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Corso di Laurea Magistrale in Matematica

Tesi di Laurea Magistrale

The quantum Teichmüller space and its representations

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Contents

Introduction

The quantum Teichmüller space is an algebraic object associated with a punctured surface, admitting an ideal triangulation. Two somewhat different versions of it have been introduced, as a quantization by deformation of the Teichmüller space of a surface, independently by Chekhov and Fock [CF99] and by Kashaev [Kas95]. As in the article [BL07], we follow the exponential version of the Chekhov-Fock approach, whose setting has been established in [Liu09]. In this way the study is focused on non-commutative algebras and their finite-dimensional representations, instead of Lie algebras and self-adjoint operators on Hilbert spaces, as in [CF99] and [Kas95].

Given S a surface admitting an ideal triangulation λ, we can produce a non-commutative $\mathbb{C}$-algebra $\mathcal{T}^q_\lambda$ generated by variables $X_i^{\pm 1}$ corresponding to the edges of λ and endowed with relations $X_i X_j = q^{2\sigma_{ij}} X_j X_i$, where σ_{ij} is an integer number, depending on the mutual position of the edges λ_i and λ_j in λ, and $q \in \mathbb{C}^*$ is a complex number. The algebra $\mathcal{T}^q_\lambda$ is called the *Chekhov-Fock algebra* associated with the surface S and the ideal triangulation λ. Varying λ in the set $\Lambda(S)$ of all the ideal triangulations of S, we obtain a collection of algebras, whose fraction rings $\widehat{\mathcal{T}}^q_\lambda$ are related by isomorphisms $\Phi^q_{\lambda\lambda'} \colon \widehat{\mathcal{T}}^q_{\lambda'} \to \widehat{\mathcal{T}}^q_\lambda$. This structure allows us to consider an object realized by "gluing" all the $\widehat{\mathcal{T}}^q_\lambda$ through the maps $\Phi^q_{\lambda\lambda'}$. The result of this procedure is an intrinsic algebraic object, called the *quantum Teichmüller space* of S and denoted by $\mathcal{T}^q_S$, which does not depend on the chosen ideal triangulation any more.

The explicit expressions of the $\Phi^q_{\lambda\lambda'}$ reveal the geometric essence of this algebraic object. These isomorphisms are designed in order to be a non-commutative generalization of the coordinate changes on the ring of rational functions on the classical Teichmüller space $\mathcal{T}(S)$ of a surface S (here $\mathcal{T}(S)$ denotes the space of isotopy classes of complete hyperbolic metrics on S). More precisely, the classical Teichmüller space $\mathcal{T}(S)$ admits a branched covering $\widetilde{\mathcal{T}}(S) \to \mathcal{T}(S)$ such that the exponential shear coordinates, associated with an ideal triangulation λ, induce a homeomorphism $\psi_\lambda \colon \mathbb{R}^n_+ \to \widetilde{\mathcal{T}}(S)$. The maps $\psi^{-1}_{\lambda'} \circ \psi_\lambda$ turn out to be rational and the isomorphisms $\Phi^q_{\lambda\lambda'}$ are constructed so that the following equality holds

$$\Phi^1_{\lambda\lambda'}(X'_i) = (\psi^{-1}_{\lambda'} \circ \psi_\lambda)^i (X_1, \dots, X_n)$$

where $(\psi^{-1}_{\lambda'} \circ \psi_\lambda)^i$ denotes the i-th component of $(\psi^{-1}_{\lambda'} \circ \psi_\lambda)$ and $\Phi^1_{\lambda\lambda'}$ is the isomorphism between $\widehat{\mathcal{T}}^1_{\lambda'}$ and $\widehat{\mathcal{T}}^1_\lambda$, the fraction rings of the Chekhov-Fock algebras in the commutative case $q = 1$ (see [Liu09] for details). This fact tells us that, for varying $q \in \mathbb{C}^*$, the quantum Teichmüller space is a non-commutative deformation of the algebra of rational functions on the space $\widetilde{\mathcal{T}}(S)$, which makes sense because the coordinate changes $(\psi^{-1}_{\lambda'} \circ \psi_\lambda)$ are rational. The classical case can be recovered by setting $q = 1$.

The main purpose of this thesis is the study of the quantum Teichmüller space and the investigation of its finite-dimensional representations. A necessary condition for the existence of a finite-dimensional representation of $\mathcal{T}^q_\lambda$ is that q^2 is a root of unity, hence we always assume that q^2 is a primitive N-th root of unity, for a certain $N \in \mathbb{N}$. The first Chapter of our work is devoted to the study of the isomorphism classes of the Chekhov-Fock algebras of closed punctured surfaces and polygons and their multiplicative centers. This analysis is crucial in Chapter 2 for the classification of irreducible finite-dimensional representations of such algebras, as exposed in [BL07]. In Chapter 2 we also define a different type of representations of $\mathcal{T}^q_\lambda$, called *local representations*, firstly introduced by Bai, Bonahon, and Liu [BBL07]. Local representations are constructed as "fusion" of irreducible representations of the Chekhov-Fock algebras associated with the triangles composing an ideal triangulation λ. These representations have a simpler set of invariants than the irreducible ones and have a good behaviour with respect to the operation of gluing surfaces along edges of ideal triangulations. In both cases, irreducible or local, the invariant of a representation $\rho\colon \mathcal{T}^q_\lambda \to \mathrm{End}(V)$ is

> a collection of numbers $x_i \in \mathbb{C}^*$, one for each edge of the ideal triangulation λ, together with a choice of a N-th (or square) root for each of certain explicit monomials in the x_i.

The set of monomials depends on the kind of representations considered: if ρ is local, then we only need to specify a N-th root of $x_1 \cdots x_n$, the product of all the weights on the edges of λ, whereas if ρ is irreducible, we also need to select N-th roots of monomial related to the punctures of S and, if N is even, square roots of monomial associated with a basis of $H_1(S; \mathbb{Z}_2)$. However, in both cases the relation between the representation $\rho\colon \mathcal{T}^q_\lambda \to \mathrm{End}(V)$ and the invariants x_i associated with the edges λ_i is that $\rho(X_i^N) = x_i \, id_V$.

In the first part of Chapter 3 we describe the construction of the isomorphisms $\Phi^q_{\lambda\lambda'}$, which are characterized by the combinatorics of the ideal triangulations and the moves conducing from one to the other. Then, following [BL07], we give a notion of finite-dimensional representation of the quantum Teichmüller space, defined as a collection

$$\rho = \{\rho_\lambda : \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)\}_{\lambda \in \Lambda(S)}$$

of representations of all the Chekhov-Fock algebras associated with the surface S, verifying a compatibility condition in terms of the isomorphisms $\Phi^q_{\lambda\lambda'}$. More precisely, two representations $\rho_\lambda\colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)$ and $\rho_{\lambda'}\colon \mathcal{T}^q_{\lambda'} \to \mathrm{End}(V_{\lambda'})$ are compatible if $\rho_{\lambda'}$ is isomorphic to $\rho_\lambda \circ \Phi^q_{\lambda\lambda'}$. A representation $\rho = \{\rho_\lambda\}_{\lambda \in \Lambda(S)}$ of the quantum Teichmüller space is *local* (or *irreducible*) if every ρ_λ is local (or irreducible).

In order to envelope a proper theory of invariants for these kinds of representations, we need to introduce another object associated with an ideal triangulation λ and a system of weights $x_i \in \mathbb{C}^*$ on the edges of λ, that is a *pleated surface* with pleating locus λ and exponential shear-bend parameters $(x_i)_i \in (\mathbb{C}^*)^n$. Every pleated surface has a *monodromy representation*, which is a homomorphism from the fundamental group of S to the group of orientation preserving isometries of $\mathbb{H}^3$. The pleated surfaces provide a bridge conducing from a local (or irreducible) representation of the Chekhov-Fock algebra to a well-defined conjugacy class of a homomorphism $r\colon \pi_1(S) \to \mathrm{Isom}_+(\mathbb{H}^3) \cong \mathbb{P}\,\mathrm{SL}(2,\mathbb{C})$. Because of the strong relation between the isomorphisms $\Phi^q_{\lambda\lambda'}$ with the coordinate changes $(\psi_{\lambda'}^{-1} \circ \psi_\lambda)$, given $\rho = \{\rho_\lambda\}_{\lambda\in\Lambda(S)}$ a local (or irreducible) representation of the quantum Teichmüller space, the conjugacy class of the homomorphism r associated to ρ_λ turns out to be independent from the choice of $\lambda \in \Lambda(S)$. In conclusion, we reach a classification theorem for local representations with the following statement:

Theorem. *Let S be a surface admitting an ideal triangulation and fix q a primitive N-th root of $(-1)^{N+1}$. Then a local representation of the quantum Teichmüller space is classified by the following data:*

- *the conjugacy class of a homomorphism $r\colon \pi_1(S) \to \mathbb{P}\,\mathrm{SL}(2,\mathbb{C})$;*
- *for each peripheral subgroup π of the fundamental group $\pi_1(S)$ (corresponding to a puncture of S), a choice of a point $\xi_\pi \in \partial\mathbb{H}^3$ fixed by π;*
- *a choice of a N-th root of the product $x_1 \cdots x_n$ (indeed the quantity $x_1 \cdots x_n$ depends only on $(r, \{\xi_\pi\}_\pi)$).*

The result in the irreducible case has the same flavour, but there are more roots of certain monomials in the x_i to be determined, just as in the classification result for representations of the Chekhov-Fock algebras. Not every enhanced homomorphism as above is realized as invariant of a representation of the quantum Teichmüller space, but a huge meaningful family is, like the injective ones. Chapter 3 ends with the proof of a theorem, due to Toulisse [Tou14], concerning the irreducible decomposition of local representations of the Chekhov-Fock algebra.

Let now

$$\rho = \{\rho_\lambda\colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)\}_{\lambda\in\Lambda(S)} \qquad \rho' = \{\rho'_\lambda\colon \mathcal{T}^q_\lambda \to \mathrm{End}(V'_\lambda)\}_{\lambda\in\Lambda(S)}$$

be two isomorphic local representations of the quantum Teichmüller space of S. By definition, for every $\lambda, \lambda' \in \Lambda(S)$, the representations $\rho_\lambda \circ \Phi^q_{\lambda\lambda'}$ and $\rho'_{\lambda'}$ are isomorphic. Therefore, there exists a linear isomorphism $L^{\rho\rho'}_{\lambda\lambda'}$ such that

$$(\rho_\lambda \circ \Phi^q_{\lambda\lambda'})(X') = L^{\rho\rho'}_{\lambda\lambda'} \circ \rho'_{\lambda'}(X') \circ (L^{\rho\rho'}_{\lambda\lambda'})^{-1} \qquad \text{for every } X' \in \mathcal{T}^q_{\lambda'}$$

Such a $L^{\rho\rho'}_{\lambda\lambda'}$ is called an *intertwining operator*. Because local representations are not irreducible in general, an intertwining operator like above is not unique (not even up to multiplicative factor - we stipulate that from now on all operators are considered up to such projective equivalence). One of the main purposes of [BBL07] was to select a unique intertwining operator $L^{\rho\rho'}_{\lambda\lambda'}$ for every ρ, ρ' local representations and for every λ, λ' ideal triangulations, by requiring that the whole system of operators (for varying $\lambda, \lambda' \in \Lambda(S)$, the representations ρ, ρ' and the surface S) verifies some natural *Fusion* and *Composition* properties, concerning their behaviour with respect to the fusion of representations and changing of triangulations. However, in our investigation of the ideas exposed in [BBL07], we have found a difficulty that compromises the original statement [BBL07, Theorem 20], in particular the possibility to select a unique intertwining operator for every choice of $\rho, \rho', \lambda, \lambda'$. Indeed, by requesting that the Fusion and Composition properties hold, we are able to select just a set $\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}$ of intertwining operators for each choice of ρ, ρ', λ and λ', instead of a unique linear isomorphism. Each set $\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}$ turns out to be endowed with a natural free and transitive action of $H_1(S;\mathbb{Z}_N)$, so its cardinality is always finite, but it goes to ∞ by increasing the complexity of the surface S and the number $N \in \mathbb{N}$ (recall that q^2 is a primitive N-th root of unity). In Chapter 4 we describe this difficulty and we provide our solution, by incorporating the affine space structure over $H_1(S;\mathbb{Z}_N)$ and its behaviour with respect to compositions and fusions. We also prove that the system of sets $\{\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}\}$ obtained in this way is minimal, hence canonical, in the sense that every set of intertwining operators which verifies "weak" Fusion and Composition properties (in practice the ones of the original statement in [BBL07]) necessarily contains such a distinguished one.

In addition, we reformulate the theory of invariants for pseudo-Anosov diffeomorphisms developed in [BBL07] in light of these facts. The final product is the conjugacy class of a set of linear isomorphisms, defined up to scalar multiplication, instead of the one described in [BBL07], which was a conjugacy class of a single linear isomorphism.

We also expose in Subsection 4.2.3 an explicit calculation of an intertwining operator when λ and λ' differ by diagonal exchange and S is an ideal square, which is basically the elementary block needed to express a generic intertwining operator.

CHAPTER 0

Preliminaries

Unless specified differently, we will always assume that a surface S is oriented and obtained, from a compact oriented surface $\overline{S}$ with genus g and b boundary components, by removing $p \geq 1$ punctures $v_1, \ldots, v_p$, with at least a puncture in each boundary component. Suppose further that $2\chi(S) \leq p_\partial - 1$, where p_∂ is the number of punctures lying in ∂S. These are the necessary and sufficient conditions in which S admits an ideal triangulation, whose definition is the following:

Definition 0.1. Let S be a surface. We define an *ideal triangulation* λ of S as a triangulation of $\overline{S}$, which has as vertices exactly the set of punctures $\{v_1, \ldots, v_p\}$, endowed with a indexing of the 1-cells $(\lambda_i)_i$. We identify two triangulations of S if they are isotopic. Also, we denote by $\Lambda(S)$ the set of all ideal triangulations of S.

Given $\lambda \in \Lambda(S)$, let n be the number of 1-cells in the ideal triangulation λ and m the number of faces of λ. Easy calculations show the following relations hold:

$$\begin{aligned} n &= -3\chi(\overline{S}) + 3p - p_\partial \\ &= -3\chi(S) + 2p_\partial \\ m &= -2\chi(S) + p_\partial \end{aligned}$$

In all the thesis n and m will always denote such quantities.

Definition 0.2. Given S a surface and λ an ideal triangulation of S, we denote by $\Gamma_{S,\lambda}$ the dual graph of λ, i. e. $\Gamma_{S,\lambda}$ is a CW-complex of dimension 1, whose vertices T_b^* correspond to the triangles T_b of λ, and, for every λ_a internal edge of λ, there is a 1-cell λ_a^* in $\Gamma_{S;\lambda}$ that connects the vertices corresponding to the triangles on the sides of λ_a, even if the triangles are the same.

It follows from the definition that all the vertices have valency ≤ 3. In particular, the valency of a vertex T_b^* in $\Gamma_{S,\lambda}$ is equal to the number of internal edges that are sides of T_b.

Given $\lambda \in \Lambda(S)$ an ideal triangulation, we can modify λ in the following ways:

- for every permutation $\tau \in \mathfrak{S}_n$, we define $\lambda' = \tau(\lambda)$ the triangulation with the same 1-cells of λ, but with the ordering $\lambda'_i := \lambda_{\tau(i)}$. This operation is called *re-indexing*;

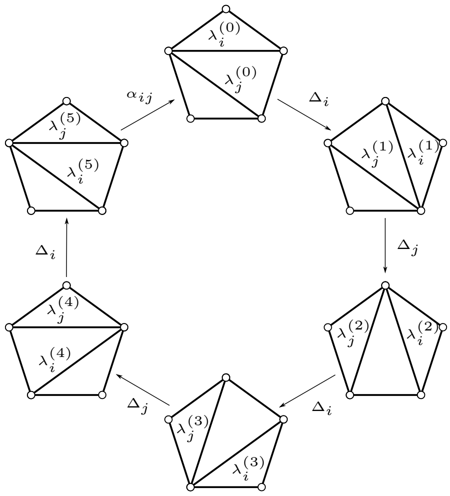

Figure 1: The Pentagon relation

- let λ_i be an edge adjacent to two distinct triangles in λ composing a square Q. Then we denote by $\Delta_i(\lambda)$ the triangulation obtained from λ by replacing the diagonal λ_i of Q with the other diagonal λ'_i. By definition, we set $\Delta_i(\lambda) = \lambda$ when the two sides of λ_i belong to the same triangle. This operation is called *diagonal exchange*.

These operations verify the following relations:

COMPOSITION RELATION: for every α, β in $\mathfrak{S}_n$ we have $\alpha(\beta(\lambda)) = (\alpha \circ \beta)(\lambda)$;

REFLEXIVITY RELATION: $(\Delta_i)^2 = id$;

RE-INDEXING RELATION: $\Delta_i \circ \alpha = \alpha \circ \Delta_{\alpha(i)}$ for every $\alpha \in \mathfrak{S}_n$;

DISTANT COMMUTATIVITY RELATION: if λ_i and λ_j do not belong to a common triangle of λ, then $(\Delta_i \circ \Delta_j)(\lambda) = (\Delta_j \circ \Delta_i)(\lambda)$;

PENTAGON RELATION: if three triangles of a triangulation λ compose a pentagon with diagonals λ_i and λ_j and if we denote by $\alpha_{ij} \in \mathfrak{S}_n$ the (ij) transposition, then we have

$$(\Delta_i \circ \Delta_j \circ \Delta_i \circ \Delta_j \circ \Delta_i)(\lambda) = \alpha_{ij}(\lambda)$$

The following results are due to Penner and the proofs can be found in [Pen87]:

Theorem 0.3. *Given two ideal triangulations $\lambda, \lambda' \in \Lambda(S)$, there exists a finite sequence of ideal triangulations $\lambda = \lambda^{(0)}, \ldots, \lambda^{(k-1)}, \lambda^{(k)} = \lambda'$ such that, for every $j = 0, \ldots, k-1$, the triangulation $\lambda^{(j+1)}$ is obtained from $\lambda^{(j)}$ by a diagonal exchange or by a re-indexing of its edges.*

Theorem 0.4. *Given two ideal triangulations $\lambda, \lambda' \in \Lambda(S)$ and given two sequences $\lambda = \lambda^{(0)}, \ldots, \lambda^{(k-1)}, \lambda^{(k)} = \lambda'$ and $\lambda = \lambda'^{(0)}, \ldots, \lambda'^{(h-1)}, \lambda'^{(h)} = \lambda'$ of* *diagonal exchanges and re-indexing connecting λ and λ', then we can obtain the second sequence from the first by the applications of a finite number of the following moves or their inverses:*

- *using the Composition relation, replace*

$$\dots, \alpha(\lambda^{(l)}), \beta(\alpha(\lambda^{(l)})), \dots$$

with

$$\dots, (\alpha \circ \beta)(\lambda^{(l)}), \dots$$

- *using the Reflexivity relation, replace*

$$\dots, \lambda^{(l)}, \dots$$

with

$$\dots, \lambda^{(l)}, \Delta_i(\lambda^{(l)}), \lambda^{(l)}, \dots$$

- *using the Re-indexing relation, replace*

$$\dots, \lambda^{(l)}, \alpha(\lambda^{(l)}), \Delta_i(\alpha(\lambda^{(l)})), \dots$$

with

$$\dots, \lambda^{(l)}, \Delta_{\alpha(i)}(\lambda^{(l)}), (\alpha \circ \Delta_{\alpha(i)})(\lambda^{(l)}), \dots$$

- *using the Distant Commutativity relation, replace*

$$\dots, \lambda^{(l)}, \dots$$

with

$$\dots, \lambda^{(l)}, \Delta_i(\lambda^{(l)}), (\Delta_j \circ \Delta_i)(\lambda^{(l)}), \Delta_j(\lambda^{(l)}), \lambda^{(l)}, \dots$$

where $(\lambda^{(l)})_i$ and $(\lambda^{(l)})_j$ are two edges of $\lambda^{(l)}$ which do not lie in a common triangle;

- *using the Pentagon relation, replace*

$$\dots, \lambda^{(l)}, \dots$$

with

$$\dots, \lambda^{(l)}, \Delta_i(\lambda^{(l)}), \dots, (\Delta_j \circ \Delta_i \circ \Delta_j \circ \Delta_i)(\lambda^{(l)}), \alpha_{ij}(\lambda^{(l)}), \lambda^{(l)}, \dots$$

where $(\lambda^{(l)})_i$ and $(\lambda^{(l)})_j$ are two diagonals of a pentagon in $\lambda^{(l)}$.

Definition 0.5. Let S be a surface and select an ideal triangulation λ of it. We can construct, starting from S and the choice of certain internal 1-cells $\lambda_{i_1}, \dots, \lambda_{i_h}$, a surface R obtained by splitting S along these selected edges $\lambda_{i_1}, \dots, \lambda_{i_h}$. In other words, R is realized by removing the identifications, along λ_{i_j}, between the ideal triangles having λ_{i_j} as side. On R we can clearly find an ideal triangulation μ and an orientation induced by λ and by the orientation on S. In this circumstances, we will say that S is obtained from R *by fusion*, and analogously that the ideal triangulation $\lambda \in \Lambda(S)$ is obtained from $\mu \in \Lambda(R)$ *by fusion*.

CHAPTER 1

The Chekhov-Fock algebra

In this Chapter we introduce a non-commutative $\mathbb{C}$-algebra $\mathcal{T}^q_\lambda$, called the *Chekhov-Fock algebra* associated with a surface S and an ideal triangulation λ. Its lack of commutativity depends on the combinatorics of the ideal triangulation λ.

The main purpose in this Chapter is to give a characterization of this algebra and of its multiplicative center, which will be key ingredients in Chapter 2 in order to classify irreducible representations of $\mathcal{T}^q_\lambda$. These results, concerning the case of a closed punctured surface, have been exposed by Bonahon and Liu [BL07] and we basically follow their presentation.

In addition, we give a similar and simpler description when S is an ideal polygon, proving a fact announced in the proof of [BBL07, Lemma 21].

1.1 First definitions

Let S be a surface (see Chapter 0 for details). Since S is oriented, on each triangle of λ we have a natural induced orientation. With respect to this orientation, it is reasonable to speak about a left and a right side of each spike of a triangle. We select an order $\lambda_1, \ldots, \lambda_n$ on the set of 1-cells of the triangulation λ. Given λ_i and λ_j two 1-cells of λ, we denote by a_{ij} the number of spikes of triangles in λ in which we find λ_i on the left side and λ_j on the right. Now we name

$$\sigma_{ij} := a_{ij} - a_{ji}$$

Definition 1.1.1. Given $q \in \mathbb{C}^*$, we define the *Chekhov-Fock algebra* associated with the ideal triangulation λ and the parameter q as the non-commutative $\mathbb{C}$-algebra $\mathcal{T}^q_\lambda$, generated by $X_1^{\pm 1}, \ldots, X_n^{\pm 1}$ and endowed with the following relations

$$X_i X_j = q^{2\sigma_{ij}} X_j X_i$$

for every $i, j = 1, \ldots, n$, where n indicates the number of 1-cells of λ.

By virtue of Corollary A.3, it can be easily seen the Chekhov-Fock algebra is always a bilateral Noetherian integral domain.

Given $\lambda \in \Lambda(S)$, we designate the free $\mathbb{Z}$-module generated by the 1-cells of the triangulation λ as $\mathcal{H}(\lambda; \mathbb{Z})$. A choice of an ordering on the 1-cells of λ gives

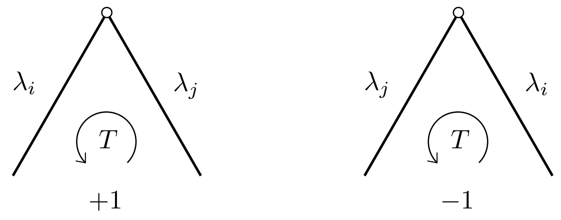

Figure 1.1: How a spike of a triangle T contributes to σ_{ij}

us a natural isomorphism of $\mathcal{H}(\lambda;\mathbb{Z})$ with $\mathbb{Z}^n$ and through it we can define a bilinear skew form on $\mathcal{H}(\lambda;\mathbb{Z})$ given by

$$\sigma\left(\sum_{i=1}^{n} a_i\,\lambda_i, \sum_{j=1}^{n} b_j\,\lambda_j\right) := \sum_{i,j=1}^{n} a_i b_j \sigma_{ij} \tag{1.1}$$

Observe σ_{ij} is determined by the mutual positions of λ_i and λ_j, hence σ is independent from the choice of an ordering on λ and it is a bilinear skew form intrinsically defined on $\mathcal{H}(\lambda;\mathbb{Z})$.

Now we fix an ordering in λ and we choose $\alpha = (\alpha_1,\ldots,\alpha_n)$ and $\beta = (\beta_1,\ldots,\beta_n)$ in $\mathbb{Z}^n \cong \mathcal{H}(\lambda;\mathbb{Z})$. We can associate with α a monomial in $\mathcal{T}^q_\lambda$, which we briefly denote by X^α, defined as

$$X_1^{\alpha_1}\cdots X_n^{\alpha_n}$$

and analogously for X^β. Introduce also the following notation

$$\underline{X}^\alpha := q^{-\sum_{i<j}\alpha_i\alpha_j\sigma_{ij}} X^\alpha = q^{-\sum_{i<j}\alpha_i\alpha_j\sigma_{ij}} X_1^{\alpha_1}\cdots X_n^{\alpha_n}$$

Lemma 1.1.2. *For every $\alpha,\beta\in\mathbb{Z}^n\cong\mathcal{H}(\lambda;\mathbb{Z})$, the following relations hold in $\mathcal{T}^q_\lambda$:*

$$X^\alpha X^\beta = q^{2\sigma(\alpha,\beta)} X^\beta X^\alpha = q^{2(\sum_{i>j}\alpha_i\beta_j\sigma_{ij})} X^{\alpha+\beta} \tag{1.2}$$

$$\underline{X}^\alpha \underline{X}^\beta = q^{2\sigma(\alpha,\beta)} \underline{X}^\beta \underline{X}^\alpha = q^{\sigma(\alpha,\beta)} \underline{X}^{\alpha+\beta} \tag{1.3}$$

Furthermore, for every permutation $\tau\in\mathfrak{S}_l$, we have

$$q^{-\sum_{h<k}\sigma_{i_h i_k}} X_{i_1}\cdots X_{i_l} = q^{-\sum_{h<k}\sigma_{i_{\tau(h)} i_{\tau(k)}}} X_{i_{\tau(1)}}\cdots X_{i_{\tau(l)}} \tag{1.4}$$

Proof. The first relations easily follow by direct calculations, it is sufficient to control how the coefficients change by the appropriate permutation. We will see how to deduce the relations in 1.3 using 1.2. The first equality is obvious, it is enough to multiply both the members of 1.2 by $q^{-\sum_{i<j}\alpha_i\alpha_j\sigma_{ij}} q^{-\sum_{i<j}\beta_i\beta_j\sigma_{ij}}$. We would like to show that $\underline{X}^\alpha\underline{X}^\beta = q^{\sigma(\alpha,\beta)}\underline{X}^{\alpha+\beta}$ holds. By virtue of 1.2, it is sufficient to prove that the exponents of q, multiplying the elements $X^\alpha X^\beta$ and $q^{2(\sum_{i>j}\alpha_i\beta_j\sigma_{ij})}X^{\alpha+\beta}$ respectively, coincide. It is simple to show we can reduce to prove the following equality

$$-\sum_{i<j}\alpha_i\alpha_j\sigma_{ij} - \sum_{i<j}\beta_i\beta_j\sigma_{ij} = -\sum_{i<j}(\alpha+\beta)_i(\alpha+\beta)_j\sigma_{ij} + \sigma(\alpha,\beta) - 2\sum_{i>j}\alpha_i\beta_j\sigma_{ij}$$

Using the fact that σ is skew-symmetric, we deduce

$$
\begin{aligned}
-\sum_{i<j}(\alpha+\beta)_i(\alpha+\beta)_j\sigma_{ij}+\sigma(\alpha,\beta)-2\sum_{i>j}\alpha_i\beta_j\sigma_{ij} = \\
&= -\sum_{i<j}(\alpha+\beta)_i(\alpha+\beta)_j\sigma_{ij}+\sum_{i<j}\alpha_i\beta_j\sigma_{ij}+\sum_{i>j}\alpha_i\beta_j\sigma_{ij}-\sum_{i>j}\alpha_i\beta_j\sigma_{ij}+ \\
&\qquad -\sum_{i>j}\alpha_i\beta_j\sigma_{ij} \\
&= -\sum_{i<j}(\alpha+\beta)_i(\alpha+\beta)_j\sigma_{ij}+\sum_{i<j}\alpha_i\beta_j\sigma_{ij}+\sum_{i<j}\beta_i\alpha_j\sigma_{ij} \\
&= -\sum_{i<j}\alpha_i\alpha_j\sigma_{ij}-\sum_{i<j}\beta_i\beta_j\sigma_{ij}
\end{aligned}
$$

that concludes the proof.

For what concerns the relation 1.4, it is simple to prove it when τ is a transposition of consecutive elements and the general case easily follows. $\square$

Remark 1.1.3. The elements X^α, for varying $\alpha \in \mathbb{Z}^n$, just as the $\underline{X}^\alpha$, compose a $\mathbb{C}$-basis of $\mathcal{T}^q_\lambda$. Denoting by $\mathcal{M}^q_\lambda$ the monomial multiplicative group of $\mathcal{T}^q_\lambda$, we have a natural projection $\mathcal{M}^q_\lambda \longrightarrow \mathbb{Z}^n$ that associates with the element aX^α the vector $\alpha \in \mathbb{Z}^n$. Identifying $\mathbb{C}^*$ with the subgroup of the elements $aX^0 \in \mathcal{M}^q_\lambda$, the following sequence is exact

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \mathcal{M}^q_\lambda \longrightarrow \mathbb{Z}^n \longrightarrow 0$$

Marking as $\mathcal{Z}^q_\lambda$ the center of $\mathcal{M}^q_\lambda$, the *monomial multiplicative center* of $\mathcal{T}^q_\lambda$, it is immediate to verify the multiplicative center of $\mathcal{T}^q_\lambda$ is the $\mathbb{C}$-span of $\mathcal{Z}^q_\lambda$. Given $X^\alpha \in \mathcal{M}^q_\lambda$, it belongs to the monomial center if and only if, for every $\beta \in \mathbb{Z}^n$, we have $q^{2\sigma(\alpha,\beta)} = 1$, in light of the relation 1.2. Therefore, there are the following possibilities:

- if q is not a root of unity, then X^α belongs to $\mathcal{Z}^q_\lambda$ if and only if, for every $\beta \in \mathbb{Z}^n$, we have $\sigma(\alpha,\beta) = 0$;
- if q^2 is a primitive N-th root of unity, then X^α belongs to $\mathcal{Z}^q_\lambda$ if and only if, for every $\beta \in \mathbb{Z}^n$, we have $\sigma(\alpha,\beta) \in N\mathbb{Z}$.

1.2 Punctured closed surfaces

Let us focus our attention on the case of a surface S obtained by removing p points from a closed compact oriented surface $\overline{S}$. We mark as $\Gamma = \Gamma_{S,\lambda}$ the non-oriented dual graph associated with the triangulation λ of S, defined in Chapter 0, and we construct from it a new oriented graph $\widehat{\Gamma}$ as follows:

- the set of vertices $\widehat{T}^*_k$ of $\widehat{\Gamma}$ coincides with the set of vertices T^*_k of Γ;
- for every 1-cell λ^*_i of Γ we take two 1-cells $\widehat{\lambda}^*_{i1}$ and $\widehat{\lambda}^*_{i2}$ that connect the vertices in $\widehat{\Gamma}$ corresponding to the ends of λ_i, with opposite orientations.

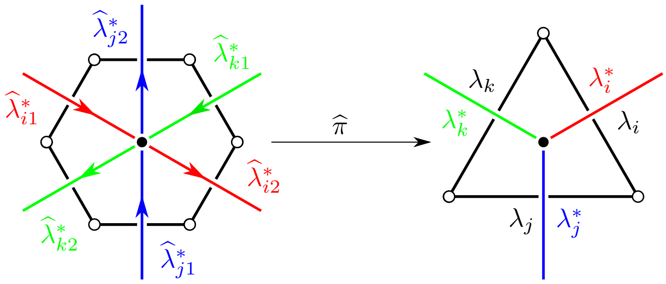

Figure 1.2: Local behaviour of $\widehat{\pi}$

There is a natural projection map $\widehat{\pi}\colon \widehat{\Gamma} \to \Gamma$ that is the identity on the vertices and that carries each 1-cell $\widehat{\lambda}^*_{i\bullet}$ of $\widehat{\Gamma}$ in the corresponding 1-cell λ^*_i of Γ. It turns out to be a bijection on the set of vertices and a $2-1$ map on the interior of the 1-cells of $\widehat{\Gamma}$. Starting from $\widehat{\pi}\colon \widehat{\Gamma} \to \Gamma$, we construct a oriented surface $\widehat{S}$ that verifies the following conditions:

- the graph $\widehat{\Gamma}$ can be identified to a deformation retract of $\widehat{S}$;
- turning around each vertex $\widehat{T}^*_k$ of $\widehat{\Gamma}$ in $\widehat{S}$, we meet the 1-cells entering in $\widehat{T}^*_k$ with alternating orientations;
- the projection map $\widehat{\pi}\colon \widehat{\Gamma} \to \Gamma$ can be extended to a branched covering $\widehat{\pi}\colon \widehat{S} \to S$ with set of ramification points that coincide with $\widehat{\Gamma}^{(0)}$, the set of vertices of $\widehat{\Gamma}$, and with every ramification point having multiplicity equal to 2. Moreover, we assume that on all the $2-1$ points of $\widehat{S}$ the map $\widehat{\pi}$ is orientation preserving.

Denoting by $\tau\colon \widehat{S} \to \widehat{S}$ the involution of the branched covering $\widehat{\pi}$, then τ acts like a π-rotation around every vertex, as in Figure 1.2.

Observe we can associate with every element $\alpha \in \mathcal{H}(\lambda;\mathbb{Z})$ a weight system $\widehat{\alpha}$ on the 1-cells of $\widehat{\Gamma}$, setting on every 1-cell $\widehat{\lambda}^*_{i\bullet}$ of $\widehat{\Gamma}$ the multiplicity given by α on $\widehat{\pi}(\widehat{\lambda}^*_{i\bullet}) = \lambda^*_i$, that is α_i. Because of the structure of $\widehat{\Gamma}$, such a $\widehat{\alpha}$ defines an element of $H_1(\widehat{S};\mathbb{Z})$ that verifies $\tau_*(\widehat{\alpha}) = -\widehat{\alpha}$. Conversely, given a in

$$\{a \in H_1(\widehat{S};\mathbb{Z}) \mid \tau_*(a) = -a\}$$

there exists a unique representative c of a belonging to $H_1(\widehat{\Gamma};\mathbb{Z})$, because $\widehat{\Gamma}$ is a deformation retract of $\widehat{S}$ and it is a graph. By construction $\tau(c)_* = -c$, as elements of $C_1(\widehat{\Gamma};\mathbb{Z})$, and this relation immediately imply that there exists a $\alpha \in \mathcal{H}(\lambda;\mathbb{Z})$ such that $\widehat{\alpha} = a$.

Lemma 1.2.1. *In the above notations, the following relation holds*

$$\sigma(\alpha,\beta) = i(\widehat{\alpha},\widehat{\beta})$$

for every $\alpha,\beta \in \mathcal{H}(\lambda;\mathbb{Z})$, where we are denoting by $i(\cdot,\cdot)$ the intersection form of the surface $\widehat{S}$.

Proof. Because both σ and i are $\mathbb{Z}$-bilinear and the association $\alpha \mapsto \widehat{\alpha}$ is $\mathbb{Z}$-linear, it is sufficient to prove the relation on couples of elements belonging to a basis of

$\mathcal{H}(\lambda;\mathbb{Z})$. Fix an ordering of the 1-cells of the triangulation λ and the resulting isomorphism $\mathcal{H}(\lambda;\mathbb{Z}) \cong \mathbb{Z}^n$. Then we can interpret $\sigma(\lambda_i, \lambda_j) = \sigma_{ij} = a_{ij} - a_{ji}$ as a sum of as many ± 1 as the spikes of triangles in λ in which λ_i and λ_j appear as sides, with positive sign if λ_i is on the left and λ_j on the right and with negative sign otherwise. If $\widehat{e}_i$ and $\widehat{e}_j$ are the elements in $H_1(\widehat{S};\mathbb{Z})$ corresponding to λ_i and λ_j in $\mathcal{H}(\lambda;\mathbb{Z})$, then $\widehat{e}_i$ and $\widehat{e}_j$ have an intersection whenever λ_i and λ_j are sides of the same triangle T. We will prove that every ± 1, coming from any such triangle T in the sum σ_{ij}, corresponds to a same ± 1 in the algebraic intersection of $\widehat{e}_i$ and $\widehat{e}_j$. We must understand which is the contribute of each triangle in both the expressions. Let us represent what happens when, in the triangle T, we have a spike with λ_i on the left and λ_j on the right, as in Figure 1.2 (in the notation of Figure 1.2, $\widehat{e}_i = \widehat{\lambda}^*_{i1} + \widehat{\lambda}^*_{i2}$ and $\widehat{e}_j = \widehat{\lambda}^*_{j1} + \widehat{\lambda}^*_{j2}$). This configuration contributes in σ_{ij} with a $+1$, exactly as in $i(\widehat{e}_i, \widehat{e}_j)$, because the couple $\widehat{e}_i, \widehat{e}_j$ has a positive intersection in $\widehat{\pi}^{-1}(T)$. Analogously, it can be observed that, in the opposite situation, we have a -1 in both the sums, fact that concludes the proof. □

Now we are going to study the behaviour of the involution τ on $\widehat{S}$. Let c be a curve in $\widehat{S}$ going counter-clockwise around a puncture of $\widehat{S}$. Analysing the local structure of $\widehat{S}$, we observe that c is uniquely represented by a chain $c_0 \in C_1(\widehat{\Gamma};\mathbb{Z})$ having coefficients with the same sign. If the coefficients of c_0 are positive, then the representative $\tau(c)_0$ of $\tau(c)$ has negative signs, and vice versa in the opposite case. This means that there is no puncture in $\widehat{S}$ that is preserved by τ and so the number of punctures in $\widehat{S}$ is equal to $2p$. Moreover, $\widehat{\pi}$ can be extended to a branched covering η of R on $\overline{S}$, where R denotes the surface obtained from $\widehat{S}$ by filling the punctures, and η has the same ramification set of $\widehat{\pi}$. The map η is determined by its restriction on the subspace $\eta^{-1}(\overline{S} \setminus V)$, where V is the union of small open discs centred in the ramification points of $\widehat{\pi}$. The restriction of η on $\eta^{-1}(\overline{S} \setminus V)$ is an ordinary covering of $\overline{S} \setminus V$ and it is identified by a homomorphism $\varphi\colon \pi_1(\overline{S} \setminus V) \to \mathbb{Z}_2$, or equivalently by a cocycle $[\varphi] \in H^1(\overline{S} \setminus V;\mathbb{Z}_2)$. Because of the Poincaré's Duality Theorem (see [Hat02, Theorem 3.43]), the class $[\varphi]$ is dual of an element $[c] \in H_1(\overline{S} \setminus V, \partial V;\mathbb{Z}_2)$. We want to show how to construct a representative K of $[c]$ realized as a disjoint union of arcs connecting distinct components of ∂V: choose any representative K of $[c]$ that is a smooth immersion with transverse self-intersections and modify it as follows:

- every component of ∂V must intersect K in a odd number of points, because the covering is not trivial along ∂V. By adding a proper relative boundary near the components of ∂V, we can assume that for every ball of V there exists exactly an arc having an end in its boundary. We still denote with K this possibly different representative of $[c]$;

- for every self-intersection, we can modify K, up to adding a boundary as described in Figure 1.3, in order to remove any self-intersection of K;

- K is now a sum of disjoint closed curves and arcs connecting two different boundary components of $\overline{S} \setminus V$. Assume that there exists a closed component in K. Because the covering is not trivial near the removed balls, there exists at least an arc component too. It is easy to see that we can found a closed component and an arc component in K that are linked in

$\overline{S}$ by a path γ whose internal part does not intersect K. By adding to K the boundary of a collar neighbourhood of γ, we obtain another representative K' of $[c]$ having a closed component less and which is still a disjoint union of arcs and closed curves. Repeating this process, we obtain a representative for which the requested condition holds.

Moreover, by perturbing K, we can assume it does not go through the punctures of S. The covering η is trivial near the punctures of S and its restriction on $\widehat{S}$ is equal to $\widehat{\pi}$. By duality between $[\varphi]$ and K, if a closed path does not meet K, then in $S \setminus V$ it has trivial monodromy for the covering $\widehat{\pi}$. Then, taken an embedded disc D in S such that $\mathring{D} \supset K \cup \overline{V}$ (here we are using the conditions requested on the representative K of $[c]$) and designated the preimage under $\widehat{\pi}$ of D as $\widehat{D}$, the restriction $\widehat{\pi}\colon \widehat{S} \setminus \widehat{D} \to S \setminus D$ is a trivial covering. Let $\widehat{S}_1$ and $\widehat{S}_2$ be the two connected components of $\widehat{S} \setminus \widehat{D}$, which are homeomorphically mapped by $\widehat{\pi}$ in $S \setminus D$.

In what follows, we are going to study $H_1(\widehat{S};\mathbb{Z})$ and the action of τ on it, by decomposing $H_1(\widehat{S};\mathbb{Z})$ in a direct sum of subgroups related to $\widehat{S}_1$, $\widehat{S}_2$ and $\widehat{D}$. We firstly focus on the branched covering $\widehat{\pi}\colon \widehat{D} \to D$: it has $m = -2\chi(S)$ ramification points of multeplicity 2 and so, by the Riemann-Hurwitz formula, the following relation holds

$$\begin{aligned} \chi(\widehat{D}) &= 2\chi(D) - (-2\chi(S)) \\ &= 2 + 2\chi(S) \end{aligned}$$

Since ∂D does not intersect K by construction, the surface $\widehat{D}$ has exactly two boundary components, which coincide with the two distinct lifts through $\widehat{\pi}$ of ∂D. Consequently, the genus of $\widehat{D}$ is equal to

$$\begin{aligned} g(\widehat{D}) &= \frac{1}{2}(2 - \chi(\widehat{D}) - p(\widehat{D}) - b(\widehat{D})) \\ &= -\frac{1}{2}\chi(\widehat{D}) = -\chi(S) - 1 \end{aligned}$$

Moreover, the restriction of the involution τ on $\widehat{D}$ is a diffeomorphism of $\widehat{D}$ with itself, having order 2 and

$$m = -2\chi(S) = 2 + 2g(\widehat{D})$$

fixed points.

The surface $\widehat{S}$ is obtained from $\widehat{S}_1$, $\widehat{S}_2$ and $\widehat{D}$ by gluing $\partial\widehat{S}_1$ and $\partial\widehat{S}_2$ along the two boundary components of $\widehat{D}$. We denote these two curves in $\widehat{D}$ by γ_1 and γ_2 respectively, endowed with the orientations induced as boundaries of $\widehat{D}$.

