UNIVERSITÀ DEGLI STUDI DI PISA

Dipartimento di Fisica
Corso di Laurea Magistrale in Fisica

TESI DI LAUREA MAGISTRALE

Asymptotic Symmetries of Gravity and Higher-Spin Theories

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Anno Accademico 2015-2016

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Introduction

Renewed interest has been recently shown in the subject of asymptotic symmetries, *i.e.*, symmetries that emerge when one studies the behavior of a physical theory "at infinity" or, better, at the "boundary" of spacetime. Reasons motivating the ongoing research concerning this topic arose in particular after observations due to A. Strominger et al. [1, 2] who noted that asymptotic gravitational symmetries, which were discovered in the sixties by Bondi, Metzner, van der Burg and Sachs, and whose underlying group is commonly referred to as the BMS group [3, 4, 5], appear to be closely related to soft theorems, *i.e.* relations among scattering amplitudes for processes involving the emission or absorption of massless particles in the low-energy limit [6, 7, 8, 9]. Other sources of interest concern the potential observable effects that such symmetries are supposed to produce [10], in relation to the so-called gravitational memory effect, together with the tantalizing, although quite speculative, possibility that they may also suggest a possible route towards the solution of the black-hole information paradox [11] as a consequence of the presence of infinitely-many conserved quantities near the horizon. Recent additional insights on their role in near-horizon geometries of extremal black holes have also been offered in [12, 13].

The interplay between asymptotic symmetries and soft theorems has been also investigated in the case of electrodynamics, where it has led to new interpretations of the link between soft photon amplitudes and asymptotic symmetries of QED and, possibly, of Yang-Mills theories [14, 15].

These results give us a hint that such an interplay should not occur as a peculiar feature of gravity alone, but rather they point to the existence of a general underlying field-theoretical mechanism, thus adding a significant piece of appeal to the subject. Indeed, the similarity between the gravitational and the electromagnetic cases can also be recognized in the nature of the asymptotic symmetry group itself, in both cases defined in terms of generators possessing an arbitrary dependence on the angular coordinates of the spheres placed at conformal infinity. More precisely, for electromagnetism it consists of gauge parameters with arbitrary dependence on those two angular coordinates, while for the gravitational field it is the asymptotic symmetry group of asymptotically flat spaces (BMS group), which is not, as maybe expectable, the Poincaré group, but rather an infinite-dimensional group given by the semidirect product of the Lorentz group and an infinite-dimensional family of direction-dependent translations, whose generators again depend arbitrarily on two angular coordinates.

In short, the main observation underlying all the recent related ongoing activity is that Weinberg's soft photon and soft graviton theorems [6] can be recast as Ward identities for infinite-dimensional families of asymptotic symmetries of electromagnetism and gravity, respectively.

In order to better frame the role to be played by these symmetries, the first step is to recall the Noether theorems [16]: classically, the conserved quantities arising from *local* symmetries vanish identically when such symmetries act only on a compact region or fall off sufficiently fast at infinity. On the other hand, they can attain nontrivial values in correspondence with transformations which do not die out "as one goes to infinity". Since these two kinds of local symmetries display very different asymptotic behavior, they are commonly referred to as "small" and "large" (gauge) transformations, respectively. Indeed, for gauge symmetries, the conserved Noether current j^μ takes the form of an "improvement", *i.e.* it is given by the divergence of an antisymmetric tensor, $j^\mu = \partial_\nu \kappa^{\mu\nu}$ with $\kappa^{\mu\nu} = -\kappa^{\nu\mu}$. This property is usually referred to as "local Gauss law" since it closely resembles the case of the Maxwell equations for electrodynamics, while also being crucial in order to characterize gauge theories even at the quantum level [17, 18, 19]. Any improvement is conserved, $\partial_\mu j^\mu \equiv 0$ (as a consequence of the antisymmetry of $\kappa_{\mu\nu}$), and the corresponding charges, obtained by integrating j^μ on a spacelike hypersurface Σ, are in fact given by integrals over the *boundary* $\sigma = \partial\Sigma$ by Stokes' theorem, and are therefore intrinsically related to the behavior of the theory near the edge of spacetime.

An additional key ingredient of the whole framework is given by the spontaneous breaking of these asymptotic or large gauge symmetries, where soft gravitons and photons are interpreted as the massless particles predicted by Goldstone's theorem.

Now, two questions in particular naturally arise in the light of the previous discussion. First of all, since Weinberg's soft theorems are valid for arbitrary spin, does the above analysis admit a nontrivial extension to spins higher than two? In other words, do these low energy results on scattering amplitudes correspond to some previously neglected symmetry group also in the case of higher-spin theories? And second, given that such theorems can be extended to any dimensions, is it possible to give a meaning to the notion of higher-dimensional asymptotic symmetries?

Another physical motivation for the extension to higher spins is the following: in view of both the impossibility of infrared effects due to massless higher-spin messengers already implied by Weinberg's results, and of the established lore against interacting higher-spin theories in flat space [20], one may hope that answering these questions could lead to nontrivial physical implications on our understanding of the infrared structure of higher-spin theories. Efforts in this direction could shed some light on the open questions concerning infrared problems in gauge theories, as the generators of asymptotic symmetries have been recognized to be closely related to soft photons and gravitons, while the infrared structure displayed by asymptotic quantization [21] bears strong resemblance to that of the infrared problem in QED [22, 23, 24]. On the other hand, Weinberg's theorem puts severe limitations to the long-range behavior of massless fields with spin higher than two, effectively ruling out the possibility that a macroscopic higher-spin interaction could emerge in the soft limit on flat space.

This issue has been shown to be even more acute in a series of no-go results, indicating that, for perturbatively local theories on flat space, no consistent interaction involving massless fields of spin higher than two appears to be allowed beyond the cubic level. Once again, the problem appears to be intimately related to the infrared behaviour of such systems, as also indicated by the two main positive solutions presently known.

As originally suggested by Fradkin and Vasiliev in their seminal paper [25], a non-vanishing cosmological constant can indeed provide the infrared regulator needed for consistent interactions among massless higher-spin particles to be possible. Their suggestions paved the way to the complete on-shell construction to all orders by Vasiliev and his school, up to the more recent developments taking place in the context of the AdS/CFT correspondence (for references, see *e.g.* [26, 27]).

Another possible way of evading the conclusions of these no-go theorems relies on string theory: in such a theory, almost all the fields in the spectrum, and in particular all the fields with spin higher than two, are lifted to being massive, the square mass of a given field being roughly proportional to its spin and to the string tension, playing the role in this context of the needed infrared regulator [28]. A long-standing conjecture relates the appearance of the string tension as a result of some higher-spin gauge symmetry breaking mechanism [29, 30], and it is conceivable that a more concrete understanding of the infrared physics of higher-spin interactions should be ultimately related to this type of picture.

It is also worth noting that the novel approach to infrared problems suggested by the interplay between asymptotic symmetries and soft theorems may also have relevant connections to the recent developments in massive gravity and its multi-metric extensions, *i.e.* theories of gravitation where the usual, massless, graviton is accompanied by one or more massive particles of spin two, producing models of gravity with different infrared behavior. (See *e.g.* [31] for a recent review.)

These arguments indicate that the tools developed in the understanding of asymptotic symmetries and their connection to higher spins have chances to be exported in other interesting sectors of the physics of gauge theories.

Results

We show that it is possible, with suitably-defined boundary conditions which essentially generalize the current notion of gravitational asymptotic flatness to the framework of higher-spin theories, to obtain an infinite-dimensional family of large gauge symmetries for any integer spin entirely analogous to those which have been found for gravity and QED; furthermore, we extend the method used in the literature to link these symmetries to Weinberg's theorems, showing that indeed Weinberg's factorization results are equivalent to Ward identities stemming from such asymptotic symmetries. Part of the strategy employed in order to achieve this goal also allows to improve the results obtained for the spin two case [1, 2] by performing the derivation with greater generality: while in the literature Weinberg's soft graviton theorem is derived from BMS Ward identities under the assumption that the equivalence principle holds, we are able to perform this proof without relying on this hypothesis, or, in other words, to prove the equivalence principle as a consequence of BMS invariance.

The extension to higher spins is again tackled in a more specific way by generalizing an old, but elegant, result concerning large gauge symmetries of QED by Ferrari and Picasso [32, 33]: working with linear large gauge symmetries, which are in fact a finite-dimensional family of asymptotic symmetry transformations, allows to recover all soft theorems of QED, gravity and higher-spin theories as consequences of the Goldstone theorem applied to the breaking of such symmetries.

As far as the question concerning higher-dimensional asymptotic symmetries is concerned, let us note that, in view of the results of [1, 2], it would appear quite bizarre if the factorization theorem, while holding for any D, admitted an interpretation as the Ward identity of an underlying symmetry group only in specific dimensions. However, in the case of gravity, a number of negative results suggested an effective trivialization of asymptotic gravitational symmetries in higher dimensions (see *e.g.* [34]). Recent investigations, differently, have pointed out that suitable choices of boundary conditions allow in fact to implement the full asymptotic symmetry group, from which the soft graviton theorem can be derived, in arbitrary even dimensions, at least. [35, 36] The strategy currently employed in this direction amounts essentially to a "change of attitude" towards these symmetries. In the past, infinite-dimensional asymptotic symmetries were regarded as a mathematical oddity and the tendency was to look for ways of getting rid of them, rather than to look for them. In four dimensions, one can in principle impose more rigid boundary conditions that shrink the asymptotic symmetry group of asymptotically flat spacetimes down to the Poincaré group, but doing so effectively rules out the possibility of gravitational radiation, which appears to be too stringent a condition. On the other hand, higher-dimensional gravitational systems admit boundary conditions which allow for gravitational radiation, while still selecting a (finite-dimensional) Poincaré asymptotic symmetry group. On account of the new understanding on the physical significance of such symmetries, authors nowadays propose to relax the boundary conditions for the metric tensor which define asymptotic flatness, in order to allow for the presence of a full, infinite-dimensional symmetry group of gravity in any dimension: the key point is to decide whether the angular part of the metric should differ from the flat, Euclidean one by corrections of order at most $\mathcal{O}\big(r^{(6-D)/2}\big)$ or rather $\mathcal{O}(r)$, where r is a suitable radial coordinate and D is the dimension of spacetime.

A more geometric approach to the definition of higher-dimensional asymptotic symmetries of gravity may offer a more convincing justification of such falloff conditions in terms of geometric properties of the conformal boundary of spacetime, as it does for the four-dimensional case: instead of defining asymptotic flatness by prescribing falloff conditions to be imposed on the metric tensor using an appropriate radial coordinate, it is in general preferred to define a four-dimensional spacetime to be asyptotically flat if it admits a conformal compactification with the same properties as the compactification of Minkowski spacetime, the latter solution being also favoured due to its explicit covariance. We show that the same procedure can be essentially carried out also for systems of dimension D, with some adjustments and some caveats. In addition, we also argue that infinite-dimensional "BMS-like" symmetries may be completely characterized by the intrinsic properties of the conformal compactification of spacetime and then, once their formal structure is defined, linked back to the asymptotic behaviour of the physical spacetime.

Plan of the work

The material is organized as follows. Chapter 1 reviews some fundamental aspects of the physics and geometry of asymptotically flat spacetimes together with their symmetry groups, while also offering a brief account of Ashtekar's strategy for performing an asymptotic quantization of gravity. In Chapter 2 we recall Weinberg's soft theorems, while the following Chapters 3, 4 are devoted to analyzing the current strategy which links these theorems, in the case of spin two and spin one fields, to asymptotic symmetries of gravity and electrodynamics respectively.

Our original results are presented in Chapters 5, 6 and 7. In Chapter 5, after a brief revisitation of the results for spin one and two, needed in order to bring them in a broader perspective, we present the main new results of this thesis, namely their generalization to higher spins, by defining higher-spin asymptotic symmetries near null infinity, and the proof of their equivalence to Weinberg's factorization results. The generalization to higher spins is then discussed from a different prespective using the Ferrari-Picasso approach in Chapter 6.

Finally, in Chapter 7, we discuss the extension of the notion of gravitational asymptotic symmetries to arbitrary-dimensional spacetimes.

1 Asymptotic Flatness

The notion of asymptotically flat spacetimes, which is thoroughly adressed in [37, Chapter XI], was introduced in order to describe ideally isolated systems in general relativity: indeed, even though no physical system truly can be isolated from the rest of the universe, while performing the study of a specific class of systems, such as compact stars or black holes for instance, it should be possible to neglect the influence of distant matter or the cosmological curvature and hence simplify the problem by assuming that the spacetime becomes flat at large distances. Furthermore, in order to give a particle interpretation to field theories, one needs to define asymptotic states and hence generators of translations "at infinity": therefore the description of gravitational radiation motivates the study of the geometry of asymptotically flat spacetimes; as we shall see, this approach allows in fact to describe the fully nonlinear regime of gravitational radiation.

The main issue, in comparison with electromagnetism in special relativity, is that in general relativity one has no natural global inertial coordinate system to define a preferred radial coordinate, r, for use in specifying falloff rates. One could define a D-dimensional spacetime to be asymptotically flat if there exists *any* system of coordinates $x^0, x^1, \dots, x^d$, for $D = d + 1$, such that the metric components behave appropriately at large coordinate values, *e.g.*

$$g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(1/r), \text{ as } r \to \infty, \tag{1.0.1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $r = [(x^1)^2 + \dots + (x^d)^2]$. However, since this definition is not coordinate-independent, the coordinate invariance of all statements obtained in this approach must be carefully checked; furthermore, from a technical point of view, it is rather difficult to specify precisely how the large distance limit $r \to \infty$ is to be taken, a notable example being the calculation of the energy flux radiating away from a system.

The main idea to solve these issues is to "add in" to the spacetime the points at infinity in a manifestly coordinate-independent way, a strategy allowing to give fully satisfactory definitions of the total energy of an isolated system and of the energy carried away from the system by gravitational radiation. Moreover, the extension makes it possible to provide a geometric description of the symmetries of asympotically flat spacetimes, which will be relevant for our discussion.

1.1 Conformal Infinity

The main technical tool allowing to give a precise definition of the notion of asymptotic flatness and to specify a meaningful notion of "limits as one goes to infinity" is the so-called "conformal infinity", which will be first reviewed in the case of flat spacetime.

1.1.1 Minkowski spacetime

In spherical coordinates, the metric of the D-dimensional Minkowski spacetime takes the form

$$ds^2 = -dt^2 + dr^2 + r^2 d\gamma^2, \tag{1.1.1}$$

where $d\gamma^2$ is the line element on the Euclidean unit $(D-2)$-sphere (*e.g.* $d\gamma^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ in $D = 4$, where θ and ϕ are the usual polar and azimuthal angles).

Suppose we are interested in expressing the energy carried to infinity by some massless field, for instance a scalar field: this requires to compute limits as one goes to null infinity; to this purpose it is convenient to introduce the light-cone coordinates

$$\begin{aligned} v &= t + r, \qquad & t &= (v+u)/2, \\ u &= t - r, \qquad & r &= (v-u)/2, \end{aligned} \tag{1.1.2}$$

and write the Minkowski metric as

$$ds^2 = -du\, dv + \frac{1}{4}(v-u)^2 d\gamma^2. \tag{1.1.3}$$

These coordinates represent affine parameters along outgoing and incoming null geodesics, respectively. Now, in order to "add in" *future* null infinity, which corresponds to letting $v \to \infty$ at fixed u, a naive idea would be to introduce the coordinate $V = 1/v$, so that null infinity be represented by the point $V = 0$. The metric in these coordinates takes the form

$$ds^2 = \frac{1}{V^2}\, du\, dV + \frac{1}{4}\left(\frac{1}{V} - u\right)^2 d\gamma^2. \tag{1.1.4}$$

However, the point $V = 0$ is singular in this spacetime. To circumvent this problem, one introduces a new *unphysical* spacetime conformally related to the original one, $d\bar{s}^2 = \Omega^2 ds^2$ where $\Omega = V$. Now the metric reads

$$d\bar{s}^2 = du\, dV + \frac{1}{4}(1 - uV)^2 d\gamma^2, \tag{1.1.5}$$

and in particular it is well behaved at $V = 0$. The conformal factor, however, blows up at the events $v = 0$ in the original spacetime and furthermore this construction does not allow to extend $\bar{g}_{\mu\nu}$ symmetrically to *past* null infinity, *i.e.* $u \to -\infty$ at fixed v.

A better idea is to perform the following conformal transformation on (1.1.3) choosing the conformal factor

$$\Omega^2(x) = \frac{4}{(1+v^2)(1+u^2)} \tag{1.1.6}$$

which treats u and v on equal footing.

Now, in order to obtain the usual compactification of Minkowski spacetime, define new coordinates T and R by

$$\begin{aligned} T &= \tan^{-1} v + \tan^{-1} u, \qquad v = \tan(\alpha/2),\ \alpha \equiv T + R, \\ R &= \tan^{-1} v - \tan^{-1} u, \qquad u = \tan(\beta/2),\ \beta \equiv T - R; \end{aligned} \tag{1.1.7}$$

note in particular that T and R have ranges restricted by the following inequalities: R is non-negative, since $v \geqslant u$, and $\tan^{-1}$ ranges from $-\pi/2$ to $\pi/2$, so

$$-\pi < T - R \leqslant T + R < \pi. \tag{1.1.8}$$

In other words, the Minkowski spacetime coordinates t, r have been mapped into the triangle of vertices $i^0(\pi, 0)$, $i^+(0, \pi)$, $i^-(0, -\pi)$ in the R, T plane as in Figure 1.1.

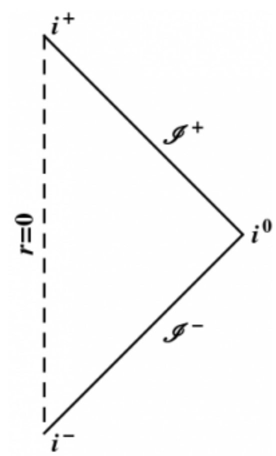


Figure 1.1: The conformal compactification of Minkowski spacetime in R, T coordinates, where the angular coordinates have been suppressed. The script capital $\mathscr{I}$ is usually pronounced "scr-I".

The components of $\tilde{g}_{\mu\nu}$ read therefore

$$d\tilde{s}^2 = -dT^2 + dR^2 + (\sin^2 R) d\gamma^2. \tag{1.1.9}$$

This is just the natural Lorentz metric on $S^3 \times \mathbb{R}$, known as the Einstein static universe, apart from the restrictions imposed on R, T. Indeed, the Friedmann-Robertson-Walker metric reads

$$d\tau^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\gamma^2\right) \tag{1.1.10}$$

and, since the Einstein universe is static $a(t) = 1$ and closed $k = +1$, letting $t = T$ and $r = \sin R$, we get (1.1.9).

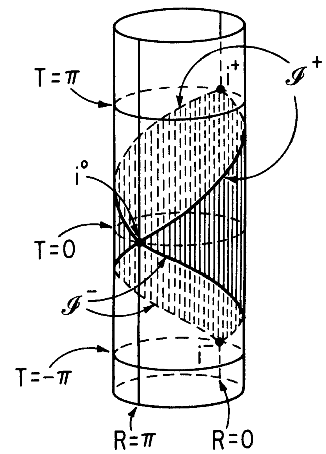


Figure 1.2: The Einstein cylinder: conformal completion of the Minkowski spacetime (shadowed region), as an embedding in Einstein's static universe (full cylinder).

We recall that, given two spacetimes (M, g) and (M', g'), *i.e.* two differentiable manifolds provided with a symmetric, non degenerate tensor of type $(0, 2)$, a *conformal isometry* [38] of (M, g) into (M', g') is a diffeomorphism $\psi : M \to M'$ such that $\psi^* g' = \Omega^2 g$.[1] Indeed in our case $\psi^* \tilde{g} = \Omega^2 g$, where $\psi : \mathbb{R}^4 \to O$, where O is an open subset (region) of $S^3 \times \mathbb{R}$.

We define the *conformal infinity of Minkowski spacetime* (figure 1.2) to be the boundary $\dot{O}$ of O. Aside from the angular coordinates, this boundary can be identified as follows:

- *Past timelike infinity* i^-, corresponding to $r = 0,\ t \to -\infty$, or $v = u \to -\infty$, *i.e.* $R = 0,\ T = -\pi$, the bottom vertex of the above described triangle;
- *Past null infinity* $\mathscr{I}^-$, which is the boundary line connecting i^- and i^0, $T = R - \pi,\ 0 < R < \pi$, the bottom right side of the triangle. Indeed, for v =constant, $u \to -\infty$, we get $T = \tan^{-1} v - \pi/2$, $R = \tan^{-1} v + \pi/2$ and hence $-T + R = \pi$.

[1] The pullback ψ^* of a diffeomorphism $\psi : M \to M'$ acts on one-forms defined on M' in the following way: let $p \in M$, $\alpha_p \in T_p^* M$ and let ψ_{*p} be the derivative of ψ at p, then for each vector $X_p \in T_p M$,

$$(\psi^* \alpha)_p(X_p) = \alpha_{\psi(p)}(\psi_{*p} X_p). \tag{1.1.11}$$

- *Spacelike infinity* i^0, corresponding to $r \to +\infty,\ t = 0$, or $v = -u \to +\infty$, i.e. $R = \pi,\ T = 0$, the right vertex of the triangle;

- *Future null infinity* $\mathscr{I}^+$, linear interpolation of i^0 and i^+, $T = -R + \pi,\ 0 < R < \pi$, the top right side of the triangle. Indeed, for u =constant, $v \to +\infty$, we get $T = \pi/2 + \tan^{-1} u$, $R = \pi/2 - \tan^{-1} u$ and hence $T + R = \pi$.

- *Future timelike infinity* i^+, corresponding to $r = 0,\ t \to +\infty$, or $v = u \to +\infty$, *i.e.* $R = 0,\ T = +\pi$, the top vertex of the triangle.

Note that the vertical side of the triangle does not belong to the boundary, since, for example, a straight line joining $(t = 0, r, \theta, \phi)$ and $(t = 0, r, \theta, -\phi)$ crosses it while obviously belonging to the original spacetime. Notice that null geodesics start at $\mathscr{I}^-$ and end at $\mathscr{I}^+$ whereas timelike geodesics start at i^- and end at i^+ and spacelike geodesics start and end at i^0.

1.1.2 Asymptotically flat curved spacetimes

Motivated by the above example, we define [21] a spacetime to be asymptotically flat at null infinity if it admits a conformal completion in which the boundary "resembles" the Minkowskian $\mathscr{I}$.

More formally, a spacetime (M, g), which may be thought of as the gravitational field of an isolated body emitting gravitational waves, or as a pure gravitational radiation field itself, is asymptotically flat if there exists a manifold $\tilde{M}$ with boundary $\partial\tilde{M} = \mathscr{I}$, equipped with a smooth metric $\tilde{g}$ of Lorentzian signature, such that the interior $\tilde{M} \smallsetminus \mathscr{I}$ of $\tilde{M}$ is diffeomorphic to M and:

- there exists a smooth function Ω on $\tilde{M}$ such that $\Omega = 0$ on $\mathscr{I}$, $\tilde{\nabla}_\mu\Omega \neq 0$ on $\mathscr{I}$, and the pullback of $\tilde{g}$ to M is $\Omega^2 g$;

- g satisfies the vacuum Einstein equations $R_{\mu\nu} = 0$ in the intersection with M of a neighborhood of $\mathscr{I}$ in $\tilde{M}$ (*i.e.*, near infinity);

- $\mathscr{I}$ is topologically $S^{D-2} \times \mathbb{R}$, the vector field $n^\mu \equiv \tilde{g}^{\mu\nu}\tilde{\nabla}_\nu\Omega$ on $\mathscr{I}$ is complete and the space of its orbits is diffeomorphic to S^{D-2}.

The intuitive idea underlying the first condition is that, since $\mathscr{I}$ represents the set of points at infinity in the physical spacetime, an "infinite amount of stretching", given by $\Omega = 0$, is needed in the conformal mapping from the physical to the unphysical spacetime. Moreover, the requirements on the derivatives of Ω allow us to use Ω itself as a coordinate in a neighborhood of $\mathscr{I}$, parametrizing how far we are from infinity (see below for the explicit construction of such a coordinate system). The second condition follows naturally from our understanding of an isolated gravitational system and can be in fact relaxed by admitting an energy-momentum tensor of matter with appropriate falloff conditions as one approaches $\mathscr{I}$, *e.g.* by requiring $\Omega^{-2}T_{\mu\nu}$ to have a smooth extension to $\mathscr{I}$. Finally, the third requirement reflects the idea that an asymptotic observer sitting at $\mathscr{I}$ should be able to characterize any observation, *e.g.* parametrize an incoming gravitatational wave packet, by its angular position and retarded time.

Notice immediately that there is a considerable arbitrariness or *gauge freedom* in the association of an unphysical spacetime $(\tilde{M}, \tilde{g})$ with an asymptotically flat physical spacetime (M, g): indeed, another spacetime $(\tilde{M}, \omega^2\tilde{g})$ satisfies the properties of the definition with a sufficiently smooth conformal factor $\omega\Omega$, provided $\omega > 0$ everywhere.

By the validity of the asymptotic field equations and the smoothness of g, we show below that $\mathscr{I}$ is a null 3-surface and n^μ is tangential to it. To further justify the above definition, we will also derive a coordinate form of the condition on the asymptotic behavior of the physical metric as one approaches future null infinity, $\mathscr{I}^+$, showing that it indeed reduces to the Minkowski metric near $\mathscr{I}^+$, as one would intuitively expect. To carry out this program we are first going to need a few technical steps, which are given below.

Recall that, under the conformal mapping $g_{\mu\nu}(x) \mapsto \tilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x)$, the components of the Ricci tensor transform as follows [37, Appendix D], Λ being a shorthand for $\log\Omega$:

$$R_{\mu\nu} \longmapsto \tilde{R}_{\mu\nu} = R_{\mu\nu} + (2-D)(\nabla_\mu\nabla_\nu\Lambda - \nabla_\mu\Lambda\nabla_\nu\Lambda) + g_{\mu\nu}g^{\rho\sigma}\left[(2-D)\nabla_\rho\Lambda\nabla_\sigma\Lambda - \nabla_\rho\nabla_\sigma\Lambda\right], \tag{1.1.12}$$

then, performing the inverse transformation amounts to interchanging quantities with and without tildes and changing the sign of Λ, obtaining [39, Appendix A]

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} + \frac{D-2}{\Omega}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\Omega + \tilde{g}_{\mu\nu}\tilde{g}^{\rho\sigma}\left(\frac{1}{\Omega}\tilde{\nabla}_\rho\tilde{\nabla}_\sigma\Omega + \frac{1-D}{\Omega^2}\tilde{\nabla}_\rho\Omega\tilde{\nabla}_\sigma\Omega\right). \tag{1.1.13}$$

The vanishing of the right-hand side of this equation is the vacuum Einstein equation expressed in terms of the new unphysical variables.

Multiplying (1.1.13) (with $R_{\mu\nu} = 0$)[2] by Ω and taking the limit $\Omega \to 0$ which brings us to $\mathscr{I}^+$, we find

$$0 = \Omega\tilde{R}_{\mu\nu} + (D-2)\tilde{\nabla}_\mu\tilde{\nabla}_\nu\Omega + \tilde{g}_{\mu\nu}\tilde{g}^{\rho\sigma}\left(\tilde{\nabla}_\rho\tilde{\nabla}_\sigma\Omega + \frac{1-D}{\Omega}\tilde{\nabla}_\rho\Omega\tilde{\nabla}_\sigma\Omega\right); \tag{1.1.14}$$

now, since $\tilde{g}_{\mu\nu}$, $\tilde{R}_{\mu\nu}$ and Ω are smooth at $\mathscr{I}^+$, the first three terms are smooth as well and hence the quantity $\Omega^{-1}\tilde{g}^{\rho\sigma}\tilde{\nabla}_\rho\Omega\tilde{\nabla}_\sigma\Omega$ can be smoothly extended to $\mathscr{I}^+$. In particular, this implies $\tilde{g}^{\rho\sigma}\tilde{\nabla}_\rho\Omega\tilde{\nabla}_\sigma\Omega = 0$ at $\mathscr{I}^+$ or, in other words, that $n^\mu = \tilde{g}^{\mu\nu}\tilde{\nabla}_\nu\Omega$ is *null* at $\mathscr{I}^+$. This result also follows from the fact that Ω is constantly zero on $\mathscr{I}^+$ and $\mathscr{I}^+$ is a null surface.

[2] Note that, if the original spacetime were asymptotically (A)dS, the left-hand side would be a constant times $g_{\mu\nu} = \Omega^{-2}\tilde{g}_{\mu\nu}$.

Now we use the gauge freedom $\Omega \mapsto \omega\Omega$ mentioned above; note that $\tilde{g}_{\mu\nu} \mapsto \omega^2\tilde{g}_{\mu\nu}$, $\tilde{g}^{\mu\nu} \mapsto \omega^{-2}\tilde{g}^{\mu\nu}$ and finally (recall that covariant derivatives are inessential since Ω and ω are scalar quantities)

$$\Omega^{-1}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\Omega\tilde{\nabla}_\nu\Omega \longmapsto \omega^{-3}\tilde{g}^{\mu\nu}\left[\Omega\tilde{\nabla}_\mu\omega\tilde{\nabla}_\nu\omega + 2\omega\tilde{\nabla}_\mu\Omega\tilde{\nabla}_\nu\omega + \omega^2\Omega^{-1}\tilde{\nabla}_\mu\Omega\tilde{\nabla}_\nu\Omega\right]. \tag{1.1.15}$$

By choosing ω to satisfy the ordinary differential equation

$$n^\mu\tilde{\nabla}_\mu \log\omega = -\frac{1}{2}\Omega^{-1}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\Omega\tilde{\nabla}_\nu\Omega \tag{1.1.16}$$

on $\mathscr{I}^+$, one gets rid of the second and third term in the previous expression, while the first one is zero since $\Omega|_{\mathscr{I}^+} = 0$. Note that the gauge-fixing condition (1.1.16) is well-defined on $\mathscr{I}^+$, since $\Omega^{-1}g^{\mu\nu}\tilde{\nabla}_\mu\Omega\tilde{\nabla}_\nu\Omega$ is smooth there, as we have just seen.

Therefore, without loss of generality, we can always assume to have chosen the conformal factor so as to ensure

$$\Omega^{-1}\tilde{g}^{\rho\sigma}\tilde{\nabla}_\rho\Omega\tilde{\nabla}_\sigma\Omega = 0 \quad \text{on } \mathscr{I}^+. \tag{1.1.17}$$

The vacuum Einstein field equation then yields

$$(D-2)\tilde{\nabla}_\mu\tilde{\nabla}_\nu\Omega + \tilde{g}_{\mu\nu}\tilde{g}^{\rho\sigma}\tilde{\nabla}_\rho\tilde{\nabla}_\sigma\Omega = 0 \quad \text{on } \mathscr{I}^+, \tag{1.1.18}$$

and finally, since tracing this equation one has $2(D-1)\tilde{g}^{\rho\sigma}\tilde{\nabla}_\rho\tilde{\nabla}_\sigma\Omega = 0$,

$$\tilde{\nabla}_\mu\tilde{\nabla}_\nu\Omega = 0 \quad \text{on } \mathscr{I}^+, \tag{1.1.19}$$

if $D \neq 1, 2$. We shall sometimes refer to this property of $\mathscr{I}^+$ by saying that it is "*divergence-free*". By the previous equation, the null tangent $n^\mu = \tilde{g}^{\mu\nu}\tilde{\nabla}_\nu\Omega$ to $\mathscr{I}^+$ is covariantly constant

$$\tilde{\nabla}_\mu n_\nu = 0, \tag{1.1.20}$$

and satisfies the geodesic equation

$$n^\mu\tilde{\nabla}_\mu n^\rho = 0 \quad \text{on } \mathscr{I}^+. \tag{1.1.21}$$

Therefore it is also called the *null geodesic generator* of $\mathscr{I}^+$.

The gauge choice (1.1.16), which led to the gauge condition (1.1.19), still permits the additional freedom of choosing ω arbitrarily on any given *cross section* of $\mathscr{I}^+$, *i.e.* on a $(D-2)$-dimensional surface $\mathscr{S}$, in $\mathscr{I}^+$, which intersects each null geodesic generator of $\mathscr{I}^+$ at precisely one point. This is because equation (1.1.19) only constrains the behavior of Ω along null geodesics.

Restricting ourselves to $D = 4$, it follows from the fact that $\mathscr{I}^+$ has the topology of $S^2 \times \mathbb{R}$ that $\mathscr{S}$ must be a topological two-sphere; since every Riemannian metric $\tilde{h}_{\mu\nu}$ on $\mathscr{S}$ is conformally equivalent to the Euclidean unit two-sphere metric $\tilde{h}_{\mu\nu} = f^2h_{\mu\nu}$ [37, Problem 3.2], we can use the residual freedom in the choice of conformal factor to make $\mathscr{S}$ a metric sphere of unit radius. As we shall discuss in Chapter 7, this is a keypoint of the discussion needed for the extension to any D of the notion of asymptotic flatness.

Now, let us introduce the following coordinates in a neighborhood of $\mathscr{I}^+$: since, by hypothesis, $\tilde{\nabla}_\mu \Omega|_{\mathscr{I}^+} \neq 0$, we may use Ω itself as one of the coordinates; we introduce the natural spherical coordinates (θ, ϕ) on a cross section, $\mathscr{S}$, and carry these coordinates to other points of $\mathscr{I}^+$ along the null geodesic generators of $\mathscr{I}^+$; we define a coordinate on $\mathscr{I}^+$ to be the affine parameter, measured from $\mathscr{S}$, along the null geodesic generators of $\mathscr{I}^+$, with u scaled so that

$$n^\mu \tilde{\nabla}_\mu u = 1; \tag{1.1.22}$$

finally, we extend these coordinates (u, θ, ϕ) off of $\mathscr{I}^+$ by holding their values fixed along each null geodesic of the family orthogonal to the 2-spheres of constant u on $\mathscr{I}^+$, except the one which generates $\mathscr{I}^+$. In such coordinates

$$d\tilde{s}^2|_{\mathscr{I}^+} = 2d\Omega\, du + d\theta^2 + \sin^2\theta\, d\phi^2. \tag{1.1.23}$$

These coordinates are particularly well-suited for studying the asymptotic behavior of the metric components in the physical spacetime, as we will see shortly. Since $\partial_\nu \Omega = \delta^\Omega_\nu$, the gauge condition (1.1.19), reads

$$\partial_\mu \partial_\nu \Omega - \tilde{\Gamma}^\rho_{\mu\nu} \partial_\rho \Omega = 0, \text{ and hence } \tilde{\Gamma}^\Omega_{\mu\nu} = 0; \tag{1.1.24}$$

in other words, since $\tilde{g}^{\Omega\rho} = \delta^\rho_u$ and $\partial_u \tilde{g}_{\mu\nu} = 0$ at $\mathscr{I}^+$,

$$0 = \tilde{\Gamma}^\Omega_{\mu\nu} = \frac{1}{2}\left(\partial_\mu \tilde{g}_{u\nu} + \partial_\nu \tilde{g}_{u\mu}\right). \tag{1.1.25}$$

Fixing $\mu = \Omega$ and $\nu = u, \theta, \phi$, thanks to $\partial_\theta \tilde{g}_{u\Omega} = 0 = \partial_\phi \tilde{g}_{u\phi}$ at $\mathscr{I}^+$, one has

$$\partial_\Omega \tilde{g}_{uu} = \partial_\Omega \tilde{g}_{u\theta} = \partial_\Omega \tilde{g}_{u\phi} = 0 \text{ on } \mathscr{I}^+, \tag{1.1.26}$$

meaning that $\tilde{g}_{uu}$, $\tilde{g}_{u\theta}$ and $\tilde{g}_{u\phi}$ must be $\mathcal{O}(\Omega^2)$ as $\Omega \to 0$. Thus, in a neighborhood of $\mathscr{I}^+$, the components of the physical metric, $g_{\mu\nu} = \Omega^{-2}\tilde{g}_{\mu\nu}$, take the form

$$\begin{aligned} ds^2 =& 2\Omega^{-2} d\Omega\, du + \Omega^{-2}(d\theta^2 + \sin^2\theta\, d\phi^2) \\ &+ \mathcal{O}(1)(du^2,\ du\, d\theta,\ du\, d\phi) \\ &+ \mathcal{O}(\Omega^{-1})(d\theta^2,\ d\theta\, d\phi,\ d\phi^2,\ d\Omega\, du,\ d\Omega^2,\ d\Omega\, d\theta,\ d\Omega\, d\phi). \end{aligned} \tag{1.1.27}$$

Now let $v = 2/\Omega$, so that

$$\begin{aligned} ds^2 =& - dv\, du + \frac{1}{4} v^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \\ &+ \mathcal{O}(1)(du^2,\ du\, d\theta,\ du\, d\phi) \\ &+ \mathcal{O}(v)(d\theta^2,\ d\theta\, d\phi,\ d\phi^2) \\ &+ \mathcal{O}(1/v)(dv\, du,\ dv\, d\theta,\ dv\, d\phi) \\ &+ \mathcal{O}(1/v^3) dv^2. \end{aligned} \tag{1.1.28}$$

A further coordinate transformation $v \mapsto v + f(u, \theta, \phi)$ can get rid of the terms $\mathcal{O}(1)du^2$ at the expense of introducing terms $\mathcal{O}(1)(dv\, d\theta,\ dv\, d\phi)$: for example if $O(1)du^2 = -du^2$, letting $v \mapsto v - u$ gives (notice the resemblance with (1.1.3))

$$ds^2 = -dv\, du + \frac{1}{4}(v-u)^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \dots \quad . \tag{1.1.29}$$

Having done this, we transform to "asymptotically cartesian coordinates" defined by

$$t = \frac{v+u}{2}, \quad x = \frac{v-u}{2}\sin\theta\,\cos\theta, \quad y = \frac{v-u}{2}\sin\theta\,\sin\theta, \quad z = \frac{v-u}{2}\cos\theta. \tag{1.1.30}$$

The components of the physical metric in these coordinates are of the form

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + \dots \tag{1.1.31}$$

and differ from $\mathrm{diag}(-1,1,1,1)$ only by terms at most of order $1/v$ as $v \to \infty$, since each angular differential behaves as $1/v$; this also clarifies the convenience of eliminating the $\mathcal{O}(1)du^2$ terms in the previous step.

We have shown that our definition of asymptotic flatness at null infinity in terms of the conformal completion of the spacetime, together with Einstein's equations, requires the physical spacetime to become asymptotically Minkowskian as one goes toward null infinity. A similar notion of flatness for spatial infinity has also been given [37, 39], but to our purposes it will not be necessary to illustrate it.

1.1.3 Universal structure and asymptotic symmetries

Minskowski spacetime $(\mathbb{R}^4, \eta)$ has a 10-parameter group of isometries: the Poincaré group. What are the corresponding asymptotic symmetries of asymptotically flat spacetimes? To answer this question, we need to introduce the concept of asymptotic symmetry.

The intuitive notion of infinitesimal asymptotic isometry at, say, future null infinity, is represented by a vector field ξ in the physical spacetime such that Killing's equation

$$\pounds_\xi g = 0 \tag{1.1.32}$$

is satisfied to "as good an approximation as possible" as one goes to $\mathscr{I}^+$.

In a more formal approach, we may require that ξ, viewed now as a vector field in the unphysical spacetime (*i.e.* is pushforward via the conformal mapping), have a smooth extension to $\mathscr{I}^+$. Then we further require that the tensor field $\Omega^2\pounds_\xi g$ also have a smooth extension to $\mathscr{I}^+$ which vanishes on $\mathscr{I}^+$. Finally we identify two vector fields ξ and ξ' on the physical spacetime as the same *infinitesimal asymptotic symmetry* if their extensions to $\mathscr{I}^+$ are equal there.

The asymptotic symmetry group arising from this definition is universal, in the sense that one gets the same abstract group for *all* asymptotically flat spacetimes. Perhaps surprisingly, this group is not the Poincaré group: it is the infinite-dimensional Bondi-Metzner-Sachs (BMS) group [3, 4, 5].

As an example, consider Minkowski spacetime in coordinates (v, u, θ, ϕ) and an arbitrary function $f = f(\theta, \phi)$; the vector

$$\xi = f(\theta, \phi)\partial_u + \frac{v}{2r^2}\frac{\partial f}{\partial \theta}\partial_\theta + \frac{v}{2r^2 \sin^2\theta}\frac{\partial f}{\partial \phi}\partial_\phi \tag{1.1.33}$$

is nonvanishing at $\mathscr{I}^+$ but

$$\Omega^2 \pounds_\xi \eta = 0, \text{ at } \mathscr{I}^+. \tag{1.1.34}$$

One can give an equivalent characterization of the BMS group in terms of the mappings of $\mathscr{I}^+$ into itself. In the unphysical spacetime, the metric $\tilde{g}_{\mu\nu}$ induces a degenerate metric $\tilde{q}_{\mu\nu}$ on the null hypersurface $\mathscr{I}^+$. Since $n^\mu = \tilde{g}^{\mu\nu}\tilde{\nabla}_\nu\Omega$ is tangent to $\mathscr{I}^+$, the vector field n^μ may be viewed as a vector field of $\mathscr{I}^+$ itself. Under a change of conformal gauge $\Omega \mapsto \omega\Omega$, one has: $\tilde{q}_{\mu\nu} \mapsto \omega^2 q_{\mu\nu}$, $n^\mu \mapsto \omega^{-1}n^\mu$. The structure $(\mathscr{I}^+, \tilde{q}, n)$, identified under such conformal rescalings, is universal: indeed, we have shown above that for any $\mathscr{I}^+$ there exists a conformal gauge such that $q_{\mu\nu}dx^\mu dx^\nu = d\theta^2 + \sin^2\theta d\phi^2$ and $n = \partial_u$.

The BMS group is the group of diffeomorphisms of $\mathscr{I}^+$ which preserves this *universal structure*, *i.e.* of the diffeomorfisms $\psi : \mathscr{I}^+ \to \mathscr{I}^+$ such that the linear maps induced by ψ correspond to a rescaling associated with a change of conformal gauge, which has no intrinsic meaning being a redundancy in the description of asymptotically flat spacetimes. This characterization turns out to be very helpful when dealing with the properties of asymptotic symmetries especially because it is intrinsically defined at $\mathscr{I}$, without any reference to the interior of the spacetime.

The infinitesimal *supertranslations* are defined as the vector fields on $\mathscr{I}^+$ of the form $\xi = \alpha n$, where α satisfies $n^\mu\tilde{\nabla}_\mu\alpha = 0$, *i.e.*, it is constant on each null generator, but otherwise is an arbitrary function. Examples of supertranslations are the vectors of the form (1.1.33), since the function $f = f(\theta, \phi)$ is clearly constant along the trajectories of n^μ and ξ reduces to $f(\theta, \phi)\partial_u$ at $\mathscr{I}^+$ giving rise to a family of direction-dependent translations; the intuitive idea as to why these supertranslations should be asymptotic symmetries is that the deformations they induce, due to their angular dependence, are eventually stretched out by the conformal factor.

The supertranslations are an infinite-dimensional Abelian normal subgroup, ST, of the BMS group, G, and the factor group obtained by quotienting the BMS group by the supertranslations is isomorphic to the Lorentz group:

$$G/ST \simeq SO(3, 1). \tag{1.1.35}$$

There exists a unique four-dimensional subgroup of supertranslations which is a normal subgroup of the BMS group. In the case of Minkowski spacetime, it consists of asymptotic symmetries associated with the exact translational symmetries of Minkowski spacetime: this motivates us to define the *asymptotic translations* of a general asymptotically flat spacetime as the elements belonging to this unique four-dimensional normal subgroup of the supertranslation group.

A similar procedure for boosts and rotations fails, as we shall see below: there is no normal subgroup of the BMS group which is isomorphic to the Poincaré group.

Another interesting feature is that in four spacetime dimensions one could, in principle, impose stronger requirements on asymptotically flat spaces at null infinity in order to recover a unique Poincaré group: this, however, would exclude the possibility of gravitational radiation, which is too stringent a condition to impose (see [37, 40] and references therein). As we shall discuss in Chapter 7, this is no longer the case in higher-dimensional spaces, where one can in fact impose more strict conditions which select a Poincaré asymptotic symmetry group while still allowing for the presence of gravitational waves.

