surface.

Definition 4.11. An equation $F \in \mathcal{PODE}$ is called *solvable for* y', respectively for y and for x, if it is of the form

$$y' = h(x, y),$$

where $h(x, y)$ is a rational function, or

$$y = h(x, y'),$$

or

$$x = h(y, y'),$$

respectively.

Remark 4.9. If an equation $F \in \mathcal{PODE}$ is solvable for a variable, it as an obvious rational proper parametrization. For instance, if F is solvable for y' its obvious proper rational paramterization is of the form

$$(s, t, h(s, t)).$$

This parametrization is proper since $h(s, t)$ is a rational function we have

$$\mathbb{K}(s, t, h(s, t)) = \mathbb{K}(s, t).$$

Note now that the equations which are solvable for y' and y are closed under the action of $\mathcal{G}$, i.e. we can not enlarge these classes by applying the transformations in $\mathcal{G}$. In fact, if $F \in \mathcal{PODE}$ can be written in the form

$$y' = h(x, y)$$

for some rational function $h(x, y)$, by applying $L \in \mathcal{G}$ we have

$$(L \cdot F)(x, y, y') = -\frac{b}{a} + \frac{1}{a}y' - h\big(x, -\frac{b}{a}x + \frac{1}{a}y - \frac{c}{a}\big).$$

Therefore, the new differential equation is of the same form. The same thing happens if F is solvable for y.

So we must check only if an equation $F \in \mathcal{PODE}$ can be transformed into an autonomous equation or into a solvable for x one.

Example 4.12. Consider the differential equation

$$F(x, y, y') = y'^2 + 3y' - 2y - 3x = 0.$$

We apply a generic $L \in \mathcal{G}$ to F to get

$$(L \cdot F)(x, y, y') = \frac{1}{a^2}y'^2 + \frac{3}{a}y' - \frac{2b}{a^2}y' - \frac{2}{a}y + \frac{2b}{a}x - 3x - \frac{3b}{a} + \frac{b^2}{a^2} + \frac{2c}{a}.$$

In this case we have the following equation

$$\frac{2b}{a} - 3 = 0, \tag{4.27}$$

so for every $a \neq 0$ and b which satisfy (4.27) we get an autonomous algebraic ODE. For $a = 1$, $b = \frac{3}{2}$ and $c = 0$ we get

$$L = \left[\begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ \frac{3}{2} \end{pmatrix} \right],$$

i.e. we obtain

$$F(L^{-1}(x, y, y')) = y'^2 - 2y - \frac{9}{4} = 0.$$

Example 4.13. Consider the differential equation

$$F(x, y, y') = -3x - 4x^2 + 4xy - y^2 + 2xy' + 2y - yy' + 8 + 8y' + 2y'^2 = 0.$$

Applying a generic $L \in \mathcal{G}$ to F we obtain

$$\begin{aligned}(L \cdot F)(x, y, y') = -3x - 4x^2 - \frac{4b}{a}x^2 + \frac{4}{a}xy - \frac{4c}{a}x - \frac{b^2}{a^2} - \frac{1}{a^2}y^2 - \frac{c^2}{a^2} + \frac{2b}{a^2}xy + \frac{2}{a}xy' \\ -\frac{2b}{a} + \frac{b}{a^2}xy' - \frac{b^2}{a^2}x - \frac{1}{a^2}yy' + \frac{b}{a^2}y + \frac{c}{a^2}y' + \frac{b}{a^2} + 8 - \frac{8}{a}y' \\ +\frac{8b}{a} + \frac{2}{a^2}y^2 + \frac{2b^2}{a^2} - \frac{4b}{a^2}y'.\end{aligned}$$

The coefficient of x^2 is

$$\frac{-(2a + b)^2}{a^2},$$

from this we get

$$2a + b = 0.$$

Putting $a = 1$, $b = -2$ and $c = 0$ we have

$$(L \cdot F)(x, y, y') = x - y^2 - yy' + 2y'^2.$$

We have the following algorithm

INPUT: A non-autonomous parametrizable algebraic differential equation $F(x, y, y') = 0$.
OUTPUT: A rational solution of $F(x, y, y') = 0$, if F can be transformed into an equation of special form and if a rational solution exists.

1. Check if F can be transformed into an equation of special form. If yes, compute the transformation L.
2. Compute a rational solution of $(L \cdot F)$.
3. (a) If $(L \cdot F)$ is autonomous:
   i. Compute a rational general solution $\eta(x)$ of $(L \cdot F)$.
   ii. Return
   $$\hat{\eta} = \pi_2(L^{-1} \circ \eta),$$
   where $\pi_2 \colon \mathbb{K}(x)^3 \to \mathbb{K}(x)$ is the projection on the second coordinate.

   (b) If $(L \cdot F)$ is solvable for x:
   i. Compute the parametrization of
   $$F(x, y, y') = 0$$
   such that its associated system and the one of $(L \cdot F)$ are equal.
   ii. Compute a rational general solution of $F(x, y, y') = 0$ following the algorithms in the previous sections and return it.

Example 4.14 (Continuing Example 4.12)**.** A rational general solution of the equation

$$F(L^{-1}(x, y, y')) = y'^2 - 2y - \frac{9}{4} = 0$$

is given by

$$\hat{y}(x) = \frac{1}{2}\Big((x + c)^2 - \frac{9}{4}\Big),$$

where c is an arbitrary constant. Then a rational general solution of the given equation is

$$y(x) = \hat{y}(x) - \frac{3}{2}x.$$

Define now $\mathcal{G}' \subset \mathcal{G}$ the set of all elements L of $\mathcal{G}$ of the form

$$L = \left[\begin{pmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ c \\ b \end{pmatrix}\right].$$

Lemma 4.20. *$\mathcal{G}'$ is a subgroup of $\mathcal{G}$.*

Proof. It is enough to verify that the subgroup is closed by composition. Consider $L_1, L_2 \in \mathcal{G}'$ wirh

$$L_i = \left[\begin{pmatrix} 1 & 0 & 0 \\ b_i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ c_i \\ b_i \end{pmatrix}\right], \qquad for \quad i = 1, 2.$$

Then

$$\begin{aligned}
(L_1 \circ L_2)\begin{pmatrix} x \\ f(x) \\ f'(x) \end{pmatrix} &= L_1\left(\begin{pmatrix} 1 & 0 & 0 \\ b_2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ f(x) \\ f'(x) \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 \\ b_2 \end{pmatrix}\right) \\
&= L_1\begin{pmatrix} x \\ b_2 x + f(x) + c_2 \\ f'(x) + b_2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ b_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ b_2 x + f(x) + c_2 \\ f'(x) + b_2 \end{pmatrix} + \begin{pmatrix} 0 \\ c_1 \\ b_1 \end{pmatrix} \\
&= \begin{pmatrix} x \\ (b_1 + b_2)x + f(x) + c_2 + c_1 \\ f'(x) + b_2 + b_1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ b_1 + b_2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ f(x) \\ f'(x) \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 + c_1 \\ b_2 + b_1 \end{pmatrix}.
\end{aligned}$$

So $L_1 \circ L_2 \in \mathcal{G}'$. □

The next proposition tells us that we can reduce to the transformations in $\mathcal{G}'$.

Proposition 4.21. *Let $F \in \mathcal{PODE}$ and $L \in \mathcal{G}$ such that $(L \cdot F)$ is of special form, i.e. autonomous or sovable for x, then there exists $L' \in \mathcal{G}'$ such that $(L' \cdot F)$ is of the same form.*

Proof. Suppose

$$L = \left[\begin{pmatrix} 1 & 0 & 0 \\ b & a & 0 \\ 0 & 0 & a \end{pmatrix} \quad \begin{pmatrix} 0 \\ c \\ b \end{pmatrix}\right],$$

and that $deg(F) = n$ then $F(x, y, y') = 0$ can be written as

$$F(x, y, y') = \sum_{i+j+k \le n} \sigma_{ijk} x^i y^j y'^k.$$

Therefore,

$$(L \cdot F) = \sum_{i+j+k \le n} \sigma_{ijk} x^i \Big(-\frac{b}{a}x + \frac{1}{a}y - \frac{c}{a}\Big)^j \Big(\frac{1}{a}y' - \frac{b}{a}\Big)^k.$$

Fix now (i,j,k) and consider the monomial $x^i y^j y'^k$, after the action of L it becomes

$$\sigma_{ijk}x^i\Big(-\frac{b}{a}x+\frac{1}{a}y-\frac{c}{a}\Big)^j\Big(\frac{1}{a}y'-\frac{b}{a}\Big)^k = \\ \sigma_{ijk}\sum_{r=0}^{j}\sum_{s=0}^{r}\sum_{t=0}^{k}\binom{j}{r}\binom{r}{s}\binom{k}{t}(-1)^{s+j+k-r-t}\frac{b^{s+k-t}\cdot c^{j-r}}{a^{j+k}}x^{i+s}y^{r-s}y'^t.$$

For a triple (r,s,t) we get the term

$$\sigma_{ijk}\binom{j}{r}\binom{r}{s}\binom{k}{t}(-1)^{s+j+k-r-t}\frac{b^{s+k-t}\cdot c^{j-r}}{a^{j+k}}x^{i+s}y^{r-s}y'^t. \tag{4.28}$$

Consider now another monomial $\sigma_{efg}x^e y^f y'^g$ such that $(i,j,k)\neq(e,f,g)$ and we do the same substitution and choose u,v,w such that

$$x^{e+u}y^{v-u}y'^w = x^{i+s}y^{r-s}y'^t, \tag{4.29}$$

if this is not possible change (e,f,g). From (4.29) we have the following equalities

$$\begin{cases} t=w\\ e+u=i+s\\ r-s=v-u\\ e+v=i+r. \end{cases} \tag{4.30}$$

The coefficient of the monomial $x^{e+u}y^{v-u}y'^w$ after the substitution is

$$\sigma_{efg}\binom{f}{v}\binom{v}{u}\binom{g}{w}(-1)^d\frac{b^{u+g-w}\cdot c^{f-v}}{a^{f+g}}x^{e+u}y^{v-u}y'^w. \tag{4.31}$$

Note that

$$j+k = s+k-t+j-r+r-s+t$$
$$f+g = u+y-w+f-v+v-u+w$$

If $b'=\frac{b}{a}$ and $c'=\frac{c}{a}$ (4.28) and (4.31) can be written as

$$\sigma_{ijk}\binom{j}{r}\binom{r}{s}\binom{k}{t}(-1)^{d'}\frac{b'^{s+k-t}\cdot c'^{j-r}}{a^{r-s+t}}x^{i+s}y^{r-s}y'^t,$$

$$\sigma_{efg}\binom{f}{v}\binom{v}{u}\binom{g}{w}(-1)^d\frac{b'^{u+g-w}\cdot c'^{f-v}}{a^{v-u+w}}x^{e+u}y^{v-u}y'^w,$$

but $a^{r-s+t}=a^{v-u+w}$. So the coefficient of $x^{i+s}y^{r-s}y'^t$ in $(L\cdot F)(x,y,y')$ can be written in the form

$$\frac{1}{a^m}\Big(\sum_l \epsilon_l b'^{h_l^b}c'^{h_l^c}\Big),$$

for some positive integers m, h_l^b, h_l^c. Since L transforms F into a special form equation if $i+s \geq 1$ (or $i+s > 1$ in the solvable for x case) we have

$$\frac{1}{a^m}\Big(\sum_l \epsilon_l b'^{h_l^b} c'^{h_l^c}\Big) = 0,$$

which is true if and only if

$$\sum_l \epsilon_l b'^{h_l^b} c'^{h_l^c} = 0.$$

From above we have $m = r - s + t$ and note that $\frac{1}{a^{r-s}}$ and $\frac{1}{a^t}$ are the contributions of y and y' respectively. So

$$\sum_l \epsilon_l b'^{h_l^b} c'^{h_l^c}$$

is the coefficient of $x^{i+s} y^{r-s} y'^t$ after the action of

$$G = \left[\begin{pmatrix} 1 & 0 & 0 \\ b' & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ c' \\ b' \end{pmatrix}\right],$$

which belongs to $\mathcal{G}'$. □

We can modify the algorithm to compute rational general solutions of non-autonomous algebraic ODEs checking if the equation given in input is or can be transformed into an equation of special form by the an element of $\mathcal{G}'$. Clearly that makes the computation easier since we have only two unknowns b and c.