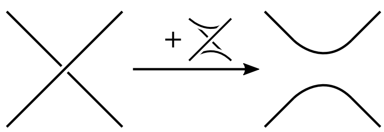

Figure 1.3: How to remove intersections in K

Analysing the Mayer-Vietoris sequence of the decomposition $\widehat{S} = (\widehat{S}_1 \sqcup \widehat{S}_2) \cup \widehat{D}$, we obtain the following

$$0 \longrightarrow H_1(\gamma_1 \sqcup \gamma_2) \longrightarrow H_1(\widehat{S}_1 \sqcup \widehat{S}_2) \oplus H_1(\widehat{D}) \longrightarrow H_1(\widehat{S}) \longrightarrow 0 \tag{1.5}$$

where we omit $\mathbb{Z}$ as coefficient ring. Now we can construct, starting from $\widehat{S}_i$, the surface $\widehat{S}_i^0$ by filling all the punctures and the boundary components of $\widehat{S}_i$, for $i = 1, 2$. Select, in $\widehat{S}_i^0$, a collection of disjoint discs $\{B_j^{(i)}\}_{j=1}^p$ centred in the punctures of $\widehat{S}_i$, which are as much as those of S, and denote by $B_\partial^{(i)}$ the disc in $\widehat{S}_i^0$ whose boundary is γ_i. We can assume that $\eta(B_j^{(1)}) = \eta(B_j^{(2)}) = B_j$ for every $j = 1, \ldots, p$, where B_j is a small disc in $\overline{S}$ that contains the j-th puncture, possibly by a re-indexing. Now select the following subspaces of $\widehat{S}_i^0$

$$U^{(i)} := \bigsqcup_{j=1}^p B_j^{(i)} \sqcup B_\partial^{(i)} \qquad V^{(i)} := \widehat{S}_i^0 \setminus (\{\textit{punctures of } \widehat{S}_i^0\} \cup \mathring{B}_\partial^{(i)})$$

and apply the Mayer-Vietoris sequence to the decomposition $\widehat{S}_i^0 = U^{(i)} \cup V^{(i)}$. Then

$$0 \longrightarrow H_2(\widehat{S}_i^0) \longrightarrow H_1(U^{(i)} \cap V^{(i)}) \longrightarrow H_1(U^{(i)}) \oplus H_1(V^{(i)}) \longrightarrow H_1(\widehat{S}_i^0) \longrightarrow 0$$

which can be rewritten as

$$0 \longrightarrow H_2(\widehat{S}_i^0) \longrightarrow \bigoplus_{j=1}^p H_1(\partial B_j^{(i)}) \oplus H_1(\gamma_i) \longrightarrow H_1(\widehat{S}_i) \longrightarrow H_1(\widehat{S}_i^0) \longrightarrow 0$$

Denote by E_i the $\mathbb{Z}$-module obtained as quotient of the second element of the exact sequence above by the relation $[\gamma_i] + \sum [\partial B_j^{(i)}] = 0$, where the $[\partial B_j^{(i)}]$ are oriented as boundaries of the $B_j^{(i)}$. Then the sequence can be expressed as

$$0 \longrightarrow E_i \longrightarrow H_1(\widehat{S}_i) \longrightarrow H_1(\widehat{S}_i^0) \longrightarrow 0 \tag{1.6}$$

where the map $E_i \to H_1(\widehat{S}_i)$ is the application on E_i induced by the inclusions of the paths γ_i and $\partial B_j^{(i)}$ in $\widehat{S}_i$. Observe that this exact sequence spits because $H_1(\widehat{S}_i^0)$ is free. The restriction of τ on $\widehat{S}_1 \sqcup \widehat{S}_2$ exchanges the two components and can be extended to a homeomorphism τ^0 of $\widehat{S}_1^0 \sqcup \widehat{S}_2^0$ with itself, switching the connected components and carrying γ_1 and $\partial B_j^{(1)}$ in γ_2 and $\partial B_j^{(2)}$ respectively. Consequently τ induces a natural isomorphism between the two exact sequences constructed from $\widehat{S}_1^0$ and $\widehat{S}_2^0$.

Denote by $\widehat{D}^0$ the surface obtained from $\widehat{D}$ by filling its boundary components. The involution $\tau|_{\widehat{D}}$ can be extended to a self-homemorphism $\bar{\tau}$ of $\widehat{D}^0$, which switches the filling discs and, in particular, has the same fixed points of $\tau|_{\widehat{D}}$. By virtue of Proposition 7.15 and corollaries in [FM11], $\bar{\tau}_*$ acts on $H_1(\widehat{D}^0; \mathbb{Z})$ as $-id$. Just as in the case of $\widehat{S}_i^0$, we have the exact sequence

$$0 \longrightarrow F \longrightarrow H_1(\widehat{D}) \longrightarrow H_1(\widehat{D}^0) \longrightarrow 0 \tag{1.7}$$

where F is the module $H_1(\gamma_1 \sqcup \gamma_2)$ with the relation $[\gamma_1] + [\gamma_2] = 0$ and the map from F to $H_1(\widehat{D})$ is induced by the inclusion of $\gamma_1 \sqcup \gamma_2$ in $\widehat{D}$. The homomorphism

τ_* is equal to $-id$ on $H_1(\widehat{D})$ too, because of the sequence 1.7 and the fact that τ_* acts on F as $-id$.

Putting together all these observations and the relations 1.5, 1.6 and 1.7, the group $H_1(\widehat{S})$ can be expressed as

$$H_1(\widehat{D}^0) \oplus H_1(\widehat{S}^0_1) \oplus \bigoplus_{j=1}^{p} H_1(\partial B_j^{(1)}) \oplus H_1(\widehat{S}^0_2) \oplus \bigoplus_{j=1}^{p} H_1(\partial B_j^{(2)})$$

with the relation $\sum_{j=1}^{p}[\partial B_j^{(1)}] + [\partial B_j^{(2)}] = 0$. Moreover, the following properties hold

- since the isomorphism is constructed with maps induced by inclusions, the intersection form of $\widehat{S}$ splits as the sum of the intersection forms on each factor;
- the action of τ_* on $H_1(\widehat{S})$ is equal to $-id$ on the term $H_1(\widehat{D}^0)$ and it isomorphically switches the $\widehat{S}^0_i$ factors, by the isomorphism described above between the exact sequences related to $\widehat{S}^0_i$.

The surfaces $\widehat{S}^0_i$ are clearly homeomorphic to $\overline{S}$ in a natural way, so the above decomposition can be rewritten as

$$H_1(\widehat{D}^0) \oplus H_1(\overline{S})^2 \oplus \frac{\mathbb{Z}[\partial B_j^{(1)}, \partial B_j^{(2)} \mid j = 1, \dots, p]}{\left(\sum_j [\partial B_j^{(1)}] + [\partial B_j^{(2)}] = 0\right)} \tag{1.8}$$

In this expression we can describe the action of τ_* as the following

$$(c; d_1, d_2; [h_1^{(1)}, h_1^{(2)}, \dots, h_p^{(1)}, h_p^{(2)}]) \xmapsto{\tau_*} (-c; d_2, d_1; [h_1^{(2)}, h_1^{(1)}, \dots, h_p^{(2)}, h_p^{(1)}])$$

Hence, the subspace of $H_1(\widehat{S})$ defined as

$$\{a \in H_1(\widehat{S}) \mid \tau_*(a) = -a\}$$

corresponds, in this decomposition, to the submodule of the elements

$$(c; d, -d; [h_1, -h_1, \dots, h_p, -h_p]) \tag{1.9}$$

for varying $c \in H_1(\widehat{D}^0)$, $d \in H_1(\overline{S})$ and $h_i \in \mathbb{Z}$. This submodule is isomorphic to

$$H_1(\widehat{D}^0) \oplus H_1(\overline{S}) \oplus \mathbb{Z}^p$$

and the restriction of the intersection form is expressed in this context as $i_{\widehat{D}} \oplus 2i_{\overline{S}} \oplus 0$ (for a more detailed analysis of the factor $\mathbb{Z}^p$ we refer to the proof of Lemma 1.2.6). Now we can select a basis of $H_1(\widehat{D}^0)$ and $H_1(\overline{S})$ such that, in their induced coordinates, the intersection forms split as the direct sum of elementary bilinear application represented by the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Thanks to Lemma 1.2.1, we have proved:

Theorem 1.2.2. *The couple $(\mathcal{H}(\lambda;\mathbb{Z}),\sigma)$ is isomorphic, through a certain $\mathbb{Z}$-linear isomorphism $A\colon \mathcal{H}(\lambda;\mathbb{Z}) \to \mathbb{Z}^n$, to $(\mathbb{Z}^n,\bar\sigma)$, where $\bar\sigma$ is represented, in the canonical basis, by the block diagonal matrix that has:*

- *$k = g(\widehat{D}^0)$ blocks of size 2×2 equal to*

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

- *$g = g(S)$ blocks of size 2×2 equal to*

$$\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

- *the remaining block $p\times p$ globally zero;*

where $k = -1-\chi(S) = 2g+p-3$.

Definition 1.2.3. Let $\mathcal{W}^\alpha = \mathcal{W}^\alpha[U,V]$ be the non-commutative $\mathbb{C}$-algebra generated by $U^{\pm 1}$ and $V^{\pm 1}$, endowed with the relation $UV = \alpha^2 VU$.

Now we are able to show an important consequence of the above analysis:

Theorem 1.2.4. *Let S be a surface, obtained by removing p punctures from an oriented closed surface $\overline{S}$ of genus g with $\chi(S)<0$, and let $\lambda\in\Lambda(S)$ be a certain ideal triangulation of S. Then the Chekhov-Fock algebra $\mathcal{T}^q_\lambda$ of S associated with λ is isomorphic to*

$$\mathcal{W}^q_{k,g,p} := \bigotimes_{i=1}^{k}\mathcal{W}^q \otimes \bigotimes_{j=1}^{g}\mathcal{W}^{q^2} \otimes \bigotimes_{h=1}^{p}\mathbb{C}[Z^{\pm 1}]$$

where $k = -1-\chi(S) = 2g+p-3$.

Proof. Let $\mathcal{U}$ be the non-commutative $\mathbb{C}$-algebra generated by $Y_1^{\pm 1},\dots,Y_n^{\pm 1}$ and endowed with the relations

$$Y_iY_j = q^{2\bar\sigma_{ij}}Y_jY_i$$

for varying $i,j=1,\dots,n$, where $\bar\sigma$ is the bilinear skew-symmetric form on $\mathbb{Z}^n$ described in Theorem 1.2.2. It is immediate to observe that $\mathcal{U}$ is isomorphic to the algebra described in the assertion. Therefore, we want to exhibit an isomorphism between $\mathcal{U}$ and $\mathcal{T}^q_\lambda$. Let $A\colon \mathbb{Z}^n \to \mathbb{Z}^n \cong \mathcal{H}(\lambda;\mathbb{Z})$ be the isomorphism of Theorem 1.2.2. Define, in the introduced notations, the following map

$$\begin{array}{rccc} f: & \mathcal{U} & \longrightarrow & \mathcal{T}^q_\lambda \\ & Y_i & \longmapsto & \underline{X}^{Ae_i} \end{array}$$

where we have labelled as e_i the i-th vector of the canonical basis of $\mathbb{Z}^n$. Thanks to the relations 1.3, we can show that this map is well defined, indeed:

$$\begin{aligned} f(Y_iY_j) &= \underline{X}^{Ae_i}\underline{X}^{Ae_j} = q^{2\sigma(Ae_i,Ae_j)}\underline{X}^{Ae_j}\underline{X}^{Ae_i} \\ &= q^{2\bar\sigma_{ij}}\underline{X}^{Ae_j}\underline{X}^{Ae_i} = q^{2\bar\sigma_{ij}}f(Y_jY_i) \end{aligned}$$

as desired. By definition of f, taken $u \in \mathbb{Z}^n$, the following holds:

$$\begin{aligned}
f(\underline{Y}^u) &= q^{-\sum_{i<j} \bar{\sigma}_{ij}} f(Y_1)^{u_1} \cdots f(Y_n)^{u_n} \\
&= q^{-\sum_{i<j} \bar{\sigma}_{ij} u_i u_j} (\underline{X}^{Ae_1})^{u_1} \cdots (\underline{X}^{Ae_n})^{u_n} \\
&= q^{-\sum_{i<j} \bar{\sigma}_{ij} u_i u_j} \underline{X}^{u_1 Ae_1} \cdots \underline{X}^{u_n Ae_n} && \text{Relation 1.3} \\
&= q^{-\sum_{i<j} \bar{\sigma}_{ij} u_i u_j + \sum_{i<j} \sigma(Ae_i, Ae_j) u_i u_j} \underline{X}^{Au} && \text{Relation 1.3} \\
&= \underline{X}^{Au}
\end{aligned}$$

Now it is easy to verify that, by defining

$$\begin{array}{rccc} g: & \mathcal{T}^q_\lambda & \longrightarrow & \mathcal{U} \\ & X_i & \longmapsto & \underline{Y}^{A^{-1}e_i} \end{array}$$

we have $f \circ g = id_{\mathcal{T}^q_\lambda}$ and $g \circ f = id_{\mathcal{U}}$, which prove the isomorphism. $\square$

Thanks to Remark 1.1.3, the study of the monomial center of $\mathcal{T}^q_\lambda$ is equivalent to the characterization of the following submodules of $\mathcal{H}(\lambda; \mathbb{Z})$:

$$\operatorname{Ker} \sigma := \{\alpha \in \mathcal{H}(\lambda; \mathbb{Z}) \mid \forall \beta \in \mathcal{H}(\lambda; \mathbb{Z}),\ \sigma(\alpha, \beta) = 0\}$$
$$\operatorname{Ker}_N \sigma := \{\alpha \in \mathcal{H}(\lambda; \mathbb{Z}) \mid \forall \beta \in \mathcal{H}(\lambda; \mathbb{Z}),\ \sigma(\alpha, \beta) \in N\mathbb{Z}\}$$

In particular the second one will have a special role in the case in which $q \in \mathbb{C}^*$ is a primitive N-th root of unity, and this will be the case of main interest for our purposes.

Definition 1.2.5. Let S be a surface, obtained by removing p points from an oriented closed surface $\overline{S}$ of genus g with $\chi(S) < 0$, and let $\lambda \in \Lambda(S)$ be an ideal triangulation of S. For every puncture p_i of S and for every λ_j 1-cell of the triangulation λ, we denote by $k_{ij} \in \{0, 1, 2\}$ the number of ends of λ_j that approach the puncture p_i. For every $i = 1, \dots, p$, define

$$k_i := (k_{i1}, \dots, k_{in}) \in \mathbb{Z}^n$$

Since every λ_j has exactly two ends counted with multiplicity, the following relation holds

$$\sum_{i=1}^{p} k_i = (2, \dots, 2)$$

Lemma 1.2.6. *In $\mathcal{H}(\lambda; \mathbb{Z})$ the subspace $\operatorname{Ker} \sigma$ is a free subgroup and the elements $(1, \dots, 1)$ and k_i, for $i = 1, \dots, p-1$, compose a basis for $\operatorname{Ker} \sigma$.*

Proof. Through the isomorphism constructed in Theorem 1.2.2, denoting by i the intersection form of $\widehat{S}$, the subspace $\operatorname{Ker} i$ corresponds to the submodule

$$W := \frac{\mathbb{Z}[\partial B_j^{(1)}, \partial B_j^{(2)} \mid j = 1, \dots, p]}{\left(\sum_j [\partial B_j^{(1)}] + [\partial B_j^{(2)}] = 0\right)} \cong \mathbb{Z}^{2p-1}$$

in the expression described in 1.8. By virtue of Theorem 1.2.2, the subspace $\operatorname{Ker} \sigma$ is identified to

$$V := \{[h_1, -h_1, \dots, h_p, -h_p] \in W \mid h_i \in \mathbb{Z}\} \subset W$$

Hence V is generated by the elements $[\partial B_j^{(1)}] - [\partial B_j^{(2)}]$, for varying $j = 1, \dots, p$. The involution τ carries $\partial B_j^{(1)}$ in $\partial B_j^{(2)}$. Moreover, the map τ switches the orientations of the edges in $\widehat{\Gamma}^{(1)}$, then exactly one curve between $\partial B_j^{(1)}$ and $\partial B_j^{(2)}$ is consistent with the orientations on $\widehat{\Gamma}^{(1)}$. Now, observing the local behaviour of $\widehat{\pi}$, it is simple to show that the vector $k_i \in \mathbb{Z}^n \cong \mathcal{H}(\lambda; \mathbb{Z})$ corresponds in $H_1(\widehat{S})$ to the element

$$\varepsilon_j \left([\partial B_j^{(1)}] - [\partial B_j^{(2)}]\right)$$

where ε_j is $+1$ or -1 if $[\partial B_j^{(1)}]$ is or is not coherent with the orientations of $\widehat{\Gamma}$.

Defining $a := \sum_j [\partial B_j^{(1)}]$, then a belongs to V, indeed

$$\tau_*(a) = \tau_* \left(\sum_j [\partial B_j^{(1)}] \right) = \sum_j [\partial B_j^{(2)}] = -\sum_j [\partial B_j^{(1)}] = -a$$

where we are using the relation $\sum_j [\partial B_j^{(1)}] + [\partial B_j^{(2)}] = 0$. Now think to W as the quotient of $\mathbb{Z}^{2p}$ by the submodule $\langle (1, \dots, 1, 1, \dots, 1) \rangle$, in which the $[\partial B_j^{(1)}]$ correspond to the first p elements of the canonical basis of $\mathbb{Z}^{2p}$ and the $[\partial B_j^{(2)}]$ to the second p. Now, taking the following basis of $\mathbb{Z}^{2p}$

$$\begin{pmatrix}
 & & & 1 & 1 & & & \\
 & I_{p-1} & & \vdots & \vdots & & I_{p-1} & \\
 & & & 1 & 1 & & & \\
0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 0 \\
 & & & 0 & 1 & & & \\
 & -I_{p-1} & & \vdots & \vdots & & 0 & \\
 & & & 0 & 1 & & & \\
0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0
\end{pmatrix}$$

we see that V, as subspace of W, is the free $\mathbb{Z}$-module having as basis the first p columns of the matrix above.

Observe that

$$\begin{aligned}
\sum_{j=1}^{p} \varepsilon_j \left([\partial B_j^{(1)}] - [\partial B_j^{(2)}]\right) &= \sum_{j=1}^{p-1} (\varepsilon_j - \varepsilon_p) \left([\partial B_j^{(1)}] - [\partial B_j^{(2)}]\right) + \\
&\qquad + \sum_{j=1}^{p} \varepsilon_p \left([\partial B_j^{(1)}] - [\partial B_j^{(2)}]\right) \qquad (1.10) \\
&= 2 \left(\sum_{j=1}^{p-1} \delta_j \left([\partial B_j^{(1)}] - [\partial B_j^{(2)}]\right) + \varepsilon_p \, a \right)
\end{aligned}$$

with $\delta_j := \frac{\varepsilon_j - \varepsilon_p}{2} \in \mathbb{Z}$. Now we can easily conclude: the first member of the equality 1.10 corresponds in $\mathbb{Z}^n \cong \mathcal{H}(\lambda; \mathbb{Z})$ to $\sum k_j = (2, \dots, 2)$, hence we have

$$(1, \dots, 1) \longleftrightarrow \sum_{j=1}^{p-1} \delta_j \left([\partial B_j^{(1)}] - [\partial B_j^{(2)}]\right) + \varepsilon_p \, a \in V \qquad (1.11)$$

Thanks to what observed and to the relation 1.11, the elements k_i, for $i = 1, \dots, p-1$, and a define a basis of $\operatorname{Ker}\sigma$, as desired. □

Now, let $\pi\colon \mathcal{H}(\lambda;\mathbb{Z}) \to \mathcal{H}(\lambda;\mathbb{Z}_N)$ be the projection on the quotient by the submodule $N\mathcal{H}(\lambda;\mathbb{Z})$. It is clear that $\operatorname{Ker}_N \sigma \supseteq N\mathcal{H}(\lambda;\mathbb{Z})$, so $\operatorname{Ker}_N \sigma$ coincides with $\pi^{-1}(\pi(\operatorname{Ker}_N \sigma))$. Denoting by $\widetilde{\sigma}$ the skew-symmetric form induced by σ on $\mathcal{H}(\lambda;\mathbb{Z}_N)$, it is easy to see that $\pi(\operatorname{Ker}_N \sigma) = \operatorname{Ker}\widetilde{\sigma}$. Following the proof of Theorem 1.2.2, we observe that $\mathcal{H}(\lambda;\mathbb{Z}_N)$ has a characterization analogous to the one found of $\mathcal{H}(\lambda;\mathbb{Z})$, i. e. we have the following isomorphism

$$\mathcal{H}(\lambda;\mathbb{Z}_N) \cong H_1(\widehat{D}^0;\mathbb{Z}_N) \oplus H_1(\overline{S};\mathbb{Z}_N) \oplus (V \otimes_{\mathbb{Z}} \mathbb{Z}_N)$$

and the form $\widetilde{\sigma}$ corresponds to $i_{\widehat{D}^0} \oplus 2i_{\widehat{S}} \oplus 0$. Note that the proof of Lemma 1.2.6 shows us that there exists a retraction of W on V, hence the inclusion map $V \longrightarrow W$ induces an inclusion $V \otimes \mathbb{Z}_N \longrightarrow W \otimes \mathbb{Z}_N$. Moreover, V is free and generated by the elements $(1, \dots, 1)$ and k_i for $i = 1, \dots, p-1$, so $V_N := V \otimes \mathbb{Z}_N$ is a free $\mathbb{Z}_N$-submodule of $W \otimes \mathbb{Z}_N$ having as basis the images of the elements k_i and $(1, \dots, 1)$.

Lemma 1.2.7. *If N is odd,* $\operatorname{Ker}_N \sigma$ *is the preimage through π of the free $\mathbb{Z}_N$-submodule V_N of $\mathcal{H}(\lambda;\mathbb{Z}_N)$, having as basis the elements* $(1, \dots, 1)$ *and* k_i *for* $i = 1, \dots, p-1$ *in* $(\mathbb{Z}_N)^n$.

Proof. We have seen that the preimage under π of $\operatorname{Ker}\widetilde{\sigma} \subseteq \mathcal{H}(\lambda;\mathbb{Z}_N)$ is equal to $\operatorname{Ker}_N \sigma$. Since 2 is invertible in $\mathbb{Z}_N$ if N is odd, Theorem 1.2.2 and the above observations tell us that $\operatorname{Ker}\widetilde{\sigma}$ is isomorphic to $V \otimes_{\mathbb{Z}} \mathbb{Z}_N$, which has as basis the elements k_i and $(1, \dots, 1)$ in $(\mathbb{Z}_N)^n$, for varying $i = 1, \dots, p-1$. □

Let $\alpha_1, \dots, \alpha_{2g}$ be a basis of $H_1(\overline{S};\mathbb{Z}_2)$. We can represent α_j as a curve a_j immersed in Γ, which passes at most one time through each 1-cell of Γ. Denote by $l_{ji} \in \{0, 1\}$ the number of times that a_j passes through λ_i.

Lemma 1.2.8. *If $N = 2M$ is even,* $\operatorname{Ker}_N \sigma$ *is the preimage through π of the direct sum $0 \oplus B \oplus V_N \subseteq \mathcal{H}(\lambda;\mathbb{Z}_N)$, where B is the submodule generated by the elements $Ml_j \in \mathcal{H}(\lambda;\mathbb{Z}_N)$, with $j = 1, \dots, 2g$.*

Proof. In these hypotheses, 2 is not invertible in $\mathbb{Z}_N$, so the form $\widetilde{\sigma}$ has a non-trivial kernel also on the summand $H_1(\overline{S};\mathbb{Z}_N)$. By virtue of the above analysis, we have $\operatorname{Ker}\widetilde{\sigma} = B' \oplus V_N$, where $B' = MH_1(\overline{S};\mathbb{Z}_N)$.

Consider the application $T\colon H_1(S;\mathbb{Z}_2) \to H_1(\widehat{S};\mathbb{Z}_2)$, induced by the chain map that associates with each singular simplex the sum of its lifts. The image of T is contained in

$$\{a \in H_1(\widehat{S};\mathbb{Z}_2) \mid \tau_*(a) = a\} \cong \mathcal{H}(\lambda;\mathbb{Z}_2)$$

If $\alpha'_i \in H_1(S;\mathbb{Z}_2)$ is represented by the curve a_i, as previously chosen, then $T(\alpha'_i)$ is identified to $l_i \in \mathcal{H}(\lambda;\mathbb{Z}_2) \subseteq H_1(\widehat{S};\mathbb{Z}_2)$. Given $\alpha \in H_1(S;\mathbb{Z}_2)$, we want to describe the element $T(\alpha)$ in the decomposition of $\mathcal{H}(\lambda;\mathbb{Z}_2)$ previously seen: representing α as a curve a lying in $S \setminus D$, the element $T(a)$ coincides, in $H_1(\widehat{S};\mathbb{Z}_2)$, with the class of $a_1 + a_2$, where a_i are paths in $\widehat{S}_i$ obtained by lifting a. In particular, $T(\alpha)$ is zero on the component $H_1(\widehat{D}^0;\mathbb{Z}_2)$ of $\mathcal{H}(\lambda;\mathbb{Z}_2)$. Moreover, starting from an element $\alpha \in H_1(\overline{S};\mathbb{Z}_2)$, and selecting a lift α' of

α in $H_1(S;\mathbb{Z}_2)$, we can consider the element $T(\alpha')$. By construction of the decomposition

$$\mathcal{H}(\lambda;\mathbb{Z}_2)\cong H_1(\widehat{D}^0;\mathbb{Z}_2)\oplus H_1(\overline{S};\mathbb{Z}_2)\oplus V_2$$

the projection of $T(\alpha')$ on the component $H_1(\overline{S};\mathbb{Z}_2)$ is exactly α. Therefore $0\oplus H_1(\overline{S};\mathbb{Z}_2)\oplus V_2\subset\mathcal{H}(\lambda;\mathbb{Z}_2)$ is isomorphic to $B''\oplus V_2$, where B'' denote the submodule $\mathcal{H}(\lambda;\mathbb{Z}_2)$ generated by the elements $T(\alpha'_i)$.

Identifying $MH_1(\overline{S};\mathbb{Z}_N)$ with the image of the M-multiplication homomorphism from $H_1(\overline{S};\mathbb{Z}_2)$ to $H_1(\overline{S};\mathbb{Z}_N)$, we have that $\operatorname{Ker}\widetilde{\sigma}=0\oplus MH_1(\overline{S};\mathbb{Z}_N)\oplus V_N$ coincides with the submodule $0\oplus B\oplus V_N$, with $B:=M\cdot B''$, generated by the elements Ml_j. $\square$

Now we introduce the following notations:

$$\begin{aligned}
H &:= \underline{X}^{(1,\dots,1)} = q^{-\sum_{i<j}\sigma_{ij}}X_1\cdots X_n\\
P_i &:= \underline{X}^{k_i} = q^{-\sum_{s<t}k_{is}k_{it}\sigma_{st}}X_1^{k_{i1}}\cdots X_n^{k_{in}}\\
A_j &:= \underline{X}^{l_j} = q^{-\sum_{s<t}l_{js}l_{jt}\sigma_{st}}X_1^{l_{i1}}\cdots X_n^{l_{in}}
\end{aligned}$$

Recalling Remark 1.1.3, by virtue of Lemma 1.2.6, the elements H and P_i belong to the monomial center $\mathcal{Z}_\lambda^q$, because $(1,\dots,1)$ and k_i are in $\operatorname{Ker}\sigma$.

Moreover, in light of relation 1.3, if $\underline{X}^\alpha$ belongs to $\mathcal{Z}_\lambda^q$ then, for every $\beta\in\mathbb{Z}^n$, we have

$$\begin{aligned}
&q^{2\sigma(\alpha,\beta)}=1\\
&\underline{X}^\alpha\underline{X}^\beta=q^{\sigma(\alpha,\beta)}\underline{X}^{\alpha+\beta}
\end{aligned}\tag{1.12}$$

Proposition 1.2.9. *If q is not a root of unity, then the monomial center $\mathcal{Z}_\lambda^q$ is isomorphic to the direct sum of $\mathbb{C}^*$ and the abelian free subgroup of $\mathcal{Z}_\lambda^q$ generated by the elements H and P_i for $i=1,\dots,p-1$.*

Proof. Remark 1.1.3 tells us that the monomial center is isomorphic to the direct sum of $\mathbb{C}^*$ and of the abelian group $\operatorname{Ker}\sigma$, by virtue of relation 1.12 and the fact that q is not a root of unity. Then the assertion follows from Lemma 1.2.6 and the definition of the elements H and P_i. $\square$

If q^2 is a primitive N-th root of unity, then, by virtue of 1.3 and 1.12, the monomial center is isomorphic to the direct sum of $\mathbb{C}^*$ and of the $\mathbb{Z}$-module $\operatorname{Ker}_N\sigma$. The last thing we need to do is to find a good description of $\operatorname{Ker}_N\sigma$ and to translate it in a description of $\mathcal{Z}_\lambda^q$.

Proposition 1.2.10. *If q^2 is a primitive N-th root of unity, with N odd, then $\mathcal{Z}_\lambda^q$ is isomorphic to the direct sum of $\mathbb{C}^*$ and of the abelian subgroup generated by the elements X_i^N with $i=1,\dots,n$, H and P_j with $j=1,\dots,p-1$, endowed with the following relations*

$$\begin{aligned}
H^N &= q^{-N^2\sum_{i<j}\sigma_{ij}}X_1^N\cdots X_n^N\\
P_i^N &= q^{-N^2\sum_{s<t}k_{is}k_{it}\sigma_{st}}(X_1^N)^{k_{i1}}\cdots(X_n^N)^{k_{in}}
\end{aligned}$$

Proof. Recall that $V=\operatorname{Ker}\sigma$ is a direct summand of $\mathcal{H}(\lambda;\mathbb{Z})$, so the matrix Q, having as columns the vectors k_i and $(1,\dots,1)$, has a minor $p\times p$ with determinant equal to ±1. Without lost of generality we can assume that the leading principal minor of order p A verifies $\det(A) = \pm 1$ (so A is invertible). Decompose Q in blocks as follows

$$\begin{pmatrix} \\ A \\ \\ K' \end{pmatrix}$$

As seen in Lemma 1.2.7, the submodule $\operatorname{Ker}_N \sigma$ can be identified with the image of the linear map represented by the following matrix:

$$T = \begin{pmatrix} NI_p & 0 & A \\ 0 & NI_{n-p} & K' \end{pmatrix}$$

The first n columns represent a basis of $N\mathcal{H}(\lambda;\mathbb{Z})$, and the others represent the basis of $\operatorname{Ker}\sigma$ selected above. Observe that the following relation holds

$$\begin{pmatrix} NI_p & 0 & A \\ 0 & NI_{n-p} & K' \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & A \\ 0 & I_{n-p} & K' \\ A^{-1} & 0 & -NI_p \end{pmatrix} = \begin{pmatrix} I_p & 0 & 0 \\ K'A^{-1} & NI_{n-p} & 0 \end{pmatrix}$$

The matrix U, by which we have multiplied T, is an automorphism of $\mathbb{Z}^n$, because A is invertible. Moreover, this equality tells us that the last p columns of U are a basis of $\operatorname{Ker} T$. So $\operatorname{Ker}_N \sigma$ is isomorphic to the free $\mathbb{Z}$-module generated by the elements Ne_i, with $i = 1, \dots, n$, by k_j with $j = 1, \dots, p-1$ and by $(1, \dots, 1)$, endowed with the relations

$$\sum_{i=1}^{n} k_{ji}(Ne_i) = Nk_j$$

$$\sum_{i=1}^{n} (Ne_i) = N(1, \dots, 1)$$

By virtue of the relation 1.12 and the definition of $\operatorname{Ker}_N \sigma$, the following map is a group isomorphism

$$\begin{array}{ccc} \mathbb{C}^* \oplus \operatorname{Ker}_N \sigma & \longrightarrow & \mathcal{Z}^q_\lambda \\ (c, \alpha) & \longmapsto & c\,\underline{X}^\alpha \end{array}$$

Hence the assertion follows from the equations found above, translated in the context of $\mathcal{Z}^q_\lambda$. We are doing the explicit calculation for the first equation, the second is analogous. Recalling the relation 1.12, we deduce

$$\begin{aligned} P_j^N &= (\underline{X}^{k_j})^N = \underline{X}^{Nk_j} = \underline{X}^{\sum_i k_{ji}(Ne_i)} \\ &= q^{-N^2 \sum_{s<t} k_{js}k_{jt}\sigma_{st}} (X_1^N)^{k_{j1}} \cdots (X_n^N)^{k_{jn}} \end{aligned}$$

□

Proposition 1.2.11. *If q^2 is a primitive N-th root of unity, with $N = 2M$ even, then $\mathcal{Z}^q_\lambda$ is isomorphic to the direct sum of $\mathbb{C}^*$ and of the abelian subgroup of $\mathcal{Z}^q_\lambda$ generated by the elements X_i^N with $i = 1, \dots, n$, H, P_j with $j = 1, \dots, p-1$ and A_k with $k = 1, \dots, 2g$, endowed with the relations*

$$\begin{aligned} H^N &= q^{-N^2 \sum_{i<j} \sigma_{ij}} X_1^N \cdots X_n^N \\ P_i^N &= q^{-N^2 \sum_{s<t} k_{is}k_{it}\sigma_{st}} (X_1^N)^{k_{i1}} \cdots (X_n^N)^{k_{in}} \\ A_k^2 &= q^{-N^2 \sum_{s<t} l_{ks}l_{kt}\sigma_{st}} (X_1^N)^{l_{k1}} \cdots (X_n^N)^{l_{kn}} \end{aligned}$$

Proof. See [BL07, Proposition 16]. □

1.3 Polygons

In this Section, we will assume S is an ideal polygon with p vertices, i. e. a surface obtained from $D = D^2$ by removing p punctures in ∂D, with $p \geq 3$. Let $\lambda \in \Lambda(S)$ be an ideal triangulation and let Γ be the dual graph of λ. In this case, the valence of a vertex T^*, associated with a triangle T, coincides with the number of edges of T that do not lie in ∂S. What still holds is that Γ is a deformation retract of S. Therefore, Γ is a tree, by virtue of the simply connectedness of S. Moreover, the leaves of Γ exactly correspond to those triangles that have two edges lying in ∂S. Our purpose is to find a presentation of $(\mathcal{H}(\lambda; \mathbb{Z}), \sigma)$ analogous to the one in Theorem 1.2.2, in order to simplify the study of $\mathcal{T}^q_\lambda$.

Firstly we must deal with the most simple case, in which S is just an ideal triangle:

Remark 1.3.1. Let T be an oriented ideal triangle, with a fixed indexing of the edges that proceeds in the opposite way of the one given by the orientation, as in Figure 1.4.

In these notations, the bilinear form σ is represented by the matrix

$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

By taking the basis $e_1' := e_1$, $e_2' := e_2$, $e_3' = e_1 + e_2 + e_3$, the matrix representing σ becomes

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The surface T clearly admits a unique ideal triangulation and its Chekhov-Fock algebra is

$$\frac{\mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}]}{(X_i X_{i+1} = q^2 X_{i+1} X_i \mid i \in \mathbb{Z}_3)}$$

Denote by H the element $\underline{X}^{(1,1,1)} = q^{-1} X_1 X_2 X_3 \in \mathcal{T}^q_T$. Then H belongs to the monomial center $\mathcal{Z}^q_T$ and, if q^2 is a primitive N-th root of unity, then the same holds for $(X_1)^N, (X_2)^N, (X_3)^N$. From the expression of σ with respect to the

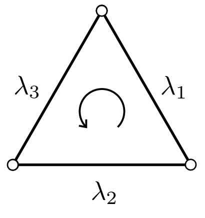

Figure 1.4: Standard indexing on a triangle

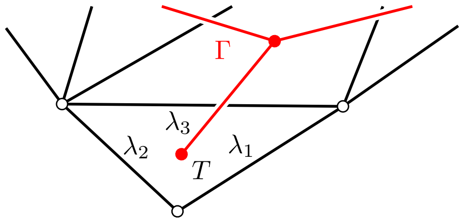

Figure 1.5: A leaf of Γ

basis $(e'_j)_j$, we observe that $\mathcal{T}^q_\lambda$ is isomorphic, through the isomorphism given by

$$\begin{array}{rcc} X_1 & \longmapsto & X'_1 \\ X_2 & \longmapsto & X'_2 \\ X_3 & \longmapsto & q(X'_2)^{-1}(X'_1)^{-1} \otimes H' \end{array}$$

to the algebra $\mathcal{W}^q[X'_1, X'_2] \otimes \mathbb{C}[(H')^{\pm 1}]$.

Going back to the generic case of an ideal polygon, with $p \geq 4$, with simple calculations we can show that the following relations hold

$$m = p - 2$$
$$n = 2p - 3$$

where n is the number of 1-cells of λ and m is the number of triangles in λ. Now, let $T = T_h$ be a triangle of λ corresponding to a leaf in Γ. In order to simplify the notations, assume that the edges of T are the 1-cells λ_1, λ_2, λ_3 of λ, ordered as in Figure 1.5. Because λ_1 and λ_2 belong to the only triangle T, the following holds

$$\begin{aligned} \sigma(e_1, e_j) &= 0 \\ \sigma(e_2, e_j) &= 0 \\ \sigma(e_1, e_2) &= 1 \end{aligned} \tag{1.13}$$

$$\sigma(e_1 + e_2 + e_3, e_j) = \sigma(e_3, e_j) \tag{1.14}$$

for every $j \geq 4$. Now define a new basis $(e'_j)_j$ of $\mathbb{Z}^n \cong \mathcal{H}(\lambda; \mathbb{Z})$ given by

$$\begin{aligned} e'_1 &:= e_1 \\ e'_2 &:= e_2 \\ e'_3 &:= e_1 + e_2 + e_3 \\ e'_j &:= e_j \end{aligned}$$

with $j \geq 4$. The block diagonal matrix that represents σ in this new basis is

$$\left(\begin{array}{cc|c} 0 & 1 & \\ -1 & 0 & \\ \hline & & \sigma' \end{array}\right)$$

thanks to relations 1.13. The equation 1.14 tells us that σ' coincides with the bilinear form associated with the surface S', obtained from S by removing the triangle T, with the obvious induced triangulation λ'. Because S' is an ideal polygon with $p-1$ punctures, we can reiterate this procedure until the $(m-1)$-th step, obtaining a surface $S^{(m-1)}$ composed of a single triangle. Analogously to what we have seen in Remark 1.3.1, we construct an isomorphism between $(\mathcal{H}(\lambda;\mathbb{Z}),\sigma)$ and $(\mathbb{Z}^n, A)$, where A is a bilinear skew-symmetric form, represented in the canonical basis of $\mathbb{Z}^n$ by the block diagonal matrix

$$\begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & 1 & \\ & & & -1 & 0 & \\ & & & & & 0 \end{pmatrix}$$

with $m = p-2$ blocks 2×2 and a block 1×1 equal to zero. By inspection of the iterative procedure, we can see that the vector in $\mathcal{H}(\lambda;\mathbb{Z})$ corresponding to $e_n \in \mathbb{Z}^n$ is the element $(1,\dots,1) \in \mathcal{H}(\lambda;\mathbb{Z})$.

Theorem 1.3.2. *Let S be an ideal polygon with $p \geq 3$ vertices and let $\lambda \in \Lambda(S)$ be an ideal triangulation. Then the Chekhov-Fock algebra $\mathcal{T}^q_\lambda$ of S associated with λ is isomorphic to*

$$\mathcal{W}^q_{m,0,1} = \bigotimes_{i=1}^{m} \mathcal{W}^q \otimes \mathbb{C}[Z^{\pm 1}]$$

where $m = p-2$. Moreover, the element $1 \otimes \cdots \otimes 1 \otimes Z \in \mathcal{W}^q_{m,0,1}$ corresponds to $H = \underline{X}^{(1,\dots,1)} \in \mathcal{T}^q_\lambda$.

Proposition 1.3.3. *The following facts hold:*

- *if q is not an N-th root of unity, then $\mathcal{Z}^q_\lambda$ is isomorphic to the direct sum of $\mathbb{C}^*$ and of the abelian subgroup generated by $H = \underline{X}^{(1,\dots,1)} \in \mathcal{T}^q_\lambda$;*
- *if q^2 is a primitive N-th root of unity, then $\mathcal{Z}^q_\lambda$ is isomorphic to the direct sum of $\mathbb{C}^*$ and of the abelian subgroup generated by the elements X_i^N with $i = 1,\dots,n$, and H, endowed with the relation*

$$H^N = q^{-N^2 \sum_{i<j} \sigma_{ij}} X_1^N \cdots X_n^N$$

Proof. Analogous to what done in Propositions 1.2.9 and 1.2.10. □

CHAPTER 2

Local and irreducible representations

In this Chapter we study finite-dimensional representations of the Chekhov-Fock algebra, with the instruments provided by the previous analysis. In the first part we focus on irreducible representations and we give a complete classification result, due to Bonahon and Liu [BL07]. We give also a similar and simpler statement when the surface S is an ideal polygon.

Later, we introduce a new kind of representations called *local representations*, firstly presented in [BBL07], which are constructed as the result of gluing irreducible representations of the Chekhov-Fock algebras of the triangles composing an ideal triangulation λ. Even in this case, we are able to prove a classification theorem, whose statement is similar than the irreducible case but simpler.

We follow the exposition in [BL07] and [BBL07] to these subjects.

2.1 Irreducible representations

Let V be a d-dimensional $\mathbb{C}$-vector space, with $d < \infty$, and let $\rho\colon \mathcal{T}^q_\lambda \to \mathrm{End}(V)$ be a representation of the Chekhov-Fock algebra associated with an ideal triangulation λ of a surface S. Applying ρ to the identity $X_i X_j = q^{2\sigma_{ij}} X_j X_i$ and calculating the determinant of both sides we obtain

$$\begin{aligned}\det(\rho(X_i))\det(\rho(X_j)) &= \det(\rho(q^{2\sigma_{ij}} X_j X_i)) \\ &= q^{2d\sigma_{ij}} \det(\rho(X_i))\det(\rho(X_j))\end{aligned}$$

On the other hand $\det(\rho(X_i))\det(\rho(X_j))$ is not zero, because both X_i and X_j have an inverse in $\mathcal{T}^q_\lambda$. Therefore, q^2 is needed to be a d-th root of unity in order to have the existence of a finite dimensional representation $\mathcal{T}^q_\lambda$. From now on, we will assume q^2 is a primitive N-th root of unity.

Definition 2.1.1. Given $\rho\colon \mathcal{A} \to \mathrm{End}(V)$ a representation of a $\mathbb{K}$-algebra $\mathcal{A}$ in a $\mathbb{K}$-vector space V, ρ is *reducible* if there exists a proper ρ-invariant subspace W of V, in other words there exists a subspace $0 \subsetneq W \subsetneq V$ such that, for every $X \in \mathcal{A}$, we have $\rho(X)(W) \subseteq W$.

Moreover, ρ is *decomposable* if there exist two ρ-invariant proper subspaces W_1 and W_2 of V such that $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$.

Remark 2.1.2. Let $\rho\colon \mathcal{A} \to \mathrm{End}(V)$ be an irreducible representation of a $\mathbb{K}$-algebra $\mathcal{A}$ in a $\mathbb{K}$-vector space V, where $\mathbb{K}$ is an algebraically closed field. Suppose further that there exists an element $X \in \mathcal{A} \setminus \{0\}$ in the multiplicative center of the algebra.

Now fix $\mu \in \mathbb{C}$ an eigenvalue of $\rho(X)$ (here we are using $\mathbb{K}$ algebraically closed) and denote by V_μ the eigenspace of $\rho(X)$ associated with μ. Because X belongs to the multiplicative center of $\mathcal{A}$, for every $Y \in \mathcal{A}$, we have

$$\rho(X)\rho(Y) = \rho(Y)\rho(X)$$

From this commutativity relation immediately follows that V_μ is a ρ-invariant subspace of V. Because ρ is irreducible, we have $V_\mu = V$ and so $\rho(X) = \mu\, id_V$. This argument shows that every element of the multiplicative center of the algebra goes, under an irreducible finite dimensional representation, necessarily in a scalar multiple of id_V.

Proposition 2.1.3. *Assume q^2 is a primitive N-th root of unity and there exists an irreducible representation $\rho\colon \mathcal{W}^q \to \mathrm{End}(V)$, where V is a $\mathbb{C}$-vector space of dimension d. Then $d = N$ and there exist $x_1, x_2 \in \mathbb{C}^*$ such that*

$$\begin{aligned} \rho(X_1^N) &= x_1\, id_V \\ \rho(X_2^N) &= x_2\, id_V \end{aligned}$$

where $\mathcal{W}^q = \mathcal{W}^q[X_1, X_2]$. Moreover, two irreducible representations $\rho\colon \mathcal{W}^q \to \mathrm{End}(V)$ and $\rho'\colon \mathcal{W}^q \to \mathrm{End}(V')$ are isomorphic if and only if $x_1 = x_1'$ and $x_2 = x_2'$, where the x_i are the scalars defined above for ρ and the x_i' for ρ'. In addition, for every choice of values of $x_1, x_2 \in \mathbb{C}^*$ there exists an irreducible representation, unique up to isomorphism, that realizes them as above.

Proof. The elements X_i^N go in scalar multiples of the identity because they belong to the center of $\mathcal{W}^q$ and because of what observed in Remark 2.1.2.