Below, we prove in detail a few of these properties of the BMS group, following mostly [21, 39]. We begin by summarizing the structures introduced until now:

- $\mathscr{I} \simeq S^2 \times \mathbb{R}$ is ruled by the trajectories of its null normal n^μ.
- The intrinsic metric q on $\mathscr{I}$ is degenerate (signature $0++$) and its pullback h on the space $\mathcal{S}$ of integral curves of n^μ is conformally related to the Euclidean 2-sphere metric.
- The permissible conformal rescalings, *i.e.* those who leave $\mathscr{I}$ divergence free, are $\Omega \mapsto \omega\Omega$ where ω is a nowhere-vanishing, smooth function on M such that

  $$\pounds_n \omega = 0 \text{ on } \mathscr{I}; \tag{1.1.36}$$

  under these rescalings, we have $q \mapsto \omega^2 q$, and $n \mapsto \omega^{-1} n$.

At the Lie algebra level, a vector field v^μ is an infinitesimal asymptotic symmetry if the diffeomorphism it generates leaves the integral curves of n^μ invariant and maps a pair (q, n) in an equivalent pair $(\omega^2 q, \omega^{-1} n)$, where $\pounds_n \omega = 0$ on $\mathscr{I}$.

Clearly, denoting by $\varphi_v^{(t)}$ the diffeomorphism generated by v in a neighborhood of the identity, letting $\varphi_{v*}^{(t)}$ stand for the push-forward it induces on vector fields and expanding $\omega(t) = e^{t\alpha} = 1 + t\alpha + \ldots$, with $\pounds_n \alpha = 0$, leads to:

$$\begin{aligned} &\varphi_{v*}^{(t)} n = \omega^{-1} n = n - t\alpha n + \ldots \\ &\frac{1}{t}\left(\varphi_{v*}^{(t)} n - n\right) = -\alpha n + \ldots \ . \end{aligned} \tag{1.1.37}$$

This proves that v^μ is an infinitesimal asymptotic symmetry if and only if

$$\pounds_v n = -\alpha n \tag{1.1.38}$$

and similarly

$$\pounds_v q = 2\alpha q. \tag{1.1.39}$$

It is clear that such vector fields form a Lie algebra $\mathfrak{g}$, with respect to the usual vector field Lie bracket:

$$\pounds_{\mu v+\mu' v'} n = \mu \pounds_v n + \mu' \pounds_{v'} n = -\left(\alpha\mu - \alpha'\mu'\right) n \tag{1.1.40}$$

$$\pounds_{[v,v']} n = [\pounds_v, \pounds_{v'}] n = \pounds_v\left(-\alpha' n\right) - \pounds_{v'}\left(-\alpha n\right) = -\left(\pounds_v \alpha' - \pounds_{v'}\alpha\right) n \equiv -\bar{\alpha} n, \tag{1.1.41}$$

where $\pounds_n \bar{\alpha} = 0$ since $[n, v] = 0 = [n, v']$. The Lie algebra $\mathfrak{g}$ admits, in particular, vectors v in the form $v = \beta n$, where β is a function on $\mathscr{I}$. Indeed, using (1.1.39), together with $q_{\mu\nu} n^\nu = 0$, and $\nabla_\mu n_\nu = 0$ on $\mathscr{I}$, we see that

$$2\alpha q_{\mu\nu} = \left(\pounds_v q\right)_{\mu\nu} = \beta\left(\pounds_n q\right)_{\mu\nu} + n^\rho q_{\rho\mu}\partial_\nu \beta + n^\rho q_{\rho\nu}\partial_\mu\beta = \beta\nabla_{(\mu} n_{\nu)} = 0, \quad \text{on } \mathscr{I}; \tag{1.1.42}$$

therefore $\alpha = 0$, which in turn implies

$$0 = \pounds_v n = -\pounds_n v = -\pounds_n\left(\beta n\right) = -(\pounds_n\beta) n \implies \pounds_n \beta = 0. \tag{1.1.43}$$

We define the space of *supertranslations* as the set $\mathfrak{st}$ of all vector fields in $\mathfrak{g}$ of the form βn, such that $\pounds_n\beta = 0$. $\mathfrak{st}$ is clearly a vector subspace of $\mathfrak{g}$; moreover, we see that, given *any* infinitesimal asymptotic symmetry $v \in \mathfrak{g}$,

$$[v, \beta n] = \pounds_v(\beta n) = \left(\pounds_v\beta\right) n - \alpha\beta n = \left(\pounds_v\beta - \alpha\beta\right) n \equiv \beta' n; \tag{1.1.44}$$

since $\mathfrak{g}$ is closed under Lie bracket, $[v, \beta n]$ is in $\mathfrak{g}$, and hence $[v, \beta n] \in \mathfrak{st}$. This shows that $\mathfrak{st}$ is in fact a Lie ideal of $\mathfrak{g}$.

Consider now the quotient algebra $\mathfrak{g}/\mathfrak{st}$. By (1.1.38), its natural projection $\pi : \mathscr{I} \to \mathcal{S}$, *i.e.* the quotient map, projects down on the space $\mathcal{S}$ of null generators of $\mathscr{I}$ and vector fields v, v' on $\mathscr{I}$ are projected in the same vector field $\pi_* v = \pi_* v'$ on $\mathcal{S}$ if and only if they differ by a supertranslation, whereas supertranslations themselves project down to zero. Furthermore, if $v \in \mathfrak{g}$, then $\pi_* v$ is a conformal Killing vector field on $\mathcal{S}$, with positive definite metric h, as a consequence of (1.1.39). It follows that $\mathfrak{g}/\mathfrak{st}$ is the Lie algebra of conformal Killing vector fields on $(\mathcal{S}, h)$ and since the conformal structure of a 2-sphere is unique, this Lie algebra is unique: it is the Lie algebra of the Lorentz group $SO(3,1)$.

We already note here that, in fact, since the conformal group of the $(D-2)$-dimensional sphere is isomorphic to the Lorentz group $SO(D-1,1)$, this part of the construction carries through in any dimension [38]; we will discuss a few developments of this observation in Chapter 7.

To move to the group level, denote by G the BMS group and by ST the group of supertranslations. Note that, since $\mathfrak{st}$ is a commutative Lie ideal of $\mathfrak{b}$, *i.e.* if $\pounds_n\beta = 0 = \pounds_n\beta'$, $v \in \mathfrak{b}$, then

$$[\beta n, \beta' n] = 0, \qquad [v, \beta n] \in \mathfrak{st}, \tag{1.1.45}$$

it follows that $\mathfrak{st}$ exponentiates to a commutative normal subgroup ST of the BMS group G. Thus, aside from the technicalities arising form the infinite dimensionality of G, this proves that, as anticipated, the BMS group is the semi-direct product of the *Lorentz* group with the infinite-dimensional, Abelian, normal subgroup ST of supertranslations. It is worth keeping in mind that ST is isomorphic to the additive group of functions on the two-sphere while the Lorentz groups acts on ST as the conformal group on the two-sphere.

Explicitly, from the above discussion it follows that a mapping $(u, \theta, \phi) \mapsto (\bar{u}, \bar{\theta}, \bar{\phi})$ is a BMS transformation if and only if letting

$$\bar{\theta} = H(\theta, \phi), \quad \bar{\phi} = I(\theta, \phi), \quad \bar{u} = K^{-1} \left[u + \alpha(\theta, \phi) \right], \tag{1.1.46}$$

one finds

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2 = K^2(\bar{\theta}, \bar{\phi}) \left(d\bar{\theta}^2 + \sin^2 \bar{\theta} d\bar{\phi}^2 \right) = K^2(\bar{\theta}, \bar{\phi}) d\bar{s}^2, \tag{1.1.47}$$

where α is a smooth angular function parametrizing supertranslations by

$$\bar{u} = u + \alpha(\theta, \phi), \tag{1.1.48}$$

whereas H and I are the expressions of a conformal transformation of conformal factor K, *i.e.*

$$\bar{\theta} = H(\theta, \phi), \quad \bar{\phi} = I(\theta, \phi). \tag{1.1.49}$$

Note that the conformal factor $K^{-1} = \omega$ appears in the transformation law of u as well, according to $n = g^{\mu\nu} \nabla_\nu$ and hence $\partial_u \mapsto \omega^{-1} \partial_u$ [41]. Historically, this was one of the first ways of defining the BMS group, as done by Sachs in [5]. The supertranslations for which

$$\alpha = \epsilon_0 + \epsilon_1 \sin\theta \cos\phi + \epsilon_2 \sin\theta \sin\phi + \epsilon_3 \cos\theta \tag{1.1.50}$$

form the translation subgroup T: this identification is apparent if one recalls the definition $u = t - r$, and the transformation laws $x^0 = t$, $x^1 = r \sin\theta \cos\phi$, $x^2 = r \sin\theta \sin\phi$, $x^3 = r \cos\theta$.

Now, any translation commutes with any supertranslation and, as can be seen from the Lorentz group, the commutator of an infinitesimal translation with an infinitesimal conformal transformation is again a translation, which proves that T is a 4-dimensional normal subgroup of the BMS group G. In [5], Sachs also proved that this is also the only possible 4-dimensional normal subgroup: the strategy consists in proving that any 4-dimensional normal subgroup of G must be contained in the supertranslation group and that assuming that T is not unique leads to a contradiction with the fact that conformal transformation "mix up" all supertranslations, which are not translations, with each other.

Rotations and boosts, instead, are not unique: consider the subgroup L of conformal transformations and let t be any finite supertranslation, then $M = tLt^{-1}$ is a new subgroup of G distinct from L and again isomorphic to the Lorentz group.

1.2 Energy

In special relativity, given the stress-energy tensor $T_{\mu\nu}$, of a classical field, the total energy is defined in terms of a time-translation Killing field t^μ on a spacelike Cauchy hypersurface[3] Σ as follows

$$E = \int_\Sigma T_{\mu\nu} n^\mu t^\nu dA, \tag{1.2.1}$$

where n^μ denotes the unit normal to Σ. The condition $\partial_\mu T^{\mu\nu} = 0$ ensures that the total energy is conserved, *i.e.* that it is independent of the choice of Σ, by Stokes' theorem.

In general relativity the energy properties of matter are, again, represented by a stress-energy tensor $T_{\mu\nu}$: it represents the local energy density as measured by a local observer. Because of general covariance this tensor satisfies $\nabla_\mu T^{\mu\nu} = 0$, which may be interpreted as expressing a local conservation law, but does not in general lead to a *global* conservation law. This apparent trouble may be actually understood on general grounds, since $T_{\mu\nu}$ represents only the energy content of matter and ignores the contribution of "gravitational field energy".

However, there is no known meaningful notion of energy density of the gravitational field in general relativity. In order to define such a quantity one usually has to abandon manifest covariance: for instance introducing a preferred coordinate system, or performing a decomposition into a background metric and dynamical metric, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

A notion of *total* energy on an *isolated* system, however, does exist: below, we will define an energy-momentum 4-vector in the case of asymptotically flat spacetimes, following the discussion in [37, Chapter 11].

In special relativity, a particle is assigned an energy momentum 4-vector P^μ; the energy of the particle is taken to be the time component of this vector, or, more covariantly, the projection $E = -P_\mu \xi^\mu$ of P^μ with respect to a time-translation Killing vector field ξ. The mass of the particle is given by $M = (-P_\mu P^\mu)^{1/2}$, so that if the particle is "at rest" with respect to ξ, *i.e.* it follows an integral curve of the Killing field, we have $E = M$.

In the Newtonian theory, the Newtonian potential φ satisfies Poisson's equation $\Delta\varphi = 4\pi\rho$ and is linked to the total mass of the system by the "Gauss's Law" formula

$$M = \frac{1}{4\pi} \oint_S \vec{\nabla}\varphi \cdot \widehat{N} dA, \tag{1.2.2}$$

where S is a topological 2-sphere which encloses all the sources and $\widehat{N}$ is the unit outward normal to S. M is independent of the choice of S since, outside the mass distribution, φ satisfies Laplace's equation $\Delta\varphi = 0$.

Note that $\vec{\nabla}\varphi$ is the force that must be exerted on a unit test mass to hold it fixed in the gravitational field generated by ρ, so, by eq. (1.2.2), $4\pi M$ is just the outward force that

[3] Intuitively, a Cauchy surface Σ is a space-like surface such that any point in the spacetime M can be influenced by or can influence points on Σ. See [37, Chapter VIII] for a more rigorous definition.

must be applied to hold in place test matter with unit surface mass density distributed over S.

1.2.1 Komar mass

The previous discussion motivates the following procedure aimed at obtaining a globally meaningful notion of energy. Let us consider a static, asymptotically flat spacetime, which is by our definition vacuum in a neighborhood of infinity (even though, as we have seen, one could allow for the presence of energy and momentum near infinity with suitable falloff conditions), and whose timelike Killing vector field ξ can be normalized so that the "redshift factor", $V = (-\xi_\mu \xi^\mu)^{1/2}$, approaches 1 at infinity. Let us also define a static observer as an observer following an orbit of the Killing vector field ξ, *i.e.* whose four-velocity reads $u^\alpha = \xi^\alpha/(-\xi_\mu\xi^\mu)^{1/2} = \xi^\alpha/V$; since the spacetime is static, there exists a (spacelike) hypersurface orthogonal to the integral curves of ξ and therefore, choosing local coordinates on this surface, static observers indeed "stay" at fixed spatial coordinates according to this definition. The acceleration of such an orbit is

$$a^\mu = u^\rho \nabla_\rho u^\mu = (\xi^\rho/V)\nabla_\rho(\xi^\mu/V) = \frac{1}{V^2}\xi^\rho\nabla_\rho\xi^\mu = -\frac{1}{V^2}\xi^\rho\nabla^\mu\xi_\rho = \nabla^\mu(\log V), \tag{1.2.3}$$

where we used

$$\xi^\rho\nabla_\rho V^2 = -\xi^\rho\nabla_\rho(\xi^\mu\xi_\mu) = -2\xi^\rho\xi^\mu\nabla_\rho\xi_\mu = -\xi^\rho\xi^\mu\nabla_{(\rho}\xi_{\mu)} = 0, \tag{1.2.4}$$

in the third equality and the Killing equation itself in the next to last equality.

Thus, the *local* force which must be exerted on a unit ($m = 1$) test mass to be held in place by a static observer is given by eq. (1.2.3):

$$F_{\text{loc}} = (-a^\mu a_\mu)^{1/2} = \frac{1}{V}(\nabla_\mu V\nabla^\mu V)^{1/2}. \tag{1.2.5}$$

The force which must be applied by a *distant* observer at infinity is computed as follows. Suppose that the particle is held fixed by a long massless string, with the other end of the string held by a stationary observer at a large distance. The energy "as measured at infinity" of a particle of unit mass moving with 4-velocity u^α by an observer who moves along the vector field ξ is given by $E = -\xi_\mu u^\mu$. So, in our case:

$$E_\infty = -\xi_\alpha\xi^\alpha/(-\xi_\mu\xi^\mu)^{1/2} = (-\xi_\mu\xi^\mu)^{1/2} = V. \tag{1.2.6}$$

The force exerted by the string on the particle is given by eq. (1.2.5), whereas the force applied by the observer on the string is therefore

$$(F_\infty)_\mu = -\nabla_\mu E_\infty = -\nabla_\mu V \implies F_\infty = (\nabla_\mu V\nabla^\mu V)^{1/2} \tag{1.2.7}$$

thus

$$F_\infty = VF_{\rm loc}. \tag{1.2.8}$$

The force at infinity differs from the local force by a redshift factor V. Now, consider a topological 2-sphere S having unit outward pointing normal $\widehat{n}$ and lying in the hypersurface orthogonal to ξ. The quantity

$$F = \oint_S (\xi^\mu/V)\nabla_\mu \xi_\nu \widehat{n}^\nu dA \tag{1.2.9}$$

is the total outward force that must be exerted by a distant observer to keep in place a unit surface mass density distributed over S. Since ξ^μ/V and $\widehat{n}^\nu$ are both normal vectors to S, we can write this surface integral as the integral of a two form over S

$$F = \oint_S \epsilon_{\alpha\beta\mu\nu}\nabla^\mu \xi^\nu dx^\alpha \wedge dx^\beta. \tag{1.2.10}$$

The integrand satisfies

$$\begin{aligned} \epsilon^{\rho\sigma\alpha\beta}\nabla_\sigma\left(\epsilon_{\alpha\beta\mu\nu}\nabla^\mu\xi^\nu\right) &= \epsilon^{\rho\sigma\alpha\beta}\epsilon_{\alpha\beta\mu\nu}\nabla_\sigma\nabla^\mu\xi^\nu \\ &= -2\nabla_\sigma(\nabla^\rho\xi^\sigma - \nabla^\sigma\xi^\rho) \\ &= 4\nabla_\sigma\nabla^\sigma\xi^\rho \\ &= -4R^\rho{}_\sigma\xi^\sigma, \end{aligned} \tag{1.2.11}$$

where we have used the identity $\epsilon^{\mu\nu\alpha\beta}\epsilon_{\alpha\beta\rho\sigma} = -2(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho)$, the Killing equation and its corollary $\Box\xi^\mu + R^\mu{}_\nu\xi^\nu = 0$. Hence, using the identity $\epsilon^{\rho\alpha\beta\gamma}\epsilon_{\rho\alpha'\beta'\gamma'} = -\delta^\alpha{}_{[\alpha'}\delta^\beta{}_{\beta'}\delta^\gamma{}_{\gamma']}$ we find

$$\begin{aligned} \epsilon_{\rho\gamma\delta\tau}\epsilon^{\rho\sigma\alpha\beta}\nabla_\sigma\left(\epsilon_{\alpha\beta\mu\nu}\nabla^\mu\xi^\nu\right) &= -4R^\rho{}_\sigma\xi^\sigma\epsilon_{\rho\gamma\delta\tau}, \\ \frac{1}{3!}\nabla_{[\gamma}(\epsilon_{\delta\tau]\mu\nu}\nabla^\mu\xi^\nu) &= \frac{2}{3}R^\rho{}_\sigma\xi^\sigma\epsilon_{\rho\gamma\delta\tau}. \end{aligned} \tag{1.2.12}$$

Hence, the 2-form $\alpha_{\rho\sigma} = \epsilon_{\rho\sigma\mu\nu}\nabla^\mu\xi^\nu$ is closed in a vacuum region:

$$d\alpha = 0, \text{ where } R_{\mu\nu} = 0. \tag{1.2.13}$$

We define the total mass of a static, asymptotically flat spacetime as

$$M = -\frac{1}{8\pi}\oint_S \epsilon_{\rho\sigma\mu\nu}\nabla^\mu\xi^\nu dx^\rho \wedge dx^\sigma. \tag{1.2.14}$$

More generally, since the key property which makes this quantity, known as the *Komar mass*, independent of the choice of the surface S is the fact that ξ is a timelike Killing field, we can adopt this expression also to define the notion of total mass in any *stationary*, and not necessarily static, asymptotically flat spacetime.

If S is the boundary of a spacelike hypersurface Σ such that $\Sigma \cup S$ is a compact manifold with boundary, then by Stokes' theorem[4]

$$\begin{aligned} M =& -\frac{1}{16\pi}\int_\Sigma \nabla_{[\gamma}(\epsilon_{\delta\tau]\mu\nu}\nabla^\mu\xi^\nu)dx^\gamma \wedge dx^\delta \wedge dx^\tau \\ =& -\frac{1}{4\pi}\int_\Sigma R^\mu{}_\nu \xi^\nu \epsilon_{\mu\alpha\beta\gamma} dx^\alpha \wedge dx^\beta \wedge dx^\gamma \\ =& \frac{1}{4\pi}\int_\Sigma R_{\mu\nu}\xi^\nu n^\mu dV, \end{aligned} \tag{1.2.16}$$

where n is the unit future-pointing normal on Σ, so that $\epsilon_{\alpha\beta\gamma} = n^\rho \epsilon_{\rho\alpha\beta\gamma}$; finally, using Einstein's equation,

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu} \implies \quad R = -8\pi T' \\ R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}T' g_{\mu\nu}\right), \end{aligned} \tag{1.2.17}$$

so one can relate the Komar mass to the energy-momentum tensor as follows

$$M = 2\int_\Gamma \left(T_{\mu\nu} - \frac{1}{2}T' g_{\mu\nu}\right) n^\mu \xi^\nu dV. \tag{1.2.18}$$

This formula allows to check whether constants and signs have been chosen correctly in the definition of the Komar mass: in the Newtonian limit the energy-momentum tensor will have $T_{00} = \rho$ as its only nonzero component and $g_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$ hence $M = \int \rho \, dV$, as desired.

1.2.2 Bondi mass

To extend the above definition to a general (even non-stationary) asymptotically flat spacetime, we will follow this strategy: first we will look for a notion of energy-momentum at a fixed "retarded" time, by selecting the behavior of the spacetime at null infinity on an asymptotically null surface, which we will again denote by Σ. Our goal is to quantify the energy carried away by the gravitational radiation, keeping in mind that only the asymptotic properties of Σ should count. Then, in order to single out a preferred time direction, we will specify an asymptotic notion of time translation using the unique four-parameter subgroup of translations provided by the BMS group.

[4]

$$\oint_{\partial\Sigma} \alpha = \int_\Sigma d\alpha. \tag{1.2.15}$$

Let $\mathscr{S}$ be a given cross section of $\mathscr{I}^+$ (or $\mathscr{I}^-$): we are looking for a way of associating $\mathscr{S}$ to a suitable linear map from the four-dimensional vector space of BMS translations into $\mathbb{R}$. The value of this map applied to a given, say, time translation would then define the notion of energy associated with this time direction at the retarded time defined by $\mathscr{S}$. To this purpose, let ξ be the generator of an asymptotic time translation symmetry. We define the *energy associated with* ξ by means of a one-parameter family of spheres $\{S_a\}$ which approaches $\mathscr{S}$ on $\mathscr{I}^+$ in the unphysical spacetime, according to the following formula:

$$E = -\lim_{S_a \to \mathscr{S}} \frac{1}{8\pi} \oint_{S_a} \epsilon_{\mu\nu\alpha\beta} \nabla^\mu \xi^\nu dx^\alpha \wedge dx^\beta. \tag{1.2.19}$$

It turns out that this limit always exists and is independent of the details of how S_a approaches $\mathscr{S}$. Indeed this limit exists whenever ξ is an arbitrary asymptotic symmetry, so that this definition also works for asymptotic spatial translations defining spatial momentum.

However, E is not invariant under a change of the choice of representative ξ in the equivalence class associated with the given BMS time translation. In order to achieve independence of the choice of representative, and thereby give unambiguous meaning to the notion of energy, one needs to impose the following gauge condition [42]:

$$\nabla_\mu \xi^\mu = 0, \tag{1.2.20}$$

in a neighborhood of $\mathscr{I}^+$. Note that this requirement is always satisfied if ξ is a Killing vector field, being the trace of the Killing equation, but that it does not hold in general for vector fields representing infinitesimal asymptotic symmetries. Similarly the four-momentum vector P_μ is defined by the action of the above linear functional on arbitrary BMS translations; we will return to this point during our computations in linearized gravity in Chapter 5.

The expression (1.2.19) of the energy of an isolated system agrees with the one given by Bondi, van der Burg, Metzner [3, 4, 5] in coordinate form, prior to the geometric formulation of the notion of asymptotic flatness, and is called the *Bondi mass*. An important result is the following: given a cross section, $\mathscr{S}_1$, and a "later"[5] one, $\mathscr{S}_2$, the energy difference between them is expressed as the integral of a function f over the region τ of $\mathscr{I}^+$ bounded by the two cross sections

$$E[\mathscr{S}_2] - E[\mathscr{S}_1] = -\int_\tau f\, dV, \tag{1.2.21}$$

where f is interpreted as the flux of energy carried away to infinity by the gravitational radiation. It can be shown that f is non-negative, *i.e.* that gravitational radiation always carries positive energy away from a radiating system [42, 39].

[5] "Later" in terms of the retarded time parametrizing the space of cross sections of $\mathscr{I}^+$.

When approaching spatial infinity, the formulas for energy and momentum become those given by Arnowitt, Deser, Misner (ADM), which in asymptotically Euclidean coordinates on a spacelike surface Σ which approaches i^0 smoothly enough, read

$$\begin{aligned} E &= \frac{1}{16\pi} \lim_{r\to\infty} \sum_{i,j=1}^{3} \int_\Sigma \left(\partial_i h_{ij} - \partial_j h_{ii}\right) n_j dA \\ P_j &= \frac{1}{8\pi} \lim_{r\to\infty} \sum_{i=1}^{3} \int_\Sigma \left(K_{ij} n_i - K_{ii} n_j\right) dA \end{aligned} \tag{1.2.22}$$

where h_{ij} is the metric fluctuation over the Minkowski background while K_{ij} is the spatial stress-energy tensor. Both these quantities are independent of the choice of asymptotically Euclidean coordinates. The ADM energy can be interpreted as the total energy available in the spacetime, whereas the Bondi energy can be thought of as the energy remaining in the spacetime at the retarded time given by the cross section $\mathscr{S}$ after the emission of gravitational radiation.

The comparison between these two quantities is made possible by the fact that each BMS translation at $\mathscr{I}^+$ (or $\mathscr{I}^-$) can be naturally associated with a tangent vector at i^0; in this way, it can be seen that the ADM energy and the Bondi energy differ precisely by the integral of the energy flux, f, over the portion of $\mathscr{I}^+$ to the causal past of $\mathscr{S}$.

For completeness we list here some relevant positivity results:

- The Bondi energy flux is always positive, as stated above [42, 39];
- The ADM energy is always positive in a nonsingular, asymptotically flat spacetime with locally nonnegative matter energy density (see [43] for the very elegant proof due to Witten);
- The Bondi energy is always positive under the same conditions, or in other words the total energy radiated away by a system is bounded by its total energy content [44].

Another way to obtain the above expression for the Bondi mass is to compute the Noether current of the Einstein-Hilbert action corresponding to the invariance under the infinitesimal diffeomorphism generated by the vector field ξ^μ: this current turns out to be the divergence of an antisymmetric tensor, in accordance with Noether's second theorem, and hence, the associated conserved charge can be computed as a boundary integral [16]. Indeed a generic variation of

$$S = \int d^D x \sqrt{-g} R \tag{1.2.23}$$

reads

$$\delta S = \int d^D x \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) \delta g^{\mu\nu} + \int d^D x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu}; \tag{1.2.24}$$

the first term gives the vacuum Einstein equations and vanishes on-shell, whereas the second term can be recast in the following form using the Palatini identity

$$\delta S = \int d^D x \sqrt{-g} \left(\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box \right) \delta g^{\mu\nu}, \tag{1.2.25}$$

where $\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$; the infinitesimal diffeomorphism generated by the vector ξ^μ induces $\delta_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$ so

$$\delta_\xi S = \int d^D x \sqrt{-g} \left(\nabla_\mu \nabla_\nu \nabla^\mu \xi^\nu + \nabla_\mu \Box \xi^\mu - 2 \Box \nabla_\mu \xi^\mu \right), \tag{1.2.26}$$

and finally using that $\Box \nabla_\mu \xi^\mu = \nabla_\alpha \nabla_\mu \nabla^\alpha \xi^\mu$ because $R_{\mu\alpha} = 0$ yields

$$\delta_\xi S = \int d^D x \sqrt{-g} \nabla_\mu \nabla_\nu \left(\nabla^\nu \xi^\mu - \nabla^\mu \xi^\nu \right) = -\frac{1}{2} \int d^D x \sqrt{-g} \epsilon^{\mu\nu\alpha\beta} \nabla_\mu \nabla_\nu \left(\epsilon_{\alpha\beta\rho\sigma} \nabla^\rho \xi^\sigma \right), \tag{1.2.27}$$

which is recast as the integral of the two form $\kappa^{\mu\nu} = \nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu$ on a surface of codimension 2, as done above in (1.2.14), and defines a conserved quantity associated with diffeomorphism invariance. Again, considering absence of matter (or an appropriate falloff of its energy-momentum tensor) is not restrictive, since this is always the case in asymptotically flat space. This quick procedure, stemming from Noether's theorem, allows to recover the Komar and Bondi masses from a field-theoretical point of view, more akin to the general language of gauge theories.

1.3 Radiative Modes

While the intrinsic metric $q_{\mu\nu}$ on $\mathscr{I}$ and the null normal n^μ to $\mathscr{I}$ describe a structure which is independent of the asymptotically flat spacetime under discussion, and may therefore be regarded as the "leading order" fields, we have yet to exhibit the dynamical, or "higher order", objects which contain the relevant dynamical information typical of a particular spacetime. We shall accomplish the task in this section, following [21].

Recall that

$$q_{\mu\nu} \approx \underleftarrow{g}_{\mu\nu} \text{ and } n^\mu \approx g^{\mu\nu} \nabla_\nu \Omega, \tag{1.3.1}$$

where $\underleftarrow{g}$ denotes the pullback of g to $\mathscr{I}$ via the embedding map and, from now on, "$\approx$" will stand for "equals, at points of $\mathscr{I}$, to".

The connection D defined intrinsically on $\mathscr{I}$, *i.e.* the second order structure on $\mathscr{I}$, is given by

$$D \underleftarrow{V} \approx \underleftarrow{\nabla V} \tag{1.3.2}$$

for any one-form V in a neighborhood of $\mathscr{I}$. Since $\Omega \approx 0$, $\underleftarrow{n}_\mu \approx 0$ and $\nabla_\mu n_\nu \approx 0$, it follows that, if for any pair of one-forms V and A one has $\underleftarrow{V} = \underleftarrow{A}$, then $D_\mu \underleftarrow{V} = D_\mu \underleftarrow{A}$.

In other words, D_μ is a well-defined covariant derivative on covectors at $\mathscr{I}$. D can be extended to any kind of tensor on $\mathscr{I}$ by exploiting linearity, Leibnitz rule, equality with partial derivative on functions and vanishing torsion. For example, if ξ is a vector field on $\mathscr{I}$ and A is a one form on $\mathscr{I}$, since the action of D_μ on the function $A(\xi)$ must be that of the usual partial derivative, then $D_\mu\xi$ is specified via the Leibnitz rule

$$\partial_\mu(A(\xi)) = D_\mu(A(\xi)) = (D_\mu A)(\xi) + A(D_\mu \xi). \tag{1.3.3}$$

The connection D has the following properties:

$$D_\mu q_{\alpha\beta} \approx 0, \qquad D_\mu n^\nu \approx 0 \tag{1.3.4}$$

which follow from $\nabla_\mu g_{\alpha\beta} = 0$ and from $n^\mu q_{\mu\nu} = 0$ respectively, and

$$D_\mu V_\nu \approx \frac{1}{2} D_{[\mu} V_{\nu]} + \frac{1}{2}\pounds_V q_{\mu\nu} \text{ if } V_\mu n^\mu = 0, \tag{1.3.5}$$

where the vector field V^μ used to compute the Lie derivative is defined by $V^\mu = q^{\mu\nu}V_\nu$, $q^{\mu\nu}$ being a generalized inverse of $q_{\mu\nu}$ in the sense that

$$q_{\mu\alpha} q^{\alpha\beta} q_{\nu\beta} = q_{\mu\nu}. \tag{1.3.6}$$

Note that $q^{\mu\nu}$ is not unique: since $q_{\alpha\beta}n^\beta = 0$, one can add to it terms of the type $n^{(\mu}w^{\nu)}$, where w^μ is any vector field on $\mathscr{I}$, but since V_μ satisfies $n^\mu V_\mu = 0$, then $(q^{\alpha\beta} + n^{(\alpha}w^{\beta)})V_\beta = q^{\alpha\beta}V_\beta + n^\alpha(w^\beta V_\beta)$, so that $\pounds_V q_{\mu\nu}$ is defined unambiguously,

$$\pounds_{(w^\beta V_\beta)n} q = w^\beta V_\beta \pounds_n q = 0. \tag{1.3.7}$$

Equations (1.3.4) imply that the induced connection D_μ on $\mathscr{I}$ is compatible with the "kinematical structure" thereon, whereas (1.3.5) states that the action of D_μ is the same on every one-form orthogonal to n^μ, and hence that its action on a one-form L_μ such that $L_\mu n^\mu = 1$ determines the connection completely.

The third order structure is obtained by pulling back to $\mathscr{I}$ the curvature tensor of g. It can be proved that the Weyl tensor $C_{\mu\nu\rho\sigma}$ of $g_{\mu\nu}$ vanishes on $\mathscr{I}$ for conformal boundaries of 4-dimensional spacetimes [39, Theorem 11]. Thus, it makes sense to consider the Ricci tensor alone: the most convenient approach is, in fact, to consider the restriction to $\mathscr{I}$ of the combination

$$S_\mu{}^\nu \equiv R_\mu{}^\nu - \frac{1}{6} R \delta_\mu{}^\nu, \tag{1.3.8}$$

in $D = 4$. The pullback of $S_{\mu\nu}$ to $\mathscr{I}$,

$$s_{\mu\nu} = \underleftarrow{S}_{\mu\nu}, \tag{1.3.9}$$

contains information about the flux of gravitational radiation across $\mathscr{I}$; however, to extract this information, one needs to remove from $s_{\mu\nu}$ a certain piece which is "pure gauge". To achieve this purpose one can use the unique tensor field $\rho_{\alpha\beta}$ on $\mathscr{I}$ given by [39, Theorem 5], which satisfies:

$$\begin{aligned} (i)\rho_{\alpha\beta} &= \rho_{\beta\alpha}, \qquad && (ii)\rho_{\mu\nu}n^{\nu} = 0, \\ (iii)\rho_{\mu\nu}q^{\mu\nu} &= \kappa, \qquad && (iv)D_{[\mu}\rho_{\alpha]\beta} = 0, \end{aligned} \tag{1.3.10}$$

where κ is the pullback to $\mathscr{I}$ of the scalar curvature of the metric h on the 2-sphere $\mathcal{S}$ of the generators of I. For example, if the conformal factor is so chosen that $h_{\mu\nu}$ is the Euclidean unit 2-sphere metric, $\rho_{\mu\nu}$ turns out to be the equal to $h_{\mu\nu}$; however, under conformal rescalings, $\rho_{\mu\nu}$ has a complicated behavior.

Define then

$$N_{\mu\nu} = s_{\mu\nu} - \rho_{\mu\nu}. \tag{1.3.11}$$

This is called the *Bondi news* tensor and is a crucial quantity in our analysis, since it is closely related to the notion of energy flux at infinity. It can be shown that

$$N_{\mu\nu}n^{\nu} = 0, \qquad N_{\mu\nu}q^{\mu\nu} = 0. \tag{1.3.12}$$

Instead of proving these propertes, we shall note that they are nicely illustrated by the conformal completion of Minkowski's spacetime in terms of a region of the Einstein static universe,[6] given above. Clearly, since this conformal completion is spatially maximally symmetric, we have (in $D = 4$) $R = 6$, whereas the intrinsic curvature of the space of generators is the usual unit sphere curvature $\kappa = 2$; in fact, also the whole Ricci tensor is not difficult to work out explicitly:[7] it must be given by

$$R_{\mu\nu} = \frac{R}{d}g_{\mu\nu} = 2g_{\mu\nu} \tag{1.3.14}$$

since $d = D - 1 = 3$ is the dimensionality of the maximally symmetric curved space, so $(i^*R)_{\mu\nu}q^{\mu\nu} = 2q_{\mu\nu}q^{\mu\nu} = 4$; finally

$$(i^*R)_{\mu\nu}q^{\mu\nu} - \frac{R}{3} - \kappa = 4 - 2 - 2 = 0. \tag{1.3.15}$$

---

[6] Note that this conformal completion, unlike all the others we are considering, is not divergence-free. This does not affect the argument, since the result is independent of the conformal gauge we adopt.

[7] The nonvanishing Christoffel symbols read:

$$\begin{aligned} \Gamma^R_{\theta\theta} &= -\sin R\cos R, \qquad && \Gamma^R_{\phi\phi} = -\sin R\cos R\sin^2\theta, \\ \Gamma^\theta_{R\theta} &= \frac{\cos R}{\sin R}, \qquad && \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \\ \Gamma^\phi_{\phi R} &= \frac{\cos R}{\sin R}, \qquad && \Gamma^\phi_{\phi\theta} = \frac{\cos\theta}{\sin\theta}. \end{aligned} \tag{1.3.13}$$

The nonvanishing components of the covariant Riemann tensor are $R_{R\theta R\theta} = \sin^2 R$, $R_{\theta\phi\theta\phi} = \sin^2 R\sin^2\theta$, $R_{\theta\phi\theta\phi} = \sin^4 R\sin^2\theta$. Finally, the diagonal elements of the Ricci tensor are $R_{RR} = 2$, $R_{\theta\theta} = 2\sin^2 R$, $R_{\phi\phi} = 2\sin^2 R\sin^2\theta$. R=6.

The Bondi news tensor is *gauge invariant*, since it can be shown that the conformal transformations of $s_{\mu\nu}$ and $\rho_{\mu\nu}$ cancel out, and its square

$$f \equiv q^{\alpha\beta} N_{\alpha\nu} N_{\beta\mu} q^{\mu\nu} \tag{1.3.16}$$

defines the local flux of energy density carried away by gravitational radiation [45, 46], as we mentioned below (1.2.21), when discussing the positivity of the Bondi mass in Section 1.2.2.

1.3.1 Properties and physical degrees of freedom

A relevant result, due to Ashtekar [21], is that it is also possible to repeat the whole construction adopting a more abstract point of view, without making any reference to any particular spacetime and using only notions defined at $\mathscr{I}$. This is a conceptually more economic procedure, that allows to single out the relevant quantities which are needed for a complete definition of asymptotic gravitation.

Even without going through the details of the construction, it is worthwhile to note that, from the point of view of an abstract $\mathscr{I}$, which is just a mathematically refined way to say "from the perspective of an observer at infinity in an asymptotically flat spacetime", one cannot distinguish between two conformal factors which agree on $\mathscr{I}$ itself. This observation leads to the following consequences. Let us consider the transformation law of the intrinsic connection D under the permitted conformal rescalings (*i.e.* those satisfying $\pounds_n \omega = 0$):

$$D'_\alpha k_\beta \approx D_\alpha k_\beta - \frac{2}{\omega} k_{(\alpha} D_{\beta)} \omega + \frac{1}{\omega} (\omega^\mu k_\mu) q_{\alpha\beta}, \tag{1.3.17}$$

where $\omega^\mu \equiv g^{\mu\nu} \nabla_\nu \omega$; set $\omega = \text{const} = 1$ at $\mathscr{I}$, so that $D_\beta \omega \approx 0$ and $\omega^\mu \approx f n^\mu$, for some function f, and hence

$$D'_\alpha k_\beta \approx D_\alpha k_\beta + f n^\mu k_\mu q_{\alpha\beta}; \tag{1.3.18}$$

since conformal factors which are equal at $\mathscr{I}$ must correspond to the same intrinsic connection, we are forced to introduce an equivalence relation among connections:

$$D \sim \tilde{D} \iff (D_\mu - \tilde{D}_\mu) k_\nu = f n^\alpha k_\alpha q_{\mu\nu}. \tag{1.3.19}$$

It follows that the true basic dynamical variables, representing the radiative modes of the (exact, non-linear) gravitational field, are the equivalence classes $\{D\}$ of connections identified by (1.3.19) and therefore satisfying:

$$\{D'_\mu\} k_\nu = \{D_\mu\} k_\nu - \frac{2}{\omega} k_{(\mu} D_{\nu)} \omega. \tag{1.3.20}$$

Denote by $\mathcal{C}$ the collection of intrinsic connections D on $\mathscr{I}$, defined abstractly as the torsion-free connections D on $\mathscr{I}$ satisfying

$$D_\mu q_{\alpha\beta} = 0, \qquad D_\mu n^\nu = 0, \tag{1.3.21}$$

and denote by Γ the space of equivalence classes $\{D\}$ subject to the equivalence relation (1.3.19).

Both $\mathcal{C}$ and Γ are affine spaces: in virtue of (1.3.21), elements of $\mathcal{C}$ are related to each other via

$$(D_\mu - D'_\mu)k_\nu = \Sigma_{\mu\nu} n^\gamma k_\gamma \tag{1.3.22}$$

where $\Sigma_{\mu\nu} = \Sigma_{\nu\mu}$ and $\Sigma_{\mu\nu} n^\nu = 0$.

Indeed, $(D_\mu - D'_\mu)k_\nu = \Gamma^\gamma_{\mu\nu} k_\gamma$, for some $\Gamma^\gamma_{\mu\nu}$; from $D_\mu n^\nu = 0$ one gets $0 = \Gamma^\gamma_{\mu\nu} n_\gamma$, hence $\Gamma^\gamma_{\mu\nu} = \Sigma_{\mu\nu} n^\gamma$, for some $\Sigma_{\mu\nu}$; clearly $\Sigma_{\mu\nu} = \Sigma_{\nu\mu}$ for torsion-free connections; finally, from $l_\mu n^\mu = 1$, we have $0 = n^\mu(D_\nu - D'_\nu)l_\mu = \Gamma^\gamma_{\mu\nu} n^\mu l_\gamma = \Sigma_{\mu\nu} n^\nu$.

Since $\{D\} = \{D'\}$ if and only if $\Sigma_{\mu\nu} = f q_{\mu\nu}$ for some f, the difference $\{D\} - \{D'\}$ of elements of Γ is completely characterized by the traceless part of $\Sigma_{\mu\nu}$:

$$\sigma_{\mu\nu} \equiv \Sigma_{\mu\nu} - \frac{1}{2} q^{\rho\sigma} \Sigma_{\rho\sigma} q_{\mu\nu}. \tag{1.3.23}$$

These $\sigma_{\mu\nu}$'s can be used to endow Γ of a set of coordinates by fixing an element $\{D'\}$ and regarding it as the origin; since $\mathscr{I}$ is a three-dimensional manifold and $\sigma_{\mu\nu}$ is a traceless symmetric 3×3 tensor bound by the three constraints $\Sigma_{\mu\nu} n^\nu = 0$, we see that the number of independent components of $\sigma_{\mu\nu}$ is

$$\frac{3(3+1)}{2} - 3 - 1 = 2, \tag{1.3.24}$$

in agreement with the number of independent polarizations of the graviton. Γ is therefore a good canidadate as phase space of physical radiative modes at infinity in exact general relativity.

1.3.2 Symplectic geometry of radiative modes

One can introduce on Γ a symplectic structure [47, 48], *i.e.* a weakly nondegenerate[8] two-form Ω (of course, not to be confused with the conformal factor Ω): given any two tangent vectors $\sigma_{\mu\nu}$ and $\sigma'_{\mu\nu}$ (which is the same as saying coordinate vectors, since the space is affine) at a point $\{D\}$ of Γ,

$$\Omega_{\{D\}}(\sigma, \sigma') \equiv \frac{1}{8\pi} \int_{\mathscr{I}} \left(\sigma_{\mu\nu} \pounds_n \sigma'_{\alpha\beta} - \sigma'_{\mu\nu} \pounds_n \sigma_{\alpha\beta} \right) q^{\mu\beta} q^{\nu\alpha} d^3\mathscr{I}, \tag{1.3.25}$$

where $d^3\mathscr{I}$ is the natural volume element on $\mathscr{I}$. Ω is conformally invariant: under a conformal rescaling $(q_{\mu\nu}, n^\mu) \mapsto (\omega^2 q_{\mu\nu}, \omega^{-1} n^\mu)$, one has $q^{\mu\nu} \mapsto \omega^{-2} q^{\mu\nu}$ and $d^3\mathscr{I} \mapsto \omega^3 d^3\mathscr{I}$; furthermore, given any two elements D and D' in $\mathcal{C}$, they will be related by (1.3.22) for

[8] An antisymmetric, bilinear form $\Omega : V \times V \to \mathbb{R}$ is weakly nondegenerate if $\Omega(u, v) = 0$ for every $v \in V$ implies $u = 0$. In addition Ω admits an inverse, *i.e.* is strongly nondegenerate, if and only if the vector space V coincides with its double dual.

some $\Sigma_{\mu\nu}$ and their images $\tilde{D}$ and $\tilde{D}'$ under the conformal rescaling are therefore related by

$$\left(\tilde{D}_\mu - \tilde{D}'_\mu\right) k_\nu = \tilde{\Sigma}_{\mu\nu}\tilde{n}^\rho k_\rho = \Sigma_{\mu\nu} n^\rho k_\rho, \tag{1.3.26}$$

where (1.3.17) has been used, so that $\tilde{\Sigma}_{\mu\nu} = \omega\Sigma_{\mu\nu}$ and $\tilde{\sigma}_{\mu\nu} = \omega\sigma_{\mu\nu}$.