As done for the other algorithms we implemented this one in Singular and CoCoA. Here some examples follow.

Example 4.15. Equation:

$$F(x, y, y') = xy'^2 - yy' + 2 = 0$$

Solution surface:

$$xz^2 - yz + 2 = 0$$

Singular[2]:

- **(1)** `-x-2`

  (2) `1`

- **(1)** `(alpha^2)*x+2`

[2]My algorithm gives in output two elements (1) and (2) respectively numerator and denominator of the rational solution.

(2) `alpha`

CoCoA:

```
-x-2
```

Example 4.16. Equation:

$$F(x, y, y') = xy'^2 - 2yy' + 2y + x = 0$$

Solution surface:

$$xz^2 - 2yz + 2y + x = 0$$

SINGULAR:

(1) `-x^2+2*x-2`

(2) `2`

CoCoA:

```
-1/2x^2+x-1
```

Example 4.17. Equation:

$$F(x, y, y') = (x + 1)y'^2 - (y + x)y' + y = 0$$

Solution surface:

$$(x + 1)z^2 - (y + x)z + y = 0$$

SINGULAR:

- **(1)** `-2x-1`
  (2) `2`
- **(1)** `(alpha^2-alpha)*x+(alpha^2)`
  (2) `(alpha-1)`

CoCoA:

```
-x-1/2
```

Chapter 5

Implementation

In this chapter I want to present the structure of the implementation of the method for computing rational general solutions of first-order algebraic ordinary differential equation described in previous chapters, analysing what can be done so far and next steps to improve the method. The algorithms have been implemented in two computer algebra system: SINGULAR, by the University of Kaiserslautern, and CoCoA, by the University of Genova. All codes of the programmes described in this chapter can be found in the Appendix A. Almost every content of this chapter is original, I point out that the algorithm for computing invariant algebraic curves of a first-order autonomous rational differential system can be found in [Man93].

5.1 The implementation in SINGULAR

In this chapter we consider as base field the algebraic closure of the rationals $\bar{\mathbb{Q}}$. The final product of the implementation in SINGULAR can be split in three parts. The first part is a programme which takes in input an autonomous algebraic ODEs, i.e. a bivariate polynomial, and it computes a rational general solution of the differential equation, if it exists. That procedure is called *fenggao.*
The SINGULAR's Library *paraplanecurves.lib* contains a function to compute rational parametrizations of homogenous algebraic curves of genus zero, we use this function in our implementations. It is based on the algorithm of Van Hoeij for computing integral basis, more details can be found in [VH94] and in [DGPS].
Since in SINGULAR rational functions are not defined instead we consider a list of two polynomials, where the first one is the numerator and the second one is the denominator of the rational function. When we need to make computations or to substitute a rational function to a variable we make a wise choice of the ring and we treat the variable of the rational function as a constant parameter.

fenggao(F(x,y));

Let $\tilde{F}(x,y,z)$ be the homogenisation of F;

if $genus(\tilde{F}) \neq 0$ then `print` *"F is not a rational curve"*;

else Compute a rational parametrization $\psi(s,t) = (\psi_1, \psi_2, \psi_3)$ of $\tilde{F}(x,y,z) = 0$; dehomogenise ψ and obtain a rational parametrization $(p(s), q(s), w(s))$ of $F(x,y) = 0$ where $w(s)$ is the common denominator.

  if $(p(s), q(s), w(s))$ is not proper then `print` *"not proper parametrization"*;

  else $A = \frac{\partial}{\partial s}\left(\frac{p(s)}{w(s)}\right) = \left(\frac{\partial}{\partial s}p(s) \cdot w(s) - \frac{\partial}{\partial s}w(s) \cdot p(s), w(s)^2\right)$;

    $B = \frac{q(s)}{w(s)} = (q(s), w(s))$;

    $R = \frac{B}{A}$;

    Change ring. We consider x and c as parameters.

    if $R = a$ for a constant a then `return` $y = s(a \cdot (x + c))$;

    else if $R = a(x-b)^2$ for some constants a and b then `return` $y = (b - \frac{1}{a(x+c)})$;

      else `print` *"no rational solution"*

Thanks to this programme we are able to compute rational general solutions of every autonomous first-order algebraic ODEs which admits a rational parametrization. The hardest step is computing such parametrizations.

The second part of the implementation in Singular covers the method presented in the first part of Chapter 4 for computing rational general solution of non-autonomous algebraic ODE. Unfortunately in Singular there is no library which contains a function to compute a proper rational parametrization of an algebraic surface. So our programme takes in input a rational proper parametrization of the solution surface $F(x,y,z) = 0$, where $F(x,y,y') = 0$ is the first-order non-autonomous algebraic equation that we want to solve. This programme is called *chauwinkler.*
The procedures *asssystem* and *invalgcurve* computes, respectively, the associated system of F with respect to the parametrization given as input in *chauwinkler* and the set of its invariant algebraic curves, an analogous of these functions will be described in details in next section.
The rational parametrization of the solution surface is given as a list of four polynomials, where the last one is the least common multiple of denominators.

chauwinkler$(P = (p_1, p_2, p_3, p_4))$;

$\Sigma = (\Sigma_{11}, \Sigma_{12}, \Sigma_{21}, \Sigma_{22}) = asssystem(p)$;
$Inv = invalgcurve(p)$;
$Sol = emptylist$;

for each $G \in Inv$ Let $\tilde{G}(x, y, z)$ be the homogenisation of G;

if $genus(\tilde{G}) \neq 0$ then change G;

else Compute a rational proper parametrization $\psi(s,t) = (\psi_1, \psi_2, \psi_3)$ of $\tilde{G}(x, y, z) = 0$; dehomogenise ψ and obtain a rational proper parametrization $(p(s), q(s), w(s))$ of $G(x, y) = 0$ where $w(s)$ is the common denominator.

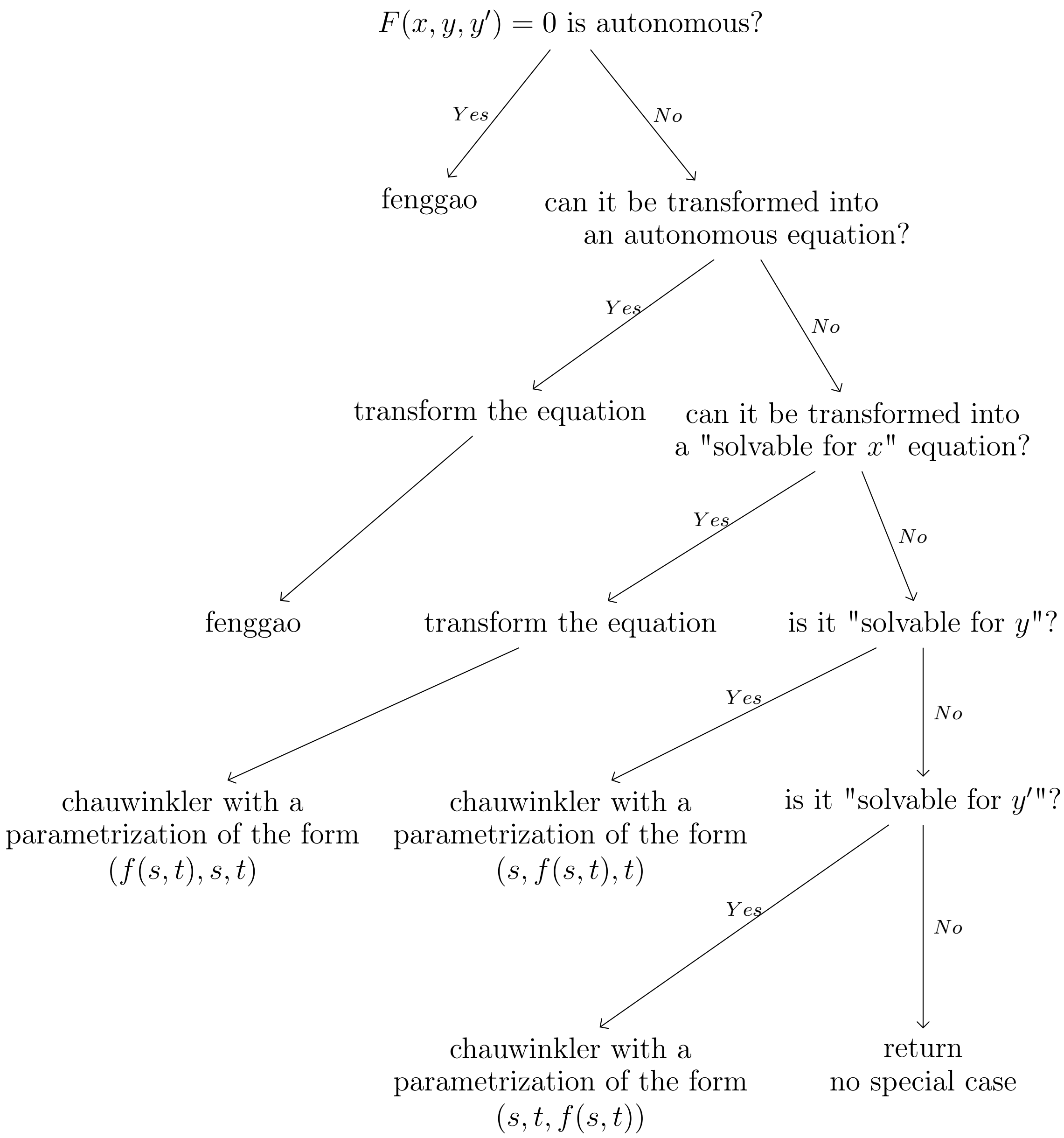
if $\left(\frac{\partial}{\partial s} q(s) \equiv 0\right)$
$T = fenggao(\Sigma_{12}(p(s), q(s), w(s)) \cdot p'(s)t - \Sigma_{11}(p(s), q(s), w(s)))$;
Add to Sol $(p_2(p(T), q(T), w(T)), p_4(p(T), q(T), W(T)))$;
Change G;

else $T = fenggao(\Sigma_{22}(p(s), q(s), w(s)) \cdot p'(s)t - \Sigma_{21}(p(s), q(s), w(s)))$;
Add to Sol $(p_2(p(T), q(T), w(T)), p_4(p(T), q(T), W(T)))$;
Change G;

end for each `return` Sol;

When we used this algorithm to compute rational solution we considered differential equations such that a proper rational parametrization of its solution surface can be found in literature, or such that it can be computed easily. The computer algebra system *MAGMA,* developed by the University of Sydney, is the only one in which is available a function to compute rational parametrizations of surfaces directly. I used this computer algebra system to compute some rational parametrizations, in Appendix A.3 can be found the lines of the programme I wrote, but unfortunately the parametrizations obatained in such way are not suitable for our method. Indeed, using rational parametrizations obtained in such way our programme returned invariant algebraic curves such that they divide the denominators of the rational functions in the associated system, so we must discard them, or the programme stick in the computation of the invariant algebraic curves of the associated system. That suggests us that if we want to improve our method we should investigate on a criterion for proper rational parametrizations of surfaces to be suitable, i.e. that can be used to find rational solutions, this can be done by a systematic study of the correspondence between associated system and its invariant algebraic curves. As well we should work on new ways to compute invariant algebraic curves of a system and on rational proper parametrizations from a computational point of view.