Let us try to select an expressive form for this kind of representations by choosing a suitable basis. Let $y_1 \in \mathbb{C}^*$ be an eigenvalue of $\rho(X_1)$ and let e_1 be a y_1-eigenvector. Because $\rho(X_1^N) = x_1\, id_V$, we have $(y_1)^N = x_1$. Now define $e_{i+1} := \rho(X_2)e_i$ for every $i = 1, \dots, N$. Thanks to what observed, we deduce that $e_{N+1} = \rho(X_2^N)e_1 = x_2\, e_1$. Moreover

$$\begin{aligned} \rho(X_1)e_{i+1} &= \rho(X_1)\rho(X_2^i)e_1 \\ &= q^{2i}\, \rho(X_2^i)\rho(X_1)e_1 \\ &= q^{2i}y_1\, \rho(X_2^i)e_1 \\ &= q^{2i}y_1\, e_{i+1} \end{aligned}$$

Then, for every $i = 0, \dots, N-1$, e_{i+1} is an eigenvector of $\rho(X_1)$ with respect to the eigenvalue $q^{2i}y_1$. If W is the subspace of V generated by the elements $e_1, \dots, e_N$, then both $\rho(X_1)$ and $\rho(X_2)$ keep W invariant, so W is a ρ-invariant non-zero subspace. By irreducibility, the equality $V = W$ must holds and the set $\{e_1, \dots, e_N\}$ composes a basis of V (because q^2 is a primitive N-th root of unity, the eigenvalues $y_1 q^{2i}$ are distinct for $i = 0, \dots, N-1$). Hence we have shown that $\dim V = N$ and we have found a basis $\{e_1, \dots, e_N\}$ in which $\rho(X_1)$

and $\rho(X_2)$ are represented, respectively, by the matrices

$$y_1 \cdot \begin{pmatrix} 1 & & & \\ & q^2 & & \\ & & \ddots & \\ & & & q^{2(N-1)} \end{pmatrix} \qquad \begin{pmatrix} 0 & \cdots & 0 & x_2 \\ & & & 0 \\ & I_{N-1} & & \vdots \\ & & & 0 \end{pmatrix}$$

We fix now $y_2 \in \mathbb{C}^*$ a N-th root of x_2. By conjugating ρ by the automorphism $A(y_2)$ of V, represented, with respect to the basis $\{e_1, \ldots, e_N\}$, by the matrix

$$\begin{pmatrix} y_2^{N-1} & & & \\ & y_2^{N-2} & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

we obtain that the elements $A(y_2)\rho(X_1)A(y_2)^{-1}$ and $A(y_2)\rho(X_2)A(y_2)^{-1}$ are represented by the matrices

$$\begin{gathered} y_1 \cdot \begin{pmatrix} 1 & & & \\ & q^2 & & \\ & & \ddots & \\ & & & q^{2(N-1)} \end{pmatrix} = y_1 \, B_1 \\ y_2 \cdot \begin{pmatrix} 0 & \cdots & 0 & 1 \\ & & & 0 \\ & I_{N-1} & & \vdots \\ & & & 0 \end{pmatrix} = y_2 \, B_2 \end{gathered} \tag{2.1}$$

A simple calculation shows us that, by conjugating the representation

$$A(y_2)\rho(\cdot)A(y_2)^{-1}$$

by the linear isomorphism represented by B_2^{-1}, we obtain

$$\begin{aligned} B_2^{-1}A(y_2)\rho(X_1)A(y_2)^{-1}B_2 &= q^2 y_1 \, B_1 \\ B_2^{-1}A(y_2)\rho(X_2)A(y_2)^{-1}B_2 &= y_2 \, B_2 \end{aligned}$$

Now we have all the tools to conclude the proof. Let $\rho \colon \mathcal{W}^q \to \mathrm{End}(V)$ and $\rho' \colon \mathcal{W}^q \to \mathrm{End}(V')$ be two irreducible representations of $\mathcal{W}^q$. It is clear that, if they are isomorphic, then their respective scalars x_i and x_i' must coincide. Vice versa, assume that $x_i = x_i'$. By virtue of what previously seen, up to considering $L\rho' L^{-1}$, with $L \colon V' \to V$ a suitable isomorphism, we can assume that $V = V'$ and that there exists a basis in which the representations ρ and ρ' are represented in coordinates as follows

$$\begin{aligned} \rho(X_1) &= y_1 \, B_1 \\ \rho(X_2) &= y_2 \, B_2 \\ \rho'(X_1) &= y_1' \, B_1 \\ \rho'(X_2) &= y_2' \, B_2 \end{aligned}$$

with $(y_i)^N = x_i = x'_i = (y'_i)^N$ for $i = 1, 2$. Because y_2 and y'_2 are both N-th roots of x_2, by conjugating ρ' by $A(y_2(y'_2)^{-1})$, we can assume $y_2 = y'_2$. Moreover, if $x_1 = q^{2k}x'_1$, then conjugating ρ' by B_2^{-k}, we can furthermore assume that $y_1 = y'_1$. What just said shows that the representations ρ and ρ' are conjugate, which concludes the proof that the x_i compose a complete set of invariants for irreducible representations of the algebra $\mathcal{W}^q$.

Finally, it is clear, by the calculations made, that every couple of values $x_1, x_2 \in \mathbb{C}^*$ can be realized as invariants of a irreducible representation of $\mathcal{W}^q$. □

In the following proposition we will make the technical hypothesis on q to be a primitive N-th root of $(-1)^{N+1}$. When N is odd, this is equivalent to ask that q^2 is a primitive N-th root of unity but, in the even case, it is a stronger assumption. This choice will make a lot of following relations concerning invariants much more pleasant. For example, with this assumption we have $q^{N^2} = (-1)^{N(N+1)} = 1$ and this simplifies the relations in Lemma 1.2.7 and Proposition 1.2.11.

Proposition 2.1.4. *Assume that there exists an irreducible representation*

$$\rho : \mathcal{T}_T^q \longrightarrow \mathrm{End}(V)$$

with V a $\mathbb{C}$-vector space of dimension d and with $\mathcal{T}_T^q$ that denotes the Chekhov-Fock algebra associated with the ideal triangle T, which admits a unique ideal triangulation. Then $d = N$ and there exist $x_1, x_2, x_3, h \in \mathbb{C}^*$ such that

$$\rho(X_i^N) = x_i\, id_V \qquad \text{for } i = 1, 2, 3$$
$$\rho(H) = h\, id_V$$

where the X_i denote the generators associated with the edges $\lambda_1, \lambda_2, \lambda_3$ of T and $H := q^{-1}X_1X_2X_3$. In addition, the following holds

$$h^N = x_1x_2x_3$$

Moreover, two irreducible representations $\rho\colon \mathcal{T}_T^q \to \mathrm{End}(V)$ and $\rho'\colon \mathcal{T}_T^q \to \mathrm{End}(V')$ are isomorphic if and only if $x_i = x'_i$ and $h = h'$, where the x_i, h are the above quantities related to ρ and the x'_i, h' related to ρ'. Furthermore, for every choice of values $x_1, x_2, x_3, h \in \mathbb{C}^*$ verifying $h^N = x_1x_2x_3$, there exists an irreducible representation ρ, unique up to isomorphism, that realizes them as invariants.

Proof. A first way conducing to the proof is to deduce this result from Proposition 2.1.3 and Remark 1.3.1. In the following, we will give a proof that does not need these results, basically because this is a good situation in order to introduce some notations that will become useful in the next Section, when we will speak about local representations.

The elements X_i^N and H belong to the multiplicative center of $\mathcal{T}_T^q$, so Remark 2.1.2 tells us that they go, under the representation, in scalar multiples of the identity. Moreover, we have

$$H^N = q^{-N^2 \sum_{i<j} \sigma_{ij}} X_1^N X_2^N X_3^N = X_1^N X_2^N X_3^N$$

which implies the relation $h^N = x_1x_2x_3$. Taken $\rho\colon \mathcal{T}_T^q \to \mathrm{End}(V)$ an irreducible representation, we select vectors $e_1, \dots, e_N$ exactly in the same way of the proof of Proposition 2.1.3, which means

$$\begin{aligned} \rho(X_2)e_i &= e_{i+1} && \text{for every } i = 1, \dots, N-1 \\ \rho(X_2)e_N &= x_2\, e_1 \\ \rho(X_1)e_{i+1} &= q^{2i} y_1\, e_{i+1} && \text{for every } i = 0, \dots, N-1 \end{aligned}$$

The element X_3 in $\mathcal{T}_T^q$ coincides with $q\, X_2^{-1}X_1^{-1}H$, so it verifies

$$\begin{aligned} \rho(X_3)e_{i+1} &= \rho(q\, X_2^{-1}X_1^{-1}H)e_{i+1} \\ &= q^{1-2i} h y_1^{-1}\, \rho(X_2)^{-1} e_{i+1} \\ &= q^{1-2i} h y_1^{-1}\, e_i \end{aligned}$$

if $i = 1, \dots, N-1$. Moreover

$$\begin{aligned} \rho(X_3)e_1 &= \rho(qX_2^{-1}X_1^{-1}H)e_1 \\ &= qhy_1^{-1}\, \rho(X_2)^{-1}e_1 \\ &= qhy_1^{-1}x_2^{-1}\, e_N \end{aligned}$$

These relations imply that $W := \langle e_1, \dots, e_N\rangle$ is invariant under $\rho(X_3)$ so, by irreducibility of ρ, we have that $V = W$ is a N-dimensional vector space, having $\{e_1, \dots, e_N\}$ as basis. Just like in Proposition 2.1.3, we see that, by conjugating ρ by $A(y_2)$, with y_2 a certain N-th root of x_2, we find

$$A(y_2)\rho(X_1)A(y_2)^{-1} = y_1 \cdot \begin{pmatrix} 1 & & & \\ & q^2 & & \\ & & \ddots & \\ & & & q^{2(N-1)} \end{pmatrix} = y_1\, B_1$$

$$A(y_2)\rho(X_2)A(y_2)^{-1} = y_2 \cdot \begin{pmatrix} 0 & \cdots & 0 & 1 \\ & & & 0 \\ & I_{N-1} & & \vdots \\ & & & 0 \end{pmatrix} = y_2\, B_2$$

$$A(y_2)\rho(X_2)A(y_2)^{-1} = y_3 \cdot \begin{pmatrix} 0 & q^{1-2(2-1)} & & \\ \vdots & & \ddots & \\ 0 & & & q^{1-2(N-1)} \\ q & 0 & \cdots & 0 \end{pmatrix} = y_3\, B_3$$

where we have labelled $y_3 := hy_1^{-1}y_2^{-1}$. The following relations hold

$$\begin{aligned} (y_i)^N &= x_i && \text{for } i = 1, 2, 3 \\ h &= y_1y_2y_3 \end{aligned}$$

With simple calculations we observe that, by conjugating ρ by B_2^{-1}, we obtain

$$\begin{aligned} B_2^{-1}A(y_2)\rho(X_1)A(y_2)^{-1}B_2 &= q^2 y_1\, B_1 \\ B_2^{-1}A(y_2)\rho(X_2)A(y_2)^{-1}B_2 &= y_2\, B_2 \\ B_2^{-1}A(y_2)\rho(X_3)A(y_2)^{-1}B_2 &= q^{-2} y_3\, B_3 \end{aligned} \tag{2.2}$$

Let $\rho\colon \mathcal{W}^q \to \operatorname{End}(V)$ and $\rho'\colon \mathcal{W}^q \to \operatorname{End}(V')$ be two irreducible representations of $\mathcal{W}^q$. Clearly if they are isomorphic, they need to have the same constants x_i and h. Assume that $x_i = x_i'$ and $h = h'$. As previously done, up to considering $L\rho' L^{-1}$, with $L\colon V' \to V$ a suitable isomorphism, we can assume that $V = V'$ and that there exists a basis in which ρ and ρ' are represented as follows

$$\rho(X_i) = y_i\, B_i$$
$$\rho'(X_i) = y_i'\, B_i$$

with $(y_i)^N = x_i = x_i' = (y_i')^N$ for $i = 1, 2, 3$. Since y_1 and y_2' are both N-th roots of x_2, up to conjugating ρ' by $A(y_2(y_2')^{-1})$, we can assume that $y_2 = y_2'$. In addition, there exist $t_1, t_3 \in \mathbb{Z}_N$ such that

$$y_1 = q^{2t_1} y_1'$$
$$y_3 = q^{2t_3} y_3'$$

Because $h = h'$, $y_2 = y_2'$, $h = y_1y_2y_3$ and $h' = y_1'y_2'y_3'$, the relation $t_1 + t_3 = 0 \in \mathbb{Z}_N$ must holds.

Now we are able to conclude that ρ and ρ' are conjugated: indeed, up to conjugate ρ' by $B_2^{t_1}$, by virtue of relation 2.2 and $t_1 + t_3 = 0$, we obtain $y_i = y_i'$ for every $i = 1, 2, 3$.

Finally, as in Proposition 2.1.3, the calculations tell us that is possible to realize every choice of values $x_i, h \in \mathbb{C}^*$ verifying $h^N = x_1x_2x_3$ as invariants. $\square$

The invariant h of an irreducible representation ρ of $\mathcal{T}_T^q$ is called the *central load* of ρ.

Remark 2.1.5. The proof of Proposition 2.1.4 shows us that every irreducible representation $\rho\colon \mathcal{T}_T^q \to \operatorname{End}(V)$ admits a basis of V in which ρ is represented as

$$\rho(X_i) = y_i\, B_i$$

with $(y_i)^N = x_i$ for $i = 1, 2, 3$ and $h = y_1y_2y_3$. Moreover, for any choice of $t_1, t_2, t_3 \in \mathbb{Z}_N$ such that $t_1 + t_2 + t_3 = 0 \in \mathbb{Z}_N$, we can conjugate ρ by a suitable linear isomorphism A, composition of the matrices B_i and their inverses, in order to obtain $A\rho(X_i)A^{-1} = y_i'B_i$, where

$$\begin{aligned} y_i' &= q^{2t_i} y_i && \text{for every } i = 1, 2, 3 \\ h' = y_1'y_2'y_3' &= q^{2(t_1+t_2+t_3)} y_1y_2y_3 = h \end{aligned} \tag{2.3}$$

Lemma 2.1.6. *Let q^2 be a primitive N-th root of unity, let $\mathcal{W}^q = \mathcal{W}^q[X_1, X_2]$ denote the algebra defined in Definition 1.2.3 and let $\mathcal{A}$ be a $\mathbb{C}$-algebra. Then every irreducible representation of $\mathcal{W}^q \otimes \mathcal{A}$ is isomorphic to the tensor product $\rho_1 \otimes \rho_2\colon \mathcal{W}^q \otimes \mathcal{A} \to \operatorname{End}(V_1 \otimes V_2)$ of two irreducible representations ρ_1 and ρ_2 of $\mathcal{W}^q$ and $\mathcal{A}$, respectively. Moreover, for every $\rho_1\colon \mathcal{W}^q \to \operatorname{End}(V_1)$ and $\rho_2\colon \mathcal{A} \to \operatorname{End}(V_2)$ irreducible representations, the tensor product $\rho_1 \otimes \rho_2$ is an irreducible representation of $\mathcal{W}^q \otimes \mathcal{A}$.*

Proof. Let $\rho\colon \mathcal{W}^q \otimes \mathcal{A} \to \operatorname{End}(W)$ be an irreducible representation of $\mathcal{W}^q \otimes \mathcal{A}$. Let y_1 be an eigenvalue of $\rho(X_1 \otimes 1)$ and W_1 its relative eigenspace. Then, with the same calculations made in the proof of Proposition 2.1.3, we observe that the element $\rho(X_2^i \otimes 1)$ carries W_1 in the eigenspace W_{i+1} of $\rho(X_1 \otimes 1)$, related to the eigenvalue $q^{2i}y_1$. Because $X_2^N \otimes 1$ belongs to the center of $\mathcal{W}^q \otimes \mathcal{A}$, its image under ρ is a scalar multiple of the identity $x_2\, id_W$, by irreducibility. In particular, $\rho(X_2 \otimes 1)$ carries W_N in W_1, as the following relation shows

$$\rho(X_2 \otimes 1)(W_N) = \rho(X_2 \otimes 1)^N(W_1) = x_2 \cdot W_1 = W_1$$

Therefore, the subspace $\bigoplus_{i=1}^N W_i$ is invariant under the action of $\mathcal{W}^q \otimes 1$. Furthermore, because $1 \otimes \mathcal{A}$ commutes with $\mathcal{W}^q \otimes 1$, this selected subspace of W is indeed ρ-invariant. Then, by irreducibility, we deduce $W = \bigoplus_{i=1}^N W_i$. More precisely, $1 \otimes \mathcal{A}$ preserves every W_i, because it commutes with $\rho(X_2 \otimes 1)$ and the subspaces W_i are eigenspaces of $\rho(X_2 \otimes 1)$. Denote by ρ_2 the restriction of ρ on the algebra $1 \otimes \mathcal{A}$, acting only on the subspace W_1. This representation is irreducible. Indeed, if there existed a proper subspace W_1' of W_1 invariant under the action of ρ_2 then, with the same observations made above, we would show that the subspace

$$\bigoplus_{i=0}^{N-1} \rho(X_2 \otimes 1)^i(W_1')$$

is ρ-invariant, which contradicts the hypothesis of irreducibility of ρ.

Fix a basis $\{v_1^1, \ldots, v_h^1\}$ of W_1 and denote by $\{v_1^i, \ldots, v_h^i\}$ the basis of W_i obtained by defining $v_j^i := \rho(X_2 \otimes 1)^{i-1}(v_j^1)$ for $i = 2, \ldots, N$. If $(e_i)_i$ is the canonical basis of $\mathbb{C}^N$, we consider the isomorphism

$$\begin{array}{rccc} L: & W & \longrightarrow & \mathbb{C}^N \otimes W_1 \\ & v_j^i & \longmapsto & e_i \otimes v_j^1 \end{array}$$

We are going to show that $L\rho L^{-1}$ is equal to the tensor product of an irreducible representation $\rho_1 \colon \mathcal{W}^q \to \mathbb{C}^N$ and of $\rho_2 \colon \mathcal{A} \to W_1$, which we have already shown to be irreducible. This fact will conclude the proof of the first part of the assertion. Now, let $v = \sum a_j v_j^1$ be an element of W_1. Observe that

$$\begin{aligned}(L \circ \rho(X_2 \otimes 1) \circ L^{-1})(e_i \otimes v) &= (L \circ \rho(X_2 \otimes 1)) \left(\sum_j a_j v_j^i \right) \\ &= L \left(\sum_j a_j v_j^{i+1} \right) \\ &= \sum_j a_j \,(e_{i+1} \otimes v_j^1) = e_{i+1} \otimes v\end{aligned}$$

Consequently, for every $U \in \mathcal{A}$, we deduce the following equalities

$$\begin{aligned}(L \circ \rho(X_1 \otimes U) \circ L^{-1})(e_i \otimes v_j^1) &= (L \circ \rho(1 \otimes U) \circ \rho(X_1 \otimes 1))(v_j^i) \\ &= (L \circ \rho(1 \otimes U))(q^{2(i-1)} y_1\, v_j^i) \\ &= (L \circ \rho(X_2^{i-1} \otimes U))(q^{2(i-1)} y_1\, v_j^1) \\ &= (L \circ \rho(X_2^{i-1} \otimes 1))(q^{2(i-1)} y_1\, \rho_2(U)(v_j^1)) \\ &= (L \circ \rho(X_2^{i-1} \otimes 1) \circ L^{-1})(q^{2(i-1)} y_1\, e_1 \otimes \rho_2(U)(v_j^1)) \\ &= q^{2(i-1)} y_1\, e_i \otimes \rho_2(U)(v_j^1)\end{aligned}$$

Analogously, we have

$$(L \circ \rho(X_2 \otimes U) \circ L^{-1})(e_i \otimes v_j^1) = e_{i+1} \otimes \rho_2(U)(v_j^1) \qquad \text{if } i = 1, \dots, N-1$$
$$(L \circ \rho(X_2 \otimes U) \circ L^{-1})(e_N \otimes v_j^1) = x_2\, e_1 \otimes \rho_2(U)(v_j^1)$$

Therefore, we have shown that ρ splits as the tensor product of ρ_2 and of a representation ρ_1 of $\mathcal{W}^q$. Observe that ρ_1 is a representation with values in $\mathbb{C}^N$, so it is necessarily irreducible, because we have shown, in Proposition 2.1.3, that every irreducible representation of $\mathcal{W}^q$ has dimension N.

Now, let $\rho_1 \colon \mathcal{W}^q \to \mathrm{End}(V_1)$ and $\rho_2 \colon \mathcal{A} \to \mathrm{End}(V_2)$ be two irreducible representations, we want to show that the tensor product $\rho := \rho_1 \otimes \rho_2$ is an irreducible representation of $\mathcal{W}^q \otimes \mathcal{A}$. We can assume that ρ_1 is in the form described in Proposition 2.1.3, relation 2.1. Let V_{y_1} be the eigenspace in V_1 of $\rho_1(X_1)$ with respect to the eigenvalue y_1. Hence $V_{y_1} \otimes V_2$ is the eigenspace of $\rho(X_1 \otimes 1)$ with respect to the eigenvalue y_1. Assume that there exists a non-trivial subspace $V' \subseteq V_1 \otimes V_2$ of $V_1 \otimes V_2$ that is ρ-invariant. In particular, it is invariant under the action of $\rho(\mathcal{W}^q \otimes 1)$, so we can find a subspace V'' of V', still invariant under $\rho(\mathcal{W}^q \otimes 1)$, such that the restriction of the representation of $\mathcal{W}^q \otimes 1$ on V'' is irreducible. Thanks to what said in the proof of Proposition 2.1.3, we can find in V'', and so in V', an eigenvector of $\rho(X_1 \otimes 1)$ with respect to the eigenvalue y_1. In particular, the subspace $W := V' \cap (V_{y_1} \otimes V_2)$ is non-zero.

On the other hand, $V_{y_1} \otimes V_2$ is invariant under the action of $\rho(1 \otimes \mathcal{A})$ because every element in $1 \otimes \mathcal{A}$ commutes with $\rho(X_1 \otimes 1)$ and $V_{y_1} \otimes V_2$ is the y_1-eigenspace of $\rho(X_1 \otimes 1)$. In addition, the representation $\rho|_{1 \otimes \mathcal{A}}$, restricted on $V_{y_1} \otimes V_2$, is isomorphic to ρ_2, because V_{y_1} has dimension 1, so it is irreducible. Therefore $W = V' \cap (V_{y_1} \otimes V_2)$ is equal to the whole $V_{y_1} \otimes V_2$ by irreducibility of ρ_2, or equivalently $V' \supseteq V_{y_1} \otimes V_2$. Because $\rho(X_2^k \otimes 1)(V_{y_1} \otimes V_2) = V_{q^{2k} y_1} \otimes V_2$, in order to be invariant V' must contains $V_{q^{2k} y_1} \otimes V_2$ for every $k \in \mathbb{Z}_N$, and so $V' = V_1 \otimes V_2$, which proves the irreducibility of ρ. □

Lemma 2.1.7. *Let $\mathcal{A}$ be a $\mathbb{C}$-algebra. Then every irreducible representation of $\mathbb{C}[Z^{\pm 1}] \otimes \mathcal{A}$ is isomorphic to the tensor product $\rho_1 \otimes \rho_2 \colon \mathbb{C}[Z^{\pm 1}] \otimes \mathcal{A} \to \mathrm{End}(V_1 \otimes V_2)$ of two irreducible representations of $\mathbb{C}[Z^{\pm 1}]$ and $\mathcal{A}$ respectively. Moreover, for every $\rho_1 \colon \mathbb{C}[Z^{\pm 1}] \to \mathrm{End}(V_1)$ and $\rho_2 \colon \mathcal{A} \to \mathrm{End}(V_2)$ irreducible representations, the tensor product $\rho_1 \otimes \rho_2$ is an irreducible representation of $\mathbb{C}[Z^{\pm 1}] \otimes \mathcal{A}$.*

Proof. It is easy to see that every irreducible representation of $\mathbb{C}[Z^{\pm 1}]$ is 1-dimensional and it is classified, up to isomorphism, by the number $z \in \mathbb{C}^*$ such that the image of Z is equal to $z\, id$. Taken $\rho \colon \mathbb{C}[Z^{\pm 1}] \otimes \mathcal{A} \to \mathrm{End}(V)$ an irreducible representation of $\mathbb{C}[Z^{\pm 1}] \otimes \mathcal{A}$, the element $Z \otimes 1$ is in the multiplicative center of the algebra, so by irreducibility there exists a scalar $z \in \mathbb{C}^*$ such that $\rho(Z \otimes 1) = z\, id_V$. Hence ρ is isomorphic to the tensor product of the evaluation homomorphism $\rho_1 \colon \mathbb{C}[Z^{\pm 1}] \to \mathbb{C}$, which carries Z in $z \in \mathbb{C}^*$, and of the representation $\frac{1}{z} \rho|_{1 \otimes \mathcal{A}}$. The second part of the assertion is obvious, because of the simple behaviour of the irreducible representations of $\mathbb{C}[Z^{\pm 1}]$. □

2.1.1 Punctured closed surfaces

Let $\rho \colon \mathcal{T}_\lambda^q \to \mathrm{End}(V)$ be an irreducible representation of the Chekhov-Fock algebra $\mathcal{T}_\lambda^q$ associated with the ideal triangulation $\lambda \in \Lambda(S)$ of a closed punctured surface S. Then every element of the monomial center $\mathcal{Z}_\lambda^q$ goes under ρ in in a scalar multiple of id_V, as seen in Remark 2.1.2. So the representation ρ induces an evaluation homomorphism, which we still denote by $\rho\colon \mathcal{Z}_\lambda^q \to \mathbb{C}^*$, sending every element Z of $\mathcal{Z}_\lambda^q$ in the number $z \in \mathbb{C}^*$ such that $\rho(Z) = z\, id_V$.

Theorem 2.1.8. *Let S be a surface, obtained by removing p punctures from a closed oriented surface $\overline{S}$ with genus g such that $\chi(S) < 0$, and let $\lambda \in \Lambda(S)$ be an ideal triangulation of S. In addition, fix $q \in \mathbb{C}^*$ such that q^2 is a primitive N-th root of unity. Then every irreducible representation $\rho\colon \mathcal{T}_\lambda^q \to \mathrm{End}(V)$ of the Chekhov-Fock algebra $\mathcal{T}_\lambda^q$ has dimension N^{3g+p-3} if N is odd, or $N^{3g+p-3}2^{-g}$ if N is even, and it is uniquely determined up to isomorphism by its induced evaluation homomorphism $\rho\colon \mathcal{Z}_\lambda^q \to \mathbb{C}^*$ on the monomial center. Moreover, every homomorphism $\rho\colon \mathcal{Z}_\lambda^q \to \mathbb{C}^*$ can be realized by a certain irreducible representation $\rho\colon \mathcal{T}_\lambda^q \to \mathrm{End}(V)$, unique up to isomorphism.*

Proof. We have shown in Theorem 1.2.4 that the Chekhov-Fock algebra $\mathcal{T}_\lambda^q$ is isomorphic to $\mathcal{W}_{k,g,p}^q$, with $k = 2g+p-3$, and that we can select an isomorphism between them sending monomials in monomials. Therefore, the assertion can be reformulated in terms of $\mathcal{W}_{k,g,p}^q$ instead of $\mathcal{T}_\lambda^q$. In light of Lemmas 2.1.6 and 2.1.7, an irreducible representation $\rho\colon \mathcal{W}_{k,g,p}^q \to \mathrm{End}(V)$ is isomorphic to a tensor product $\rho_1 \otimes \cdots \otimes \rho_{k+g+p}$, where ρ_i is an irreducible representation of $\mathcal{W}^q$ if $i = 1, \ldots, k$, of $\mathcal{W}^{q^2}$ if $i = k+1, \ldots, k+g$, of $\mathbb{C}[Z^{\pm 1}]$ if $i = k+g+1, \ldots, k+g+p$. Denote by U_i, V_i the generators of the i-th factor in the tensor product

$$\mathcal{W}_{k,g,p}^q := \bigotimes_{i=1}^{k} \mathcal{W}^q \otimes \bigotimes_{j=1}^{g} \mathcal{W}^{q^2} \otimes \bigotimes_{h=1}^{p} \mathbb{C}[Z^{\pm 1}]$$

if $i = 1, \ldots, k+g$, and by Z_{k+g+i} the generator of the i-th factor $\mathbb{C}[Z^{\pm 1}]$.

Assume that N is odd. In this case $\mathcal{W}^{q^2}$ is isomorphic to $\mathcal{W}^q$, because q^4 is sill a primitive N-th root of unity. Consequently, the elements U_i^N, V_i^N with $i = 1, \ldots, k+g$ and Z_j with $j = 1, \ldots, p$ generate the monomial center of $\mathcal{W}_{k,g,p}^q$ (it is sufficient to study $\mathrm{Ker}_N\, \bar\sigma$, where $\bar\sigma$ is the one in Theorem 1.2.2). By virtue of the analysis made concerning irreducible representations of $\mathcal{W}^q$ and $\mathbb{C}[Z^{\pm 1}]$ in Propositions 2.1.3 and 2.1.4, every irreducible factor ρ_i is determined, up to isomorphism, by the images of U_i^N, V_i^N if $i = 1, \ldots, k+g$, or Z_j otherwise. Then the isomorphism class of ρ is determined by the evaluation homomorphism $\rho\colon \mathcal{Z}_\lambda^q \to \mathbb{C}^*$. The dimension of an irreducible representation $\rho_1 \otimes \cdots \otimes \rho_{k+g+p}$ is equal to

$$N^k N^g 1^p = N^{3g+p-3}$$

If N is even, then q^4 is a primitive $N/2$-th root of unity. In this situation, the monomial center is generated by the elements U_i^N, V_i^N with $i = 1, \ldots, k$, $U_i^{N/2}, V_i^{N/2}$ with $i = k+1, \ldots, k+g$, and Z_j with $j = 1, \ldots, p$. Because of Propositions 2.1.3 and 2.1.4, every irreducible factor ρ_i is determined, up to isomorphism, by the images of the elements U_i^N, V_i^N if $i = 1, \ldots, k$, of $U_i^{N/2}, V_i^{N/2}$ if $i = k+1, \ldots, k+g$, and of Z_j otherwise. So even in this case the evaluation homomorphism on the monomial center determines, up to isomorphism, the representation ρ and its dimension is equal to

$$N^k (N/2)^g 1^p = N^{3g+p-3} 2^{-g}$$

Now we are going to prove the second part of the assertion. Assume that N is odd, with the same argument can be proved the even case too. Let $\rho\colon \mathcal{Z}_\lambda^q \to \mathbb{C}^*$ be a certain homomorphism. The map ρ induces a homomorphism ρ' on the monomial center of $\mathcal{W}_{k,g,p}^q$. Denote by $u_i, v_i \in \mathbb{C}^*$ the images under ρ' of the generators $U_i, V_i \in \mathcal{W}^q$ and by z_i the images of $Z_i \in \mathbb{C}[Z_i^{\pm 1}]$. Because of what seen in Propositions 2.1.3 and 2.1.4, we know that, for every $i = 1, \dots, k+g$, there exists an irreducible representation $\rho_i\colon \mathcal{W}^q[U_i, V_i] \to W_i$ such that $\rho_i(U_i^N) = u_i\, id_{W_i}$ and $\rho_i(V_i^N) = v_i\, id_{W_i}$. Defining $\overline{\rho}' := \rho_1 \otimes \dots \otimes \rho_{k+g+p}$, we just have to check that its evaluation homomorphism is equal to ρ', but this follows from the fact that the monomial center of $\mathcal{W}_{k,g,p}^q$ is generated, as already observed, by the elements U_i^N, V_i^N if $i = 1, \dots, k+g$ and Z_j. □

Denote by $\mathscr{R}_{irr}(\mathcal{A})$ the set of isomorphism classes of finite-dimensional irreducible representations of the algebra $\mathcal{A}$. We can summarize all the achieved results in the following theorem:

Theorem 2.1.9. *Let S be a surface, obtained by removing p punctures from a closed oriented surface $\overline{S}$ with genus g and such that $\chi(S) < 0$, and let $\lambda \in \Lambda(S)$ be an ideal triangulation of S, with n 1-cells $\lambda_1, \dots, \lambda_n$. Fix $q \in \mathbb{C}^*$ such that q^2 is a primitive N-th root of unity.*

If N is odd, $\mathscr{R}_{irr}(\mathcal{T}_\lambda^q)$ is in bijection with the elements $((x_i)_i; (p_j)_j; h)$ of $(\mathbb{C}^*)^n \times (\mathbb{C}^*)^{p-1} \times \mathbb{C}^*$ verifying:

- *for every $j = 1, \dots, p-1$ the number p_j is an N-th root of*

$$q^{-N^2 \sum_{s<t} k_{js} k_{jt} \sigma_{st}} x_1^{k_{j1}} \cdots x_n^{k_{jn}}$$

- *the number h is an N-th root of*

$$q^{-N^2 \sum_{s<t} \sigma_{st}} x_1 \cdots x_n$$

and the correspondence associates, with the isomorphism class of an irreducible representation $\rho\colon \mathcal{T}_\lambda^q \to \mathrm{End}(V)$, the element $((x_i)_i; (p_j)_j; h)$ defined by the following relations

$$\rho(X_i^N) = x_i\, id_V$$
$$\rho(P_j) = p_j\, id_V$$
$$\rho(H) = h\, id_V$$

with $i = 1, \dots, n$ and with $j = 1, \dots, p-1$.

If N is even, $\mathscr{R}_{irr}(\mathcal{T}_\lambda^q)$ is in bijection with the elements $((x_i)_i; (p_j)_j; h; (a_k)_k)$ of $(\mathbb{C}^*)^n \times (\mathbb{C}^*)^{p-1} \times \mathbb{C}^* \times (\mathbb{C}^*)^k$ verifying:

- *for every $j = 1, \dots, p-1$ the number p_j is an N-th root of*

$$q^{-N^2 \sum_{s<t} k_{js} k_{jt} \sigma_{st}} x_1^{k_{j1}} \cdots x_n^{k_{jn}}$$

- *the number h is an N-th root of*

$$q^{-N^2 \sum_{s<t} \sigma_{st}} x_1 \cdots x_n$$

- *for every $k = 1, \dots, 2g$ the number a_k is an N-th root of*

$$q^{-\frac{N^2}{4}\sum_{s<t} l_{ks} l_{kt} \sigma_{st}} x_1^{l_{k1}} \cdots x_n^{l_{kn}}$$

and the correspondence associates, with the isomorphism class of an irreducible representation $\rho\colon \mathcal{T}_\lambda^q \to \mathrm{End}(V)$, the element $((x_i)_i; (p_j)_j; h; (a_k)_k)$ defined by the following relations

$$\begin{aligned} \rho(X_i^N) &= x_i \, id_V \\ \rho(P_j) &= p_j \, id_V \\ \rho(H) &= h \, id_V \\ \rho(A_k) &= a_k \, id_V \end{aligned}$$

with $i = 1, \dots, n$, with $j = 1, \dots, p-1$ and with $k = 1, \dots, 2g$.

Proof. By virtue of Theorem 2.1.8, the set $\mathscr{R}_{irr}(\mathcal{T}_\lambda^q)$ is in bijection with the set of homomorphisms $\rho\colon \mathcal{Z}_\lambda^q \to \mathbb{C}^*$ on the monomial center of $\mathcal{T}_\lambda^q$. Thanks to Propositions 1.2.10 and 1.2.11, every homomorphism $\rho\colon \mathcal{Z}_\lambda^q \to \mathbb{C}^*$ is uniquely determined by the images of the elements X_i^N for $i = 1, \dots, n$, P_j for $j = 1, \dots, p-1$, H and, when N is even, A_k for $k = 1, \dots, 2g$. In light of the relations exposed in Propositions 1.2.10 and 1.2.11, the numbers $\rho(P_j), \rho(H)$ (and $\rho(A_k)$) are identified by a choice of an N-th (and square) roots of the values

$$q^{-\sum_{s<t} k_{js} k_{jt} \sigma_{st}} x_1^{k_{j1}} \cdots x_n^{k_{jn}} \qquad\qquad q^{-N^2 \sum_{s<t} \sigma_{st}} x_1 \cdots x_n$$

(and $q^{-\sum_{s<t} l_{ks} l_{kt} \sigma_{st}} x_1^{l_{k1}} \cdots x_n^{l_{kn}}$). Vice versa, an element $((x_i)_i; (p_j)_j; h)$ (or $((x_i)_i; (p_j)_j; h; (a_k)_k)$), verifying the conditions exposed in the assertion, induces a unique evaluation homomorphism $\rho\colon \mathcal{Z}_\lambda^q \to \mathbb{C}^*$, because of what observed in Propositions 1.2.10 and 1.2.11, and hence it determines a unique isomorphism class of irreducible representation, in light of Theorem 2.1.8. □

2.1.2 Polygons

Theorem 2.1.10. *Let S be an ideal polygon with $p \geq 3$ vertices and let $\lambda \in \Lambda(S)$ be an ideal triangulation of S. Fix $q \in \mathbb{C}^*$ such that q^2 is a primitive N-th root of unity. Then every irreducible representation $\rho\colon \mathcal{T}_\lambda^q \to \mathrm{End}(V)$ has dimension N^{p-2} and it is uniquely determined, up to isomorphism, by the induced evaluation homomorphism $\rho\colon \mathcal{Z}_\lambda^q \to \mathbb{C}^*$ on the monomial center. Moreover, every homomorphism $\rho\colon \mathcal{Z}_\lambda^q \to \mathbb{C}^*$ is realized by a certain irreducible representation $\rho\colon \mathcal{T}_\lambda^q \to \mathrm{End}(V)$, unique up to isomorphism.*

Proof. Analogous to what done in the proof of Theorem 2.1.8, using Theorem 1.3.2, Proposition 2.1.3, Lemmas 2.1.6 and 2.1.7. □

Theorem 2.1.11. *Let S be an ideal polygon with $p \geq 3$ vertices and let $\lambda \in \Lambda(S)$ be an ideal triangulation of it, with n 1-cells $\lambda_1, \dots, \lambda_n$. Fix $q \in \mathbb{C}^*$ such that q^2 is a primitive N-th root of unity. Then $\mathscr{R}_{irr}(\mathcal{T}_\lambda^q)$ is in bijection with the set of elements $((x_i)_i; h) \in (\mathbb{C}^*)^n \times \mathbb{C}^*$, where h is an N-th root of*

$$q^{-N^2 \sum_{s<t} \sigma_{st}} x_1 \cdots x_n$$

and the correspondence associates, with the isomorphism class of an irreducible representation $\rho\colon \mathcal{T}^q_\lambda \to \mathrm{End}(V)$*, the element* $((x_i)_i; h)$ *defined by the relations*

$$\rho(X_i^N) = x_i \, id_V$$
$$\rho(H) = h \, id_V$$

for $i = 1, \ldots, n$.

Proof. Analogous to what done in the proof of Theorem 2.1.9, using Theorem 2.1.10 and Proposition 1.3.3. □

2.2 Local representations

In this Subsection we will always assume that q is a primitive N-th root of $(-1)^{N+1}$.

Let R be a surface, μ an ideal triangulation of R and denote by S a certain surface obtained from R by fusion (see Chapter 0 for details), with λ the ideal triangulation of S obtained by fusion from μ. We can construct a map $j_{\mu\lambda}\colon \mathcal{H}(\lambda;\mathbb{Z}) \to \mathcal{H}(\mu;\mathbb{Z})$, which associates with each $\lambda_i \in \mathcal{H}(\lambda;\mathbb{Z})$ the vector $v_i = \sum_{j=1}^s v_{ij}\, \mu_j \in \mathcal{H}(\mu;\mathbb{Z})$, defined by

$$v_{ij} := \begin{cases} 1 & \textit{if } \mu_j \textit{ goes by fusion in } \lambda_i \\ 0 & \textit{otherwise} \end{cases}$$

It is easy to verify that map $j_{\mu\lambda}$ is an inclusion. Indeed, every element v_i has at most two non-zero components on the μ_j and, if $i \neq j$, the supports of vectors v_i and v_j are disjoint. Denote by σ and η the skew-symmetric bilinear forms on $\mathcal{H}(\lambda;\mathbb{Z})$ and $\mathcal{H}(\mu;\mathbb{Z})$, respectively, defined as in the beginning of the first Chapter. Then the following holds

$$\sigma(v, w) = \eta(j_{\mu\lambda}(v), j_{\mu\lambda}(w))$$

In order to prove it, it is sufficient to verify the equality on a set of generators, so we just need to prove that, for every i, j, we have $\sigma_{ij} = \eta(v_i, v_j)$. Recalling the definition, we can express σ_{ij} as follows

$$\sum_{\substack{T \textit{ triangle} \\ \textit{of } \lambda}} \sum_{\substack{c \textit{ spike} \\ \textit{of } T}} s_\lambda(c, \lambda_i, \lambda_j)$$

where $s_\lambda(c, \lambda_i, \lambda_j)$ is equal to $+1$ if c has λ_i on the left and λ_j on the right, -1 if c has λ_j on the left and λ_i on the right, and is equal to 0 otherwise. Suppose that the edge λ_i is the result of the identification of the edges μ_{i_1}, μ_{i_2} in μ, and that analogously λ_j is the result of the identification of the edges μ_{j_1}, μ_{j_2}. The faces of the ideal triangulation λ are in natural bijection with the faces of μ, and consequently the respective spikes too. Hence it is sufficient to prove that

$$s_\lambda(c, \lambda_i, \lambda_j) = s_\mu(c, \mu_{i_1}, \mu_{j_1}) + s_\mu(c, \mu_{i_1}, \mu_{j_2}) + s_\mu(c, \mu_{i_2}, \mu_{j_1}) + s_\mu(c, \mu_{i_2}, \mu_{j_2})$$

for every spike c of the ideal triangulation. In the right member there exists at most one non-zero term and it is immediate to see that, thanks to the coherent choice of orientations on the faces of both triangulations, the equality holds.

Denote now by $X_1, \dots, X_n$ the generators of $\mathcal{T}^q_\lambda$ associated with the edges of λ and by $Y_1, \dots, Y_s$ the ones of $\mathcal{T}^q_\mu$ associated with the edges of μ. Define the map

$$\begin{array}{rccc} \iota_{\mu\lambda}: & \mathcal{T}^q_\lambda & \longrightarrow & \mathcal{T}^q_\mu \\ & \underline{X}^\alpha & \longmapsto & \underline{Y}^{j_{\mu\lambda}(\alpha)} \end{array}$$

Using the relation 1.3 and what just shown, we can verify that

$$\begin{aligned} \iota_{\mu\lambda}(\underline{X}^\alpha \underline{X}^\beta) &= \iota_{\mu\lambda}(q^{\sigma(\alpha,\beta)}\underline{X}^{\alpha+\beta}) = q^{\sigma(\alpha,\beta)}\underline{Y}^{j_{\mu\lambda}(\alpha+\beta)} \\ &= q^{\eta(j_{\mu\lambda}(\alpha),j_{\mu\lambda}(\beta))}\underline{Y}^{j_{\mu\lambda}(\alpha)+j_{\mu\lambda}(\beta)} = \underline{Y}^{j_{\mu\lambda}(\alpha)}\underline{Y}^{j_{\mu\lambda}(\beta)} \\ &= \iota_{\mu\lambda}(\underline{X}^\alpha)\iota_{\mu\lambda}(\underline{X}^\beta) \end{aligned}$$

The injectivity of $j_{\mu\lambda}: \mathcal{H}(\lambda;\mathbb{Z}) \to \mathcal{H}(\mu;\mathbb{Z})$ immediately implies the injectivity of $\iota_{\mu\lambda}$. Hence we have proved that, if (S,λ) is obtained from (R,μ) by fusion, then there exists an inclusion $\iota_{\mu\lambda}: \mathcal{T}^q_\lambda \to \mathcal{T}^q_\mu$ between the respective Chekhov-Fock algebras.

These maps, for varying S, R, λ and μ as above, verify a sort of composition property. More precisely, assume that (S,λ), (S',λ') and (S'',λ'') are three surfaces endowed with ideal triangulations, with (S,λ) obtained from (S',λ') by fusion and with (S',λ') obtained from (S'',λ'') by fusion. Then, by definition, the homomorphisms $j_{\lambda''\lambda}: \mathcal{H}(\lambda;\mathbb{Z}) \to \mathcal{H}(\lambda'';\mathbb{Z})$ and $j_{\lambda''\lambda'} \circ j_{\lambda'\lambda}: \mathcal{H}(\lambda;\mathbb{Z}) \to \mathcal{H}(\lambda'';\mathbb{Z})$, coincide, so, on the Chekhov-Fock algebras, the following relation holds

$$\iota_{\lambda''\lambda} = \iota_{\lambda''\lambda'} \circ \iota_{\lambda'\lambda} \tag{2.4}$$

Now let us focus on a more specific situation. Given S a surface and $\lambda \in \Lambda(S)$ an ideal triangulation of it, S can be obtained by fusion from a surface S' realized by splitting S along all its internal edges. S' admits a unique ideal triangulation λ' and its Chekhov-Fock algebra associated with λ' is naturally isomorphic to

$$\mathcal{T}^q_{T_1} \otimes \cdots \otimes \mathcal{T}^q_{T_m}$$

In this case we will denote simply by ι_λ the inclusion map $\iota_{\lambda'\lambda}: \mathcal{T}^q_\lambda \to \bigotimes_i \mathcal{T}^q_{T_i}$, defined as above.