It can be shown that a BMS transformation, *i.e.* a one-parameter diffeomorphism $\psi^{(\lambda)}$ on $\mathscr{I}$ that preserves the universal structure, not only induces a natural isomorphisms from Γ to itself, indicating that Γ is indeed the right phase space of asymptotic radiative modes, but its action on Γ is also a symplectomorphism, *i.e.*

$$\Omega_{\{D\}}(\sigma, \sigma') = \Omega_{\{\psi_*^{(-\lambda)} D\}}(\psi_*^{(\lambda)}\sigma, \psi_*^{(\lambda)}\sigma'). \tag{1.3.27}$$

For simplicity, we restrict ourselves to infinitesimal BMS transformations satisfying (1.1.38) and (1.1.39) for $\alpha = 0$, namely $\pounds_\xi q = 0 = \pounds_\xi n$. The symplectic vector field $\xi_\psi \simeq \sigma_{\mu\nu}$ on Γ that generates the one-parameter family of symplectomorphisms $\psi^{(\lambda)}$ is given by the equation

$$\sigma_{\mu\nu}|_{\{D\}} = \left(\pounds_\xi D_\mu - D_\mu \pounds_\xi\right) l_\nu, \tag{1.3.28}$$

up to terms with non-vanishing trace (l_μ is any covector satisfying $l_\mu n^\mu = 0$).

It can be further shown that such a vector field is also Hamiltonian: the function

$$H_{(\xi)}(\{D\}) = -\frac{1}{2}\int_{\mathscr{I}} N_{\alpha\beta}\left(\pounds_\xi D_\mu - D_\mu \pounds_\xi\right) l_\nu q^{\alpha\mu} q^{\beta\nu} d^3\mathscr{I} \tag{1.3.29}$$

is its Hamilton function, which means that for any vector field X on Γ,

$$\pounds_X H_{(\xi)} = \Omega(\xi_\psi, X). \tag{1.3.30}$$

The result is independent on the particular choice of l_μ, on the conformal frame (q, n) and on the specific inverse $q^{\mu\nu}$ under consideration. Note that BMS supertranslations $\xi = \alpha n$, for $\pounds_n \alpha = 0$, automatically fall under the additional assumption $\pounds_\xi q = 0 = \pounds_\xi n$. In particular, for BMS translations, which are singled out by the relation $D_\mu D_\nu \alpha + \alpha\rho_{\mu\nu} = (\text{const.}) \times q_{\mu\nu}$ (see [39]), one has

$$H_{(\alpha n)}(\{D\}) = \frac{1}{32\pi}\int_{\mathscr{I}} \alpha N_{\mu\nu} N_{\alpha\beta} q^{\mu\alpha} q^{\nu\beta} d^3\mathscr{I}, \tag{1.3.31}$$

which is the Bondi four-momentum.

1.3.3 Classical vacua and Poincaré reduction of the BMS group

Borrowing some terminology from gauge theories, we shall call an element $\{D^0\}$ of Γ a *classical vacuum* if the corresponding field $N_{\mu\nu}$ vanishes identically. This definition is made reasonable by the fact that, at these points $\{D^0\}$, the corresponding Hamiltonians vanish for each BMS symmetry: in particular, in a classical vacuum, there is no flux of energy across $\mathscr{I}$. Let D^0 and D'^0 be two connections giving rise to classical vacua. It can be shown that they must be related by

$$(D_\mu^0 - D'^0_\mu)K_\nu = \Sigma_{\mu\nu} n^\rho K_\rho \tag{1.3.32}$$

where

$$\Sigma_{\mu\nu} = D_\mu D_\nu f + f\rho_{\mu\nu} \tag{1.3.33}$$

for some angular function f, *i.e.* $\pounds_n f = 0$. We immediately recognize that $\xi = fn$ defines an infinitesimal supertranslation; furthermore, it also defines a BMS translation if in addition $D_\mu D_\nu f + f\rho_{\mu\nu} = (\text{const.}) \times q_{\mu\nu}$, but this is just the equivalence relation (1.3.19) identifying different intrinsic connections as representative of the same physical configuration. We have obtained the following important result:

> Two, a priori distinct, classical vacua D^0 and D'^0 are always related by a *supertranslation* and define, in fact, the same classical vacuum if and only if they are related by a BMS *translation*.

The action of BMS supertranslations on Γ maps each classical vacuum into a classical vacuum, meaning that it leaves stable the affine space Γ^0 of classical vacua. Under the action of the translation group T, each classical vacuum is invariant, whereas under a general supertranslation it is mapped to a different classical vacuum. This means that the group ST/T acts simply[9] and transitively[10] on Γ^0.

There is another interesting result: for a fixed classical vacuum $\{D^0\}$, the subgroup of the BMS group G which leaves $\{D^0\}$ invariant (a sort of little group) is precisely a Poincaré subgroup of G; furthermore, ST/T acts simply and transitively also on the space S_P of Poincaré subgroups of G and Γ^0 is isomorphic to S_P in a natural way.

1.4 Asymptotic Quantization

In order to decide which classical observables are to be selected and promoted to quantum operators [49], we are led to look to the generators of canonical transformations that preserve the affine structure of Γ; in the infinitesimal form they correspond to constant vector fields on Γ, *i.e.* symmetric tensor fields $f_{\mu\nu}$ on $\mathscr{I}$ satisfying $f_{\mu\nu}n^\nu = 0$ and $f_{\mu\nu}q^{\mu\nu} = 0$. Let $f_{\mu\nu}$ be such a tensor and let it be rapidly decreasing in $u \in (-\infty, +\infty)$; we then define the smeared news observable by

$$N(f)_{\{D\}} = -\frac{1}{8\pi}\int_{\mathscr{I}} N_{\mu\nu} f_{\alpha\beta} q^{\mu\alpha} q^{\nu\beta} d^3\mathscr{I} \tag{1.4.1}$$

[9] That is, with trivial stabilizers.

[10] An action ψ of a group G on a manifold M is transitive if for any x, $y \in M$ there exists $g \in G$ such that $\psi_g x = y$.

which is nothing but the Hamiltonian for the vector field $f_{\mu\nu}$ on Γ. The Poisson brackets satisfy

$$\{N(f), N(g)\} = \Omega(f, g). \tag{1.4.2}$$

Now, we construct the quantum C^*-algebra[11] $\mathfrak{A}$ as follows: introduce the operator-valued distribution $N_{\mu\nu}$, the news operator, subject to the canonical commutation relations

$$[N(f), N(g)] = -i\hbar\Omega(f, g)\mathbb{1}, \tag{1.4.3}$$

where $N(f) = \int_{\mathscr{I}} N_{\mu\nu} f^{\mu\nu} d^3\mathscr{I}$ denotes the smeared out news operator. In analogy with electromagnetism, $N_{\mu\nu}$ corresponds to the field strength and $\{D\}$ plays the same role as the vector potential.

The algebra $\mathfrak{A}$ is the C^*-algebra generated by the smeared out news operators $N(f)$ with the above specified commutation relations.

1.4.1 The Fock representation

It is rather straightforward to construct the Fock representation of $\mathfrak{A}$. Given an $f_{\mu\nu}(u, \theta, \phi)$, rapidly decreasing in u, its Fourier transform $\tilde{f}_{\mu\nu}(\omega, \theta, \phi)$ with respect to u is again rapidly decreasing in ω.[12] The positive-frequency and negative-frequency parts of the test field $f_{\mu\nu}$ are given by

$$\begin{aligned} f^+(u, \theta, \phi) &= \frac{1}{2\pi} \int_0^{+\infty} \tilde{f}_{\mu\nu}(\omega, \theta, \phi) e^{+i\omega u} d\omega \\ f^-(u, \theta, \phi) &= \frac{1}{2\pi} \int_{-\infty}^{0} \tilde{f}_{\mu\nu}(\omega, \theta, \phi) e^{+i\omega u} d\omega = \frac{1}{2\pi} \int_0^{+\infty} \tilde{f}^*_{\mu\nu}(\omega, \theta, \phi) e^{-i\omega u} d\omega \\ &= \left(f^+(u, \theta, \phi)\right)^* \end{aligned} \tag{1.4.4}$$

and the operation

$$\begin{aligned} \langle f^+, g^+ \rangle &\equiv \frac{1}{i\hbar} \Omega(f^-, g^+) \\ &= \frac{1}{32\hbar\pi^3} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{+\infty} \omega \tilde{f}^*_{\mu\nu} \tilde{g}^{\mu\nu} d\omega \end{aligned} \tag{1.4.5}$$

[11] A C^*-algebra $\mathfrak{A}$ is a linear associative algebra over the field $\mathbb{C}$ where in addition a norm $||\cdot|| : \mathfrak{A} \to \mathbb{R}^+$ and an involution $* : \mathfrak{A} \to \mathfrak{A}$ are defined, which satisfy the following properties:

- the product is continuous $||AB|| \leqslant ||A||\,||B||$ and $\mathfrak{A}$ is complete with respect to the topology defined by the norm;
- $(A + B)^* = A^* + B^*$, $(\lambda A)^* = \bar{\lambda} A^*$, $(AB)^* = B^* A^*$ and $(A^*)^* = A$;
- $||A^* A|| = ||A||^2$ (called C^*-condition).

The algebraic approach to quantization is especially relevant when discussing different possible phases of a physical system and, hence, symmetry breaking.

[12] Since u has the interpretation of retarded time, it seems appropriate to keep the symbol ω for its corresponding angular frequency; no confusion should arise with the conformal factor ω.

defines a Hermitian inner product on the space of positive-frequency test fields. Denote by $\mathcal{H}$ the Cauchy completion of the so-obtained pre-Hilbert space. This is the one-particle space, or the one-graviton space of states. Denote by $\mathfrak{F}$ the symmetric Fock space based on $\mathcal{H}$: to each one-graviton state f^+ in $\mathcal{H}$ there corresponds a creation operator $C(f^+)$ and an annihilation operator $A(f^+)$ acting as ladder operators on $\mathfrak{F}$, *i.e.* increasing or decreasing the graviton occupation number by 1. These are densely defined (anti-)linear operators

$$A(f^+ + \lambda g^+) = A(f^+) + \lambda^* A(g^+) \qquad C(f^+ + \lambda g^+) = C(f^+) + \lambda C(g^+) \tag{1.4.6}$$

satisfying $C(f^+) = A(f^+)^*$,

$$[A(f^+), A(g^+)] = 0 = [C(f^+), C(g^+)] \tag{1.4.7}$$

and

$$[A(f^+), C(g^+)] = \langle f^+, g^+ \rangle = \frac{1}{i\hbar}\Omega(f^-, g^+). \tag{1.4.8}$$

The Fock representation of $\mathfrak{A}$ is the linear map Λ defined as

$$\Lambda(N(f)) = \hbar \left[A(f^+) + C(f^+)\right]. \tag{1.4.9}$$

More formally, define the compatible complex structure[13] $Jf = if^+ - if^-$: one has $N(f) = N(f^+) + N(f^-)$ and

$$\begin{aligned} A(f^+) &= \frac{1}{2}\Lambda\left[N(f) + iN(Jf)\right] = \Lambda(N(f^-)), \\ C(f^+) &= \frac{1}{2}\Lambda\left[N(f) - iN(Jf)\right] = \Lambda(N(f^+)), \end{aligned} \tag{1.4.10}$$

and also $N(Jf) = iN(f^+) - iN(f^-) \overset{\Lambda}{\mapsto} iC(f^+) - iA(f^+)$. Let us show that Λ preserves the canonical commutation relations:

$$\begin{aligned} [\Lambda(N(f)), \Lambda(N(g))] &= \hbar^2\left[C(f^+), A(g^+)\right] + \hbar^2\left[A(f^+), C(g^+)\right] \\ &= -i\hbar\left[\Omega(f^-, g^+) - \Omega(g^-, f^+)\right]\mathbb{1} \\ &= -i\hbar\left[\Omega(f, g^+) + \Omega(f, g^-)\right]\mathbb{1} \\ &= -i\hbar\Omega(f, g)\mathbb{1} \\ &= \Lambda\left([N(f), N(g)]\right), \end{aligned} \tag{1.4.11}$$

where we have used the fact that $\Omega(f^-, g^-) = 0$ for any two negative-frequency fields.

[13] A complex structure on a vector space V is a linear map $J : V \to V$ satisfying $J^2 = -\mathbb{1}$. The complex structure J is said to be compatible with a symplectic structure Ω on V if $G_J(u, v) \equiv \Omega(u, Jv)$ is a positive inner product on V. The symmetry of G_J requires J to be a symplectomorphism $\Omega(Ju, Jv) = \Omega(u, v)$ and its positivity imposes $\Omega(u, Ju) > 0$, for all $u \neq 0$.

Now, the action of the BMS group G leaves the Schwartz space and the symplectic structure invariant, so that $\mathcal{H}$ provides a unitary representation of G and hence of any Poincaré subgroup of G.

Using this action, one finds for example that the square-mass, given by $m^2 = -p_0^2 + p_1^2 + p_2^2 + p_3^2$, is expressed in terms of the translation generators $X_\mu = \alpha_\mu n$, where

$$\alpha_0 = 1, \quad \alpha_1 = \sin\theta\cos\phi, \quad \alpha_2 = \sin\theta\sin\phi, \quad \alpha_3 = \cos\theta, \tag{1.4.12}$$

by

$$m^2 f_{\mu\nu}^+ = \underbrace{\left(-\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2\right)}_{=0} \pounds_n \pounds_n f_{\mu\nu}^+ = 0. \tag{1.4.13}$$

Another computation shows that this representation splits into two irreducible representations of helicity $+2$ and -2.

The action of the BMS group G on Γ preserves the symplectic structure Ω and we gave the expressions of the corresponding generating functionals, *i.e.* the Hamilton functions generating the corresponding canonical transformations. These classical observables can be readily promoted to quantum operators on the Fock space $\mathfrak{F}$, using the normal ordering prescription to regularize the product of operator-valued distributions that appear in the expressions. Hence the BMS group can be realized as a symmetry group also in the quantum theory.

1.4.2 Infrared sectors

The finiteness of energy requires any radiative mode $\{D\}$ to approach some classical vacua $\{D^0\}^\pm$ in the limits $u \to \pm\infty$. Fix an origin $\{D^0\}$ in Γ and represent any $\{D\} \in \Gamma$ by the tensor field $f_{\mu\nu} = \{D\} - \{D^0\}$; let the corresponding news tensor be given by

$$F_{\mu\nu} = -2\pounds_n f_{\mu\nu}, \tag{1.4.14}$$

from which $\tilde{F}_{\mu\nu} = 2i\omega\tilde{f}_{\mu\nu}$. The requirement needed for f^+ to be in the one-particle space is that its norm does not diverge: therefore, substituting in (1.4.5), we need

$$\int_0^{+\infty} \left|\tilde{F}_{\mu\nu}(\omega,\theta,\phi)\right|^2 \frac{d\omega}{\omega} < +\infty. \tag{1.4.15}$$

This integral is infrared divergent unless $\tilde{F}(0,\theta,\phi) = 0$, or, equivalently, the zero-mode of the news tensor vanishes

$$Q_{\mu\nu}(\theta,\phi) \equiv \int_{-\infty}^{+\infty} F_{\mu\nu}(u,\theta,\phi)du = 0, \tag{1.4.16}$$

or again, in terms of $f_{\mu\nu}$,

$$f_{\mu\nu}(+\infty,\theta,\phi) = f_{\mu\nu}(-\infty,\theta,\phi). \tag{1.4.17}$$

In turn, this implies, from a classical viewpoint, that $\{D\}$ should tend to the same classical vacuum in the limit $u \to +\infty$ as it does in the limit $u \to -\infty$:

$$\{D^0\}^+ = \{D^0\}^-. \tag{1.4.18}$$

For the linearized theory, one can show that this condition is equivalent to having trivial scattering.

Introduce the following equivalence relation

$$\{D\} \sim \{D'\} \text{ if and only if } Q_{\mu\nu} = Q'_{\mu\nu}, \tag{1.4.19}$$

which identifies modes with the same "phase" between the distant past and the distance future classical vacua; as we have seen, only the subspace $Q_{\mu\nu} = 0$ gives rise to one-graviton states in the Fock representation. Are there other representations, other than the Fock one? Consider the following ("charged") automorphism Π_f defined on the C^*-algebra $\mathfrak{A}$:

$$\Pi_f(N_{\mu\nu}(u,\theta,\phi)) = N_{\mu\nu}(u,\theta,\phi) + f_{\mu\nu}(u,\theta,\phi)\mathbb{1}. \tag{1.4.20}$$

We now prove that this automorphism, in general, cannot be implemented by any unitary operator in the Fock representation: if it were, denote by U such an operator, which acts therefore by

$$U\Lambda(B)U^{-1} = \Lambda(\Pi_f(B)) \tag{1.4.21}$$

for any $B \in \mathfrak{A}$; applying the previous equation to $U|0\rangle$, where $|0\rangle$ is the Fock vacuum,

$$U\Lambda(B)|0\rangle = \Lambda(\Pi_f(B))U|0\rangle, \tag{1.4.22}$$

choosing $B = N(h^-)$ so that $\Lambda(B) = A(h^+)$, for some one-graviton state h^+,

$$0 = \left(A(h^+) + \langle h^+, f^+\rangle\right) U|0\rangle. \tag{1.4.23}$$

On the other hand, the canonical commutation relations and the Baker-Hausdorff formula[14] imply that

$$e^{-C(f^+)}A(h^+)e^{C(f^+)} = A(h^+) - \langle h^+, f^+\rangle, \tag{1.4.26}$$

[14] The Baker-Hausdorff formula states that

$$e^{-\lambda C}Ae^{\lambda C} = A + \lambda[A,C] + \frac{\lambda^2}{2!}[[A,C],C] + \dots \tag{1.4.24}$$

and it is easily derived as follows: letting $f(\lambda) \equiv e^{-\lambda C}Ae^{\lambda C}$, by direct computation $f(\lambda)$ satisfies the Cauchy problem

$$\begin{aligned} \dot{f}(\lambda) &= [A, f(\lambda)] \\ f(0) &= A \end{aligned} \tag{1.4.25}$$

which is also satisfied by $g(\lambda) = e^{\lambda \mathrm{ad}_C}A$, where $\mathrm{ad}_C A = [A,C]$. Hence $f(\lambda) = g(\lambda)$ by uniqueness of the solution to the Cauchy problem; note that this result holds for any λ since one is dealing with everywhere-convergent power series.

so that the solution of (1.4.23) is

$$U|0\rangle = \mathcal{N} e^{C(f^+)}|0\rangle; \tag{1.4.27}$$

by the same formula, $|f^+\rangle \equiv e^{C(f^+)}|0\rangle$ is a coherent state $A(f^+)|f^+\rangle = \langle f^+, f^+\rangle|f^+\rangle$ and hence

$$\left|\left|e^{C(f^+)}|0\rangle\right|\right|^2 = e^{\langle f^+, f^+\rangle} \tag{1.4.28}$$

so that this state exists if and only if f^+ belongs to the $Q_{\mu\nu} = 0$ sector, since above we have seen that the norm $\langle f^+, f^+\rangle$ is finite if and only if $Q_{\mu\nu} = 0$. Non-Fock representations are built as follows: let

$$\langle B\rangle_f \equiv \langle 0|\Lambda(\Pi_f(B))|0\rangle \tag{1.4.29}$$

be the expectation value on the f-vacuum; using the Gelfand-Naimark-Segal (GNS) [50] construction, one then recovers an irreducible representation Λ_f of $\mathfrak{A}$ which is inequivalent to the Fock representation. Furthermore, if f and g belong to the same sector, then Λ_{f-g} is unitarily equivalent to the Fock representation, by the above argument, meaning that the representations are labelled by the value of $Q_{\mu\nu}(\theta, \phi)$.

Even though Λ_f admits a cyclic vector $|0\rangle_f$ which plays the role of the vacuum, the zero value is not a proper, discrete eigenvalue of the Hamiltonian in this representation, but rather the infimum of its continuous spectrum; another possibility is to work, instead, in the (non-separable) Hilbert space made up by the direct sum (or integral) of all the $Q_{\mu\nu}$ sectors.

The $Q_{\mu\nu}$ commute with any N and hence form the elements of the center $\mathcal{Z}$ of the observable algebra $\mathfrak{A}$ [47]. Another way to prove that the automorphism Π_f is not unitarily implementable is to note that it does not commute with $\mathcal{Z}$, and hence it must be broken in any irreducible representation of the algebra $\mathfrak{A}$. In fact, as we have seen above, Π_f plays the role of an intertwiner between various inequivalent representations obtained via the GNS construction.

Now, what about the BMS transformations? Let us consider the subgroup of BMS supertranslations ST. Clearly, ST commutes with $\mathcal{Z}$: the action of a supertranslation $\psi_{(\lambda)}$ generated by αn, for $\pounds_n \alpha = 0$, of parameter λ is given, in a Bondi frame, by [51]

$$f_{\mu\nu} \longmapsto f_{\mu\nu} + \lambda\left(D_\mu D_\nu \alpha + q_{\mu\nu}\frac{1}{2}q^{\rho\sigma}D_\rho D_\sigma \alpha\right) \equiv f_{\mu\nu} + \lambda\Delta_{\mu\nu}, \tag{1.4.30}$$

which does not alter the zero mode $Q_{\mu\nu}$ since α is u-independent. It follows that any representation of $\mathfrak{A}$ obtained by acting with ST on a given irreducible representation will be unitarily equivalent to the starting one. Therefore, ST constitutes a Wigner symmetry in any irreducible representation of $\mathfrak{A}$. An ordinary translation is in particular characterized by the fact that $\Delta = 0$.

This proves that the BMS group is a Wigner symmetry, *i.e.* it can be implemented unitarily in any irreducible representation of the observable algebra $\mathfrak{A}$.

As we shall see in Chapters 3, 4, 5 and 6, a key ingredient in the discussion of asymptotic symmetries will be given by their spontaneous breaking, in apparent contradiction with the conclusions we have just drawn concerning the BMS group. The difference in the two approaches is given by the field algebra one chooses to consider: here we were dealing with observable quantities exclusively, whereas in the following chapters we will consider larger spaces, also comprising some non-radiative modes.

2 Weinberg's Soft Theorems

2.1 Soft Theorems from Gauge Invariance

In his celebrated 1964 paper [6], Weinberg showed that, using only the Lorentz invariance and the pole structure of the S matrix, together with masslessness and spins of the photon and the graviton, it is possible to derive the conservation of electric charge and the equality of gravitational and inertial mass. On the same grounds, he gave a possible explanation as to why we observe no macroscopic fields corresponding to massless particles of spin 3 or higher.

In particular, exploiting the S-matrix pole structure and Lorentz covariance only, he could prove the following two properties:

(1) The S matrix for the emission of a photon or a graviton can be written as the product of a polarization "vector" ε^μ or "tensor" $\varepsilon^\mu\varepsilon^\nu$ with a covariant vector or tensor amplitude, and it vanishes if any of the ε^μ's is replaced by the photon or graviton momentum q^μ;

(2) Charge, defined dynamically by the strenght of soft-photon interactions, is additively conserved in all reactions. Gravitational mass, defined by the strength of soft graviton interactions, is equal to inertial mass for all nonrelativistic particles (and is twice the total energy for relativistic or massless particles).

For the moment we shall allow ourselves to think in terms of fields and to rely on the usual notion of gauge invariance as well, in order to give a simple overview of Weinberg's more ambitious work.

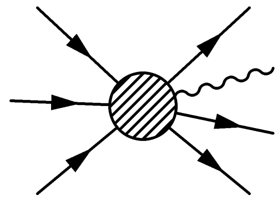
2.1.1 Charge conservation

Let us consider a transition amplitude involving N external particles, distributed between a set of *in* states $|\alpha\rangle$ and a set of *out* states $|\beta\rangle$, together with an additional external particle with mass $m = 0$, spin (helicity) 1 and momentum q^μ:

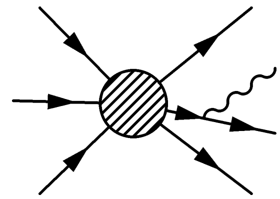
$$S_{\alpha,\beta;q^\mu}. \tag{2.1.1}$$

With respect to the same process without the additional massless particle, one can consider two different types of diagrams, illustrated by the following cartoons.

$$S_{\alpha,\beta;q^\mu} = \qquad (2.1.2)$$



$+$



In the soft limit $q^\mu \to 0$, diagrams of the second type display a pole singularity: letting p_i be the momentum of the outgoing external particle interacting with the photon,

$$\frac{1}{(p_i+q)^2 - m^2} = \frac{1}{2p_i \cdot q} \qquad (2.1.3)$$

since the external particles are on shell and thus $p_i^2 - m^2 = 0$. Therefore as $q^\mu \to 0$ the amplitude will be dominated by second contribution in (2.1.2). Weinberg showed that under these conditions the full amplitude for the process factorizes as follows

$$S_{\alpha\beta;q} = \sum_i e_i p_i^\mu \varepsilon_\mu(q) S(p_1, \dots, p_N) \frac{1}{2p_i \cdot q}, \qquad (2.1.4)$$

where ε_μ is the polarization vector for the spin-1 massless particle, while e_i defines the electric charge of the other external particles. The sum runs over all external particles, incoming and outgoing, other than the soft one. Now, gauge invariance imposes the requirement that when ε_μ is substituted with $q_\mu \alpha$ (the Fourier transform of the gauge parameter) the amplitude must vanish, implying

$$q_\mu \sum_i e_i p_i^\mu S(p_1, \dots, p_N) \frac{1}{2p_i \cdot q} = 0 \implies \sum_i e_i = 0. \qquad (2.1.5)$$

This is the equation of charge conservation.

2.1.2 The equivalence principle and higher spins

In the case of a spin-2 soft massless particle, following the same line of reasoning and denoting by f_i the gravitational couplings, one finds

$$S_{\alpha\beta;q} = \sum_i f_i p_i^\mu p_i^\nu \varepsilon_{\mu\nu}(q) S(p_1, \dots, p_N) \frac{1}{2p_i \cdot q}. \qquad (2.1.6)$$

Again gauge invariance requires that substituting $\varepsilon_{\mu\nu}$ with $q_\mu \varepsilon_\nu + q_\nu \varepsilon_\nu$, the amplitude vanishes,

$$- q_{i\mu} \varepsilon_\nu \sum_i f_i p_i^\mu p_i^\nu S(p_1, \dots, p_N) \frac{1}{2p_i \cdot q} = 0 \implies \varepsilon_\mu \sum_i f_i p_i^\mu = 0 \qquad (2.1.7)$$

and thus, by the arbitrariness of ε_μ, one finds the following condition on the momenta

$$\sum_i f_i p_i^\mu = 0. \tag{2.1.8}$$

Now, recalling that energy-momentum conservation also requires $\sum_i p_i^\mu = 0$, the only case for which the interaction is nontrivial is

$$f_i = \text{constant}, \tag{2.1.9}$$

which is nothing but a form of the equivalence principle: every type of matter must couple to gravity with the same coupling constant. We have deduced the equivalence principle from the requirement of gauge invariance in the limit of small energies. Notice that in our derivation the matter particles can be either massive or massless, with any spin.

In a similar manner, one sees that there can be no gauge interaction surviving the soft limit for higher spins, $s \geqslant 3$, since by the factorization result for the amplitude

$$S^{(s)}_{\alpha\beta;q} = \sum_i g_i^{(s)} p_i^{\mu_1} p_i^{\mu_2} \dots p_i^{\mu_2} \varepsilon_{\mu_1\mu_2\dots\mu_s}(q) S(p_1, \dots, p_N) \frac{1}{2p_i \cdot q}, \tag{2.1.10}$$

which should vanish when $\varepsilon_{\mu_1\mu_2\dots\mu_s}(q) = q_{(\mu_1}\varepsilon_{\mu_2\dots\mu_s)}(q)$, one gets

$$\sum_i g_i^{(s)} p_i^{\mu_2} \dots p_i^{\mu_2} \varepsilon_{\mu_2\dots\mu_s}(q) = 0, \tag{2.1.11}$$

and this is incompatible with energy-momentum conservation, unless the interaction is trivial $g_i^{(s)} = 0$.

2.2 Weinberg's Covariant S-matrix Approach

We turn now to the proof of Weinberg's theorems in the spirit of the original paper mentioned above. For some important technical results on the implications of covariance on the S-matrix structure, which are here assumed to hold, we refer to the appendices of Weinberg's paper [6].

2.2.1 Amplitudes for massless particles of integer spin

Consider a process in which a massless particle is emitted with momentum $\mathbf{q}$ and helicity $\pm s$.[1] The Lorentz transformation property of the S matrix can be inferred from the

[1] We limit ourselves to integer spins.

transformation law for one-particle states; using p as a shorthand for the p_i's of the previous section, we find

$$S_{\pm s}(\mathbf{q},p) = \left(\frac{|\Lambda \mathbf{q}|}{|\mathbf{q}|}\right)^{1/2} e^{\pm is\Theta(\mathbf{q},\Lambda)} S_{\pm s}(\Lambda\mathbf{q},\Lambda p), \tag{2.2.1}$$

where Θ is a function of the massless particle momentum $\mathbf{q}$ and of the Lorentz transformation Λ. It is always possible to write $S_{\pm s}$ as a product of a "polarization tensor" and an "M function" in the following way:

$$S_{\pm s}(\mathbf{q},p) = \frac{1}{\sqrt{2\mathbf{q}}}\varepsilon_{\pm}^{\mu_1 *}(\mathbf{q})\dots\varepsilon_{\pm}^{\mu_s *}(\mathbf{q}) M_{\pm\mu_1\dots\mu_s}(\Lambda\mathbf{q},\Lambda p), \tag{2.2.2}$$

where M denotes a symmetric Lorentz tensor. The "polarization vector" $\varepsilon_{\pm}^{\nu}$ follows the transformation rule

$$\left(\Lambda_{\nu}{}^{\mu} - \frac{q^{\mu}}{|\mathbf{q}|}\Lambda_{\nu}{}^{0}\right)\varepsilon_{\pm}^{\nu}(\Lambda q) = e^{\pm i\Theta(\mathbf{q},\Lambda)}\varepsilon_{\pm}^{\mu}(q). \tag{2.2.3}$$

Since apparently $\varepsilon_{\pm}^{\mu}$ is not a vector, an auxiliary condition will be needed to make sure that $S_{\pm s}$ satisfies Lorentz invariance. The S-matrix transformation law is then

$$\begin{aligned} S_{\pm s}(\mathbf{q},p) =& \frac{1}{\sqrt{2\mathbf{q}}} e^{\pm is\Theta(\mathbf{q},\Lambda)} \left[\varepsilon_{\pm}^{\mu_1}(\Lambda q) - \frac{(\Lambda q)^{\mu_1}}{|\mathbf{q}|}\Lambda_{\nu}{}^{0}\varepsilon_{\pm}^{\nu}\right]\dots \\ &\times \left[\varepsilon_{\pm}^{\mu_s}(\Lambda q) - \frac{(\Lambda q)^{\mu_s}}{|\mathbf{q}|}\Lambda_{\nu}{}^{0}\varepsilon_{\pm}^{\nu}\right] M_{\pm\mu_1\dots\mu_s}(\Lambda\mathbf{q},\Lambda p). \end{aligned} \tag{2.2.4}$$

For an infinitesimal Lorentz transformation $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$, we can use (2.2.2) and the symmetry of M to put the previous equation in the form

$$\begin{aligned} S_{\pm s}(\mathbf{q},p) =& \left(\frac{|\Lambda \mathbf{q}|}{|\mathbf{q}|}\right)^{1/2} e^{\pm is\Theta(\mathbf{q},\Lambda)} S_{\pm s}(\Lambda\mathbf{q},\Lambda p) \\ &- s\frac{1}{\sqrt{2|q|^3}}\left(\omega_{\nu}{}^{0}\varepsilon_{\pm}^{\nu *}(q)\right)\left[q^{\mu_1}\varepsilon_{\pm}^{\mu_2 *}(q)\dots\varepsilon_{\pm}^{\mu_s *}(q) M_{\pm\mu_1\dots\mu_s}(\mathbf{q},p)\right]. \end{aligned} \tag{2.2.5}$$

Hence the necessary and sufficient condition for this transformation law not to contradict the first one is that $S_{\pm}$ vanishes when one of the $\varepsilon_{\pm}^{\mu}$ is replaced with q^{μ}:

$$q^{\mu_1}\varepsilon_{\pm}^{\mu_2 *}(q)\dots\varepsilon_{\pm}^{\mu_s *}(q) M_{\pm\mu_1\dots\mu_s}(\mathbf{q},p) = 0. \tag{2.2.6}$$

2.2.2 Dynamic definition of charge e and gravitational mass f

Considering the vertex amplitude for a very-low-energy massless particle of integer helicity $\pm s$, emitted by a particle of spin 0 and mass m (perhaps zero), and momentum $p^{\mu} = (\mathbf{p}, E)$, the only tensor which can be used to form $M_{\pm}^{\mu_1\dots\mu_s}$ is $p^{\mu_1}\dots p^{\mu_s}$, since terms involving $g^{\mu\mu'}$

do not contribute to the S matrix because of $\varepsilon^{\mu}_{\pm}\varepsilon_{\pm\mu} = 0$,[2] so that the vertex amplitude must be of the form

$$\frac{1}{2E(\mathbf{p})\sqrt{2|\mathbf{q}|}} p_{\mu_1} \dots p_{\mu_s} \varepsilon^{\mu_1 *}_{\pm}(q) \dots \varepsilon^{\mu_s *}_{\pm}(q). \tag{2.2.7}$$

In the next section we will see that, even for emitting particles with spin greater than 0, the S matrix elements will be given by this expression, times $\delta_{\sigma\sigma'}$ where σ and σ' are respectively the initial and final helicity of the emitting particle.

We define the soft photon coupling constant e by the statement that the $s = 1$ vertex amplitude is

$$\frac{2ie(2\pi)^4 \delta_{\sigma\sigma'} p_\mu \varepsilon^{\mu *}_{\pm}(q)}{(2\pi)^{9/2} 2E(\mathbf{q})\sqrt{2|\mathbf{q}|}}, \tag{2.2.8}$$

and similarly for the "gravitational charge" we state that the $s = 2$ vertex amplitude is

$$\frac{2if(8\pi)^{1/2}(2\pi)^4 \delta_{\sigma\sigma'} \left(p_\mu \varepsilon^{\mu *}_{\pm}(q)\right)^2}{(2\pi)^{9/2} 2E(\mathbf{q})\sqrt{2|\mathbf{q}|}}. \tag{2.2.9}$$

2.2.3 Conservation of e and universality of f

Let $S_{\beta\alpha}$ be the S matrix for some reaction $\alpha \to \beta$, the states α and β consisting of various charged and uncharged particles, perhaps including gravitons and photons. The same reaction can occur with emission of a very soft extra photon or graviton of momentum $\mathbf{q}$ and helicity ± 1, or ± 2, and we will denote the corresponding S-matrix element as $S^{\pm 1}_{\beta\alpha}(\mathbf{q})$ or $S^{\pm 2}_{\beta\alpha}(\mathbf{q})$.

As illustrated above, these emission matrix elements will have poles at $\mathbf{q} = 0$, corresponding to the Feynman diagrams in which the extra photon or graviton is emitted by one of the incoming or outgoing particles in states α or β, since then the n-th outgoing, respectively incoming, particle of mass m_n and momentum p_n gives rise to a term of the form

$$\frac{1}{(p_n \pm q)^2 - m_n^2} = \pm \frac{1}{2p_n \cdot q}. \tag{2.2.10}$$

In the limit $\mathbf{q} \to 0$ we will get, denoting by $S_{\beta\alpha}$ the remainder of the diagram once we have factored out the pole and tensor structure,

$$\begin{aligned} S^{\pm 1}_{\beta\alpha}(\mathbf{q}) &\approx \frac{1}{(2\pi)^{3/2}\sqrt{2|\mathbf{q}|}} \left[\sum_n \eta_n e_n \frac{p_n \cdot \varepsilon^*_{\pm}(q)}{p_n \cdot q} \right] S_{\beta\alpha} \\ S^{\pm 2}_{\beta\alpha}(\mathbf{q}) &\approx \frac{(8\pi)^{1/2}}{(2\pi)^{3/2}\sqrt{2|\mathbf{q}|}} \left[\sum_n \eta_n f_n \frac{(p_n \cdot \varepsilon^*_{\pm}(q))^2}{p_n \cdot q} \right] S_{\beta\alpha}, \end{aligned} \tag{2.2.11}$$

[2] This follows from the fact that external particles are on-shell and the corresponding tensors live in traceless representations of the stability group of p^μ.

η_n being $+1$ or -1 according to whether the particle n is outgoing or incoming.

As we have learned in the previous sections, Lorentz invariance requires the vanishing of the vertex amplitude when a polarization is substituted with the corresponding four-momentum. This yields for $s = 1$

$$\sum_n \eta_n e_n = 0 \tag{2.2.12}$$

and for $s = 2$

$$\sum_n \eta_n f_n p_n^\mu = 0. \tag{2.2.13}$$

The first one is precisely the conservation of the electric charge, whereas the second, when compared with the equation of momentum conservation $\sum \eta_n p_n = 0$, yields the universality of the gravitational coupling constant $f_n = 1$, for all n.

Weinberg exploits these calculations also in order to justify the choice of $p_n^{\mu_1} \dots p_n^{\mu_s}$ in (2.2.7) as the only possible form of the M function emitting particles with spin 1 or higher, since any other helicity-dependent term could never give rise to cancellations between different poles needed to satisfy the Lorentz invariance condition.

2.2.4 Higher-spin soft emission

For higher helicities $s = 3, 4, \dots$ one still has a factorization of the form

$$S_{\beta\alpha}^{\pm s}(\mathbf{q}) \approx \frac{1}{(2\pi)^{3/2}\sqrt{2|\mathbf{q}|}} \left[\sum_n \eta_n g_n^{(s)} \frac{(p_n \cdot \varepsilon_\pm^*(q))^s}{p_n \cdot q} \right] S_{\beta\alpha}, \tag{2.2.14}$$

and the requirement

$$\sum_n \eta_n g_n^{(s)} \left[p_n \cdot \varepsilon_\pm^*(q) \right]^{s-1} = 0, \tag{2.2.15}$$

which contradicts momentum conservation unless $g_n^{(s)} = 0$. This tells us that the low energy interaction for higher spins is trivial or, in other words, that massless higher-spin particles cannot propagate long-range forces. On the Lagrangian side, this implies that higher-spin interactions should be of multipolar type, *i.e.* the vertices should contain enough derivatives so that they vanish in the soft limit.

An expression like (2.1.11) or (2.2.15) looks like some sort of conservation law, and one may wonder whether it can be derived from some underlying symmetry. This question was addressed long ago for $s = 1$ in a couple of papers [32, 33] by Ferrari and Picasso, and received recently a renewed attention after the contribution of Strominger et al., who linked it to the BMS symmetry presented in the previous chapter. We shall illustrate the corresponding findings in the next two chapters, to then move to face the main issue at stake in this work: investigating the higher-spin symmetry underlying (2.1.11).

3 BMS Ward Identities and the Soft Graviton Theorem

Weinberg's soft graviton theorem, which was described in the previous chapter, has been recast in [2] as the Ward identity following from BMS supertraslation symmetry. To build a bridge between these two results, which are naturally expressed in two very different languages, we will need to find a way to link the boundary data at null infinity, expressed in terms of the Bondi news, to the properties of scattering of momentum-space plane waves.

We will also have to define the physical phase space of gravitational modes, which needs to include the Bondi news as well as all soft graviton degrees of freedom which do not decouple from the S matrix. Notice that this contrasts with Ashtekar's asymptotic quantization, where only observable quantities were considered.

This space will turn out to contain the usual radiative modes plus the Goldstone modes of spontaneously broken supertranslation invariance.

3.1 Vacuum-to-Vacuum Geometries

We start by fixing the notation and by introducing a useful set of local coordinates near $\mathscr{I}^{\pm}$, called the Bondi coordinates.

3.1.1 Bondi coordinates for asymptotically flat spacetimes

As derived in [4, 5], a general Lorentzian metric can be written in local coordinates $u = x^0$, $t = x^1$, $\theta = x^2$, $\phi = x^3$ as

$$ds^2 = \frac{Ve^{2\beta}}{r}du^2 - 2e^{2\beta}dudr + r^2 h_{AB}\left(dx^A - U^A du\right)\left(dx^B - U^B du\right), \tag{3.1.1}$$

where A and B take the values $2, 3$, and $\det(h_{AB}) \equiv b(u, \theta, \phi)$, where $b(u, \theta, \phi)$ is an arbitrary but *fixed* function; V, β, U^A and h_{AB} are any six functions of the coordinates. This result relies only on the properties of the above coordinates, in the sense that such a component expansion for ds^2 is allowed if and only if the following conditions hold:

(i) the hypersurfaces $u =$ const. are tangent to the local light cone at each point;

(ii) θ and ϕ are constant along each ray, where a ray is defined as the line with tangent $k^\mu = -g^{\mu\nu}\partial_\nu u$;

(iii) r is the corresponding luminosity distance given by[1]

$$r^4 = \left\{\left[\theta_{,\mu}g^{\mu\nu}\theta_{,\nu}\,\phi_{,\alpha}g^{\alpha\beta}\phi_{,\beta} - (\theta_{,\alpha}g^{\alpha\mu}\phi_{,\mu})^2\right]\sin^2\theta\right\}^{-1}. \tag{3.1.2}$$

By analyzing the field equations it was shown (see [4, 5] and references therein) that, in the case of asymptotically flat spacetimes, the asymptotic behavior of the quantities in the above component expansion compatible with the presence of gravitational radiation is the following: denoting by $d\sigma^2 \equiv h_{AB}dx^A dx^B$ the two-dimensional line element,

$$\begin{aligned}
V &= -r + 2M(u,\theta,\phi) + \mathcal{O}(r^{-1}),\\
\beta &= -c(u,\theta,\phi)c^*(u,\theta,\phi)(2r)^{-2} + \mathcal{O}(r^{-4}),\\
U^A &= \mathcal{O}(r^{-2}),\\
d\sigma^2 &= d\theta^2 + \sin^2\theta d\phi^2 + \mathcal{O}(r^{-1})\\
&\equiv \gamma_{AB}dx^A dx^B + r^{-1}A_{AB}dx^A dx^B + \mathcal{O}(r^{-2}),
\end{aligned} \tag{3.1.3}$$

where $A_{AB} = \mathcal{O}(1)$. The function $M(u,\theta,\phi)$ is called the *Bondi mass aspect* and its integral on the two-sphere coincides with the Bondi mass defined in Section 1.2.2. This allows to expand the metric in the following form:

$$\begin{aligned}
ds^2 =& - du^2 - 2dudr + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\\
&+ \frac{2M}{r}du^2 + rA_{AB}dx^A dx^B - 2r^2U^A\gamma_{AB}dudx^B + \dots
\end{aligned} \tag{3.1.4}$$

the dots denoting subleading terms in r^{-1} with respect to those explicitly written (note that $r^2U^A = \mathcal{O}(1)$). Using the standard complex coordinates[2] $z = \tan(\theta/2)e^{i\phi}$, in place of

[1] Recall that the area element on the sphere is given by $\sqrt{e_\theta^2 e_\phi^2 - (e_\theta\cdot e_\phi)^2} = r^2\sin\theta$, where $e^\mu_\theta = \partial x^\mu/\partial\theta$ and $e^\mu_\phi = \partial x^\mu/\partial\phi$.

[2] The transformation rules from the usual spherical coordinates to these complex coordinates read

$$z = e^{i\phi}\tan\frac{\theta}{2} \qquad z\bar z = \tan^2\frac{\theta}{2} \equiv t^2 \qquad \theta = 2\tan^{-1}\left(\sqrt{z\bar z}\right) \tag{3.1.5}$$

$$\bar z = e^{-i\phi}\tan\frac{\theta}{2} \qquad \frac{z}{\bar z} = e^{i2\phi} \qquad \phi = \frac{1}{2i}\ln\frac{z}{\bar z} \tag{3.1.6}$$

and hence

$$x^1 + ix^2 = r\sin\theta e^{i\phi} = r\frac{2z}{1+z\bar z}, \qquad x^3 = r\cos\theta = r\frac{1-z\bar z}{1+z\bar z}. \tag{3.1.7}$$

The line and surface elements are given by

$$d\theta^2 + \sin^2\theta d\phi^2 = 2\frac{2}{(1+z\bar z)^2}dzd\bar z, \qquad d\theta\sin\theta d\phi = \frac{2}{(1+z\bar z)^2}dzd\bar z, \tag{3.1.8}$$

whereas the non-vanishing Christoffel symbols read

$$\Gamma^z_{zz} = -\frac{2\bar z}{1+z\bar z} = \overline{\Gamma^{\bar z}_{\bar z\bar z}}, \tag{3.1.9}$$

so that $R_{z\bar z} = \gamma_{z\bar z}$ and $[D_{\bar z}, D_z]X^z = X_{\bar z}$.

the usual spherical coordinates θ, ϕ and relabelling the last terms appropriately, we finally obtain:

$$\begin{aligned} ds^2 = &- du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\ &+ \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 - 2U_z dudz - 2U_{\bar{z}}dud\bar{z} + \dots \end{aligned} \tag{3.1.10}$$

where we have defined $\gamma_{z\bar{z}} = 2/(1+z\bar{z})^2$ and $m_B(u,z,\bar{z}) = M(u,\theta,\phi)$; the quantities C_{zz}, $C_{\bar{z}\bar{z}}$, U_z, $U_{\bar{z}}$ are independent of r.