In the last part of Chapter 4 we presented a way to compute rational solutions of algebraic ODE avoiding the computation of proper rational parametrizations of algebraic surfaces. We use a group $\mathcal{G}$ of affine transformations of the space such that an element of $\mathcal{G}$ sends rational solutions into rational solutions to transform an algebraic ODE into an autonomous equation or into an equation such that it has an obvious rational proper parametrization. The last part of the implementation in SINGULAR concerns such affine transformations. The procedures *specialAODEratsolver* follows the scheme:



This programme allows us to compute almost every rational solution of special form first-order algebraic ODE. We must say almost because it may happen that during *chauwinkler* the programme gets stuck in the computation of invariant algebraic curves of the system, since in that computation we need to compute a Gröbner basis and we know it can be really expensive. I used this procedure to compute the examples in Section 4.6.

If a User want to compute a rational solution of a first-order algebraic ODE $F(x, y, y') = 0$, He or She should follow

1. **if** $F = 0$ is autonomous then use *fenggao.*

   else Use *specialAODEratsolver.*

2. **if** *specialAODEratsolver* does not give a solution, then find a rational proper parametrization of $F(x, y, z) = 0$ and use *chauwinkler*

5.2 Implementation in CoCoA

In CoCoA there are no default functions such that we can compute proper rational parametrizations of curves or surfaces. So we write a programme to compute rational general solutions of algebraic ODE like an interpreted language, i.e. it is asked to the user to insert some informations in intermediate steps, for instance the parametrizations of algebraic objects. I created a CoCoAprocedure for each step of the algorithm described in previous steps. I used CoCoA*4.7*, and not the latest CoCoA*5,* since the rational functions are not defined in the version 5.

FengGao

The function to compute rational general solutions of autonomous first-order algebraic ODEs is called *FengGao* and it takes in input a bivariate polynomial F, which defines the equation and two rational functions R, S which are the rational parametrization of the curve $F = 0$. It clearly follows the algorithm given in Chapter 3.

FengGao(F,R,S)

1. **If** the parametrization (R, S) is not proper **Then** `Return` "not proper parametrization":
2. **Else If** $(Deg(F, x) < Deg(F, y) - 1$ or $Deg(F, x) > 2Deg(F, y))$ **Then** `Return` "no rational general solution exists";
3. **Else** Compute $H = \frac{\partial}{\partial x} R$.
4. Discuss the form of
$$A = \frac{S}{H}.$$
5. `Return` the rational general solution, if it exists.

AssSystem

Consider the non-autonomous first-order algebraic ODE $F(x, y, y') = 0$. The function to compute the associated system of $F(x, y, y') = 0$ with respect to a parametrization $\phi = (\phi_1, \phi_2, \phi_3)$ of the solution surface is called *AssSystem* and it takes in input three rational function $C1, C2$ and $C3$ such that $Ci = \phi_i(s, t)$. The function returns a list of two rational functions which are the right hand sides of the associated system of $F(x, y, y') = 0$ with respect to the parametrization ϕ.

AssSystem(C1,C2,C3)

1. Compute $G = \frac{\partial}{\partial t} C2 \cdot \frac{\partial}{\partial s} C1 - \frac{\partial}{\partial t} C1 \cdot \frac{\partial}{\partial s} C2$.
2. Compute $H = \frac{\partial}{\partial t} C2 - \frac{\partial}{\partial t} C1 \cdot C3$.
3. **If** ($G = 0$ And $num(H)$ non constant) `Return` $L = [num(G), num(H)]$.
4. **Else** Compute $K = -\frac{\partial}{\partial s} C2 + \frac{\partial}{\partial s} C1 \cdot C3$.
5. `Return` $L = [H/G, K/G]$.

By $num(\cdot)$ we mean the numerator of the argument, usually a rational function.

InvAlgCurve

To compute invariant algebraic curves of a differential system of the form

$$\begin{cases} s' = \frac{N_1(s,t)}{M_1(s,t)}, \\ t' = \frac{N_2(s,t)}{M_2(s,t)}. \end{cases}$$

I use an improvement of the algorithm present in [Man93]. That function, named *InvAlgCurve*, takes in input two rational functions S and T, which are the righthand sides of the system above. It returns the set of irreducible invariant curves of the system up to a fixed degree k. I used $k = 3$ because for a greater k the computation can be really heavy, since we need to compute a Gröbner basis of an ideal with a large number of generators.

By $LT(\cdot)$ and $LC(\cdot)$ we mean respectively leading term and leading coefficient of the argument. By *interreduce* we mean that since in $feqns$ there are polynomial relations among the undetermined coefficients of f_i. If there is a linear relation we can use that relation to describe a coefficient in terms of the other ones. This allows us to reduce the number of unknowns and to reach a zero dimensional ideal.

InvAlgCurve(S,T) $S_f = \emptyset$;
$k = 1$;
$P = num(S) \cdot den(T)$;
$Q = num(T) \cdot den(S)$;
While ($k \leq degreebound$) **do**

1. Construct all monic polynomials with undetermined coefficients f_i of degree $\leq k$.
2. **For each** f_i **Do**
   - Compute $Df_i = P\frac{\partial}{\partial x}f_i + Q\frac{\partial}{\partial y}f_i$;
   - $g_i = 0$;
   - $indivisible = false$;
   - $feqns = \emptyset$;

- **While** ($indivisible = false$ and $Df_i \neq 0$) **Do**
  - **If** $LT(f_i)$ divides $LT(Df_i)$ **Then**
    * $g_i = g_i + \frac{LT(Df_i)}{LT(f_i)}$;
    * $Df_i = Df_i - f_i \cdot \frac{LT(Df_i)}{LT(f_i)}$;
  - **Else If** $LC(Df_i)$ is a constant **Then** $indivisible = true$;
  - **Else If** $LC(Df_i)$ contains one f_i variable (say var) only **Then**
    * $s = Solve(LC(Df_i), var)$;
    * $f_i = Subst(s, f_i)$;
    * $g_i = Subst(s, g_i)$;
    * $feqns = Subst(s, feqns)$;
  - **Else**
    * $feqns = LC(Df_i) \cup feqns$;
    * $Df_i = Df_i - LT(Df_i)$;
- **End While**
- **If** ($indivisible = false$ and $feqns \neq 0$) **Then**
  - Interreduce $feqns$;
  - Compute Gröbner basis of $feqns$ with respect to undetermined coefficients of f_i;
  - **If** the system $feqns$ is consistent **Then** all unknown coefficients in f_i can be determined;
  - **Else** Try next f_i;
- **If** ($indivisible = false$ and f_i is irreducible) **Then** $S_f = S_f \cup f_i$;

3. End For each

$k = k + 1$;
End While
`Return` S_f;

Finalstep

The function *Finalstep* computes a rational solution of an algebraic ODE $F(x, y, y') = 0$ starting from a rational parametrization of an invariant algebraic curve of the associate system with respect to a proper rational parametrization of $F(x, y, z) = 0$, i.e. it computes at first a rational solution of the system and then it returns the solution of the equation. It takes in input three lists P, L, Q, the list Q is a rational proper parametrization of the solution surface $F(x, y, z) = 0$, L is a list which contains the two righthand sides of the associated system of $F(x, y, y') = 0$ with respect to the parametrization given in Q, finally P is a proper rational parametrization of an invariant algebraic curve of the system given in L.

Finalstep(P, L, Q);

1. - **If** $\frac{\partial}{\partial x}P[1] = 0$ **Then** Compute a rational solution Sol of
   $$T = \frac{1}{\frac{\partial}{\partial x}P[2]} \cdot L[2](P[1], P[2]);$$
   - **Else** Compute a rational solution Sol of
   $$T = \frac{1}{\frac{\partial}{\partial x}P[1]} \cdot L[1](P[1], P[2]);$$
2. Compute $C = x - Q[1](Sol[1], Sol[2])$;
3. Compute $\tilde{Y} = Q[2](Sol[1], Sol[2])$;
4. `Return` $Y = \tilde{Y}(x - C)$;

Check

The last function concerns the last section of Chapter 4. The function *Check* takes in input a polynomial F of three or two variables, and it decides if the polynomial is of a special form, i.e., autonomous, solvable for x, for y or for y', and if it is not of one of these forms, it tries to transform the polynomial into one autonomous or a solvable for x one and, in this case, it returns the new polynomial. Moreover, If the equation is of the form solvable for an indeterminate the functions also returns the obvious parametrization.

Check(F);

1. - **If** $Deg(F, y') = 1$ **Then** Compute $G(x, y)$ such that $y' = G(x, y)$;
   - `Return` "Solvable for y'" and `Return` $(x, y, G(x, y))$;
2. - **Else If** $Deg(F, y) = 1$ **Then** Compute $G(x, y')$ such that $y = G(x, y')$;
   - `Return` "Solvable for y" and `Return` $(x, G(x, y'), y')$;
3. - **Else If** $Deg(F, x) = 1$ **Then** Try to transform F into an autonomous equation;
     - **If** F is in the same class of an autonomous equation **Then** `Return` the new equation;
     - **Else** `Return` "Solvable for x" and `Return` $(G(y, y'), y, y')$;
4. **Else** Try to transform F into an autonomous equation;
   - **If** F is in the same class of an autonomous equation **Then** `Return` the new equation;
   - **Else** `Return` "no special case"

Appendix A

Codes

All the codes in this Appendix are available at the following links:

- SINGULAR:

  https://dl.dropboxusercontent.com/u/49368831/Singular_F_Zucca_Library

- CoCoA:

  https://dl.dropboxusercontent.com/u/49368831/CoCoA_F_Zucca_Library

A.1 SINGULAR

Prologue: before using these codes declare the following

```
LIB "normal.lib";
LIB "zeroset.lib";
LIB "solve.lib";
LIB "paraplanecurves.lib";
LIB "monomialideal.lib";
ring q=0,(x,y,z),dp;
ring r=(0,alpha,h,c),(x,y,z,s,t),dp;
ring ri=0,(x,y,s,t),(Dp(2),Dp(2));
ring qbc=(0,a),(x,y,z,b,c), (lp);
setring q;
int B=0;
```