Each triangle T_i is endowed with an orientation, determined by the one on S. Order the edges $\lambda^{(i)}_1, \lambda^{(i)}_2, \lambda^{(i)}_3$ of $T_i \subseteq S'$ clockwise, as in Figure 1.4, and denote the generators of $\mathcal{T}^q_\lambda$ associated with these edges by $X^{(i)}_1, X^{(i)}_2, X^{(i)}_3$ respectively. For the sake of simplicity, given $F^{(1)} \otimes \cdots \otimes F^{(m)}$ a monomial in $\bigotimes_i \mathcal{T}^q_{T_i}$, we will omit the tensor product by those terms verifying $F^{(i)} = 1$. Recalling the definition of ι_λ, we want to give an explicit description of the image, under this inclusion, of the generators X_i associated with the edges of the ideal triangulation λ of S:

- if λ_i is a boundary edge, then it is side of a single ideal triangle T_{k_i}, so we have $\iota_\lambda(X_i) = X^{(k_i)}_{a_i} \in \bigotimes_i \mathcal{T}^q_{T_i}$, where a_i is the index of the edge of T_{k_i} identified in S with λ_i;

- if λ_i is an internal edge and it is side of two distinct triangles T_{l_i} and T_{r_i}, then we have $\iota_\lambda(X_i) = X^{(l_i)}_{a_i} \otimes X^{(r_i)}_{b_i} \in \bigotimes_i \mathcal{T}^q_{T_i}$, where a_i and b_i are the indices of the edges of T_{l_i} and T_{r_i}, respectively, identified in S with λ_i;

- if λ_i is an internal edge and it is side of a single triangle T_{k_i}, then we have $\iota_\lambda(X_i) = q^{-1} X_{a_i}^{(k_i)} X_{b_i}^{(k_i)} = q X_{b_i}^{(k_i)} X_{a_i}^{(k_i)} \in \bigotimes_i \mathcal{T}_{T_i}^q$, where a_i and b_i are the indices of the edges of T_{k_i} identified in S with λ_i and $\lambda_{a_i}^{(k_i)}$, $\lambda_{b_i}^{(k_i)}$ lie on the left and on the right, respectively, of their common spike.

Now we are ready to give the definition of local representation, whose study will be the main topic of this Section.

Definition 2.2.1. Let S be a surface and select an ideal triangulation λ of it. A *local representation* of $\mathcal{T}_\lambda^q$ is an equivalence class of m-tuples $[\rho_1, \ldots, \rho_m]$, where $\rho_j \colon \mathcal{T}_{T_j}^q \to \mathrm{End}(V_j)$ is an irreducible representation of the Chekhov-Fock algebra of the triangle T_j in the ideal triangulation λ for every $j = 1, \ldots, m$. We will say that $(\rho_1, \ldots, \rho_m)$ is *locally equivalent* to $(\rho_1', \ldots, \rho_m')$ if and only if the following relations hold:

- for every $j = 1, \ldots, m$, if ρ_j has values in $\mathrm{End}(V_j)$ and ρ_j' has values in $\mathrm{End}(V_j')$, then $V_j = V_j'$;
- for every $i = 1, \ldots, n$ we have:
  - if λ_i is a boundary edge, side of a single triangle T_{k_i}, then
  $$\rho_{k_i}(X_{a_i}^{(k_i)}) = \rho_{k_i}'(X_{a_i}^{(k_i)})$$
  where a_i is the index of the edge of T_{k_i} identified in S with λ_i;
  - if λ_i is an internal edge and it is side of two distinct triangles T_{l_i} and T_{r_i}, then there exists $t \in \mathbb{C}^*$ such that the following hold
  $$\rho_{l_i}(X_{a_i}^{(l_i)}) = t\, \rho_{l_i}'(X_{a_i}^{(l_i)})$$
  $$\rho_{r_i}(X_{b_i}^{(r_i)}) = t^{-1}\, \rho_{r_i}'(X_{b_i}^{(r_i)})$$
  where a_i and b_i are the indices of the edges of T_{l_i} and T_{r_i}, respectively, identified in S to λ_i;
  - if λ_i is an internal edge and it is side of a single triangle T_{k_i}, then there exists $t \in \mathbb{C}^*$ such that the following hold
  $$\rho_{k_i}(X_{a_i}^{(k_i)}) = t\, \rho_{k_i}'(X_{a_i}^{(k_i)})$$
  $$\rho_{k_i}(X_{b_i}^{(k_i)}) = t^{-1}\, \rho_{k_i}'(X_{b_i}^{(k_i)})$$
  where a_i and b_i are the indices of the edges of T_{k_i} identified in S with λ_i and $\lambda_{a_i}^{(k_i)}$, $\lambda_{b_i}^{(k_i)}$ lie on the left and on the right, respectively, of their common spike.

Given $[\rho_1, \ldots, \rho_m]$ a local representation of $\mathcal{T}_\lambda^q$, we can define a representation of $\mathcal{T}_\lambda^q$ as follows

$$\rho := (\rho_1 \otimes \cdots \otimes \rho_m) \circ \iota_\lambda : \mathcal{T}_\lambda^q \longrightarrow \mathrm{End}(V_1 \otimes \cdots \otimes V_m)$$

By definition of the locally equivalence relation between m-tuples of representations, this ρ does not depend on the choice of the representative of the equivalence class $[\rho_1, \ldots, \rho_m]$.

Definition 2.2.2. If S is obtained by fusion of R and $\lambda \in \Lambda(S)$ is obtained by fusion of $\mu \in \Lambda(R)$, then every local representation of $\mathcal{T}^q_\mu$ leads to a local representation of $\mathcal{T}^q_\lambda$. Indeed, denoting with $\sim_S$ and $\sim_R$ the equivalence relations that define local representations on (S, λ) and on (R, μ) respectively, then $\sim_R$ is finer than $\sim_S$. This implies in particular that a local representation $[\rho_1, \dots, \rho_m]_R$ on R with respect to the ideal triangulation μ determines a unique local representation $[\rho_1, \dots, \rho_m]_S$ of S with respect to λ, where we are implicitly using the natural identification between the triangles in λ and in μ. In this case we will say that $[\rho_1, \dots, \rho_m]_S$ is obtained from $[\rho_1, \dots, \rho_m]_R$ *by fusion* or that $[\rho_1, \dots, \rho_m]_R$ *represents* $[\rho_1, \dots, \rho_m]_S$.

Remark 2.2.3. Note that requiring that $[\rho_1, \dots, \rho_m]_R$ represents $[\rho'_1, \dots, \rho'_m]_S$ is stronger than saying that the representations

$$\rho_\lambda := (\rho'_1 \otimes \cdots \otimes \rho'_m) \circ \iota_\lambda \qquad\qquad \rho_\mu := (\rho_1 \otimes \cdots \otimes \rho_m) \circ \iota_\mu$$

are related by the relation $\rho_\mu \circ \iota_{\mu\lambda} = \rho_\lambda$. Indeed, take R equal to S_0, the surface obtained by splitting S along every edge of λ, and $\rho_1 \otimes \cdots \otimes \rho_m$, $\rho'_1 \otimes \cdots \otimes \rho'_m$ two local representations of S_0. If there are identified couples of edges that belong to the same triangle, the fact that $(\rho_1 \otimes \cdots \otimes \rho_m) \circ \iota_\lambda = (\rho'_1 \otimes \cdots \otimes \rho'_m) \circ \iota_\lambda$ does not provide sufficient conditions to show that they are equivalent (See the definition of local representation).

Definition 2.2.4. Given S a surface, $\lambda \in \Lambda(S)$ an ideal triangulation and two local representations $[\rho_1, \dots, \rho_m]$, $[\rho'_1, \dots, \rho'_m]$ of $\mathcal{T}^q_\lambda$, with

$$\begin{aligned} \rho_j &: \mathcal{T}^q_{T_j} \longrightarrow \mathrm{End}(V_j) \\ \rho'_j &: \mathcal{T}^q_{T_j} \longrightarrow \mathrm{End}(V'_j) \end{aligned}$$

we will say that $[\rho_1, \dots, \rho_m]$ and $[\rho'_1, \dots, \rho'_m]$ are *isomorphic* if there exist representatives $(\rho_1, \dots, \rho_m)$ and $(\rho'_1, \dots, \rho'_m)$ of them, respectively, and there exist linear isomorphisms $L_j \colon V_j \to V'_j$ such that, for every $j = 1, \dots, m$ and for every $X \in \mathcal{T}^q_{T_j}$, we have

$$L_j \circ \rho_j(X) \circ L_j^{-1} = \rho'_j(X)$$

Assume that $[\rho_1, \dots, \rho_m]$ and $[\rho'_1, \dots, \rho'_m]$ are isomorphic and let $(\rho_1, \dots, \rho_m)$ and $(\rho'_1, \dots, \rho'_m)$ be two representatives of them such that there exist linear isomorphisms $L_j \colon V_j \to V'_j$ with $L_j \circ \rho_j \circ L_j^{-1} = \rho'_j$. Then, for any other choice of a representative $(\bar{\rho}_1, \dots, \bar{\rho}_m)$ of $[\rho_1, \dots, \rho_m]$, the m-tuple of representations $\bar{\rho}'_j := L_j \circ \bar{\rho}_j \circ L_j^{-1}$ is $\sim_S$-locally equivalent to $(\rho'_1, \dots, \rho'_m)$. From this fact immediately follows that the isomorphism relation defined above is indeed an equivalence relation.

Lemma 2.2.5. *Let $[\rho_1, \dots, \rho_m]$ be a local representation of $\mathcal{T}^q_\lambda$. Then, for every generator $X_i \in \mathcal{T}^q_\lambda$ associated with the edge λ_i, the representation $\rho := (\rho_1 \otimes \cdots \otimes \rho_m) \circ \iota_\lambda$ verifies*

$$\rho(X_i^N) = x_i \, id_{V_1 \otimes \cdots \otimes V_m}$$

for a certain $x_i \in \mathbb{C}^*$. In addition, there exists $h \in \mathbb{C}^*$ such that

$$\rho(H) = h \, id_{V_1 \otimes \cdots \otimes V_m}$$

Proof. Consider the following cases:

- if λ_i is a boundary edge, side of a single triangle T_{k_i}, we have $\iota_\lambda(X_i^N) = (X_{a_i}^{(k_i)})^N \in \bigotimes_i \mathcal{T}_{T_i}^q$, where a_i is the index of the edge of T_{k_i} identified in S with λ_i. Therefore $\rho(X_i^N) = \rho_{k_i}((X_{a_i}^{(k_i)})^N) = x_{a_i}^{(k_i)}\, id_{V_1\otimes\cdots\otimes V_m}$, thanks to what observed about irreducible representations of $\mathcal{T}_{T_{k_i}}^q$.
- if λ_i is an internal edge, side of two distinct triangles T_{l_i} and T_{r_i}, then $\iota_\lambda(X_i) = X_{a_i}^{(l_i)} \otimes X_{b_i}^{(r_i)} \in \bigotimes_i \mathcal{T}_{T_i}^q$, where a_i and b_i are the indices of the edges of T_{l_i} and T_{r_i}, respectively, identified in S with λ_i. So
$$\rho(X_i^N) = \rho_{l_i}((X_{a_i}^{(l_i)})^N) \otimes \rho_{r_i}((X_{b_i}^{(r_i)})^N) = x_{a_i}^{(l_i)} x_{b_i}^{(r_i)}\, id_{V_1\otimes\cdots\otimes V_m}$$
- if λ_i is an internal edge, side of a single triangle T_{k_i}, then $\iota_\lambda(X_i) = q^{-1}X_{a_i}^{(k_i)}X_{b_i}^{(k_i)} = qX_{b_i}^{(k_i)}X_{a_i}^{(k_i)} \in \bigotimes_i \mathcal{T}_{T_i}^q$, where a_i and b_i are the indices of the edges of T_{k_i} identified in S with λ_i and $\lambda_{a_i}^{(k_i)}$, $\lambda_{b_i}^{(k_i)}$ lie on the left side and on the right side, respectively, of their common spike. So
$$\begin{aligned}\rho(X_i^N) &= \rho_{k_i}(q^{-N}(X_{a_i}^{(k_i)}X_{b_i}^{(k_i)})^N)\\ &= \rho_{k_i}(q^{-N-\frac{N(N-1)}{2}2\sigma_{a_ib_i}}(X_{a_i}^{(k_i)})^N(X_{b_i}^{(k_i)})^N)\\ &= q^{-N^2}x_{a_i}^{(k_i)}x_{b_i}^{(k_i)}\, id_{V_1\otimes\cdots\otimes V_m}\\ &= x_{a_i}^{(k_i)}x_{b_i}^{(k_i)}\, id_{V_1\otimes\cdots\otimes V_m}\end{aligned}$$
where we have used $q^N = (-1)^{N+1}$ in the last equality.

For what concerns the second part of the assertion, we observe that, denoting by $j_\lambda\colon \mathcal{H}(\lambda;\mathbb{Z}) \to \mathcal{H}(\lambda';\mathbb{Z})$ the map through which is defined ι_λ, we have
$$\mathcal{H}(\lambda;\mathbb{Z}) \cong \mathbb{Z}^n \ni (1,\dots,1) \overset{j_\lambda}{\longmapsto} (1,\dots,1) \in \mathbb{Z}^{3m} \cong \mathcal{H}(\lambda';\mathbb{Z})$$
Label as $H_i \in \mathcal{T}_{T_i}^q$ the element $q^{-1}X_1^{(i)}X_2^{(i)}X_3^{(i)}$. Because each H_i, as element of $\bigotimes_i \mathcal{T}_{T_i}^q$, lies in the multiplicative center, it is simple to verify that
$$H_1\otimes\cdots\otimes H_m = \underline{Y}^{(1,\dots,1)} \in \bigotimes_{i=1}^m \mathcal{T}_{T_i}^q$$
Hence we conclude
$$\iota_\lambda(H) = \iota_\lambda(\underline{X}^{(1,\dots,1)}) = \underline{Y}^{(1,\dots,1)} = H_1\otimes\cdots\otimes H_m \tag{2.5}$$
and then
$$\rho(H) = \rho_1(H_1)\otimes\cdots\otimes\rho_m(H_m) = h_1\cdots h_m\, id_{V_1\otimes\cdots\otimes V_m}$$
as desired. □

We will designate the number $h \in \mathbb{C}^*$ as the *central load* of the local representation ρ.

Denote by $\mathscr{R}_{loc}(\mathcal{T}_\lambda^q)$ the set of isomorphism classes, as local representations, of local representations of $\mathcal{T}_\lambda^q$.

Theorem 2.2.6. *Let S be a surface (see Chapter 0 for details) and fix $q \in \mathbb{C}^*$ a primitive N-th root of $(-1)^{N+1}$. Then the set $\mathscr{R}_{loc}(\mathcal{T}^q_\lambda)$ is in bijection with the set of elements $((x_i)_i; h)$ in $(\mathbb{C}^*)^n \times \mathbb{C}^*$ verifying*

$$h^N = x_1 \cdots x_n$$

and the correspondence associates with the isomorphism class as local representations of $[\rho_1, \ldots, \rho_m]$, the element $((x_i)_i; h)$ defined by the relations

$$\begin{aligned} \rho(X_i^N) &= x_i \, id_{V_1 \otimes \cdots \otimes V_m} \\ \rho(H) &= h \, id_{V_1 \otimes \cdots \otimes V_m} \end{aligned}$$

for $i = 1, \ldots, n$, where $\rho = (\rho_1 \otimes \cdots \otimes \rho_m) \circ \iota_\lambda$.

Proof. The map $[\rho_1, \ldots, \rho_m] \longmapsto ((x_i)_i; h)$ is clearly well defined on the isomorphism classes, we firstly want to show its injectivity. Let $[\rho_1, \ldots, \rho_m]$, $[\rho'_1, \ldots, \rho'_m]$ be two local representations with the same invariants $((x_i)_i; h)$. Select two representatives $(\rho_1, \ldots, \rho_m)$, $(\rho'_1, \ldots, \rho'_m)$ of them, with $\rho_j \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_j)$, $\rho'_j \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V'_j)$. Up to conjugating the representations ρ_j and ρ'_j and recalling the notations introduced in the proof of Proposition 2.1.4, we can assume that the following relations hold:

- for every $j = 1, \ldots, m$ we have $V_j = V'_j = \mathbb{C}^N$;
- for every $j = 1, \ldots, m$ the representations ρ_j and ρ'_j have the following form

$$\begin{aligned} \rho_j(X_h^{(j)}) &= y_h^{(j)} \, B_h && \text{for } h = 1, 2, 3 \\ \rho'_j(X_h^{(j)}) &= z_h^{(j)} \, B_h && \text{for } h = 1, 2, 3 \end{aligned}$$

  for certain $y_h^{(j)}, z_h^{(j)} \in \mathbb{C}^*$, where $X_h^{(j)}$ denote the generators of the algebra $\mathcal{T}^q_{T_j}$.

Let us find the relations holding between $y_i^{(j)}, z_i^{(j)}$ and the invariants x_i, h. In the notations introduced in the proof of Lemma 2.2.5, because $B_h^N = I$, we have

- if λ_i is a boundary edge, then

$$\begin{aligned} \rho(X_i^N) &= \rho_{k_i}((X_{a_i}^{(k_i)})^N) \\ &= (y_{a_i}^{(k_i)})^N \, id_{V_1 \otimes \cdots \otimes V_m} \\ &= x_i \, id_{V_1 \otimes \cdots \otimes V_m} \end{aligned}$$

- if λ_i is an internal edge and it is side of two distinct triangles T_{l_i} and T_{r_i}, then

$$\begin{aligned} \rho(X_i^N) &= \rho_{l_i}((X_{a_i}^{(l_i)})^N) \otimes \rho_{r_i}((X_{b_i}^{(r_i)})^N) \\ &= (y_{a_i}^{(l_i)} y_{b_i}^{(r_i)})^N \, id_{V_1 \otimes \cdots \otimes V_m} \\ &= x_i \, id_{V_1 \otimes \cdots \otimes V_m} \end{aligned}$$

- if λ_i is an internal edge and it is side of a single triangle T_{k_i}, then

$$\begin{aligned}
\rho(X_i^N) &= \rho_{k_i}(q^{-N}(X_{a_i}^{(k_i)}X_{b_i}^{(k_i)})^N) \\
&= \rho_{k_i}(q^{-N-\frac{N(N-1)}{2}2\sigma_{a_ib_i}}(X_{a_i}^{(k_i)})^N(X_{b_i}^{(k_i)})^N) \\
&= q^{-N^2}(y_{a_i}^{(k_i)}y_{b_i}^{(k_i)})^N\, id_{V_1\otimes\cdots\otimes V_m} \\
&= (y_{a_i}^{(k_i)}y_{b_i}^{(k_i)})^N\, id_{V_1\otimes\cdots\otimes V_m} \\
&= x_i\, id_{V_1\otimes\cdots\otimes V_m}
\end{aligned}$$

With the same calculations for $(\rho'_1, \dots, \rho'_m)$, we find the following identities

$$\begin{aligned}
(y_{a_i}^{(l_i)}y_{b_i}^{(r_i)})^N &= x_i = x'_i = (z_{a_i}^{(l_i)}z_{b_i}^{(r_i)})^N \\
(y_{a_i}^{(k_i)})^N &= x_i = x'_i = (z_{a_i}^{(k_i)})^N
\end{aligned} \tag{2.6}$$

depending on whether λ_i is internal or in the boundary. Furthermore, recalling the relation 2.5, we deduce also that

$$\prod_{j=1}^m y_1^{(j)}y_2^{(j)}y_3^{(j)} = h = h' = \prod_{j=1}^m z_1^{(j)}z_2^{(j)}z_3^{(j)} \tag{2.7}$$

The equations 2.6 tell us that, for every $i = 1, \dots, n$, there exists a $t_i \in \mathbb{Z}_N$ such that

$$\begin{aligned}
y_{a_i}^{(l_i)}y_{b_i}^{(r_i)} &= q^{2t_i}z_{a_i}^{(l_i)}z_{b_i}^{(r_i)} \\
y_{a_i}^{(k_i)} &= q^{2t_i}z_{a_i}^{(k_i)}
\end{aligned}$$

depeding on whether λ_i is internal or in the boundary. Moreover, from relation 2.7, we observe

$$\prod_{j=1}^m y_1^{(j)}y_2^{(j)}y_3^{(j)} = \prod_{j=1}^m z_1^{(j)}z_2^{(j)}z_3^{(j)} = q^{-2\sum_{i=1}^n t_i}\prod_{j=1}^m y_1^{(j)}y_2^{(j)}y_3^{(j)} \tag{2.8}$$

So we must have $\sum_{i=1}^n t_i = 0 \in \mathbb{Z}_N$. Our purpose is now to prove that, up to conjugating in a suitable way the representations ρ'_j, we can assume that $t_i = 0$ for every $i = 1, \dots, n$. Temporarily accepting this fact, let us conclude the prove of the injectivity of the map. If the relation $y_{a_i}^{(l_i)}y_{b_i}^{(r_i)} = z_{a_i}^{(l_i)}z_{b_i}^{(r_i)}$ holds for every internal λ_i and $y_{a_i}^{(l_i)} = z_{a_i}^{(r_i)}$ for every λ_i in the boundary, then the n-tuples of representations $(\rho_1, \dots, \rho_m)$ and $(\rho'_1, \dots, \rho'_m)$ are locally equivalent. Indeed, fixed $i = 1, \dots, n$ with λ_i internal, we observe

$$\begin{aligned}
\rho_{l_i}(X_{a_i}^{(l_i)}) &= y_{a_i}^{(l_i)}\, B_{a_i} = \frac{y_{a_i}^{(l_i)}}{z_{a_i}^{(l_i)}} z_{a_i}^{(l_i)}\, B_{a_i} = \frac{y_{a_i}^{(l_i)}}{z_{a_i}^{(l_i)}}\, \rho'_{l_i}(X_{a_i}^{(l_i)}) = \alpha_i\, \rho'_{l_i}(X_{a_i}^{(l_i)}) \\
\rho_{r_i}(X_{b_i}^{(r_i)}) &= y_{b_i}^{(r_i)}\, B_{b_i} = \frac{y_{b_i}^{(r_i)}}{z_{b_i}^{(r_i)}} z_{b_i}^{(r_i)}\, B_{b_i} = \frac{y_{b_i}^{(r_i)}}{z_{b_i}^{(r_i)}}\, \rho'_{r_i}(X_{b_i}^{(r_i)}) = \beta_i\, \rho'_{r_i}(X_{b_i}^{(r_i)})
\end{aligned}$$

Assuming $y_{a_i}^{(l_i)} y_{b_i}^{(r_i)} = z_{a_i}^{(l_i)} z_{b_i}^{(r_i)}$, we obtain exactly $\alpha_i = \beta_i^{-1}$. If λ_i is in the boundary, the relation is obviously verified, so we can conclude that the n-tuples of representations are locally equivalent.

We just need to prove that, by conjugating the representations ρ'_j, we can suppose that for every $i = 1, \ldots, n$ we have $t_i = 0$ (observe that conjugating the representations ρ'_j we do not change the isomorphism class, as local representation, of ρ'). Recall that, as observed in Remark 2.1.5, for every choice of $u_1, u_2, u_3 \in \mathbb{Z}_N$ with $u_1 + u_2 + u_3 = 0$ and for every $j = 1, \ldots, m$, we can conjugate the representation ρ'_j in order to change the numbers $z_i^{(j)}$ as follows

$$\begin{aligned} z_1^{(j)} &\longrightarrow \bar{z}_1^{(j)} = q^{2u_1} z_1^{(j)} \\ z_2^{(j)} &\longrightarrow \bar{z}_2^{(j)} = q^{2u_2} z_2^{(j)} \\ z_3^{(j)} &\longrightarrow \bar{z}_3^{(j)} = q^{2u_3} z_3^{(j)} \end{aligned} \tag{2.9}$$

The vector $t = (t_1, \ldots, t_n)$ can be thought as an element of $\mathcal{H}(\lambda; \mathbb{Z})$. Define

$$M := \left\{ t = (t_1, \ldots, t_n) \in \mathcal{H}(\lambda; \mathbb{Z}) \,\middle|\, \sum_{i=1}^{n} t_i = 0 \right\}$$

$$M' := \langle \lambda_i - \lambda_j \in \mathcal{H}(\lambda; \mathbb{Z}) \mid \lambda_i, \lambda_j \textit{ sides of a common triangle } \rangle$$

The submodule M' is obviously contained in M, because each generator of M' belongs to M. Assume that the equality holds, i. e. $M = M'$. Then, t could be expressed as a $\mathbb{Z}$-linear combination $\sum_{s,t} c_{st} (\lambda_s - \lambda_t)$ in M'. Taken λ_s, λ_t a couple of edges that are sides of a common triangle T_j then, by conjugating the representation ρ_j as in relation 2.9 with a suitable choice of u_i, we can obtain a new m-tuple of representations such that its corresponding vector t differ from the previous one by $c_{st} (\lambda_s - \lambda_t)$. In particular, iterating this process, we could reduce the proof to the case $t = 0$ and so conclude.

Let $\Gamma = \Gamma_{S,\lambda}$ be the dual graph of the ideal triangulation λ of S (see Chapter 0 for details about notations) and select A a tree in Γ maximal with respect to the inclusion. Let λ_i be a 1-cell of λ that does not correspond to any element of the 1-skeleton $A^{(1)}$ of A. Then we have two possibilities:

- λ_i lies in the boundary, so it is side of a single triangle T_k in λ;
- λ_i is an internal edge, so it is side of one or two triangles in λ. Denote by T_k one of these triangles, as in the previous case;

In both cases, the vertex T_k^* must belong to A by maximality. The triangle T_k has λ_i as a side and we can assume that it has at least another side λ_j that is internal and corresponding to a 1-cell λ_j^* in Γ belonging to $A^{(1)}$ (if there did not be any such λ_j, then the surface S would be a single triangle and this situation has already been proved in Proposition 2.1.4). Taking $t' := t - t_i (\lambda_i - \lambda_j)$, we obtain another element of M verifying: $t'_h = t_h$ if $h \notin \{i, j\}$ and $t'_i = 0$. So we have modified t on the component corresponding to $\lambda_j^* \in A^{(1)}$ obtaining another element of M that has the i-th component equal to zero. Iterating this procedure, we can assume that t has support contained in the set of internal 1-cells of λ corresponding to $A^{(1)}$, the 1-skeleton of the fixed tree A.

Now we take a leaf T_s^* of A, which is a vertex with valence equal to 1. Because A is a tree, there exists at least one leaf. The vertex T_s^* is an end of

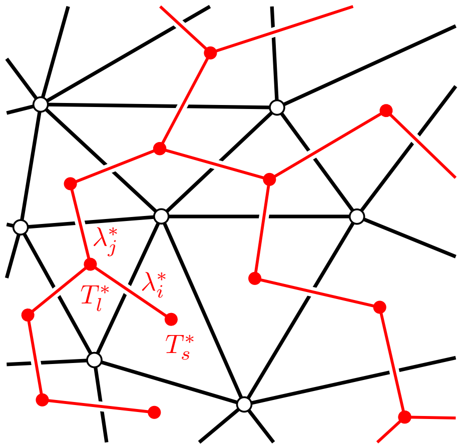

Figure 2.1: A leaf v_k in the tree A

a single 1-cell λ_i^* in A. If the other end $T_l^* \neq T_s^*$ of λ_i^* is not a vertex of other 1-cells in $A^{(1)}$, then the tree is composed of the vertices T_s^*, T_l^* and the single 1-cell λ_i^* between them. Because t has support in $A^{(1)}$ and $\sum_l t_l = 0$, we must have $t = 0$, so we conclude.

Therefore, we assume that there exists another 1-cell $\lambda_j^* \neq \lambda_i^*$ in $A^{(1)}$ having T_l^* as end. Now consider $t' := t - t_i(\lambda_i - \lambda_j)$. This element has support in $A^{(1)} \setminus \{\lambda_i^*\}$, because we have erased the component related to λ_i possibly by modifying the one on λ_j, where $\lambda_j^* \in A^{(1)} \setminus \{\lambda_i^*\}$. The subgraph A' of Γ, defined by $(A')^{(0)} := A^{(0)} \setminus \{T_s^*\}$ and $(A')^{(1)} := A^{(1)} \setminus \{\lambda_i^*\}$, is still a tree, so we can repeat the procedure just described on A'. After a number of these steps bounded by n, the number of edges in λ, we will have reduced the support of t enough to obtain $t = 0$.

We have shown the injectivity of the map associating with every isomorphism class of local representation the collection of its invariants. The surgectivity is simpler. Select numbers $x_i, h \in \mathbb{C}^*$, with $h^N = x_1 \cdots x_n$. For every $i = 1, \ldots, n$ we choose a N-th root y_i of x_i so that the relation $h = y_1 \cdots y_n$ holds. We also choose a square root $y_i^{1/2}$ of y_i for every i such that λ_i is an internal edge. Fixed T_h a triangle of λ, assume that its edges are labelled as $\lambda_i, \lambda_j, \lambda_k$ clockwise, possibly with coincidences. Now define $\rho_h \colon \mathcal{T}^q_{T_h} \to \mathrm{End}(\mathbb{C}^N)$ as follows

$$\rho_h(X_1^{(h)}) := y_i^{1/2} B_1$$
$$\rho_h(X_2^{(h)}) := y_j^{1/2} B_2$$
$$\rho_h(X_3^{(h)}) := y_k^{1/2} B_3$$

if all the edges $\lambda_i, \lambda_j, \lambda_k$ are internal, otherwise replace the number $y_t^{1/2}$ with y_t if λ_t is in the boundary, for every $t \in \{i, j, k\}$. By inspection, it is easy to see that the invariants of the local representation $[\rho_1, \ldots, \rho_m]$, defined in this way, are exactly the numbers $x_i, h \in \mathbb{C}^*$ previously selected. □

Corollary 2.2.7. *Let $[\rho_1, \ldots, \rho_m]$ and $[\rho'_1, \ldots, \rho'_m]$ be two local representations of $\mathcal{T}^q_\lambda$. Then they are isomorphic to each other as local representations if and* *only if* $\rho := (\rho_1 \otimes \cdots \otimes \rho_m) \circ \iota_\lambda$ *and* $\rho' := (\rho'_1 \otimes \cdots \otimes \rho'_m) \circ \iota_\lambda$ *are isomorphic as representations.*

Proof. Obviously, if $[\rho_1, \ldots, \rho_m]$ and $[\rho'_1, \ldots, \rho'_m]$ are isomorphic as local representations, then ρ e ρ' are isomorphic, in particular through a tensor-split linear isomorphism $L_1 \otimes \cdots \otimes L_m$. Vice versa, if ρ and ρ' are isomorphic, then $x_i = x'_i$, $h = h'$. So, thanks to Theorem 2.2.6, $[\rho_1, \ldots, \rho_m]$ and $[\rho'_1, \ldots, \rho'_m]$ are isomorphic as local representations. □

Proposition 2.2.8. *Let* S *be an ideal polygon with* $p \geq 3$ *vertices. Then, for every* λ *triangulation of* S *and for every local representation* $[\rho_1, \ldots, \rho_m]$ *of the Chekhov-Fock algebra* $\mathcal{T}^q_\lambda$*, the representation* $\rho := (\rho_1 \otimes \cdots \otimes \rho_m) \circ \iota_\lambda$ *is irreducible.*

Proof. It is sufficient to observe that the representation ρ has dimension N^m, just as any irreducible representation, as seen in Theorem 2.1.11. Therefore, if ρ had a proper invariant subspace $0 \subsetneq W \subsetneq V_1 \otimes \cdots \otimes V_m$, then we would find an irreducible representation of $\mathcal{T}^q_\lambda$ with dimension strictly lower than N^m, which is absurd by virtue of Theorem 2.1.11. □

CHAPTER 3

The quantum Teichmüller space

In this Chapter we deal with the construction of an intrinsic algebraic object, depending only on the surface S, starting from the collection of the Chekhov-Fock algebras $\{\mathcal{T}^q_\lambda\}_{\lambda\in\Lambda(S)}$. The most important instrument of this procedure is a family of isomorphisms $\Phi^q_{\lambda\lambda'}$ between the fraction rings $\widehat{\mathcal{T}}^q_{\lambda'}$ and $\widehat{\mathcal{T}}^q_\lambda$ of $\mathcal{T}^q_{\lambda'}$ and $\mathcal{T}^q_\lambda$, respectively. These isomorphisms can be thought as non-commutative deformations of the coordinate changes of the classical Teichmüller space, induced by different choices of ideal triangulations.

Let us try to be more detailed on this fact. Given S a closed punctured surface (with $\chi(S) < 0$) and $g \in \mathcal{T}(S)$ a complete hyperbolic metric of S, the ends of (S, g) can be finite-area cusps, bounded by horocycles, or funnels, bounded by a simple closed geodesic. If $\mathrm{Conv}(S, g)$ denotes the *convex core* of (S, g) (see [Rat06] for the definition), then the cusps ends are the ones contained in $\mathrm{Conv}(S, g)$, whereas the funnels are in bijection with the simple closed geodesics in $\partial \mathrm{Conv}(S, g)$. We can endow the metric $g \in \mathcal{T}(S)$ with an additional choice of orientations on the simple closed geodesics in $\partial \mathrm{Conv}(S, g)$. The set of these enhanced isotopy classes of metrics is called the *enhanced Teichmüller space* of S and it is denoted by $\widetilde{\mathcal{T}}(S)$. This space can be endowed with a natural topology, simply by asking that the forgetful map $\widetilde{\mathcal{T}}(S) \longrightarrow \mathcal{T}(S)$ is a branched covering. Now, given $\lambda \in \Lambda(S)$ an ideal triangulation and $x_i \in \mathbb{R}_+$ real positive numbers, one for each edge λ_i of λ, we can define a hyperbolic metric $\psi_\lambda((x_i)_i)$ on S setting on each triangle an hyperbolic structure and defining the coordinate changes in terms of this choice of parameters $(x_i)_i$. The numbers $(x_i)_i$ are called the *shear coordinates* of the hyperbolic metric $\psi_\lambda((x_i)_i)$: each number $x_i \in \mathbb{R}_+$ measures how the triangles are slided along λ_i (we refer to [Bon96] for details). This give us a possibly uncompleted metric, but its completion, which is homeomorphic to S, can be identified with the enhanced convex core of a certain enhanced complete metric on S (see [Liu09] for details). In this way we can define a map $\psi_\lambda \colon \mathbb{R}^n_+ \to \widetilde{\mathcal{T}}(S)$, which turns out to be a homeomorphism between these topological spaces. These applications are called the exponential shear parametrizations of $\widetilde{\mathcal{T}}(S)$. The main point is that the coordinate changes $\psi^{-1}_{\lambda'} \circ \psi_\lambda$ turn out to be rational and the isomorphisms $\Phi^q_{\lambda\lambda'}$ are constructed in order to verify, in the commutative case $q = 1$, the identity

$$\Phi^1_{\lambda\lambda'}(x'_i) = (\psi^{-1}_{\lambda'} \circ \psi_\lambda)(x_i) \tag{3.1}$$

The first part of our work in this Chapter is devoted to a proof of the existence of such isomorphisms, involving less case-to-case arguments than the original one exposed in [Liu09], following a suggestion in [BBL07].

Later, we give a notion of finite dimensional representations of the quantum Teichmüller space, in particular we study its irreducible and local representations. This part requests some efforts because of certain technical obstructions, due to the non-commutativity of the Chekhov-Fock algebras and the fact that the isomorphisms are not defined on the rings $\mathcal{T}^q_\lambda$ but only on their fraction rings. Once we have this notion, we envelop a classification statement for local and irreducible representations. The resulting invariants are complex non-zero scalars x_i associated with the edges λ_i of the ideal triangulations λ, together with a choice of N-th roots of certain functions of them. A crucial point is that the numbers $x_i \in \mathbb{C}^*$ associated with a certain λ are related to the ones of another triangulation λ' by the isomorphisms $\Phi^1_{\lambda\lambda'}$. This fact, together with the relation 3.1, implies that the algebraic invariants of local (or irreducible) representations can be related with a more geometric object, in particular with a *peripherally generic enhanced homomorphism*. This class of homomorphisms, from the fundamental group of S to the orientation-preserving isometries of the 3-dimensional hyperbolic space, contains a huge set of geometrically meaningful homomorphisms, like the injective ones. In order to show this link, we need to define the notion of pleated surface, which makes the bridge between the algebraic and the geometric aspects. All these facts are exposed in [BL07] and [BBL07], focusing on the irreducible and local case, respectively.

In the last part, we describe the irreducible decomposition of a local representation of a closed punctured surface, result due to Toulisse [Tou14].

3.1 Coordinate change isomorphisms

We have already observed that the Chekhov-Fock algebra is a bilateral Noetherian integral domain and so, by virtue of Proposition A.14, it is a Ore integral domain. Therefore, we can construct $\widehat{\mathcal{T}}^q_\lambda$, the classical right quotient ring of $\mathcal{T}^q_\lambda$, as in Remark A.6.

Recall what we have described in Chapter 0 about operations on ideal triangulations.

Theorem 3.1.1. *Let S be a surface (see Chapter 0 for details). Then there exists a unique family $(\Phi^q_{\lambda\lambda'})_{\lambda,\lambda'}$, where λ and λ' are varying in the set of all ideal triangulations of S and $\Phi^q_{\lambda\lambda'}\colon \widehat{\mathcal{T}}^q_{\lambda'} \to \widehat{\mathcal{T}}^q_\lambda$ are algebra isomorphisms, which satisfies the following properties:*

Composition relation: *for every $\lambda, \lambda', \lambda'' \in \Lambda(S)$ ideal triangulations, we have*

$$\Phi^q_{\lambda\lambda''} = \Phi^q_{\lambda\lambda'} \circ \Phi^q_{\lambda'\lambda''}$$

Re-indexing: *if $\lambda' = \alpha(\lambda)$, then $\Phi^q_{\lambda\lambda'}(X_i) = X_{\alpha(i)}$;*

Naturality: *let $\varphi\colon S \to R$ be a diffeomorphism that sends the triangulations $\lambda, \lambda' \in \Lambda(S)$ in $\mu, \mu' \in \Lambda(R)$, respectively. It induces an isomorphism $\hat{\varphi}^q_{\mu\lambda}\colon \widehat{\mathcal{T}}^q_\lambda \to \widehat{\mathcal{T}}^q_\mu$, defined by sending each generator X_i of $\mathcal{T}^q_\lambda$, associated with the edge λ_i of λ, in the generator Y_i of $\mathcal{T}^q_\mu$, associated with the edge $\varphi(\lambda_i) = \mu_i$ of μ, and extended to the quotient rings. Analogously we* *define $\hat{\varphi}_{\mu'\lambda'}$. Then, the isomorphisms $\Phi^q_{\mu\mu'}$ and $\Phi^q_{\lambda\lambda'}$ respect the following relation*

$$\Phi^q_{\mu\mu'} \circ \hat{\varphi}^q_{\mu'\lambda'} = \hat{\varphi}^q_{\mu\lambda} \circ \Phi^q_{\lambda\lambda'}$$

Disjoint Union: *let S be the disjoint union of S_1 and S_2, let λ_1, λ'_1 be triangulations of S_1 and λ_2, λ'_2 triangulations of S_2. Setting $\lambda := \lambda_1 \sqcup \lambda_2$ and $\lambda' = \lambda'_1 \sqcup \lambda'_2$, with $\lambda, \lambda' \in \Lambda(S)$, the isomorphism $\Phi^q_{\lambda\lambda'}$ is the extension to $\widehat{T}^q_{\lambda'}$ of the following algebra homomorphism*

$$\mathcal{T}^q_{\lambda'} = \mathcal{T}^q_{\lambda'_1} \otimes \mathcal{T}^q_{\lambda'_2} \longrightarrow \widehat{\mathcal{T}}^q_{\lambda'_1} \otimes \widehat{\mathcal{T}}^q_{\lambda'_2} \xrightarrow{\Phi^q_{\lambda_1\lambda'_1} \otimes \Phi^q_{\lambda_2\lambda'_2}} \widehat{\mathcal{T}}^q_{\lambda_1} \otimes \widehat{\mathcal{T}}^q_{\lambda_2} \longrightarrow \widehat{\mathcal{T}}^q_{\lambda}$$

where the first and the third maps are the natural inclusions.

Fusion: *if S is obtained by fusing a surface R and if $\lambda, \lambda' \in \Lambda(S)$ are obtained by fusing $\mu, \mu' \in \Lambda(R)$ respectively, then*

$$\hat{\iota}_{\mu\lambda} \circ \Phi^q_{\lambda\lambda'} = \Phi^q_{\mu\mu'} \circ \hat{\iota}_{\mu'\lambda'}$$

where $\hat{\iota}_{\mu\lambda}$ and $\hat{\iota}_{\mu'\lambda'}$ are the inclusions induced on the quotient rings by the maps $\iota_{\mu\lambda}$ and $\iota_{\mu'\lambda'}$ defined in Section 2.2;

Diagonal Exchange: *let $S = Q$ be the ideal square and let $\lambda, \lambda' \in \Lambda(Q)$ be the two possible ideal triangulations of Q, with edges labelled as in Figure 3.1. Then*

$$\begin{aligned}
\Phi^q_{\lambda\lambda'}(X'_i) &= X_i^{-1} \\
\Phi^q_{\lambda\lambda'}(X'_j) &= (1 + qX_i)X_j \\
\Phi^q_{\lambda\lambda'}(X'_k) &= (1 + qX_i^{-1})^{-1}X_k \\
\Phi^q_{\lambda\lambda'}(X'_l) &= (1 + qX_i)X_l \\
\Phi^q_{\lambda\lambda'}(X'_m) &= (1 + qX_i^{-1})^{-1}X_m
\end{aligned}$$

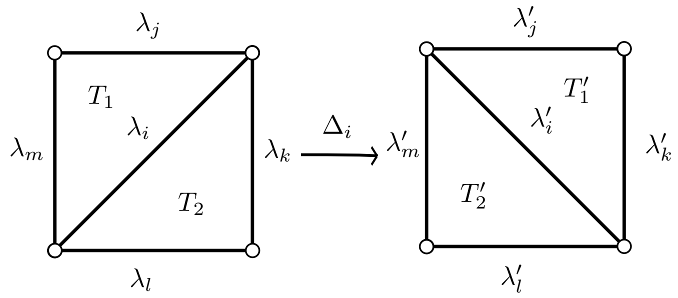

Figure 3.1: The ideal triangulation λ, λ', respectively, of Q

Proof. In what follows, we will ignore the re-indexing property, because from the construction it is quite clear that this relation holds, but it would be very annoying to carry on all the indices in order to verify it.

Given S a surface like above, we will firstly define the isomorphisms $\Phi^q_{\lambda\lambda'}$ in the case in which λ and λ' differ by an elementary move. We need to introduce some notations: we denote by Q' the square in S in which there is the diagonal exchange and we designate T_1, T_2 and T'_1, T'_2 the triangles in λ and λ', respectively, that compose the square Q'. Furthermore, let S_0 be the surface obtained from S by splitting it along all the edges of λ (or λ') except for the diagonal λ_i of Q' (or λ'_i). Then S_0 is the disjoint union of an embedded square Q and triangles $T_i = T'_i$ for $i > 2$. S_0 is endowed with two triangulations λ_0, λ'_0, and λ, λ' are obtained from them, respectively, by fusion on S. We clearly have that λ_0 and λ'_0 coincide on all the triangles $T_i = T'_i$ for $i > 2$, except on Q, where they coincide with the only two possible triangulations λ_Q and λ'_Q, respectively, that Q admits. By the Naturality, Disjoint Union and Diagonal Exchange properties, the isomorphism $\Phi_{\lambda_0\lambda'_0}$ is forced to be the extension to the quotient ring $\widehat{\mathcal{T}}^q_{\lambda'_0}$ of the following injective map

$$\mathcal{T}^q_{\lambda'_0} = \mathcal{T}^q_{\lambda'_Q} \bigotimes_{j\neq 1,2} \mathcal{T}^q_{T'_j} \longrightarrow \widehat{\mathcal{T}}^q_{\lambda'_Q} \bigotimes_{j\neq 1,2} \widehat{\mathcal{T}}^q_{T'_j} \xrightarrow{\Phi^q_{\lambda_Q\lambda'_Q} \bigotimes_j id_{T_j}} \widehat{\mathcal{T}}^q_{\lambda_Q} \bigotimes_{j\neq 1,2} \widehat{\mathcal{T}}^q_{T_j} \longrightarrow \widehat{\mathcal{T}}^q_{\lambda_0}$$

It can be easily verified that $(\Phi^q_{\lambda_Q\lambda'_Q})^{-1} = \Phi_{\lambda'_Q\lambda_Q}$ by explicit calculations on the formulae expressed in the Diagonal Exchange property. Hence we deduce that $\Phi^q_{\lambda_0\lambda'_0}$ is an isomorphism and $(\Phi^q_{\lambda_0\lambda'_0})^{-1} = \Phi^q_{\lambda'_0\lambda_0}$. We would like to define $\Phi^q_{\lambda\lambda'}$ as $\hat{\iota}^{-1}_{\lambda_0\lambda} \circ \Phi_{\lambda_0\lambda'_0} \circ \hat{\iota}_{\lambda'_0\lambda'}$ but, in the first place, we must verify that the image of $\Phi_{\lambda_0\lambda'_0} \circ \hat{\iota}_{\lambda'_0\lambda'}$ is contained in $\hat{\iota}_{\lambda_0\lambda}(\widehat{\mathcal{T}}^q_{\lambda})$. In order to prove this assertion, we have to discuss all the possible configurations of the square Q' in S. Denote by X_i, $\overline{X}_i$, X'_i, $\overline{X}'_i$ the generators of the algebras $\mathcal{T}^q_{\lambda}$, $\mathcal{T}^q_{\lambda_0}$, $\mathcal{T}^q_{\lambda'}$, $\mathcal{T}^q_{\lambda'_0}$, respectively, and assume that the edges in $\lambda_Q \subseteq \lambda_0$ and $\lambda'_Q \subseteq \lambda'_0$ are indexed as their identifications in λ and λ', respectively. We refer to the cases of [Liu09] in the following discussion:

Case 1 λ_j, λ_k, λ_l, λ_m are all distinct.