The coordinates $(u,r,z,\bar{z})$ are called retarded Bondi coordinates: the retarded time u parametrizes the null generators of $\mathscr{I}^+$ whereas z and $\bar{z}$ parametrize the conformal two-sphere, whose metric is $2\gamma_{z\bar{z}}dzd\bar{z}$. D will denote the γ-covariant derivative.

Furthermore, using an appropriate Ansatz for the angular terms of the metric, Barnich and Troessaert [52] also derived

$$U_z = -\frac{1}{2}D^z C_{zz}. \tag{3.1.11}$$

A similar expansion is available near past null infinity $\mathscr{I}^-$:

$$\begin{aligned} ds^2 = &- dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\ &+ \frac{2m_B^-}{r}dv^2 + rD_{zz}dz^2 + rD_{\bar{z}\bar{z}}d\bar{z}^2 - 2V_z dvdz - 2V_{\bar{z}}dvd\bar{z} + \dots \end{aligned} \tag{3.1.12}$$

(m_B will always denote the Bondi mass aspect at $\mathscr{I}^+$, unless otherwise specified), where

$$V_z = \frac{1}{2}D^z D_{zz}. \tag{3.1.13}$$

We use the following shorthand notation for the causal future and past of $\mathscr{I}^\pm$: $J_\pm(\mathscr{I}^\pm) \equiv \mathscr{I}^\pm_\pm$. The outgoing and incoming Bondi news are, respectively, (see [1, 52])

$$N_{zz} = \partial_u C_{zz}, \qquad M_{zz} = \partial_v D_{zz}. \tag{3.1.14}$$

Their physical meaning is given by the following relation, which follows from the Einstein equations: assuming no matter fields,

$$\partial_u m_B = -\frac{1}{4}N_{zz}N^{zz} - \frac{1}{2}\partial_u\left(D^z U_{\bar{z}} + D^{\bar{z}}U_z\right) \tag{3.1.15}$$

or, in other words, the news tensor controls the mass loss (see [4, 53]), since integrating on a cross-section $\mathscr{S}$ yields

$$\frac{\partial}{\partial u}\int_{\mathscr{S}} m_B\, d^2\Omega = -\frac{1}{4}\int_{\mathscr{S}} N_{zz}N^{zz}d^2\Omega, \tag{3.1.16}$$

analogous to the expression (1.2.21) in Section 1.2.2. This confirms

$$T_{uu} = \frac{1}{4}N_{zz}N^{zz} \tag{3.1.17}$$

as the total outgoing energy flux of gravitational radiation.

3.1.2 BMS supertranslations

As we have seen, BMS$^+$ supertranslations are generated by the infinite-dimensional family of vector fields

$$\xi_f \equiv f\partial_u - \frac{1}{r}(f^{,z}\partial_{\bar z} + f^{,\bar z}\partial_z) + D^z D_z f\partial_r, \tag{3.1.18}$$

labeled by the functions $f = f(z, \bar z)$ defined on the conformal two-sphere. BMS$^+$ acts on $\mathscr{I}^+$ by Lie derivative and asymptotic Killing vector fields form a faithful representation of the BMS algebra when equipped with the standard Lie bracket [52].

For example, the (zz)-component of the variation of the metric tensor under the infinitesimal supertranslation ξ_f is given by (letting $\pounds_f \equiv \pounds_{\xi_f}$)

$$\begin{aligned}(\pounds_f g)_{zz} &= f\partial_u g_{zz} - \frac{1}{r}\left(f^{,z}\partial_{\bar z} g_{zz} + f^{,\bar z}\partial_z g_{zz}\right) + \frac{1}{2}\left(\Delta f\right)\partial_r g_{zz} + 2g_{z\mu}\partial_z \xi^\mu \\ &= r\left(f\partial_u C_{zz} - 2D_z^2 f\right) + \mathcal{O}(1),\end{aligned} \tag{3.1.19}$$

and since by confrontation with the original form of the metric $(\pounds_f g)_{zz} \equiv r\pounds_\xi C_{zz} + \mathcal{O}(1)$, we finally have:

$$\pounds_f C_{zz} = f\partial_u C_{zz} - 2D_z^2 f. \tag{3.1.20}$$

Similarly BMS$^-$ transformations act on $\mathscr{I}^-$ and contain the supertranslations labelled by $f^-(z, \bar z)$

$$f^-\partial_v - \frac{1}{r}(D^{\bar z} f^- \partial_{\bar z} + D^z f^- \partial_z) - D^z D_z f^- \partial_r \tag{3.1.21}$$

under which

$$\pounds_{f^-} D_{zz} = f^-\partial_u D_{zz} + 2D_z^2 f^-. \tag{3.1.22}$$

3.2 Supertranslation Generators

In this section we construct the physical phase space, the symplectic form and the canonical generators of supertranslations at $\mathscr{I}^\pm$, following [1, 2].

3.2.1 Poisson brackets on $\mathscr{I}$

The Ashtekar symplectic form (see [47, 48, 21]) on the space Γ of radiative modes defined in Chapter 1, *i.e.* the space of equivalence classes of connections identified under suitable relations due to the conformal structure of null infinity, is defined as follows with our choice of normalization

$$\Omega_{\{D\}}(\sigma, \sigma') = \frac{1}{16\pi}\int_{\mathscr{I}^+} (\sigma_{\bar z\bar z}\partial_u \sigma'_{zz} - \sigma'_{zz}\partial_u \sigma_{\bar z\bar z})\gamma^{z\bar z} du d^2 z \tag{3.2.1}$$

and

$$\{N_{\bar z\bar z}(u, z, \bar z), N_{zz}(u', w, \bar w)\} = -16\pi\partial_u \delta(u - u')\delta^2(z - w)\gamma_{z\bar z}^2. \tag{3.2.2}$$

The generator of BMS$^+$ supertranslations on these physical modes is [1, 2]

$$T^+(f) = \frac{1}{4\pi}\int_{\mathscr{I}^+_-} \gamma_{z\bar{z}} f m_B d^2z. \tag{3.2.3}$$

Note that this generator reduces to the Arnowitt-Deser-Misner Hamiltonian when $f \equiv 1$, *i.e.* $T^+(1) = M$. Using the (3.1.15), assuming no matter fields and dropping a term proportional to the late time mass aspect $m_B|_{\mathscr{I}^+_+}$, which vanishes for the classical solutions, we find

$$T^+(f) = -\frac{1}{16\pi}\int_{\mathscr{I}^+} f\left[\gamma^{z\bar{z}} N_{zz} N_{\bar{z}\bar{z}} + 2\partial_u(\partial_z U_{\bar{z}} + \partial_{\bar{z}} U_z)\right] du\, d^2z \tag{3.2.4}$$

leading to

$$\{T^+(f), N_{zz}\} = f\partial_u N_{zz}, \tag{3.2.5}$$

since the second term of $T^+(f)$ can be recast as a boundary term. The analogous expressions for $\mathscr{I}^-$ read

$$\begin{aligned}
\partial_v m_B^- &= \frac{1}{4} M_{zz} M^{zz} + \frac{1}{2}\partial_v\left(D^z V_z + D^{\bar{z}} V_{\bar{z}}\right)\\
T^-(f) &= \frac{1}{4\pi}\int_{\mathscr{I}^-_+} d^2z\, \gamma_{z\bar{z}} f m_B^-\\
&= \frac{1}{16\pi}\int_{\mathscr{I}^-_+} dv\, d^2z\, f\left[\gamma_{z\bar{z}} M_{zz} M^{zz} + 2\partial_v(\partial_z V_{\bar{z}} + \partial_{\bar{z}} V_z)\right].
\end{aligned} \tag{3.2.6}$$

Since these generators involve non-radiative modes, however, we would like to define their action on an elarged phase space Γ^+ which includes some non-radiative modes but which is able to generate the BMS symmetry transformations. That is to say: we enlarge our field algebra from the algebra of observables to a bigger one which ensures that the symmetry automorphisms are inner automorphisms.

One way to do this is to identify this larger space by the one parametrized by C_{zz} and hence to integrate the above bracket (3.2.2) with respect to u and u', obtaining thus

$$\{C_{\bar{z}\bar{z}}(u, z, \bar{z}), C_{ww}(u', w, \bar{w})\} = 8\pi\, \mathrm{sign}(u - u')\delta(z - w)\gamma_{z\bar{z}}, \tag{3.2.7}$$

where the antisymmetry of the bracket fixes the constant in the integration of $\partial_u \delta(u - u')$ and requires the sign function.[3]

If we use this result, together with

$$T^+(f) = \frac{1}{16\pi}\int_{\mathscr{I}^+} f\left[\gamma^{z\bar{z}}\partial_u C_{zz}\partial_u C_{\bar{z}\bar{z}} + 2\partial_u(\partial_z U_{\bar{z}} + \partial_{\bar{z}} U_z)\right] du\, d^2z\,, \tag{3.2.8}$$

we obtain

$$\begin{aligned}
\{T^+(f), C_{zz}\} &= f\partial_u C_{zz} + \gamma_{z\bar{z}}\partial_z D^{\bar{z}} f\\
&= f\partial_u C_{zz} - D^z f \neq \pounds_f C_{zz},
\end{aligned} \tag{3.2.9}$$

[3] Note that this bracket already appears in [21, (C.17)], but with a different interpretation.

where (3.1.11) and (3.1.14) have been taken into account: a factor of two appears to be missing with respect to (3.1.20). To solve this problem one introduces the auxiliary boundary conditions

$$[\partial_z U_{\bar{z}} - \partial_{\bar{z}} U_z]_{\mathscr{I}^+_\pm} = 0, \tag{3.2.10}$$

$$N_{zz}\big|_{\mathscr{I}^+_\pm} = 0. \tag{3.2.11}$$

Equivalently, by (3.1.11), the first one may be written

$$[D_z^2 C_{\bar{z}\bar{z}} - D_{\bar{z}}^2 C_{zz}]_{\mathscr{I}^+_\pm} = 0. \tag{3.2.12}$$

The general solution of these additional constraints can be expressed as follows:

$$\begin{aligned} C_{zz}\big|_{\mathscr{I}^+_-} &= D_z^2 C, \\ \int_{-\infty}^{\infty} N_{zz} du &= D_z^2 N, \end{aligned} \tag{3.2.13}$$

where the boundary fields C, N are real.

We may then take as our coordinates on the phase space the boundary and bulk fields:

$$\Gamma^+ \equiv \{C(z,\bar{z}), N(z,\bar{z}), C_{zz}(u,z,\bar{z}), C_{\bar{z}\bar{z}}(u,z,\bar{z})\}\,. \tag{3.2.14}$$

Now, equation (3.2.7) remains valid as the bulk-bulk Dirac bracket; we impose (3.2.13) for bulk-bulk or bulk-boundary brackets in the following manner

$$D_z^2\left\{N(z,\bar{z}), C_{\bar{w}\bar{w}}(u,w,\bar{w})\right\} = \int_{-\infty}^{+\infty} du' \left\{N_{zz}(u',z,\bar{z}), C_{\bar{w}\bar{w}}(u,w,\bar{w})\right\} \tag{3.2.15}$$

and then we constrain the boundary-boundary bracket by requiring continuity

$$D_{\bar{w}}^2\{N(z,\bar{z}), C(w,\bar{w})\} = \lim_{u\to-\infty}\{N(z,\bar{z}), C_{\bar{w}\bar{w}}(u,w,\bar{w})\}. \tag{3.2.16}$$

This is a nontrivial request: there may be other, inequivalent, extensions of the symplectic form to the enlarged phase space, corresponding to inequivalent quantizations of the boundary sector. These conditions determine the brackets uniquely. A similar construction is available at $\mathscr{I}^-$.

3.2.2 Canonical generators

The supertraslation generator can be recast as follows, using (3.1.11) and the boundary constraints,

$$\begin{aligned} T^+(f) =& \frac{1}{16\pi}\int_{\mathscr{I}^+} f\left[\gamma^{z\bar{z}}\partial_u C_{zz}\partial_u C_{\bar{z}\bar{z}} + 2\partial_u(\partial_z U_{\bar{z}} + \partial_{\bar{z}} U_z)\right] du\, d^2z \\ =& \frac{1}{16\pi}\int_{\mathscr{I}^+} f\left[\gamma^{z\bar{z}}\partial_u C_{zz}\partial_u C_{\bar{z}\bar{z}} - \gamma^{z\bar{z}}(D_z^2 N_{\bar{z}\bar{z}} + D_{\bar{z}}^2 N_{zz})\right] du\, d^2z \\ =& \frac{1}{16\pi}\int_{\mathscr{I}^+} f\gamma^{z\bar{z}}\partial_u C_{zz}\partial_u C_{\bar{z}\bar{z}}\, du\, d^2z - \frac{1}{8\pi}\int_{\mathscr{I}^+_\pm}\gamma^{z\bar{z}} D_z^2 D_{\bar{z}}^2 N d^2z. \end{aligned} \tag{3.2.17}$$

The non-vanishing brackets are

$$\begin{aligned}\{T^+(f), N_{zz}\} &= f\partial_u N_{zz},\\ \{T^+(f), C_{zz}\} &= f\partial_u C_{zz} - 2D_z^2 f,\\ \{T^+(f), N\} &= 0,\\ \{T^+(f), C\} &= -2f,\end{aligned} \tag{3.2.18}$$

so that, as desired, the realization of supertranslations is canonical on $\Gamma^\pm$. We refer to [1] for the details of the analogous construction for $\mathscr{I}^-$.

These transformation laws indicate that supertranslations do not leave the *in* or *out* vacua invariant and are spontaneously broken in our enlarged phase space: in particular the last bracket in (3.2.18) identifies $-\frac{1}{2}C$ as the Goldstone mode associated with this symmetry breaking.

As was mentioned during the discussion of asymptotic quantization, however, the soft graviton zero mode,

$$Q_{zz}(z,\bar{z}) = \int_{-\infty}^{+\infty} N_{zz}(u,z,\bar{z})du = D_z^2 N, \tag{3.2.19}$$

commutes with all physical observables, and hence is an element of the center of the observable [47] algebra, its different values labelling different, physically inequivalent sectors of the theory. In the quantum case [21, Section II.C.3], the Fock representation is selected by the value $Q_{zz} = 0$, whereas non-vanishing values of the central element correspond to non-Fock representations intimately related to infrared problems: the non-Fock states are determined by "clouds" of coherent soft gravitons. Then it is also clear that supertranslations are unbroken in the representations of the observable algebra, only, as made apparent by the third relation in (3.2.18).

3.3 Supertranslations and the Soft Graviton Theorem

3.3.1 Supertranslation invariance of the S matrix

In the conformal compactification of asymptotically flat spacetimes, the sphere at spatial infinity is identified as i^0 (see figure 1.1 on page 11): the null generator n^μ of $\mathscr{I}^\pm$ flows from $\mathscr{I}^-$ to $\mathscr{I}^+$ through i^0; carrying the coordinates $(z,\bar{z})$ along this flow we identify points on the conformal spheres at $\mathscr{I}^-$ with those at $\mathscr{I}^+$ and this procedure allows to define a matching or continuity relation between final and initial boundary data. We call BMS^0 the diagonal subgroup of $\mathrm{BMS}^+ \times \mathrm{BMS}^-$ which preserves this continuity relation. In particular, the diagonal supertranslation generators are those which are constant on the null generators of $\mathscr{I}$, *i.e.*

$$f^-(z,\bar{z}) = f(z,\bar{z}). \tag{3.3.1}$$

Christodoulou-Klainerman spaces are just smooth and regular enough [1] as to make these matchings possible: their properties ensure that in weakly gravitating systems the null generators going from $\mathscr{I}^-$ to $\mathscr{I}^+$ are suitable for the above identification. As we will see in Chapter 5, this choice can be also justified by thinking the action of the BMS group on the whole physical spacetime, and not only on its null boundary: this allows to argue that the condition $f^-(z,\bar{z}) = f(z,\bar{z})$ follows naturally by computing the charge for the same supertranslation on both $\mathscr{I}^+$ and $\mathscr{I}^-$. [16]

One may remark that, since BMS^0 supertranslation generators commute and obey $T^+(f) = T^-(f)$, and since the S matrix is constructed from exponentials of the Hamiltonian $T(1) = M$, infinitesimal BMS^0 transformations should commute with the S matrix. The BMS^0 subgroup is therefore conjectured to be an exact symmetry of both classical gravitational scattering *and* of the quantum gravity S matrix. More precisely, the conjecture states that the S matrix obeys

$$T^+(f)\, S - S\, T^-(f) = 0. \tag{3.3.2}$$

The Ward identity corresponding to this relation is obtained by taking the matrix elements of the previous equation between Fock states (as stated above, the Fock representation is only allowed for vanishing Q_{zz}) with n incoming and m outgoing particles at z_k^{in}, respectively z_k^{out}, on the conformal sphere at $\mathscr{I}$, denoted by $|\text{in}\rangle \equiv |z_1^{\text{in}}, \ldots, z_n^{\text{in}}\rangle$ and $\langle\text{out}| \equiv \langle z_1^{\text{out}}, \ldots, z_m^{\text{out}}|$. These carry energies E_k^{in} and E_k^{out}, where $\sum E_k^{\text{in}} = \sum E_k^{\text{out}}$ by total energy conservation.

Choosing, for fixed direction z on the outgoing sphere, the function

$$f(w,\bar{w}) = \frac{1}{z-w} \tag{3.3.3}$$

and the "soft graviton current"

$$P_z \equiv \frac{1}{2G}\left(\int_{-\infty}^{+\infty} \partial_v V_z dv - \int_{-\infty}^{+\infty} \partial_u U_z du\right) = \frac{1}{2G}\left(V_z\Big|_{\mathscr{I}^-_-}^{\mathscr{I}^-_+} - U_z\Big|_{\mathscr{I}^+_-}^{\mathscr{I}^+_+}\right), \tag{3.3.4}$$

we can show that the matrix elements of (3.3.2) between the above states are

$$\langle\text{out}| : P_z S : |\text{in}\rangle = \langle\text{out}|S|\text{in}\rangle \left[\sum_{k=1}^{m} \frac{E_k^{\text{out}}}{z - z_k^{\text{out}}} - \sum_{k=1}^{n} \frac{E_k^{\text{in}}}{z - z_k^{\text{in}}}\right], \tag{3.3.5}$$

where the symbol : : denotes time-ordering. The argument [1] goes as follows: the operator

$$T^-(f) = \frac{1}{4\pi}\int_{\mathscr{I}^-_+} d^2z \gamma_{z\bar{z}} f m_B^- = \frac{1}{4\pi}\int dv d^2z f\left[\gamma_{z\bar{z}} T_{vv} + \frac{1}{2}\partial_v\left(\partial_z V_{\bar{z}} + \partial_{\bar{z}} V_z\right)\right] \tag{3.3.6}$$

generates supertranslations on $\mathscr{I}^-$, where T_{vv} is the total incoming radiation energy flux, rescaled by 4π, given by $M_{zz}M^{zz}$, and hence obeys the relation

$$T^-(f)|\text{in}\rangle = F^-|\text{in}\rangle + \sum_{k=1}^{n} E_k^{\text{in}} f(z_k)|\text{in}\rangle \tag{3.3.7}$$

where F^- denotes the incoming soft graviton operator with polarization tensor proportional to $D_z^2 f$:

$$F^-(f) = \frac{1}{4\pi}\int_{\mathscr{I}^-} dvd^2z f\partial_v\partial_{\bar{z}}V_z = \frac{1}{8\pi}\int_{\mathscr{I}^-} dvd^2z D_{\bar{z}}^2 f M_z^{\bar{z}}. \tag{3.3.8}$$

Note that we have used the boundary condition

$$\partial_z V_{\bar{z}} = \partial_{\bar{z}} V_z \text{ at } \mathscr{I}_\pm^-, \tag{3.3.9}$$

which is the counterpart of (3.2.10) on $\mathscr{I}^-$, in order to get rid of $V_{\bar{z}}$. Combing with the analogous result for $\mathscr{I}^+$, one can write

$$F \equiv F^+ - F^- = -\frac{1}{8\pi}\int d^2z\gamma^{z\bar{z}}D_{\bar{z}}^2 f\left[\int_{\mathscr{I}^-} dvM_{zz} + \int_{\mathscr{I}^+} duN_{zz}\right]. \tag{3.3.10}$$

Exploiting the supertranslation invariance of the S matrix:

$$\langle\text{out}| : FS : |\text{out}\rangle = \left[\sum_{k=1}^n E_k^{\text{in}} f(z_k^{\text{in}}) - \sum_{k=1}^m E_k^{\text{out}} f(z_k^{\text{out}})\right]\langle\text{out}|S|\text{out}\rangle. \tag{3.3.11}$$

Computing F for $f = (z-w)^{-1}$, in particular, gives

$$P_z = -\frac{1}{8\pi}\int d^2z\partial_z\left(\partial_{\bar{z}}\frac{1}{z-w}\right)\left[\int_{\mathscr{I}^-} dvM_{zz} + \int_{\mathscr{I}^+} duN_{zz}.\right] \tag{3.3.12}$$

Integrating by parts, using equations (3.1.11), (3.1.13) and $\partial_{\bar{z}}\frac{1}{z} = 2\pi\delta^2(z)$, this reverts to the above soft graviton current and its Ward identity is precisely the desired one.

3.3.2 From momentum space to position space

Much of the work needed, in order to make apparent the connection between (3.3.5) and Weinberg's result

$$S_{\beta\alpha}^{\pm 2}(\mathbf{q}) \approx \frac{\kappa}{2}\left\{\sum_n \eta_n \frac{\left[p_n\cdot\varepsilon_\pm^*(q)\right]^2}{p_n\cdot q}\right\} S_{\beta\alpha} \text{ for } \kappa^2 = 32\pi, \tag{3.3.13}$$

consists in translating the latter formula, which is written in momentum space, into the language of position space. To begin with, recall that the usual Minkowski coordinates $(t, x^1, x^2, x^3) = (t, \mathbf{x})$, in which the Minkowski metric is written as

$$ds^2 = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = -dt^2 + d\mathbf{x}\cdot d\mathbf{x}, \tag{3.3.14}$$

are related to the retarded Bondi coordinates used above by the transformation

$$t = u + r, \qquad x^1 + ix^2 = \frac{2rz}{1+z\bar{z}}, \qquad x^3 = \frac{r(1-z\bar{z})}{1+z\bar{z}}, \tag{3.3.15}$$

where $r = \left[(x^1)^2 + (x^2)^2 + (x^3)^2\right]^{1/2} = |\mathbf{x}|$. Consider now a wave packet for a massless particle with spatial momentum centered around $\mathbf{p}$ and satisfying the mass-shell relation $\mathbf{p}^2 = \omega^2$ or, in other words, with four-momentum $p = (\omega, \mathbf{p})$. At large times and large r this wave packet becomes localized on the conformal sphere near the point

$$\mathbf{p} = \omega\hat{\mathbf{x}} = \frac{\omega}{1+z\bar{z}}(z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z}), \tag{3.3.16}$$

so that the momentum of massless particles may be equivalently characterized by p^μ or $(\omega, z, \bar{z})$.

At late times $t \to \infty$ the gravitational field becomes free and can be approximated by the mode expansion

$$h^{\text{out}}_{\mu\nu}(x) = \sum_{\alpha=\pm}\int \frac{d^3q}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{q}}}\left[\varepsilon^{\alpha *}_{\mu\nu}(\mathbf{q})a^{\text{out}}_\alpha(\mathbf{q})e^{iq\cdot x} + \varepsilon^{\alpha}_{\mu\nu}(\mathbf{q})a^{\text{out}}_\alpha(\mathbf{q})^* e^{-iq\cdot x}\right], \tag{3.3.17}$$

where $q^0 = \omega_{\mathbf{q}} = |\mathbf{q}|$ for brevity, $\alpha = \pm$ are the two helicities and

$$[a^{\text{out}}_\alpha(\mathbf{q}), a^{\text{out}}_{\alpha'}(\mathbf{q}')^*] = 2\omega_{\mathbf{q}}(2\pi)^3\delta_{\alpha\alpha'}\delta(\mathbf{q}-\mathbf{q}'). \tag{3.3.18}$$

The polarization tensors can be chosen [54] such that $\varepsilon^{\pm\mu\nu} = \varepsilon^{\pm\mu}\varepsilon^{\pm\nu}$ where

$$\begin{aligned}\varepsilon^+(\mathbf{q}) &= \frac{1}{\sqrt{2}}(\bar{w}, 1, -i, -\bar{w})\\ \varepsilon^-(\mathbf{q}) &= \frac{1}{\sqrt{2}}(w, 1, i, -w) = \overline{\varepsilon^+(\mathbf{q})};\end{aligned} \tag{3.3.19}$$

note that, by direct calculation using the Minkowski metric, $\varepsilon^{\pm\mu\nu}q_\nu = 0$ and $\varepsilon^{\pm\mu}{}_\mu = 0$. Transforming to retarded Bondi coordinates, we get

$$\begin{aligned}\varepsilon^+_z(\mathbf{q}) &= \frac{\partial x^\mu}{\partial z}\varepsilon^+_\mu(\mathbf{q}) = \frac{\sqrt{2}r\bar{z}(\bar{w}-\bar{z})}{(1+z\bar{z})^2}\\ \varepsilon^-_z(\mathbf{q}) &= \frac{\partial x^\mu}{\partial z}\varepsilon^-_\mu(\mathbf{q}) = \frac{\sqrt{2}r(1+\bar{z}w)}{(1+z\bar{z})^2}\end{aligned} \tag{3.3.20}$$

By comparison with the expression of the metric in retarded Bondi coordinates, one has

$$C_{zz}(u, z, \bar{z}) = \kappa\lim_{r\to\infty}\frac{1}{r}h^{\text{out}}_{zz}(r, u, z, \bar{z}). \tag{3.3.21}$$

Using the transformation rule $h_{zz} = \partial_z x^\mu \partial_z x^\nu h_{\mu\nu}$, we get

$$C_{zz}(u, z, \bar{z}) = \kappa\lim_{r\to\infty}\frac{1}{r}\partial_z x^\mu\partial_z x^\nu h^{\text{out}}_{\mu\nu}(r, u, z, \bar{z}), \tag{3.3.22}$$

where $h_{\mu\nu}$ is given by the above mode expansion. Denoting by θ the angle between $\widehat{\mathbf{x}}$ and $\mathbf{q}$, recalling that $t = u + r$ and hence $iq \cdot x = -i\omega_{\mathbf{q}}(u+r) + i\omega_{\mathbf{q}} r\cos\theta = -i\omega_{\mathbf{q}}[u + r(1-\cos\theta)]$, and using the above properties of the polarization tensors, we have

$$\begin{aligned} C_{zz} &= \kappa \lim_{r\to\infty} \frac{1}{r} \left(\partial_z x^\mu \partial_z x^\nu\right) \sum_{\alpha=\pm} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{q}}} \left[\varepsilon^{\alpha *}_{\mu\nu}(\mathbf{q}) a^{\text{out}}_\alpha(\mathbf{q}) e^{-i\omega_{\mathbf{q}}[u+r(1-\cos\theta)]} + \text{h.c.}\right] \\ &= \kappa \lim_{r\to\infty} \frac{1}{r} \sum_{\alpha=\pm} \int_0^{+\infty} d\omega_{\mathbf{q}} \int_{-1}^{+1} \frac{\omega_{\mathbf{q}}}{4\pi^2} \left[\frac{2r^2 A^{*2}_\alpha}{(1+z\bar{z})^4} a^{\text{out}}_\alpha(\mathbf{q}) e^{-i\omega_{\mathbf{q}}[u+r(1-x)]} + \text{h.c.}\right] dx, \end{aligned} \tag{3.3.23}$$

where

$$A_\alpha \equiv \begin{cases} \bar{z}(\bar{w} - \bar{z}) & \text{if } \alpha = + \\ 1 + w\bar{z} & \text{if } \alpha = - \end{cases} \quad \text{and } A^*_\alpha \equiv \overline{A_{-\alpha}}. \tag{3.3.24}$$

Integrals of this form can be treated as follows: integrating by parts, the integral is split into the sum of three terms

$$\begin{aligned} &\int_0^{+\infty} d\omega_{\mathbf{q}} f(\omega_{\mathbf{q}}) \omega_{\mathbf{q}} \int_{-1}^{+1} g(x) r e^{i\omega_{\mathbf{q}} r(x-1)} dx \\ =& - i \int_0^{+\infty} d\omega_{\mathbf{q}} f(\omega_{\mathbf{q}}) g(1) \\ &+ i \int_0^{+\infty} d\omega_{\mathbf{q}} f(\omega_{\mathbf{q}}) g(-1) e^{-i2\omega_{\mathbf{q}} r} + i \int_0^{+\infty} d\omega_{\mathbf{q}} f(\omega_{\mathbf{q}}) \int_{-1}^{+1} g'(x) e^{i\omega_{\mathbf{q}} r(x-1)} dx; \end{aligned} \tag{3.3.25}$$

in the large r limit, the second and third terms tend to zero by the Riemann-Lebesgue lemma, and only the first one contributes. Thus only the contribution from $x = 1$, *i.e.* $\theta = 0$, is relevant in the $r \to \infty$ limit, corresponding to $w = z$, $\bar{w} = \bar{z}$ and

$$A_+ = 0 = A^*_-, \tag{3.3.26}$$

$$A^*_+ = 1 + z\bar{z} = A_-, \tag{3.3.27}$$

finally yielding

$$C_{zz} = -\frac{i\kappa}{4\pi^2(1+z\bar{z})^2} \int_0^\infty d\omega_{\mathbf{q}} \left[a^{\text{out}}_+(\omega_{\mathbf{q}}\widehat{\mathbf{x}}) e^{-i\omega_{\mathbf{q}} u} - a^{\text{out}}_-(\omega_{\mathbf{q}}\widehat{\mathbf{x}})^* e^{i\omega_{\mathbf{q}} u}\right]. \tag{3.3.28}$$

Defining

$$N^\omega_{zz}(z,\bar{z}) \equiv \int_{-\infty}^{+\infty} e^{i\omega u} \partial_u C_{zz} du, \tag{3.3.29}$$

and using the previous limiting expression for C_{zz}, we find

$$N^\omega_{zz} = -\frac{\kappa}{2\pi(1+z\bar{z})^2} \int_0^\infty d\omega_{\mathbf{q}}\, \omega_{\mathbf{q}} \left[a^{\text{out}}_+(\omega_{\mathbf{q}}\widehat{\mathbf{x}}) \delta(\omega_{\mathbf{q}} - \omega) + a^{\text{out}}_-(\omega_{\mathbf{q}}\widehat{\mathbf{x}})^* \delta(\omega_{\mathbf{q}} + \omega)\right], \tag{3.3.30}$$

so that, letting ω be a positive quantity,

$$N_{zz}^{+\omega}(z,\bar{z}) = -\frac{\kappa\omega a_+^{\text{out}}(\omega\hat{\mathbf{x}})}{2\pi(1+z\bar{z})^2}, \qquad N_{zz}^{-\omega}(z,\bar{z}) = -\frac{\kappa\omega a_-^{\text{out}}(\omega\hat{\mathbf{x}})^*}{2\pi(1+z\bar{z})^2}. \tag{3.3.31}$$

We regulate the zero mode by computing the Hermitian combination

$$\begin{aligned} N_{zz}^0 &\equiv \lim_{\omega\to 0^+}\frac{1}{2}(N_{zz}^{+\omega}+N_{zz}^{-\omega}) \\ &= -\frac{\kappa}{4\pi(1+z\bar{z})^2}\lim_{\omega\to 0^+}\left[\omega a_+^{\text{out}}(\omega\hat{\mathbf{x}})+\omega a_-^{\text{out}}(\omega\hat{\mathbf{x}})^*\right]; \end{aligned} \tag{3.3.32}$$

we shall see in Chapter 6 how this issue can be dealt with in a more rigorous fashion by smearing with suitable test functions. A parallel construction on $\mathscr{I}^-$, where

$$M_{zz}^{\omega}(z,\bar{z}) \equiv \int_{-\infty}^{+\infty} e^{i\omega v}\partial_v D_{zz}dv, \tag{3.3.33}$$

gives, for $\omega > 0$,

$$M_{zz}^{+\omega}(z,\bar{z}) = -\frac{\kappa\omega a_+^{\text{in}}(\omega\hat{\mathbf{x}})}{2\pi(1+z\bar{z})^2}, \qquad M_{zz}^{-\omega}(z,\bar{z}) = -\frac{\kappa\omega a_-^{\text{in}}(\omega\hat{\mathbf{x}})^*}{2\pi(1+z\bar{z})^2}, \tag{3.3.34}$$

and, for the zero mode,

$$M_{zz}^0(z,\bar{z}) = -\frac{\kappa}{4\pi(1+z\bar{z})^2}\lim_{\omega\to 0^+}\left[\omega a_+^{\text{in}}(\omega\hat{\mathbf{x}})+\omega a_-^{\text{in}}(\omega\hat{\mathbf{x}})^*\right]. \tag{3.3.35}$$

It follows from the definitions of N_{zz}^{ω} and M_{zz}^{ω}, and from the constraint solutions (3.2.13), that

$$N_{zz}^0(z,\bar{z}) = D_z^2 N, \qquad M_{zz}^0(z,\bar{z}) = D_z^2 M. \tag{3.3.36}$$

Defining $\mathcal{O}_{zz} \equiv N_{zz}^0(z,\bar{z}) + M_{zz}^0(z,\bar{z})$, the soft graviton current (3.3.4) can be written as

$$P_z = \frac{1}{2G}\left(\int_{-\infty}^{+\infty}\partial_v V_z dv - \int_{-\infty}^{+\infty}\partial_u U_z du\right) = \frac{1}{4G}\gamma^{z\bar{z}}\partial_{\bar{z}}\mathcal{O}_{zz}, \tag{3.3.37}$$

where again (3.1.11) and (3.1.13) have been taken into account.

3.3.3 Weinberg's soft graviton theorem as a BMS Ward identity

Using the expressions for the zero modes given above, we find

$$\langle\text{out}| :\mathcal{O}_{zz}S: |\text{in}\rangle = -\frac{\kappa}{4\pi(1+z\bar{z})^2}\lim_{\omega\to 0^+}\left[\omega\langle\text{out}|a_+^{\text{out}}(\omega\hat{\mathbf{x}})S|\text{in}\rangle + \omega\langle\text{out}|Sa_-^{\text{in}}(\omega\hat{\mathbf{x}})^*|\text{in}\rangle\right], \tag{3.3.38}$$

where we have used the time ordering prescription together with the fact that $a_-^{\text{out}}(\omega\hat{\mathbf{x}})^*$ (respectively $a_+^{\text{in}}(\omega\hat{\mathbf{x}})$) annihilates the out (in) state for $\omega \to 0$; note that this holds even when the asymptotic states contain soft gravitons because of the ω factor in the commutation relations.

The two matrix elements in the previous formula are equal by crossing symmetry: they describe symmetric processes involving a positive helicity outgoing graviton or a negative helicity incoming graviton.

Weinberg's soft graviton theorem for a positive helicity outgoing graviton reads

$$\lim_{\omega\to 0^+}\left[\omega\langle \text{out}|a_-^{\text{out}}(\omega\hat{\mathbf{x}})S|\text{in}\rangle\right] = \frac{\kappa}{2}\lim_{\omega\to 0^+}\left[\sum_{k=1}^{m}\frac{\omega[p_k'\cdot\varepsilon^+(\mathbf{q})]^2}{p_k'\cdot q} - \sum_{k=1}^{n}\frac{\omega[p_k\cdot\varepsilon^+(\mathbf{q})]^2}{p_k\cdot q}\right]\langle\text{out}|S|\text{in}\rangle. \tag{3.3.39}$$

We now employ the above described reparametrization of momenta:

$$\begin{aligned} p_k &= E_k^{\text{in}}\left(1, \frac{z_k^{\text{in}}+\bar{z}_k^{\text{in}}}{1+z_k^{\text{in}}\bar{z}_k^{\text{in}}}, \frac{1}{i}\frac{z_k^{\text{in}}-\bar{z}_k^{\text{in}}}{1+z_k^{\text{in}}\bar{z}_k^{\text{in}}}, \frac{1-z_k^{\text{in}}\bar{z}_k^{\text{in}}}{1+z_k^{\text{in}}\bar{z}_k^{\text{in}}}\right),\\ p_k' &= E_k^{\text{out}}\left(1, \frac{z_k^{\text{out}}+\bar{z}_k^{\text{out}}}{1+z_k^{\text{out}}\bar{z}_k^{\text{out}}}, \frac{1}{i}\frac{z_k^{\text{out}}-\bar{z}_k^{\text{out}}}{1+z_k^{\text{out}}\bar{z}_k^{\text{out}}}, \frac{1-z_k^{\text{out}}\bar{z}_k^{\text{out}}}{1+z_k^{\text{out}}\bar{z}_k^{\text{out}}}\right),\\ q &= \omega\left(1, \frac{z+\bar{z}}{1+z\bar{z}}, \frac{1}{i}\frac{z-\bar{z}}{1+z\bar{z}}, \frac{1-z\bar{z}}{1+z\bar{z}}\right),\\ \varepsilon^+(\mathbf{q}) &= \frac{1}{\sqrt{2}}(\bar{z}, 1, -i, -\bar{z}), \end{aligned} \tag{3.3.40}$$

with respect to which

$$\begin{aligned} \frac{\omega\left[p_k'\cdot\varepsilon^+(\mathbf{q})\right]^2}{p_k'\cdot q} &= -(1+z\bar{z})\frac{E_k^{\text{out}}(\bar{z}-\bar{z}_k^{\text{out}})}{(z-z_k^{\text{out}})(1+z_k^{\text{out}}\bar{z}_k^{\text{out}})},\\ \frac{\omega\left[p_k\cdot\varepsilon^+(\mathbf{q})\right]^2}{p_k\cdot q} &= -(1+z\bar{z})\frac{E_k^{\text{in}}(\bar{z}-\bar{z}_k^{\text{in}})}{(z-z_k^{\text{in}})(1+z_k^{\text{in}}\bar{z}_k^{\text{in}})}, \end{aligned} \tag{3.3.41}$$

leading to

$$\begin{aligned} \langle\text{out}|:\mathcal{O}_{zz}S:|\text{in}\rangle =& \frac{8G}{1+z\bar{z}}\langle\text{out}|S|\text{in}\rangle\\ &\times\left[\sum_{k=1}^{m}\frac{E_k^{\text{out}}(\bar{z}-\bar{z}_k^{\text{out}})}{(z-z_k^{\text{out}})(1+z_k^{\text{out}}\bar{z}_k^{\text{out}})} - \sum_{k=1}^{n}\frac{E_k^{\text{in}}(\bar{z}-\bar{z}_k^{\text{in}})}{(z-z_k^{\text{in}})(1+z_k^{\text{in}}\bar{z}_k^{\text{in}})}\right]. \end{aligned} \tag{3.3.42}$$

Plugging this expression into (3.3.37) we can relate the insertion of P_z to that of $\mathcal{O}_{zz}$:

$$\langle\text{out}|:P_zS:|\text{in}\rangle = \frac{1}{4G}\gamma^{z\bar{z}}\frac{\partial}{\partial\bar{z}}\langle\text{out}|:\mathcal{O}_{zz}S:|\text{in}\rangle; \tag{3.3.43}$$

the contribution of the sum over outgoing states yields

$$
\begin{aligned}
&\langle \mathrm{out}|S|\mathrm{in}\rangle(1+z\bar{z})^2\frac{\partial}{\partial\bar{z}}\left\{\frac{1}{1+z\bar{z}}\left[\sum_{k=1}^{m}\frac{E_k^{\mathrm{out}}(\bar{z}-\bar{z}_k^{\mathrm{out}})}{(z-z_k^{\mathrm{out}})(1+z_k^{\mathrm{out}}\bar{z}_k^{\mathrm{out}})}\right]\right\}\\
=&\langle \mathrm{out}|S|\mathrm{in}\rangle\left[\sum_{k=1}^{m}E_k^{\mathrm{out}}\frac{-z(\bar{z}-\bar{z}_k^{\mathrm{out}})+(1+z\bar{z})}{(z-z_k^{\mathrm{out}})(1+z_k^{\mathrm{out}}\bar{z}_k^{\mathrm{out}})}\right]\\
=&\langle \mathrm{out}|S|\mathrm{in}\rangle\left[\sum_{k=1}^{m}\left(\frac{E_k^{\mathrm{out}}}{z-z_k^{\mathrm{out}}}+\frac{E_k^{\mathrm{out}}\bar{z}_k^{\mathrm{out}}}{1+z_k^{\mathrm{out}}\bar{z}_k^{\mathrm{out}}}\right)\right],
\end{aligned}
\tag{3.3.44}
$$

then, combining with the analogous result for incoming states,

$$
\begin{aligned}
\langle \mathrm{out}|:P_z S:|\mathrm{in}\rangle =&\langle \mathrm{out}|S|\mathrm{in}\rangle\left[\sum_{k=1}^{m}\frac{E_k^{\mathrm{out}}}{z-z_k^{\mathrm{out}}}-\sum_{k=1}^{n}\frac{E_k^{\mathrm{in}}}{z-z_k^{\mathrm{in}}}\right]\\
&+\langle \mathrm{out}|S|\mathrm{in}\rangle\underbrace{\left[\sum_{k=1}^{m}\frac{E_k^{\mathrm{out}}\bar{z}_k^{\mathrm{out}}}{1+z_k^{\mathrm{out}}\bar{z}_k^{\mathrm{out}}}-\sum_{k=1}^{n}\frac{E_k^{\mathrm{in}}\bar{z}_k^{\mathrm{in}}}{1+z_k^{\mathrm{in}}\bar{z}_k^{\mathrm{in}}}\right]}_{\equiv\Delta}.
\end{aligned}
\tag{3.3.45}
$$

On the other hand, Δ is zero due to total momentum conservation, since

$$
\frac{E_k\bar{z}_k^{\mathrm{out}}}{1+z_k^{\mathrm{out}}\bar{z}_k^{\mathrm{out}}}=\frac{1}{2}\left({p'_k}^1-i{p'_k}^2\right),\qquad \frac{E_k\bar{z}_k^{\mathrm{in}}}{1+z_k^{\mathrm{in}}\bar{z}_k^{\mathrm{in}}}=\frac{1}{2}\left({p_k}^1-i{p_k}^2\right);
\tag{3.3.46}
$$

it should be stressed that this cancellation is possible only thanks to the constancy of the gravitational coupling (equivalence principle): we will discuss in Chapter 5 a slightly improved method which allows to avoid using this fact and we will see what role this improvement plays in the extension to higher spins.

We have shown that Weinberg's soft graviton theorem implies the BMS supertranslation Ward identity (3.3.5); as already stressed, under the *a priori* assumption of the equivalence principle, we can also run the above argument backwards to show that this supertranslation Ward identity implies Weinberg's soft graviton theorem, thus eventually proving the equivalence of the statements under given assumptions.