The function *degiac* returns the degree of `poly f` with respect to the monomial `poly v`, which usually is a variable. It is written to work inside the function *invalgcurve.*

```
proc degiac(poly f, poly v)
{
```

```
int i;
int dd=0;
poly h=f;
int u=univariate(v);
for (i=1;i<=nvars(w);i=i+1) { //(for#1)
if (i!=u) { //(if#1)
h=subst(h,var(i),a);
}} //(end if#1,end for#1)
dd=deg(h);
return(dd);
}
```

The function *chauwinkler* is described in Section 5.1.

```
proc chauwinkler(list p)
{
setring ri;
list P=imap(r,p);
list Inv=invalgcurve(P);

if (size(Inv)==0) { //(if#1)
print("no invariant invariant algebraic curves with coefficients in QQ");}
//(end if#1);
else { //(else if #1)
setring r;
list Inv=imap(ri,Inv);
list sigma=asssystem(p);
def Sol=finalstep(Inv,sigma,p);
poly Sgcd;
list S1,S;
for (int i=1;i<=size(Sol);i=i+1) { //(for#1)
S=subst(numerator(number(Sol[i])),h,x),subst(denominator(number(Sol[i])),h,x);
Sgcd=gcd(S[1],S[2]);
S=S[1]/Sgcd,S[2]/Sgcd;
S1=insert(S1,S,size(S1)); } //(end for#1)
S1;
} //(end else if#1)
}
```

The function *Deg* returns the degree of `poly f` with respect to the monomial `poly v`, which usually is a variable.

```
proc Deg(poly f, poly v)
{
```

```
int dd=0;
int u=univariate(v);
ring qa=(0,a),(x,y,z,b,c),dp;
setring qa;
poly f=imap(qbc,f);
if (u==1) { //(if#1)
poly h=subst(f,y,a,z,a,b,a,c,a);
dd=deg(h);
setring q;
return(dd)} //(end if#1)
else { if (u==2) { //(if#2)
 poly h=subst(f,x,a,z,a,b,a,c,a);
dd=deg(h);
setring q;
return(dd)} //(end if#2)
else {
poly h=subst(f,y,a,x,a,b,a,c,a);
dd=deg(h);
setring q;
return(dd)}} //(end else if#2,end else if#1)
}
```

The function *Degree* is the same as *Deg,* but it is written to work in the function *SpecialAODEratsolver.*

```
proc Degree(poly f, poly v)
{
int dd=0;
int u=univariate(v);
ring qa=(0,a),(x,y,z),dp;
setring qa;
poly f=imap(q,f);
if (u==1) { //(if#1)
poly h=subst(f,y,a,z,a);
dd=deg(h);
setring q;
return(dd)} //(end if#1)
else { if (u==2) { //(if#2)
 poly h=subst(f,x,a,z,a);
dd=deg(h);
setring q;
return(dd)} //(end if#2)
else {
poly h=subst(f,y,a,x,a);
```

```
dd=deg(h);
setring q;
return(dd)}} //(end else if#2,end else if#1)
}
```

The function *fenggao* is described in Section 5.1.

```
proc fenggao(poly f)
{
f=subst(f,y,x,z,y);
f=homog(f,z);
if (genus(f)==0)  { //(if#1)
def Rp = paraPlaneCurve(f);
setring Rp;
PARA[1]=subst(PARA[1],t,1);
PARA[2]=subst(PARA[2],t,1);
PARA[3]=subst(PARA[3],t,1);
ring F= (0,c,h),(x,s,t),lp;
setring F;
def L=imap(Rp,PARA);
ideal C=L[1],L[2],L[3];
if (tracindex(C)=="not proper") { //(if#2)
string j="not proper parametrization";
return(j); } //(end if#2)
else { //(else if#2)
def P=diff(L[1],s)*L[3]-L[1]*diff(L[3],s);
poly Q=L[2]*L[3];
def d=Div(Q,P);
if (d[2]==0) { //(if#3)
if (deg(d[1])==0) { //(if#4)
list S=subst(L[1],s,d[1]*(x)),subst(L[3],s,d[1]*(x));
poly Gcd=gcd(S[1],S[2]);
S=S[1]/Gcd,S[2]/Gcd;
setring q;
def S=imap(F,S);
return(S); } //(end if#4)
else  { //(else if#4)
if (deg(d[1])==2) { //(if#5)
def v=sqrfree(d[1]);
if (deg(v[1][2])==2) { //(if#6)
print("no rational solution"); }//(end if#6)
else { //(else if#6)
def a=v[1][1];
```

```
def b=subst(v[1][2],s,0)/leadcoef(v[1][2]);
b=-b;
list S=subst(L[1],s,(a*b*(h)-1)/(a*(h))),subst(L[3],s,(a*b*(h)-1)/(a*(h)));
def S0=S[1]/S[2];
S=numerator(number(S0)),denominator(number(S0));
S[1]=subst(S[1],h,x);
S[2]=subst(S[2],h,x);
poly Gcd=gcd(S[1],S[2]);
S=S[1]/Gcd,S[2]/Gcd;
setring q;
def S=imap(F,S);
return(S); }} //(end else if#6, end if#5)
else { //(else if#5)
print("no rational solution"); }}} //(end else if#5, end else if#4,end if#3)
else  { //(else if#3)
print("no rational solution"); }}} //(end else if#3, emd else if#2, end if#1)
else { //(else if#1)
print("no rational curve"); } //(end else if#1)
}
```

The function *fenggaoX* is just *fenggao* modified to work in the function *specialAODErat-solver.*

```
proc fenggaoX(poly f,poly lambda,poly gamma)
{
def s1=fenggao(f);
s1=s1[1]-lambda*x*s1[2]-gamma*s1[2],s1[2];
return(s1);
}
```

The function *finalstep*†is the version in SINGULARof the function *Finalstep* in CoCoAdescribed in Section 5.2.

```
proc finalstep(list b, list sigma, list p)
{

list S,F;
int i=1;
int j;
poly X,c1,f,phi1,phi2,g,T;
list a=1,1,1;

for (i=1;i<=size(b);i=i+1) { //(for#1)
for (j=1;j<=size(b[i]);j=j+1) { //(for#2)
```

```
def k=Para(b[i][j]);
if (k!=0) { //(if#1)
k[1]=subst(k[1],s,y,t,1);
k[2]=subst(k[2],s,y,t,1);
k[3]=subst(k[3],s,y,t,1);
phi1=diff(k[1],y)*k[3]-k[1]*diff(k[3],y);
phi2=(diff(k[2],y)*k[3]-k[2]*diff(k[3],y));
def E1=subst(k[1],y,h)/(subst(k[3],y,h));
def E2=subst(k[2],y,h)/(subst(k[3],y,h));
def den1=(subst(phi1,y,h)*subst(sigma[1][2],s,E1,t,E2));
def den2=(subst(phi2,y,h)*subst(sigma[2][2],s,E1,t,E2));

if ((phi1==0)and(den2!=0)) { //(if#2)

def theta1=(subst(sigma[2][1],s,E1,t,E2)*(subst(k[3],y,h))^2)/den2;
number theta=number(theta1);
F=subst(numerator(theta),h,y),subst(denominator(theta),h,y);
g=gcd(F[1],F[2]);
F=F[1]/g,F[2]/g;
if ((deg(F[1])==0)and(deg(F[2])==0)and(F[2]!=0)) { //(if#3.1)
T=F[1]/F[2]*(h);
a[1]=subst(k[1],y,T);
a[2]=subst(k[2],y,T);
a[3]=subst(k[3],y,T);
def dom=subst(p[4],s,a[1]/a[3],t,a[2]/a[3]);
if (dom!=0) { //(if#1*)
c1=(subst(p[1],s,a[1]/a[3],t,a[2]/a[3]))/(subst(p[4],s,a[1]/a[3],t,a[2]/a[3]))-h;
a[1]=subst(numerator(number(a[1])),h,h-c1)/subst(denominator(number(a[1])),h,h-c1);
a[2]=subst(numerator(number(a[2])),h,h-c1)/subst(denominator(number(a[2])),h,h-c1);
a[3]=subst(numerator(number(a[3])),h,h-c1)/subst(denominator(number(a[3])),h,h-c1);
X=(subst(p[2],s,a[1]/a[3],t,a[2]/a[3]))/(subst(p[4],s,a[1]/a[3],t,a[2]/a[3]));
S=insert(S,X,size(S));
}//(end if#1*)
}//(end if#3)
else { if ((deg(F[1])==2)and(deg(F[2])==0)and(F[2]!=0)) { //(else if#3.1)(if#3.1.1)
def v=sqrfree(F[1]);
if (deg(v[1][2])==1) { // (if#3.1.2)

def a1=v[1][1]/F[2];
def b1=subst(v[1][2],y,0)/leadcoef(v[1][2]);
b1=-b1;
T=b1-1/(a1*h);
```

```
a[1]=subst(k[1],y,T);
a[2]=subst(k[2],y,T);
a[3]=subst(k[3],y,T);
def dom=subst(p[4],s,a[1]/a[3],t,a[2]/a[3]);
if (dom!=0) { //(if#2*)
c1=(subst(p[1],s,a[1]/a[3],t,a[2]/a[3]))/(subst(p[4],s,a[1]/a[3],t,a[2]/a[3]))-h;
a[1]=subst(numerator(number(a[1])),h,h-c1)/subst(denominator(number(a[1])),h,h-c1);
a[2]=subst(numerator(number(a[2])),h,h-c1)/subst(denominator(number(a[2])),h,h-c1);
a[3]=subst(numerator(number(a[3])),h,h-c1)/subst(denominator(number(a[3])),h,h-c1);
X=(subst(p[2],s,a[1]/a[3],t,a[2]/a[3]))/(subst(p[4],s,a[1]/a[3],t,a[2]/a[3]));
S=insert(S,X,size(S));
}//(end if#2*)
}//(end if#3.1.2)
}//(end if#3.1.1)
}//(end(else if#3.1)
}//(end if#2)
else { if (den1!=0){//(else if#2,if#@)
def theta1=(subst(sigma[1][1],s,E1,t,E2)*(subst(k[3],y,h))^2)/den1;
number theta=number(theta1);
F=subst(numerator(theta),h,y),subst(denominator(theta),h,y);
g=gcd(F[1],F[2]);
F=F[1]/g,F[2]/g;
if ((deg(F[1])==0)and(deg(F[2])==0)and(F[2]!=0)) { //(if#3.2)
T=F[1]/F[2]*(h);
a[1]=subst(k[1],y,T);
a[2]=subst(k[2],y,T);
a[3]=subst(k[3],y,T);
def dom=subst(p[4],s,a[1]/a[3],t,a[2]/a[3]);
if (dom!=0) { //(if#3*)
c1=(subst(p[1],s,a[1]/a[3],t,a[2]/a[3]))/(subst(p[4],s,a[1]/a[3],t,a[2]/a[3]))-h;
a[1]=subst(numerator(number(a[1])),h,h-c1)/subst(denominator(number(a[1])),h,h-c1);
a[2]=subst(numerator(number(a[2])),h,h-c1)/subst(denominator(number(a[2])),h,h-c1);
a[3]=subst(numerator(number(a[3])),h,h-c1)/subst(denominator(number(a[3])),h,h-c1);
X=(subst(p[2],s,a[1]/a[3],t,a[2]/a[3]))/(subst(p[4],s,a[1]/a[3],t,a[2]/a[3]));
S=insert(S,X,size(S));
}//(end if#3*)
}//(end if#3.2)
else { if ((deg(F[1])==2)and(deg(F[2])==0)and(F[2]!=0)) { //(else if#3.1)(if#3.2.1)
def v=sqrfree(F[1]);
if (deg(v[1][2])==1) { // (if#3.2.2)

def a1=v[1][1]/F[2];
def b1=subst(v[1][2],y,0)/leadcoef(v[1][2]);
```

```
b1=-b1;
T=b1-1/(a1*h);
a[1]=subst(k[1],y,T);
a[2]=subst(k[2],y,T);
a[3]=subst(k[3],y,T);
def dom=subst(p[4],s,a[1]/a[3],t,a[2]/a[3]);
if (dom!=0) { //(if#4*)
c1=(subst(p[1],s,a[1]/a[3],t,a[2]/a[3]))/(subst(p[4],s,a[1]/a[3],t,a[2]/a[3]))-h;
a[1]=subst(numerator(number(a[1])),h,h-c1)/subst(denominator(number(a[1])),h,h-c1);
a[2]=subst(numerator(number(a[2])),h,h-c1)/subst(denominator(number(a[2])),h,h-c1);
a[3]=subst(numerator(number(a[3])),h,h-c1)/subst(denominator(number(a[3])),h,h-c1);

X=(subst(p[2],s,a[1]/a[3],t,a[2]/a[3]))/(subst(p[4],s,a[1]/a[3],t,a[2]/a[3]));
S=insert(S,X,size(S));
}//(end if#4*)
}//(end if#3.2.2)
}//(end if#3.2.1)
}//(end else if#3)
}//(end if#@)
}//(end else if#2)
}//(end if#1)
}//(end for#2)
}//(end for#1)
if (size(S)==0) { //(if#4)
print("no rational solution found");
}//(end if#4)
else { return(S); } //(else if#4)(end else if#4)
}
```