Suppose that the edge $\lambda_l \in \lambda$ is the result of the identification of the edge $(\lambda_0)_l \in \lambda_Q$ and of an edge $(\lambda_0)_n$, belonging to a certain triangle different from T_1 and T_2 in λ. Then observe

$$\begin{aligned}(\Phi^q_{\lambda_0\lambda'_0} \circ \hat{\iota}_{\lambda'_0\lambda'})(X'_l) &= \Phi_{\lambda_0\lambda'_0}(\overline{X}'_l\overline{X}'_n) = (1 + q\overline{X}_i)\overline{X}_l\overline{X}_n \\ &= \hat{\iota}_{\lambda_0\lambda}((1 + qX_i)X_l)\end{aligned}$$

Notice that the polynomial $(1 + qX_i)X_l$ coincides with the image of X'_l in the case of an embedded square. This situation arise for every external edge of the square that is not identified to an other side of it. In the following, we will always omit the calculations for these cases and we will focus on the identified couples of sides of the square, if there is any. In conclusion, when the edges λ_j, λ_k, λ_l, λ_m are all distinct, the expressions of the images of the elements X'_s under $\Phi^q_{\lambda_0\lambda'_0} \circ \hat{\iota}_{\lambda'_0\lambda'}$ are the following

$$\begin{aligned}\Phi^q_{\lambda\lambda'}(X'_i) &= \hat{\iota}_{\lambda_0\lambda}(X_i^{-1}) \\ \Phi^q_{\lambda\lambda'}(X'_j) &= \hat{\iota}_{\lambda_0\lambda}((1 + qX_i)X_j) \\ \Phi^q_{\lambda\lambda'}(X'_k) &= \hat{\iota}_{\lambda_0\lambda}((1 + qX_i^{-1})^{-1}X_k) \\ \Phi^q_{\lambda\lambda'}(X'_l) &= \hat{\iota}_{\lambda_0\lambda}((1 + qX_i)X_l) \\ \Phi^q_{\lambda\lambda'}(X'_m) &= \hat{\iota}_{\lambda_0\lambda}((1 + qX_i^{-1})^{-1}X_m)\end{aligned}$$

CASE 2 $\lambda_j = \lambda_k$ and $\lambda_l \neq \lambda_m$.

Studying the case of $\lambda_j = \lambda_k$, we obtain

$$\begin{aligned}(\Phi^q_{\lambda_0\lambda_0'} \circ \hat{\iota}_{\lambda_0'\lambda'})(X_j') &= \Phi_{\lambda_0\lambda_0'}(q^{-1}\overline{X}_j'\overline{X}_k') \\ &= q^{-1}(1+q\overline{X}_i)\overline{X}_j(1+q\overline{X}_i^{-1})^{-1}\overline{X}_k \\ &= q^{-1}(1+q\overline{X}_i)(1+q^{-1}\overline{X}_i^{-1})^{-1}\overline{X}_j\overline{X}_k \\ &= q^{-1}q\overline{X}_i(\overline{X}_j\overline{X}_k) \\ &= \hat{\iota}_{\lambda_0\lambda}(X_iX_j)\end{aligned}$$

The image of the other elements have the same appearance of the ones in the Case 1.

CASE 3 $\lambda_j = \lambda_m$ and $\lambda_k \neq \lambda_l$.

In same spirit as in the previous case, we have

$$(\Phi^q_{\lambda_0\lambda_0'} \circ \hat{\iota}_{\lambda_0'\lambda'})(X_j') = \hat{\iota}_{\lambda_0\lambda}(X_iX_j)$$

The image of the other elements have the same appearance of the ones in the Case 1.

CASE 4 $\lambda_j = \lambda_l$ and $\lambda_k \neq \lambda_m$.

We observe

$$(\Phi^q_{\lambda_0\lambda_0'} \circ \hat{\iota}_{\lambda_0'\lambda'})(X_j') = \hat{\iota}_{\lambda_0\lambda}((1+qX_i)(1+q^3X_i)X_j)$$

The image of the other elements have the same appearance of the ones in the Case 1.

CASE 5 $\lambda_k = \lambda_m$ and $\lambda_j \neq \lambda_l$.

We observe

$$(\Phi^q_{\lambda_0\lambda_0'} \circ \hat{\iota}_{\lambda_0'\lambda'})(X_k') = \hat{\iota}_{\lambda_0\lambda}((1+qX_i^{-1})^{-1}(1+q^3X_i^{-1})^{-1}X_k)$$

The image of the other elements have the same appearance of the ones in the Case 1.

CASE 6 $\lambda_j = \lambda_k$ and $\lambda_m = \lambda_l$.

We observe

$$\begin{aligned}(\Phi^q_{\lambda_0\lambda_0'} \circ \hat{\iota}_{\lambda_0'\lambda'})(X_j') &= \hat{\iota}_{\lambda_0\lambda}(X_iX_j) \\ (\Phi^q_{\lambda_0\lambda_0'} \circ \hat{\iota}_{\lambda_0'\lambda'})(X_l') &= \hat{\iota}_{\lambda_0\lambda}(X_iX_l)\end{aligned}$$

CASE 7 $\lambda_j = \lambda_m$ and $\lambda_k = \lambda_l$.

We observe

$$\begin{aligned}(\Phi^q_{\lambda_0\lambda_0'} \circ \hat{\iota}_{\lambda_0'\lambda'})(X_j') &= \hat{\iota}_{\lambda_0\lambda}(X_iX_j) \\ (\Phi^q_{\lambda_0\lambda_0'} \circ \hat{\iota}_{\lambda_0'\lambda'})(X_k') &= \hat{\iota}_{\lambda_0\lambda}(X_iX_k)\end{aligned}$$

CASE 8 $\lambda_j = \lambda_l$ and $\lambda_k = \lambda_m$.

We observe

$$(\Phi^q_{\lambda_0\lambda'_0} \circ \hat{\iota}_{\lambda'_0\lambda'})(X'_j) = \hat{\iota}_{\lambda_0\lambda}((1+qX_i)(1+q^3X_i)X_j)$$
$$(\Phi^q_{\lambda_0\lambda'_0} \circ \hat{\iota}_{\lambda'_0\lambda'})(X'_j) = \hat{\iota}_{\lambda_0\lambda}((1+qX_i^{-1})^{-1}(1+q^3X_i^{-1})^{-1}X_k)$$

The above discussion allows us to define $\Phi^q_{\lambda\lambda'}$ as $(\hat{\iota}_{\lambda_0\lambda})^{-1} \circ \Phi_{\lambda_0\lambda'_0} \circ \hat{\iota}_{\lambda'_0\lambda'}$, for every S and for every $\lambda, \lambda' \in \Lambda(S)$ that differ by a diagonal exchange. By definition, we have

$$\Phi^q_{\lambda_0\lambda'_0} \circ \hat{\iota}_{\lambda'_0\lambda'} = \hat{\iota}_{\lambda_0\lambda} \circ \Phi^q_{\lambda\lambda'} \tag{3.2}$$

Now, let R be a surface with $\mu, \mu' \in \Lambda(R)$ triangulations that differ by a diagonal exchange along μ_i. Suppose that S is obtained from R by fusion and that λ, λ' are the triangulations of S induced by μ, μ' respectively. We want to prove that, in this situation, the following holds

$$\Phi^q_{\mu\mu'} \circ \hat{\iota}_{\mu'\lambda'} = \hat{\iota}_{\mu\lambda} \circ \Phi^q_{\lambda\lambda'} \tag{3.3}$$

Denoting by μ_0 and μ'_0 the triangulations of R_0 and R'_0 that appear in the definition of $\Phi^q_{\mu\mu'}$, as above, then we clearly have $R_0 = S_0$ and $\mu_0 = \lambda_0$, $\mu'_0 = \lambda'_0$, by construction. Because of the injectivity of $\hat{\iota}_{\mu_0\mu}$, it is sufficient to prove

$$\hat{\iota}_{\mu_0\mu} \circ \hat{\iota}_{\mu\lambda} \circ \Phi^q_{\lambda\lambda'} = \hat{\iota}_{\mu_0\mu} \circ \Phi^q_{\mu\mu'} \circ \hat{\iota}_{\mu'\lambda'}$$

Now observe

$$\begin{aligned}
\hat{\iota}_{\mu_0\mu} \circ \hat{\iota}_{\mu\lambda} \circ \Phi^q_{\lambda\lambda'} &= \hat{\iota}_{\mu_0\lambda} \circ \Phi_{\lambda\lambda'} && \text{Relation 2.4}\\
&= \hat{\iota}_{\lambda_0\lambda} \circ \Phi^q_{\lambda\lambda'} && \mu_0 = \lambda_0\\
&= \Phi^q_{\lambda_0\lambda'_0} \circ \hat{\iota}_{\lambda'_0\lambda'} && \text{Relation 3.2}\\
&= \Phi^q_{\mu_0\mu'_0} \circ \hat{\iota}_{\mu'_0\lambda'} && \mu_0 = \lambda_0 \text{ and } \mu'_0 = \lambda'_0\\
&= \Phi^q_{\mu_0\mu'_0} \circ \hat{\iota}_{\mu'_0\mu'} \circ \hat{\iota}_{\mu'\lambda'} && \text{Relation 2.4}\\
&= \hat{\iota}_{\mu_0\mu} \circ \Phi^q_{\mu\mu'} \circ \hat{\iota}_{\mu'\lambda'} && \text{Relation 3.2}
\end{aligned}$$

and hence the relation 3.3, that is a "baby" version of the general Fusion property, holds whenever we are in the above situation.

Now we are going to define the $\Phi^q_{\lambda\lambda'}$ in the general case and to prove that the Composition relation holds. In order to do this, it is necessary to show that the isomorphisms, defined in the elementary cases, respect the Pentagon relation (the other relations between ideal triangulations in Theorem 0.4 are easier and can be verified in the same way).

Select in S a triangulation $\lambda \in \Lambda(S)$ and λ_i, λ_j two diagonals of a certain pentagon in λ. Designate also with $\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(5)}$ the following sequence of triangulations

$$\lambda,\ \Delta_i(\lambda),\ \Delta_j\Delta_i(\lambda),\ \Delta_i\Delta_j\Delta_i(\lambda),\ \Delta_j\Delta_i\Delta_j\Delta_i(\lambda),\ \Delta_i\Delta_j\Delta_i\Delta_j\Delta_i(\lambda) = \tau_{ij}(\lambda)$$

Then we have to prove that

$$\Phi^q_{\lambda\lambda^{(5)}} \circ \Phi^q_{\lambda^{(5)}\lambda^{(4)}} \circ \cdots \circ \Phi^q_{\lambda^{(1)}\lambda} = \Phi^q_{\lambda\lambda}$$

Assuming for a moment that this relation holds for every $\lambda, \lambda' \in \Lambda(S)$, we can select a sequence $\lambda = \lambda^{(0)}, \dots, \lambda^{(k)} = \lambda'$, by virtue of the Theorem 0.3, and define $\Phi^q_{\lambda\lambda'}$ as

$$\Phi^q_{\lambda\lambda^{(1)}} \circ \cdots \circ \Phi^q_{\lambda^{(k-1)}\lambda'}$$

Now, by virtue of the Theorem 0.4 and the assumed Pentagon relation (together with the others, on which we will not focus), it easy to verify that this is a good definition and that the Composition relation naturally holds.

Let S be a surface and $\lambda \in \Lambda(S)$ a certain triangulation. Select also λ_i and λ_j diagonals of a pentagon in S. Let R be the surface obtained by splitting S along all the edges of λ except for λ_i and λ_j. Hence R is the disjoint union of an embedded pentagon P and some triangles. Let $\mu = \mu^{(0)}, \dots, \mu^{(5)}$ the triangulations of R such that their fusions induce the triangulations $\lambda = \lambda^{(0)}, \dots, \lambda^{(5)}$ of S. Suppose that the following holds

$$\Phi^q_{\mu\mu^{(1)}} \circ \cdots \circ \Phi^q_{\mu^{(5)}\mu} = id_{\widetilde{\mathcal{T}}^q_\mu}$$

and observe

$$\begin{aligned}
\hat{\iota}_{\mu\lambda} \circ \Phi_{\lambda\lambda^{(1)}} \circ \cdots \circ \Phi_{\lambda^{(1)}\lambda} &= \Phi_{\mu\mu^{(5)}} \circ \hat{\iota}_{\mu^{(5)}\lambda^{(5)}} \circ \Phi_{\lambda^{(1)}\lambda^{(2)}} \circ \cdots \circ \Phi_{\lambda^{(5)}\lambda} && \text{Rel. } 3.2 \\
&\ \ \vdots \\
&= \Phi^q_{\mu\mu^{(1)}} \circ \cdots \circ \Phi^q_{\mu^{(5)}\mu} \circ \hat{\iota}_{\mu\lambda} && \text{Rel. } 3.2 \\
&= \hat{\iota}_{\mu\lambda}
\end{aligned}$$

Then, because of the injectivity of $\hat{\iota}_{\mu\lambda}$, the assumption of $\Phi^q_{\mu\mu^{(1)}} \circ \cdots \circ \Phi^q_{\mu^{(5)}\mu} = id_{\widetilde{\mathcal{T}}^q_\mu}$ implies that the Pentagon relation holds in the general case. From the definition given of the $\Phi_{\mu^{(i+1)}\mu^{(i)}}$ it is clear that the identity $\Phi^q_{\mu\mu^{(1)}} \circ \cdots \circ \Phi^q_{\mu^{(5)}\mu} = id_{\widetilde{\mathcal{T}}^q_\mu}$ follows from the proof of the pentagon relation in case in which S is an embedded pentagon. For the proof of this case we refer to [Liu09, Proposition 9].

Finally we have defined the isomorphisms $\Phi^q_{\lambda\lambda'}$ in the general case and we have proved that the Composition relation holds. It remains to verify that, with this definition, all the properties hold. The validity of the Naturality property is clear from the definition, just as the Disjoint Union property in case of λ_i and λ'_i that differs by a diagonal exchange. Now, by applying the Composition property, it is straightforward to prove the general case of the Disjoint Union property.

In the matter of the fusion property, we we have already shown it when μ and μ' differ by an elementary move in 3.3. In what follows, we will see how to deduce the general case from 3.3 and from the Composition relation. Suppose that S is obtained by fusing a surface R and that $\lambda, \lambda' \in \Lambda(S)$ are constructed as fusion of $\mu, \mu' \in \Lambda(R)$, respectively. Connect the triangulations μ and μ' with a sequence $\mu = \mu^{(0)}, \mu^{(1)}, \dots, \mu^{(k)} = \mu'$, in which $\mu^{(l+1)}$ is obtained from $\mu^{(l)}$ by a diagonal exchange. Then we can define an induced sequence $\lambda = \lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(k)} = \lambda'$, where $\lambda^{(l)} \in \Lambda(S)$ is obtained by fusion of $\mu^{(l)}$. Now, using the Composition property, we see

$$\begin{aligned}
\hat{\iota}_{\mu\lambda} \circ \Phi^q_{\lambda\lambda'} &= \hat{\iota}_{\mu\lambda} \circ \Phi^q_{\lambda\lambda^{(1)}} \circ \cdots \circ \Phi^q_{\lambda^{(k-1)}\lambda'} \\
\Phi^q_{\mu\mu'} \circ \hat{\iota}_{\mu'\lambda'} &= \Phi^q_{\mu\mu^{(1)}} \circ \cdots \circ \Phi^q_{\mu^{(k-1)}\mu'} \circ \hat{\iota}_{\mu'\lambda'}
\end{aligned}$$

Applying the relation 3.3 to $\lambda^{(i)}$, $\lambda^{(i+1)}$, $\mu^{(i)}$ and $\mu^{(i+1)}$ for every $i = 0, \dots, k-1$ we observe

$$\begin{aligned}
\hat{\iota}_{\mu\lambda} \circ \Phi^q_{\lambda\lambda'} &= \hat{\iota}_{\mu\lambda} \circ \Phi^q_{\lambda\lambda^{(1)}} \circ \cdots \circ \Phi^q_{\lambda^{(k-1)}\lambda'} \\
&= \Phi^q_{\mu\mu^{(1)}} \circ \hat{\iota}_{\mu^{(1)}\lambda^{(1)}} \circ \Phi^q_{\lambda^{(1)}\lambda^{(2)}} \circ \cdots \circ \Phi^q_{\lambda^{(k-1)}\lambda'} \\
&\ \vdots \\
&= \Phi^q_{\mu\mu^{(1)}} \circ \cdots \circ \Phi^q_{\mu^{(k-1)}\mu'} \circ \hat{\iota}_{\mu'\lambda'} \\
&= \Phi^q_{\mu\mu'} \circ \hat{\iota}_{\mu'\lambda'}
\end{aligned}$$

as desired.

As said in the very beginning of this construction, the definition of the isomorphisms $\Phi^q_{\lambda_0\lambda_0'}$, when S_0 is a disjoint union of triangles and a square, is obliged by the Naturality, the Disjoint Union and the Diagonal Exchange properties. Furthermore, retracing the above discussion, we see that from the uniqueness of this base case, follows the uniqueness of the $\Phi^q_{\lambda\lambda'}$ in the general case and so we conclude the proof of the assertion. □

Lemma 3.1.2. *Let S be a surface (see Chapter 0 for details) and let λ, λ' be two ideal triangulations of S. Then the isomorphism $\Phi^q_{\lambda\lambda'} \colon \widehat{\mathcal{T}}^q_{\lambda'} \to \widehat{\mathcal{T}}^q_{\lambda}$ sends the central element H' of $\widehat{\mathcal{T}}^q_{\lambda'}$ in the central elements H of $\widehat{\mathcal{T}}^q_{\lambda}$ and, if S is a closed punctured surface (with $\chi(S) < 0$), then $\Phi^q_{\lambda\lambda'}$ sends also $P_1', \dots, P_p' \in \widehat{\mathcal{T}}^q_{\lambda'}$ in $P_1, \dots, P_p \in \widehat{\mathcal{T}}^q_{\lambda}$ respectively.*

Proof. See [Liu09, Proposition 14]. □

The quantum Teichmüller space $\mathcal{T}^q_S$ is defined as the quotient

$$\bigsqcup_{\lambda \in \Lambda(S)} \widehat{\mathcal{T}}^q_{\lambda} \Big/ \sim$$

where $\sim$ is an equivalence relation that identifies two elements $X \in \widehat{\mathcal{T}}^q_{\lambda}$ and $X' \in \widehat{\mathcal{T}}^q_{\lambda'}$ if and only if $\Phi^q_{\lambda\lambda'}(X) = X'$. By virtue of the Composition relation, it is clear that this is an equivalence relation. We have natural bijections $i_\lambda \colon \widehat{\mathcal{T}}^q_{\lambda} \to \mathcal{T}^q_S$, which satisfy $i_\lambda^{-1} \circ i_{\lambda'} = \Phi^q_{\lambda\lambda'}$ for every $\lambda, \lambda' \in \Lambda(S)$. As a consequence, the set $\mathcal{T}^q_S$ can be naturally endowed with an algebra structure that makes the bijections i_λ algebra isomorphisms. Therefore, the maps $\Phi^q_{\lambda\lambda'}$ can be seen as coordinate changes, determined by the ideal triangulations λ, λ', of the intrinsic object $\mathcal{T}^q_S$.

3.2 Representations of the quantum Teichmüller space

We would like to give sense to a notion of finite-dimensional representation of the quantum Teichmüller space $\mathcal{T}^q_S$. The first obstruction is that $\mathcal{T}^q_S$, as algebra, does not admit any finite-dimensional representation in the usual sense. Indeed, suppose that there exists a representation $\rho \colon \widehat{\mathcal{T}}^q_{\lambda} \to \mathrm{End}(V)$, then the homomorphism ρ should have a huge kernel, because $\widehat{\mathcal{T}}^q_{\lambda}$ is an infinite-dimensional

$\mathbb{C}$-vector space. But every element in $\widehat{\mathcal{T}}^q_\lambda \setminus \{0\}$ has an inverse in $\widehat{\mathcal{T}}^q_\lambda \setminus \{0\}$, so though ρ should go in an invertible endomorphism, fact that is clearly absurd. The space $\widehat{\mathcal{T}}^q_\lambda$ is isomorphic to $\mathcal{T}^q_S$, then the same can be observed for $\mathcal{T}^q_S$.

Hence the first idea does not work for our purpose. Nevertheless, we have seen that $\mathcal{T}^q_\lambda$ has a rich finite-dimensional representation theory, so we could try to define a representation of $\mathcal{T}^q_S$ as a collection $\rho = \{\rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V)\}_{\lambda \in \Lambda(S)}$ of representations of all the Chekhov-Fock algebras that match up in some sense. Because of the existence of the coordinate change $\Phi^q_{\lambda\lambda'}$, it would be reasonable to ask that, for every $\lambda, \lambda' \in \Lambda(S)$ the corresponding representations ρ_λ and $\rho_{\lambda'}$ verify a condition

$$\rho_\lambda \circ \Phi^q_{\lambda\lambda'} = \rho_{\lambda'}$$

as applications from $\mathcal{T}^q_{\lambda'}$ in $\mathrm{End}(V)$. The problem of this relation is that the image of $\Phi^q_{\lambda\lambda'}$ of $\mathcal{T}^q_{\lambda'}$ is not necessarily contained in $\mathcal{T}^q_\lambda$, so $\rho_\lambda \circ \Phi^q_{\lambda\lambda'}$ is not well defined a priori. Because of this remark, we introduce the following definition:

Definition 3.2.1. Given $\lambda, \lambda' \in \Lambda(S)$ two ideal triangulations of S and

$$\rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V) \qquad\qquad \rho_{\lambda'} \colon \mathcal{T}^q_{\lambda'} \to \mathrm{End}(V)$$

two representations in the same finite-dimensional vector space, we say that $\rho_{\lambda'}$ is *compatible* with ρ_λ, and we write $\rho_{\lambda'} = \rho_\lambda \circ \Phi^q_{\lambda\lambda'}$, if, for every generator $X'_i \in \mathcal{T}^q_{\lambda'}$, the element $\Phi^q_{\lambda\lambda'}(X'_i)$ can be written as $P_i Q_i^{-1} \in \widehat{\mathcal{T}}^q_\lambda$, with $P_i, Q_i \in \mathcal{T}^q_\lambda$, in such a way that $\rho_\lambda(Q_i)$ is invertible and $\rho_{\lambda'}(X'_i) = \rho_\lambda(P_i)\rho_\lambda(Q_i)^{-1}$.

Observe that, by considering $\rho_{\lambda'}((X'_i)^{-1})$, the element $\rho_\lambda(P_i)$ has to be invertible too.

Lemma 3.2.2. *Let $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)}$ be a sequence of triangulations of S, in which $\lambda^{(i+1)}$ is obtained by a re-indexing or a diagonal exchange from $\lambda^{(i)}$ for every $i = 1, \ldots, k-1$ and let $\rho_i := \rho_{\lambda^{(i)}} \colon \mathcal{T}^q_{\lambda^{(i)}} \to \mathrm{End}(V)$ be a finite-dimensional representation for every $i = 1, \ldots, k$. If ρ_i is compatible with ρ_{i+1} for every $i = 1, \ldots, k-1$, then $\rho_1 = \rho_k \circ \Phi^q_{\lambda^{(k)}\lambda^{(1)}}$ and $\rho_k = \rho_1 \circ \Phi^q_{\lambda^{(1)}\lambda^{(k)}}$.*

Consequently, the compatibility relation is symmetric, transitive and obviously reflexive.

Proof. We will prove $\rho_1 = \rho_h \circ \Phi^q_{\lambda^{(h)}\lambda^{(1)}}$ by induction on h. Given $X^{(1)}_i$ a generator of $\mathcal{T}^q_{\lambda^{(1)}}$, we need to show that there exist $P^{(h)}, Q^{(h)} \in \mathcal{T}^q_{\lambda^{(h)}}$ such that $\rho_h(P^{(h)})$, $\rho_h(Q^{(h)})$ are invertible, the relation $\Phi^q_{\lambda^{(h)}\lambda^{(1)}}(X^{(1)}_i) = P^{(h)}\,(Q^{(h)})^{-1}$ holds and $\rho_h(P^{(h)})\,\rho_h(Q^{(h)})^{-1} = \rho_1(X^{(1)}_i)$. When $\lambda^{(h)}$ and $\lambda^{(h-1)}$ differ by a re-indexing, then the property follows from the inductive hypothesis and the fact that $\Phi^q_{\lambda^{(h)}\lambda^{(h-1)}}$ is just a reordering. We will suppose from now on that $\lambda^{(h)}$ and $\lambda^{(h-1)}$ differ by a diagonal exchange along $\lambda^{(h-1)}_{i_0}$.

Let $\mathbb{C}\{Z_1^{\pm 1}, \ldots, Z_n^{\pm 1}\}$ denote the algebra of non-commutative polynomials in the variables $Z_1^{\pm 1}, \ldots, Z_n^{\pm 1}$. So, by inductive hypothesis, there exist $P, Q \in \mathbb{C}\{Z_1^{\pm 1}, \ldots, Z_n^{\pm 1}\}$ such that

$$\Phi^q_{\lambda^{(h-1)}\lambda^{(1)}}(X^{(1)}_i) = P((X^{(h-1)}_s)_s)\, Q((X^{(h-1)}_s)_s)^{-1}$$

with $\rho_{h-1}(P((X^{(h-1)}_s)_s))$, $\rho_{h-1}(Q((X^{(h-1)}_s)_s))$ invertible and such that

$$\rho_1(X^{(1)}_i) = \rho_{h-1}(P((X^{(h-1)}_s)_s))\, \rho_{h-1}(Q((X^{(h-1)}_s)_s))^{-1}$$

By virtue of the Composition relation, we have

$$\begin{aligned}\Phi^q_{\lambda^{(h)}\lambda^{(1)}}(X_i^{(1)}) &= \Phi^q_{\lambda^{(h)}\lambda^{(h-1)}}(P((X_s^{(h-1)})_s)\,Q((X_s^{(h-1)})_s)^{-1}) \\ &= P((\Phi^q_{\lambda^{(h)}\lambda^{(h-1)}}(X_s^{(h-1)}))_s)\,Q((\Phi^q_{\lambda^{(h)}\lambda^{(h-1)}}(X_s^{(h-1)}))_s)^{-1}\end{aligned}$$

From the discussion in the proof of Theorem 3.1.1 we observe that the elements $P((\Phi^q_{\lambda^{(h)}\lambda^{(h-1)}}(X_s^{(h-1)}))_s)$ and $Q((\Phi^q_{\lambda^{(h)}\lambda^{(h-1)}}(X_s^{(h-1)}))_s)$ are polynomials in $(X_j^{(h)})^{\pm1}$, $(1+q(X_{i_0}^{(h)})^{\pm1})^{-1}$ and $(1+q^3(X_{i_0}^{(h)})^{\pm1})^{-1}$. The terms $(1+q(X_{i_0}^{(h)})^{\pm1})^{-1}$ and $(1+q^3(X_{i_0}^{(h)})^{\pm1})^{-1}$ appear in relations of this kind

$$\begin{aligned}\Phi^q_{\lambda^{(h)}\lambda^{(h-1)}}((X_j^{(h-1)})^{-1}) &= (X_j^{(h)})^{-1}(1+qX_{i_0}^{(h)})^{-1}\\ \Phi^q_{\lambda^{(h)}\lambda^{(h-1)}}(X_j^{(h-1)}) &= (1+q(X_{i_0}^{(h)})^{-1})^{-1}X_j^{(h)}\\ \Phi^q_{\lambda^{(h)}\lambda^{(h-1)}}((X_j^{(h-1)})^{-1}) &= (X_j^{(h)})^{-1}(1+q^3X_{i_0}^{(h)})^{-1}(1+qX_{i_0}^{(h)})^{-1}\\ \Phi^q_{\lambda^{(h)}\lambda^{(h-1)}}(X_j^{(h-1)}) &= (1+q(X_{i_0}^{(h)})^{-1})^{-1}(1+q^3(X_{i_0}^{(h)})^{-1})^{-1}X_j^{(h)}\end{aligned}$$

Now, using the fact that $\rho_{h-1}=\rho_h\circ\Phi_{\lambda^{(h)}\lambda^{(h-1)}}$ and the above expressions, we deduce the following equalities

$$\begin{aligned}\rho_h(1+qX_{i_0}^{(h)}) &= \rho_h(X_j^{(h)})^{-1}\rho_{h-1}(X_j^{(h-1)})\\ \rho_h(1+q(X_{i_0}^{(h)})^{-1}) &= \rho_h(X_j^{(h)})\rho_{h-1}(X_j^{(h-1)})^{-1}\\ \rho_h(1+qX_{i_0}^{(h)})\rho_h(1+q^3X_{i_0}^{(h)}) &= \rho_h(X_j^{(h)})^{-1}\rho_{h-1}(X_j^{(h-1)})\\ \rho_h(1+q^3(X_{i_0}^{(h)})^{-1})\rho_h(1+q(X_{i_0}^{(h)})^{-1}) &= \rho_h(X_j^{(h)})\rho_{h-1}(X_j^{(h-1)})^{-1}\end{aligned}$$

Since $\rho_h(X_j^{(h)})$ and $\rho_{h-1}(X_j^{(h-1)})$ are invertible and V is finite-dimensional, we conclude that

FACT 1: $\rho_h(1+q(X_{i_0}^{(h)})^{\pm1})$ and $\rho_h(1+q^3(X_{i_0}^{(h)})^{\pm1})$ are invertible endomorphisms of V.

Expressing P as a sum of monomials $\sum_\alpha P_\alpha$, we obtain

$$P((\Phi^q_{\lambda^{(h)}\lambda^{(h-1)}}(X_s^{(h-1)}))_s)=\sum_\alpha P_\alpha((\Phi^q_{\lambda^{(h)}\lambda^{(h-1)}}(X_s^{(h-1)}))_s)$$

In each monomial, by using the relation

$$(1+q^{2k+1}(X_{i_0}^{(h)})^{\pm1})^{-1}X_j^{(h)}=(1+q^{2k+1\pm2\sigma_{ji_0}}(X_{i_0}^{(h)})^{\pm1})^{-1}X_j^{(h)} \tag{3.4}$$

we can push all the $(1+q(X_{i_0}^{(h)})^{\pm1})^{-1}$ and $(1+q^3(X_{i_0}^{(h)})^{\pm1})^{-1}$ on the right, in order to obtain an expression like

$$P((\Phi^q_{\lambda^{(h)}\lambda^{(h-1)}}(X_s^{(h-1)}))_s)=\sum_\alpha P'_\alpha((X_s^{(h)})_s)\,R_\alpha(X_{i_0}^{(h)})^{-1}$$

where P'_α and R_α are Laurent polynomials. Recalling the definition of the sum in $\widehat{\mathcal{T}}^q_{\lambda^{(h)}}$, we see that $P((\Phi^q_{\lambda^{(h)}\lambda^{(h-1)}}(X_s^{(h-1)}))_s)$ can be expressed in the following form

$$P((\Phi^q_{\lambda^{(h)}\lambda^{(h-1)}}(X_s^{(h-1)}))_s)=P'((X_s^{(h)})_s)R(X_{i_0}^{(h)})^{-1} \tag{3.5}$$

where $P'((X_s^{(h)})_s)$ is a Laurent polynomial in the variables $X_1^{(h)}, \dots, X_n^{(h)}$ and $R(X_{i_0}^{(h)})$ is a Laurent polynomial in $X_{i_0}^{(h)}$.

Now, applying ρ_h to the relation

$$(1 + q^{2k+1}(X_{i_0}^{(h)})^{\pm 1})X_j^{(h)} = X_j^{(h)}(1 + q^{2k+1 \pm 2\sigma_{i_0 j}}(X_{i_0}^{(h)})^{\pm 1})$$

we see that each term of the form $(1 + q^{2k'+1}(X_{i_0}^{(h)})^{\pm 1})$, appearing in the expression 3.5, has invertible image in $\mathrm{End}(V)$, so

> FACT 2: the endomorphism $\rho_h(R(X_{i_0}^{(h)}))$ is invertible

because it is product of factors of this kind. Using Facts 1 and 2 and the relation obtained by applying ρ_h to 3.4, we can repeat the operations done before on the $\rho_h(X_s^{(h)})$ instead of $X_s^{(h)}$ obtaining

$$\begin{aligned}
\rho_{h-1}(P((X_s^{(h-1)})_s)) &= P((\rho_{h-1}(X_s^{(h-1)}))_s) \\
&= P((\rho_h(\Phi_{\lambda^{(h)}\lambda^{(h-1)}}(X_s^{(h-1)})))_s) && \rho_{h-1} = \rho_h \circ \Phi^q_{\lambda^{(h)}\lambda^{(h-1)}} \text{ and Fact 1} \\
&= P'((\rho_h(X_s^{(h)}))_s)R(\rho_h(X_{i_0}^{(h)}))^{-1} && \rho_h(\text{Relation 3.4}) \\
&= \rho_h(P'((X_1^{(h)})_s))\, \rho_h(R(X_{i_0}^{(h)}))^{-1} && \text{Fact 2}
\end{aligned}$$

By definition of P and Q the endomorphism $\rho_{h-1}(P((X_s^{(h-1)})_s))$ is invertible, hence the previous equation and Fact 2 imply that $\rho_h(P'((X_s^{(h)})_s))$ is invertible too. By the same arguments we observe that there exist a Laurent polynomial Q' in the variables $X_1^{(h)}, \dots, X_n^{(h)}$ and a Laurent polynomial S in $X_{i_0}^{(h)}$ such that

$$\begin{aligned}
Q((\Phi^q_{\lambda^{(h)}\lambda^{(h)}}(X_s^{(h-1)}))_i) &= Q'((X_s^{(h)})_s)\, S(X_{i_0}^{(h)})^{-1} \\
\rho_{h-1}(Q((X_s^{(h-1)})_s)) &= \rho_h(Q'((X_s^{(h)})_s))\, \rho_h(S(X_{i_0}^{(h)}))^{-1}
\end{aligned}$$

Now, we have found the following decomposition

$$\begin{aligned}
\Phi^q_{\lambda^{(h)}\lambda^{(1)}}(X_i^{(1)}) &= P((\Phi^q_{\lambda^{(h)}\lambda^{(h-1)}}(X_s^{(h-1)}))_s)\, Q((\Phi^q_{\lambda^{(h)}\lambda^{(h-1)}}(X_s^{(h-1)}))_s)^{-1} \\
&= P'((X_s^{(h)})_s)R(X_{i_0}^{(h)})^{-1}\,(Q'((X_s^{(h)})_s)S(X_{i_0}^{(h)})^{-1})^{-1} \\
&= (P'((X_s^{(h)})_s)S(X_{i_0}^{(h)}))\,(Q'((X_s^{(h)})_s)R(X_{i_0}^{(h)}))^{-1}
\end{aligned}$$

where the equality holds because $R(X_{i_0}^{(h)})$ and $S(X_{i_0}^{(h)})$ commute, and analogously

$$\begin{aligned}
\rho_1(X_i^{(1)}) &= \rho_{h-1}(P((X_i^{(h-1)})_i)\, \rho_{h-1}(Q((X_i^{(h-1)})_i)^{-1} \\
&= \rho_h(P'((X_i^{(h)})_i))\rho_h(R(X_{i_0}^{(h)}))^{-1}\,(\rho_h(Q'((X_i^{(h)})_i))\rho_h(S(X_{i_0}^{(h)}))^{-1})^{-1} \\
&= \rho_h(P'((X_i^{(h)})_i)S(X_{i_0}^{(h)}))\,(\rho_h(Q'((X_i^{(h)})_i)R(X_{i_0}^{(h)})))^{-1}
\end{aligned}$$

so we have finally proved that $\rho_1 = \rho_h \circ \Phi^q_{\lambda^{(1)}\lambda^{(h)}}$, by taking the decomposition of $\Phi^q_{\lambda^{(h)}\lambda^{(1)}}(X_i^{(1)})$ given by $(P'S)(RQ')^{-1}$.

With similar arguments and the explicit form of the isomorphisms $\Phi^q_{\lambda^{(h)}\lambda^{(h-1)}}$ and $\Phi^q_{\lambda^{(h-1)}\lambda^{(h)}}$ we can see that $\rho_{h-1} = \rho_h \circ \Phi_{\lambda^{(h)}\lambda^{(h-1)}}$ implies $\rho_h = \rho_{h-1} \circ \Phi_{\lambda^{(h-1)}\lambda^{(h)}}$, so with the same argument we conclude also that $\rho_h = \rho_1 \circ \Phi_{\lambda^{(1)}\lambda^{(h)}}$. □

Definition 3.2.3. A *finite-dimensional representation* of the quantum Teichmüller space $\mathcal{T}^q_S$ is a collection $\rho = \{\rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V)\}_{\lambda \in \Lambda(S)}$, in which for every $\lambda, \lambda' \in \Lambda(S)$ $\rho_{\lambda'}$ is compatible with ρ_λ.

Because of Lemma 3.2.2, in order to verify that a collection $\rho = \{\rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V)\}_{\lambda \in \Lambda(S)}$ is a finite-dimensional representation of the quantum Theichmüller space, it is sufficient to control the compatibility for the couples of triangulations λ, λ' that differ by a re-indexing or a diagonal exchange.

Lemma 3.2.4. *Let $\rho = \{\rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V)\}_{\lambda \in \Lambda(S)}$ be a finite-dimensional representation of $\mathcal{T}^q_S$. Then, for every λ, λ' triangulations of S, and for every $X' \in \mathcal{T}^q_{\lambda'}$ there exist $P, Q, R, S \in \mathcal{T}^q_\lambda$ such that*

- $\Phi^q_{\lambda\lambda'}(X') = PQ^{-1} = S^{-1}R$;
- $\rho_\lambda(Q)$ *and* $\rho_\lambda(S)$ *are invertible;*

and for every such decomposition

$$\rho_{\lambda'}(X') = \rho_\lambda(P)\rho_\lambda(Q)^{-1} = \rho_\lambda(S)^{-1}\rho_\lambda(R)$$

Proof. See [BL07, Lemma 26]. □

Now we state a fact that will be useful for what follows:

Lemma 3.2.5 (Quantum Binomial Formula)**.** *Let $\mathcal{W}$ be the non-commutative $\mathbb{C}$-algebra $\mathcal{W} = \mathcal{W}^\alpha[U, V]$ (see Definition 1.2.3). If $\alpha \in \mathbb{C}^*$ verifies $\alpha^i \neq 1$ for every $k \in \{1, \dots, N-1\}$, then the following relation holds*

$$(U+V)^N = \sum_{k=0}^{N} \binom{N}{k}_\alpha V^{N-k}U^k$$

where $\binom{N}{k}_\alpha$ is defined as

$$\binom{N}{k}_\alpha := \begin{cases} \prod_{i=0}^{k-1} \frac{1-\alpha^{N-i}}{1-\alpha^{i+1}} & \text{if } k \in \{1, \dots, N-1\} \\ 1 & \text{otherwise} \end{cases}$$

for every $N, k \in \mathbb{N}$ with $k \leq N$. Moreover, we have

$$\lim_{\alpha \to 1} \binom{N}{k}_\alpha = \binom{N}{k}$$

where $\binom{N}{k}$ is the usual binomial coefficient.

Proof. The proof of the relation can be straightforwardly done by induction on N and we omit it. For what concern the second assertion it is sufficient to observe that, for $\alpha \to 1$, we have

$$\begin{aligned}\binom{N}{k}_\alpha &= \prod_{i=0}^{k-1} \frac{1-\alpha^{N-i}}{1-\alpha^{i+1}} = \prod_{i=0}^{k-1} \frac{-(N-i)(\alpha-1)+o(\alpha-1)}{-(i+1)(\alpha-1)+o(\alpha-1)} \\ &= \frac{\left(\prod_{i=0}^{k-1}(N-i)\right)(q-1)^k + o((q-1)^k)}{\left(\prod_{i=0}^{k-1}(i+1)\right)(q-1)^k + o((q-1)^k)} \\ &= \binom{N}{k} + o(1)\end{aligned}$$

□

Observe in particular that, if α is a primitive N-th root of unity, then $\binom{N}{k}_\alpha = 1$ for every $k \neq 0, N$, so we have $(U+V)^N = U^N + V^N$.

3.2.1 Irreducible representations

Definition 3.2.6. A finite-dimensional representation

$$\rho = \{\rho_\lambda \colon \mathcal{T}_\lambda^q \to \mathrm{End}(V)\}_{\lambda \in \Lambda(S)}$$

of the quantum Teichmüller space is said to be *irreducible* if, for every $\lambda \in \Lambda(S)$, ρ_λ is irreducible.

Definition 3.2.7. Given

$$\rho = \{\rho_\lambda \colon \mathcal{T}_\lambda^q \to \mathrm{End}(V)\}_{\lambda \in \Lambda(S)} \qquad \rho' = \{\rho'_\lambda \colon \mathcal{T}_\lambda^q \to \mathrm{End}(V')\}_{\lambda \in \Lambda(S)}$$

two representations of the quantum Teichmüller space, we say that ρ and ρ' are *isomorphic* if there exists an isomorphism of vector spaces $L \colon V \to V'$ such that, for every $\lambda \in \Lambda(S)$ and for every $X \in \mathcal{T}_\lambda^q$, we have

$$L \circ \rho_\lambda(X) \circ L^{-1} = \rho'_\lambda(X)$$

Remark 3.2.8. In order to prove that two representations

$$\rho = \{\rho_\lambda \colon \mathcal{T}_\lambda^q \to \mathrm{End}(V)\}_{\lambda \in \Lambda(S)} \qquad \rho' = \{\rho'_\lambda \colon \mathcal{T}_\lambda^q \to \mathrm{End}(V')\}_{\lambda \in \Lambda(S)}$$

are isomorphic, it is sufficient to show that for a fixed $\lambda \in \Lambda(S)$ the representations ρ_λ and ρ'_λ are isomorphic through a certain isomorphism $L \colon V \to V'$. Indeed, if this happens, by virtue of Lemma 3.2.4 and of the compatibility of $\rho_{\lambda'}$, ρ_λ and $\rho'_{\lambda'}$, ρ'_λ, for every $\lambda' \in \Lambda(S)$ we have

$$\begin{aligned} L \circ \rho_{\lambda'}(X') \circ L^{-1} &= L \circ (\rho_\lambda \circ \Phi^q_{\lambda\lambda'})(X') \circ L^{-1} \\ &= (\rho'_\lambda \circ \Phi^q_{\lambda\lambda'})(X') \\ &= \rho'_{\lambda'}(X') \end{aligned}$$

for every $X' \in \mathcal{T}^q_{\lambda'}$. Moreover, if ρ_λ and ρ'_λ are irreducible, then the linear isomorphisms carrying ρ_λ in ρ'_λ are equal to each other up to scalar multiplication.