3.3.4 A higher-spin formula

We conclude this chapter by employing the same "momentum space"-"position space" dictionary, as above, to recast the Weinberg factorization result in terms of z and $\bar{z}$ in the more general case of arbitrary integer helicity s, *i.e.*

$$
\lim_{\omega\to 0}\langle \mathrm{out}|\omega a_+^{\mathrm{out}}(\mathbf{q})S|\mathrm{in}\rangle=-\lim_{\omega\to 0}\left[\sum_n \eta_n g_n^{(s)}\frac{(p_n\cdot\varepsilon_+(\mathbf{q}))^s}{p_n\cdot q}\right]\langle \mathrm{out}|S|\mathrm{in}\rangle,
\tag{3.3.47}
$$

where η_n is defined to be + when incoming, − when outgoing. Using

$$\omega \frac{(p_n \cdot \varepsilon_+(\mathbf{q}))^s}{p_n \cdot q} = -(-1)^s 2^{s/2-1}(1+z\bar{z})\frac{(E_n)^{s-1}(\bar{z}-\bar{z}_n)^{s-1}}{(z-z_n)(1+z_n\bar{z}_n)^{s-1}}, \tag{3.3.48}$$

one has

$$\begin{aligned}&\lim_{\omega\to 0}\langle \text{out}|\omega a_+^{\text{out}}(\mathbf{q})S|\text{in}\rangle \\ =&\lim_{\omega\to 0}(-1)^s 2^{s/2-1}(1+z\bar{z})\left[\sum_n \eta_n g_n^{(s)} \frac{(E_n)^{s-1}(\bar{z}-\bar{z}_n)^{s-1}}{(z-z_n)(1+z_n\bar{z}_n)^{s-1}}\right]\langle \text{out}|S|\text{in}\rangle.\end{aligned} \tag{3.3.49}$$

4 $U(1)$ Large Gauge Symmetries and the Soft Photon Theorem

What are the asymptotic symmetries at $\mathscr{I}^+$ of electrodynamics with massless charged particles? [14]

At first sight, this appears to be just a toy version of the question asked by BMS. It is however of interest in its own right as well as for a warm-up to the higher-spin case: part of this problem is defining both what is meant by asymptotic symmetries and how they act; this allows us to illustrate the concept of asymptotic or "large" gauge symmetry, which will be crucial in the sequel.

We will also see how Weinberg's result for spin one (2.1.5) is equivalent to suitable large gauge symmetries for massless electrodynamics.

4.1 Electrodynamics in "Radial Gauge"

In a recent paper [15], the physical relevance of $U(1)$ asymptotic symmetries has been stressed, using a strategy similar to the one adopted for gravity in the previous chapter.

4.1.1 Action and asymptotic equations of motion in radial gauge

One starts from Minkowski spacetime, whose metric in retarded coordinates reads

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}, \tag{4.1.1}$$

where, as already mentioned, $\gamma_{z\bar{z}} = 2/(1+z\bar{z})^2$ is the Euclidean metric of the sphere in conformal coordinates, while the transformation rules from the usual Minkowski coordinates are given by

$$t = u + r, \qquad x^1 + ix^2 = \frac{2rz}{1+z\bar{z}}, \qquad x^3 = \frac{r(1-z\bar{z})}{1+z\bar{z}}. \tag{4.1.2}$$

The Christoffel symbols compatible with $g_{\mu\nu}$ are

$$\Gamma^z_{rz} = \frac{1}{r}, \qquad \Gamma^z_{zz} = \partial_z \log\gamma_{z\bar{z}}, \qquad \Gamma^u_{z\bar{z}} = r\gamma_{z\bar{z}}, \qquad \Gamma^r_{z\bar{z}} = -r\gamma_{z\bar{z}}. \tag{4.1.3}$$

Consider now $U(1)$ electrodynamics coupled to an external source, subsuming the role of the matter sector, thus given by the action

$$S = -\frac{1}{4e^2}\int d^4x\sqrt{-g}\,\mathcal{F}_{\mu\nu}\mathcal{F}_{\alpha\beta}g^{\mu\alpha}g^{\nu\beta} - \int d^4x\sqrt{-g}\,\mathcal{J}^\mu\mathcal{A}_\mu, \tag{4.1.4}$$

where $\mathcal{F}_{\mu\nu} = \nabla_\mu\mathcal{A}_\nu - \nabla_\nu\mathcal{A}_\mu$; the $U(1)$ gauge invariance under

$$\delta_{\hat{\varepsilon}}\mathcal{A}_\mu(u,r,z,\bar{z}) = \nabla_\mu\hat{\varepsilon}(u,r,z,\bar{z}), \tag{4.1.5}$$

which is automatically satisfied by the first term, requires $\nabla_\mu\mathcal{J}^\mu = 0$ from the second term:

$$\begin{aligned} -\int d^4x\sqrt{-g}\,\mathcal{J}^\mu\delta_{\hat{\varepsilon}}A_\mu &= -\int d^4x\sqrt{-g}\,\mathcal{J}^\mu\nabla_\mu\hat{\varepsilon} \\ &= -\int d^4x\partial_\mu\left(\sqrt{-g}\mathcal{J}^\mu\hat{\varepsilon}\right) + \int d^4x\sqrt{-g}\,\left(\nabla_\mu\mathcal{J}^\mu\right)\hat{\varepsilon}, \end{aligned} \tag{4.1.6}$$

Where we used the standard identity $\sqrt{-g}\nabla_\mu V^\mu = \partial_\mu(\sqrt{-g}V^\mu)$. The first contribution can be reduced to integrals over spacelike hypersurfaces in the far past and in the far future, where $\mathcal{J}^\mu$ is supposed to vanish, and the second requires covariant conservation of $\mathcal{J}^\mu$, by the arbitrariness of $\hat{\varepsilon}$. Since we will be interested in writing down Ward identities at $\mathscr{I}^+$, at least for now $\mathcal{J}^\mu$ will be thought of as a massless charged current.

The equations of motion are the Maxwell equations $\nabla^\mu\mathcal{F}_{\mu\nu} = e^2\mathcal{J}^\mu$ in curved coordinates. The energy-momentum tensor $T_{\mu\nu}$ of the electromagnetic field is on the other hand obtained by varying the action with respect to $g^{\mu\nu}$:

$$\delta_{g^{\mu\nu}}S = -\frac{1}{2e^2}\int d^4x\,\underbrace{\sqrt{-g}\left[\mathcal{F}_{\mu\alpha}\mathcal{F}_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}\left(\mathcal{F}^2 - 4e^2\mathcal{J}\cdot\mathcal{A}\right)\right]}_{=T_{\mu\nu}}\delta g^{\mu\nu}, \tag{4.1.7}$$

where $\mathcal{F}^2 \equiv \mathcal{F}_{\mu\nu}\mathcal{F}_{\alpha\beta}g^{\mu\alpha}g^{\nu\beta}$ and $\mathcal{J}\cdot\mathcal{A} = \mathcal{J}^\mu\mathcal{A}_\mu$. The energy stored at future null infinity $\mathscr{I}^+$ is given as follows: recall that here the conformal factor defining $\mathscr{I}^+$ is $\Omega = r$, hence $n^\mu = g^{\mu\nu}\nabla_\nu r = g^{\mu r}$, and that the asymptotic time translation vector at null infinity is given by $t^\nu = (\partial_u)^\nu = \delta^\nu_u$, thus, recalling equation (1.2.1), we find

$$E(\mathscr{I}^+) = \int_{\mathscr{I}^+} T_{\mu\nu}n^\mu t^\nu = \int_{\mathscr{I}^+} T^r{}_u = \int_{\mathscr{I}^+}\mathcal{F}^r{}_\alpha\mathcal{F}_u{}^\alpha, \tag{4.1.8}$$

where the natural measure element is understood; note that the terms proportional to $g_{\mu\nu}$ vanish in this step. Now, we fix the gauge by imposing

$$\begin{aligned} \mathcal{A}_r &= 0 \\ \mathcal{A}_u\Big|_{\mathscr{I}^+} &= 0, \end{aligned} \tag{4.1.9}$$

which is called *retarded radial gauge*. Explicitly, in this gauge,

$$E(\mathscr{I}^+) = \int_{-\infty}^{+\infty} du \int d^2z \gamma_{z\bar{z}} \sum_{a=z,\bar{z}} \left(-\partial_u \mathcal{A}_a + \partial_a \mathcal{A}_u + \partial_r\right) \left(\partial_u \mathcal{A}_{\bar{a}} - \partial_{\bar{a}} \mathcal{A}_u\right), \tag{4.1.10}$$

where the r^2 coming from the metric determinant gets cancelled in the non-vanishing contractions. As for the gravitational case, a delicate point in the discussion of asymptotic symmetries consists in the assignment of proper falloff conditions on the remaining field components; the very existence and meaning of asymptotic symmetries depends on the choice of the functional class of solutions to the dynamics one considers to be physically sensible, in relation to this behavior at large distances.

In view of our gauge choice and of the expression of the energy flux at future null infinity, a sensible falloff choice for the components of the gauge potential in this case appears to be

$$\begin{aligned} \mathcal{A}_z(r,u,z,\bar{z}) =& A_z(u,z,\bar{z}) + \frac{1}{r} A_z^{(1)}(u,z,\bar{z}) + \dots \\ \mathcal{A}_u(r,u,z,\bar{z}) =& \frac{1}{r} A_u(u,z,\bar{z}) + \dots \end{aligned} \tag{4.1.11}$$

and hence

$$\begin{aligned} \mathcal{F}_{z\bar{z}} =& \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + \dots \\ \mathcal{F}_{uz} =& \partial_u A_z + \dots \\ \mathcal{F}_{rz} =& -\frac{1}{r^2} A_z^{(1)} + \dots \\ \mathcal{F}_{ur} =& \frac{1}{r^2} A_u + \dots \ . \end{aligned} \tag{4.1.12}$$

We shall denote the leading order coefficients in (4.1.12) as $F_{z\bar{z}}$, F_{uz}, F_{rz} and F_{ur}, respectively. Using the Christoffel symbols written above, the equation of motion for $\nu = u$ at leading order reads

$$\gamma_{z\bar{z}} \partial_u A_u = \partial_u(\partial_z A_{\bar{z}} + \partial_{\bar{z}} A_z) + e^2 j_u \gamma_{z\bar{z}}, \tag{4.1.13}$$

where $j_u(u,z,\bar{z}) = \lim_{r\to\infty} \left[r^2 \mathcal{J}_u(r,u,z,\bar{z})\right]$; using this equation and imposing the boundary conditions

$$F_{ur}\Big|_{\mathscr{I}^+_+} = 0 = F_{uz}\Big|_{\mathscr{I}^+_+} \tag{4.1.14}$$

for A_u and A_z, we can express A_u in terms of A_z and $A_{\bar{z}}$, given j_u. Therefore A_z and $A_{\bar{z}}$ play the role of coordinates of the asymptotic phase space at $\mathscr{I}^+$.

4.1.2 Large gauge transformations

The gauge fixing (4.1.9) can be operatively defined in the following way: starting from an unconstrained $\mathcal{A}_\mu(r,u,z,\bar{z})$, one can choose a gauge parameter $\hat{\varepsilon}(r,u,z,\bar{z})$ satisfying the boundary problem

$$\begin{aligned} \partial_r\hat{\varepsilon}(r,u,z,\bar{z})+\mathcal{A}_r(r,u,z,\bar{z}) =&0,\\ \partial_u\hat{\varepsilon}(r=\infty,u,z,\bar{z})+\mathcal{A}_u(r=\infty,u,z,\bar{z}) =&0. \end{aligned} \tag{4.1.15}$$

Performing a gauge transformation with this gauge parameter $\hat{\varepsilon}$ indeed enforces the retarded radial gauge, as requested. We see now that $\hat{\varepsilon}$ is still determined only up to an arbitrary function $\varepsilon(z,\bar{z})$ of the angular coordinates on the sphere at null infinity: we have therefore *residual* or *large*[1] gauge transformations acting at $\mathscr{I}^+$ via

$$\delta_\varepsilon A_z(u,z,\bar{z})=\partial_z\varepsilon(z,\bar{z}), \tag{4.1.16}$$

and similarly for $\bar{z}$. The conserved charge associated to this symmetry can be computed by the Noether procedure. The Lagrangian is invariant up to a total divergence

$$\begin{aligned} \delta_\varepsilon\mathcal{L}=&-\frac{1}{e^2}\sqrt{-g}\left(\nabla_\mu\delta_\varepsilon\mathcal{A}_\nu\right)\mathcal{F}^{\mu\nu}-\sqrt{-g}\mathcal{J}^\mu\delta_\varepsilon\mathcal{A}_\mu\\ =&-\frac{1}{e^2}\sqrt{-g}\left(\nabla_\mu\nabla_\nu\varepsilon\right)\mathcal{F}^{\mu\nu}-\sqrt{-g}\mathcal{J}^\mu\nabla_\mu\varepsilon=-\partial_\mu\left(\sqrt{-g}\mathcal{J}^\mu\varepsilon\right), \end{aligned} \tag{4.1.17}$$

and, on the other hand, integration by parts gives

$$\begin{aligned} \delta_\varepsilon\mathcal{L}=&-\frac{1}{e^2}\sqrt{-g}\left(\nabla_\mu\delta_\varepsilon\mathcal{A}_\nu\right)\mathcal{F}^{\mu\nu}-\sqrt{-g}\mathcal{J}^\mu\delta_\varepsilon\mathcal{A}_\mu\\ =&-\frac{1}{e^2}\partial_\mu\left(\sqrt{-g}\mathcal{F}^{\mu\nu}\nabla_\nu\varepsilon\right)+\frac{1}{e^2}\sqrt{-g}\underbrace{\left(\nabla_\mu\mathcal{F}^{\mu\nu}-e^2\mathcal{J}^\nu\right)}_{=0\text{ on shell}}\delta_\varepsilon\mathcal{A}_\nu, \end{aligned} \tag{4.1.18}$$

so that the current

$$j^\mu=\frac{1}{e^2}\sqrt{-g}\left(-\mathcal{F}^{\mu\nu}\nabla_\nu\varepsilon+e^2\mathcal{J}^\mu\varepsilon\right)\text{ satisfies }\partial_\mu j^\mu=0; \tag{4.1.19}$$

again integrating by parts, employing the equations of motion and using the antisymmetry of $\mathcal{F}_{\mu\nu}$, one also gets

$$j^\mu=\frac{1}{e^2}\partial_\mu\left(\sqrt{-g}\mathcal{F}^{\mu\nu}\varepsilon\right). \tag{4.1.20}$$

---

[1] Gravity can be understood as a gauge theory of diffeomorphism symmetry. One therefore expects to identify states that differ by a diffeomorphism; however, as we have seen, there is a class of diffeomorphisms that fall off slowly enough at large r to affect the radiative data and give finite asymptotically conserved charges (*e.g.* the Bondi mass). These "large diffeomorphisms" should not be quotiented out of the space of states, in contradistinction to the familiar "small diffeomorphisms".

In gauge theories involving spin-one fields, one can make similar consideration and therefore distinguish between small and large $U(1)$ (local) transformations, also in analogy with the fact that one does not quotient out the global part of the gauge group. This issue only arises because we have a manifold with boundary, and boundary conditions, meaning that there can also be forbidden diffeomorphisms that violate the boundary conditions. The asymptotic symmetry group can be thought of as the small diffeomorphism equivalence classes of the allowed diffeomorphisms. The classification of large versus small diffeomorphisms is part of the definition of the theory that depends on the boundary conditions, and is determined by which states are to be regarded as physically equivalent and which are not [36, 55].

This is the local Gauss law, a consequence of Noether's second theorem: the local invariance of the action under gauge transformations ensures that (when the gauge fields are on shell) the corresponding Noether current is equal to the divergence of an antisymmetric tensor. The Noether charge is computed as the integral of j^μ on a three-dimensional Cauchy hypersurface, in our case $\mathscr{I}^+$, and can be recast as the integral over the boundary $\mathscr{I}^+_\pm$ of $\mathscr{I}^+$ as a consequence of the local Gauss law:[2]

$$\begin{aligned} Q^+_\varepsilon =& \int_{\mathscr{I}^+} j^\nu n_\nu = \frac{1}{e^2}\int_{\mathscr{I}^+} \partial_\mu \left(\sqrt{-g}\mathcal{F}^{\mu\nu}\varepsilon\right) n_\nu \\ =& \frac{1}{e^2}\left(\int_{\mathscr{I}^+_+} - \int_{\mathscr{I}^+_-}\right) t_\mu \sqrt{-g}\mathcal{F}^{\mu\nu}\varepsilon n_\nu . \end{aligned} \tag{4.1.22}$$

Using the appropriate unit normal vectors $n_\nu = \nabla_\nu r = \delta^r_\nu$ and $t_\mu = \delta^u_\mu$, recalling that $\sqrt{-g} = r^2\gamma_{z\bar{z}}$ and $\mathcal{F}_{ur} = r^{-2}F_{ur}$, with F_{ur} defined below (4.1.12), we find

$$Q^+_\varepsilon = \frac{1}{e^2}\left(\int_{\mathscr{I}^+_+} - \int_{\mathscr{I}^+_-}\right)\gamma_{z\bar{z}}F_{ur}\varepsilon d^2z \tag{4.1.23}$$

and given the assumption $F_{ur}\big|_{\mathscr{I}^+_+} = 0$, finally,

$$Q^+_\varepsilon = \frac{1}{e^2}\int d^2z\gamma_{z\bar{z}}F_{ru}\varepsilon(z,\bar{z}). \tag{4.1.24}$$

Using (4.1.13) and since $F_{ru} = -A_u$, one gets

$$\begin{aligned} Q^+_\varepsilon =& -\frac{1}{e^2}\int_{\mathscr{I}^+_-} d^2z\left(\partial_z A_{\bar{z}} - \partial_{\bar{z}}A_z\right)\varepsilon(z,\bar{z}) + \int_{\mathscr{I}^+} dud^2z\gamma_{z\bar{z}}j_u\varepsilon(z,\bar{z}) \\ =& \frac{1}{e^2}\int du\, d^2z\left[\partial_u\left(\partial_z A_{\bar{z}} - \partial_{\bar{z}}A_z\right) + e^2\gamma_{z\bar{z}}j_u\right]\varepsilon(z,\bar{z}). \end{aligned} \tag{4.1.25}$$

For $\varepsilon(z,\bar{z}) = 1$, noting that total derivatives of angular variables vanish, one recovers the total electric charge accumulated at $\mathscr{I}^+$,

$$Q^+_1 = \int_{\mathscr{I}^+} dud^2z\gamma_{z\bar{z}}j_u, \tag{4.1.26}$$

[2] This is again a general consequence of Noether's second theorem: since the conserved current j^μ satisfies $j^\mu = \partial_\nu\kappa^{\mu\nu}$, where $\kappa^{\mu\nu}$ is antisymmetric, the associated conserved charge, defined as the integral of j^μ over a $(D-1)$-dimensional Cauchy hypersurface Σ, can be recast as the integral of $\kappa^{\mu\nu}$ over the $(D-2)$-dimensional boundary σ of Σ:

$$Q = \int_\Sigma j^\mu d\Sigma_\mu = \int_\sigma \kappa^{\mu\nu}d\sigma_{\mu\nu}. \tag{4.1.21}$$

whereas for functions $\varepsilon(z,\bar{z})$ peaked at some generator $(w,\bar{w})$, approximating $\delta^2(z-w)$, one has the radiated electric charge at a given angle

$$Q^+_{w\bar{w}} = -\frac{1}{e^2}\left(\partial_w A_{\bar{w}} - \partial_{\bar{w}} A_w\right)\Big|_{u=-\infty} + \int_{-\infty}^{+\infty} du \gamma_{w\bar{w}} j_u. \tag{4.1.27}$$

The charge Q^+_ε is assumed to act on a massless matter field Φ carrying charge q in the usual way

$$[Q^+_\varepsilon, \Phi(u,z,\bar{z})] = -e\varepsilon(z,\bar{z})\Phi(u,z,\bar{z}). \tag{4.1.28}$$

4.2 Asymptotic Phase Space and Canonical Formulation

The commutators on the physical radiative phase space $\{F_{uz}, F_{u\bar{z}}\}$ at $\mathscr{I}^+$ were found by Ashtekar [21]

$$[F_{uz}(u,z,\bar{z}), F_{u\bar{z}}(u',w,\bar{w})] = \frac{ie^2}{2}\partial_u \delta(u-u')\delta^2(z-w), \tag{4.2.1}$$

where $\delta^2(z)$ is a shorthand notation for $\delta(z,\bar{z})$; just like in the gravitational case, however, one wishes to enlarge this asymptotic phase space so as to take the role of soft modes into account. Again note that the zero-mode of F_{uz}, *i.e.* the soft photon given by

$$\int_{-\infty}^{+\infty} du\, F_{uz}, \tag{4.2.2}$$

has vanishing commutator with the physical phase space, or, in other words, it generates the center of this observable algebra.

Such an enlargement is achieved as follows. First, one considers the fields A_z and $A_{\bar{z}}$, with the following *bulk* commutator

$$[A_z(u,z,\bar{z}), A_{\bar{z}}(u',w,\bar{w})] = -\frac{ie^2}{4}\text{sign}(u-u')\delta^2(z-w), \tag{4.2.3}$$

obtained by integrating the previous one and fixing the integration constants by antisymmetry. Then, motivated by the fact that the charge Q^+_ε does *not* generate the correct transformation law on A_z with this symplectic form (in particular, it is off once again by a factor of two as in Section 3.2), one introduces the boundary fields

$$A_z^{\pm}(z,\bar{z}) \equiv \lim_{u\to\pm\infty} A_z(u,z,\bar{z}), \tag{4.2.4}$$

with the constraint that there should be no long-range magnetic fields at $\mathscr{I}^+$, *i.e.*

$$F_{z\bar{z}}\Big|_{\mathscr{I}^+_\pm} = 0. \tag{4.2.5}$$

This can be also read as an integrability condition for $A_z^{\pm}$, which can therefore be expressed as

$$A_z^{\pm}(z,\bar{z}) = e^2 \partial_z \phi_{\pm}(z,\bar{z}), \tag{4.2.6}$$

for some scalar fields $\phi_{\pm}(z,\bar{z})$ living on the boundary $\mathscr{I}^+_{\pm}$ of $\mathscr{I}^+$. For the boundary fields $A_z^{\pm}$, we impose

$$\begin{aligned}[A_z^{\pm}(z,\bar{z}), A_{\bar{z}}(u',w,\bar{w})] &= \lim_{u\to\pm\infty}[A_z(u,z,\bar{z}), A_{\bar{z}}(u',w,\bar{w})] \\ &= \mp \frac{ie^2}{4}\delta^2(z-w)\end{aligned} \tag{4.2.7}$$

and

$$\begin{aligned}[A_z^{+}(z,\bar{z}) - A_z^{-}(z,\bar{z}), A_{\bar{z}}^{\pm}(w,\bar{w})] &= \lim_{u'\to\pm\infty}[A_z^{+}(z,\bar{z}) - A_z^{-}(z,\bar{z}), A_{\bar{z}}(u',w,\bar{w})] \\ &= -\frac{ie^2}{2}\delta^2(z-w).\end{aligned} \tag{4.2.8}$$

In terms of the boundary fields $\phi_{\pm}(z,\bar{z})$, these commutators reads

$$\begin{aligned}[\phi_{\pm}(z,\bar{z}), A_z(u',w,\bar{z})] &= \mp \frac{i}{8\pi}\frac{1}{z-w} \\ [\phi_{+}(z,\bar{z}), \phi_{-}(w,\bar{w})] &= \frac{i}{8\pi e^2}\log|z-w|^2.\end{aligned} \tag{4.2.9}$$

Now the charge, rewritten as

$$Q_{\varepsilon}^{+} = 2\int_{S^2} \partial_z \partial_{\bar{z}}(\phi_+ - \phi_-)\varepsilon(z,\bar{z})d^2z + \int_{\mathscr{I}^+} dud^2z\varepsilon(z,\bar{z})j_u\gamma_{z\bar{z}}, \tag{4.2.10}$$

clearly generates the correct transformation on the gauge field A_z (thanks to the factor of two appearing in front)

$$[Q_{\varepsilon}^{+}, A_z(u,z,\bar{z})] = i\partial_z\varepsilon(z,\bar{z}), \tag{4.2.11}$$

and in general satisfies the following commutation relations

$$\begin{aligned}[Q_{\varepsilon}^{+}, \phi_-(z,\bar{z})] &= \frac{i}{e^2}\varepsilon(z,\bar{z}), \\ [Q_{\varepsilon}^{+}, e^{ine^2\phi_-}] &= -n\varepsilon e^{ine^2\phi_-},\end{aligned} \tag{4.2.12}$$

where the last one is easily derived using the Baker-Hausdorff formula. The so-derived algebra is Abelian

$$[Q_{\varepsilon}^{+}, Q_{\eta}^{+}] = 0. \tag{4.2.13}$$

Notice in particular that, in a vacuum $|0\rangle$, defined by $\phi_-(z,\bar{z})|0\rangle = 0$, we have

$$\langle\delta_{\varepsilon}\phi_-\rangle = -i\langle[Q_{\varepsilon}^{+}, \phi_-(z,\bar{z})]\rangle = \frac{1}{e^2}\varepsilon(z,\bar{z}) \neq 0 \tag{4.2.14}$$

which indicates the spontaneous breaking of residual gauge symmetry in $|0\rangle$. This discussion shows how the observable algebra generated by F_{uz} and $F_{u\bar{z}}$ can be consistently enlarged to include the soft modes $\phi_{\pm}(z,\bar{z})$ as Goldstone modes of spontaneously broken large $U(1)$ symmetry.

4.2.1 Ward identities for large $U(1)$ gauge symmetry

After performing at $\mathscr{I}^-$ a construction similar to the one discussed in the previous section, with large gauge symmetries generated by charges Q^-_ε, labelled by angular functions $\varepsilon^-(z,\bar{z})$, one identifies generators at $\mathscr{I}^+$ with those at $\mathscr{I}^-$ under the requirement

$$\varepsilon(z,\bar{z}) = \varepsilon^-(z,\bar{z}), \tag{4.2.15}$$

which is the usual antipodal identification. Notice that now $\varepsilon(z,\bar{z})$ is a function of the space of null generators of the whole $\mathscr{I}$.

Large gauge symmetries are now postulated to be symmetries of the S matrix, *i.e.*, employing the same notation as the one in the previous chapter, one assumes that

$$\langle \text{out}|(SQ^-_\varepsilon - Q^+_\varepsilon S)|\text{in}\rangle = 0, \tag{4.2.16}$$

where

$$Q^+_\varepsilon = \underbrace{\frac{1}{e^2}\int d^2z du\, \partial_u \left(\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z\right) \varepsilon(z,\bar{z})}_{\equiv F^+(\varepsilon)} + \int d^2z du\, \gamma_{z\bar{z}} j_u \varepsilon(z,\bar{z}), \tag{4.2.17}$$

$$Q^-_\varepsilon = \underbrace{\frac{1}{e^2}\int d^2z dv\, \partial_v \left(\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z\right) \varepsilon(z,\bar{z})}_{\equiv F^-(\varepsilon)} + \int d^2z dv\, \gamma_{z\bar{z}} j_v \varepsilon(z,\bar{z}). \tag{4.2.18}$$

Assuming the semiclassical identity

$$\langle \text{out}|Q^+_\varepsilon = \langle \text{out}|F^+(\varepsilon) + \langle \text{out}|\sum_n e_n \varepsilon(z_n,\bar{z}_n) \tag{4.2.19}$$

together with a similar one for Q^-_ε, and denoting

$$F(\varepsilon) = F^+(\varepsilon) - F^-(\varepsilon), \tag{4.2.20}$$

we can recast the large gauge symmetry of the S matrix in the following form

$$\langle \text{out}| : SF(\varepsilon) : |\text{in}\rangle = \sum_n \eta_n q_n \varepsilon(z_n,\bar{z}_n) \langle \text{out}|S|\text{in}\rangle, \tag{4.2.21}$$

where η_n is $+1$ (resp. -1) for incoming (resp. outgoing) particles in the process.

Now, expanding the vector potential near $\mathscr{I}^{\pm}$ and using a stationary phase approximation analogous to the one employed in the previous chapter, one is able to express $F(\varepsilon)$ in terms of soft photon creation operators in the following manner: near $\mathscr{I}^{+}$, writing $\mathcal{A}_{\mu}(x)$ in terms of a free-field mode decomposition and using

$$\varepsilon_z^{+}(\mathbf{q}) = \frac{\sqrt{2}r\bar{z}(\bar{w}-\bar{z})}{(1+z\bar{z})^2}, \qquad \varepsilon_z^{-}(\mathbf{q}) = \frac{\sqrt{2}r(1+\bar{z}w)}{(1+z\bar{z})^2}, \tag{4.2.22}$$

one has

$$\begin{aligned} A_z(u,z,\bar{z}) &= \lim_{r\to\infty}\mathcal{A}_z(u,r,z,\bar{z}) \\ &= -\frac{i}{8\pi^2}\frac{\sqrt{2}e}{1+z\bar{z}}\int_0^{\infty} d\omega_{\mathbf{q}}\left[a_+^{\text{out}}(\omega_{\mathbf{q}}\hat{x})e^{-i\omega_{\mathbf{q}}u} - a_-^{\text{out}}(\omega_{\mathbf{q}}\hat{x})^{*}e^{-i\omega_{\mathbf{q}}u}\right]. \end{aligned} \tag{4.2.23}$$

Upon defining

$$N_z^{\omega}(z,\bar{z}) \equiv \int_{-\infty}^{+\infty} du\, e^{i\omega u}\partial_u A_z \tag{4.2.24}$$

and

$$N_z^{0}(z,\bar{z}) \equiv \lim_{\omega\to 0^+}\left(N_z^{\omega} + N_z^{-\omega}\right) \tag{4.2.25}$$

one finds in addition

$$N_z^{0}(z,\bar{z}) = -\frac{1}{8\pi}\frac{\sqrt{2}e}{1+z\bar{z}}\lim_{\omega\to 0^+}\left[\omega a_+^{\text{out}}(\omega\hat{x}) + \omega a_-^{\text{out}}(\omega\hat{x})^{*}\right]. \tag{4.2.26}$$

On the other hand, including the corresponding term M_z^0 from $\mathscr{I}^{-}$, we have, recalling that $F_{uz} = \partial_u A_z$ and $\partial_{\bar{z}}(1/z) = 2\pi\delta^2(z)$,

$$N_z^0 - M_z^0 = \frac{e^2}{4\pi}F\left[\frac{1}{z-w}\right]. \tag{4.2.27}$$

Notice that the auxiliary boundary condition

$$\partial_z A_{\bar{z}} = \partial_{\bar{z}} A_z \text{ at } \mathscr{I}^{+}_{\pm} \tag{4.2.28}$$

has been used. Substituting in the Ward identity (4.2.21) this reads

$$\lim_{\omega\to 0^+}\left[\omega\langle\text{out}|a_+^{\text{out}}(\omega\hat{x})S|\text{in}\rangle\right] = -\frac{1+z\bar{z}}{\sqrt{2}}\sum_n \eta_n\frac{q_n}{z-z_n}\langle\text{out}|S|\text{in}\rangle, \tag{4.2.29}$$

which we eventually recognize as Weinberg's soft photon theorem in position space as summarized in (3.3.49) for $s=1$.

5 Asymptotic Symmetries and Soft Theorems for Arbitrary Spin

This chapter is mainly devoted to the extension of the results which we presented in Chapters 3 and 4 to higher spins.

The main new result is the proof that Weinberg's factorization theorem follows as the Ward identity of spin-s large gauge symmetries. Moreover, as a byproduct of our approach, we shall also be able to provide a slight improvement of the proof of the soft graviton theorem from supertranslation symmetry, without assuming the equivalence principle from the start.

As an intermediate step, it is worthwhile to carefully revise both the spin-one and spin-two cases in order to bring out some elements which will be necessary for the subsequent extension to all spins; in particular, we provide a treatment of $U(1)$ large gauge symmetries more similar to the one given in [16] for electromagnetism, together with a thorough analysis of the BMS group from the perspective of the linearized theory.

Eventually we shall deal with the extension to higher spins. For the sake of clarity, we shall first illustrate the spin-3 case, where all the new ingredients are already present in a relatively simpler setting, to then move to illustrating the general case of arbitrary integer spin s.

5.1 Electromagnetism Revisited

As we have seen, the action for electromagnetism coupled to a locally conserved current $\mathcal{J}^\mu$,

$$S = -\frac{1}{4}\int \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} d^D x - \int \mathcal{A}_\mu \mathcal{J}^\mu d^D x, \tag{5.1.1}$$

being invariant under $\delta\mathcal{A}_\mu = \partial_\mu \varepsilon$ up to the boundary term, possesses the canonical current

$$j^\mu = \mathcal{F}^{\nu\mu}\partial_\nu \varepsilon + \mathcal{J}^\mu \varepsilon. \tag{5.1.2}$$

5.1.1 Large $U(1)$ gauge charge

In Bondi coordinates, near $\mathscr{I}^+$, in the case $\mathcal{J} = 0$,

$$Q^+ = \int_{\mathscr{I}^+} j^r \gamma_{z\bar{z}} r^2 du d^2 z. \tag{5.1.3}$$

Choosing retarded radial gauge

$$\begin{aligned}\mathcal{A}_u =& A_u(u,z,\bar{z})/r + \dots \\ \mathcal{A}_r =& 0 \\ \mathcal{A}_z =& A_z(u,z,\bar{z}) + \dots \\ \mathcal{A}_{\bar{z}} =& A_{\bar{z}}(u,z,\bar{z}) + \dots\end{aligned} \tag{5.1.4}$$

where the dots denote further subleading terms in $1/r$, and using (5.1.2), the charge associated to the residual gauge freedom given by angular functions $\varepsilon(z,\bar{z})$ computed at $\mathscr{I}^+$ reads

$$Q^+ = \int_{\mathscr{I}^+} \varepsilon(z,\bar{z}) \left[\partial_u (D^z A_z + D^{\bar{z}} A_{\bar{z}}) + \mathcal{J}\right] \gamma_{z\bar{z}} du d^2 z, \tag{5.1.5}$$

where

$$\mathcal{J}(u,z,\bar{z}) \equiv \lim_{r\to\infty} r^2 \mathcal{J}^r(u,r,z,\bar{z}). \tag{5.1.6}$$

Since this charge acts on matter fields by $\delta\Phi(x) = i[Q,\Phi(x)] = ie\varepsilon(x)\Phi(x)$, any correlation function will satisfy

$$\begin{aligned}\langle \delta \prod_{n=1}^{N} \Phi_n(x_n)\rangle =& i\langle 0| \left(Q^+ \prod_{n=1}^{N} \Phi_n(x_n) - \prod_{n=1}^{N} \Phi_n(x_n) Q^- \right) |0\rangle \\ =& i \sum_{n=1}^{N} e_n \varepsilon(x_n) \langle \prod_{n=1}^{N} \Phi_n(x_n)\rangle.\end{aligned} \tag{5.1.7}$$

Performing LSZ reduction of the previous formula yields the Ward identity (4.2.21)

$$\langle \text{out}|(Q^+ S - S Q^-)|\text{in}\rangle = \sum_{n=1}^{N} \eta_n e_n \varepsilon(z_n,\bar{z}_n) \langle \text{out}|S|\text{in}\rangle. \tag{5.1.8}$$

This derivation is given in [16], who also noted that Strominger's antipodal identification essentially consists in choosing the same gauge transformation for $\mathscr{I}^+$ and $\mathscr{I}^-$, and that, since the charge is computed on a surface approximating $\mathscr{I}^\pm$ which necessarily cuts through time-like infinity, the results also hold for massive fields. Using the auxiliary boundary condition $\partial_z A_{\bar{z}} = \partial_{\bar{z}} A_z$ at $\mathscr{I}^+_\pm$, choosing $\varepsilon(z,\bar{z}) = \frac{1}{w-z}$, where w is a fixed complex parameter, and exploiting $\partial_{\bar{z}} \frac{1}{z-w} = 2\pi\delta^2(z-w)$ gives

$$4\pi \langle \text{out}| \left[\left(\int du \partial_u A_z \right) S - S \left(\int dv \partial_v A_z \right) \right] |\text{in}\rangle = \sum_{n=1}^{N} \frac{\eta_n e_n}{z - z_n} \langle \text{out}|S|\text{in}\rangle, \tag{5.1.9}$$

where we have used that $\mathcal{J}$ annihilates the vacuum, since the global $U(1)$ symmetry is unbroken. Using the free mode expansion for A_z near $\mathscr{I}$ and the usual stationary phase approximation, we obtain

$$\int du e^{i\omega u} \partial_u A_z = -\frac{i}{8\pi^2} \frac{\sqrt{2}}{1+z\bar{z}} \int_0^\infty d\omega_{\mathbf{q}} \left[a_+^{\text{out}}(\omega_q \hat{x}) e^{-i\omega_{\mathbf{q}} u} - a_-^{\text{out}\dagger}(\omega_q \hat{x}) e^{i\omega_{\mathbf{q}} u} \right] \tag{5.1.10}$$

so that

$$\int_{-\infty}^{+\infty} du \partial_u A_z = -\frac{1}{8\pi} \frac{\sqrt{2}}{1+z\bar{z}} \lim_{\omega \to 0^+} \left[\omega a_+^{\text{out}}(\omega \hat{x}) + \omega a_-^{\text{out}\dagger}(\omega \hat{x}) \right]. \tag{5.1.11}$$

Substituting this result into (5.1.9), together with the analogous one for $\mathscr{I}^-$, and using crossing symmetry yields

$$\lim_{\omega \to 0} \left[\omega \langle \text{out} | a_+^{\text{out}} S | \text{in} \rangle \right] = -\frac{1+z\bar{z}}{\sqrt{2}} \sum_n \frac{\eta_n e_n}{z - z_n} \langle \text{out} | S | \text{in} \rangle, \tag{5.1.12}$$

which is Weinberg's theorem (3.3.49).

5.2 Linearized Gravity Revisited

The action for a massless Fierz-Pauli field $h_{\mu\nu}$, describing a linear perturbation of the Minkowski metric tensor, is

$$S = \frac{1}{2} \int \mathcal{E}^{\mu\nu} h_{\mu\nu} d^D x - \int J^{\mu\nu} h_{\mu\nu} d^D x, \tag{5.2.1}$$

where $\mathcal{E}^{\mu\nu}$ is the linearized Einstein tensor

$$\mathcal{E}_{\mu\nu} = \Box h_{\mu\nu} - \partial_{(\mu} \partial \cdot h_{\nu)} - \partial_\mu \partial_\nu h' + \eta_{\mu\nu} (\partial \cdot \partial \cdot h - \Box h'), \tag{5.2.2}$$

and $J^{\mu\nu}$ is a conserved "energy-momentum tensor", $\partial_\mu J^{\mu\nu} = 0$. The action (5.2.1) is invariant under $\delta h_{\mu\nu} = \partial_{(\mu} \xi_{\nu)}$ up to the boundary term

$$\int \partial_\mu \left[\left(\mathcal{E}^{\mu\nu} - 2J^{\mu\nu} \right) \xi_\nu \right] d^D x, \tag{5.2.3}$$

since $\mathcal{E}^{\mu\nu}$ satisfies the linearized Bianchi identity $\partial \cdot \mathcal{E}^\nu = 0$ and $J^{\mu\nu}$ is conserved. The equations of motion are $\mathcal{E}^{\mu\nu} = J^{\mu\nu}$.

The variational derivatives needed for the computation of the current are

$$\begin{aligned} \frac{\delta S}{\delta h_{\alpha\beta,\mu\nu}} =& \frac{1}{2} \Big[\eta^{\alpha\beta} h^{\mu\nu} + \eta^{\mu\nu} h^{\alpha\beta} \\ & - \frac{1}{2} \left(\eta^{\mu\beta} h^{\nu\alpha} + \eta^{\mu\alpha} h^{\nu\beta} - \eta^{\nu\alpha} h^{\mu\beta} - \eta^{\nu\beta} h^{\alpha\mu} \right) \\ & - \left(\eta^{\alpha\beta} \eta^{\mu\nu} - \frac{1}{2} (\eta^{\mu\beta} \eta^{\nu\alpha} + \eta^{\mu\alpha} \eta^{\nu\beta}) \right) h' \Big] \\ =& \frac{1}{2} \left[\frac{1}{2} H^{\mu\alpha\nu\beta} + \frac{1}{2} H^{\mu\beta\nu\alpha} \right], \end{aligned} \tag{5.2.4}$$

where $H^{\mu\alpha\nu\beta}$ is defined by

$$H^{\mu\alpha\nu\beta} \equiv \eta^{\mu\nu}h^{\alpha\beta} + \eta^{\alpha\beta}h^{\mu\nu} - \eta^{\mu\beta}h^{\nu\alpha} - \eta^{\nu\alpha}h^{\mu\beta} - (\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\beta}\eta^{\nu\alpha})h'. \tag{5.2.5}$$

This tensor has the same symmetries of $R^{\mu\alpha\nu\beta}$,

$$H^{\mu\alpha\nu\beta} = -H^{\alpha\mu\nu\beta} = -H^{\mu\alpha\beta\nu} = H^{\nu\beta\mu\alpha}, \tag{5.2.6}$$

satisfies the cyclic identity,

$$H^{\mu\alpha\nu\beta} + H^{\mu\nu\beta\alpha} + H^{\mu\beta\alpha\nu} = 0, \tag{5.2.7}$$

and works as a superpotential for the linearized Einstein tensor, meaning

$$\mathcal{E}^{\mu\nu} = \partial_\alpha\partial_\beta H^{\mu\alpha\nu\beta}. \tag{5.2.8}$$

Defining the trace-reversed[1] tensor $\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h'$, one gets the simpler form

$$H^{\mu\alpha\nu\beta} = \eta^{\mu\nu}\bar{h}^{\alpha\beta} + \eta^{\alpha\beta}\bar{h}^{\mu\nu} - \eta^{\mu\beta}\bar{h}^{\nu\alpha} - \eta^{\nu\alpha}\bar{h}^{\mu\beta}. \tag{5.2.9}$$

The on-shell Noether current is given by

$$j^\mu = \frac{\delta S}{\delta h_{\alpha\beta,\mu\nu}}\delta h_{\alpha\beta,\nu} - \partial_\nu \frac{\delta S}{\delta h_{\alpha\beta,\mu\nu}}\delta h_{\alpha\beta} + J^{\mu\nu}\xi_\nu, \tag{5.2.10}$$

where $\delta h_{\alpha\beta} = \partial_{(\alpha}\xi_{\beta)}$. The contribution $J^{\mu\nu}\xi_\nu$ is given by the boundary term in the variation of the action. Thus

$$j^\mu = \frac{1}{2}\left(H^{\mu\alpha\nu\beta}\partial_\nu\partial_\alpha\xi_\beta - \partial_\nu H^{\mu\alpha\nu\beta}(\partial_\alpha\xi_\beta + \partial_\beta\xi_\alpha)\right) + J^{\mu\nu}\xi_\nu, \tag{5.2.11}$$

where we have used the antisymmetry of $H^{\mu\alpha\nu\beta}$ in $\nu\beta$ and the symmetry of $\partial_\nu\partial_\beta$ (or analogous considerations for similar contributions) for the first term and symmetrized in $\alpha\beta$ the second term.

It is instructive to recover the Noether tensor $\kappa^{\mu\nu}$ satisfying $j^\mu = \partial_\nu\kappa^{\mu\nu}$, whose existence is ensured by Noether's second theorem; for this purpose we can set $J^{\mu\nu} = 0$ without loss of generality. Integrating by parts each term in (5.2.11), employing the equations of motion $\partial_\alpha\partial_\beta H^{\mu\alpha\nu\beta} = 0$, and renaming the indices appropriately we get

$$j^\mu = \frac{1}{2}\left\{\partial_\alpha\left[H^{\mu\alpha\nu\beta}\partial_\nu\xi_\beta - \partial_\nu\left(H^{\mu\alpha\nu\beta} + H^{\mu\nu\alpha\beta} + H^{\mu\beta\nu\alpha}\right)\xi_\beta\right]\right\}, \tag{5.2.12}$$

so that thanks to the cyclic identity

$$j^\mu = \partial_\alpha\kappa^{\mu\alpha}, \qquad \kappa^{\mu\alpha} = \frac{1}{2}H^{\mu\alpha\nu\beta}\partial_\nu\xi_\beta + \xi_\nu\partial_\beta H^{\mu\alpha\nu\beta}. \tag{5.2.13}$$

[1] The name is due to the fact that $\bar{h}^\alpha{}_\alpha = h' - 2h' = -h'$, *i.e.*, the trace of $\bar{h}^{\mu\nu}$ is opposite to the trace of $h^{\mu\nu}$.