The function *lin* takes in input a polynomial `poly f`, it returns the action of a generic element of $\mathcal{G}'$ on `poly f`. For the definition of $\mathcal{G}'$ see Section 4.6.

```
proc lin(poly f)
{
f=subst(f,y,-bx+y-c,z,z-b);
return(f);
}
```

The function *genpoly* takes in input an integer `int d` and it returns the set of monic polynomials up to degree `int d` with undetermined coefficients. It is used in the function *invalgcurve.*

```
proc genpoly(int d)
```

```
{
int i,j,l,m;
poly h1,h2,f,h;
poly p=c(1)(1);
for  (i=1; i<=d; i=i+1) {
l=i;
m=0;
while ((l+m==i)&&(l>=0)) {
h=c(i+1)(m+1);
h1=x^l;
h2=y^m;
p=p+h*h1*h2;
l=l-1;
m=m+1;
}}
list X;
for (i=0; i<=d; i=i+1){
f=p;
for (j=0; j<=i; j=j+1){
if (i==j) {f=subst(f,c(d+1)(i+1),1);}
else { f=subst(f,c(d+1)(j+1),0);}
}
X=insert(X,f,size(X));
}
return(X);
}
```

The function *invalgcurve* is the version in SINGULAR of the function *InvAlgCurve* in CoCoAdescribed in Section 5.2. The difference is that this function takes in input the rational proper parametrization of the solution surface of the equation studied and the associated system is computed by this function.

```
proc invalgcurve(list p) {
int k=1;
list Curves;
list sigma,b1,f,l1,l2;
ring tau=0,x,dp;
setring ri;
int Ind=0;
sigma=asssystem(p);
sigma[1][1]=subst(sigma[1][1],s,x,t,y);
sigma[1][2]=subst(sigma[1][2],s,x,t,y);
sigma[2][1]=subst(sigma[2][1],s,x,t,y);
```

```
sigma[2][2]=subst(sigma[2][2],s,x,t,y);
while (k<=2) { // (while #1) ring w=(0,a),(x,y,c(1..(k+1))(1..(k+1))),(Dp(2),lp);
setring w;
list b,b1;
def sigma=imap(ri,sigma);
poly D,ltD,lcD,ltl,alpha;
int k1,n,u,i,j;
def l=genpoly(k);
for (n=1; n<=size(l); n=n+1) { //(for #1)
D=sigma[2][2]*sigma[1][1]*diff(l[n],x)+sigma[1][2]*sigma[2][1]*diff(l[n],y);
list f,l1;
b1=list();
Ind=0;
while ((Ind==0)and(D!=0)) {// (while #2)
ltD=lterm(D);
lcD=lcoef(D);
ltl=lterm(l[n]);
if (membershipMon(ltD,ideal(ltl))==1) { D=D-l[n]*ltD/ltl;} //(if #1, end if#1)
else { if (deg(lcD)==0) { Ind=1;} //(else if#1, if#2, end if#2)
else { //(else if#2)
u=univariate(lcD);
if (u!=0) //(if#3)
{ def r1=Ratroot(lcD);
  if (r1!=(a)) { //(if#4)
  l[n]=subst(l[n],var(u),r1[1]);
for (i=1;i<=size(f);i=i+1) { //(for#1', end for#1')
  f[i]=subst(f[i],var(u),r1[1]);}
  D=subst(D,var(u),r1[1]); }//(end if#4)
else {Ind=1;} //(else if#4, end else if#4)
} //(end if#3)
else { f=insert(f,lcD,size(f)); //(else if#3)
D= D-ltD; }}}} //(end else if#3, end else if#2, end else if#1, end while #2)
list DD;
poly DDQ, F1;
int o,XX;
for (i=1;i<=(k+1);i=i+1) { //(for#1+)
for (j=1;j<=(i);j=j+1) {  //(for#2+)
XX=1;
while (XX<=size(f)) { //(while#1+)
if (f[XX]!=0) {//(if#1+)
DD=Div(f[XX],c(i)(j));
DDQ=DD[1];
if ((degiac(f[XX],c(i)(j))==1)and(deg(DDQ)==0)) { //(if#2+)
```

```
F1=subst(f[XX],c(i)(j),0);
for (o=1;o<=size(f);o=o+1) { //(for#3+)
f[o]=subst(f[o],c(i)(j),-1/DDQ*F1);
if (f[o]==0) {f=delete(f,o); } //(if#1@, end if#1@)
} //(end for#3+)
l[n]=subst(l[n],c(i)(j),-1/DDQ*F1);
XX=1+size(f); } //(end if#2+)
else{ //(else if #2+);
XX=XX+1;
}}//(end else if#2+, end if#1+)
else{ //(else if#1+)
XX=XX+1;
} //(end else if#1+)
}}}  //(ed while#1+, end for#2+, end for#1+)
if ((Ind==0)and(size(f)!=0)) { //(if#5)
for (i=1; i<=k+1; i=i+1){ //(for #2)
for (j=1; j<=i; j=j+1){ //(for #3)
k1=1;
while (k1<=size(f)) { //(while#3)
if (f[k1]!=subst(f[k1],c(i)(j),a)) { //(if#6)
l1=insert(l1,var(univariate(c(i)(j))),size(l1));
k1=size(f)+1;}//(end if#6)
else{ k1=k1+1; }}}} //(else if#6, end else if#6, end while#3, end for#3)
//(end for#2)
list l2=ringlist(tau);
l2[2]=l1;
tau=ring(l2);
setring tau;
ideal Id;
def f=imap(w,f);
for (i=1;i<=size(f);i=i+1) { //(for #7)
Id=Id+ideal(f[i]);} //(end for #7)
def F=std(Id);
if (dim(F)==0) { //(if#7)

def e=zeroSet(F);
def A=imap(e,newA);
if (A==0) { //(if#8)
setring w;
def Z=imap(e,theZeroset);
alpha=l[n];
for (i=1;i<=size(Z); i=i+1) { //(for#4)
for (j=1;j<=size(l1); j=j+1){ //(for#5)
```

```
alpha=subst(alpha,l1[j],Z[i][j]);} //(end for #5)
b1=insert(b1,alpha,size(b1));
 } } //(end for#4, end if#8)
else {Ind = 1;}} //(else if#8,end else if#8, end if#7)
else {Ind = 1;}} //(else if#7, end else if#7, end if#5)
else { if (Ind==0) { //(else if#5, if#11)
b1=insert(b1,l[n],size(b1));
}} //(end if#11, end else if#5)
setring w;
if (Ind==0) { //(if#9)
for (i=1;i<=size(b1); i=i+1) { //(for#6)
if (isIrr(b1[i])==1) { //(if#10)
b=insert(b,b1[i],size(b));
}}}}//(end if#10, end for#6, end if#9, end for#1)
if (size(b)==0) { //(if#12)
k=k+1;
setring ri;
} //(end if#12)
else { //(else if#12)
for (n=1;n<=size(b);n=n+1) { //(for#7)
for (i=1;i<=k+1;i=i+1) { //(for#8)
for (j=1;j<=k+1;j=j+1) { //(for#9)
b[n]=subst(b[n],c(i)(j),1);}}} //(end for#7,end for#8,end for#9)
setring ri;
k=k+1;
def b=imap(w,b);
Curves=insert(Curves,b,size(Curves));
Curves;
}} //(end else if#12, end while#1)
return(Curves);
}
```

The function *isAut* takes in input a polynomial `poly f` and if it is possible it returns an autonomous equation of the same class of `poly f`, i.e. it can be obtained from `poly f` by the action of an element of $\mathcal{G}'$.

```
proc isAut(poly f)
{
int u,No,i;
poly lambda,gamma;
f=lin(f);
list L=lcoef1(f);
ideal Id;
for (i=1;i<=size(L);i=i+1) { //(for #1)
```

```
Id=Id+ideal(L[i]);} //(end for#1)
Id=groebner(Id);
if (Id==1) { //(if#1)
return(No);} //(end if#1)
else { //(else if#1)
i=1;
while (i<=size(Id)){//(while#1)
u=univariate(Id[i]);
if (u>3) { //(if#2)
def root=Ratroot(Id[i]);
if (root==a) { return(No); } //(if#3)
else  { f=subst(f,var(u),root); //(else if#3)
Id=subst(Id,var(u),root);
if (u==4) {lambda=root;} //(if#4, end if#4)
else {gamma=root;} }} //(else if#4, end else if4end else if#3, end if#2)
else { i=i+1; } } //(end(else if#2), end while#1)
if (groebner(ideal(subst(f,x,1,y,1,z,1)))==1) { return(f); } //(if#5, end if#5)
else {  f=subst(f,var(4),0,var(5),0); } //(else if#5, end else if#5)
list res=1,f,lambda,gamma;
return(res);
} //(end else if#1)
}
```

The function *isIrr* decides if the polynomial `poly f` given in input is irreducible or not.

```
proc isIrr(poly f)
{
f=f/leadcoef(f);
def e=factorize(f,2);
if (size(e[1])>1) { return(0);}
else { if (e[2]!=1) { return(0);}
else { return(1);}}
}
```