As seen in Theorem 2.1.9, any irreducible representation $\rho\colon \mathcal{T}^q_\lambda \to \mathrm{End}(V)$ of the Chekhov-Fock algebra associated with a triangulation of a closed punctured surface S is classified by the numbers $x_i \in \mathbb{C}^*$ and by some N-th roots of certain functions of the same x_i. Starting from ρ, we can define a representation $\rho^1\colon \mathcal{T}^1_\lambda \to \mathrm{End}(\mathbb{C})$ of the commutative algebra $\mathcal{T}^1_\lambda$ setting

$$\rho^1(X_i) := x_i \in \mathrm{End}(\mathbb{C})$$

We will say that ρ^1 is the *non-quantum shadow* of the irreducible representation ρ.

Starting from an irreducible representation of the quantum Teichmüller space $\rho = \{\rho_\lambda\colon \mathcal{T}^q_\lambda \to \mathrm{End}(V)\}_{\lambda\in\Lambda(S)}$, we can take, for each triangulation $\lambda \in \Lambda(S)$, the non-quantum shadow $\rho^1_\lambda\colon \mathcal{T}^1_\lambda \to \mathrm{End}(\mathbb{C})$ of the representation ρ_λ. We denote by ρ^1 the collection of representation

$$\{\rho^1_\lambda\colon \mathcal{T}^1_\lambda \to \mathrm{End}(\mathbb{C})\}_{\lambda\in\Lambda(S)}$$

In what follows, it is crucial the hypothesis that q is a primitive N-th root of $(-1)^{N+1}$.

Lemma 3.2.9. *Let λ, λ' be two ideal triangulations, which differ by an elementary move, of a closed punctured surface S with $\chi(S) < 0$. Consider $\rho_\lambda\colon \mathcal{T}^q_\lambda \to \mathrm{End}(V)$ and $\rho_{\lambda'}\colon \mathcal{T}^q_{\lambda'} \to \mathrm{End}(V)$ two irreducible representations and suppose that $\rho_{\lambda'} = \rho_\lambda \circ \Phi^q_{\lambda\lambda'}$. Then the corresponding non-quantum shadows are compatible, that is $\rho^1_{\lambda'} = \rho^1_\lambda \circ \Phi^1_{\lambda\lambda'}$.*

Proof. Denote by x_i and x'_i, respectively, the invariants of the representations ρ_λ and $\rho_{\lambda'}$, which define the non-quantum shadows ρ^1_λ, $\rho^1_{\lambda'}$. If λ is obtained from λ' by re-indexing, then the assertion is obvious. Suppose for the moment that λ and λ' differ by a diagonal axchange along a diagonal λ_i of an embedded square Q, with edges labelled as in Figure 3.1.

Through Lemma 3.2.4, we have

$$\begin{aligned} x'_i\, id_V &= \rho_{\lambda'}((X'_i)^N) = \rho_\lambda(\Phi^q_{\lambda\lambda'}((X'_i)^N)) \\ &= \rho_\lambda(X_i^{-N}) = x_i^{-1}\, id_V \end{aligned}$$

so $x'_i = x_i^{-1}$. Now we analyse the case of x'_j. We have that $X_jX_i = q^2X_iX_j$ then, by virtue of the quantum Binomial formula, we obtain

$$\begin{aligned} \Phi^q_{\lambda\lambda'}((X'_j)^N) &= \Phi^q_{\lambda\lambda'}(X'_j)^N = ((1+qX_i)X_j)^N \\ &= (X_i + qX_iX_j)^N = q^N(X_iX_j)^N + X_j^N \\ &= q^{N+N(N-1)}X_i^NX_j^N + X_j^N = (1+X_i^N)X_j^N \end{aligned}$$

where we are taking advantage of the facts that

- $\binom{N}{k}_{q^2}$ is equal to 0 for every $k = 1, \dots, N-1$, because q^2 is a primitive N-th root of unity;
- by hypothesis, we know that q is a primitive N-th root of $(-1)^{N+1}$, so $q^{N^2} = (-1)^{N(N-1)} = 1$.

Hence we conclude immediately that $x'_j = (1+x_i)x_j$. With the same calculations we obtain the following relations

$$\begin{cases} x'_i = x_i^{-1} \\ x'_j = (1+x_i)x_j \\ x'_k = (1+x_i^{-1})^{-1}x_k \\ x'_l = (1+x_i)x_l \\ x'_m = (1+x_i^{-1})^{-1}x_m \\ x'_h = x_h \quad \text{if } h \notin \{i,j,k,l,m\} \end{cases} \tag{3.6}$$

By inspection of the definition of $\Phi^1_{\lambda\lambda'}$, we see that these are exactly the relations needed in order to prove $\rho^1_{\lambda'} = \rho_\lambda \circ \Phi^1_{\lambda\lambda'}$.

Now we will discuss the non embedded case. We are going to analyse only the Case 2 (as in the proof of Theorem 3.1.1), in which $\lambda_j = \lambda_l$ and $\lambda_k \neq \lambda_m$, the other possibility can be treated in the same way.

In this context we have $X_jX_i = q^4X_iX_j$. We have to discriminate the case in which N is odd or not in order to apply the quantum Binomial formula.

Firstly, suppose that N is odd. In this case, q^4 is still a primitive N-th root of unity so, denoting with $U := (1+q^3X_i)X_j$, we observe that

$$\begin{aligned} \Phi^q_{\lambda\lambda'}((X'_j)^N) &= \Phi^q_{\lambda\lambda'}(X'_j)^N = (U + qX_iU)^N \\ &= (qX_iU)^N + U^N = q^{N+4\frac{(N-1)N}{2}} X_i^N U^N + U^N \\ &= (X_i^N + 1)U^N \end{aligned}$$

where we are using that $U(qX_iU) = q^4(qX_iU)U$ and $q^N = (-1)^{N+1} = 1$ because N is odd. Similarly, we have

$$\begin{aligned} U^N &= ((1+q^3X_i)X_j)^N = (q^3X_iX_j)^N + X_j^N \\ &= q^{3N+4\frac{(N-1)N}{2}} X_i^N X_j^N + X_j^N \\ &= (1+X_i^N)X_j \end{aligned}$$

So, in conclusion,

$$\Phi^q_{\lambda\lambda'}((X'_j)^N) = (1+X_i^N)^2 X_j^N$$

which implies $x'_j = (1+x_i)^2x_j$. Moreover, with the same calculations of the embedded case, we obtain the relations:

$$\begin{cases} x'_i = x_i^{-1} \\ x'_j = (1+x_i)^2x_j \\ x'_k = (1+x_i^{-1})^{-1}x_k \\ x'_m = (1+x_i^{-1})^{-1}x_m \\ x'_h = x_h \quad \text{if } h \notin \{i,j,k,l,m\} \end{cases} \tag{3.7}$$

When N is even, we have to be a little more careful, because q^4 is a primitive $\frac{N}{2}$-th root of unity, instead of the previous case. With analogous arguments, remembering that N is even and that $q^N = (-1)^{N+1} = -1$, we obtain the relation

$$\begin{aligned}\Phi^q_{\lambda\lambda'}((X'_j)^{N/2}) &= \Phi^q_{\lambda\lambda'}(X'_j)^{N/2} = (U + qX_iU)^{N/2} \\ &= (qX_iU)^{N/2} + U^{N/2} = q^{N/2+4\frac{(N/2-1)N/2}{2}} X_i^{N/2} U^{N/2} + U^{N/2} \\ &= (1 + q^{N/2}(-1)^{\frac{N-2}{2}} X_i^{N/2}) U^{N/2}\end{aligned}$$

Furthermore,

$$\begin{aligned}U^{N/2} &= (q^3 X_i X_j)^{N/2} + X_j^{N/2} \\ &= q^{3N/2+4\frac{(N-1)N}{2}} X_i^N X_j^N + X_j^N \\ &= (1 + (-1)^{N/2} q^{N/2} X_i^{N/2}) X_j^{N/2}\end{aligned}$$

So we have that $\Phi^q_{\lambda\lambda'}((X'_j)^{N/2}) = (1 + X_i^N) X_j^{N/2}$. By virtue of the fact that $X_i^{N/2}$ and $X_j^{N/2}$ commute, we deduce finally that

$$\Phi^q_{\lambda\lambda'}((X'_j)^N) = \Phi^q_{\lambda\lambda'}((X'_j)^{N/2})^2 = (1 + X_i^N)^2 X_j^N$$

Therefore, in this case too the relation $x'_j = (1 + x_i)^2 x_j$ holds, and in the same way can be proved all 3.7. □

Theorem 3.2.10. *Given $\rho = \{\rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V)\}_{\lambda \in \Lambda(S)}$ an irreducible representation of the quantum Teichmüller space $\mathcal{T}^q_S$, the collection $\rho^1 = \{\rho^1_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(\mathbb{C})\}$ of the non-quantum shadows of ρ is a representation of the non-quantum space $\mathcal{T}^1_S$, called the* non-quantum shadow *of ρ.*

Proof. It is sufficient to apply Lemmas 3.2.9 and 3.2.2. □

Lemma 3.2.11. *Let λ and λ' be two triangulations of a closed punctured S, which differ by a diagonal exchange or a re-indexing. Consider $\rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V)$ an irreducible representation of the Chekhov-Fock algebra $\mathcal{T}^q_\lambda$, with non-quantum shadow $\rho^1_\lambda \colon \mathcal{T}^1_\lambda \to \mathrm{End}(\mathbb{C})$. Suppose that there exists a non-quantum representation $\rho^1_{\lambda'} \colon \mathcal{T}^1_{\lambda'} \to \mathrm{End}(\mathbb{C})$ such that $\rho^1_{\lambda'} = \rho^1_\lambda \circ \Phi^1_{\lambda\lambda'}$, then there exists a unique irreducible representation $\rho_{\lambda'} \colon \mathcal{T}^q_{\lambda'} \to \mathrm{End}(V)$ with non-quantum shadow $\rho^1_{\lambda'}$ and such that $\rho_{\lambda'} = \rho_\lambda \circ \Phi^q_{\lambda\lambda'}$.*

Proof. The assertion is obvious in the case of re-indexing, so in the following we will assume that λ' is obtained from λ by a diagonal exchange along λ_i. We will focus on the case in which λ_i is the diagonal of an embedded square, the other cases can be proved trough the calculations of Lemma 3.2.9.

In the notation of Figure 3.1, observe that

$$\begin{aligned}\rho_\lambda((1 + qX_i)X_j)^N &= \rho_\lambda((1 + X_i^N) X_j^N) \\ &= (1 + x_i) x_j \, id_V = \rho^1_{\lambda'}(X'_j) \, id_V\end{aligned}$$

Because $\rho^1_{\lambda'}(X'_j) = x'_j \neq 0$, we conclude that $\rho_\lambda((1 + X_i)X_j)$ is invertible. With exactly the same computations we obtain that $\rho_\lambda((1 + X_i)X_l)$ is invertible too. Moreover, we have

$$\begin{aligned}\rho_\lambda(X_k^{-1}(1 + qX_i^{-1}))^N &= \rho_\lambda(X_k^{-N}(1 + X_i^{-N})) \\ &= (1 + x_i^{-1}) x_k^{-1} \, id_V = \rho^1_{\lambda'}((X'_k)^{-1}) id_V\end{aligned}$$

and in the same spirit we obtain that $\rho_\lambda(X_m^{-1}(1+qX_i^{-1}))^N = \rho^1_{\lambda'}((X'_m)^{-1})id_V$. Hence we have shown that $X_m^{-1}(1+qX_i^{-1})$ and $X_m^{-1}(1+qX_i^{-1})$ have invertible images through ρ_λ. Now we define $\rho_{\lambda'}$ on the generators X_s as follows

$$\begin{array}{ccc} X'_i & \longmapsto & \rho_\lambda(X_i)^{-1} \\ X'_j & \longmapsto & \rho_\lambda((1+qX_i)X_j) \\ X'_k & \longmapsto & \rho_\lambda(X_k^{-1}(1+qX_i^{-1}))^{-1} \\ X'_l & \longmapsto & \rho_\lambda((1+qX_i)X_l) \\ X'_m & \longmapsto & \rho_\lambda(X_m^{-1}(1+qX_i^{-1}))^{-1} \end{array}$$

It can be proved that the above definition respects the relations

$$X'_s X'_t = q^{2\sigma'_{st}} X'_t X'_s$$

so $\rho_{\lambda'}$ is indeed a representation of $\mathcal{T}^q_{\lambda'}$, it is irreducible because of the dimension of V, and by construction verifies $\rho_{\lambda'} = \rho_\lambda \circ \Phi^q_{\lambda\lambda'}$. □

Let $\mathscr{R}_{irr}(\mathcal{T}^q_S)$ denote the set of the isomorphism classes of irreducible representations of the quantum Teichmüller space $\mathcal{T}^q_S$. If $\mathrm{Repr}(\mathcal{T}^1_S, \mathbb{C})$ is the set of all the (irreducible) representations of the non-quantum Teichmüller space $\mathcal{T}^1_S$ in $\mathrm{End}(\mathbb{C})$, then $\mathrm{Repr}(\mathcal{T}^1_S, \mathbb{C})$ is clearly a natural set of representatives for $\mathscr{R}_{irr}(\mathcal{T}^1_S)$.

Now we have all the elements to state the following classification theorem for irreducible representations of the quantum Teichmüller space:

Theorem 3.2.12. *Let S be a closed punctured surface with genus g and p punctures such that $\chi(S) < 0$ and let $q \in \mathbb{C}^*$ be a primitive N-th root of $(-1)^{N+1}$. Then, the application*

$$\begin{array}{ccc} \mathscr{R}_{irr}(\mathcal{T}^q_S) & \longrightarrow & \mathrm{Repr}(\mathcal{T}^1_S, \mathbb{C}) \\ [\rho] & \longmapsto & \rho^1 \end{array}$$

sending an isomorphism class of a representation ρ in its non-quantum shadow ρ^1 is well defined and onto. Moreover, the fibre on every element of $\mathrm{Repr}(\mathcal{T}^1_S, \mathbb{C})$ *is composed of N^p classes in $\mathscr{R}_{irr}(\mathcal{T}^q_S)$ if N is odd, and by $2^{2g}N^p$ classes if N is even. Fixed $\lambda \in \Lambda(S)$ a triangulation, each element of the fibre on ρ^1 is determined by the choices an N-th root of the following functions of the $x_i = \rho^1_\lambda(X_i) \in \mathbb{C}^*$:*

- *$x_1^{k_{j1}} x_2^{k_{j2}} \cdots x_n^{k_{jn}}$ for $j = 1, \ldots, p-1$, where k_j is the vector associated with the j-th puncture, as in Definition 1.2.5;*
- *$x_1 x_2 \cdots x_n$.*

and, if N is even, also by a square root of the $x_1^{l_{k1}} x_2^{l_{k2}} \cdots x_n^{l_{kn}}$ for $k = 1, \ldots, 2g$, where the vectors $l_k = (l_{k1}, \ldots, l_{kn})$ are defined before Lemma 1.2.8.

Proof. It is immediate to verify that this application is well defined, in view of the definition of isomorphism between representations and of Theorem 3.2.10. It is onto by virtue of the combinations of Theorem 2.1.9 and Lemma 3.2.11. Also the assertion on the cardinality of the fibre follows immediately from Theorem 2.1.9 and Lemma 3.2.11. Indeed, given ρ^1 a representation of $\mathcal{T}^1_\lambda$ and fixed $\lambda \in \Lambda(S)$, we can construct an irreducible representation ρ_λ of $\mathcal{T}^q_\lambda$ with non-quantum shadow ρ_λ^1 by choosing certain roots as in the statement and, thanks to Lemma 3.2.11, we can extend it to an irreducible representation of $\mathcal{T}_S^q$. Different choices of ρ_λ, endowed with the same roots, conduce to isomorphic representations. Moreover, all the possible choices of roots can be realized as invariants of a ρ_λ that is part of a representation of $\mathcal{T}_\lambda^q$, still because of Theorem 2.1.9 and Lemma 3.2.11, and this concludes the proof. □

3.2.2 Local representations

Given S a surface and λ a certain triangulation of it, recall that a local representation of the Chekhov-Fock algebra $\mathcal{T}_\lambda^q$ is an equivalence class $[\rho_1, \dots, \rho_m]$, where $\rho_j \colon \mathcal{T}_\lambda^q \to \mathrm{End}(V_{\lambda,j})$ is an irreducible representation of $\mathcal{T}_{T_j}^q$ for each triangle T_j in λ. We have seen that a local representation $[\rho_1, \dots, \rho_m]$ induces a representation of $\mathcal{T}_\lambda^q$ in the ordinary sense, by defining

$$\rho := (\rho_1 \otimes \cdots \otimes \rho_m) \circ \hat{\imath}_\lambda : \mathcal{T}_\lambda^q \longrightarrow \mathrm{End}(V_{\lambda,1} \otimes \cdots \otimes V_{\lambda,m}) = \mathrm{End}(V_\lambda)$$

Hereafter, with abuse, we will denote a local representation $[\rho_1, \dots, \rho_m]$ by the representation $\rho \colon \mathcal{T}_\lambda^q \to \mathrm{End}(V_\lambda)$.

Definition 3.2.13. A *local representation* of the quantum Teichmüller space is a collection $\rho = \{\rho_\lambda \colon \mathcal{T}_\lambda^q \to \mathrm{End}(V_\lambda)\}_{\lambda \in \Lambda(S)}$, where

- for every $\lambda \in \Lambda(S)$ the map $\rho_\lambda \colon \mathcal{T}_\lambda^q \to \mathrm{End}(V_\lambda)$ is a local representation of $\mathcal{T}_\lambda^q$;
- for every $\lambda, \lambda' \in \Lambda(S)$ there exists a linear isomorphism $L_{\lambda\lambda'} \colon V_{\lambda'} \to V_\lambda$ such that the representation $L_{\lambda\lambda'} \circ \rho_{\lambda'}(\cdot) \circ (L_{\lambda\lambda'})^{-1}$ is compatible with ρ_λ.

The difference between this definition and the one given in 3.2.3 is motivated by the fact that the vector space in which a local representation of $\mathcal{T}_\lambda^q$ arrives is naturally endowed with a decomposition as tensor product of vector spaces associated with the triangles of λ, and there is not a canonical way to identify decompositions associated with different triangulations.

Definition 3.2.14. Two local representations $\rho = \{\rho_\lambda \colon \mathcal{T}_\lambda^q \to \mathrm{End}(V_\lambda)\}_{\lambda \in \Lambda(S)}$ and $\rho' = \{\rho'_\lambda \colon \mathcal{T}_\lambda^q \to \mathrm{End}(V'_\lambda)\}_{\lambda \in \Lambda(S)}$ are said to be *isomorphic* if, for every $\lambda \in \Lambda(S)$ the representations ρ_λ and ρ'_λ are isomorphic (as local representations, recall Corollary 2.2.7).

Observe that, in the proof of Lemma 3.2.9, the only property of irreducible representations used is the fact that $\rho_\lambda(X_i^N)$ is a multiple of the identity map, so, by replacing $\rho_{\lambda'}$ with $L_{\lambda\lambda'} \circ \rho_{\lambda'}(\cdot) \circ L_{\lambda\lambda'}^{-1}$, the same proof leads us to an equivalent statement for local representations (in [BBL07, Proposition 10] can be found a proof of this fact for local representations, which is less intimidating that the one of Lemma 3.2.9, but it is specifically for the local case). Hence, we immediately deduce the following theorem for local representations:

Theorem 3.2.15. *Given $\rho = \{\rho_\lambda \colon \mathcal{T}_\lambda^q \to \mathrm{End}(V_\lambda)\}_{\lambda \in \Lambda(S)}$ a local representation of the quantum Teichmüller space $\mathcal{T}_S^q$, the collection $\rho^1 = \{\rho_\lambda^1 \colon \mathcal{T}_\lambda^q \to \mathrm{End}(\mathbb{C})\}$ of the non-quantum shadows of ρ is a representation of the non-quantum space $\mathcal{T}_S^1$, called the non-quantum shadow of ρ.*

The non-quantum shadow of a local representation is defined just as in the irreducible case. We define the *central load* of a local representation $\rho = \{\rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)\}_{\lambda \in \Lambda(S)}$ of the quantum Teichmüller space as the central load of ρ_λ, for any $\lambda \in \Lambda(S)$. This is a good definition, in light of Lemma 3.1.2.

Definition 3.2.16. Let R and S be surfaces, with S obtained by fusion from R. Given $\eta = \{\eta_\mu \colon \mathcal{T}^q_\mu \to \mathrm{End}(W_\mu)\}_{\mu \in \Lambda(R)}$ a local representation of $\mathcal{T}^q_R$ and $\rho = \{\rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)\}_{\lambda \in \Lambda(S)}$ a local representation of $\mathcal{T}^q_S$, ρ is said to be obtained *by fusion* from η if η_μ represents ρ_λ (recall Definition 2.2.2) for every ideal triangulation $\mu \in \Lambda(R)$, where λ denote the ideal triangulation on S obtained by fusion from μ.

Remark 3.2.17. It is possible that there exist representations η of $\mathcal{T}^q_R$ such that there is not any local representation ρ of $\mathcal{T}^q_S$ that is obtained by fusion from η. For example, take $R = T_1 \sqcup T_2$ and $S = Q$, the square obtained by identifying a certain couple of edges in T_1 and T_2. R admits only one ideal triangulation μ_0, so a local representation of $\mathcal{T}^q_R$ is just a local representation of $\mathcal{T}^q_{\mu_0}$. Now choose a local representation η_{μ_0} such that its fusion ρ_λ on λ, the induced triangulation on Q, has -1 as invariant of the diagonal in λ of Q. Such a η_{μ_0} can be clearly constructed. Now it is evident that ρ_λ can not be extended to a whole representation $\mathcal{T}^q_S$, because $\rho_\lambda \circ \Phi^q_{\lambda\lambda'}$ does not make sense (we are denoting by λ' the triangulation on Q obtained by diagonal exchange from λ), see Theorem 3.2.15. The point is that η^1, the non-quantum shadow of η, can lead to a collection of non-quantum shadows that can not be extended to a non-quantum representation of $\mathcal{T}^1_S$.

Lemma 3.2.18. *Let R be a surface (see Chapter 0 for details), endowed with an ideal triangulation $\mu \in \Lambda(R)$. Suppose that S is a surface constructed by fusion of R, and denote by λ the triangulation of S induced by μ. If $\rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)$ is a local representation of $\mathcal{T}^q_\lambda$ constructed as fusion of a local representation $\rho_\mu \colon \mathcal{T}^q_\mu \to \mathrm{End}(V_\mu)$ of $\mathcal{T}^q_\mu$, with $V_\lambda = V_\mu$, then the non-quantum shadow ρ^1_λ of ρ_λ is the fusion of the non-quantum shadow ρ^1_μ of ρ_μ, i. e. $\rho^1_\lambda = \rho^1_\mu \circ \iota_{\lambda\mu}$.*

Proof. We will denote by X_i the generators of $\mathcal{T}^1_\lambda$, $\mathcal{T}^q_\lambda$ (it will be clear which is from the contest) and by Y_i the generators of $\mathcal{T}^1_\mu$, $\mathcal{T}^q_\mu$. Recall the definition of the maps $\iota_{\mu\lambda}$ and observe

- if λ_i is an edge in the boundary of S, then it comes from only one edge μ_{a_i} of μ in R, so $\iota_{\mu\lambda}(X_i^N) = (Y_{a_i})^N \in \mathcal{T}^q_\mu$. Hence

$$\begin{aligned}\rho^1_\lambda(X_i)\, id_{V_1 \otimes \cdots \otimes V_m} &= \rho_\lambda(X_i^N) = \rho_\mu(Y_{a_i}^N)\\ &= \rho^1_\mu(Y_{a_i})\, id_{V_1 \otimes \cdots \otimes V_m}\\ &= (\rho^1_\mu \circ \Phi^1_{\mu\lambda})(X_i)\, id_{V_1 \otimes \cdots \otimes V_m}\end{aligned}$$

  The same can be said in the case in which λ_i is internal, but is not the result of a fusion of a couple of edges in μ.

- if λ_i is an internal edge of λ, which is the fusion of a couple of edges μ_{a_i}, μ_{b_i}, then we have

$$\begin{aligned}\rho^1_\lambda(X_i)\, id_{V_1 \otimes \cdots \otimes V_m} &= \rho_\lambda(X_i^N)\\ &= \rho_\mu((q^{-\sigma_{a_i b_i}} Y_{a_i} Y_{b_i})^N)\end{aligned}$$

$$
\begin{aligned}
&= \rho_\mu(q^{-N\sigma'_{a_ib_i}-\sigma'_{a_ib_i}N(N-1)}Y_{a_i}^NY_{b_i}^N) \\
&= \rho_\mu^1(Y_{a_i}Y_{b_i})\, id_{V_1\otimes\cdots\otimes V_m} \\
&= (\rho_\mu^1 \circ \Phi_{\mu\lambda}^1)(X_i)\, id_{V_1\otimes\cdots\otimes V_m}
\end{aligned}
$$

where we are using the hypothesis $q^N = (-1)^{N+1}$.

□

Lemma 3.2.19. *Let $\rho^1 = \{\rho_\lambda^1 \colon \mathcal{T}_\lambda^1 \to \mathrm{End}(\mathbb{C})\}_{\lambda\in\Lambda(S)}$ be a representation of the non-quantum Teichmüller space $\mathcal{T}_S^1$. Fix $\lambda_0 \in \Lambda(S)$ and consider h an N-th root of $x_1\cdots x_n$. Then, there exists a local representation $\rho = \{\rho_\lambda \colon \mathcal{T}_\lambda^q \to \mathrm{End}(V_\lambda)\}_{\lambda\in\Lambda(S)}$ with non-quantum shadow ρ^1 and such that, for every $\lambda \in \Lambda(S)$, h is the invariant associated with the element $H \in \mathcal{T}_\lambda^q$, as in Theorem 2.2.6. In addition, such a ρ is unique up to isomorphism of local representations of $\mathcal{T}_S^q$.*

Proof. Thanks to Lemma 3.1.2, if $\rho = \{\rho_\lambda \colon \mathcal{T}_\lambda^q \to \mathrm{End}(V_\lambda)\}_{\lambda\in\Lambda(S)}$ is a local representation having h as invariant for a fixed triangulation λ_0, then for every $\lambda \in \Lambda(S)$ h is the invariant associated with ρ_λ.

Fix, for every $\lambda \in \Lambda(S)$, a local representation $\rho_\lambda \colon \mathcal{T}_\lambda^q \to \mathrm{End}(V_\lambda)$ with non-quantum shadow ρ_λ^1 and with central load h, which exists by Theorem 2.2.6. In order to prove that this collection of representations is a local representation of the quantum Teichmüller space $\mathcal{T}_S^q$, Lemma 3.2.2 tells us that it is sufficient to show the compatibility for λ and λ' that differ by a diagonal exchange (the re-indexing case is, as usual, obvious). Split S along the sides of the square in which the diagonal exchange occurs, obtaining the disjoint union of an embedded square Q and of a surface R, possibly disconnected. Let λ_R denote the triangulation induced by λ (and λ') on R and λ_Q, λ'_Q denote the two possible triangulations of the square Q, so that λ is obtained by fusion of the triangulation $\mu := \lambda_Q \sqcup \lambda_R$ on $Q \sqcup R$, and λ' of $\mu' := \lambda'_Q \sqcup \lambda_R$. Without loss of generality, we can assume that $Q = T_1 \cup T_2 = T'_1 \cup T'_2$, where the T_i and T'_i are the faces of the triangulations λ and λ', respectively. Fixed a representative $(\rho_1, \ldots, \rho_m)$ of ρ_λ, the representation ρ_λ can be realized as the fusion of $\rho_{\lambda_Q} \otimes \rho_{\lambda_R}$, where $\rho_{\lambda_Q} := \rho_1 \otimes \rho_2$ and $\rho_{\lambda_R} := \rho_3 \otimes \cdots \otimes \rho_m$. Assume for the moment that there exists a representation

$$\rho_{\lambda'_Q} : \mathcal{T}_{\lambda'_Q}^q \longrightarrow \mathrm{End}(V_{\lambda'_Q})$$

such that $\rho_{\lambda'_Q}$ is isomorphic to $\rho_{\lambda_Q} \circ \Phi_{\lambda_Q\lambda'_Q}^q$ and let us conclude the proof.

Denote by $\rho'_{\lambda'}$ the fusion of $\rho_{\lambda'_Q} \otimes \rho_{\lambda_R}$ on S. By construction, the central load of $\rho'_{\lambda'}$ is equal to the central load of ρ_λ and so to the one of $\rho_{\lambda'}$. In addition, the following relations hold

$$
\begin{aligned}
\rho_{\lambda'}^1 &= \rho_\lambda^1 \circ \Phi_{\lambda\lambda'}^1 \\
&= (\rho_{\lambda_Q}^1 \otimes \rho_{\lambda_R}^1) \circ \hat{\iota}_{\mu\lambda} \circ \Phi_{\lambda\lambda'}^1 && \text{Lemma 3.2.18} \\
&= (\rho_{\lambda_Q}^1 \otimes \rho_{\lambda_R}^1) \circ \Phi_{\mu\mu'}^1 \circ \hat{\iota}_{\mu'\lambda'} && \text{Fusion property} \\
&= (\rho_{\lambda_Q}^1 \circ \Phi_{\lambda_Q\lambda'_Q}^1 \otimes \rho_{\lambda_R}^1) \circ \hat{\iota}_{\mu'\lambda'} && \text{Disjoint Union property} \\
&= (\rho_{\lambda'_Q}^1 \otimes \rho_{\lambda_R}) \circ \hat{\iota}_{\mu'\lambda'} && \text{Theorem 3.2.15} \\
&= (\rho'_{\lambda'})^1 && \text{Lemma 3.2.18}
\end{aligned}
$$

Hence, having the same non-quantum shadow and central load, the representations $\rho_{\lambda'}$ and $\rho'_{\lambda'}$ are isomorphic. On the other hand, the representation $\rho'_{\lambda'}$ is isomorphic to $\rho_\lambda \circ \Phi^q_{\lambda\lambda'}$ too, indeed

$$\begin{aligned}
\rho'_{\lambda'} &= (\rho_{\lambda'_Q} \otimes \rho_{\lambda_R}) \circ \iota_{\mu'\lambda'} \\
&\cong (\rho_{\lambda_Q} \otimes \rho_{\lambda_R}) \circ \Phi^q_{\mu\mu'} \circ \hat{\iota}_{\mu'\lambda'} && \text{by contruction + Disjoint Union property} \\
&\cong (\rho_{\lambda_Q} \otimes \rho_{\lambda_R}) \circ \hat{\iota}_{\mu\lambda} \circ \Phi^q_{\lambda\lambda'} && \text{Fusion property} \\
&= \rho_\lambda \circ \Phi^q_{\lambda\lambda'}
\end{aligned}$$

Therefore, we have proved that ρ_λ and $\rho_{\lambda'} \circ \Phi^q_{\lambda\lambda'}$ are isomorphic, as desired.

The last thing we need to check is that there exists a local representation $\rho_{\lambda'_Q}$ such that $\rho_{\lambda_Q} \circ \Phi^q_{\lambda_Q\lambda'_Q}$ is isomorphic to $\rho_{\lambda'_Q}$. The first obstruction for the existence of such a $\rho_{\lambda'_Q}$ is the well definition of the composition $\rho^1_{\lambda_Q} \circ \Phi^1_{\lambda_Q\lambda'_Q}$.

The map $\rho^1_\lambda \circ \Phi^1_{\lambda\lambda'}$ makes sense, because ρ^1 is a non-quantum representation of $\mathcal{T}^1_S$ and, by inspection of the relations defining $\Phi^1_{\lambda\lambda'}$, this implies $\rho^1_\lambda(X_i) \neq -1$. On the other hand, Lemma 3.2.18 tells us that $\rho^1_{\lambda_Q}(X_i)$ is equal to $\rho^1_\lambda(X_i)$. Therefore we have $\rho_{\lambda_Q}(X_i) \neq -1$ and this implies the composition $\rho^1_{\lambda'_Q} := \rho^1_{\lambda_Q} \circ \Phi^1_{\lambda_Q\lambda'_Q}$ is well defined, still by inspection of $\Phi^1_{\lambda_Q\lambda'_Q}$. Now it is sufficient to choose a representation $\rho_{\lambda'_Q}$ having $\rho^1_{\lambda'_Q}$ as non-quantum shadow and the central load of ρ_{λ_Q} as its own central load, choice that can be done by virtue of Theorem 2.2.6. The uniqueness follows immediately from Theorem 2.2.6. □

Let $\mathscr{R}_{loc}(\mathcal{T}^q_S)$ denote the set of the isomorphism classes of local representations of the quantum Teichmüller space $\mathcal{T}^q_S$.

Theorem 3.2.20. *Let S be a surface (see Chapter 0 for details) and let $q \in \mathbb{C}^*$ be a primitive N-th root of $(-1)^{N+1}$. Then, the application*

$$\begin{array}{ccc}
\mathscr{R}_{loc}(\mathcal{T}^q_S) & \longrightarrow & \mathrm{Repr}(\mathcal{T}^1_S, \mathbb{C}) \\
[\rho] & \longmapsto & \rho^1
\end{array}$$

that sends an isomorphism class of a local representation ρ in its non-quantum shadow ρ^1 is well defined and onto. Moreover, the fibre on every element of $\mathrm{Repr}(\mathcal{T}^1_S, \mathbb{C})$ is composed of N classes in $\mathscr{R}_{loc}(\mathcal{T}^q_S)$ and each element of the fibre on ρ^1 is determined by the choice an N-th root of the $x_1 x_2 \cdots x_n$, where $x_i = \rho^1_\lambda(X_i)$, for a certain $\lambda \in \Lambda(S)$.

Proof. It is analogous to the proof of Theorem 3.2.12, we only need to make use of Lemma 3.2.19 and Theorem 2.2.6 instead of Lemma 3.2.11 and Theorem 2.1.9. □

3.3 Pleated surfaces

In this Section we restrict our attention to the case in which S is a closed punctured surface, i. e. it is obtained from a closed orientable surface $\overline{S}$ by removing $p \geq 1$ punctures, with $\chi(S) < 0$. We will always denote by $p\colon \widetilde{S} \to S$ the universal covering of S.

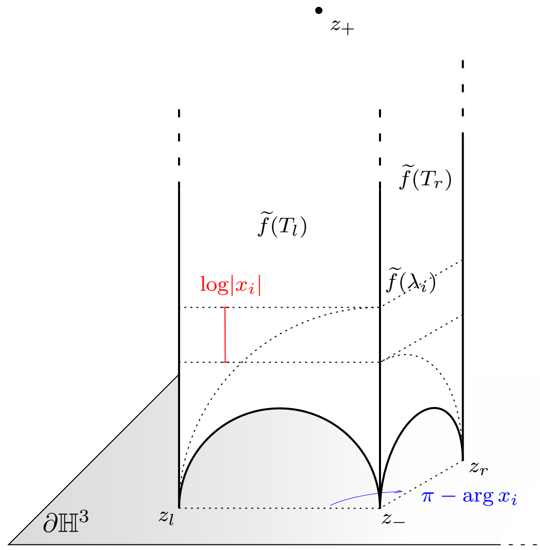

Figure 3.2: Shear-bend coordinates

Definition 3.3.1. Given $\lambda \in \Lambda(S)$ an ideal triangulation, a *pleated surface* with pleating locus λ is a pair (f, r), where $\widetilde{f}\colon \widetilde{S} \to \mathbb{H}^3$ is a continuous map from $\widetilde{S}$ to the 3-dimensional hyperbolic space $\mathbb{H}^3$ and $r\colon \pi_1(S) \to \mathbb{P}\,\mathrm{SL}(2,\mathbb{C})$ is a group homomorphism from the fundamental group of S in the group of orientation-preserving isometries of $\mathbb{H}^3$, such that:

- if $\widetilde{\lambda}$ is the preimage in $\widetilde{S}$ of the ideal triangulation λ of S, then $\widetilde{f}$ homeomorphically sends each edge of $\widetilde{\lambda}$ in a complete geodesic of $\mathbb{H}^3$;
- $\widetilde{f}$ homeomorphically sends the closure of each component of $\widetilde{S} \setminus \widetilde{\lambda}$ in an ideal triangle of $\mathbb{H}^3$;
- $\widetilde{f}$ is r-equivariant, i. e. for every $\gamma \in \pi_1(S)$ and for every $\widetilde{x} \in \widetilde{S}$ we have $\widetilde{f}(\gamma(\widetilde{x})) = r(\gamma)(\widetilde{f}(\widetilde{x}))$.

Let $(\widetilde{f}, r)$ be a pleated surface with pleating locus λ, let λ_i be an edge of λ and choose $\widetilde{\lambda}_i$ a preimage of λ_i through the universal covering $p\colon \widetilde{S} \to S$. Fix arbitrarily an orientation on λ_i and the induced one on $\widetilde{\lambda}_i$. Denote by T_l and T_r the components of $\widetilde{S} \setminus \widetilde{\lambda}$ on the left and on the right, respectively, of $\widetilde{\lambda}_i$ (remember that S is oriented). Let $z_+, z_- \in \partial\mathbb{H}^3$ be the ends of the geodesic $\widetilde{f}(\widetilde{\lambda}_i)$, the one in which $\widetilde{f}(\widetilde{\lambda})$ converges and the one from which it comes from respectively. Moreover, let z_l be the vertex of the triangle $\widetilde{f}(T_l)$ opposite to the edge $\widetilde{f}(\widetilde{\lambda}_i)$ and analogously z_r for $\widetilde{f}(T_r)$ (see Figure 3.2). Now we define x_i, namely the *exponential shear-bend parameter* of the pleated surface $(\widetilde{f}, r)$ along λ_i, as the complex number

$$x_i = -\frac{(z_l - z_+)(z_r - z_-)}{(z_l - z_-)(z_r - z_+)} \tag{3.8}$$

It is clear from the definition that x_i does not depend on the choice of the orientation of λ_i, because of its symmetries as function of z_-, z_+, z_l, z_r. It does not depend on the lift $\widetilde{\lambda}_i$ of λ_i either, because $\widetilde{f}$ is r-equivariant and r acts by isometries, so any automorphism in $\pi_1(S)$ preserves x_i (recall that orientation preserving isometries of $\mathbb{H}^3$ act like projective isomorphisms of $\partial\mathbb{H}^3$ and hence they preserve cross ratios). Observe that, by post-composing $\tilde{f}$ with a certain isometry, we can suppose that, in the half-space model of $\mathbb{H}^3$, the boundary points z_+, z_-, z_l are equal to $\infty, 0, -1 \in \widehat{C} := \mathbb{C} \cup \{\infty\} = \partial\mathbb{H}^3$, respectively. Then the shear-bend parameter associated with λ_i assumes the following form

$$x_i = -\frac{(-1-\infty)(z_r - 0)}{(-1-0)(z_r - \infty)} = z_r$$

Consequently, the argument of x_i is equal to the external dihedral angle of the ridge of $\widetilde{f}(\widetilde{S})$ along $\widetilde{\lambda}_i$, and $\log|x_i|$ is the signed distance between the intersections of $\widetilde{f}(\widetilde{\lambda}_i)$ and the two geodesics in $\mathbb{H}^3$ that are perpendicular to $\widetilde{f}(\widetilde{\lambda}_i)$ and that arrive in the points z_l and z_r of $\partial\mathbb{H}^3$, respectively (see Figure 3.2).

Definition 3.3.2. Two pleated surfaces $(\widetilde{f}, r)$ and $(\widetilde{f}', r')$ are *isometric* if there exist a hyperbolic isometry $A \in \mathbb{P}\mathrm{SL}(2,\mathbb{C})$ and a lift $\widetilde{\varphi}\colon \widetilde{S} \to \widetilde{S}$ of a isotopically trivial diffeomorphism of S such that, for every $\gamma \in \pi_1(S)$, we have $r'(\gamma) = A \circ r(\gamma) \circ A^{-1} \in \mathbb{P}\mathrm{SL}(2,\mathbb{C})$ and $\widetilde{f}' = A \circ \widetilde{f} \circ \widetilde{\varphi}$.

Proposition 3.3.3. *Let $\lambda \in \Lambda(S)$ be an ideal triangulation. Two pleated surfaces $(\widetilde{f}, r)$ and $(\widetilde{f}', r')$ with pleating locus λ are isometric if and only if they have the same shear-bend parameters $x_i \in \mathbb{C}^*$ for every edge λ_i in λ. Conversely, every set of weights $x_i \in \mathbb{C}^*$ on the edges of λ can be realized as the shear-bend parameters of a pleated surface with pleating locus λ.*

Proof. Fix λ_i an edge of λ whose shear-bend coordinate is $x_i \neq -1$, and $\widetilde{\lambda}_i$ a lift of λ_i to the universal covering $\widetilde{S}$. Moreover, choose an arbitrary orientation on $\widetilde{\lambda}_i$. Let z_-, z_+, z_l and z_r be the points in $\partial\mathbb{H}^3$ associated with $(\widetilde{f}, r)$ as described above, and analogously z'_-, z'_+, z'_l and z'_r for $(\widetilde{f}', r')$. Then we can find an orientation-preserving isometry $A \in \mathrm{Isom}^+(\mathbb{H}^3)$ that takes the ordered triple (z_l, z_-, z_+) in (z'_l, z'_-, z'_+). Now it is immediate to check that, since $(\widetilde{f}, r)$ and $(\widetilde{f}', r')$ have the same shear-bend coordinates, the maps $A \circ \widetilde{f}$ and $\widetilde{f}'$ sends each edge of $\widetilde{\lambda}$ and each component of $\widetilde{S} \setminus \widetilde{\lambda}$ in the same geodesic and in the same ideal triangle. Moreover, the following holds

$$A \circ r(\gamma) \circ A^{-1} = r'(\gamma)$$

for every $\gamma \in \pi_1(S)$. In conclusion, we can easily construct a $\pi_1(S)$-equivariant isotopy $\widetilde{\varphi}\colon \widetilde{S} \to \widetilde{S}$ with $\widetilde{\varphi}_0 = id_{\widetilde{S}}$ and such that $A \circ \widetilde{f} \circ \widetilde{\varphi}$ is equal to $\widetilde{\varphi}$.

To prove the second assertion, it is clear how to construct $\widetilde{f}$ on a fundamental domain for S in order to obtain the candidate shear-coordinates, then it is sufficient to extend it by r-equivariance on all $\widetilde{S}$. □

Definition 3.3.4. A *peripheral subgroup* π of $\pi_1(S)$, the fundamental group of S, is a subgroup obtained in the following way: there exist a puncture of S, a small neighbourhood A of it and a choice of base points and paths such that π is the image of the map induced by the inclusion $\pi_1(A) \to \pi_1(S)$.

We will denote by Π the set of all the peripheral subgroups of $\pi_1(S)$.

Let $\widetilde{A}$ be the preimage of A through the universal covering p. Then each component of $\widetilde{A}$ corresponds to a certain peripheral subgroup. If $\widetilde{A}_\pi$ is a component corresponding to the subgroup π, then the images under $\widetilde{f}$ of the triangles of $\widetilde{S} \setminus \widetilde{\lambda}$ that meet $\widetilde{A}_\pi$ all have a vertex $z_\pi \in \widehat{\mathbb{C}} = \partial\mathbb{H}^3$ in common, and this vertex is fixed by all the isometries in $r(\pi)$. So $\widetilde{f}$ associates with each peripheral subgroup π a point $z_\pi \in \widehat{\mathbb{C}}$ in the stabilizer of $r(\pi)$. This association is r-equivariant, in the sense that $z_{\gamma\pi\gamma^{-1}} = r(\gamma)(z_\pi)$ for every $\gamma \in \pi_1(S)$ and π peripheral subgroup.

Definition 3.3.5. An *enhanced homomorphism* $(r, \{z_\pi\}_{\pi\in\Pi})$ from $\pi_1(S)$ to the group $\mathbb{P}\,\mathrm{SL}(2,\mathbb{C})$ consists of a group homomorphism $r\colon \pi_1(S) \to \mathbb{P}\,\mathrm{SL}(2,\mathbb{C})$ and a r-equivariant assignment of a fixed point $z_\pi \in \widehat{\mathbb{C}}$ to each peripheral subgroup π of $\pi_1(S)$.

An enhanced homomorphism $(r, \{z_\pi\}_{\pi\in\Pi})$ *realizes* the ideal triangulation λ if it is associated with a pleated surface $(\widetilde{f}, r)$ with pleating locus λ, as above. It is *peripherally generic* if it realizes every ideal triangulation λ of S.