Now, we may think to have obtained these expressions in a given locally inertial frame: to covariantize them we simply replace ordinary derivatives with covariant derivatives and note that no ambiguity arises in their ordering, since the connection defining them is given by the flat background metric and hence such derivatives commute; thus respectively

$$j^\mu = \frac{1}{2}\Big(H^{\mu\alpha\nu\beta}\nabla_\nu\nabla_\alpha\xi_\beta - \nabla_\nu H^{\mu\alpha\nu\beta}(\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha)\Big) + J^{\mu\nu}\xi_\nu, \tag{5.2.14}$$

and

$$\kappa^{\mu\alpha} = \frac{1}{2}H^{\mu\alpha\nu\beta}\nabla_\nu\xi_\beta + \xi_\nu\nabla_\beta H^{\mu\alpha\nu\beta}. \tag{5.2.15}$$

The overall normalization constant can be checked by comparison with the ADM four-momentum itself (see *e.g.* [59, Chapter 6.7]): from the Einstein equations

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi G T^{\mu\nu} \tag{5.2.16}$$

one splits the left-hand side in linear and non-linear order[2] $G^{\mu\nu} = \frac{1}{2}\mathcal{E}^{\mu\nu} + G^{\mu\nu}_{\mathrm{NL}}$ obtaining

$$\frac{1}{2}\mathcal{E}^{\mu\nu} = 8\pi G\left[T^{\mu\nu} - \frac{1}{8\pi G}G^{\mu\nu}_{\mathrm{NL}}\right] \equiv 8\pi G t^{\mu\nu}, \tag{5.2.19}$$

which defines the energy pseudo-tensor $t^{\mu\nu}$. The ADM four-momentum is defined by integrating on surfaces Σ of constant time, *i.e.*,

$$\begin{aligned} P^\mu &= \int_\Sigma t^{\mu 0} d^3x = \frac{1}{16\pi G}\int_\Sigma \mathcal{E}^{\mu 0} d^3x = \frac{1}{16\pi G}\int_\Sigma \partial_\alpha\partial_j H^{\mu\alpha 0 j} d^3x \\ &= \frac{1}{16\pi G}\int_{\partial\Sigma} \partial_\alpha H^{\mu\alpha 0 j} n_j d^2x. \end{aligned} \tag{5.2.20}$$

In our framework, using constant symmetry generators $(\xi_\alpha)^\mu = \delta^\mu_\alpha$ and integrating over $\partial\Sigma$ gives

$$Q = \int_{\partial\Sigma} \partial_\alpha H^{\mu\alpha 0 j} n_j d^2x = 16\pi G P^\mu. \tag{5.2.21}$$

[2] The extra factor of 1/2 comes from the normalization in the definition of $\mathcal{E}_{\mu\nu}$: expanding with respect to $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric, the Christoffel symbols read

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}\left(\partial_\mu h^\rho{}_\nu + \partial_\nu h^\rho{}_\mu - \partial^\rho h_{\mu\nu}\right) + \mathcal{O}(h^2), \tag{5.2.17}$$

where indices are raised and lowered by $\eta_{\mu\nu}$, and hence the Ricci tensor is given by

$$R_{\mu\nu} = \frac{1}{2}\left(\Box h_{\mu\nu} - \partial_\mu\partial\cdot h_\nu - \partial_\nu\partial\cdot h_\mu + \partial_\mu\partial_\nu h'\right) + \mathcal{O}(h^2). \tag{5.2.18}$$

Finally, $R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \frac{1}{2}\mathcal{E}_{\mu\nu} + \mathcal{O}(h^2)$.

5.2.1 Bondi gauge and residual freedom: the BMS algebra

From the perspective of linearized gravity (*i.e.* of a generic spin-2 massless field) the Bondi gauge is fixed by the following choice of boundary conditions, which as we saw stems from considerations in the non-linear theory:

$$h_{\mu\nu}dx^\mu dx^\nu = \frac{2m_B}{r}du^2 - 2U_z du dz - 2U_{\bar z} du d\bar z + rC_{zz}dz^2 + rC_{\bar z\bar z}d\bar z^2, \tag{5.2.22}$$

or, equivalently,

$$h^{\mu\nu}\partial_\mu\partial_\nu = \frac{2m_B}{r}\partial_r^2 + 2\frac{U_{\bar z}}{\gamma_{z\bar z}r^2}\partial_r\partial_z + 2\frac{U_z}{\gamma_{z\bar z}r^2}\partial_r\partial_{\bar z} + \frac{C_{\bar z\bar z}}{\gamma_{z\bar z}^2 r^3}\partial_z^2 + \frac{C_{zz}}{\gamma_{z\bar z}^2 r^3}\partial_{\bar z}^2, \tag{5.2.23}$$

where $u, r, z, \bar z$ are the Bondi coordinates defined in (3.3.15) and indices are raised and lowered by the Minkowski metric (4.1.1).

For higher spins we cannot rely on the knowledge of a full non-linear theory to the goal of setting the asymptotic gauge conditions, as a complete interacting theory in (asymptotically) flat space is not known. Rather, we will take inspiration from the chain of boundary conditions (5.2.22), implied from the full Bondi gauge on the linear spin-two theory.

Notice that

$$h' \equiv h^\alpha{}_\alpha = 0. \tag{5.2.24}$$

Forgetting for the moment what we know about the BMS group, we can then look for the residual gauge freedom leaving this form of $h_{\mu\nu}$ invariant.

For simplicity, we shall start by restricting ourselves to gauge parameters ξ_μ which are u-independent and which have power-like dependence on r: as we shall see, this will allow us to recover the subgroup of supertranslations. Recall $\delta h_{\mu\nu} = \xi_{(\mu;\nu)} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - 2\Gamma^\rho_{\mu\nu}\xi_\rho$ and the Christoffel symbols for Minkowski space given in (4.1.3). From

$$\xi_{(r;r)} = 0 \implies \xi_{r,r} = 0 \tag{5.2.25}$$

we deduce $\xi_r = -T(z,\bar z)$ for some angular function T, since we do not allow any u-dependence. Similarly

$$\xi_{(u;r)} = 0 \implies \xi_{u,r} = 0 \tag{5.2.26}$$

and hence $\xi_u = -S(z,\bar z)$ for some S. In principle, we could allow for some non-trivial transformation of the uu component, but, since our parameter does not depend on u, for now,

$$\xi_{(u;u)} = \frac{1}{r}p(z,\bar z) = 0 \implies p(z,\bar z) = 0. \tag{5.2.27}$$

Now we have

$$\xi_{(r;z)} = 0 \implies \xi_{z,r} - \frac{2}{r}\xi_z - \partial_z T(z,\bar z) = 0, \tag{5.2.28}$$

which is solved by $\xi_z = -r\partial_z T$ and

$$\xi_{(z;\bar{z})} = 0 \implies \xi_{z,\bar{z}} + \xi_{\bar{z},z} - 2r\gamma_{z\bar{z}}(T - S) = 0, \tag{5.2.29}$$

which gives $S = T + D^z D_z T$ or, equivalently, $\xi_u = -T - D^z D_z T$, recalling that mixed z and $\bar{z}$ connection symbols on the sphere are zero. In addition, the non-vanishing gauge variations are:

$$\delta h_{uz} = \xi_{(u;z)} = -D_z(T + D^z D_z T), \tag{5.2.30}$$

$$\delta h_{zz} = \xi_{(z;z)} = -2r D_z^2 T, \tag{5.2.31}$$

consistently with Bondi gauge.

This computation leads therefore to supertranslations, generated by

$$\begin{aligned} \xi_\mu dx^\mu &= -(T + D^z D_z T)du - T(z,\bar{z})dr - rD_z T dz - rD_{\bar{z}} T d\bar{z}, \\ \xi^\mu \partial_\mu &= T(z,\bar{z})\partial_u + D^z D_z T \partial_r - \frac{1}{r}(D^z T \partial_z + D^{\bar{z}} T \partial_{\bar{z}}) \end{aligned} \tag{5.2.32}$$

which indeed leave the "Bondi gauge" defined by (5.2.22) invariant.

Incidentally, the divergence of this supertranslation parameter vanishes:

$$\nabla \cdot \xi = \frac{1}{\gamma_{z\bar{z}} r^2}(\partial_{\bar{z}}\xi_z + \partial_z \xi_{\bar{z}}) - \frac{2}{r}(\xi_u - \xi_r) = 0; \tag{5.2.33}$$

therefore, we see that our representatives of the BMS supertranslations defined intrinsically at $\mathscr{I}^+$ satisfy condition (1.2.20), as needed for them to produce well-defined conserved quantities. In particular, BMS translation representatives selected by $T_\mu(z,\bar{z})$

$$T_0 = 1, \quad T_1 + iT_2 = \frac{2z}{1 + z\bar{z}}, \quad T_3 = \frac{1 - z\bar{z}}{1 + z\bar{z}} \tag{5.2.34}$$

give rise to well-defined notions of energy and momentum. However, even though the condition $h^\alpha{}_\alpha = 0$ is invariant, supertranslations are *not* in general a symmetry of the on-shell fields satisfying the other Fierz conditions $\Box h_{\mu\nu} = 0$ and $\nabla \cdot h_\mu = 0$, as can be seen by the fact that $\Box \xi_\mu \neq 0$. (Further remarks on this point can be found in [36].)

We can now ask ourselves whether enlarging the functional type of allowed gauge parameters allows us to recover the full BMS group. Indeed, this can be achieved by considering the most general form of the residual gauge parameters ξ_μ, expanded in powers of r^{-1}, and solving the above equations as follows. From the rr and ur equations we obtain

$$\partial_r \xi_r = 0 \implies \xi_r = \xi_r(u, z, \bar{z}), \tag{5.2.35}$$

and

$$\partial_r \xi_u + \partial_u \xi_r = 0 \implies \partial_u^2 \xi_r = -\partial_r \partial_u \xi_u. \tag{5.2.36}$$

From the uu equation we also require

$$\partial_u \xi_u = \mathcal{O}\left(\frac{1}{r}\right), \tag{5.2.37}$$

which, together with the previous equation, implies

$$\partial_u^2 \xi_r = \mathcal{O}\left(\frac{1}{r^2}\right). \tag{5.2.38}$$

Hence

$$\xi_r = -T(z,\bar{z}) - uF(z,\bar{z}) \tag{5.2.39}$$

whereas, integrating $\partial_r \xi_u + \partial_u \xi_r = 0$ in r, we get

$$\xi_u = -S(z,\bar{z}) + rF(z,\bar{z}), \tag{5.2.40}$$

where S cannot depend on u, by (5.2.37). Then the zr equation

$$\partial_r \xi_z + \partial_z \xi_r - \frac{2}{r}\xi_z = 0 \tag{5.2.41}$$

reads

$$\partial_r \xi_z - \frac{2}{r}\xi_z - D_z T - uD_z F = 0; \tag{5.2.42}$$

looking for a solution of the type $\xi_z = r^\alpha \psi_\alpha(u,z,\bar{z})$, for some integer α, one readily sees that only

$$\psi_1 = -D_z T - uD_z F, \qquad \psi_2 = Y_z(u,z,\bar{z}) \tag{5.2.43}$$

are allowed, where $Y_A(u,z,\bar{z})$ for $A = 1,2$ is some one-form ($Y^A \equiv \gamma^{AB} Y_B$), hence

$$\xi_z = -rD_z T - ruD_z F + r^2 Y_z(u,z,\bar{z}). \tag{5.2.44}$$

From the uz equation

$$\partial_u \xi_z + \partial_z \xi_u = \mathcal{O}(1) \implies \mathcal{O}(1) + r^2 \partial_u Y_z(u,z,\bar{z}) = \mathcal{O}(1) \tag{5.2.45}$$

one infers $Y_A = Y_A(z,\bar{z})$. Now, the zz equation

$$D_z \xi_z = \mathcal{O}(r) \implies D_z Y_z = 0, \tag{5.2.46}$$

implies that $Y^A(z,\bar{z})$ is a conformal Killing vector on the sphere: the conformal Killing equation $D_{(A}Y_{B)} = \alpha\gamma_{AB}$ is equivalent to $D_{(A}Y_{B)} = \gamma_{AB} D \cdot Y$ on the sphere, where $D \cdot Y \equiv D_A Y^A$, but since γ_{AB} is off-diagonal

$$D_z Y_z = 0 = D_{\bar{z}} Y_{\bar{z}}, \tag{5.2.47}$$

whereas $D_z Y_{\bar z} + D_{\bar z} Y_z = \gamma_{z\bar z} D \cdot Y$ is identically satisfied. Notice that this property also implies $D_z D_{\bar z} D \cdot Y = -\gamma_{z\bar z} D \cdot Y$ by using $[D_z, D_{\bar z}] Y_z = -\gamma_{z\bar z} Y_z$.

Up to now we have

$$\begin{aligned}
\xi_r &= - T(z,\bar z) - uF(z,\bar z)\\
\xi_u &= - S(z,\bar z) + rF(z,\bar z)\\
\xi_z &= - rD_z T(z,\bar z) - ruD_z F(z,\bar z) + r^2 Y_z(z,\bar z),
\end{aligned} \tag{5.2.48}$$

where Y^A is a conformal Killing vector on the sphere. We substitute in the $z\bar z$ equation

$$\partial_z \xi_{\bar z} + \partial_{\bar z}\xi_z - 2\gamma_{z\bar z} r\left(\xi_u - \xi_r\right) = 0 \tag{5.2.49}$$

and we get

$$- 2r\gamma_{z\bar z}\left(D_z D^z T + T - S\right) + r^2\left(D_z Y_{\bar z} + D_{\bar z} Y_z - 2\gamma_{z\bar z} F\right) - 2ur\left(D_z D_{\bar z} F + \gamma_{z\bar z} F\right) = 0. \tag{5.2.50}$$

This equation can be satisfied only if the coefficient of each independent monomial r, r^2 and ur is zero: this requires

$$S = T + D_z D^z T \tag{5.2.51}$$

and

$$F = \frac{1}{2} D \cdot Y, \tag{5.2.52}$$

so that also the last condition is automatically satisfied. To sum up, the residual gauge freedom is parametrized by $\xi_\mu dx^\mu$ whose components are

$$\begin{aligned}
\xi_r &= - T - \frac{u}{2} D \cdot Y(z,\bar z),\\
\xi_u &= - \left(T + D_z D^z T\right) + \frac{r}{2} D \cdot Y,\\
\xi_A &= - rD_A T - \frac{ru}{2} D_A D \cdot Y + r^2 Y_A.
\end{aligned} \tag{5.2.53}$$

Equivalently, the residual symmetry vector is

$$\begin{aligned}
\xi^u &= T + \frac{u}{2} D \cdot Y,\\
\xi^r &= D_z D^z T - \frac{1}{2}(u + r) D \cdot Y,\\
\xi^A &= - \frac{1}{r} D^A T + Y^A - \frac{u}{2r} D^A D \cdot Y.
\end{aligned} \tag{5.2.54}$$

The corresponding vector acting on $\mathscr{I}^+$ is therefore

$$\xi = T(z,\bar z)\partial_u + Y^A(z,\bar z)\partial_A, \tag{5.2.55}$$

which identifies the (full) BMS algebra (compare with [52]): an infinite-dimensional family of direction-dependent translations $T(z,\bar z)\partial_u$ together with the conformal Killing vectors on the sphere $Y^A(z,\bar z)\partial_A$.

5.2.2 Supertranslation charge

We may now compute the charge associated with this residual supertranslation gauge symmetry, starting either with the Noether tensor $k^{\mu\alpha}$ or from the current j^μ itself. In any case, the explicit computation of the non-vanishing components of the tensor $H^{\mu\alpha\nu\beta}$ is quite useful:

$$\begin{aligned} H^{urzr} &= \frac{U_{\bar z}}{\gamma_{z\bar z} r^2}, & H^{uzrz} &= -\frac{C_{\bar z\bar z}}{\gamma_{z\bar z}^2 r^3}, & H^{rzrz} &= \frac{C_{\bar z\bar z}}{\gamma_{z\bar z}^2 r^3}, \\ H^{rzz\bar z} &= \frac{U_{\bar z}}{(\gamma_{z\bar z} r^2)^2}, & H^{rzr\bar z} &= \frac{2m_B}{\gamma_{z\bar z} r^3}, \end{aligned} \tag{5.2.56}$$

where the components with $\bar z$ and z interchanged are obtained by formal conjugation of all indices. It is also convenient to compute the "commutators" $\xi_{[\mu;\nu]} = \xi_{[\mu,\nu]}$:

$$\xi_{[u,r]} = 0, \qquad \xi_{[u,z]} = D_z(T + D^z D_z T), \qquad \xi_{[r,z]} = 0, \qquad \xi_{[z,\bar z]} = 0. \tag{5.2.57}$$

We start computing κ^{ur} from (5.2.15), since this component is selected by the measure element of $\mathscr{I}^+_-$. Observe that $\frac{1}{2} H^{ur\nu\beta}\nabla_\nu \xi_\beta = \frac{1}{4} H^{ur\nu\beta}\xi_{[\beta,\nu]}$ by the antisymmetry of $H^{ur\nu\beta}$ in $\nu\beta$; by (5.2.57) the only potentially surviving term would be $\frac{1}{4}H^{uruz}\xi_{[u,z]}$, which vanishes anyway since H^{uruz} is itself zero. The other contribution to the $\kappa^{\mu\nu}$ form from (5.2.15) is

$$\xi_\nu \nabla_\beta H^{ur\nu\beta} = \xi_\nu \partial_\beta H^{ur\nu\beta} + \xi_\nu \Gamma^u_{\rho\beta} H^{\rho r\nu\beta} + \xi_\nu \Gamma^r_{\rho\beta} H^{u\rho\nu\beta} + \xi_\nu \Gamma^\nu_{\rho\beta} H^{ur\rho\beta} + \xi_\nu \Gamma^\beta_{\beta\rho} H^{ur\nu\rho}; \tag{5.2.58}$$

the fourth term on the right-hand side vanishes by symmetry/antisymmetry in the summed indices while $\Gamma^\beta_{\beta\rho} = \partial_\rho \log\sqrt{g}$ in the last term. Taking into account the non-vanishing Christoffel symbols and $H^{\mu\alpha\nu\beta}$ components, we get

$$\begin{aligned} k^{ur} = &\left[2\partial_z T \frac{U_{\bar z}}{\gamma_{z\bar z} r^2} + T\partial_z \left(\frac{U_{\bar z}}{\gamma_{z\bar z} r^2} \right) + z \leftrightarrow \bar z \right] \\ &+ \left[\left(\partial_z T \frac{U_{\bar z}}{\gamma_{z\bar z} r^2} + z \leftrightarrow \bar z \right) + 2T \frac{2m_B}{r^2} \right] \\ &+ \left[-2\partial_z T \frac{U_{\bar z}}{\gamma_{z\bar z} r^2} + T \frac{U_{\bar z}}{\gamma_{z\bar z}^2 r^2} \partial_z \gamma_{z\bar z} + z \leftrightarrow \bar z \right], \end{aligned} \tag{5.2.59}$$

were "$z \leftrightarrow \bar z$" refers to formal complex conjugation in the z and $\bar z$ indices. Hence, after expanding the derivative in the second term,

$$\kappa^{ur} = 2T\,\frac{2m_B}{r^2} + \frac{1}{\gamma_{z\bar z} r^2}\left[\partial_z (T U_{\bar z}) + z \leftrightarrow \bar z \right]; \tag{5.2.60}$$

integrating this expression as

$$Q^+ = \int_{\mathscr{I}^+_-} k^{ur} \gamma_{z\bar z} r^2 d^2 z, \tag{5.2.61}$$

and recalling that the sphere has no boundary, we obtain

$$Q^+ = 4\int_{\mathscr{I}^+_-} T(z,\bar{z}) m_B(u,z,\bar{z})\gamma_{z\bar{z}} d^2z. \tag{5.2.62}$$

Again, the factor r^2 from the measure element gets canceled and the charge is meaningfully expressed as an integral over the boundary of null infinity. This agrees with the supertranslation charge used in Chapter 3.

The computation of j^r from (5.2.14), instead, goes as follows. Note that

$$H^{r\alpha\nu\beta}\nabla_\nu\nabla_\alpha\xi_\beta = H^{r\alpha\nu\beta}\partial_\alpha\partial_\nu\xi_\beta = \frac{1}{2}H^{r\alpha\nu\beta}\partial_\alpha\xi_{[\beta,\nu]} \tag{5.2.63}$$

by the vanishing of the Riemann tensor and by antisymmetry in $\nu\beta$. Therefore, due to (5.2.56), the only relevant component is $H^{rzuz} \sim 1/r^3$: this term gives a sub-leading contribution. Altogether, always taking (5.2.56) and (5.2.57) into account, one finds that the only leading contribution to j^μ comes from the following term

$$\partial_u H^{rzuz}\nabla_{(z}\xi_{z)} + z \leftrightarrow \bar{z} = \frac{2}{\gamma_{z\bar{z}}^2 r^2}\left[\partial_u C_{\bar{z}\bar{z}} D_z^2 T(z,\bar{z}) + z \leftrightarrow \bar{z}\right]. \tag{5.2.64}$$

Thus,

$$j^r = -\frac{1}{\gamma_{z\bar{z}}^2 r^2}\left[\partial_u C_{zz} D_{\bar{z}}^2 T(z,\bar{z}) + \partial_u C_{\bar{z}\bar{z}} D_z^2 T(z,\bar{z})\right] - J^{rr}(u,r,z,\bar{z})T(z,\bar{z}) \tag{5.2.65}$$

and

$$Q^+ = \int_{\mathscr{I}^+} T(z,\bar{z})\left[-\partial_u\left(D^zD^zC_{zz} + D^{\bar{z}}D^{\bar{z}}C_{\bar{z}\bar{z}}\right) - J(u,z,\bar{z})\right]\gamma_{z\bar{z}}d^2z du, \tag{5.2.66}$$

where

$$J(u,z,\bar{z}) \equiv \lim_{r\to\infty} r^2 J^{rr}(u,r,z,\bar{z}). \tag{5.2.67}$$

Since supertranslations act on matter fields by $iT(z,\bar{z})\partial_u$ at $\mathscr{I}^+$, we get by LSZ reduction

$$\langle \text{out}|(Q^+S - SQ^-)|\text{in}\rangle = \sum_{n=1}^{N}\eta_n f_n E_n T(z_n,\bar{z}_n)\langle\text{out}|S|\text{in}\rangle, \tag{5.2.68}$$

where f_n is the gravitational coupling of each field. Using the auxiliary boundary condition

$$D^zD^zC_{zz} = D^{\bar{z}}D^{\bar{z}}C_{\bar{z}\bar{z}} \quad \text{at } \mathscr{I}^+_\pm, \tag{5.2.69}$$

we have

$$Q^+ = -2\int_{\mathscr{I}^+} T(z,\bar{z})\partial_u D^zD^zC_{zz}\gamma_{z\bar{z}}d^2z du. \tag{5.2.70}$$

Now, in order to make contact with Weinberg's soft theorem, instead of choosing the simplest possible $T(z, \bar{z})$, as we did in Chapter 3, let us try with an angular function of the following type:

$$T(z, \bar{z}) = \frac{1}{w - z} \frac{1 + w\bar{z}}{1 + z\bar{z}}. \tag{5.2.71}$$

Then the left-hand side of (5.2.68), after an integration by parts in $\partial_{\bar{z}}$, involves computing

$$\begin{aligned} \partial_{\bar{z}} \left(\frac{1}{w - z} \frac{1 + w\bar{z}}{1 + z\bar{z}} \right) &= - 2\pi\delta^2(z - w) \frac{1 + w\bar{z}}{1 + z\bar{z}} + \frac{1}{w - z} \frac{w(1 + z\bar{z}) - (1 + w\bar{z})z}{(1 + z\bar{z})^2} \\ &= - 2\pi\delta^2(z - w) + \frac{1}{(1 + z\bar{z})^2} \\ &= - 2\pi\delta^2(z - w) + \frac{1}{2}\gamma_{z\bar{z}}. \end{aligned} \tag{5.2.72}$$

Therefore

$$Q^+ = -4\pi \int du D^w C_{ww} + \int D^z C_{zz} \gamma_{z\bar{z}} d^2 z du, \tag{5.2.73}$$

where the second term is a boundary contribution on the sphere and hence gives zero. To sum up:

$$- 4\pi D^z \langle \text{out} | \left[\left(\int du \partial_u C_{zz} \right) S - S \left(\int dv \partial_v C_{zz} \right) \right] | \text{in} \rangle = \sum_{n=1}^{N} \eta_n \frac{f_n E_n}{z - z_n} \frac{1 + z\bar{z}_n}{1 + z_n \bar{z}_n} \langle \text{out} | S | \text{in} \rangle. \tag{5.2.74}$$

Now we need to perform the usual stationary phase approximation to express C_{zz} in terms of soft graviton creation and annihilation operators. As we already saw in Chapter 3, the result is

$$C_{zz} = - \frac{i}{8\pi^2} \frac{2}{(1 + z\bar{z})^2} \int_0^{+\infty} d\omega_{\mathbf{q}} \left[a_+^{\text{out}}(\omega_q \hat{x}) e^{-i\omega_{\mathbf{q}} u} - a_-^{\text{out}\dagger}(\omega_q \hat{x}) e^{i\omega_{\mathbf{q}} u} \right], \tag{5.2.75}$$

and

$$\int du \partial_u C_{zz} = - \frac{1}{8\pi} \frac{2}{(1 + z\bar{z})^2} \lim_{\omega \to 0^+} \left[\omega a_+^{\text{out}}(\omega \hat{x}) + \omega a_-^{\text{out}\dagger}(\omega \hat{x}) \right]. \tag{5.2.76}$$

Thus, using crossing symmetry, we also have

$$- 4\pi \langle \text{out} | \left[\left(\int du \partial_u C_{zz} \right) S - S \left(\int dv \partial_v C_{zz} \right) \right] | \text{in} \rangle = \frac{2}{(1 + z\bar{z})^2} \lim_{\omega \to 0} \langle \text{out} | \omega a_+^{\text{out}}(\omega \hat{x}) | \text{in} \rangle, \tag{5.2.77}$$

and this implies, by comparison with (5.2.74),

$$\lim_{\omega \to 0} \langle \text{out} | \omega a_+^{\text{out}}(\omega \hat{x}) | \text{in} \rangle = \lim_{\omega \to 0} (1 + z\bar{z}) \sum_n \eta_n f_n \frac{E_n(\bar{z} - \bar{z}_n)}{(z - z_n)(1 + z_n \bar{z}_n)}, \tag{5.2.78}$$

since

$$\gamma_{z\bar{z}}\partial_{\bar{z}}\frac{2}{1+z\bar{z}}\sum_n \eta_n f_n \frac{E_n(\bar{z}-\bar{z}_n)}{(z-z_n)(1+z_n\bar{z}_n)} = \sum_n \eta_n f_n \frac{E_n(1+z\bar{z}_n)}{(z-z_n)(1+z_n\bar{z}_n)}; \tag{5.2.79}$$

note that we omitted the $\partial_{\bar{z}}\frac{1}{z-z_n}$ term, since here the delta multiplies a function which vanishes when $\bar{z}=\bar{z}_n$.

This shows the supertranslation Ward identity to be fully equivalent to Weinberg's factorization formula (3.3.49), without assuming from the beginning f_n =constant. Notice also that our choice (5.2.70) of T is not restrictive, since we may always write

$$f(z,\bar{z}) = \int \frac{d^2w}{2\pi} f(w,\bar{w})\partial_{\bar{w}}\frac{1}{w-z}\frac{1+w\bar{z}}{1+z\bar{z}} \tag{5.2.80}$$

and then use the linearity of the Ward identity to recover the full supertranslation invariance from Weinberg's theorem.

5.3 Weinberg's Factorization from Spin-Three Large Gauge Symmetry

Free spin-three gauge fields are described by the Fronsdal action [56] (see also [57] for a more recent review)

$$S[\varphi] = \frac{1}{2}\int \mathcal{E}^{\mu\nu\rho}\varphi_{\mu\nu\rho}d^Dx - \int J^{\mu\nu\rho}\varphi_{\mu\nu\rho}d^Dx, \tag{5.3.1}$$

where the "Einstein" tensor is given by

$$\mathcal{E}^{\mu\nu\rho} = \mathcal{F}^{\mu\nu\rho} - \frac{1}{2}\eta^{(\mu\nu}\mathcal{F}'^{\rho)} \tag{5.3.2}$$

and the Fronsdal or "Ricci" tensor reads

$$\mathcal{F}^{\mu\nu\rho} = \Box\varphi^{\mu\nu\rho} - \partial^{(\mu}\partial\cdot\varphi^{\nu\rho)} + \partial^{(\mu}\partial^{\nu}\varphi'^{\rho)}; \tag{5.3.3}$$

$J^{\mu\nu\rho}$ is a symmetric source tensor whose traceless part is locally conserved. In compact notation, where all spacetime indices are suppressed, [58]

$$\mathcal{E} = \mathcal{F} - \frac{1}{2}\eta\mathcal{F}', \qquad \mathcal{F} = \Box\varphi - \partial\partial\cdot\varphi + \partial^2\varphi', \tag{5.3.4}$$

which is in fact the form of these tensors for arbitrary integer spin. The equations of motion are $\mathcal{E}=J$ and the explicit trace of $\mathcal{F}$ is $\mathcal{F}'_\rho = 2\Box\varphi'_\rho - 2\partial\cdot\partial\cdot\varphi_\rho + \partial_\rho\partial\cdot\varphi'$. The "Einstein" tensor satisfies the "anomalous" Bianchi identity:

$$\partial_\mu\mathcal{E}^{\mu\nu\rho} = -\frac{1}{2}\eta^{\nu\rho}\partial\cdot\mathcal{F}' \quad \text{or, equivalently,} \quad \partial_\mu\left(\mathcal{E}^{\mu\nu\rho} + \frac{1}{2}\eta^{\nu\rho}\mathcal{F}'^{\mu}\right) = 0. \tag{5.3.5}$$

The field φ is subject to the gauge symmetry

$$\varphi_{\mu\nu\rho} \sim \varphi_{\mu\nu\rho} + \partial_{(\mu}\varepsilon_{\nu\rho)} \text{ where } \varepsilon' \equiv \varepsilon^{\alpha}{}_{\alpha} = 0. \tag{5.3.6}$$

Indeed, under such transformations, the Fronsdal tensor is gauge-invariant

$$\delta\mathcal{F}^{\mu\nu\rho} = 3\partial_\mu\partial_\nu\partial_\rho\varepsilon' = 0. \tag{5.3.7}$$

The variation of the action gives, for the first term,

$$\frac{1}{2}\delta\int \mathcal{E}^{\mu\nu\rho}\varphi_{\mu\nu\rho}d^Dx = \frac{3}{2}\int \mathcal{E}^{\mu\nu\rho}\partial_\mu\varepsilon_{\nu\rho}d^Dx = \frac{3}{2}\int \partial_\mu\left(\mathcal{E}^{\mu\nu\rho}\varepsilon_{\nu\rho}\right) + \frac{3}{4}\int \partial\cdot\mathcal{F}'\varepsilon' d^Dx, \tag{5.3.8}$$

i.e. a boundary term plus a vanishing contribution, thanks to the anomalous Bianchi identity and the trace constraint. For the second term we have

$$-3\int \partial_\mu\left(J^{\mu\nu\rho}\varepsilon_{\nu\rho}\right)d^Dx, \tag{5.3.9}$$

since the traceless projection of $J^{\mu\nu\rho}$ is conserved:

$$\partial\cdot J - \frac{1}{D}\eta\partial\cdot J' = 0. \tag{5.3.10}$$

The symmetrized derivatives needed for the computation of the Noether current are, in compact notation,

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta\varphi_{,\alpha\beta}} =& \frac{1}{2}\Big\{\eta^{\alpha\beta}\varphi - \frac{1}{2}\left(\eta^{\alpha}\varphi^{\beta} + \eta^{\beta}\varphi^{\alpha}\right) + \eta\varphi^{\alpha\beta} - \eta^{\alpha\beta}\varphi'\eta \\ &+\frac{1}{2}\left(\eta^{\alpha}\eta^{\beta}\varphi' + \eta^{\beta}\eta^{\alpha}\varphi'\right) - \frac{1}{4}\left(\varphi'^{\alpha}\eta^{\beta}\eta + \varphi'^{\beta}\eta^{\alpha}\eta\right)\Big\}.\end{aligned} \tag{5.3.11}$$

We also define

$$\frac{\delta\mathcal{S}}{\delta\varphi_{\mu\nu\rho,\alpha\beta}} \equiv \frac{1}{2}\mathcal{K}^{\mu\nu\rho\alpha\beta}. \tag{5.3.12}$$

Then

$$j^{\alpha} = \frac{1}{2}\left(\mathcal{K}^{\mu\nu\rho\alpha\beta}\nabla_\beta\delta\varphi_{\mu\nu\rho} - \nabla_\beta\mathcal{K}^{\mu\nu\rho\alpha\beta}\delta\varphi_{\mu\nu\rho}\right) + \frac{3}{2}J^{\alpha\nu\rho}\varepsilon_{\nu\rho}. \tag{5.3.13}$$

5.3.1 Bondi-like gauge for spin 3 and residual symmetry

Following the pattern displayed by the spin 1 and 2 cases, we choose our "Bondi-like gauge" near $\mathscr{I}^+$ to be the following set of boundary/falloff conditions: in Bondi coordinates,

- $\varphi_{r\alpha\beta} = 0$, for all α, β;
- $\varphi_{z\bar{z}\mu} = 0$, for all $\mu \neq r$;

- the other components scale in the following manner as $r \to \infty$:

$$\varphi_{uuu} = \frac{\varphi}{r}, \qquad \varphi_{uuz} = -U_z, \qquad \varphi_{uzz} = rC_{zz}, \qquad \varphi_{zzz} = r^2 B_{zzz}, \tag{5.3.14}$$

  and similarly for $z \leftrightarrow \bar{z}$ components, where φ, U_z, C_{zz} and B_{zzz} are r-independent functions. Notice that with our choices $\varphi'^{\mu} = 0$.

Again, we ask ourselves if there are residual gauge transformations leaving this structure invariant: we look for gauge parameters $\varepsilon_{\mu\nu}$, subject to $\varepsilon' = 0$, such that the variation

$$\begin{aligned}\delta\varphi_{\mu\nu\rho} =& \nabla_\mu \varepsilon_{\nu\rho} + \nabla_\nu \varepsilon_{\rho\mu} + \nabla_\rho \varepsilon_{\mu\nu} \\ =& \partial_\mu \varepsilon_{\nu\rho} + \partial_\nu \varepsilon_{\rho\mu} + \partial_\rho \varepsilon_{\mu\nu} - 2\Gamma^{\alpha}_{\mu\nu}\varepsilon_{\alpha\rho} - 2\Gamma^{\alpha}_{\rho\nu}\varepsilon_{\alpha\mu} - 2\Gamma^{\alpha}_{\mu\rho}\varepsilon_{\alpha\nu}\end{aligned} \tag{5.3.15}$$

does not alter our gauge-fixing conditions.

From

$$\varepsilon_{(rr;r)} = 0 \implies \varepsilon_{rr,r} = 0 \tag{5.3.16}$$

we deduce

$$\varepsilon_{rr} = -T(z, \bar{z}) \tag{5.3.17}$$

for some angular function $T(z, \bar{z})$, since we do not allow any u-dependence. Similarly

$$\varepsilon_{(uu;r)} = 0 \implies \varepsilon_{uu,r} = 0, \tag{5.3.18}$$

whence

$$\varepsilon_{uu} = -S(z, \bar{z}) \tag{5.3.19}$$

for some S, and

$$\varepsilon_{(ur;r)} = 0 \implies \varepsilon_{ur,r} = 0, \tag{5.3.20}$$

thus

$$\varepsilon_{ur} = -A(z, \bar{z}). \tag{5.3.21}$$

The uuu component of the field has to be invariant too (no u-dependence is considered)

$$\varepsilon_{(uu;u)} = \frac{1}{r} p(z, \bar{z}) = 0 \implies p(z, \bar{z}) = 0. \tag{5.3.22}$$

Now we have

$$\varepsilon_{(rr;z)} = 0 \implies 2\partial_r \varepsilon_{rz} - \frac{4}{r}\varepsilon_{rz} - \partial_z T(z, \bar{z}) = 0, \tag{5.3.23}$$

which is solved by

$$\varepsilon_{rz} = -\frac{r}{2}\partial_z T \tag{5.3.24}$$

and

$$\varepsilon_{(rz;z)} = 0 \implies \partial_r \varepsilon_{zz} - \frac{4}{r}\varepsilon_{zz} - rD_z^2 T = 0, \tag{5.3.25}$$

where we used $D_z^2 T = \partial_z^2 T - \partial_z \log \gamma_{z\bar{z}} \partial_z T$, and which gives

$$\varepsilon_{zz} = -\frac{r^2}{2} D_z^2 T. \tag{5.3.26}$$

The trace[3] constraint reads

$$\varepsilon' = 0 \implies \varepsilon_{z\bar{z}} = \gamma_{z\bar{z}} \frac{r^2}{2} (T - 2A). \tag{5.3.28}$$

Substituting the trace constraint into

$$\varepsilon_{(rz;\bar{z})} = 0 \implies \partial_r \varepsilon_{z\bar{z}} + \partial_z \varepsilon_{r\bar{z}} + \partial_{\bar{z}} \varepsilon_{rz} - \frac{4}{r} \varepsilon_{z\bar{z}} - 2r\gamma_{z\bar{z}}(T - A) = 0 \tag{5.3.29}$$

gives

$$A = \frac{3}{4} T + \frac{1}{4} D^z D_z T \implies \varepsilon_{ur} = -\left(\frac{3}{4} T + \frac{1}{4} D^z D_z T\right) \tag{5.3.30}$$

$$\varepsilon_{z\bar{z}} = -\gamma_{z\bar{z}} \frac{r^2}{4} (T + D^z D_z T). \tag{5.3.31}$$

By

$$\varepsilon_{(ru;z)} = 0 \implies \partial_r \varepsilon_{uz} - \frac{2}{r} \varepsilon_{uz} - \partial_z A = 0, \tag{5.3.32}$$

we have $\varepsilon_{uz} = -r D_z A(z, \bar{z})$ or

$$\varepsilon_{uz} = -r \left(\frac{3}{4} D_z T + \frac{1}{4} D_z^2 D^z T\right). \tag{5.3.33}$$

From

$$\varepsilon_{(uz;\bar{z})} = 0 \implies \partial_z \varepsilon_{u\bar{z}} + \partial_{\bar{z}} \varepsilon_{uz} - 2\gamma_{z\bar{z}} r (A - S) = 0, \tag{5.3.34}$$

we can determine S and thus get from (5.3.19)

$$\varepsilon_{uu} = -\left(\frac{3}{4} T + D^z D_z T + \frac{1}{4} (D^z D_z)^2 T\right). \tag{5.3.35}$$

Finally, the remaining consistency condition is identically satisfied[4]

$$\varepsilon_{(zz;\bar{z})} = \partial_{\bar{z}} \varepsilon_{zz} + 2D_z \varepsilon_{z\bar{z}} - 4\gamma_{z\bar{z}} r (\varepsilon_{uz} - \varepsilon_{rz}) = -\frac{r^2}{2} \Big([D_z, D_{\bar{z}}] D_z T + \gamma_{z\bar{z}} D_z T\Big) = 0. \tag{5.3.36}$$

[3] Using the Minkowski metric, expanded in Bondi coordinates, the trace is given by

$$\varepsilon' = \varepsilon_{rr} - 2\varepsilon_{ru} + 2 \frac{\varepsilon_{z\bar{z}}}{r^2 \gamma_{z\bar{z}}}. \tag{5.3.27}$$

[4] To check this result, it may be useful to recall that $\partial_z \partial_{\bar{z}} \log \gamma_{z\bar{z}} = -\gamma_{z\bar{z}}$.

The non-vanishing gauge variations are:

$$
\begin{aligned}
\delta\varphi_{uuz} &= -D_z\left(\frac{3}{4}T + D^z D_z T + \frac{1}{4}(D^z D_z)^2 T\right)\\
\delta\varphi_{uzz} &= -\frac{r}{2}\left(3D_z^2 T + D_z^2 D^z D_z T\right)\\
\delta\varphi_{zzz} &= -\frac{3}{2}r^2 D_z^3 T,
\end{aligned}
\tag{5.3.37}
$$

and similarly for $z \leftrightarrow \bar{z}$, consistently with the chosen scaling behaviors.

Therefore, the answer to our question is yes: there is residual gauge freedom, given by the following family of tensors, parametrized by the angular function $T(z,\bar{z})$,

$$
\begin{aligned}
\varepsilon_{\mu\nu}dx^\mu dx^\nu = &-\left(\frac{3}{4}T + D^z D_z T + \frac{1}{4}(D^z D_z)^2 T\right)du^2 - 2\left(\frac{3}{4}T + \frac{1}{4}D^z D_z T\right)dudr\\
&- 2r\left(\frac{3}{4}D_z T + \frac{1}{4}D_z^2 D^z T\right)dudz - T(z,\bar{z})dr^2\\
&- rD_z T drdz - \frac{r^2}{2}D_z^2 T dz^2 - \frac{r^2}{2}\gamma_{z\bar{z}}\left(T + D^z D_z T\right)dzd\bar{z} + z \leftrightarrow \bar{z}
\end{aligned}
\tag{5.3.38}
$$

and the corresponding contravariant tensor is given by

$$
\varepsilon^{\mu\nu}\partial_\mu\partial_\nu = -T(z,\bar{z})\partial_u^2 + \mathcal{O}\left(\frac{1}{r}\right). \tag{5.3.39}
$$

5.3.2 Charge for spin 3 residual symmetry

Like for $s=2$, the only leading contribution to the Noether charge comes from $\partial_u \mathcal{K}^{zzzru}\delta\varphi_{zzz}$, with $\mathcal{K}^{\mu\nu\rho\alpha\beta}$ defined in (5.3.12), where

$$
\mathcal{K}^{zzzru} = -\varphi^{zzz} = -\frac{B_{\bar{z}\bar{z}\bar{z}}}{\gamma_{z\bar{z}}^3 r^4} = -\frac{1}{r^4}B^{zzz} \tag{5.3.40}
$$

and

$$
\partial_u \mathcal{K}^{zzzru}\delta\varphi_{zzz} = \frac{3}{2r^2}\partial_u B^{zzz} D_z^3 T(z,\bar{z}). \tag{5.3.41}
$$

Hence

$$
Q^+ = \int_{\mathscr{I}^+} T(z,\bar{z})\left[\frac{3}{4}\partial_u\left((D^z)^3 B_{zzz} + z \leftrightarrow \bar{z}\right) - \frac{3}{2}J(u,z,\bar{z})\right]\gamma_{z\bar{z}}d^2z du, \tag{5.3.42}
$$

where

$$
J(u,z,\bar{z}) \equiv \lim_{r\to\infty} r^2 J^{rrr}(u,r,z,\bar{z}). \tag{5.3.43}
$$

Since the residual symmetry acts on matter fields by $-T(z,\bar{z})\partial_u^2$ at $\mathscr{I}^+$, *i.e.*, since $J(u,z,\bar{z})$ generates the global group, we can write

$$[Q^+,\Phi] = \frac{3}{2}g_n^{(3)}T(i\partial)_u^2\Phi, \tag{5.3.44}$$

where $g_n^{(3)}$ is the coupling of each matter field to the spin-three gauge field. By LSZ reduction, we get

$$\langle\text{out}|(Q^+S - SQ^-)|\text{in}\rangle = \frac{3}{2}\sum_{n=1}^{N}\eta_n g_n^{(3)}E_n^2 T(z_n,\bar{z}_n)\langle\text{out}|S|\text{in}\rangle, \tag{5.3.45}$$

Using the auxiliary boundary condition

$$(D^z)^3 B_{zzz} = (D^{\bar{z}})^3 B_{\bar{z}\bar{z}\bar{z}} \text{ at } \mathscr{I}^+_\pm, \tag{5.3.46}$$

we have

$$Q^+ = \frac{3}{2}\int_{\mathscr{I}^+} T(z,\bar{z})\left[\partial_u(D^z)^3B_{zzz} - J(u,z,\bar{z})\right]\gamma_{z\bar{z}}d^2z du. \tag{5.3.47}$$

In order to get a more explicit expression we employ a suitable generalization of (5.2.71):

$$T(z,\bar{z}) = \frac{1}{w-z}\left(\frac{1+w\bar{z}}{1+z\bar{z}}\right)^2. \tag{5.3.48}$$

The left-hand side of (5.3.45), after an integration by parts in $\partial_{\bar{z}}$, involves the following computation

$$\partial_{\bar{z}}\left(\frac{1}{w-z}\left(\frac{1+w\bar{z}}{1+z\bar{z}}\right)^2\right) = -2\pi\delta^2(z-w) + \frac{1}{2}\gamma_{z\bar{z}}\frac{1+w\bar{z}}{1+z\bar{z}}. \tag{5.3.49}$$

Therefore, leaving the J term aside, since it does not contribute to the left-hand side of the Ward identity,

$$Q^+ = 3\pi\int du D^w D^w B_{www} - \frac{3}{4}\int D^z D^z B_{zzz}\gamma_{z\bar{z}}\frac{1+w\bar{z}}{1+z\bar{z}}d^2z du; \tag{5.3.50}$$

but integrating again by parts the second term one has

$$\frac{3}{8}\int \gamma_{z\bar{z}}D^z B_{zzz}(w-z)d^2z, \tag{5.3.51}$$

which is a boundary contribution on the sphere and hence gives zero. To sum up:

$$\begin{aligned}
&2\pi(D^z)^2\langle\text{out}|\left[\left(\left(\int du\partial_u B_{zzz}\right)S - S\left(\int dv\partial_v B_{zzz}\right)\right]|\text{in}\rangle \\
&= \sum_{n=1}^{N}\eta_n\frac{g_n E_n^2}{z-z_n}\left(\frac{1+z\bar{z}_n}{1+z_n\bar{z}_n}\right)^2\langle\text{out}|S|\text{in}\rangle.
\end{aligned} \tag{5.3.52}$$

The usual stationary phase approximation for B_{zzz} in turn gives:

$$B_{zzz} = -\frac{i}{8\pi^2}\frac{2^{3/2}}{(1+z\bar{z})^3}\int_0^{+\infty} d\omega_{\mathbf{q}}\left[a_+^{\text{out}}(\omega_q\hat{x})e^{-i\omega_{\mathbf{q}}u} - a_-^{\text{out}\dagger}(\omega_q\hat{x})e^{i\omega_{\mathbf{q}}u}\right] \tag{5.3.53}$$

and

$$\int du\partial_u B_{zzz} = -\frac{1}{8\pi}\frac{2^{3/2}}{(1+z\bar{z})^3}\lim_{\omega\to 0^+}\left[\omega a_+^{\text{out}}(\omega\hat{x}) + \omega a_-^{\text{out}\dagger}(\omega\hat{x})\right]. \tag{5.3.54}$$

Thus, using crossing symmetry, we also have

$$-4\pi\langle\text{out}|\left[\left(\left(\int du\partial_u B_{zzz}\right)S - S\left(\int dv\partial_v B_{zzz}\right)\right)\right]|\text{in}\rangle = \frac{2^{3/2}}{(1+z\bar{z})^3}\lim_{\omega\to 0}\langle\text{out}|\omega a_+^{\text{out}}(\omega\hat{x})|\text{in}\rangle, \tag{5.3.55}$$

which implies, by confrontation with (5.3.52),

$$\lim_{\omega\to 0}\langle\text{out}|\omega a_+^{\text{out}}(\omega\hat{x})|\text{in}\rangle = -\lim_{\omega\to 0}\sqrt{2}(1+z\bar{z})\sum_n \eta_n g_n \frac{E_n^2(\bar{z}-\bar{z}_n)^2}{(z-z_n)(1+z_n\bar{z}_n)^2}, \tag{5.3.56}$$

since

$$(D^z)^2\frac{4}{(1+z\bar{z})^2}\sum_n \eta_n g_n \frac{E_n^2(\bar{z}-\bar{z}_n)^2}{(z-z_n)(1+z_n\bar{z}_n)^2} = 2\sum_n \eta_n g_n \frac{E_n^2(1+z\bar{z}_n)^2}{(z-z_n)(1+z_n\bar{z}_n)^2}. \tag{5.3.57}$$

This shows that the residual gauge symmetry Ward identity for our Bondi-like spin-three gauge is equivalent to Weinberg's factorization formula (3.3.49).