The function *isSx* takes in input a polynomial `poly f` and if it is possible it returns a solvable for x equation of the same class of `poly f`, i.e. it can be obtained from `poly f` by the action of an element of $\mathcal{G}'$.

```
proc isSx(poly f)
{
int u,No,i;
poly lambda,gamma;
f=lin(f);
```

```
list L=lcoef2(f);
ideal Id;
for (i=1;i<=size(L);i=i+1) { //(for #1)
Id=Id+ideal(L[i]);} //(end for#1)
Id=groebner(Id);
if (Id==1) { //(if#1)
return(No);} //(end if#1)
else { //(else if#1)
i=1;
while (i<=size(Id)){//(while#1)
u=univariate(Id[i]);
if (u>3) { //(if#2)
def root=Ratroot(Id[i]);
if (root==a) { return(No); } //(if#3)
else  { f=subst(f,var(u),root); //(else if#3)
Id=subst(Id,var(u),root);
if (u==4) {lambda=root;} //(if#4, end if#4)
else {gamma=root;} //(else if#4,end else if#4}
}} //(end else if#3, end if#2)
else { i=i+1; }} //(end(else if#2), end while#1)
if (groebner(ideal(subst(f,x,1,y,1,z,1)))==1) {
list res=1,f,lambda,gamma;
return(res); } //(if#4, end if#4)
else {  f=subst(f,var(4),0,var(5),0); } //(else if#4, end else if#4)
list res=1,f,lambda,gamma;
return(res);} //(end else if#1)
```

Next functions: *lcoef, lcoef1, lcoef2, lterm, lterm1, lterm2* are used inside other functions to compute the leading coefficient or the leading term of `poly f` given in input.

```
proc lcoef1(poly f)
{
list coeff;
while (Deg(leadmonom(f),x)>=1) { //(while#1)
int m=leadexp(f)[1];
int n=leadexp(f)[2];
int l=leadexp(f)[3];
poly j=x^m;
poly h=y^n;
poly k=z^l;
j=j*h*k;
def e=Div(f,j);
f=f-e[1]*j;
```

```
coeff=insert(coeff,e[1],size(coeff));
} //(end while#1)
return(coeff);
}


proc lterm1(poly f)
{
list terms;
while (Deg(leadmonom(f),x)>=1) { //(while#1)
int m=leadexp(f)[1];
int n=leadexp(f)[2];
int l=leadexp(f)[3];
poly j=x^m;
poly h=y^n;
poly k=z^l;
j=j*h*k;
def e=Div(f,j);
f=f-e[1]*j;
terms=insert(terms,e[1]*j,size(terms));
} //(end while#1)
return(terms);
}

proc lcoef2(poly f)
{
list coeff;
while (Deg(leadmonom(f),x)>=2) { //(while#1)
int m=leadexp(f)[1];
int n=leadexp(f)[2];
int l=leadexp(f)[3];
poly j=x^m;
poly h=y^n;
poly k=z^l;
j=j*h*k;
def e=Div(f,j);
f=f-e[1]*j;
coeff=insert(coeff,e[1],size(coeff));
} //(end while#1)
return(coeff);
}
```

```
proc lterm2(poly f)
{
list terms;
while (Deg(leadmonom(f),x)>=2) { //(while#1)
int m=leadexp(f)[1];
int n=leadexp(f)[2];
int l=leadexp(f)[3];
poly j=x^m;
poly h=y^n;
poly k=z^l;
j=j*h*k;
def e=Div(f,j);
f=f-e[1]*j;
terms=insert(terms,e[1]*j,size(terms));
} //(end while#1)
return(terms);
}

proc lcoef(poly f)
{
int m=leadexp(f)[1];
int n=leadexp(f)[2];
poly j=x^m;
poly h=y^n;
j=j*h;
def e=Div(f,j);
return(e[1]);
}

proc lterm(poly f)
{
int m=leadexp(f)[1];
int n=leadexp(f)[2];
poly j=x^m;
poly h=y^n;
j=j*h;
def e=Div(f,j);
return(e[1]*j);
}
```

The function *Para* computes a rational proper parametrization of the curve defined by `poly f` given in input. In our program `poly f` is an invariant algebraic curve for a system and the function is used in the function *finalstep.*

```
proc Para(poly f) //f is an invariant algebraic curve for the system.
{
ideal k;
poly g=homog(f,z);
if (univariate(f)==0) { //(if#1)
if (deg(f)==1) { //(if#2)
if (g==f) { //(if#3)
k=s,-subst(f,x,0,y,s)/leadcoef(f),1;
} //(end if#3)
else { //(else if#3)
k=s,-((coef(g,x)[1][2])*s+coef(g,z)[1][2])/(coef(g,y)[1][2]),1;
}}//(end else if#3, end if#2)
else { //(else if#2)
if (g==f) { //(if#4)
return(k);
} //(end if#4)
else { //(else if#4)
if (genus(g)==0) {   //(if#5)
ring q=0,(x,y,z),dp;
setring q;
poly f=imap(r,g);
def R=paraPlaneCurve(f);
setring r;
k=imap(R,PARA);
} //(end if#5)
else { //(else if#5)
return(0);
} //(end else if#5)
}} //(end else if#4,end else if#2)
} //(end if#1)
else { //(else if#1)
k=1,1,1;
int u=univariate(f);
if (u==1) { //(if#6)
k[1]=Ratroot1(f);
k[2]=s; } //(end if#6)
else { //(else if#6)
k[1]=s;
k[2]=Ratroot1(f);
}} //(end else if#6, end else if#1)
return(k);
}
```

The function *Div* is a polynomial division between `poly f` and `poly g`

```
proc Div(poly f, poly g)
{
list qr=f/g;
qr=insert(qr, f-qr[1]*g,1);
return(qr);
}
```

The functions *Ratroot* and *Ratroot1* computes a rational root of the polynomial `poly f` given in input.

```
proc Ratroot(poly f) {
def l=factorize(f,1);
string b="yes";
number c=a;
int i=1;
while ((i<=size(l))and(b=="yes")) {
if (deg(l[i])==1) { b="not";
def c=-subst(l[i],var(univariate(f)),0)/leadcoef(l[1]);
}
else {i=i+1;}
}
return(c);
}


proc Ratroot1(poly f) {
def l=factorize(f,1);
string str="yes";
number c=alpha;
int i=1;
while ((i<=size(l))and(str=="yes")) {
if (deg(l[i])==1) {str="not";
def c=subst(l[i],var(univariate(f)),0)/leadcoef(l[1]);
}
else {i=i+1;}
}
return(c);
}
```

The function *specialAODEratsolver* is described in Section 5.1.

```
proc SpecialAODEratsolver(poly f)
{
```

```
if (Degree(f,x)==0) { //(if#1)
f=subst(f,y,x,z,y);
return(fenggao(f)); } //(end if#1)
else {//(else if#1)
if (Degree(f,y)==1) { //(if#2)
setring r;
poly f=imap(q,f);
def P=Div(f,y);
poly den=subst(P[1],x,s,z,t);
list p=s*den,subst(-P[2],x,s,z,t),t*den,den;
return(chauwinkler(p));} //(end if#2)
else { //(else if#2)
if (Degree(f,z)==1) { //(if#3)
setring r;
poly f=imap(q,f);
def P=Div(f,z);
poly den=subst(P[1],x,s,y,t);
list p=s*den,t*den,subst(-P[2],x,s,y,t),den;
return(chauwinkler(p));} //(end if#3)
else { setring qbc;  //(else if#3)
poly f=imap(q,f);
def g=isAut(f);
if (g[1]!=0) { setring q; //(if#4)
def gg=imap(qbc,g);
return(fenggaoX(gg[2],gg[3],gg[4]));} //(end if#4)
else { //(else if#4)
def g1=isSx(f);
if (g1[1]!=0) { //(if#5)
setring r;
def f=imap(qbc,g1);
def P=Div(f[2],x);
poly den=subst(P[1],y,s,z,t);
list p=subst(-P[2],y,s,z,t),s*den-f[3]*subst(-P[2],y,s,z,t)-f[4]*den,  ---
t*den-f[3]*den,den;
return(chauwinkler(p));} //(end if#5)
else{   //(else if#5)
return("no special case") }}}}} //(end else if #5, end else if#4, end else if#3)
//(end else if#2, end else if#1)
}
```

The function *tracindex* takes an ideal `ideal L` in input which is the parametrization of a curve, and verify if it is a proper rational paramerization.

```
proc tracindex(ideal L)
{
poly g1=subst(L[1],s,t)*L[3]-subst(L[3],s,t)*L[1];
poly g2=subst(L[2],s,t)*L[3]-subst(L[3],s,t)*L[2];

poly g=gcd(g1,g2);

if (deg(subst(leadmonom(g),t,1))==1) {
string j="proper";
return(j);
}
else {
string j="not proper";
return(j);
}
}
```

The function *asssystem* computes the associated system of an equaiton $F = 0$ with solution parametrized by `list P`.

```
proc asssystem(list P)
{
poly Ds=diff(P[4],s);
poly Dt=diff(P[4],t);
list g=(diff(P[1],s)*P[4]-Ds*P[1])*(diff(P[2],t)*P[4]-Dt*P[2])-     ---
(diff(P[2],s)*P[4]-Ds*P[2])*(diff(P[1],t)*P[4]-Dt*P[1]),P[4]^4;
g=g[1]/gcd(g[1],g[2]),g[2]/gcd(g[1],g[2]);
list f1=P[4]*(diff(P[2],t)*P[4]-Dt*P[2])-P[3]*(diff(P[1],t)*P[4]-Dt*P[1]),P[4]^3;
f1=f1[1]/gcd(f1[1],f1[2]),f1[2]/gcd(f1[1],f1[2]);
if (g[1]==0) {
list S=g[1],f1[1];
return(S);
}
else
{
list f2=-P[4]*(diff(P[2],s)*P[4]-Ds*P[2])+(diff(P[1],s)*P[4]-Ds*P[1])*P[3],P[4]^3;
f2=f2[1]/gcd(f2[1],f2[2]),f2[2]/gcd(f2[1],f2[2]);
list s1=f1[1]*g[2],f1[2]*g[1];
s1=s1[1]/gcd(s1[1],s1[2]),s1[2]/gcd(s1[1],s1[2]);
list s2=f2[1]*g[2],f2[2]*g[1];
s2=s2[1]/gcd(s2[1],s2[2]),s2[2]/gcd(s2[1],s2[2]);
list S=s1,s2;
return(S);
}
```

```
}
```