Lemma 3.3.6. *Every enhanced homomorphism $(r, \{z_\pi\}_{\pi\in\Pi})$ in which r is injective is peripherally generic.*

Proof. We want to show that, because of the injectivity of r, there can not exist π, π' peripheral subgroup such that $z_\pi = z_{\pi'}$. Suppose, by contradiction that this happens, then we would have that both $r(\pi)$ and $r(\pi')$ are contained in the stabilizer in $\mathbb{P}\,\mathrm{SL}(2,\mathbb{C})$ of the point $z_\pi = z_{\pi'} \in \partial\mathbb{H}^3$, which is isomorphic to $\mathrm{Aff}(\mathbb{C})$, and so solvable. Hence we would have that the subgroup generated by π and π', which is free of rank 2, would be embbedded through r in a solvable group, and so it would be solvable too, which is absurd. So the association $\pi \mapsto z_\pi \in \widehat{\mathbb{C}}$ is injective.

Therefore, we can construct a pleated surface $(\widetilde{f}, r)$ with pleating locus λ by sending each $\widetilde{\lambda}_i$, lift of some λ_i, in the complete geodesic in $\mathbb{H}^3$ joining the points $z_\pi, z_{\pi'} \in \partial\mathbb{H}^3$ associated with the peripheral subgroup corresponding to the ends of $\widetilde{\lambda}_i$. This operation can be done r-equivariantly, and leads us to a $\widetilde{f}$ defined on $\widetilde{\lambda}$. Now we send every component T of $\widetilde{S} \setminus \widetilde{\lambda}$, with boundary $\widetilde{\lambda}_i \cup \widetilde{\lambda}_j \cup \widetilde{\lambda}_k$, in the unique triangle in $\mathbb{H}^3$ that has as edges the geodesics $\widetilde{f}(\widetilde{\lambda}_i), \widetilde{f}(\widetilde{\lambda}_j), \widetilde{f}(\widetilde{\lambda}_k)$. Even in this case the definition can be done in order to obtain a r-equivariant map $\widetilde{f}$ defined on all $\widetilde{S}$. $\square$

Lemma 3.3.6 tells us that the set of peripherally generic enhanced homomorphisms contains a large class of geometrically interesting homomorphisms.

Observe that a generic homomorphism $r\colon \pi_1(S) \to \mathbb{P}\,\mathrm{SL}(2,\mathbb{C})$ admits a few possible enhancements. Indeed, if the subgroup $r(\pi)$ is generated by a hyperbolic or elliptic transformation, then it has exactly two fixed points in $\partial\mathbb{H}^3$, the ends of the axis of $r(\pi)$. If $r(\pi)$ is parabolic, then it has only one fixed point in $\partial\mathbb{H}^3$, so there is a unique enhancement that can be realized. The unique case in which we have infinite possibilities is the one in which $r(\pi)$ is the trivial group, but it is clear that this is non-generic.

Definition 3.3.7. Two enhanced homomorphisms $(r, \{z_\pi\}_{\pi\in\Pi})$, $(r', \{z'_\pi\}_{\pi\in\Pi})$ are said to be *conjugated* if there exists an isometry $A \in \mathbb{P}\,\mathrm{SL}(2,\mathbb{C})$ such that $r(\gamma) = A \circ r'(\gamma) \circ A^{-1}$ for every $\gamma \in \pi_1(S)$ and $z_\pi = A(z'_\pi)$ for every peripheral subgroup $\pi \in \Pi$.

We can associate, with each non-quantum representation $\rho^1_\lambda \colon \mathcal{T}^1_\lambda \to \mathrm{End}(\mathbb{C})$, the isometry class of a pleated surface $(\widetilde{f}_\lambda, r_\lambda)$ with pleating locus $\lambda \in \Lambda(S)$ realizing the numbers $x_i := \rho^1(X_i) \in \mathbb{C}^*$ as shear-bending coordinates, and this map is well defined by virtue of Proposition 3.3.3. It is clear that two isometric pleated surfaces provide two conjugated enhanced homomorphisms, so we have described a way to associate with a non-quantum representation $\rho^1_\lambda \colon \mathcal{T}^1_\lambda \to \mathrm{End}(\mathbb{C})$ the conjugation class of an enhanced homomorphism $(r, \{z_\pi\}_{\pi \in \Pi})$.

Lemma 3.3.8. *Let $\lambda, \lambda' \in \Lambda(S)$ be two ideal triangulations of a surface S, which differ by a diagonal exchange or a re-indexing. Given $\rho^1_\lambda \colon \mathcal{T}^1_\lambda \to \mathrm{End}(\mathbb{C})$ and $\rho^1_{\lambda'} \colon \mathcal{T}^1_{\lambda'} \to \mathrm{End}(\mathbb{C})$ two compatible non-quantum representations, i. e. $\rho^1_{\lambda'} = \rho^1_\lambda \circ \Phi^1_{\lambda\lambda'}$, then the pleated surfaces $(\widetilde{f}_\lambda, r_\lambda)$ and $(\widetilde{f}_{\lambda'}, r_{\lambda'})$, associated with ρ^1_λ and $\rho^1_{\lambda'}$ respectively, define the same conjugation class of enhanced homomorphisms $(r, \{z_\pi\}_{\pi \in \Pi})$, with $r = r_\lambda = r_{\lambda'}$.*

Proof. When λ' is obtained by re-indexing from λ, the assertion is obvious. Hence we assume that λ' is obtained from λ by diagonal exchange along λ_i. Choose a component $\widetilde{\lambda}_i$ of the preimage of λ_i, with an arbitrary orientation, and denote by $\widetilde{T}_l$ and $\widetilde{T}_r$ the triangles on the left and on the right of $\widetilde{\lambda}_i$, respectively, and by $z_-, z_+, z_l, z_r \in \mathbb{C}^*$ the vertices as previously done. We label as Q_i the square $\widetilde{\lambda}_i \cup \widetilde{T}_l \cup \widetilde{T}_r$ in $\widetilde{S}$, having $\widetilde{\lambda}_i$ as diagonal in $\widetilde{\lambda}$ and $\widetilde{\lambda}'_i$ as diagonal in $\widetilde{\lambda}'$, where $\widetilde{\lambda}'_i$ is the component of the preimage $\widetilde{\lambda}'$ of λ' in $\widetilde{S}$ that is contained in Q_i. Q_i is also equal to $\widetilde{\lambda}'_i \cup \widetilde{T}'_l \cup \widetilde{T}'_r$, where $\widetilde{T}'_l$ and $\widetilde{T}'_r$ are the triangles in $\widetilde{S}$ on the left and on the right, respectively, of $\widetilde{\lambda}'_i$ in Q_i.

By hypothesis, the composition $\rho^1_{\lambda'} \circ \Phi^1_{\lambda\lambda'}$ is well-defined, hence $x_i \neq -1$ or, equivalently, the points z_l and z_r are distinct. Then it is reasonable to consider the geodesic s joining z_- and z_+. Now we modify $\widetilde{f}_\lambda$ in order to obtain a map $\widetilde{f}'_\lambda$, which sends the diagonal $\widetilde{\lambda}'_i$ in the geodesic s and the triangles $\widetilde{T}'_l$ and $\widetilde{T}'_r$ in the ideal triangles $\mathrm{Conv}(z_-, z_l, z_r)$ and $\mathrm{Conv}(z_+, z_l, z_r)$. Now we perform the modification of $\widetilde{f}_\lambda$ on all the components of the preimage of λ_i in a r_λ-equivariant way by acting with the deck transformations. The resulting pleated surface, with pleating locus λ', is composed of the new map $\widetilde{f}'_\lambda$ and the homomorphism r_λ, as $(\widetilde{f}_\lambda, r_\lambda)$, and its enhanced homomorphism $(r_\lambda, \{z_\pi\}_{\pi \in \Pi})$ is exactly the same of $(\widetilde{f}_\lambda, r_\lambda)$, by construction.

Now, showing that the shear-bend coordinates of $(\widetilde{f}'_\lambda, r_\lambda)$ are the ones defined by $\rho^1_\lambda \circ \Phi^1_{\lambda\lambda'}$, then we will conclude that $(\widetilde{f}'_\lambda, r_\lambda)$ and $(\widetilde{f}'_\lambda, r'_\lambda)$ are isometric and in particular that they lead to the same conjugation class of enhanced homomorphisms, as desired.

We should consider all the possible configurations of the square in λ with diagonal λ_i as in the proof of Theorem 3.1.1. We will focus on Case 4 only, i. e. when $\lambda_j = \lambda_l$ and $\lambda_k \neq \lambda_m$, as in Figure 3.1. The other cases can be treated in the same way. Q_i is an embedded square and the condition $\lambda_j = \lambda_l$ tells us that there exists a deck transformation $\gamma \in \pi_1(S)$ such that $\gamma(\widetilde{\lambda}_l) = \widetilde{\lambda}_j$. If the vertices of $\widetilde{f}_\lambda(Q_i)$ are labelled as $z_-, z_l, z_+, z_r \in \widehat{\mathbb{C}}$, starting clockwise from the corner $\widetilde{\lambda}_j \cap \widetilde{\lambda}_k$, then the shear-bend parameter of $\widetilde{f}_\lambda$ associated with λ_i and the one of $\widetilde{f}'_\lambda$ associated with $\widetilde{\lambda}'_i$ are related in the following way

$$x_i = -\frac{(z_r - z_-)(z_l - z_+)}{(z_r - z_+)(z_l - z_-)} \longrightarrow -\frac{(z_- - z_l)(z_+ - z_l)}{(z_- - z_r)(z_+ - z_l)} = x_i^{-1} = x'_i$$

The edge $\widetilde{\lambda}_k$ is mapped under $\widetilde{f}_\lambda$ in a diagonal of the ideal square having vertices z_+, z_-, z_l and z_k, for some $z_k \in \widehat{\mathbb{C}}$. The original coordinate relative to λ_k is

$$x_k = -\frac{(z_k - z_l)(z_+ - z_-)}{(z_k - z_-)(z_+ - z_l)}$$

and the new one is

$$x'_k = -\frac{(z_k - z_l)(z_r - z_-)}{(z_k - z_-)(z_r - z_l)} = (1 + x_i^{-1})^{-1} x_k$$

and in the same spirit $x'_m = (1 + x_i^{-1})^{-1} x_m$. The case of $\lambda_j = \lambda_l$ is different because of the identification. In $\widetilde{\lambda}$ the edge $\widetilde{\lambda}_j$ is the diagonal of a square whose vertices go under $\widetilde{f}_\lambda$ in $z_- = r(\gamma)(z_l)$, z_+, $r(\gamma)(z_r) = z_r$ and $r(\gamma)(z_-)$, whereas $\widetilde{\lambda}'_j$ is the diagonal of a square whose vertices go under $\widetilde{f}_\lambda$ in $z_- = r(\gamma)(z_l)$, z_l, $z_r = r(\gamma)(z_+)$ and $r(\gamma)(z_r)$. Then we have

$$\begin{aligned}
x_j &= -\frac{(r(\gamma)(z_-) - r(\gamma)(z_l))(z_+ - z_r)}{(r(\gamma)(z_-) - r(\gamma)(z_+))(z_+ - z_-)} \\
x'_j &= -\frac{(r(\gamma)(z_r) - r(\gamma)(z_l))(z_l - z_r)}{(r(\gamma)(z_r) - r(\gamma)(z_+))(z_l - z_-)} \\
&= \frac{(r(\gamma)(z_r) - r(\gamma)(z_l))(z_l - z_r)}{(r(\gamma)(z_r) - r(\gamma)(z_+))(z_l - z_-)} \frac{(r(\gamma)(z_-) - r(\gamma)(z_+))(z_+ - z_-)}{(r(\gamma)(z_-) - r(\gamma)(z_l))(z_+ - z_r)} x_j \\
&= \left(\frac{(z_l - z_r)(z_+ - z_-)}{(z_l - z_-)(z_+ - z_r)} \right)^2 x_j = (1 + x_i)^2 x_j
\end{aligned}$$

and this relation concludes the proof, in light of the explicit relations exhibited in the proof of Theorem 3.1.1. □

Proposition 3.3.9. *Every representation $\rho^1 = \{\rho^1_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(\mathbb{C})\}_{\lambda \in \Lambda(S)}$ of the non-quantum Teichmüller space $\mathcal{T}^1_S$ uniquely determines a conjugation class of peripherally generic enhanced homomorphisms $(r, \{z_\pi\}_{\pi \in \Pi})$ such that, for every $\lambda \in \Lambda(S)$ $(r, \{z_\pi\}_{\pi \in \Pi})$ is the enhanced homomorphism associated with a pleated surface $(\widetilde{f}_\lambda, r)$ with pleating locus λ and shear-bend coordinates $x_i := \rho^1_\lambda(X_i) \in \mathbb{C}^*$. Moreover, two representations of $\mathcal{T}^1_S$ leading to the same conjugation class of enhanced homomorphisms must be equal.*

Proof. The first part of the assertion is an immediate consequence of Lemma 3.3.8. In order to prove the second statement, let ρ and ρ' be two non-quantum representations that lead to two conjugated enhanced homomorphisms. Fix $\lambda \in \Lambda(S)$ an ideal triangulation and construct $(\widetilde{f}_\lambda, r_\lambda)$ and $(\widetilde{f}'_\lambda, r'_\lambda)$ two pleated surfaces with shear-bend coordinates given by ρ_λ and ρ'_λ, respectively. Possibly by replacing $(\widetilde{f}_\lambda, r_\lambda)$ with $(A \circ \widetilde{f}_\lambda, A \circ r_\lambda \circ A^{-1})$ for some $A \in \mathrm{Isom}^+(\mathbb{H}^3)$, we can assume that $r = r_\lambda = r'_\lambda$ and that the set $\{z_\pi\}_{\pi \in \Pi}$ is the enhancement induced on r by both $\widetilde{f}_\lambda$ and $\widetilde{f}'_\lambda$.

Now $\widetilde{f}$ and $\widetilde{f}'$ must send each component of $\widetilde{\lambda}$ in the same geodesics, because the ends are obliged to go in the same points z_- and z_+, determined by certain peripheral subgroups. Then it is sufficient to repeat the procedure described in the proof of Proposition 3.3.3 in order to prove that $(\widetilde{f}_\lambda, r_\lambda)$ and $(\widetilde{f}'_\lambda, r'_\lambda)$ are isometric, and in particular that they have the same shear-bend coordinates, which clearly implies the equality of ρ_λ and ρ'_λ. □

Proposition 3.3.10. *Let $(r, \{z_\pi\}_{\pi\in\Pi})$ be an enhanced homomorphism from $\pi_1(S)$ to $\mathrm{Isom}^+(\mathbb{H}^3)$ which is peripherally generic. In particular, for any two ideal triangulations $\lambda, \lambda' \in \Lambda(S)$, it associates weights $x_i \in \mathbb{C}^*$ with the edges of λ, and weights $x'_i \in \mathbb{C}^*$ to the edges of λ'. If we denote by $\rho^1_\lambda \colon \mathcal{T}^1_\lambda \to \mathrm{End}(\mathbb{C})$ and $\rho^1_{\lambda'} \colon \mathcal{T}^1_{\lambda'} \to \mathrm{End}(\mathbb{C})$ the non-quantum representations defined by*

$$\rho^1_\lambda(X_i) = x_i\, id_{\mathbb{C}}$$
$$\rho^1_{\lambda'}(X'_i) = x'_i\, id_{\mathbb{C}}$$

then the two representations are compatible, i. e. $\rho^1_{\lambda'} = \rho^1_\lambda \circ \Phi^1_{\lambda\lambda'}$.

Proof. The proof of the previous Proposition shows that, given a peripherally generic enhanced homomorphism $(r, \{z_\pi\}_{\pi\in\Pi})$ and taken two pleated surfaces with pleating locus λ that both lead to the enhanced homomorphism $(r, \{z_\pi\}_{\pi\in\Pi})$, then they are isometric. Therefore, the induced non-quantum representation ρ^1_λ is well-defined and independent from the choice of the pleated surface.

Now, to prove the assertion, it is sufficient to consider the case of λ and λ' that differ by a diagonal exchange along the edge λ_i, thanks to the Composition relation for $\Phi^1_{\lambda\lambda'}$. Fix a pleated surface $(\widetilde{f}_\lambda, r)$ with pleating locus λ that realizes $(r, \{z_\pi\}_{\pi\in\Pi})$ as induced enhanced homomorphism, and assume for a moment that $x_i \neq -1$. Then we can perform the modification of $(\widetilde{f}_\lambda, r)$ done in the proof of Lemma 3.3.8, in order to obtain a pleated surface $(\widetilde{f}'_\lambda, r)$, with pleating locus λ', shear-bend coordinates $\rho^1_\lambda \circ \Phi^1_{\lambda\lambda'}$ and having $(r, \{z_\pi\}_{\pi\in\Pi})$ as induced enhanced homomorphism, and this concludes.

It remains to prove that $(\widetilde{f}_\lambda, r)$ has shear-bend coordinate $x_i \neq -1$. As before, fix a lift $\widetilde{\lambda}_i$ of λ_i and an arbitrary orientation on it, and let z_-, z_+, z_l, z_r be the corresponding points. $x_i = -1$ if and only if $z_l = z_r$, but this can not happen because $(r, \{z_\pi\}_{\pi\in\Pi})$ is peripherally generic, so in particular it can be realized by the ideal triangulation $\Delta_i(\lambda)$. □

Theorem 3.3.11. *There exists a bijection between the set of conjugation classes of peripherally generic enhanced homomorphisms $(r, \{z_\pi\}_{\pi\in\Pi})$, from $\pi_1(S)$ to $\mathrm{Isom}^+(\mathbb{H}^3)$, and the set of non-quantum representations*

$$\rho^1 = \{\rho^1_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(\mathbb{C})\}_{\lambda\in\Lambda(S)}$$

of the non-quantum Teichmüller space $\mathcal{T}^1_S$, which sends the conjugation class of a peripherally generic enhanced homomorphism $(r, \{z_\pi\}_{\pi\in\Pi})$ in the non-quantum representation of $\mathcal{T}^1_S$ in which, for every $\lambda \in \Lambda(S)$, ρ^1_λ is defined by the relation

$$\rho^1_\lambda(X_i) = x_i\, id_{\mathbb{C}}$$

for every X_i generator of $\mathcal{T}^q_\lambda$, where x_i is the shear-bend coordinate associated with λ_i for a certain pleated surface $(\widetilde{f}, r)$ having $(r, \{z_\pi\}_{\pi\in\Pi})$ as enhanced homomorphism.

Proof. In order to obtain the assertion, it is sufficient to combine Propositions 3.3.10 and 3.3.9. □

Denote by $\mathscr{E}\mathscr{H}(S)$ the set of conjugation classes of peripherally generic enhanced homomorphisms $(r, \{z_\pi\}_\pi)$, from $\pi_1(S)$ to $\mathrm{Isom}^+(\mathbb{H}^3)$. Starting from a quantum irreducible (resp. local) representation ρ, we can associate with it its non-quantum shadow ρ^1, and consequently construct the corresponding conjugation class of peripherally generic enhanced homomorphisms $[r, \{z_\pi\}_\pi]$, which we will call the *hyperbolic shadow* of the irreducible (resp. local) representation ρ. Now we can combine this result with our Classification Theorems for local and irreducible representations and obtain the following statements:

Theorem 3.3.12. *Let S a closed punctured surface, with $\chi(S) < 0$ and let $q \in \mathbb{C}^*$ be a primitive N-th root of $(-1)^{N+1}$. Then the application*

$$\begin{array}{ccc} \mathscr{R}_{irr}(\mathcal{T}_S^q) & \longrightarrow & \mathscr{E}\mathscr{H}(S) \\ [\rho] & \longmapsto & [r, \{z_\pi\}_\pi] \end{array}$$

which sends an isomorphism class of an irreducible representation ρ in its hyperbolic shadow $[r, \{z_\pi\}_\pi]$ is well defined and onto. Moreover, the fibre on every element of $\mathscr{E}\mathscr{H}(S)$ is composed of N^p classes in $\mathscr{R}_{irr}(\mathcal{T}_S^q)$ if N is odd, and by $2^{2g}N^p$ classes if N is even. Fixed $\lambda \in \Lambda(S)$ a triangulation, each element of the fibre on $[r, \{z_\pi\}_\pi]$ is determined by the choices an N-th root of the following functions of the shear-bend coordinates $x_i \in \mathbb{C}^*$, associated with a certain pleated surface $(\widetilde{f}_\lambda, r)$ with pleating locus λ realizing $[r, \{z_\pi\}_\pi]$ as enhanced homomorphism:

- *$x_1^{k_{j1}} x_2^{k_{j2}} \cdots x_n^{k_{jn}}$ for $j = 1, \ldots, p-1$, where k_j is the vector associated with the j-th puncture, as in Definition 1.2.5;*
- *$x_1 x_2 \cdots x_n$.*

and, if N is even, also by a square root of the $x_1^{l_{k1}} x_2^{l_{k2}} \cdots x_n^{l_{kn}}$ for $k = 1, \ldots, 2g$, where the vectors $l_k = (l_{k1}, \ldots, l_{kn})$ are defined before Lemma 1.2.8.

Theorem 3.3.13. *Let S be a surface (see Chapter 0 for details) and let $q \in \mathbb{C}^*$ be a primitive N-th root of $(-1)^{N+1}$. Then, the application*

$$\begin{array}{ccc} \mathscr{R}_{loc}(\mathcal{T}_S^q) & \longrightarrow & \mathscr{E}\mathscr{H}(S) \\ [\rho] & \longmapsto & [r, \{z_\pi\}_\pi] \end{array}$$

that sends an isomorphism class of a local representation ρ in its hyperbolic shadow $[r, \{z_\pi\}_\pi]$ is well defined and onto. Moreover, the fibre on every element of $\mathscr{E}\mathscr{H}(S)$ is composed of N classes in $\mathscr{R}_{loc}(\mathcal{T}_S^q)$. Fixed $\lambda \in \Lambda(S)$, each element of the fibre on ρ^1 is determined by the choice an N-th root of the $x_1 x_2 \cdots x_n$, where the x_i are the shear-bend coordinates associated with a certain pleated surface $(\widetilde{f}_\lambda, r)$ with pleating locus λ realizing $[r, \{z_\pi\}_\pi]$ as enhanced homomorphism.

3.4 Irreducible decomposition for local representations

In this Section we will always assume N odd. Moreover, we will consider closed punctured surfaces S, with genus $g \geq 1$ and with $s+1$ punctures $v_0, \ldots, v_s$. Recall that, in these hypotheses, the following relation holds

$$m = -2\chi(\overline{S}) + 2(s+1) = 4g - 2 + 2s$$

where m denotes the number of ideal triangles composing an ideal triangulation of S. Fix an ideal triangulation $\lambda \in \Lambda(S)$ and $\rho\colon \mathcal{T}^q_\lambda \to \mathrm{End}(V)$ a local representation of the Chekhov-Fock algebra $\mathcal{T}^q_\lambda$, with invariants $x_i \in \mathbb{C}^*$ for every $i = 1, \ldots, n$ and $h \in \mathbb{C}^*$. In Theorem 2.1.9 we have proved that every irreducible representation η of $\mathcal{T}^q_\lambda$ is classified up to isomorphism by the following data:

- for every $i = 1, \ldots, n$ a number $x_i \in \mathbb{C}^*$ such that $\eta(X_i^N) = x_i \, id$;
- for every $j = 1, \ldots, s$ (observe that we are not considering the 0-th puncture) an N-th root $t_j^{1/N}$ of
$$t_j(x_1, \ldots, x_n) := x_1^{k_{j1}} \cdots x_n^{k_{jn}}$$
such that $\eta(P_j) = t_j^{1/N} \, id$;
- an N-th root h of
$$x_1 \cdots x_n$$
such that $\eta(H) = h \, id$.

Given $\alpha \in \mathbb{C}$ a complex number, we denote by $\mathcal{U}(N, \alpha)$ the set of the N-th roots of α. We are going to prove the following result, exposed in [Tou14]:

Theorem 3.4.1. *Let S be a surface with genus $g \geq 1$ and with $s+1$ punctures. Moreover, fix q a primitive N-th root of unity, with N odd, an ideal triangulation $\lambda \in \Lambda(S)$ and $\rho\colon \mathcal{T}^q_\lambda \to \mathrm{End}(V)$ a local representation of the Chekhov-Fock algebra $\mathcal{T}^q_\lambda$ having invariants $x_i \in \mathbb{C}^*$ for every $i = 1, \ldots, n$ and $h \in \mathbb{C}^*$ as central load. Then, for every $\underline{d} = (d_1, \ldots, d_s)$ in*
$$\mathcal{U} := \prod_{j=1}^{s} \mathcal{U}(N, t_j(x_1, \ldots, x_n))$$
the subspace
$$V_{\underline{d}} := \{v \in V \mid \rho(P_j)(v) = d_j \, v \text{ for every } j = 1, \ldots, s\}$$
is invariant for the representation ρ. For varying $\underline{d}$, we obtain a direct sum decomposition of V on which the representation ρ splits as $\rho = \bigoplus_{\underline{d} \in \mathcal{U}} \rho^{\underline{d}}$, with
$$\rho^{\underline{d}} : \mathcal{T}^q_\lambda \longrightarrow \mathrm{End}(V_{\underline{d}})$$
Moreover, for every $\underline{d} \in \mathcal{U}$, the subspace $V_{\underline{d}}$ has dimension equal to N^{4g-2+p} and the representation $\rho^{\underline{d}}$ is isomorphic to a direct sum of N^g irreducible representations $\rho_h^{\underline{d}}$, all having central load equal to h, invariants for the j-th puncture equal to d_j for every $j = 1, \ldots, s$ and invariants for the edges equal to x_i for every $i = 1, \ldots, n$.

Observe that the decomposition of ρ given by $\bigoplus_{\underline{d}} \rho^{\underline{d}}$ is natural and it does depend only on the representation ρ. Nevertheless, the irreducible decomposition of each $\rho^{\underline{d}}$ is not unique.

We will firstly focus on the case in which the ideal triangulation is constructed with the following procedure: fix $\widetilde{\lambda}$ an ideal triangulation of $S \cup \{v_1, \ldots, v_s\} = \overline{S} \setminus \{v_0\}$, where $v_0, \ldots, v_s$ are the punctures of S. Now choose a triangle of $\widetilde{\lambda}$ such that $\mathring{T} \supset \{v_1, \dots, v_s\}$ and define the triangulation $\overline{\lambda}$ on S produced by taking $\widetilde{\lambda}$ on $S \setminus T$ and the triangulation in Figure 3.3 on T. The number of triangles in $\overline{\lambda}$ composing $S \setminus T$ is $4g - 3$ and the triangles composing the triangulation on T are $2s + 1$.

Now, given ρ a local representation of $\mathcal{T}^q_{\overline{\lambda}}$, we choose a representative of ρ and we label it as follows

- we denote by ρ_0 the representation on the algebra of the triangle T_0, with values in $\mathrm{End}(W^0)$;
- we denote by ρ_k and ρ'_k the representations on the algebras of the triangles T_k and T'_k respectively, with values in $\mathrm{End}(W^k)$ and $\mathrm{End}((W^k)')$, for $k = 1, \dots, s$;
- we denote by $\widetilde{\rho}$ the tensor product of the representations on the triangles not contained in T, with values in $\mathrm{End}(\widetilde{W})$.

Consequently, the representation ρ is the equivalence class of the representation

$$\rho_0 \otimes \bigotimes_{k=1}^{s} (\rho_k \otimes \rho'_k) \otimes \widetilde{\rho}$$

with values in

$$V = W^0 \otimes \bigotimes_{k=1}^{s} (W^k \otimes (W^k)') \otimes \widetilde{W} = W \otimes \widetilde{W}$$

Having fixed a representative of ρ, we are able to define an action of the elements $\rho(P_j)$ on the single terms W^0, W^k and $(W^k)'$. Indeed, fixed $j \in \{1, \dots, s\}$ and $k \in \{0, \dots, s\}$, we define the element $Y_{jk} \in \mathcal{T}^q_{T_k}$ as follows: denoting with $X_1^{(k)}, X_2^{(k)}, X_3^{(k)}$ the generators of $\mathcal{T}^q_{T_k}$ associated with the edges of T_k, we define $Y_{jk} := \underline{X}^{a^{(k)}}$ (we are using the notation introduced in relation 1.1). The vector $a^{(k)} \in \{0, 1, 2\}^3$ has components $a_h^{(k)}$ for $h = 1, 2, 3$ and $a_h^{(k)}$ is equal to the numbers of ends of the side of T_k corresponding to $X_h^{(k)}$ that are identified in

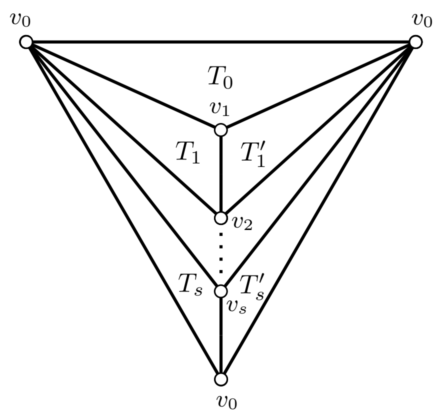

Figure 3.3: The triangulation on T

$\overline{\lambda}$ with v_j. Analogously we construct the elements $Y'_{jk} \in \mathcal{T}^q_{T'_k}$ for every $k \in \{1, \ldots, s\}$. In this way, for every $j = 1, \ldots, p$ we have constructed a monomial

$$Y_{j0} \otimes Y_{j1} \otimes Y'_{j1} \otimes \cdots \otimes Y_{js} \otimes Y'_{js} \otimes 1$$

that belongs to the algebra

$$\mathcal{T}^q_{\overline{\lambda}_0} = \mathcal{T}^q_{T_0} \otimes \bigotimes_{k=1}^{s} (\mathcal{T}^q_{T_k} \otimes \mathcal{T}^q_{T'_k}) \otimes \mathcal{T}^q_{\widetilde{\lambda} \setminus T}$$

where $\overline{\lambda}_0$ is the ideal triangulation on the surface obtained by splitting S along all the edges of $\overline{\lambda}$. This tensor split element has the property that

$$\iota_{\overline{\lambda}}(Y_{j0} \otimes Y_{j1} \otimes Y'_{j1} \otimes \cdots \otimes Y_{js} \otimes Y'_{js} \otimes 1) = P_j \in \mathcal{T}^q_{\overline{\lambda}}$$

for every $j = 1, \ldots, s$. Hence we have

$$\rho(P_j) = \rho_0(Y_{j0}) \otimes \rho_1(Y_{j1}) \otimes \rho'_1(Y'_{j1}) \otimes \cdots \otimes \rho_1(Y_{js}) \otimes \rho'_1(Y'_{js}) \otimes 1$$

With this explicit tensor split decomposition of $\rho(P_j)$ we can define the following actions

$$\begin{aligned} P_j \cdot v_0 &= \rho_k(Y_{j0})(v_0) && \forall v_0 \in W^0 \\ P_j \cdot (v_k \otimes v'_k) &= (\rho_k(Y_{jk}) \otimes \rho'_k(Y'_{jk}))(v_k \otimes v'_k) && \forall v_k \otimes v'_k \in W^k \otimes (W^k)' \end{aligned}$$

For the moment assume further that ρ has invariants $x_i = 1$ for every $i = 1, \ldots, n$ and central load $h \in \mathcal{U}(N) := \mathcal{U}(N, 1)$. Then we can choose a representative

$$\rho_0 \otimes \bigotimes_{k=1}^{s} (\rho_k \otimes \rho'_k) \otimes \widetilde{\rho}$$

of ρ having all the invariants on the edges of $\overline{\lambda}_0$ equal to 1 and all the central loads of the triangles in $\mathcal{U}(N)$. Now, given $\underline{c} \in \mathcal{U}(N)^s$, we define the following sets

$$\begin{aligned} W^0_{\underline{c}} &:= \{x \in W^0 \mid P_j \cdot x = c_j\, x \ \ \textit{for every } j = 1, \ldots, s\} \\ W^k_{\underline{c}} &:= \{x \in W^k \otimes (W^k)' \mid P_j \cdot x = c_j\, x \ \ \textit{for every } j = 1, \ldots, s\} \end{aligned}$$

for every $k = 1, \ldots, s$.

Lemma 3.4.2. *In the assumptions above, the following relations hold:*

1.
$$\dim W^0_{\underline{c}} = \begin{cases} 1 & \textit{if } c_j = 1 \textit{ for every } j \neq 1 \\ 0 & \textit{otherwise} \end{cases}$$

2.
$$\dim W^k_{\underline{c}} = \begin{cases} 1 & \textit{if } c_j = 1 \textit{ for every } j \notin \{k, k+1\} \\ 0 & \textit{otherwise} \end{cases}$$

for every $k = 1, \ldots, s-1$*;*

3.

$$\dim W^s_{\underline{c}} = \begin{cases} N & \textit{if } c_j = 1 \textit{ for every } j \neq s \\ 0 & \textit{otherwise} \end{cases}$$

Proof. 1. If $j \neq 1$, then the vertex v_j does not belong to T_0, so $Y_{j0} = 1 \in \mathcal{T}^q_{T_0}$. This means that, for every $j \neq 1$, the element P_j acts on W^0 as the identity. In particular, if there exists an index $j \neq 1$ such that $c_j \neq 1$, then we have $W^0_{\underline{c}} = 0$. Hence we can assume $c_j = 1$ for every $j \neq 1$. In this assumption the following equality holds

$$W^0_{\underline{c}} = \{x \in W^0 \mid P_1 \cdot x = c_1\, x\}$$

Labelling the variables corresponding to the edges of T_0 as in Figure 3.4, we see that $Y_{10} = q^{-1}YZ$. If H_0 is the central element of $\mathcal{T}^q_{T_0}$ and h_0 the central load of ρ_0, then we have

$$\begin{aligned} \rho_0(Y_{10}) &= \rho_0(q^{-1}YZ) \\ &= \rho_0(q^{-1}YZH_0^{-1}H_0) \\ &= h_0\, \rho_0(q^{-1}YZqZ^{-1}Y^{-1}X^{-1}) \\ &= h_0\, \rho_0(X)^{-1} \end{aligned}$$

As seen in Proposition 2.1.4, the spectrum of $\rho_0(X)$ is equal to $\mathcal{U}(N)$ and every eigenvalue has multiplicity 1. Hence we have $\dim W^0_{\underline{c}} = 1$ for every $\underline{c}$ such that $c_j = 1$ for all $j \neq 1$.

2. Given $k \in \{1, \ldots, s-1\}$ and $j \notin \{k, k+1\}$, the vertex v_j does not belong to neither T_k nor T'_k, so $Y_{jk} = 1 \in \mathcal{T}^q_{T_k}$ and $Y'_{jk} = 1 \in \mathcal{T}^q_{T'_k}$. This means that, for every $j \notin \{k, k+1\}$, the element P_j acts on $W^k \otimes (W^k)'$ as the identity. In particular, if there exists an index $j \notin \{k, k+1\}$ such that $c_j \neq 1$, then we have $W^k_{\underline{c}} = 0$. Hence we can assume $c_j = 1$ for every $j \notin \{k, k+1\}$. In this case, the following equality holds

$$W^k_{\underline{c}} = \{x \in W^k \otimes (W^k)' \mid P_k \cdot x = c_k\, x,\;\; P_{k+1} \cdot x = c_{k+1}\, x\}$$

We want to study how P_k and P_{k+1} act on W^k and $(W^k)'$. We label the variables corresponding to the edges as in Figure 3.4.

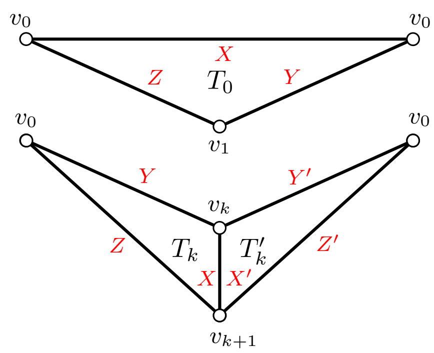

Figure 3.4: Notations for T_0, T_k and T'_k.

With calculations similar to the ones done in the previous part of the proof, we can prove that

$$P_k \text{ acts on } \begin{cases} W^k & \text{as } h_k Z^{-1} \\ (W^k)' & \text{as } h'_k (Z')^{-1} \end{cases}$$
$$P_{k+1} \text{ acts on } \begin{cases} W^k & \text{as } h_k Y^{-1} \\ (W^k)' & \text{as } h'_k (Y')^{-1} \end{cases}$$

Recalling what seen in Proposition 2.1.4, we can select a basis $\{e'_0, \ldots, e'_{N-1}\}$ of $(W^k)'$ such that

$$\begin{aligned} \rho'_k(X') &= h'_k \, B_1 \\ \rho'_k(Y') &= B_2 \\ \rho'_k(Z') &= B_3 \end{aligned}$$

where the equality holds in the coordinates induced by this choice of basis $\{e_0, \ldots, e'_{N-1}\}$, the B_i are the matrices introduced in the proof of Proposition 2.1.4 and h'_k is the central load of ρ'_k. In the same way, we can select a basis $\{e_0, \ldots, e_{N-1}\}$ of W^k such that

$$\begin{aligned} \rho_k(X) &= h_k \, \overline{B}_1 \\ \rho_k(Y) &= \overline{B}_2 \\ \rho_k(Z) &= \overline{B}_3 \end{aligned}$$

where $\overline{B}$ is the matrix $\overline{B}_{ij} := \overline{B_{ij}}$ for every i, j ($\overline{B_{ij}}$ is denoting the complex conjugate of B_{ij}) and h_k is the central load of ρ'_k. The reason why we must consider the conjugate matrices is that the edges X, Y, Z are ordered counterclockwise instead of clockwise. From these equalities we deduce the following

$$\begin{aligned} P_k \cdot e_l &= h_k q^{-1-2l} \, e_{l+1} && \forall l \in \{0, \ldots, N-1\} \\ P_k \cdot e'_l &= h'_k q^{1+2l} \, e_{l+1} && \forall l \in \{0, \ldots, N-1\} \\ P_{k+1} \cdot e_l &= h_k \, e_{l-1} && \forall l \in \{0, \ldots, N-1\} \\ P_{k+1} \cdot e'_l &= h'_k \, e_{l-1} && \forall l \in \{0, \ldots, N-1\} \end{aligned}$$

Denote by $e_{k,l} \in W^k \otimes (W^k)'$ the element $e_k \otimes e'_l$, for $k, l \in \{0, \ldots, N-1\}$, and by $\alpha_{n,m}$ the vector

$$\alpha_{n,m} := \sum_{h=0}^{N-1} q^{2hm} \, e_{h,h+n}$$

Then, with simple calculations, we see that

$$\begin{aligned} P_k \cdot \alpha_{n,m} &= h_k h'_k q^{2(n-m)} \alpha_{n,m} \\ P_{k+1} \cdot \alpha_{n,m} &= h_k h'_k q^{2m} \alpha_{n,m} \end{aligned}$$

The set $\{\alpha_{n,m} \mid n, m = 0, \ldots, N-1\}$ is a basis of $W^k \otimes (W^k)'$ and, thanks to the achieved relations, it is a basis of eigenvalues for both P_k and P_{k+1}

on $W^k \otimes (W^k)'$. The number $h_k h'_k$ is an N-th root of unity and for every $c_k, c_{k+1} \in \mathcal{U}(N)$ there exists a unique couple $(n,m) \in \mathbb{Z}_N^2$ such that

$$\begin{cases} c_j & = h_k h'_k q^{2(n-m)} \\ c_{j+1} & = h_k h'_k q^{2m} \end{cases}$$

So, for every $\underline{c} \in \mathcal{U}(N)^s$ such that $c_j = 1$ for every $j \notin \{k, k+1\}$, there exists a unique $\alpha_{n,m}$ which is both a c_k-eigenvector of P_k and a c_{k+1}-eigenvector of P_{k+1}, and so

$$\dim W^j_{\underline{c}} = 1$$

as desired.

3. If there exists a $j \in \{1, \ldots, s-1\}$ such that $c_j \neq 1$, then $W^s_{\underline{c}}$ is equal to 0, because all the P_j acts on $W^s \otimes (W^s)'$ as the identity when $j \neq s$. It remains the case $c_j = 1$ for every $j \neq s$. In these assumptions, we have

$$W^s_{\underline{c}} := \{x \in W^s \otimes (W^s)' \mid P_s \cdot x = c_s \, x\}$$

Thanks to the previous calculations, we see that

$$P_s \cdot \alpha_{n,m} = h_s h'_s q^{2(n-m)} \alpha_{n,m}$$

where the $\alpha_{n,m}$ are defined as in the previous Section. Hence we have that $\dim W^s_{\underline{c}} \geq N$ for every $c_s \in \mathcal{U}(N)$. On the other hand, the following holds

$$N^2 = \dim(W^s \otimes (W^s)') = \sum_{c_s \in \mathcal{U}(N)} \dim W^s_{\underline{c}} \geq \sum_{c_s \in \mathcal{U}(N)} N = N^2$$

So $\dim W^s_{\underline{c}} = N$ for every $\underline{c}$ such that $c_j = 1$ for every $j \neq s$. □

Proof of Theorem 3.4.1. We firstly deal with the case in which ρ has invariants $x_i = 1$ for every $i = 1, \ldots, n$. Fixed $\underline{c} \in \mathcal{U}(N)^s$, we have the following inclusion

$$V_{\underline{c}} \supseteq \bigoplus_{\underline{c}^0 \cdots \underline{c}^s = \underline{c}} (W^0_{\underline{c}^0} \otimes \cdots \otimes W^s_{\underline{c}^s} \otimes \widetilde{W})$$

A $(s+1)$-tuple of $\underline{c}^i$ verifying $\prod_i \underline{c}^i = \underline{c}$ is associated with a non-zero subspace $W^0_{\underline{c}^0} \otimes \cdots \otimes W^s_{\underline{c}^s} \otimes \widetilde{W}$ if and only if the following relations hold

$$\begin{cases} c^0_j = 1 & \text{for every } j \neq 1 \\ c^k_j = 1 & \text{for every } j \notin \{k, k+1\} \text{ and } k \in \{1, \ldots, s-1\} \\ c^s_j = 1 & \text{for every } j \neq s \\ c_1 = c^0_1 c^1_1 & \\ c_2 = c^1_2 c^2_2 & \\ \vdots & \\ c_s = c^{s-1}_s c^s_s & \end{cases}$$

The number of solutions of this system of equations is exactly N^s. Hence there are exactly N^s non-trivial addends in the above expression and each of them has dimension

$$1 \cdot \cdots \cdot 1 \cdot N \cdot N^{4g-3} = N^{4g-2}$$

Therefore, we have shown that $\dim V_{\underline{c}} \geq N^{4g-2+s}$ for every $\underline{c} \in \mathcal{U}(N)^s$. On the other hand

$$N^{4g-2+2s} = \dim V \geq \sum_{\underline{c}\in\mathcal{U}(N)^s} V_{\underline{c}} \geq \sum_{\underline{c}\in\mathcal{U}(N)^s} N^{4g-2+s} = N^{4g-2+2s}$$

So $V = \bigoplus_{\underline{c}} V_{\underline{c}}$ and $\dim V_{\underline{c}} = 4g-2+s$ for every $\underline{c}$. Every $V_{\underline{c}}$ is obviously invariant for the representation ρ: indeed, every eigenspace of a central element is invariant under the action of the representation ρ; the subspaces $V_{\underline{c}}$ are intersections of eigenspaces of the elements $\rho(P_j)$, so they are invariant. Being V the direct sum of these subspaces, we have proved the existence of the decomposition $\rho = \bigoplus_{\underline{c}} \rho^{\underline{c}}$ when ρ has invariants $x_i = 1$ for every i.

If $\widetilde{\rho}$ is a generic representation of $\mathcal{T}^q_{\overline{\lambda}}$, we can choose, for every $i = 1, \ldots, n$, an N-th root y_i on x_i, the invariant of $\widetilde{\rho}$ associated with the i-th edge of $\overline{\lambda}$, and define

$$\rho(X_i) := y_i^{-1}\widetilde{\rho}(X_i)$$

The map ρ is indeed a representation of the algebra $\mathcal{T}^q_{\lambda_0}$ and has invariants on the edges equal to 1. Moreover, the images of the elements P_j under $\widetilde{\rho}$ are equal to $\beta_j\, \rho(P_j)$, where β_j is

$$q^{-\sum_{l<m}\sigma_{lm}k_{jl}k_{jm}}\, y_1^{k_{j1}} \cdots y_n^{k_{jn}}$$

that is a certain N-th root of $t_j(x_1, \ldots, x_n) = x_1^{k_{j1}} \cdots x_n^{k_{jn}}$. Now, fixed $\underline{d}$ an element of

$$\mathcal{U} := \prod_{j=1}^{s} \mathcal{U}(N, t_j(x_1, \ldots, x_n))$$

we observe

$$\begin{aligned} V^{\widetilde{\rho}}_{\underline{d}} &= \{x \in V \mid \widetilde{\rho}(P_j)(x) = d_j\, x \text{ for every } j = 1, \ldots, N\} \\ &= \{x \in V \mid \rho(P_j)(x) = \beta_j^{-1} d_j\, x \text{ for every } j = 1, \ldots, N\} \\ &= V^{\rho}_{\underline{c}} \end{aligned}$$

where $\underline{c} \in \mathcal{U}(N)^s$ is defined by $c_j := \beta_j^{-1} d_j \in \mathcal{U}(N)$. From what proved in the first part we deduce that

$$V = \bigoplus_{\underline{d}\in\mathcal{U}} V^{\widetilde{\rho}}_{\underline{d}}$$

Notice that $V^{\widetilde{\rho}}_{\underline{d}}$ is invariant under the action of $\widetilde{\rho}$, because it is an intersection of eigenspaces of central elements, just as in the previous case. This proves the first part of the assertion and the relation $\dim V_{\underline{d}} = N^{4g-2+s}$ when $\lambda = \overline{\lambda}$. Furthermore, the observations above prove that:

Remark 3.4.3. Fixed $\lambda \in \Lambda(S)$ a certain triangulation, knowing that the first part of the assertion holds for a certain isomorphism class of representation on $\mathcal{T}^q_\lambda$, we can deduce that the same holds for every local representation on $\mathcal{T}^q_\lambda$.