Let us stress the role played by our choices (5.2.71) and (5.3.48) for the function $T(z,\bar{z})$. For the spin-two case it allowed us to avoid assuming the equivalence principle in the form of universality of matter-graviton couplings. Prior knowledge of the latter, on the other hand, allows for alternative, simple choices of $T(z,\bar{z})$ as the one employed in [2]. For the spin-three case, on the other hand, no sum rule is expected to hold for the corresponding charges and we could not have hoped to reproduce the result using cancellations justifiable for $s=2$ on account of the equivalence principle.

It should also be stressed that, just like for spin two, our residual gauge symmetry is not a symmetry of the Fierz system identifying irreducible representations of arbitrary spin and, although the choice of falloff conditions allowing for such a symmetry seems reasonable, the requirements of our Bondi-like gauge are not pure gauge-fixing conditions, but rather involve nontrivial assumptions on the behavior of the gauge fields on the boundary.

5.4 Weinberg's Factorization from Higher-Spin Large Gauge Symmetry

We now move to the generalization of the above program to any integer spin s. The main logical steps required to achieve this goal are essentially the same as in the spin-three case: first we need to look for a reasonable definition of "Bondi-like gauge" ammitting some suitable residual gauge freedom, and then we have to make explicit the connection between this gauge freedom and Weinberg's factorization theorem.

We will organize this material in a slightly different order, with respect to our discussion of the spin-three case: after a synthetic review of the formulation of the spin-s free theory, we will state the falloff conditions defining our asymptotic Bondi-like gauge and prove that Weinberg's soft theorems can be recast as Ward identities for residual gauge symmetry associated with such a gauge choice, postponing the explicit computation of the asymptotic symmetry group to the last section.

5.4.1 Free spin-s gauge fields

We recall now a few basic elements of the description of free massless fields with arbitrary integer spin s. A spin-s field is usually described in terms of a totally symmetric tensor $\varphi_{\mu_1\dots\mu_s}$ of rank s. Its dynamics is encoded in the Fronsdal action [56, 57]

$$S[\varphi] = \frac{1}{2}\int \mathcal{E}^{\mu_1\dots\mu_s}\varphi_{\mu_1\dots\mu_s}d^Dx \tag{5.4.1}$$

where the "Einstein" tensor is given by

$$\mathcal{E}^{\mu_1\dots\mu_s} = \mathcal{F}^{\mu_1\dots\mu_s} - \frac{1}{2}\eta^{(\mu_1\mu_2}\mathcal{F}'^{\mu_3\dots\mu_s)} \tag{5.4.2}$$

and the Fronsdal or "Ricci" tensor reads

$$\mathcal{F}^{\mu_1\dots\mu_s} = \Box\varphi^{\mu_1\dots\mu_s} - \partial^{(\mu_1}\partial\cdot\varphi^{\mu_2\dots\mu_s)} + \partial^{(\mu_1}\partial^{\mu_2}\varphi'^{\mu_3\dots\mu_s)}. \tag{5.4.3}$$

Instead of writing down the indices explicitly, it is most convenient to suppress all indices in the following compact notation, where the symmetrization of all free indices is understood [58]:

$$\mathcal{E} = \mathcal{F} - \frac{1}{2}\eta\mathcal{F}', \qquad \mathcal{F} = \Box\varphi - \partial\partial\cdot\varphi + \partial^2\varphi'. \tag{5.4.4}$$

The equations of motion are $\mathcal{E} = 0$.

For massless fields φ the following gauge symmetry is introduced,

$$\varphi_{\mu_1\dots\mu_s} \sim \varphi_{\mu_1\dots\mu_s} + \partial_{(\mu_1}\varepsilon_{\mu_2\dots\mu_s)}, \qquad \text{or} \qquad \delta\varphi = \partial\varepsilon, \tag{5.4.5}$$

and, for $s \geqslant 4$, the fields are subject to the double-trace constraint $\varphi^{\alpha\ \beta\ \mu_5\dots\mu_s}_{\ \alpha\ \beta} = \varphi'' = 0$; furthermore, the gauge parameter ε must be traceless $\varepsilon^{\alpha\ \mu_3\dots\mu_s}_{\ \alpha} = \varepsilon' = 0$ for all $s \geqslant 3$.

The Fronsdal and "Einstein" tensors possess two very important properties: the Fronsdal tensor is gauge-invariant,

$$\delta\mathcal{F} = 3\partial^3\varepsilon' = 0, \tag{5.4.6}$$

thanks to the trace constraint $\varepsilon' = 0$, whereas the "Einstein" tensor satisfies an "anomalous" Bianchi identity,

$$\partial \cdot \mathcal{E} = -\frac{1}{2}\eta\partial \cdot \mathcal{F}' - \frac{3}{2}\partial^3\varphi'', \tag{5.4.7}$$

where the second term on the right-hand side shows up for $s \geqslant 4$ only and vanishes due to the double-trace constraint $\varphi'' = 0$. Using these relations, the variation of the action gives

$$\begin{aligned}\frac{1}{2}\delta\int \mathcal{E}\varphi d^D x = \frac{1}{2}\int \mathcal{E}\partial\varepsilon d^D x = \frac{s}{2}\int \mathcal{E}^{\alpha\mu_2\ldots\mu_s}\partial_\alpha\varepsilon_{\mu_2\ldots\mu_s}d^D x \\ = \frac{s}{2}\int \partial_\alpha\left(\mathcal{E}^{\alpha\mu_2\ldots\mu_s}\varepsilon_{\mu_2\ldots\mu_s}\right)d^D x,\end{aligned} \tag{5.4.8}$$

i.e. a boundary term.

The symmetrized derivatives needed for the computation of the Noether current are, in compact notation,

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta\varphi_{,\alpha\beta}} =& \frac{1}{2}\left\{\eta^{\alpha\beta}\varphi - \frac{1}{2}\left(\eta^\alpha\varphi^\beta + \eta^\beta\varphi^\alpha\right) + \eta\varphi^{\alpha\beta} - \eta^{\alpha\beta}\varphi'\eta\right.\\ &\left.+\frac{1}{2}\left(\eta^\alpha\eta^\beta\varphi' + \eta^\beta\eta^\alpha\varphi'\right) - \frac{1}{4}\left(\varphi'^\alpha\eta^\beta\eta + \varphi'^\beta\eta^\alpha\eta\right)\right\}.\end{aligned} \tag{5.4.9}$$

We also define

$$\frac{\delta\mathcal{S}}{\delta\varphi_{\mu_1\ldots\mu_s,\alpha\beta}} \equiv \frac{1}{2}\mathcal{K}^{\mu_1\ldots\mu_s\alpha\beta}. \tag{5.4.10}$$

The interaction with matter is introduced via the following coupling to a symmetric source $J^{\mu_1\ldots\mu_s}$, whose traceless projection is divergence free:

$$S = \frac{1}{2}\int \mathcal{E}^{\mu_1\ldots\mu_s}\varphi_{\mu_1\ldots\mu_s}d^D x - \int J^{\mu_1\ldots\mu_s}\varphi_{\mu_1\ldots\mu_s}d^D x. \tag{5.4.11}$$

Now the equations of motion are $\mathcal{E} = J$ and the variation of the action reads

$$-\frac{s}{2}\int \partial_\alpha\left(J^\alpha\varepsilon\right)d^D x. \tag{5.4.12}$$

Hence, the Noether current corresponding to the spin-s gauge invariance is given by

$$j^\alpha = \frac{1}{2}\left(\mathcal{K}^{\alpha\beta}\nabla_\beta\delta\varphi - \nabla_\beta\mathcal{K}^{\alpha\beta}\delta\varphi\right) + \frac{s}{2}J^\alpha\varepsilon. \tag{5.4.13}$$

5.4.2 Bondi-like gauge for spin s

Our Bondi-like gauge for any integer spin s is given by the obvious generalization of the conditions we chose for spin three:

$$\varphi_{r\mu_2\ldots\mu_s} = 0 = \varphi_{z\bar{z}\mu_3\ldots\mu_s} \tag{5.4.14}$$

and

$$\varphi_{uu\ldots u\underbrace{zz\ldots z}_{d}} = r^{d-1}B_{zz\ldots z}. \tag{5.4.15}$$

for $d = 1, \ldots, s$. Notice that, in this gauge, any number of traces of these fields vanishes: this is consistent with the fact that for spin four or higher the vanishing of the double trace of φ must be enforced to recover the correct Lagrangian equations.

Assuming that there are in fact spin-s residual gauge transformations, we would like to compute their charge, given by the variation of the $zz\ldots z$ component, postponing the (more technical) task of proving that our Bondi-like gauge-fixing indeed admits such residual freedom to the next section: this will allow us to immediately get to Weinberg's result for spin s, representing the main result of this thesis.

First, let us recall that the explicit gauge variation for a spin s field reads

$$\begin{aligned}\delta\varphi_{\mu_1\ldots\mu_s} =& \nabla_{(\mu_1}\varepsilon_{\mu_2\ldots\mu_s)} \\ =& \partial_{(\mu_1}\varepsilon_{\mu_2\ldots\mu_s)} - 2\sum_{i<j}\Gamma^{\alpha}_{\mu_i\mu_j}\varepsilon_{\alpha\mu_1\ldots\widehat{\mu_i}\ldots\widehat{\mu_j}\ldots\mu_s},\end{aligned} \tag{5.4.16}$$

where the summed indices i, j take values from 1 to s and $\widehat{\mu}_i$ means that the corresponding index has been omitted. From the invariance, in Bondi-like gauge, of the $rr\ldots r$ component, we get the following definition of $T(z,\bar{z})$:

$$\varepsilon_{(r\ldots r;r)} = 0 \implies \partial_r\varepsilon_{r\ldots r} = 0, \tag{5.4.17}$$

i.e.,

$$\varepsilon_{r\ldots r} = -T(z,\bar{z}). \tag{5.4.18}$$

Now,

$$\varepsilon_{(r\ldots r;z)} = 0 \implies (s-1)\partial_r\varepsilon_{r\ldots rz} - \partial_z T - 2(s-1)\frac{1}{r}\varepsilon_{r\ldots rz} = 0, \tag{5.4.19}$$

which is solved by

$$\varepsilon_{r\ldots rz} = -\frac{r}{s-1}D_zT; \tag{5.4.20}$$

again,

$$\varepsilon_{(r\ldots rz;z)} = 0 \implies (s-2)\partial_r\varepsilon_{r\ldots zz} + 2D_z\varepsilon_{r\ldots rz} - 4(s-2)\frac{1}{r}\varepsilon_{r\ldots zz} = 0, \tag{5.4.21}$$

hence

$$\varepsilon_{r\ldots zz} = -\frac{r^2}{(s-1)(s-2)}D_z^2T. \tag{5.4.22}$$

Proceeding this way, by induction we get

$$\varepsilon_{z\ldots zz} = -\frac{r^{s-1}}{(s-1)!}D_z^{s-1}T, \tag{5.4.23}$$

so that the desired variation takes the form

$$\delta\varphi_{z\ldots zz} = -\frac{sr^{s-1}}{(s-1)!}D_z^s T. \tag{5.4.24}$$

The Noether current will be given by the relevant contribution to (5.4.9)

$$\begin{aligned} -\frac{1}{2}\partial_u \mathcal{K}^{z\ldots zru}\delta\varphi_{z\ldots zz} &= -\frac{1}{2}\frac{-r^{s-1}\partial_u B_{z\ldots zz}}{(\gamma_{z\bar{z}}r^2)^s}\left[-\frac{sr^{s-1}}{(s-1)!}D_z^s T\right] \\ &= -\frac{s}{2r^2(s-1)!}(D^z)^s T\partial_u B_{z\ldots zz} \end{aligned} \tag{5.4.25}$$

so, using the auxiliary boundary condition

$$(D^z)^s B_{z\ldots zz} = (D^{\bar{z}})^s B_{\bar{z}\ldots\bar{z}\bar{z}}, \tag{5.4.26}$$

and integrating by parts we arrive at

$$Q^+ = (-1)^s \frac{s}{2(s-1)!}\int_{\mathscr{I}^+}\partial_{\bar{z}}T(D^z)^{s-1}\partial_u B_{z\ldots zz}d^2z du - \frac{s}{2}\int_{\mathscr{I}^+}\gamma_{z\bar{z}}J(u,z,\bar{z})d^2z du, \tag{5.4.27}$$

where

$$J(u,z,\bar{z}) \equiv \lim_{r\to\infty} r^2 J^{r\ldots r}(u,r,z,\bar{z}). \tag{5.4.28}$$

Once again, the next important step is the choice of the function $T(z,\bar{z})$; following our spin-two and spin-three Ansätze, we choose

$$T(z,\bar{z}) = \frac{1}{w-z}\left(\frac{1+w\bar{z}}{1+z\bar{z}}\right)^{s-1}, \tag{5.4.29}$$

which yields

$$\begin{aligned} &\langle\text{out}|(Q^+S - SQ^-)|\text{in}\rangle \\ =& -4\pi\frac{(-1)^s}{(s-1)!}(D^z)^{s-1}\langle\text{out}|\left[\left(\int du\partial_u B_{z\ldots zz}\right)S - S\left(\int dv\partial_v B_{z\ldots zz}\right)\right]|\text{in}\rangle \\ =& \sum_n \eta_n \frac{g_n^{(s)}E_n^{s-1}}{z-z_n}\left(\frac{1+z\bar{z}_n}{1+z_n\bar{z}_n}\right)^{s-1}\langle\text{out}|S|\text{in}\rangle, \end{aligned} \tag{5.4.30}$$

where we have used the usual action of Q^+ on the matter fields,

$$[Q^+,\Phi] = \frac{s}{2}g_n^{(s)}T(i\partial_u)^{s-1}\Phi. \tag{5.4.31}$$

The stationary phase approximation gives

$$\begin{aligned} &-4\pi\langle\text{out}|\left[\left(\int du\partial_u B_{z\ldots zz}\right)S - S\left(\int dv\partial_v B_{z\ldots zz}\right)\right]|\text{in}\rangle \\ =&\frac{2^{s/2}}{(1+z\bar{z})^s}\lim_{\omega\to 0^+}\left[\omega\langle\text{out}|a_+^{\text{out}}S|\text{in}\rangle\right] \end{aligned} \tag{5.4.32}$$

and hence

$$\lim_{\omega\to 0^+}\left[\omega\langle\text{out}|a_+^{\text{out}}S|\text{in}\rangle\right]=(-1)^s 2^{s/2-1}(1+z\bar z)\sum_n \eta_n\frac{g_n^{(s)}E_n^{s-1}}{z-z_n}\left(\frac{\bar z-\bar z_n}{1+z_n\bar z_n}\right)^{s-1}, \tag{5.4.33}$$

due to

$$\begin{aligned}&\frac{1}{(s-1)!}(D^z)^{s-1}\frac{2^{s-1}}{(1+z\bar z)^{s-1}}\sum_n\eta_n\frac{g_n^{(s)}E_n^{s-1}}{z-z_n}\left(\frac{\bar z-\bar z_n}{1+z_n\bar z_n}\right)^{s-1}\\ &=\sum_n\eta_n\frac{g_n^{(s)}E_n^{s-1}}{z-z_n}\left(\frac{1+z\bar z_n}{1+z_n\bar z_n}\right)^{s-1}\langle\text{out}|S|\text{in}\rangle.\end{aligned} \tag{5.4.34}$$

Indeed, proving this formula amounts to proving that

$$\frac{1}{n!}[(1+z\bar z)^2\partial_{\bar z}]^n\left(\frac{\bar z-\bar\zeta}{1+z\bar z}\right)^n=(1+z\bar\zeta)^n,\ \text{for all } n\in\mathbb{N}, \tag{5.4.35}$$

and this follows by induction: we have already checked the formula for $n=1,2$ in the previous sections and if the formula holds for $n-1$, then

$$\begin{aligned}\frac{1}{n!}[(1+z\bar z)^2\partial_{\bar z}]^n\left(\frac{\bar z-\bar\zeta}{1+z\bar z}\right)^n&=\frac{1}{n!}[(1+z\bar z)^2\partial_{\bar z}]^{n-1}\left[n\left(\frac{\bar z-\bar\zeta}{1+z\bar z}\right)^{n-1}(1+z\bar\zeta)\right]\\ &=(1+z\bar\zeta)\frac{1}{(n-1)!}[(1+z\bar z)^2\partial_{\bar z}]^{n-1}\left(\frac{\bar z-\bar\zeta}{1+z\bar z}\right)^{n-1}\\ &=(1+z\bar\zeta)(1+z\bar\zeta)^{n-1}=(1+z\bar\zeta)^n.\end{aligned} \tag{5.4.36}$$

5.4.3 Residual gauge freedom for spin s

We now turn to the problem of verifying whether or not the presence of residual gauge transformations parametrized by an angular function $T(z,\bar z)$ is allowed by our Bondi-like gauge for any spin. It will turn out that there is indeed one such a family, in complete analogy with the lower-spin cases. The keypoint of this discussion will be to identify, on the one hand, which equations can be used to restrict the allowed gauge parameters and, on the other hand, which should be identically satisfied as consistency conditions, if we are to have any nontrivial symmetry left.

Recall that our Bondi-like gauge is summarized by the following conditions

$$\varphi_{r\mu_2\ldots\mu_s}=0=\varphi_{z\bar z\mu_3\ldots\mu_s} \tag{5.4.37}$$

and

$$\varphi_{uu\ldots u\underbrace{zz\ldots z}_{d}}=r^{d-1}B_{zz\ldots z}. \tag{5.4.38}$$

for $d = 1, \ldots, s$. The equations defining our residual gauge freedom are precisely those encoding the preservation of these scaling behaviors:

$$\delta\varphi_{r\mu_2\ldots\mu_s} = 0 = \delta\varphi_{z\bar{z}\mu_3\ldots\mu_s} \tag{5.4.39}$$

and

$$\delta\varphi_{uu\ldots u\underbrace{zz\ldots z}_{d}} = \mathcal{O}(r^{d-1}). \tag{5.4.40}$$

Let us observe that the independent equations of this system are labelled by the following numbers:

- the number p of "u" indices appearing,
- the number d of "z" indices appearing without $\bar{z}$ counterpart,
- the number c of pairs "$z\bar{z}$", counted ignoring their order.

For simplicity of notation, when useful, we shall indicate by $\varphi^p_{d,c}$ and $\varepsilon^p_{d,c}$ the field components and gauge parameter components, respectively, labelled according to these counting criteria. The remaining number of "r" indices is understood to be $s - p - d - 2c$ for the field components and $(s-1) - p - d - 2c$ for the gauge parameter components.

The equations with $d = c = 0$, with increasing number p of "u" indices, are

$$\begin{aligned}
\partial_r \varepsilon_{rr\ldots r} =& 0 \implies \varepsilon_{rr\ldots r} = -T(z,\bar{z}) \equiv -T_0(z,\bar{z}),\\
\partial_r \varepsilon_{ur\ldots r} =& 0 \implies \varepsilon_{ur\ldots r} = -T_1(z,\bar{z})\\
&\vdots\\
\partial_r \varepsilon_{uu\ldots u} =& 0 \implies \varepsilon_{uu\ldots u} = -T_{s-1}(z,\bar{z})
\end{aligned} \tag{5.4.41}$$

or, briefly, $\varepsilon^p_{0,0} = -T_p(z,\bar{z})$, for some set of angular functions T_p. The first relation in the set, $\varepsilon_{rr\ldots r} = -T(z,\bar{z})$, plays a special role, since we will see that it is in fact this function which determines all the others. The equations with $p = c = 0$, with increasing number of "z" indices, read

$$\begin{aligned}
(s-1)\partial_r \varepsilon_{zr\ldots r} - (s-1)\frac{2}{r}\varepsilon_{zr\ldots r} - D_z T =& 0 \implies \varepsilon_{zr\ldots r} = -\frac{r}{s-1} D_z T\\
(s-2)\partial_r \varepsilon_{zz\ldots r} - (s-2)\frac{4}{r}\varepsilon_{zz\ldots r} - \frac{r}{s-1} D_z^2 T =& 0 \implies \varepsilon_{zz\ldots r} = -\frac{r^2}{(s-1)(s-2)} D_z^2 T\\
&\vdots\\
\partial_r \varepsilon_{zz\ldots z} - (s-1)\frac{2}{r}\varepsilon_{\varepsilon_z z\ldots z} - \frac{(s-1)r^{s-2}}{(s-1)(s-2)\cdots} D_z^{s-1} T =& 0 \implies \varepsilon_{zz\ldots z} = -\frac{r^{s-1}}{(s-z)!} D_z^{s-1} T.
\end{aligned} \tag{5.4.42}$$

Thus, in short,

$$\varepsilon^0_{d,0} = -\frac{r^d}{(s-1)\cdots(s-d)} D^d_z T. \tag{5.4.43}$$

The equations with mixed u and z indices, for fixed $c=0$, are analyzed as follows: consider first $p=1=d$,

$$(s-2)\partial_r \varepsilon_{uzr\dots r} - (s-2)\frac{2}{r}\varepsilon_{uzr\dots r} - D_z T_1 = 0 \implies \varepsilon_{uzr\dots r} = -\frac{r}{s-2} D_z T_1 \tag{5.4.44}$$

then $p=2$, $d=1$,

$$(s-3)\partial_r \varepsilon_{uuzr\dots r} - (s-3)\frac{2}{r}\varepsilon_{uuzr\dots r} - D_z T_2 = 0 \implies \varepsilon_{uuzr\dots r} = -\frac{r}{s-3} D_z T_p \tag{5.4.45}$$

and so on, which gives

$$\varepsilon^p_{1,0} = -\frac{r}{s-1-p} D_z T_p; \tag{5.4.46}$$

also, the case for arbitrary p and $d=2$ is easily studied,

$$(s-p-2)\partial_r \varepsilon^p_{2,0} - \frac{4}{r}\varepsilon^p_{2,0} - \frac{r}{s-p-1} D^2_z T_p \implies \varepsilon^p_{2,0} = -\frac{r^2}{(s-p-1)(s-p-2)} D^2_z T_p, \tag{5.4.47}$$

and by increasing d one sees that in general

$$\varepsilon^p_{d,0} = -\frac{r^d D^d_z T_p}{\prod_{k=1}^d (s-p-k)}. \tag{5.4.48}$$

This formula is extended to non-zero c by means of the trace constraints

$$\varepsilon_{z\bar{z}\mu_3\dots\mu_{s-1}} = -\frac{1}{2}\gamma_{z\bar{z}} r^2 \left(\varepsilon_{rr\mu_3\dots\mu_{s-1}} - 2\varepsilon_{ru\mu_3\dots\mu_{s-1}}\right), \tag{5.4.49}$$

or

$$\varepsilon^p_{d,c+1} = -\frac{1}{2}\gamma_{z\bar{z}} r^2 \left(\varepsilon^p_{d,c} - 2\varepsilon^{p+1}_{d,c}\right). \tag{5.4.50}$$

For instance, the relation for $p=0=d$ and $c=1$,

$$\varepsilon_{z\bar{z}r\dots r} = -\frac{1}{2} r^2 \gamma_{z\bar{z}}(-T + 2T_1) \tag{5.4.51}$$

allows to eliminate $\varepsilon_{z\bar{z}r\dots r}$ from the corresponding equation

$$-\frac{2r}{s-1} D_z D_{\bar{z}} T + \partial_{\bar{z}} \varepsilon_{zr\dots r} + (s-2)\left[\partial_r \varepsilon_{z\bar{z}r\dots r} - \frac{4}{r}\varepsilon_{z\bar{z}r\dots r}\right] - 2\gamma_{z\bar{z}} r (T - T_1) = 0 \tag{5.4.52}$$

yielding

$$T_1 = \frac{s}{2(s-1)} T + \frac{1}{(s-1)^2} D^z D_z T, \tag{5.4.53}$$

and substituting back

$$\varepsilon_{z\bar{z}r\ldots r} = -\frac{1}{2}r^2\gamma_{z\bar{z}}\left(\frac{1}{s-1}T + 2(s-1)^2 D_z D^z T\right). \tag{5.4.54}$$

Considering, more generally, $d = 0$, arbitrary p and $c = 1$, one has, exactly by the same strategy,

$$T_{p+1} = \frac{s-p}{s[s-(p+1)]}T_p + \frac{1}{[s-(p+1)]^2}D^z D_z T_p \tag{5.4.55}$$

which determines every T_p recursively as a combination of T and its derivatives. In fact, equations (5.4.48), (5.4.50), and (5.4.55) completely specify the components of $\varepsilon_{\mu_1\ldots\mu_{s-1}}$ in terms of T, in a recursive way. Let us stress that, while at this point our residual gauge freedom is completely fixed, so far we have only used a part of our Bondi-like conditions, namely all the equations where either $d = 0$ or $c = 0$. As a consequence, either the leftover equations with $d, c \geqslant 1$ and arbitrary p are identically satisfied or new conditions would arise on the gauge parameters that may well kill the infinite-dimensional asymptotic symmetry encoded in particular in the function $T(z, \bar{z})$.

The equation with arbitrary p and with $d, c \geqslant 1$ can be expressed as the vanishing of the following quantity

$$\begin{aligned} C^p_{d,c} =&(s-p-d-2c)\left[\partial_r\varepsilon^p_{d,c} - (d+2c)\frac{2}{r}\varepsilon^p_{d,c}\right] \\ &+ (d+c)D_z\varepsilon^p_{d-1,c} + cD_{\bar{z}}\varepsilon^p_{d+1,c-1} \\ &- 2\gamma_{z\bar{z}}rc(d+c)\left(\varepsilon^{p+1}_{d,c-1} - \varepsilon^p_{d,c-1}\right). \end{aligned} \tag{5.4.56}$$

The following Newton-like formula is an immediate consequence of the recurrence relation (5.4.50):

$$\varepsilon^p_{d,c} = \gamma^c_{z\bar{z}}r^{2c}\sum_{l=0}^{c}\binom{c}{l}\left(-\frac{1}{2}\right)^{c-l}\varepsilon^{p+l}_{d,0}; \tag{5.4.57}$$

by extracting the r-dependence from $\varepsilon^{p+l}_{d,0}$, we obtain

$$\varepsilon^p_{d,c} = r^{d+2c}\hat{\varepsilon}^p_{d,c} \tag{5.4.58}$$

and

$$\hat{\varepsilon}^p_{d,c} = \gamma^c_{z\bar{z}}\sum_{l=0}^{c}\binom{c}{l}\left(-\frac{1}{2}\right)^{c-l}\hat{\varepsilon}^{p+l}_{d,0}. \tag{5.4.59}$$

This allows to rewrite (5.4.50) as

$$\varepsilon^p_{d,c} = -\frac{1}{2}\gamma_{z\bar{z}}r^{d+2c}\left(\hat{\varepsilon}^p_{d,c-1} - 2\hat{\varepsilon}^{p+1}_{d,c-1}\right) \tag{5.4.60}$$

and to rewrite $C^p_{d,c}$, suitably rescaled, as follows

$$\begin{aligned} C^p_{d,c} r^{1-d-2c} =& \frac{1}{2}\gamma_{z\bar{z}}(s-p-d-2c)(d+2c)\left(\hat{\varepsilon}^p_{d,c-1} - 2\hat{\varepsilon}^{p+l}_{d,c-1}\right) \\ & - \frac{1}{2}\gamma_{z\bar{z}}(d+c)\left(D_z\hat{\varepsilon}^p_{d-1,c-1} - 2D_z\hat{\varepsilon}^{p+l}_{d-1,c-1}\right) + cD_{\bar{z}}\hat{\varepsilon}^p_{d+1,c-1} \\ & + 2\gamma_{z\bar{z}} c(d+c)\left(\hat{\varepsilon}^{p+l}_{d,c-1} - \hat{\varepsilon}^p_{d,c-1}\right). \end{aligned} \tag{5.4.61}$$

Having in mind (5.4.57), we can expand $C^p_{d,c} r^{1-d-2c}$ with respect to l as follows

$$C^p_{d,c} r^{1-d-2c} = \gamma^{c-1}_{z\bar{z}} \sum_{l=0}^{c-1} \binom{c-1}{l} \left(-\frac{1}{2}\right)^{c-1-l} \hat{C}^p{}_{d,c}{}^{(l)}, \tag{5.4.62}$$

where the components $\hat{C}^p{}_{d,c}{}^{(l)}$ are obtained by inserting

$$\hat{\varepsilon}^p_{d,c-1} = -\gamma^{c-1}_{z\bar{z}} \sum_{l=0}^{c-1} \binom{c-1}{l} \left(-\frac{1}{2}\right)^{c-1-l} \frac{D^d_z T_{p+l}}{\prod_{k=1}^d (s-p-l-k)}, \tag{5.4.63}$$

and read

$$\begin{aligned} \hat{C}^p{}_{d,c}{}^{(l)} =& -\frac{\gamma_{z\bar{z}}}{2}(s-p-d-2c)(d+2c) \\ & \times \left[\frac{D^d_z T_{p+l}}{\prod_{k=1}^d (s-p-l-k)} - \frac{2D^d_z T_{p+l+1}}{\prod_{k=1}^d (s-p-l-k-1)}\right] \\ & + \frac{\gamma_{z\bar{z}}}{2}(d+c)\left[\frac{D^d_z T_{p+l}}{\prod_{k=1}^{d-1} (s-p-l-k)} - \frac{2D^d_z T_{p+l+1}}{\prod_{k=1}^{d-1} (s-p-l-k-1)}\right] \\ & - c\frac{D_{\bar{z}} D^{d+1}_z T_{p+l}}{\prod_{k=1}^{d+1} (s-p-l-k)} \\ & + 2\gamma_{z\bar{z}} c(d+c)\left[\frac{D^d_z T_{p+l+1}}{\prod_{k=1}^d (s-p-l-k-1)} - \frac{D^d_z T_{p+l}}{\prod_{k=1}^d (s-p-l-k)}\right]. \end{aligned} \tag{5.4.64}$$

The recursion relation (5.4.55), with l-dependence included,

$$T_{p+l+1} = \frac{s-p-l}{2(s-p-l-1)} T_{p+l} + \frac{1}{(s-p-l-1)^2} D^z D_z T_{p+l} \tag{5.4.65}$$

allows to rewrite everything in terms of T_{p+l} while the curvature relation

$$D_{\bar{z}} D^{d+1}_z T_{p+l} = D^d_z D_{\bar{z}} D_z T_{p+l} + \frac{d(d+1)}{2}\gamma_{z\bar{z}} D^d_z T_{p+l} \tag{5.4.66}$$

allows to eliminate the $D_{\bar z} D_z^{d+1} T_{p+l}$ yielding

$$\begin{aligned} \hat{C}^{p}{}_{d,c}{}^{(l)} = & \frac{d+2c}{\prod_{k=1}^{d+1}(s-p-l-k)} \\ & \times \left\{ \frac{l+1-c}{s-p-l-1} D_z^d D_{\bar z} D_z T_{p+l} + \frac{\gamma_{z\bar z}}{2} \left[l(d+c) + (1-c)(s-p) \right] D_z^d T_{p+l} \right\} ; \end{aligned} \tag{5.4.67}$$

using (5.4.65) to eliminate $D_{\bar z} D_z T_{p+l}$, instead, we have

$$\hat{C}^{p}{}_{d,c}{}^{(l)} = \frac{\gamma_{z\bar z}(d+2c)}{\prod_{k=2}^{d+1}(s-p-l-k)} (l+1-c) D_z^d T_{p+l+1} - \frac{\gamma_{z\bar z}(d+2c)}{2\prod_{k=1}^{d}(s-p-l-k)}\, l\, D_z^d T_{p+l}. \tag{5.4.68}$$

Two successive terms in the sum (5.4.62) combine as follows

$$\frac{(c-1)\dots(c-1-l)}{l!}(-1)^{c-1-l} 2^l \left(\frac{\hat{C}^{p}{}_{d,c}{}^{(l)}}{c-1-l} - 2\frac{\hat{C}^{p}{}_{d,c}{}^{(l+1)}}{l+1} \right), \tag{5.4.69}$$

where in this expression the terms involving T_{p+l+1} are

$$-\frac{\gamma_{z\bar z}(d+2c)}{\prod_{k=2}^{d+1}(s-p-l-k)} D_z^d T_{p+l+1} + \frac{\gamma_{z\bar z}(d+2c)}{\prod_{k=1}^{d}(s-p-l-k-1)} D_z^d T_{p+l+1} = 0; \tag{5.4.70}$$

therefore the sum defining $C^p_{d,c}$ is telescopic. But clearly $\hat{C}^{p}{}_{d,c}{}^{(0)}$ gives no contribution involving T_p and neither does $\hat{C}^{p}{}_{d,c}{}^{(c-1)}$ involve T_{p+c-1} since those terms have vanishing coefficient by (5.4.68). Thus $C^p_{d,c} = 0$ whenever d or c do not vanish, and every consistency condition is satisfied, showing that the existence of the infinite-dimensional asymptotic higher-spin symmetry survives this rather nontrivial stress test.

Furthermore for $s = p + d$,

$$\delta\varphi^p_{d,0} = d\, D_z \varepsilon^p_{d-1,0} = -\frac{d r^{d-1} D_z^d T_p}{\prod_{k=1}^{d-1}(s-p-k)} \tag{5.4.71}$$

which indeed respects the scaling behavior with r^{d-1} imposed on $\varphi^p_{d,0}$.

6 Soft Theorems and Goldstone Theorem

At the beginning of the seventies [32, 33], it had already been observed in quantum electrodynamics that all soft theorems of Weinberg [6, 7] and Low [8] type can be seen to arise from the spontaneous breaking of linear large gauge symmetries, constituting a subgroup of the usual $U(1)$ gauge group of electrodynamics. In this section, we first review and then extend this approach to soft theorems for any spin. For a recent review of various aspects of symmetry breaking in QFT from a rigorous perspective see [50].

6.1 Spontaneous Symmetry Breaking in Quantum Electrodynamics

We work in the Feynman-Gupta-Bleuler formulation of QED, corresponding to a local gauge where

$$\Box A_\mu(x) = j_\mu(x), \qquad \partial^\mu j_\mu(x) = 0, \tag{6.1.1}$$

where $j_\mu(x)$ is the conserved current associated to the global $U(1)$ symmetry transformation $g(\lambda)$, implemented by $e^{i\lambda Q}$. Consider the following family of local gauge transformations, given by the linear gauge parameter $\varepsilon(x) = -l_\mu x^\mu$:

$$\alpha(l) : A_\mu(x) \longmapsto A_\mu(x) - l_\mu, \tag{6.1.2}$$

$$\psi(x) \longmapsto \psi(x)\exp\{-iel_\mu x^\mu\}. \tag{6.1.3}$$

This $\alpha(l)$, parametrized by the co-vector l_μ, is indeed a *large* gauge transformation, in that its gauge parameter $\varepsilon(x)$ grows linearly as one approaches infinity. Denoting by $\tau(a, \Lambda)$ the action of the Poincaré group, where a^μ is a translation vector and $\Lambda \in SO(3,1)$, we immediately see that $\alpha(l)$ is not an internal symmetry: for example, considering its commutation with translations

$$\begin{aligned}
\tau(a)\alpha(l)\psi(x) &= \tau(a)\psi(x)\exp\{-iel_\mu x^\mu\} \\
&= \psi(x+a)\exp\{-iel_\mu x^\mu\} \\
&= \psi(x+a)\exp\{-iel_\mu (x+a)^\mu\}\exp\{iel_\mu a^\mu\} \\
&= \alpha(l)\psi(x+a)\exp\{iel_\mu a^\mu\} \\
&= \alpha(l)\tau(a)g(l\cdot a)\psi(x)
\end{aligned} \tag{6.1.4}$$

thus

$$\tau(a)\alpha(l) = \alpha(l)\tau(a)g(l\cdot a). \tag{6.1.5}$$

6.1.1 Broken large gauge symmetries and Goldstone's theorem

Let us suppose that $\alpha(l)$ could be implemented by some unitary operator $V(l)$. We we can then translate (6.1.5) into the following operator relation,

$$U(a)V(l) = V(l)U(a)e^{il\cdot aQ}, \tag{6.1.6}$$

where $U(a)$ is the unitary operator implementing translations. Applying this identity to the vacuum $|0\rangle$, which is translationally invariant and has zero charge, one has

$$U(a)V(l)|0\rangle = V(l)|0\rangle, \tag{6.1.7}$$

which shows that $V(l)|0\rangle$ is itself invariant under translations and, since the vacuum is the unique translationally invariant state, it must be

$$V(l)|0\rangle = |0\rangle. \tag{6.1.8}$$

On the other hand, if we compute the vacuum expectation value of

$$V(l)A_\mu(x)V(l)^{-1} = A_\mu(x) - l_\mu, \tag{6.1.9}$$

we easily obtain a contradiction: $l_\mu\langle 0|0\rangle = 0$, for every l_μ. Therefore, the symmetry $\alpha(l)$ cannot be implemented by a unitary operator: it is a broken symmetry, as may be expected, since vacuum expectations of $A_\mu(x)$ are not left invariant under $\alpha(l)$, unless $l_\mu = 0$.

We turn now to the discussion of the implications of this spontaneous breaking on the spectrum of the theory. It is well-known that the breaking of an internal symmetry gives rise to massless Goldstone excitations, but since $\alpha(l)$ does not commute with translations, we may wonder whether the Goldstone theorem still holds. In this case, the non-covariance of the current $J^{(l)}_\rho(x)$ which generates $\alpha(l)$ is explicit, thanks to (6.1.5):

$$U(a)J^{(l)}_\rho(x)U(a)^{-1} = J^{(l)}_\rho(x+a) + l_\mu a^\mu j_\rho(x+a) \tag{6.1.10}$$

or

$$J^{(l)}_\rho(x) = K^{(l)}_\rho(x) - l_\mu x^\mu j_\rho(x), \tag{6.1.11}$$

where $K^{(l)}_\rho(x) \equiv U(x)J^{(l)}_\rho(0)U(x)^{-1}$ is by definition translationally covariant. The smeared charge is given by

$$Q^{(l)}_{R,\alpha} \equiv \int f_R(\mathbf{x})\alpha(x_0)J^{(l)}_0(x)d^4x, \tag{6.1.12}$$

where the test functions f_R and α satisfy

$$f_R(\mathbf{x}) \equiv f\left(\frac{|\mathbf{x}|}{R}\right), \qquad f(x) = \begin{cases} 1 & \text{if } x < 1 \\ 0 & \text{if } x > 1+\epsilon \end{cases} \tag{6.1.13}$$

and

$$\int \alpha(x_0)dx_0 = 1, \tag{6.1.14}$$

so that the infinitesimal variation of a local operator B is

$$i \lim_{R\to\infty} [Q^{(l)}_{R,\alpha}, B] \equiv \delta^{(l)} B. \tag{6.1.15}$$

The Goldstone theorem states that, if the symmetry is broken, then there are massless one-particle modes in the Fourier transform of $\langle 0|\delta^{(l)}B|0\rangle$, or more precisely

$$\lim_{R\to\infty} \langle 0|[Q^{(l)}_{R,\alpha}, B]|0\rangle = \lim_{R\to\infty} \langle 0|Q^{(l)}_{R,\alpha} E_1 B - B E_1 Q^{(l)}_{R,\alpha}|0\rangle, \tag{6.1.16}$$

where E_1 denotes the projection on zero-mass one-particle states. Notice that the left-hand side of the previous equation is non-vanishing if and only if the symmetry is broken.

The key-point in the extension of the usual proof of Goldstone's theorem [50] is that the non-covariant piece of (6.1.11) involves the generator $j_\mu(x)$ of the unbroken global symmetry.