A.2 CoCoA

The function *Finalstep* is described in Section 5.2.

```
Define Finalstep(P,L,Q)

If Der(P[1],x)=0 Then
F:=1/(Subst(Der(P[2],x),x,y))*Subst(L[2],[[x,Subst(P[1],x,y)],  ----
[y,Subst(P[2],x,y)]]);
If F=0 Or Type(F)=RATFUN Then
Print "Bad Surface parametrization";
Elsif Deg(F)=0 And Type(F)=POLY Then
T:=F*x;
Sol:=Subst(P,x,T);
CC1:=x-Subst(Q[1],[[s,Sol[1]],[t,Sol[2]]]);
Chi:=Subst(Q[2],[[s,Sol[1]],[t,Sol[2]]]);
Y:=Subst(Chi,x,x-CC1);
Return Y;
Elsif Deg(F)=2 Then
V:=Factor(F);
I:=1;
K:=[ ];
While (I<=Len(V)) Do
Append(K,V[I][2]);
I:=I+1;
EndWhile;
D:=Count(K,2);
If D=1 Then
If Len(V)=1 Then A:=1;
Else A:=V[2][1];
EndIf;
B:=V[1][1];
B:=Subst(B,x,0)/LC(B);
B:=-B;
T:=B-1/(A*x);
Sol:=Subst(P,x,T);
CC1:=x-Subst(Q[1],[[s,Sol[1]],[t,Sol[2]]]);
Chi:=Subst(Q[2],[[s,Sol[1]],[t,Sol[2]]]);
```

```
Y:=Subst(Chi,x,x-CC1);
Return Y;
Else Print "bad invariant algebraic curve";
EndIf;
Else Print "bad invariant algebraic curve";
EndIf;
Else
F:=1/(Subst(Der(P[1],x),x,y))*Subst(L[1],[[x,Subst(P[1],x,y)],---
[y,Subst(P[2],x,y)]]);
If F=0 Or Type(F)=RATFUN Then
Print "Bad Surface parametrization";
Elsif Deg(F)=0 Then
T:=F*x;
Sol:=Subst(P,x,T);
CC1:=x-Subst(Q[1],[[s,Sol[1]],[t,Sol[2]]]);
Chi:=Subst(Q[2],[[s,Sol[1]],[t,Sol[2]]]);
Y:=Subst(Chi,x,x-CC1);
Return Y;
Elsif Deg(F)=2 Then
V:=Factor(F);
I:=1;
K:=[ ];
While (I<=Len(V)) Do
Append(K,V[I][2]);
I:=I+1;
EndWhile;
D:=Count(K,2);
If D=1 Then
If Len(V)=1 Then A:=1;
Else A:=V[2][1];
EndIf;
B:=V[1][1];
B:=Subst(B,x,0)/LC(B);
B:=-B;
T:=B-1/(A*x);
Sol:=Subst(P,x,T);
CC1:=x-Subst(Q[1],[[s,Sol[1]],[t,Sol[2]]]);
Chi:=Subst(Q[2],[[s,Sol[1]],[t,Sol[2]]]);
Y:=Subst(Chi,x,x-CC1);
Return Y;
Else Print "bad invariant algebraic curve";
EndIf;
Else Print "bad invariant algebraic curve";
```

```
EndIf;
EndIf;
EndDefine;
```

The function *FengGao* is described in Section 5.2.

```
Define FengGao(F,R,S)
If Tracindex(R,s)="not proper" Then
Return "not proper parametrization";
Else
If (Deg(F,y)<Deg(F,z)-1) Then
Print "no rational solution";
Elsif 2*Deg(F,z) < Deg(F,y) Then
Print "no rational solution";
Else
H:=Der(R,x);
A:=S/H;
If Type(A)=POLY Or Type(A)=INT Or Type(A)=RAT Then
If Deg(A)=0 Then
RS:=Subst(R,x,A*(x+c));
Print(RS);
Elsif Deg(A)=2 Then
V:=Factor(A);
I:=1;
L:=[ ];
While (I<=Len(V)) Do
Append(L,V[I][2]);
I:=I+1;
EndWhile;
D:=Count(L,2);
If D=1 Then
If Len(V)=1 Then B:=1;
Else B:=V[2][1];
EndIf;
N:=V[1][1];
N:=Subst(N,x,0)/LC(N);
N:=-N;
RS:=Subst(R,x,(B*N*(x+c)-1)/(B*(x+c)));
Print(RS);
Else Print "no rational solution";
EndIf;
Else Print "no rational solution";
EndIf;
Else Print "bad parametrization";
```

```
EndIf;
EndIf;
EndIf;
EndDefine;
```

The function *Check* is described in Section 5.2.

```
Define Check(F)


If Deg(F,z)=1 Then

D:=DivAlg(F,[z]);
G:=-D.Remainder/D.Quotients[1];
Par:=[x,y,G];
Par:=Subst(Par,[[x,s],[y,t]]);
Print "Solvable for y'";
Return Par;
Elsif Deg(F,y)=1 Then

D:=DivAlg(F,[y]);
G:=-D.Remainder/D.Quotients[1];
Par:=[x,G,z];
Par:=Subst(Par,[[x,s],[z,t]]);
Print "Solvable for y";
Return Par;

Elsif Deg(F,x)=1 Then
A:=AutTransform(F);
If A[1]<>0 Then
Print "Autonomous equations associated:";
Return A;
Else
D:=DivAlg(F,[x]);
G:=-D.Remainder/D.Quotients[1];
Par:=[G,y,z];
Par:=Subst(Par,[[y,s],[z,t]]);
Print "Solvable for x";
Return Par;
EndIf;
Else A:=AutTransform(F);
If A[1]<>0 Then
Print "Autonomous equations associated:";
Return A;
```

```
Else T:=Transform(F);
If T[1]<>0 Then
Return T;
Else Print "no special case";
EndIf;
EndIf;
EndIf;
EndDefine;
```

The function *AutTransform* transforms the equation defined by the polynomial `F` given in input into an autonomous equation, if it is possible. It returns the new polynomial and the constants b and c which define the transformation.

```
Define AutTransform(F)
Lambda:=0;
Gamma:=0;
F:=Lin(F);
L:=LCoef1(F);
L:=GBasis(Ideal(L));
If L=[1] Then
Return [0];
Else
I:=1;
While I<=Len(L) Do
U:=UnivariateIndetIndex(L[I]);
If U>5 Then
RR:=RealRootsApprox(L[I]);
If Len(RR)<>0 Then
F:=Subst(F,Indet(U),RR[1]);
L:=Subst(L,Indet(U),RR[1]);
If U=6 Then
Lambda:=RR[1];
Else Gamma:=RR[1];
EndIf;
I:=1;
Else
I:=I+1;
EndIf;
Else
I:=I+1;
EndIf;
EndWhile;
```

```
If GBasis(Ideal(Eval(F,[1,1,1])))=[1] Then
Return F;
Else
F:=Subst(F,[[Indet(6),0],[Indet(7),0]]);
Return [F,Lambda,Gamma];

EndIf;
EndIf;
EndDefine;
```

The function *AssSys* is described in Section 5.2.

```
Define AssSys(C1,C2,C3)
L:=[ ];
G:=Der(C2,t)*Der(C1,s)-Der(C1,t)*Der(C2,s);
H:=Der(C2,t)-C3*Der(C1,t);
If G=0 Then
L:=[Num(G),Num(H)];
If Deg(L[2],s)=0 And Deg(L[2],t)=0 Then
Print "bad parametrization";
Else Return L;
EndIf;
Else
K:=C3*Der(C1,s)-Der(C2,s);
L:=[H/G,K/G];
Return L;

EndIf;
EndDefine;
```

The functions *LCoef, LCoef1, LCoef2, LTerm, LTerm1, LTerm2* are used inside other functions to compute the leading coefficient or the leading term of `F` given in input.

```
Define LCoef2(F)

L:=[ ];
While Deg(LM(F),x)>=2 Do
X:=Deg(LM(F),x);
Y:=Deg(LM(F),y);
Z:=Deg(LM(F),z);
Q:=x^X*y^Y*z^Z;
```

```
E:=DivAlg(F,[Q]);
E:=E.Quotients;
F:=F-E[1]*Q;
Append(L,E[1])
EndWhile;
Return L;
EndDefine;


Define LTerm2(F)
L:=[ ];
While Deg(LM(F),x)>=2 Do
X:=Deg(LM(F),x);
Y:=Deg(LM(F),y);
Z:=Deg(LM(F),z);
Q:=x^X*y^Y*z^Z;
E:=DivAlg(F,[Q]);
E:=E.Quotients;
E:=E[1]*Q;
F:=F-E;
Append(L,E)
EndWhile;
Return L;
EndDefine;

Define LCoef1(F)

L:=[ ];
While Deg(LM(F),x)>=1 Do
X:=Deg(LM(F),x);
Y:=Deg(LM(F),y);
Z:=Deg(LM(F),z);
Q:=x^X*y^Y*z^Z;
E:=DivAlg(F,[Q]);
E:=E.Quotients;
F:=F-E[1]*Q;
Append(L,E[1])
EndWhile;
Return L;
EndDefine;


Define LTerm1(F)
```

```
L:=[ ];
While Deg(LM(F),x)>=1 Do
X:=Deg(LM(F),x);
Y:=Deg(LM(F),y);
Z:=Deg(LM(F),z);
Q:=x^X*y^Y*z^Z;
E:=DivAlg(F,[Q]);
E:=E.Quotients;
E:=E[1]*Q;
F:=F-E;
Append(L,E)
EndWhile;
Return L;
EndDefine;

Define LCoef(F)

X:=Deg(LM(F),x);
Y:=Deg(LM(F),y);
Q:=x^X*y^Y;
E:=DivAlg(F,[Q]);
E:=E.Quotients;
Return E[1];
EndDefine;

Define LTerm(F)

X:=Deg(LM(F),x);
Y:=Deg(LM(F),y);
Q:=x^X*y^Y;
E:=DivAlg(F,[Q]);
E:=E.Quotients;
E:=E*Q;
Return E[1];
EndDefine;
```

The function *Coeff* takes in input a polynomial `F` and returns the vector of its coefficient, the ones which are zero included.

```
Define Coeff(F)

S:=[LC(F)];
While F<> 0 Do
```

```
Q:= F- LC(F)*LT(F);
If Q=0 Then
For K:=1 To  Deg(F) Do
Append(S, 0);
K:= K+1;
EndFor;
F:=0;

Elsif Deg(F)-Deg(Q)>1 Then
For K:=1 To  Deg(F)-Deg(Q)-1 Do
Append(S, 0);
K:= K+1;
EndFor;
F:=Q;
Append(S,LC(F));
Else
F:=Q;
Append(S,LC(F));
EndIf;
EndWhile;
Return S;
EndDefine;
```

The function *Transform* transforms the equation defined by the polynomial `F` given in input into a solvable for x equation, if it is possible. It returns the new polynomial and the constants b and c which define the transformation.

```
Define Transform(F)
Lambda:=0;
Gamma:=0;
F:=Lin(F);
L:=LCoef2(F);
L:=GBasis(Ideal(L));
If L=[1] Then   //(If#1)
Return [0];
Else
I:=1;
While I<=Len(L) Do  //(while#1)
U:=UnivariateIndetIndex(L[I]);
If U>5 Then   //(If#2)
RR:=RealRootsApprox(L[I]);
If Len(RR)<>0 Then //(If#3)
F:=Subst(F,Indet(U),RR[1]);
```

```
L:=Subst(L,Indet(U),RR[1]);
If U=6 Then //(If#4)
Lambda:=RR[1];
Else //(Else#4)
Gamma:=RR[1];
EndIf; //(EndIf#4)
I:=1;
Else //(Else#3)
I:=I+1;
EndIf; //(EndIf#3)
Else   //(Else#2)
I:=I+1;
EndIf; //(EndIf#2)
EndWhile; //(EndWhile#1)

If GBasis(Ideal(Eval(F,[1,1,1])))=[1] Then  //(If#5)
Return F;
Else //(Else#5)
F:=Subst(F,[[Indet(6),0],[Indet(7),0]]);
Return [F,Lambda,Gamma];
EndIf;//(EndIf#5)
EndIf;//(EndIf#2)
EndDefine;
```