An explicit form of the current can be obtained as follows: we know that

$$J_\mu = F_{\nu\mu}\partial^\nu \varepsilon + j_\nu \varepsilon, \tag{6.1.17}$$

from the Noether theorem, thus, using $\Box A_\mu = j_\mu$ and integrating by parts

$$J_\mu = \partial_\nu(A_\mu \partial^\nu \varepsilon) - A_\mu \Box \varepsilon - \partial_\mu A_\nu \partial^\nu \varepsilon + \varepsilon \Box A_\mu; \tag{6.1.18}$$

but $\Box\varepsilon = 0$ and, up to a boundary term,

$$J^{(l)}_\rho(x) = l^\mu \partial_\rho A_\mu(x) - l^\mu x_\mu \Box A_\rho. \tag{6.1.19}$$

Luckily, the non-covariant piece $l^\mu x_\mu \Box A_0$ gives no contribution to the right-hand side of (6.1.16) thanks to the spectral projector E_1, which imposes $k^2 = 0$. Hence, we can write

$$\lim_{R\to\infty} \langle 0|[Q^{(l)}_{R,\alpha}, B]|0\rangle = \int d^4x f_R(\mathbf{x})\alpha(x_0) l^\mu \langle 0|\dot{A}_\mu(x) E_1 B - B E_1 \dot{A}_\mu(x)|0\rangle \tag{6.1.20}$$

and letting

$$H_\mu(\mathbf{x}) \equiv \int dx_0 \alpha(x_0) \langle 0|\dot{A}_\mu(x) E_1 B - B E_1 \dot{A}_\mu(x)|0\rangle, \tag{6.1.21}$$

we see that by locality $H_\mu(\mathbf{x})$ has compact support and therefore its Fourier transform $\widetilde{H}_\mu(\mathbf{k})$ is an entire analytic function. Also

$$\lim_{R\to\infty} \langle 0|[Q^{(l)}_{R,\alpha}, B]|0\rangle = \lim_{R\to\infty} \int d^3x f_R(\mathbf{x}) l^\mu H_\mu(\mathbf{x}) = (2\pi)^{3/2} \lim_{\mathbf{k}\to 0} l^\mu \widetilde{H}_\mu(\mathbf{k}). \tag{6.1.22}$$

In fact, the two pieces making up $\widetilde{H}_\mu(\mathbf{k})$ on the right-hand side give the same contribution: letting

$$G_\mu(\mathbf{x}) \equiv \int dx_0 \alpha(x_0) \langle 0| \dot{A}_\mu(x) E_1 B + B E_1 \dot{A}_\mu(x) |0\rangle \tag{6.1.23}$$

and

$$L_\mu(\mathbf{x}) \equiv \int dx_0 \alpha(x_0) \langle 0| A_\mu(x) E_1 B - B E_1 A_\mu(x) |0\rangle, \tag{6.1.24}$$

we see that thanks to the spectral representation and to the projector E_1, which selects $\omega(\mathbf{k}) = |\mathbf{k}|$, one gets

$$\widetilde{G}_\mu(\mathbf{k}) = -i|\mathbf{k}| \widetilde{L}_\mu(\mathbf{k}), \tag{6.1.25}$$

a quantity that tends to zero as $|\mathbf{k}| \to 0$ by the analiticity of $\widetilde{L}_\mu(\mathbf{k})$. To sum up:

$$\lim_{R\to\infty} \langle 0| [Q^{(l)}_{R,\alpha}, B] |0\rangle = 2(2\pi)^{3/2} \lim_{\mathbf{k}\to 0} l^\mu \widetilde{H}^+_\mu(\mathbf{k}), \tag{6.1.26}$$

where

$$\widetilde{H}^+_\mu(\mathbf{k}) = \int dx_0 \alpha(x_0) \langle 0| \dot{A}_\mu(x) E_1 B |0\rangle. \tag{6.1.27}$$

For convenience, let us rewrite this identity as follows: denote $l_\mu Q^\mu_{R,\alpha} = Q^{(l)}_{R,\alpha}$, then

$$\lim_{R\to\infty} \langle 0| [Q^\mu_{R,\alpha}, B] |0\rangle = (2\pi)^{3/2} \lim_{\mathbf{k}\to 0} \langle \mathbf{k}, \mu | B |0\rangle, \tag{6.1.28}$$

where

$$|\mathbf{k}, \mu\rangle = -2i \int dx_0 \alpha(x_0) E_1 \widetilde{\dot{A}}_\mu(\mathbf{k}, x_0) |0\rangle. \tag{6.1.29}$$

The infinitesimal transformations induced by $\alpha(l)$ are

$$i \lim_{R\to\infty} [Q^\mu_{R,\alpha}, A^\nu(x)] = -\eta^{\mu\nu}, \qquad i \lim_{R\to\infty} [Q^\mu_{R,\alpha}, \psi(x)] = -iex^\mu \psi(x). \tag{6.1.30}$$

6.1.2 Soft theorems of QED

Using $B = A^\nu(x)$ in (6.1.28) yields

$$(2\pi)^{3/2} \lim_{\mathbf{k}\to 0} \langle \mathbf{k}, \mu | A^\nu(x) |0\rangle = -\eta^{\mu\nu}. \tag{6.1.31}$$

Using instead $B = T(\psi(x)\bar{\psi}(0))$, where now T denotes time ordering, allows to recover

$$(2\pi)^{3/2} \lim_{\mathbf{k}\to 0} \langle \mathbf{k}, \mu | T(\psi(x)\bar{\psi}(0)) |0\rangle = -iex^\mu \langle 0| T(\psi(x)\bar{\psi}(0)) |0\rangle, \tag{6.1.32}$$

which is nothing but the Ward identity

$$S(p)\Gamma^\mu(p,0)S(p) = -ie \frac{\partial}{\partial p_\mu} S(p), \tag{6.1.33}$$

where $S(p)$ is the electron propagator and $\Gamma^\mu(p,k)$ is the electron vertex function.

Again, upon choosing $B = T(A^{\mu_1}(x_1)\dots A^{\mu_n}(x_n))$, one obtains

$$\begin{aligned}&(2\pi)^{3/2}\lim_{\mathbf{k}\to 0}\langle \mathbf{k},\mu|T(A^{\mu_1}(x_1)\dots A^{\mu_n}(x_n))|0\rangle\\ =&-\sum_{i=1}^{n}\eta^{\mu\mu_i}\langle 0|T(A^{\mu_1}(x_1)\dots\widehat{A^{\mu_i}(x_i)},\dots A^{\mu_n}(x_n))|0\rangle\end{aligned}\tag{6.1.34}$$

where the hat indicates that the corresponding factor has been omitted. Using the previous identity (6.1.31), one sees that the right-hand side reconstructs the disconnected part of the left hand side, leaving as a consequence

$$\lim_{\mathbf{k}\to 0}\langle \mathbf{k},\mu|T(A^{\mu_1}(x_1)\dots A^{\mu_n}(x_n))|0\rangle_{\text{connected}} = 0. \tag{6.1.35}$$

Weinberg's [6] and Low's [8] soft theorems are instead obtained by taking insertions of n photon fields and $2m$ charged fields $B = T(A^{\mu_1}(x_1)\dots\psi(y_1)\dots\bar\psi(z_1)\dots)$, since then, again reconstructing the disconnected photon contributions thanks to (6.1.31), we can write

$$\begin{aligned}&(2\pi)^{3/2}\lim_{\mathbf{k}\to 0}\langle \mathbf{k},\mu|T(A^{\mu_1}(x_1)\dots\psi(y_1)\dots\bar\psi(z_1)\dots|0\rangle_{\text{connected}}\\ =&-\sum_{j=1}^{m}(y_j-z_j)^\mu\langle 0|T(A^{\mu_1}(x_1)\dots\psi(y_1)\dots\bar\psi(z_1)\dots|0\rangle.\end{aligned}\tag{6.1.36}$$

After Fourier-transforming, and using $\overset{*}{\sum}$ to denote the sum with respect to the $2m+n-1$ independent momenta, we get

$$\begin{aligned}&\prod_{r,s}D(q_r)S(p'_s)K^\mu(p,p',q)S(p_s)\\ =&\overset{*}{\sum_{j}}\left(\frac{\partial}{\partial p'_{j\mu}}+\frac{\partial}{\partial p_{j\mu}}\right)\prod_{rs}D(q_r)S(p'_s)K(p,p',q)S(p_s),\end{aligned}\tag{6.1.37}$$

where $D(q)$ is the photon propagator, and $K^\mu(p,p',q)$ denotes the amputated amplitude for the process $K(p,p',q)$ with the addition of an extra soft photon with momentum k^μ. Taking into account the Ward identity (6.1.33) when applying the derivatives on the right-hand side gives, in its turn,

$$K^\mu(p,p',q) = i\sum_{j=1}^{m}\left\{\Gamma^\mu(p'_j,0)S(p'_j)+S(p_j)\Gamma^\mu(p_j,0)\right\}K(p,p',q) \tag{6.1.38}$$

$$+\, e\overset{*}{\sum_{j}}\left(\frac{\partial}{\partial p'_{j\mu}}+\frac{\partial}{\partial p_{j\mu}}\right)K(p,p',q). \tag{6.1.39}$$

The first line (6.1.38) encodes the Weinberg poles as can be easily seen by considering, for example,

$$\begin{aligned}\bar{u}(p')\Gamma^{\mu}(p',k)S(p'+k) &= \bar{u}(p')(ie\gamma^{\mu}+i\lambda\sigma^{\mu\nu}k_{\nu})\frac{1}{i\gamma^{\rho}(p'_{\rho}+k_{\rho})+m} \\ &= \bar{u}(p')ie\frac{p'^{\mu}}{p'\cdot k}+\dots,\end{aligned} \tag{6.1.40}$$

and is associated with those diagrams where the soft photon interaction occurs after (or before) all other interactions, whereas the second line (6.1.39) gives finite corrections [8, 9] to the leading singular behavior, corresponding to the other diagrams.

This shows that Weinberg's soft photon theorem, together with the corresponding subleading corrections, can be seen as consequences of the spontaneous breaking of linear large gauge symmetries of QED. With respect to the treatment given in the previous chapters, the asymptotic symmetry group considered in this case is much simpler since it can be parametrized by a co-vector l_{μ} instead of an angular function $T(z,\bar{z})$.

6.2 Spontaneous Symmetry Breaking in Linearized Gravity

We work in the harmonic or "De Donder" gauge of linearized gravity

$$\Box h_{\mu\nu}(x) = j_{\mu\nu}(x), \qquad \partial^{\mu}j_{\mu\nu}(x) = 0; \tag{6.2.1}$$

here $j_{\mu\nu}(x)$ encodes both the conserved stress-energy tensor of matter and the non-linear terms of the Einstein equations in the Arnowitt-Deser-Misner formulation (see *e.g.* [59, Chapter 6.7]. The tensor $j_{\mu\nu}(x)$ also generates global space-time translations via the ADM energy-momentum tensor P_{μ}. Consider the following family of infinitesimal local gauge transformations, given by the linear gauge parameter $\varepsilon_{\mu}(x) = -l_{\mu\nu}x^{\nu}$:

$$\alpha(l) : h_{\mu\nu}(x) \longmapsto h_{\mu\nu}(x) - 2l_{\mu\nu} \tag{6.2.2}$$

$$\psi(x) \longmapsto \Phi(x) - if l_{\mu\nu}x^{\mu}\partial^{\nu}\Phi(x), \tag{6.2.3}$$

where f denotes the coupling to gravity. Thus, $\alpha(l)$ is parametrized by the constant symmetric tensor $l_{\mu\nu}$. Denoting by $\tau(a,\Lambda)$, for a^{μ} a translation vector and $\Lambda \in SO(3,1)$, the infinitesimal action of the Poincaré group, we see that $\alpha(l)$ is not an internal symmetry: considering its commutation with translations

$$[\tau(a),\alpha(l)]\Phi(x) = -fa^{\rho}l_{\rho\sigma}\partial^{\sigma}\Phi(x) = -if\tau(l_{\mu\nu}a^{\nu})\Phi(x) \tag{6.2.4}$$

thus, in terms of generators, denoting $Q^{(l)} = l_{\mu\nu}\tilde{Q}^{\mu\nu}$,

$$[P^{\mu},\tilde{Q}^{\rho\sigma}] = -if\eta^{\mu\rho}P^{\sigma}. \tag{6.2.5}$$

6.2.1 Broken linear large diffeomorphisms

Supposing now that $\alpha(l)$ could be implemented by some self-adjoint generators, which is equivalent to the weak continuity of the corresponding unitary exponentials, we get, applying the previous commutation relation to the vacuum $|0\rangle$, which is translationally invariant,

$$P^{\mu}\tilde{Q}^{\rho\sigma}|0\rangle = 0; \tag{6.2.6}$$

this shows that $\tilde{Q}^{\rho\sigma}|0\rangle$ is itself invariant under translations and therefore, in clear analogy with the $U(1)$ generators of the previous section, it must coincide with the vacuum

$$\tilde{Q}^{\rho\sigma}|0\rangle = |0\rangle. \tag{6.2.7}$$

Taking now the vacuum expectation value of the following expression

$$i[\tilde{Q}^{\rho\sigma}, h^{\mu\nu}(x)] = -\eta^{\rho\mu}\eta^{\sigma\nu} \tag{6.2.8}$$

easily allows to derive a contradiction. Consequently, as before, the symmetry $\alpha(l)$ is a broken symmetry. Again, the Goldstone theorem still holds, in spite of $\alpha(l)$ not commuting with translations, since the non-covariance of the current $J_{\rho}^{(l)}(x)$ which generates $\alpha(l)$ is explicit, thanks to (6.2.5):

$$U(a)J_{\rho}^{(l)}(x)U(a)^{-1} = J_{\rho}^{(l)}(x+a) + l^{\mu\nu}a_{\mu}j_{\nu\rho}(x+a) \tag{6.2.9}$$

or

$$J_{\rho}^{(l)}(x) = K_{\rho}^{(l)}(x) - l^{\mu\nu}x_{\mu}j_{\nu\rho}(x), \tag{6.2.10}$$

where $K_{\rho}^{(l)}(x) \equiv U(x)J_{\rho}^{(l)}(0)U(x)^{-1}$ is by definition translationally covariant. Indeed, we will explicitly verify below that this is the case for the Noether current we will choose. The charge $Q_{R,\alpha}^{(l)}$ is regulated with the test functions f_R and α like those appearing in the previous section, so that the infinitesimal variation of a local operator B is

$$i\lim_{R\to\infty}[Q_{R,\alpha}^{(l)}, B] \equiv \delta^{(l)}B, \tag{6.2.11}$$

independently of α. The Goldstone theorem states that, if the symmetry is broken, then there are massless one-particle modes in the Fourier transform of $\langle 0|\delta^{(l)}B|0\rangle$, according to (6.1.16), which we report here for simplicity:

$$\lim_{R\to\infty}\langle 0|[Q_{R,\alpha}^{(l)}, B]|0\rangle = \lim_{R\to\infty}\langle 0|Q_{R,\alpha}^{(l)}E_1 B - BE_1 Q_{R,\alpha}^{(l)}|0\rangle. \tag{6.2.12}$$

The extension of the usual proof of Goldstone's theorem [50] is the same as in the case of electromagnetism, since the non-covariant piece of (6.2.10) still involves the generator $j_{\mu\nu}(x)$ of the unbroken global symmetry.

An explicit form of the current can be obtained as follows: from the Noether theorem, using $\Box h_{\mu\nu} = j_{\mu\nu}$ and integrating by parts,

$$J^{(l)}_{\rho}(x) = 2l^{\mu\nu}\partial_{\rho}h_{\mu\nu}(x) - l^{\mu\nu}x_{\mu}\Box h_{\nu\rho}. \tag{6.2.13}$$

Also in this case, the non-covariant piece $l^{\mu\nu}x_{\mu}\Box h_{\nu 0}$ gives no contribution to the right-hand side of (6.2.12) thanks to the spectral projector E_1, which imposes $k^2 = 0$. Hence, we can write

$$\lim_{R\to\infty}\langle 0|[Q^{(l)}_{R,\alpha}, B]|0\rangle = \int d^4x f_R(\mathbf{x})\alpha(x_0)2l^{\mu\nu}\langle 0|\dot{h}_{\mu\nu}(x)E_1 B - BE_1\dot{h}_{\mu\nu}(x)|0\rangle. \tag{6.2.14}$$

Using exactly the same locality and analyticity arguments of the previous section, one arrives at:

$$\lim_{R\to\infty}\langle 0|[Q^{\mu\nu}_{R,\alpha}, B]|0\rangle = (2\pi)^{3/2}\lim_{\mathbf{k}\to 0}\langle \mathbf{k}, \mu\nu|B|0\rangle, \tag{6.2.15}$$

where

$$|\mathbf{k}, \mu\nu\rangle = -4i\int dx_0\alpha(x_0)E_1\dot{\tilde{h}}_{\mu\nu}(\mathbf{k}, x_0)|0\rangle. \tag{6.2.16}$$

The infinitesimal transformations induced by $\alpha(l)$ are

$$i\lim_{R\to\infty}[Q^{\mu\nu}_{R,\alpha}, h^{\rho\sigma}(x)] = -2\eta^{\mu\rho}\eta^{\nu\sigma}, \qquad i\lim_{R\to\infty}[Q^{\mu\nu}_{R,\alpha}, \psi(x)] = -\frac{i}{2}fx^{(\mu}\partial^{\nu)}\psi(x). \tag{6.2.17}$$

6.2.2 Soft graviton theorems

Using $B = h^{\rho\sigma}(x)$ in (6.2.15) yields

$$(2\pi)^{3/2}\lim_{\mathbf{k}\to 0}\langle \mathbf{k}, \mu\nu|h^{\rho\sigma}(x)|0\rangle = -2\eta^{\mu\rho}\eta^{\nu\sigma}. \tag{6.2.18}$$

Using instead $B = T(\Phi(x)\bar{\Phi}(0))$, where T denotes time ordering, allows to recover

$$(2\pi)^{3/2}\lim_{\mathbf{k}\to 0}\langle \mathbf{k}, \mu\nu|T(\Phi(x)\bar{\Phi}(0))|0\rangle = -\frac{i}{2}fx^{(\mu}\partial^{\nu)}\langle 0|T(\Phi(x)\bar{\Phi}(0))|0\rangle, \tag{6.2.19}$$

and the Ward identity

$$S(p)\Gamma^{\mu\nu}(p,0)S(p) = -\frac{i}{2}f\frac{\partial}{\partial p_{(\mu}}p^{\nu)}S(p), \tag{6.2.20}$$

where $S(p)$ is the "charged" field propagator and $\Gamma^{\mu\nu}(p,k)$ is the graviton vertex function.

Again, if one chooses $B = T(h^{\mu_1\nu_1}(x_1)\ldots h^{\mu_n\nu_n}(x_n))$,

$$\begin{aligned}&(2\pi)^{3/2}\lim_{\mathbf{k}\to 0}\langle \mathbf{k}, \mu\nu|T(h^{\mu_1\nu_1}(x_1)\ldots h^{\mu_n\nu_n}(x_n))|0\rangle \\ &= -\sum_{i=1}^{n}2\eta^{\mu\mu_i}\eta^{\nu\nu_i}\langle 0|B = T(h^{\mu_1\nu_1}(x_1)\ldots \widehat{h^{\mu_i\nu_i}(x_i)}\ldots h^{\mu_n\nu_n}(x_n))|0\rangle\end{aligned} \tag{6.2.21}$$

where the hat indicates that the factor has been omitted. Using the previous identity (6.2.18), one sees that the right-hand side reconstructs the disconnected part of the left hand side, yielding

$$\lim_{\mathbf{k}\to 0}\langle \mathbf{k}, \mu\nu | T(h^{\mu_1\nu_1}(x_1)\dots h^{\mu_n\nu_n}(x_n))|0\rangle_{\text{connected}} = 0. \tag{6.2.22}$$

Weinberg's soft graviton theorem [6] and the subleading corrections (corresponding to those found in [8] for QED) are instead obtained by taking insertions of n graviton fields and $2m$ charged fields $B = T(h^{\mu_1\nu_1}(x_1)\dots\Phi(y_1)\dots\bar{\Phi}(z_1)\dots)$, since then, again reconstructing the disconnected photon contributions thanks to (6.2.18),

$$\begin{aligned}&(2\pi)^{3/2}\lim_{\mathbf{k}\to 0}\langle \mathbf{k}, \mu\nu | T(h^{\mu_1\nu_1}(x_1)\dots\Phi(y_1)\dots\bar{\Phi}(z_1)\dots|0\rangle_{\text{connected}}\\ =&-\frac{1}{2}\sum_{j=1}^{m}\left(y_j^{(\mu}\frac{\partial}{\partial y_{j\nu)}} - z_j^{(\mu}\frac{\partial}{\partial z_{j\nu)}}\right)\langle 0|T(h^{\mu_1\nu_1}(x_1)\dots\Phi(y_1)\dots\bar{\Phi}(z_1)\dots|0\rangle.\end{aligned} \tag{6.2.23}$$

Upon Fourier-transforming, and denoting by $\overset{*}{\sum}$ the sum with respect to the $2m+n-1$ independent momenta, one finds

$$\begin{aligned}&\prod_{r,s} D(q_r)S(p'_s)K^{\mu\nu}(p,p',q)S(p_s)\\ =&\frac{1}{2}\overset{*}{\sum_{j}}\left(\frac{\partial}{\partial p'_{j(\mu}}p_j'^{\nu)} + \frac{\partial}{\partial p_{j(\mu}}p_j^{\nu)}\right)\prod_{rs} D(q_r)S(p'_s)K(p,p',q)S(p_s),\end{aligned} \tag{6.2.24}$$

where $D(q)$ is the graviton propagator, and $K^{\mu\nu}(p,p',q)$ denotes the amputated amplitude for the process $K(p,p',q)$ with the addition of an extra soft graviton with momentum k^μ; taking into account the Ward identity (6.2.20) when applying the derivatives on the right-hand side gives

$$K^{\mu\nu}(p,p',q) = i\sum_{j=1}^{m}\left\{\Gamma^{\mu\nu}(p'_j,0)S(p'_j) + S(p_j)\Gamma^{\mu\nu}(p_j,0)\right\}K(p,p',q) \tag{6.2.25}$$

$$+\frac{f}{2}\overset{*}{\sum_{j}}\left(\frac{\partial}{\partial p'_{j(\mu}}p_j'^{\nu)} + \frac{\partial}{\partial p_{j(\mu}}p_j^{\nu)}\right)K(p,p',q). \tag{6.2.26}$$

In complete analogy with the spin-1 case, the first line (6.2.25) encodes the Weinberg poles

$$\Gamma^{\mu\nu}(p',k)S(p'+k) \sim \frac{p'^\mu p'^\nu}{-(p'+k)^2+m^2} \sim \frac{p'^\mu p'^\nu}{p'\cdot k} \tag{6.2.27}$$

and is associated with those diagrams where the soft graviton interaction occurs after (or before) all other interactions, whereas the second line (6.2.26) gives finite corrections to the leading singular behavior, corresponding to the subleading diagrams.

This completes the extension of the results presented in Section 6.1 to the case of spin-two fields, showing that soft graviton theorems emerge as consequences of the spontaneous breaking of large gauge transformations comprising a subgroup of linearized diffeomorphisms, parametrized by a symmetric tensor $l_{\mu\nu}$.

6.3 Spontaneous Symmetry Breaking in Higher-Spin Theories

The discussion done in the previous two sections can be naturally extended the context of spin-s gauge theories, by choosing the higher-spin De Donder gauge,

$$\partial \cdot \varphi_{\mu_2 \dots \mu_s} = \frac{1}{2} \partial_{(\mu_2} \varphi'_{\mu_2 \dots \mu_s)}, \tag{6.3.1}$$

in which the equations of motion for the gauge field $\varphi_{\mu_1 \dots \mu_s}$ are again

$$\Box \varphi_{\mu_1 \dots \mu_s} = j_{\mu_1 \dots \mu_s}, \tag{6.3.2}$$

where $j_{\mu_1 \dots \mu_s}$ is a totally symmetric current whose traceless projection is locally conserved. Then, linear large gauge transformations with parameter $\varepsilon_{\mu_1 \dots \mu_{s-1}} = -l_{\mu_1 \dots \mu_{s-1} \mu_s} x^{\mu_s}$,

$$\delta \varphi_{\mu_1 \dots \mu_s} = -s l_{\mu_1 \dots \mu_s}, \qquad \delta \psi = -i f x^{\mu_1} l_{\mu_1 \dots \mu_s} \partial^{\mu_2} \cdots \partial^{\mu_s} \psi, \tag{6.3.3}$$

where $l_{\mu_1 \dots \mu_s}$ is a totally symmetric, traceless, constant tensor, ψ denotes an elementary matter field, and f is the spin-s coupling, are broken in each irreducible representation of the field algebra, as one can infer from the shifts induced on gauge fields and hence on their vacuum expectation values.

With this gauge choice, the Noether current associated to such transformations is, after an integration by parts,

$$J_\rho^{(l)}(x) = s l^{\mu_1 \dots \mu_s} \partial_\rho \varphi_{\mu_1 \dots \mu_s}(x) - l^{\mu_1 \dots \mu_s} x_{\mu_1} j_{\mu_2 \dots \mu_s \rho}. \tag{6.3.4}$$

Note that the form of this current is again given by a translationally covariant piece plus a non-covariant one, but the latter involves the (unbroken) matter current $j_{\mu_1 \dots \mu_s}$ and hence gives no contribution to the Goldstone theorem, as it was described in the spin-one and spin-two cases.

Now, any difficulty arising from the structure of higher-spin symmetries has been taken care of, and hence one can perform the same computations done in Sections 6.1 and 6.2, with straightforward modifications in the index structure of the above formulas, to extend the link between the spontaneous breaking of linear large gauge symmetries and higher-spin soft theorems: the latter, emerging as the rewriting of the Goldstone theorem when applied to such a symmetry breaking.

More explicitly, limiting ourselves to the main formulae, the Ward identity linking the matter propagator $S(p)$ to the s-field vertex function $\Gamma^{\mu_1 \cdots \mu_s}$ reads

$$S(p) \Gamma^{\mu_1 \cdots \mu_s}(p, 0) S(p) = -\frac{i}{s} f \frac{\partial}{\partial p_{(\mu_1}} p^{\mu_2} \dots p^{\mu_s)} S(p), \tag{6.3.5}$$

and the soft theorem expressing the amputated amplitude $K^{\mu_1\cdots\mu_s}(p,p',q)$ for the process $K(p,p',q)$ with the addition of an extra soft spin-s particle with momentum k^μ is encoded in the following expression,

$$K^{\mu_1\cdots\mu_s}(p,p',q) = i\sum_{j=1}^{m}\left\{\Gamma^{\mu_1\cdots\mu_s}(p'_j,0)S(p'_j) + S(p_j)\Gamma^{\mu_1\cdots\mu_s}(p_j,0)\right\}K(p,p',q) \tag{6.3.6}$$

$$+\frac{f}{2}\sum_{j}^{*}\left(\frac{\partial}{\partial p'_{j(\mu_1}}p'^{\mu_2}_j\dots p'^{\mu_s)}_j + \frac{\partial}{\partial p_{j(\mu_1}}p^{\mu_2)}_j\dots p^{\mu_s)}_j\right)K(p,p',q). \tag{6.3.7}$$

7 BMS Group in D dimensions

In the same way as the validity for any spin of Weinberg's factorization theorem motivated, among other considerations, our investigations of higher-spin soft theorems and their connection with infinite-dimensional asymptotic symmetry groups, the observation that the factorization itself holds independently of the dimension of spacetime has aroused interest in the study of asymptotic symmetries, in particular of those of gravity, in spacetimes of arbitrary dimension D.

In Chapter 1, we have already reviewed the precise definition of asymptotically flat four-dimensional spacetimes and its implications for the enlargement of the symmetry group from the Poincaré group to the BMS group. We devote this final chapter to describing an extension of the approach to asymptotic symmetries that still relies on the tool of conformal compactification but extends, in principle, to spacetimes of any dimension D; this approach both accounts for the presence of an infinite-dimensional, BMS-like asymptotic symmetry group in any dimension and gives indications for the falloff requirements that are to be imposed on the metric tensor in order describe asymptotic flatness.

7.1 Abstract Null Infinity

It is worthwhile to stress the elementary abstract properties that enter the definition of the BMS group: after isolating such fundamental elements, we will be able to propose a definition of this group in arbitrary dimension, a generalization which proves rewarding, for instance, due to the implications that this symmetry group has shown on the infrared properties of gravity. Indeed, the generators of the asymptotic symmetry transformations have been interpreted as creation operators of gravitons with vanishing momentum, and, furthermore, we have seen in the previous chapters how such symmetries are related to soft theorems and to the equivalence principle itself.

In [41], the authors have identified the BMS group as the group of diffeomorphisms preserving certain geometrical structure at three-dimensional null infinity (thought of as the conformal boundary of a four-dimensional spacetime). Their strategy admits a simple and natural generalization to D-dimensional spacetimes, which is given below.

Given the manifold $\mathscr{N} = \mathbb{R} \times S^{D-2}$, consider on it a conformal class $\mathscr{C}$ of degenerate metrics of signature $0 + + \ldots +$. A vector field n is called isotropic if $q(n, \cdot) = 0$ for some $q \in \mathscr{C}$ (and hence for all $q \in \mathscr{C}$, since these metric tensors are conformally related). The integral curves of an isotropic vector field are called isotropic lines. Examples of such objects were given by $q_{\mu\nu}$ and n^μ below equation (1.1.34) in Chapeter 1.

The first property we require is:

> **S1**: The isotropic lines define a product decomposition of $\mathscr{N}$ into $\mathbb{R} \times S^{D-2}$, the lines being transversal to the hyperspheres.

Physically, this property selects the parameter $u \in \mathbb{R}$ as the retarded (or advanced) time of some light wavefronts, and implies that the space $\mathscr{Q}$ of isotropic lines (*i.e.* light rays) is the $(D-2)$-sphere; $\pi : \mathscr{N} \to \mathscr{Q}$ will denote the corresponding natural projection.

Assume also:

> **S2**: Any diffeomorphism $\varphi : \mathscr{N} \to \mathscr{N}$ which maps each isotropic line into itself preserves $\mathscr{C}$, *i.e.* it is a conformal transformation for each $q \in \mathscr{C}$.

By the latter requirement, $\mathscr{Q}$ is naturally endowed with a conformal structure of signature $+ + \ldots +$ in the following way: picking any section $\tau : \mathscr{Q} \to \mathscr{N}$, then $\tilde{q} = \tau^* q$ is positive definite on $\mathscr{Q}$ and all metrics on $\mathscr{Q}$ constructed in this way are conformally related. Let us denote this conformal structure by $\tilde{\mathscr{C}}$.

We shall define the pair $(\mathscr{N}, \mathscr{C})$, together with the axioms **S1** and **S2**, as pre-$\mathscr{I}$ ("pre-scr-I").

We will now investigate under which conditions the geometric structure defining pre-$\mathscr{I}$ is invariant. Let φ be a diffeomorphism of $\mathscr{N}$ which preserves $\mathscr{C}$. Then φ maps isotropic lines into isotropic lines, and consequently defines a mapping $\tilde{\varphi}$ on the quotient space $\mathscr{Q}$; but $\tilde{\varphi}$ preserves the conformal structure $\tilde{\mathscr{C}}$ because φ preserves $\mathscr{C}$. Conversely, if $\tilde{\varphi}$ is conformal on $\mathscr{Q}$, then any lift of $\tilde{\varphi}$ to a diffeomorphism φ on $\mathscr{N}$ with $\pi \circ \varphi = \tilde{\varphi} \circ \pi$ preserves $\mathscr{C}$. The crucial point is that the group preserving such a conformal structure on S^{D-2} is isomorphic to the orthochronous, proper Lorentz group $SO(D-1,1)$ (see *e.g.* [38]). Thus we have proved the following statement:

Proposition. *The automorphisms of* pre-$\mathscr{I}$ *are all the (orientation-preserving) lifts to* $\mathscr{N}$ *of the action of the proper, orthochronous Lorentz group* $SO(D-1,1)$ *on* $\mathscr{Q}$.

In particular, choosing hyperspherical coordinates θ^A, for $A = 1, \ldots, D-2$, on the $(D-2)$-hypersphere and any smooth parametrization $u : \mathscr{N} \to \mathbb{R}$ of isotropic lines, the automorphisms of pre-$\mathscr{I}$ are given by

$$\theta^A \mapsto H_j(\theta^1, \ldots, \theta^{D-2}) \tag{7.1.1}$$

and

$$u \mapsto F(u, \theta), \tag{7.1.2}$$

for a conformal transformation H on the hypersphere, and F being an arbitrary smooth function such that $\partial F / \partial u > 0$.

Let us add:

S3: There exists a nowhere-vanishing tensor field $S : \mathscr{N} \to T^2_2(\mathscr{N})$ of type $(2,2)$ such that:

(i) S is symmetric in both pair of arguments;

(ii) S contracted twice with any one-form in $\mathscr{N}$ either vanishes or is in $\mathscr{C}$;

(iii) any contraction of S with itself vanishes.

The structure of pre-$\mathscr{I}$ together with the axiom **S3** will be simply called $\mathscr{I}$. This axiom is equivalent to requiring that there is a rule which associates an isotropic vector field n with a given degenerate metric q, and viceversa, so that

$$S = q \otimes n \otimes n. \tag{7.1.3}$$

Of course, from the perspective of introducing $\mathscr{I}$ as the conformal boundary of an asymptotic spacetime, this is totally reasonable since $n = (d\Omega)^\sharp$, where $\mathscr{I}$ is defined by the vanishing of the conformal factor $\Omega = 0$ and the one-form $d\Omega$ is sharpened into a vector field using the full D-dimensional metric $\tilde{g}$; in components, $n^\mu = \tilde{g}^{\mu\nu}\tilde{\nabla}_\nu\Omega$. Note that, under a change of conformal parameter $\Omega \mapsto \omega\Omega$, we get $q \mapsto \omega^2 q$ and $d\Omega \mapsto \omega d\Omega$, meaning that $n \mapsto \omega^{-1}n$ and hence $S = q \otimes n \otimes n$ is conformally invariant. In fact, this tensor was also introduced in the definition of asymptotic geometry given by Geroch [39] (where it appears as $\Gamma^{ab}{}_{cd}$).

Let us now look to the automorphisms of $\mathscr{I}$, that is, the mappings of $\mathscr{N}$ to itself preserving all of our three axioms: the additional requirement, with respect to the previous proposition, is that S must be preserved, which restricts the arbitariness of the lifts to $\mathscr{N}$ of conformal transformations of $\mathscr{Q}$.

Let $h \in \tilde{\mathscr{C}}$ and $q = \pi^* h$ the pullback of h to $\mathscr{N}$. Let n be the isotropic vector field corresponding to q via S. Let $\varphi : \mathscr{Q} \to \mathscr{Q}$ preserve $\tilde{\mathscr{C}}$, with $\tilde{\varphi}^* h = \omega^2 h$. Any lift φ of $\tilde{\varphi}$ to $\mathscr{N}$ will map q to $\omega^2 q$, but since $\varphi_* S = \varphi_* q \otimes \varphi_* n \otimes \varphi_* n = \omega^2 q \otimes \varphi_* n \otimes \varphi_* n$, *only* those lifts for which

$$\varphi_* n = \omega^{-1} n \tag{7.1.4}$$

will preserve S in addition to $\mathscr{C}$. In the usual coordinates, chosen so that $n = \partial/\partial u$, since $\varphi_* n = \omega^{-1} n$, we have $\partial/\partial u \mapsto \omega^{-1}\partial/\partial u$. Since ω is a function only of the angular coordinates, this can be integrated to give

$$u \longmapsto \omega^{-1}(u + \alpha(\theta)), \tag{7.1.5}$$

where α is any smooth function on S^{D-2}. This completes the proof of the following statement:

Proposition. *The group of automorphisms of $\mathscr{I}$ is the (analog of the) BMS group, in D dimensions.*

The transformations (7.1.1) and (7.1.5) indeed provide the analog of the BMS group for D-dimensional spacetimes, in that they give rise to a group which is the semi-direct product of the Lorentz group with an infinite-dimensional normal subgroup of angular-dependent translations.

7.2 Boundary Expansion of the Metric Tensor

On top of the properties discussed in the previous section, it is also reasonable to assume our abstract $\mathscr{I}$ to have vanishing Weyl tensor given by the connection compatible with $q_{\mu\nu}$: the reason is that this manifold is meant as the conformal equivalent of the *flat* boundary of the physical spacetime. Indeed, this property has been in fact *deduced* [39, Theorem 11] for $D = 4$ from the definition of asymptotic flatness given above. By doing so, we also obtain that each cross section inherits, as an embedding in $\mathscr{I}$, an induced metric $h_{\mu\nu}$ whose compatible connection also has vanishing Weyl tensor; this follows immediately by noting that $q_{\mu\nu} = h_{\mu\nu}$ since by definition $q_{\mu\nu}n^\nu = 0$, where n^μ is the isotropic normal to the given cross section.

On the other hand, any topological $(D-2)$-sphere with this property is conformally equivalent to the Euclidean $(D-2)$-sphere: in the equivalence class of metrics on a given cross section we can always choose the metric of the standard unit sphere [60]. Now the procedure linking $\mathscr{I}$ to the asymptotic behavior of the metric in the physical spacetime can proceed as discussed after (1.1.22), without imposing $D = 4$. Then, the structure of such falloff rates turns out to be identical to the one given in (1.1.28) with θ and ϕ replaced by $D-2$ Euclidean angular coordinates θ^A, as done below.

One first introduces coordinates $(\Omega, u, \theta^1, \dots, \theta^{D-2})$ in a neighborhood of $\mathscr{I}^+$ in the usual way. In such coordinates

$$d\tilde{s}^2|_{\mathscr{I}^+} = 2d\Omega\, du + d\gamma^2, \tag{7.2.1}$$

where $d\gamma^2$ is the line element on the Euclidean $(D-2)$-sphere.[1] To study the asymp-

[1] The line element on the Euclidean unit $(n-1)$-sphere, in coordinates

$$\begin{aligned}
x^1 &= \cos\phi \sin\theta_1 \sin\theta_2 \dots \sin\theta_{n-3} \sin\theta_{n-2} \\
x^2 &= \sin\phi \sin\theta_1 \sin\theta_2 \dots \sin\theta_{n-3} \sin\theta_{n-2} \\
x^3 &= \cos\theta_1 \sin\theta_2 \dots \sin\theta_{n-3} \sin\theta_{n-2} \\
&\vdots \\
x^{n-1} &= \cos\theta_{n-3} \sin\theta_{n-2} \\
x^n &= \cos\theta_{n-2},
\end{aligned} \tag{7.2.2}$$

reads

$$d\sigma^2 = d\theta^2_{n-2} + \sin^2\theta_{n-2}\, d\theta^2_{n-3} + \sin^2\theta_{n-2} \sin^2\theta_{n-3}\, d\theta^2_{n-4} + \dots\ . \tag{7.2.3}$$

totic behavior of the metric components in the physical spacetime, one recalls the gauge condition (1.1.19), which was already discussed in arbitrary dimension D in Chapter 1,

$$\partial_\mu \partial_\nu \Omega - \tilde{\Gamma}^\rho_{\mu\nu} \partial_\rho \Omega = 0, \text{ and hence } \tilde{\Gamma}^\Omega_{\mu\nu} = 0; \tag{7.2.4}$$

but, since $\tilde{g}^{\Omega\rho} = \delta^\rho_u$ and $\partial_u \tilde{g}_{\mu\nu} = 0$ at $\mathscr{I}^+$,

$$0 = \tilde{\Gamma}^\Omega_{\mu\nu} = \frac{1}{2}\left(\partial_\mu \tilde{g}_{u\nu} + \partial_\nu \tilde{g}_{u\mu}\right); \tag{7.2.5}$$

fixing $\mu = \Omega$ and selecting $\nu = u, \theta^1, \theta^2, \ldots, \theta^{D-2}$, thanks to $\partial_A \tilde{g}_{u\Omega} = 0 = \partial_B \tilde{g}_{uA}$ at $\mathscr{I}^+$, for $A, B = 1, \ldots, D-2$, one has

$$\partial_\Omega \tilde{g}_{uu} = \partial_\Omega \tilde{g}_{uA} = 0 \text{ on } \mathscr{I}^+, \tag{7.2.6}$$

meaning that $\tilde{g}_{uu}$ and $\tilde{g}_{uA}$ must be $\mathcal{O}(\Omega^2)$ as $\Omega \to 0$. Thus, in a neighborhood of $\mathscr{I}^+$, the components of the physical metric, $g_{\mu\nu} = \Omega^{-2}\tilde{g}_{\mu\nu}$, take the form

$$\begin{aligned} ds^2 =& 2\Omega^{-2} d\Omega\, du + \Omega^{-2}(\gamma_{AB} d\theta^A d\theta^B) \\ &+ \mathcal{O}(1)(du^2,\ du\, d\theta^A) \\ &+ \mathcal{O}(\Omega^{-1})(d\theta^A d\theta^B, d\Omega\, du,\ d\Omega^2,\ d\Omega\, d\theta^A). \end{aligned} \tag{7.2.7}$$

Now let $r = 2/\Omega$, so that

$$\begin{aligned} ds^2 =& -dr\, du + \frac{1}{4} r^2 (\gamma_{AB} d\theta^A d\theta^B) \\ &+ \mathcal{O}(1)(du^2,\ du\, d\theta^A) \\ &+ \mathcal{O}(r)(d\theta^A d\theta^B) \\ &+ \mathcal{O}(1/r)(dr\, du,\ dr\, d\theta^A) \\ &+ \mathcal{O}(1/r^3) dr^2. \end{aligned} \tag{7.2.8}$$

Compare with [36, Eq. (21)] and in particular with [35, Eq.(2.9)], where the authors have noted that the crucial point allowing the presence of supertranslations is to impose that g_{AB} should to deviate from the Euclidean metric at most by $\mathcal{O}(r)$, as in our case, rather than at most by $\mathcal{O}\big(r^{\frac{6-D}{2}}\big)$, as was required in less recent literature.

This result shows that our conformal approach agrees with the strategy recently employed [35] in order to extend the infinite-dimensional BMS group to spacetimes with dimension higher than four: instead of adopting the traditional, more restrictive, boundary conditions for the metric tensor (see *e.g.* [34]) which allow for the presence of gravitational radiation but spoil the structure of the asymptotic symmetry group, it is preferable to impose the weaker falloff requirements of type (7.2.8), which in fact allow for the presence of the full BMS group.

Therefore, we hope that the present discussion may shed new light on these seemingly ad-hoc weaker boundary conditions (aside from slight differences in the subleading behaviors), whose ultimate justification up to now lied in the fact that they allow to recover an infinite-dimensional asymptotic symmetry group and to derive Weinberg's graviton theorem from supertranslation symmetry. Here we have reinterpreted these falloff behaviors in terms of the geometric properties of null infinity, in any dimension. It is then the link of the geometric structures on $\mathscr{I}$ to the physical spacetime that selects the weaker boundary condition.

One ought to mention that analyticity problems arise near $\mathscr{I}$, for perturbations around Minkowski spacetime in *odd* ($D = 2n+1$) spacetime dimensions, thus pointing out a potential flaw in the conformal approach to null infinity: in particular, it has been shown that, in such solutions, the leading order behavior of the unphysical Weyl tensor in a neighborhood of $\mathscr{I}$ always begins with a half-integral power of Ω [61]. The perspective we adopted in this section is slightly different: first we studied the automorphism group of abstract $\mathscr{I}$, which turned out to be the D-dimensional BMS group, and then we "attached" this $\mathscr{I}$ as conformal boundary to a physical asymptotically flat spacetime (*assuming* its existence) thus recovering the falloff conditions (7.2.8). Since the discussion of small perturbations of flat spacetime, describing for instance weak gravitational radiation and whose conformal description is ruled out by the discussion in [61], is very important for the physical interpretation of null infinity, these problems should not be ignored; it would therefore be desirable to perform further investigations on the matter, in order to inquire if such analyticity problems persist when studying asymptotically flat spacetimes which are not small perturbations of odd-dimensional Minkowski spacetimes.

Nevertheless it is our opinion that the above described strategy still provides a convincing procedure for justifying the falloff conditions adopted in the current (less formal) treatment of the BMS group in higher dimensions.

Conclusions and Outlook

In order to look at the topics treated in this thesis from a broader perspective, let us review and analyze a few relevant points of our previous discussions.

First of all, a first step towards a deeper understanding of the higher-spin asymptotic symmetries we propose clearly entails a critical analysis of our "Bondi-like gauge", which consists of both gauge-fixing and falloff conditions, and of its meaning for higher spins: for instance, how much freedom do we have in the choice of such an asymptotic gauge? How unique may it be? Or is it conceivable that a weaker set of asymptotic conditions may give rise to an even bigger higher-spin asymptotic symmetry group, while still being physically acceptable?

As far as the link between Weinberg's theorem and our asymptotic symmetries is concerned, some *ad hoc* assumptions were made in the specific properties of the higher-spin gauge field (in particular, the auxiliary boundary conditions (5.4.26)), which were not justified, unlike in the spin-two case, by a discussion of the canonical relations of asymptotic degrees of freedom.

Possible further explorations on this topic include its extension to asymptotically (Anti-) de Sitter spaces, which should be in principle physically realizable, for instance if the correspondence between asymptotic symmetries and soft theorems is meant to be compatible with cosmological observations, even though there are indications that the corresponding analysis on cosmological backgrounds may lead to drastically different results [62]. Another relevant direction to be considered is the analysis of the enlarged asymptotic symmetry algebra proposed in [63], where the authors suggest that, in the case of four-dimensional gravity, by looking at infinitesimal transformations rather than finite, globally defined ones, one may recover an enhanced algebra, namely the semidirect sum of supertranslations and the Virasoro algebra.

Finally, as we already mentioned at the end of Chapter 7, the conformal approach to asymptotically flat spacetimes still elicits some questions: one cannot prove that the unphysical Weyl tensor must vanish at $\mathscr{I}$, when $D \neq 4$, and one may therefore inquire as to how general it may be to require this property, from a mathematical point of view; furthermore, analyticity problems arise around Minkowski spacetime in the case of odd-dimensional spacetimes, and this points to the need for further investigations on the physical meaning of such singularities, a tempting possibility being the analogy with the failure of Huygens' principle [61, Footnote 2].

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