The function *Tracindex* takes two rational function`R,S` in input which are the parametrization of a curve, and verify if it is a proper rational paramerization.

```
Define Tracindex(R,S)

G1:=Subst(Num(R),s,t)*Den(R)-Subst(Den(R),s,t)*Num(R);
G2:=Subst(Num(S),s,t)*Den(S)-Subst(Den(S),s,t)*Num(S);

G:=GCD(G1,G2);

If Deg(G,s)=1 Then
Return "Proper";
Else Return "not proper";
EndIf;
EndDefine;
```

The function *Solve* computes the zeros of the univariate polynomial `V` given in input.

```
Define Solve(V)
```

```
K:=[ ];

S:= "Good";

While Len(V)<>0 Do

If UnivariateIndetIndex(V[Len(V)])<=1 Then

S:="Bad";
Return S;
Else

R:=RealRootsApprox(V[Len(V)],10^(-15));

If Len(R)<>0 Then
Append(K,[Indet(UnivariateIndetIndex(V[Len(V)])),R[1]]);

V:=Subst(V,Indet(UnivariateIndetIndex(V[Len(V)])),R[1]);

Remove(V,Len(V));
Else Remove(V,Len(V));
EndIf;
EndIf;

EndWhile;

Return [S,K];

EndDefine;
```

The function *IsIrr* decides if the polynomial `F` given in input is irreducible or not.

```
Define IsIrr(F)
F:=Monic(F);
E:=Factor(F);
If Len(E)>1 Then
Return FALSE;
Elsif E[1][2]<>1 Then
Return FALSE;
Else
Return TRUE;
EndIf;
```

```
EndDefine;
```

The function *InvAlgCurve* is described in Section 5.2.

```
Define InvAlgCurve(S,T)

B:=[ ];
K:=1;
While K<=3  Do
L:=GenPoly(K);
Using W Do
B:=BringIn(B);
S:=BringIn(S);
T:=BringIn(T);
ForEach N In 1..Len(L) Do
D:=Num(S)*Den(T)*Der(L[N],x)+Den(S)*Num(T)*Der(L[N],y);
Ind:=FALSE;
F:=[ ];
While Ind=FALSE And D<>0 Do
If IsIn(LTerm(D), Ideal(LTerm(L[N])))=True Then
D:=D-L[N]*LTerm(D)/LTerm(L[N]);
Elsif Deg(LCoef(D))=0 Then
Ind:=TRUE;
Elsif UnivariateIndetIndex(LCoef(D))<>0 Then
RR:=RealRootsApprox(LCoef(D));
If Len(RR)<>0 Then
L[N]:=Subst(L[N],Indet(UnivariateIndetIndex(LCoef(D))),RR[1]);
F:=Subst(F,Indet(UnivariateIndetIndex(LCoef(D))),RR[1]);
D:=Subst(D,Indet(UnivariateIndetIndex(LCoef(D))),RR[1]);
Else
Ind:=TRUE;
EndIf;
Else
Append(F,LCoef(D));
D:=D-LTerm(D);
EndIf;
EndWhile;
For I:=1 To (K+1) Do
For J:=1 To I Do
X:=1;
While X<=Len(F) Do
If F[X]<>0 Then
DD:=DivAlg(F[X],[c[I,J]]);
DDQ:=DD.Quotients[1];
```

```
If Deg(F[X],c[I,J])=1 And Deg(DDQ)=0 Then
F1:=Subst(F[X],c[I,J],0);
F:=Subst(F,c[I,J],-1/DDQ*F1);
L[N]:=Subst(L[N],c[I,J],-1/DDQ*F1);
X:=1+Len(F);
Else
X:=X+1;
EndIf;
Else
X:=X+1;
EndIf;
EndWhile;
EndFor;
EndFor;
If Ind=FALSE And F<>[ ] Then
V:=GBasis(Ideal(F));
E:=Solve(V);
If E[1]="Good" Then
Remove(E,1);
For I:=1 To Len(E[1]) Do
L[N]:=Subst(L[N], E[1][I][1],E[1][I][2]);
EndFor;
Else Ind:=TRUE;
EndIf;
EndIf;
For I:=1 To (K+1) Do
For J:=1 To (K+1) Do
L[N]:=Subst(L[N],c[I,J],1);
EndFor;
EndFor;

If Ind=FALSE And IsIrr(L[N])=True Then
Append(B,L[N]);
EndIf;

EndForEach;
EndUsing;
K:=K+1;
B:=BringIn(B);
EndWhile;
Print B;
EndDefine;
```

The function *GenPoly* takes in input an integer `int d` and it returns the set of monic polynomials up to degree `int d` with undetermined coefficients. It is used in the function *invalgcurve.*

```
Define GenPoly(D)
A:=NewMat(3+(D+1)^2,3+(D+1)^2,0);
A[1][1]:=1;
A[1][2]:=1;
A[2][1]:=1;
For Y:=3 To 3+(D+1)^2 Do
A[Y][Y]:=1;
EndFor;
W::=QQ[x,y,z,c[1..(D+1),1..(D+1)]], Ord(A);
Using W Do
P:= c[1,1];
For I:=1 To D Do
L:= I;M:=0;
While L+M=I And L>=0 Do
P:=P+c[I+1,M+1]*x^L*y^M;
L:=L-1;
M:=M+1;
EndWhile;
EndFor;
X:=[ ];
For O:=0 To D Do
F:=P;
For Z:=0 To O Do
If Z=O Then
F:=Subst(F,c[D+1,O+1],1);
Else F:=Subst(F,c[D+1,Z+1],0);
EndIf;
EndFor;
Append(X,F);
EndFor;
EndUsing;
Return X;
Return W;
EndDefine;
```

The function *Lin* takes in input a polynomial `F`, it returns the action of a generic element of $\mathcal{G}'$ on `F`. For the definition of $\mathcal{G}'$ see Section 4.6.

```
Define Lin(F)

F:=Subst(F,[[y,-bx+y-c],[z,z-b]]);
```

```
Return F;
EndDefine;
```

A.3 MAGMA

How to compute a rational parametrization of an algebraic surface of $\mathbb{Q}^3$ in MAGMA.

```
P<x,y,z,w>:=ProjectiveSpace(Rationals( ),3);
P2<X,Y,Z>:=ProjectiveSpace(Rationals( ),2);
f:= [the polynomial defining the surface];
ParametrizeProjectiveHypersurface(Surface(P,f),P2);
```

Bibliography

[CoCoA] Abbott, J., Bigatti, A.M., Lagorio, G., 2009. *CoCoA-4.7: a system for doing Computations in Commutative Algebra.* Available at http://cocoa.dima.unige.it. CoCoA manual Available at http://cocoa.dima.unige.it/download/doc/help.pdf.

[Bay15] Bayer, T., 2015. `zeroset.lib`. *A* SINGULAR 4-0-2 *library for computing roots and factorizations.*

[BGLG] Bermejo, I, Garcia-Llorente, E., Gimenez, Ph., 2015. `monomialideal.lib`. *A* SINGULAR 4-0-2 *library for computing primary and irreducible decompositions of monomial ideals.*

[BDLS] Boehm, J., Decker, W. Laplagne, S., Seelisch, F., 2015. `paraplanecurves.lib`. *A* SINGULAR 4-0-2 *library for computing rational parametrizations of rational plane curves.*

[Car94] Carnicer, M. M., 1994. *The Poincaré problem in the nondicritical case*, Annal of Mathematics, 140 (2), 289-294.

[CNSW12] Ngô, L. X. C., Sendra, J.R., Winkler, F., 2012. *Classification of algebraic ODEs with respect to rational solvability*, Contemporary Mathematics Volume 572.

[CNW10] Ngô, L. X. C., Winkler, F., 2010. *Rational general solutions of first order non-autonomous parametrizable ODEs.*, Journal of Symbolic Computation 45, 1426-1441.

[CNW11H] Ngô, L. X. C., Winkler, F., 2011. *Rational general solutions of parametrizable aODEs.*, Publicationes Mathematicae 79 3/4, 573-587, Debrecen.

[CNW11] Ngô, L. X. C., Winkler, F., 2011. *Rational general solutions of planar rational systems of autonomous ODEs.*, Journal of Symbolic Computation 46, 1173-1186.

[DGPS] Decker, W., Greuel, G.-M., Pfister, G.,Schönemann, H., 2015. SINGULAR 4-0-2 - *A computer algebra system for polynomial computations.* http://www.singular.uni-kl.de. The manual is available at https://www.singular.uni-kl.de/index.php/singular-manual.html.

[FG04] Feng, R., Gao, X.-S., 2004. *Rational general solutions of algebraic ordinary differential equations.*, Proceedings ISSAC2004. ACM Press, New York, pp. 155-162.

[FG06] Feng, R., Gao, X.-S., 2006. *A Polynomial time algorithm for finding rational general solution of first order autonomous ODEs.*, Journal of Symbolic Computation 41, 739-762.

[GLP] Greuel, G.-M., Laplagne, S., Pfister, G., 2015. `normal.lib`. *A* SINGULAR 4-0-2 *library for computing the normalization of affine rings,*

[GP02] Greuel, G.-M., Pfister, G., 2015. *A* SINGULAR*introduction to commutative algebra,* Springer.

[Hub96] Hubert, E., 1996. *The general solution of an ordinary differential equation,* Proceedings ISSAC1996. ACM Press, New York, pp. 189-195.

[Kam59] Kamke, E., 1959. *Differentialgleichungen*, New York, Chelsea Publishing.

[Kol73] Kolchin, E.R., 1973. *Differential Algebra and Algebraic Groups*, Academic Press.

[MAGMA] *MAGMA's Handbook.* Available at http://magma.maths.usyd.edu.au/magma/handbook/.

[Man93] Man, Y.-K., 1993. *Computing closed form solutions of first order ODEs using the Prelle-Singer procedure*, Journal of Symbolic Computation 45, 1426-1441.

[Ngo11] Ngô, L.X. C., 2011. *Rational general solutions of first order algebraic ODEs.*, Dissertation.

[PoWe] Pohl, W., Wenk, M., 2015. `solve.lib`. *A* SINGULAR 4-0-2 *library for complex solving of polynomial systems.*

[PS83] Prelle, M.J., Singer, M.F., 1983. *Elementary first integrals of differential equations*, Trans. Amer. Math. Soc. 279 (1), 215-229.

[Ritt] Ritt, J.F., 1950. *Differential Algebra*, American Mathematical Society, vol. 33. Colloquium Publications.

[SW97] Sendra, J. R., Winkler, F., 1997. *Parametrizations of algebraic curves over optimal field extensions* Journal of Symbolic Computation, 23, 191-2-7.

[SW01] Sendra, J. R., Winkler, F., 2001. *Tracing index of rational curve parametrizations.* Computer Aided Geometric Design 18, 771-795.

[SWPD08] Sendra, R., Winkler, F.,Perez-Diaz, S., 2008. *Rational Algebraic Curves - A Computer algebra approach.*, Springer.

[Sin92] Singer, M.F., 1992. *Liouvillian first integrals of differential equations*, Trans. Amer. Math. Soc. 333 (2), 673-688.

[VDW1] Van der Waerden, B. L., 1991. *Algebra: Volume I*, Springer.

[VDW2] Van der Waerden, B. L., 1991. *Algebra: Volume II*, Springer.

[VH94] Van Hoeij, M., 1994. *An Algorithm for Computing an Integral Basis in an Algebraic Function Field,* Journal of Symbolic Computation 18, p. 353-363