In the following we will prove that the first part of the assertion holds for every $\lambda \in \Lambda(S)$ and finally we will deal with the irreducible decomposition. By virtue of what said above, it is sufficient to prove that for every $\lambda \in \Lambda(S)$ there exists a local representation $\mathcal{T}^q_\lambda$ verifying the condition. S is a closed surface having genus ≥ 1 and at least a puncture. Hence there exists a complete finite-volume hyperbolic structure, having holonomy $r\colon \pi_1(S) \to \mathbb{P}\,\mathrm{SL}(2,\mathbb{R})$. By Theorem 3.3.13 there exists a local representation $\eta = \{\eta_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)\}_{\lambda\in\Lambda(S)}$ of the whole quantum Teichmüller space having hyperbolic shadow r. Therefore, for every $\lambda \in \Lambda(S)$ the composition $\eta_{\overline{\lambda}} \circ \Phi^q_{\overline{\lambda}\lambda}$ makes sense and it is isomorphic to η_λ. So there exists a linear isomorphism $L\colon V_\lambda \to V_{\overline{\lambda}}$ such that, for every $X \in \mathcal{T}^q_\lambda$

$$(\eta_{\overline{\lambda}} \circ \Phi^q_{\overline{\lambda}\lambda})(X) = L \circ \eta_\lambda(X) \circ L^{-1} \tag{3.9}$$

The element $\eta_{\overline{\lambda}}$ is a local representation of $\mathcal{T}^q_{\overline{\lambda}}$, so we know that the decomposition

$$V_{\overline{\lambda}} = \bigoplus_{\underline{d}\in\mathcal{U}} (V_{\overline{\lambda}})_{\overline{d}}$$

is preserved by $\eta_{\overline{\lambda}}$ and, by virtue of relation 3.9, every subspace $L^{-1}((V_{\overline{\lambda}})_{\overline{d}})$ is invariant under η_λ. Moreover, as observed in Lemma 3.1.2, the elements P_j in $\mathcal{T}^q_{\overline{\lambda}}$ go under $\Phi^q_{\lambda\overline{\lambda}}$ in the P_j in $\mathcal{T}^q_\lambda$, so it is immediate to see that the spaces $L^{-1}((V_{\overline{\lambda}})_{\overline{d}})$ are exactly the $(V_\lambda)_{\overline{d}}$ related to η_λ. This proves that, for every $\lambda \in \Lambda(S)$ there exists a local representation of $\mathcal{T}^q_\lambda$ that verifies the first part of the assertion. Thanks to the previous remark, we have concluded the proof of the first part of the assertion and the dimension relation.

For what concerns the last part of the statement, fixed ρ a local representation of $\mathcal{T}^q_\lambda$, we can choose a representative $\rho_1 \otimes \cdots \otimes \rho_m$, with $\rho_k\colon \mathcal{T}^q_{T_k} \to \mathrm{End}(V_k)$. For every $k = 1, \ldots, m$, we can fix a basis of V_k in which ρ_k is represented by multiples of the B_i, introduced in Proposition 2.1.4. By taking the tensor product of these bases, we obtain a standard presentation of ρ. If we endow the space $V_1 \otimes \cdots \otimes V_m$ with the hermitian product in which this selected basis is orthonormal, we obtain a conformal class that is preserved by the action of the generators $X_i \in \mathcal{T}^q_\lambda$ via the representation ρ. This is an immediate corollary of the fact that the matrices B_i are unitary. Consequently, if V' is a subspace of $V_1 \otimes \cdots \otimes V_m$ invariant under the action of ρ, then the same holds for $(V')^\perp$. Moreover, if V' is a subspace of $V_{\underline{d}}$ invariant under the action of $\rho^{\underline{d}}$, then the same holds for $(V')^\perp$, where now we are taking the orthogonal in $V_{\underline{d}}$ instead of the whole V. From this fact immediately follows that every $\rho^{\underline{d}}$ is the direct sum of irreducible representations $\rho_i^{\underline{d}}$. The numbers of these representations is $N^{4g-2+s-(3g-2+s)} = N^g$, because every irreducible representation has dimension N^{3g-2+s}, as seen in Theorem 2.1.9. Moreover, because on $V_{\underline{d}}$ every P_j goes under $\rho^{\underline{d}}$ in $d_j\, id$, all the $\rho_i^{\underline{d}}$ have the same invariants and so they are isomorphic to each other. This concludes the proof of Theorem 3.4.1. $\square$

CHAPTER 4

Intertwining operators

Let $\rho = \{\rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)\}_{\lambda \in \Lambda(S)}$ and $\rho' = \{\rho'_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V'_\lambda)\}_{\lambda \in \Lambda(S)}$ be two isomorphic local representations of the quantum Teichmüller space $\mathcal{T}^q_S$. By definition, for every $\lambda, \lambda' \in \Lambda(S)$, the representations $\rho_\lambda \circ \Phi^q_{\lambda\lambda'}$ and $\rho_{\lambda'}$ are isomorphic and $\rho_{\lambda'}$ itself is isomorphic to $\rho'_{\lambda'}$. Therefore, there exists a linear isomorphism $L^{\rho\rho'}_{\lambda\lambda'} \colon V'_{\lambda'} \to V_\lambda$ such that

$$(\rho_\lambda \circ \Phi^q_{\lambda\lambda'})(X') = L^{\rho\rho'}_{\lambda\lambda'} \circ \rho'_{\lambda'}(X') \circ (L^{\rho\rho'}_{\lambda\lambda'})^{-1} \tag{4.1}$$

for every $X' \in \mathcal{T}^q_{\lambda'}$. Such a $L^{\rho\rho'}_{\lambda\lambda'}$ is called an *intertwining operator*. In general, fixed ρ, ρ' and λ, λ', there is not a unique intertwining operator. For example, by multiplying a certain $L^{\rho\rho'}_{\lambda\lambda'}$ by a non-zero scalar, we clearly obtain another isomorphism that verifying the condition above. When two linear isomorphisms A, B differ by a non-zero scalar, we will briefly write $A \doteq B$. Actually this is not the only difference that can be observed between two intertwining operators verifying 4.1. Indeed, we have seen in Section 3.4 that, in the most interesting situations, local representations are far from being irreducible, so they admit a lot of non-trivial automorphisms (see for example the case $\rho = \rho'$ and $\lambda = \lambda'$). One of the main purposes of [BBL07] was to select a unique intertwining operator $\widehat{L}^{\rho\rho'}_{\lambda\lambda'}$ for every $\rho, \rho', \lambda, \lambda$, by requesting some additional properties on them. More precisely, one of the results stated in [BBL07] was the following theorem:

Theorem ([BBL07, Theorem 20])**.** *For every surface S (see Chapter 0 for details) there exists a unique family of intertwining operators $\widehat{L}^{\rho\rho'}_{\lambda\lambda'}$, indexed by couples of isomorphic local representations of $\mathcal{T}^q_S$ and by couples of ideal triangulations $\lambda, \lambda' \in \Lambda(S)$, individually defined up to scalar multiplication, such that:*

Composition relation: *for every $\lambda, \lambda', \lambda'' \in \Lambda(S)$ and for every triple of isomorphic local representations ρ, ρ', ρ'', we have $\widehat{L}^{\rho\rho''}_{\lambda\lambda''} \doteq \widehat{L}^{\rho\rho'}_{\lambda\lambda'} \circ \widehat{L}^{\rho'\rho''}_{\lambda'\lambda''}$;*

Fusion relation: *let S be a surface obtained from another surface R by fusion, and let λ, λ' be two triangulations of S obtained by fusion of* *two triangulations μ, μ' of R. If η, η' are two isomorphic local representations of $\mathcal{T}_R^q$ and ρ, ρ' are two isomorphic local representations of $\mathcal{T}_S^q$ obtained by fusion (recall Definition 3.2.16) from η, η' respectively, then we have $\widehat{L}_{\mu\mu'}^{\eta\eta'} \doteq \widehat{L}_{\lambda\lambda'}^{\rho\rho'}$.*

However, in the investigation of the ideas exposed in [BBL07], we have found a problem that compromises this statement, in particular the possibility to select a unique intertwining operator for every choice of $\rho, \rho', \lambda, \lambda'$.

Let us try to describe this obstruction. Let λ be an ideal triangulation of S. Denote by S_0 the surface obtained by splitting S along all the edges of the ideal triangulation λ. S_0 admits a unique ideal triangulation λ_0, being a disjoint union of ideal triangles. In [BBL07] the procedure to select the isomorphism $L_{\lambda\lambda'}^{\rho\rho'}$ was the following: we fix two representatives $(\rho_j)_{j=1}^m$ and $(\rho'_j)_{j=1}^m$ of ρ_λ and ρ'_λ, respectively, that are isomorphic to each other as local representations of the Chekhov-Fock algebra $\mathcal{T}_{\lambda_0}^q$. As observed in the proof of [BBL07, Lemma 21], a local representation of $\mathcal{T}_{\lambda_0}^q$ is irreducible, so there exists a unique isomorphism $M_{\lambda\lambda}^{\rho\rho'}$ between $(\rho_j)_{j=1}^m$ and $(\rho'_j)_{j=1}^m$, up to scalar multiplication. Then the isomorphism $\widehat{L}_{\lambda\lambda'}^{\rho\rho'}$ was defined in [BBL07, Lemma 22] as $\widehat{L}_{\lambda\lambda'}^{\rho\rho'} := M_{\lambda\lambda}^{\rho\rho'}$. The problem we will observe is that this choice does depend on the selected representatives. In other words, the representation ρ_λ has representatives that are non-trivially isomorphic to each other, so different choices of $(\rho_j)_{j=1}^m$ and $(\rho'_j)_{j=1}^m$ lead us to a (finite) collection of intertwining operators, in general not to a unique element. We will focus on this problem in the first Section and in particular in Remark 4.2.2.

The main purpose of this Chapter is to understand this phenomenon and to try to recover a result on intertwining operators similar to the one in [BBL07, Theorem 20]. A few steps will be necessary: we will produce the fundamental objects in Section 4.2, we will investigate on their properties in Section 4.3 and finally we will complete the procedure in Section 4.4, where we will prove the following Theorem:

Theorem (Existence Theorem)**.** *For every surface S there exists a collection $\{(\mathscr{L}_{\lambda\lambda'}^{\rho\rho'}, \psi_{\lambda\lambda'}^{\rho\rho'})\}$, indexed by couples of isomorphic local representations ρ, ρ' of the quantum Teichmüller space $\mathcal{T}_S^q$ and by couples of ideal triangulations $\lambda, \lambda' \in \Lambda(S)$ such that*

Intertwining: *for every couple of isomorphic local representations*

$$\rho = \{\rho_\lambda \colon \mathcal{T}_\lambda^q \to \mathrm{End}(V_\lambda)\}_{\lambda\in\Lambda(S)} \quad \rho' = \{\rho'_\lambda \colon \mathcal{T}_\lambda^q \to \mathrm{End}(V'_\lambda)\}_{\lambda\in\Lambda(S)}$$

and for every $\lambda, \lambda' \in \Lambda(S)$, $\mathscr{L}_{\lambda\lambda'}^{\rho\rho'}$ is a set of linear isomorphisms $L_{\lambda\lambda'}^{\rho\rho'}$ from $V'_{\lambda'}$ to V_λ such that

$$(\rho_\lambda \circ \Phi_{\lambda\lambda'}^q)(X') = L_{\lambda\lambda'}^{\rho\rho'} \circ \rho'_{\lambda'}(X') \circ (L_{\lambda\lambda'}^{\rho\rho'})^{-1}$$

for every $X' \in \mathcal{T}_{\lambda'}^q$;

Action: *every set* $\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}$ *is endowed with a transitive and free action* $\psi^{\rho\rho'}_{\lambda\lambda'}$ *of* $H_1(S;\mathbb{Z}_N)$*;*

Fusion property: *let* R *be a surface and* S *be obtained by fusion from* R*. Fix*

$$\eta = \{\eta_\mu \colon \mathcal{T}^q_\mu \to \mathrm{End}(W_\mu)\}_{\mu\in\Lambda(R)} \quad \eta' = \{\eta'_\mu \colon \mathcal{T}^q_\mu \to \mathrm{End}(W'_\mu)\}_{\mu\in\Lambda(R)}$$

two isomorphic local representations of $\mathcal{T}^q_R$ *and*

$$\rho = \{\rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)\}_{\lambda\in\Lambda(S)} \quad \rho' = \{\rho'_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V'_\lambda)\}_{\lambda\in\Lambda(S)}$$

two isomorphic local representations of $\mathcal{T}^q_S$*, with* ρ *and* ρ' *obtained by fusion from* η *and* η'*, respectively. Then, for every* $\mu,\mu' \in \Lambda(R)$*, if* $\lambda,\lambda' \in \Lambda(S)$ *are the corresponding ideal triangulations on* S*, there exists a natural inclusion* $j \colon \mathscr{L}^{\eta\eta'}_{\mu\mu'} \to \mathscr{L}^{\rho\rho'}_{\lambda\lambda'}$ *such that for every* L *in* $\mathscr{L}^{\eta\eta'}_{\mu\mu'}$ *the following holds*

$$(j \circ \psi^{\eta\eta'}_{\mu\mu'})(c, L) = \psi^{\rho\rho'}_{\lambda\lambda'}(\pi_*(c), j(L))$$

for every $c \in H_1(R;\mathbb{Z}_N)$*, where* $\pi \colon R \to S$ *is the projection map;*

Composition property: *for every* ρ,ρ',ρ'' *isomorphic local representations of* $\mathcal{T}^q_S$ *and for every* $\lambda,\lambda',\lambda'' \in \Lambda(S)$*, the composition map*

$$\begin{array}{ccc} \mathscr{L}^{\rho\rho'}_{\lambda\lambda'} \times \mathscr{L}^{\rho'\rho''}_{\lambda'\lambda''} & \longrightarrow & \mathscr{L}^{\rho\rho''}_{\lambda\lambda''} \\ (L, M) & \longmapsto & L \circ M \end{array}$$

is well defined and it verifies

$$(c \cdot L) \circ (d \cdot M) = (c + d) \cdot (L \circ M)$$

Therefore, we will produce a collection of families of intertwining operators, endowed with a transitive and free action of $H_1(S;\mathbb{Z}_N)$, verifying a Fusion and a Composition properties similar to the ones stated in [BBL07, Theorem 20]. Moreover, we will observe that this collection verifies a uniqueness property, as described in the following Theorem that we are going to prove in Section 4.4:

Theorem (Uniqueness Theorem)**.** *Suppose that* $\{\mathscr{M}^{\rho\rho'}_{\lambda\lambda'}\}$ *is a collection indexed by couples of isomorphic local representations* ρ,ρ' *of the quantum Teichmüller space* $\mathcal{T}^q_S$ *and by couples of ideal triangulations* $\lambda,\lambda' \in \Lambda(S)$ *such that*

Intertwining: *for every couple of isomorphic local representations*

$$\rho = \{\rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)\}_{\lambda\in\Lambda(S)} \quad \rho' = \{\rho'_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V'_\lambda)\}_{\lambda\in\Lambda(S)}$$

and for every $\lambda, \lambda' \in \Lambda(S)$, $\mathscr{M}^{\rho\rho'}_{\lambda\lambda'}$ *is a non-empty set of linear isomorphisms* $M^{\rho\rho'}_{\lambda\lambda'}$ *from* $V'_{\lambda'}$ *to* V_λ *such that*

$$(\rho_\lambda \circ \Phi^q_{\lambda\lambda'})(X') = M^{\rho\rho'}_{\lambda\lambda'} \circ \rho'_{\lambda'}(X') \circ (M^{\rho\rho'}_{\lambda\lambda'})^{-1}$$

for every $X' \in \mathcal{T}^q_{\lambda'}$*;*

Weak Fusion property: *let* R *be a surface and* S *obtained by fusion from* R*. Fix*

$$\eta = \{\eta_\mu \colon \mathcal{T}^q_\mu \to \mathrm{End}(W_\mu)\}_{\mu\in\Lambda(R)} \quad \eta' = \{\eta'_\mu \colon \mathcal{T}^q_\mu \to \mathrm{End}(W'_\mu)\}_{\mu\in\Lambda(R)}$$

two isomorphic local representations of $\mathcal{T}^q_R$ *and*

$$\rho = \{\rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)\}_{\lambda\in\Lambda(S)} \quad \rho' = \{\rho'_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V'_\lambda)\}_{\lambda\in\Lambda(S)}$$

two isomorphic local representations of $\mathcal{T}^q_S$*, with* ρ *and* ρ' *obtained by fusion from* η *and* η'*, respectively. Then, for every* $\mu, \mu' \in \Lambda(R)$*, if* $\lambda, \lambda' \in \Lambda(S)$ *are the corresponding ideal triangulations on* S*, the set* $\mathscr{M}^{\eta\eta'}_{\mu\mu'}$ *is contained in* $\mathscr{M}^{\rho\rho'}_{\lambda\lambda'}$ *(it makes sense because they are both contained in* $\mathrm{Hom}(V'_{\lambda'}, V_\lambda)$*);*

Weak Composition property: *for every* ρ, ρ', ρ'' *isomorphic local representations of* $\mathcal{T}^q_S$ *and for every* $\lambda, \lambda', \lambda'' \in \Lambda(S)$*, the composition map*

$$\begin{array}{ccc} \mathscr{M}^{\rho\rho'}_{\lambda\lambda'} \times \mathscr{M}^{\rho'\rho''}_{\lambda'\lambda''} & \longrightarrow & \mathscr{M}^{\rho\rho''}_{\lambda\lambda''} \\ (M, N) & \longmapsto & M \circ N \end{array}$$

is well defined.

Then, for every ρ *and* ρ' *isomorphic local representations and for every* $\lambda, \lambda' \in \Lambda(S)$ *we have*

$$\mathscr{L}^{\rho\rho'}_{\lambda\lambda'} \subseteq \mathscr{M}^{\rho\rho'}_{\lambda\lambda'}$$

where $\{\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}\}$ *is the family described in the previous theorem.*

Hence the collection $\{\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}\}$ of intertwining operators we will exhibit is minimal in the family of collections of intertwining operators verifying the Weak Fusion and the Weak Composition properties. Observe that, if the statement of [BBL07, Theorem 20] was true and so we were able to select a unique $\widehat{L}^{\rho\rho'}_{\lambda\lambda'}$, then the collection of families $\{\widehat{\mathscr{L}}^{\rho\rho'}_{\lambda\lambda'}\}$ defined by $\widehat{\mathscr{L}}^{\rho\rho'}_{\lambda\lambda'} := \{\widehat{L}^{\rho\rho'}_{\lambda\lambda'}\}$ would verify the Weak Fusion and the Weak Composition properties of the previous Theorem. This would lead us to a contradiction, because each set $\widehat{\mathscr{L}}^{\rho\rho'}_{\lambda\lambda'}$ would contain $\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}$, but $\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}$ has cardinality equal to $|H_1(S;\mathbb{Z}_N)|$, instead of $\widehat{\mathscr{L}}^{\rho\rho'}_{\lambda\lambda'}$, which is composed of a single element.

These results tell us that the collection $\{\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}\}$ is the best object we can produce by imposing the Weak Fusion and the Weak Composition properties. Moreover, each $\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}$ is endowed with a transitive and free action of $H_1(S;\mathbb{Z}_N)$, which will clarify which are the linear isomorphisms composing the sets $\mathscr{L}_{\lambda\lambda'}^{\rho\rho'}$.

Finally, in Section 4.5 we will focus on the development of a theory of invariants of pseudo-Anosov diffeomorphisms, as done in [BL07] and [BBL07], with the tools achieved in the previous Sections. The resulting invariant is more complicated to the ones described in the mentioned works, but the way we will produce is the same.

4.1 Preliminary observations

We will assume that q is a primitive N-th root of $(-1)^{N+1}$ (in particular q^2 is a primitive N-th root of unity).

First notations Fix $\lambda \in \Lambda(S)$ an ideal triangulation of S and, for every edge λ_i of λ, choose an arbitrary orientation on it. Now orient the edges of the dual graph $\Gamma = \Gamma_{S,\lambda}$ (see Chapter 0 for details) as follows: the 1-cell λ_i^*, dual of the internal edge λ_i, is oriented in such a way that the intersection number $i(\lambda_i, \lambda_i^*)$ in S is equal to $+1$ (remember that we are considering oriented surfaces). Moreover, we assume that all the vertices T_l^* have positive sign.

Let n be the number of 1-cells of λ and m the number of triangles composing λ. Given $a = 1, \dots, n$ with λ_a an internal edge having two different triangles on its sides, we define

$$\varepsilon(a,b) := \begin{cases} +1 & \textit{if } T_b \textit{ is on the left of } \lambda_a \\ -1 & \textit{if } T_b \textit{ is on the right of } \lambda_a \\ 0 & \textit{otherwise} \end{cases}$$

for every $b = 1, \dots, m$. If λ_a is internal and it has the same triangle on its sides, we define $\varepsilon(a,b) = 0$ for every $b = 1, \dots, m$. Observe that a triangle T_b is on the left of λ_a if and only if the previously fixed orientation on λ_a coincides with the orientation determined as boundary of T_b. Moreover, it is immediate to see that the definition of $\varepsilon(a,b)$ can be reformulated as follows

$$\varepsilon(a,b) := \begin{cases} +1 & \textit{if } \lambda_a^* \textit{ goes towards } T_b^* \\ -1 & \textit{if } \lambda_a^* \textit{ comes from } T_b^* \\ 0 & \textit{otherwise} \end{cases}$$

if λ_a^* has different ends, otherwise $\varepsilon(a,b) = 0$ for every $b = 1, \dots, m$. Denote by $(C_\bullet(\Gamma; \mathbb{Z}_N), \partial_\bullet)$ the cellular chain complex of Γ. Then $C_0(\Gamma; \mathbb{Z}_N)$ is the $\mathbb{Z}_N$-module freely generated by the vertices T_b^* of Γ and $C_1(\Gamma; \mathbb{Z}_N)$ is the $\mathbb{Z}_N$-module freely generated by the oriented 1-cells λ_a^* of Γ. Because Γ has dimension 1, all the other $C_i(\Gamma; \mathbb{Z}_N)$ are equal to zero. Thanks to what observed, we can describe the boundary ∂_1 in terms of the triangulation λ. Given λ_a^* a 1-cell of Γ, the boundary $\partial_1(\lambda_a^*)$ verifies

$$\partial_1(\lambda_a^*) = \sum_{b=1}^{m} \varepsilon(a,b)\, T_b^* \in C_0(\Gamma; \mathbb{Z}_N)$$

Hence the first group of cellular homology $H_1(\Gamma; \mathbb{Z}_N)$ is equal to the subgroup $\operatorname{Ker} \partial_1$ of $C_1(S; \mathbb{Z}_N)$, whose elements are the $\mathbb{Z}_N$-combinations $\sum_{a=1}^{n} c_a\, \lambda_a^*$ such that, for every $b = 1, \ldots, m$, the following relation holds

$$\sum_{a=1}^{n} \varepsilon(a,b)\, c_a = 0 \in \mathbb{Z}_N$$

The dual graph Γ is a deformation retract of S, so the group $H_1(\Gamma; \mathbb{Z}_N)$ can be identified to $H_1(S; \mathbb{Z}_N)$ via a certain inclusion of Γ in S, which is well defined up to homotopy.

4.1.1 Local representations

Fixed an orientation on an ideal triangulation λ, the definition of local representation of $\mathcal{T}^q_\lambda$ can be reformulated as follows:

Definition 4.1.1. Let $\lambda \in \Lambda(S)$ be an ideal triangulation, with triangles $T_1, \ldots, T_m$, and let $(\rho_1, \ldots, \rho_m)$, $(\bar{\rho}_1, \ldots, \bar{\rho}_m)$ be two m-tuples in which, for every $j = 1, \ldots, m$, $\rho_j \colon \mathcal{T}^q_{T_j} \to \mathrm{End}(V_j)$ and $\bar{\rho}_j \colon \mathcal{T}^q_{T_j} \to \mathrm{End}(W_j)$ are irreducible representations of the triangle algebra $\mathcal{T}^q_{T_j}$. The elements $(\rho_1, \ldots, \rho_m)$, $(\bar{\rho}_1, \ldots, \bar{\rho}_m)$ are *locally equivalent* if the following hold

- for every $j = 1, \ldots, m$ the vector spaces V_j and W_j are equal;
- for every $i = 1, \ldots, n$, we have:
  - if λ_i is a boundary edge, then there is a unique triangle T_{s_i} that has λ_i on its side. In this case we ask that
  $$\rho_{s_i}(X^{(s_i)}_{a_i}) = \bar{\rho}_{s_i}(X^{(s_i)}_{a_i})$$
  where a_i is the index of the edge in T_{s_i} that is identified to λ_i;
  - if λ_i is an internal edge and T_{l_i}, T_{r_i} are distinct triangles on the left and on the right, respectively, of λ_i (recall that we have fixed orientations on the edges λ_i), then there exists $\alpha_i \in \mathbb{C}^*$ such that
  $$\rho_{l_i}(X^{(l_i)}_{a_i}) = \alpha_i\, \bar{\rho}_{l_i}(X^{(l_i)}_{a_i}) = \alpha_i^{\varepsilon(i,l_i)}\, \bar{\rho}_{l_i}(X^{(l_i)}_{a_i})$$
  $$\rho_{r_i}(X^{(r_i)}_{b_i}) = \alpha_i^{-1}\, \bar{\rho}_{r_i}(X^{(r_i)}_{b_i}) = \alpha_i^{\varepsilon(i,r_i)}\, \bar{\rho}_{r_i}(X^{(r_i)}_{b_i})$$
  where a_i and b_i are the indices of the edges in T_{l_i} and T_{r_i}, respectively, that are identified to λ_i;
  - if λ_i is an internal edge and it has the triangle T_{k_i} on both its sides, then there exists $\alpha_i \in \mathbb{C}^*$ such that
  $$\rho_{k_i}(X^{(k_i)}_{a_i}) = \alpha_i^{+1}\, \bar{\rho}_{k_i}(X^{(k_i)}_{a_i})$$
  $$\rho_{r_i}(X^{(k_i)}_{b_i}) = \alpha_i^{-1}\, \bar{\rho}_{k_i}(X^{(k_i)}_{b_i})$$
  where a_i and b_i are the indices of the edges in T_{k_i} that are identified to λ_i and the a_i-th side, unlike the b_i-th one, has the orientation as boundary of T_{k_i} coherent with the orientation of λ_i.

A *local representation* ρ of $\mathcal{T}^q_\lambda$ is a local equivalence class of m-tuples of representations $(\rho_1, \ldots, \rho_m)$ as above.

We will confuse a representative $(\rho_1, \dots, \rho_m)$ of a local representation ρ with the obvious corresponding representation $\rho_1 \otimes \cdots \otimes \rho_m$ on the Chekhov-Fock algebra $\mathcal{T}^q_{\lambda_0}$ of the surface S_0, where S_0 is the surface obtained by splitting S along all its edges and λ_0 is its ideal triangulation, which is unique since S_0 is a disjoint union of triangles.

Definition 4.1.2. Let $\lambda \in \Lambda(S)$ be an ideal triangulation of S and $\rho\colon \mathcal{T}^q_\lambda \to \mathrm{End}(V)$ a local representation of $\mathcal{T}^q_\lambda$. We denote by $\mathscr{F}_{S_0}(\rho)$ the set of representatives of ρ as local representation, which are local (and irreducible) representations of the Chekhov-Fock algebra $\mathcal{T}^q_{\lambda_0}$ of the surface S_0, obtained by splitting S along λ. Moreover, fixed an orientation on λ and given $\rho_1 \otimes \cdots \otimes \rho_m$ and $\bar\rho_1 \otimes \cdots \otimes \bar\rho_m$ two elements of $\mathscr{F}_{S_0}(\rho)$, we will write

$$\bar\rho_1 \otimes \cdots \otimes \bar\rho_m \xrightarrow{\alpha_i} \rho_1 \otimes \cdots \otimes \rho_m$$

if $\rho_1 \otimes \cdots \otimes \rho_m$ and $\bar\rho_1 \otimes \cdots \otimes \bar\rho_m$ are related by the numbers $(\alpha_i)_i$ as described in Definition 4.1.1. The $(\alpha_i)_i$ are called the *transition constants* from $\bar\rho_1 \otimes \cdots \otimes \bar\rho_m$ to $\rho_1 \otimes \cdots \otimes \rho_m$.

It is very simple to see that, with these notations introduced, the following hold:

Lemma 4.1.3. *Let $\zeta = \bigotimes_i \rho_i$, $\zeta' = \bigotimes_i \rho'_i$ and $\zeta'' = \bigotimes_i \rho''_i$ be three representatives of a local representation ρ. Then the following properties hold:*

1. *there exists a unique collection of transition constants, depending on the chosen orientation on λ, such that $\zeta \xrightarrow{\alpha_i} \zeta'$;*
2. *if $\zeta \xrightarrow{\alpha_i} \zeta'$ and $\zeta' \xrightarrow{\beta_i} \zeta''$, then $\zeta \xrightarrow{\alpha_i\beta_i} \zeta''$*
3. *if $\zeta \xrightarrow{\alpha_i} \zeta'$, then $\zeta' \xrightarrow{\alpha_i^{-1}} \zeta$.*

Given $\lambda \in \Lambda(S)$ and $\rho\colon \mathcal{T}^q_\lambda \to \mathrm{End}(V)$ a local representation, we can define an action of $H_1(S;\mathbb{Z}_N)$ on the set $\mathscr{F}_{S_0}(\rho)$ as follows:

Definition 4.1.4. Given $c = \sum_i c_i\, \lambda_i^*$ an element of $H_1(\Gamma;\mathbb{Z}_N) \cong H_1(S;\mathbb{Z}_N)$ and fixed a representative $\rho_1 \otimes \cdots \otimes \rho_m$ of ρ, we can produce another m-tuple of representations $\bar\rho_1 \otimes \cdots \otimes \bar\rho_m$ of ρ defined as follows: for every $l = 1, \dots, m$

- if T_l has distinct sides in λ, labelled clockwise as λ_i, λ_j, λ_k, then $\bar\rho_l$ is equal to

$$\begin{aligned} \bar\rho_l(X_1^{(l)}) &:= q^{2c_i\varepsilon(i,l)}\, \rho_l(X_1^{(l)}) \\ \bar\rho_l(X_2^{(l)}) &:= q^{2c_j\varepsilon(j,l)}\, \rho_l(X_2^{(l)}) \\ \bar\rho_l(X_3^{(l)}) &:= q^{2c_k\varepsilon(k,l)}\, \rho_l(X_3^{(l)}) \end{aligned} \tag{4.2}$$

  where the edges λ_i, λ_j, λ_k of T_l correspond to the variables $X_1^{(l)}$, $X_2^{(l)}$, $X_3^{(l)} \in \mathcal{T}^q_{T_l}$, respectively;

- if T_l has two sides that are identified to λ_i in λ and λ_j is the other one, then $\bar{\rho}_l$ is equal to

$$\begin{aligned}\bar{\rho}_l(X_1^{(l)}) &:= q^{2c_i}\,\rho_l(X_1^{(l)})\\ \bar{\rho}_l(X_2^{(l)}) &:= q^{-2c_i}\,\rho_l(X_2^{(l)})\\ \bar{\rho}_l(X_3^{(l)}) &:= q^{2c_j\varepsilon(j,l)}\,\rho_l(X_3^{(l)})\end{aligned} \tag{4.3}$$

where the variables $X_1^{(l)}, X_2^{(l)}$ correspond to the sides of T_l identified to λ_i and $X_1^{(l)}$ is associated with the side having its boundary orientation coherent with the orientation of λ_i (the other cases are treated in the same way). Effectively the number c_j is necessarily equal to zero, because c is a cycle in $C_1(\Gamma;\mathbb{Z}_N)$.

It is immediate to see that, by construction, the representation $\bar{\rho}_1\otimes\cdots\otimes\bar{\rho}_m$ is a representative of ρ and, in the notations introduced above, the relations 4.2 can be summarized as

$$\rho_1\otimes\cdots\otimes\rho_m \xrightarrow{q^{2c_i}} \bar{\rho}_1\otimes\cdots\otimes\bar{\rho}_m$$

We will denote by $c\cdot(\rho_1\otimes\cdots\otimes\rho_m)$ the representation $\bar{\rho}_1\otimes\cdots\otimes\bar{\rho}_m$ constructed in this way.

Now we are able to enunciate the main result of this Section:

Proposition 4.1.5. *Let ρ be a local representation of $\mathcal{T}^q_\lambda$. Then there exists an action of $H_1(S;\mathbb{Z}_N)$ on $\mathscr{F}_{S_0}(\rho)$, which verifies:*

- *two elements $\rho_1\otimes\cdots\otimes\rho_m$ and $\bar{\rho}_1\otimes\cdots\otimes\bar{\rho}_m$ of $\mathscr{F}_{S_0}(\rho)$ are isomorphic as representations of $\mathcal{T}^q_{\lambda_0}$ if and only if there exists an element $c\in H_1(S;\mathbb{Z}_N)$ such that*

$$c\cdot(\rho_1\otimes\cdots\otimes\rho_m)=\bar{\rho}_1\otimes\cdots\otimes\bar{\rho}_m$$

- *the action is free, i. e. $c\cdot(\rho_1\otimes\cdots\otimes\rho_m)$ is equal to $\rho_1\otimes\cdots\otimes\rho_m$ if and only if $c=0\in H_1(S;\mathbb{Z}_N)$.*

Proof. Let $\rho_1\otimes\cdots\otimes\rho_m$ and $\bar{\rho}_1\otimes\cdots\otimes\bar{\rho}_m$ be two representatives of ρ. Let us focus on a single triangle T_l and observe how the representations $\rho_l\colon\mathcal{T}^q_{T_l}\to\mathrm{End}(V_l)$ and $\bar{\rho}_l\colon\mathcal{T}^q_{T_l}\to\mathrm{End}(V_l)$ differ. Label clockwise as λ_i, λ_j and λ_k the edges of T_l in λ and as $X_1^{(l)}$, $X_2^{(l)}$, $X_3^{(l)}$ the corresponding variables in $\mathcal{T}^q_{T_l}$. The relations between ρ_l and $\bar{\rho}_l$ are the following

$$\begin{aligned}\rho_l(X_1^{(l)}) &= \alpha_i^{\varepsilon(i,l)}\,\bar{\rho}_l(X_1^{(l)})\\ \rho_l(X_2^{(l)}) &= \alpha_j^{\varepsilon(j,l)}\,\bar{\rho}_l(X_2^{(l)})\\ \rho_l(X_3^{(l)}) &= \alpha_k^{\varepsilon(k,l)}\,\bar{\rho}_l(X_3^{(l)})\end{aligned}$$

where we are assuming that the sides of T_l are distinct, the other cases can be treated in a similar way. Denote by $x_1^{(l)}$, $x_2^{(l)}$, $x_3^{(l)}$, $h^{(l)}$ the invariants of the irreducible representation ρ_l, and by $\bar{x}_1^{(l)}$, $\bar{x}_2^{(l)}$, $\bar{x}_3^{(l)}$, $\bar{h}^{(l)}$ the ones of $\bar{\rho}_l$. Then we deduce the following relations

$$\begin{aligned}
x_1^{(l)} &= \alpha_i^{N\varepsilon(i,l)} \bar{x}_1^{(l)} \\
x_2^{(l)} &= \alpha_j^{N\varepsilon(j,l)} \bar{x}_2^{(l)} \\
x_3^{(l)} &= \alpha_k^{N\varepsilon(k,l)} \bar{x}_3^{(l)} \\
h^{(l)} &= \alpha_i^{\varepsilon(i,l)} \alpha_j^{\varepsilon(j,l)} \alpha_k^{\varepsilon(k,l)} \bar{h}^{(l)}
\end{aligned}$$

Now assume further that the representations ρ_l and $\bar{\rho}_l$ are isomorphic. By virtue of Proposition 2.1.4, this is equivalent to ask that the invariants coincide. Then, for every triangle T_l with edges labelled as before, we must have

$$\begin{aligned}
\alpha_i^{N\varepsilon(i,l)} &= 1 \\
\alpha_j^{N\varepsilon(j,l)} &= 1 \\
\alpha_k^{N\varepsilon(k,l)} &= 1 \\
\alpha_i^{\varepsilon(i,l)} \alpha_j^{\varepsilon(j,l)} \alpha_k^{\varepsilon(k,l)} &= 1
\end{aligned}$$

The first three equations can be rewritten as $\alpha_i^N = \alpha_j^N = \alpha_k^N = 1$ because the appearing terms $\varepsilon(a,l)$ are all equal to ± 1. Hence, there exist $c_i, c_j, c_k \in \mathbb{Z}_N$ such that

$$\begin{aligned}
\alpha_i &= q^{2c_i} \\
\alpha_j &= q^{2c_j} \\
\alpha_k &= q^{2c_k}
\end{aligned}$$

Then the last condition can be rewritten as

$$c_i\,\varepsilon(i,l) + c_j\,\varepsilon(j,l) + c_k\,\varepsilon(k,l) = 0 \in \mathbb{Z}_N \tag{4.4}$$

Observe that the number $c_i\,\varepsilon(i,l)+c_j\,\varepsilon(j,l)+c_k\,\varepsilon(k,l)$ is exactly the coefficient of T_l^* of the combination $\partial_1(\sum_a c_a\,\lambda_a^*)$. This relation must hold for every triangle T_l in the ideal triangulation λ, so the element $\sum_a c_a\,\lambda_a^*$ is a cycle in $C_1(\Gamma;\mathbb{Z}_N)$, or equivalently it belongs to $H_1(\Gamma;\mathbb{Z}_N)$, and the representation $\rho_1 \otimes \cdots \otimes \rho_m$ coincides with $c \cdot (\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_m)$.

Vice versa, if $\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_m$ is equal to $c \cdot (\rho_1 \otimes \cdots \otimes \rho_m)$, then $\bar{\rho}_l$ is defined as in relation 4.2 or 4.3. Assume that the edges of T_l are distinct, the other situations are analogous. Then we can easily see that ρ_l is isomorphic to $\bar{\rho}_l$ for every $l = 1, \ldots, m$. Indeed

$$\begin{aligned}
\bar{\rho}_l((X_1^{(l)})^N) &= \rho_l((X_1^{(l)})^N) & q^{2Nc_i\varepsilon(i,l)} &= 1 \\
\bar{\rho}_l((X_2^{(l)})^N) &= \rho_l((X_2^{(l)})^N) & q^{2Nc_j\varepsilon(j,l)} &= 1 \\
\bar{\rho}_l((X_3^{(l)})^N) &= \rho_l((X_3^{(l)})^N) & q^{2Nc_k\varepsilon(k,l)} &= 1 \\
\bar{\rho}_l(H^{(l)}) &= \rho_l(H^{(l)}) & q^{2(c_i\,\varepsilon(i,l)+c_j\,\varepsilon(j,l)+c_k\,\varepsilon(k,l))} &= 1
\end{aligned}$$

where $H^{(l)}$ is the central element $q^{-1}X_1^{(l)}X_2^{(l)}X_3^{(l)}$ of $\mathcal{T}_{T_l}^q$ and $c_i\,\varepsilon(i,l)+c_j\,\varepsilon(j,l)+c_k\,\varepsilon(k,l) = 0$ holds because c is a cycle. These relations tell us that the invariants of ρ_l and $\bar{\rho}_l$ are the same, and so that the representations are isomorphic. This concludes the proof of the first part of the assertion.

The second part is obvious because, if c is not equal to zero, then the representation $c \cdot (\rho_1 \otimes \cdots \otimes \rho_m)$ is different from $\rho_1 \otimes \cdots \otimes \rho_m$. □

The transition constants between two representatives of a local representation ρ clearly depend on the choice of the orientation on λ, but the described action of $H_1(S;\mathbb{Z}_N)$ does not, it depends only on the orientation of S. In order to justify this assertion, observe that, by changing the orientation of an edge λ_i we change firstly the coefficient c_i of λ_i^* in $-c_i$ of every cycle c, but we swap also the left with the right, so the resulting modification on the representations is the same. So the action is intrinsic and does not depend on the choices we made.

Remark 4.1.6. If two elements $\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_m$ and $\rho_1 \otimes \cdots \otimes \rho_m$ of $\mathscr{F}_{S_0}(\rho)$ are isomorphic, then there exists a linear isomorphism that leads from one to the other. We can give a quite explicit description of this application, which is unique up to scalar multiplication by virtue of Proposition 2.2.8.

Recall that an irreducible representation $\rho_l \colon \mathcal{T}_{T_l}^q \to \mathrm{End}(V_l)$ of the triangle T_l admits a basis $\mathcal{B}$ such that, if $L \colon V_l \to \mathbb{C}^N$ is the coordinate isomorphism induced by $\mathcal{B}$, we have

$$\begin{aligned}
L \circ \rho_l(X_1^{(l)}) \circ L^{-1} &= y_1^{(l)}\, B_1 \\
L \circ \rho_l(X_2^{(l)}) \circ L^{-1} &= y_2^{(l)}\, B_2 \\
L \circ \rho_l(X_3^{(l)}) \circ L^{-1} &= y_3^{(l)}\, B_3
\end{aligned}$$

where $y_i^{(l)}$ is a N-th root of $x_i^{(l)}$ for every $i = 1,2,3$, $y_1^{(l)} y_2^{(l)} y_3^{(l)} = h^{(l)}$ is the central load of ρ_l and the B_i are defined as

$$B_1 := \begin{pmatrix} 1 & & & \\ & q^2 & & \\ & & \ddots & \\ & & & q^{2(N-1)} \end{pmatrix}$$

$$B_2 := \begin{pmatrix} 0 & \cdots & 0 & 1 \\ & & & 0 \\ & I_{N-1} & & \vdots \\ & & & 0 \end{pmatrix}$$

$$B_3 := \begin{pmatrix} 0 & q^{1-2(2-1)} & & \\ \vdots & & \ddots & \\ 0 & & & q^{1-2(N-1)} \\ q & 0 & \cdots & 0 \end{pmatrix}$$

Moreover, it is immediate to verify that the conjugation homomorphisms $C \mapsto A \circ C \circ A^{-1}$, with $A = B_1, B_2, B_3$, applied to $L \circ \rho \circ L^{-1}$ change the $y_i^{(l)}$ respectively as follows

$$\begin{aligned}
y_1^{(l)} &\longmapsto y_1^{(l)} \\
y_2^{(l)} &\longmapsto q^2 y_2^{(l)} \\
y_3^{(l)} &\longmapsto q^{-2} y_3^{(l)}
\end{aligned}$$

$$\begin{aligned} y_1^{(l)} &\longmapsto q^{-2} y_1^{(l)} \\ y_2^{(l)} &\longmapsto y_2^{(l)} \\ y_3^{(l)} &\longmapsto q^{2} y_3^{(l)} \end{aligned}$$

$$\begin{aligned} y_1^{(l)} &\longmapsto q^{2} y_1^{(l)} \\ y_2^{(l)} &\longmapsto q^{-2} y_2^{(l)} \\ y_3^{(l)} &\longmapsto y_3^{(l)} \end{aligned}$$

Then, defining $M_i := L^{-1} \circ B_i \circ L$ for $i = 1, 2, 3$, we have construct automorphisms M_i of V_l such that

$$\begin{aligned} M_1 \circ \rho_l(X_1^{(l)}) \circ M_1^{-1} &= \rho_l(X_1^{(l)}) \\ M_1 \circ \rho_l(X_2^{(l)}) \circ M_1^{-1} &= q^2\, \rho_l(X_2^{(l)}) \\ M_1 \circ \rho_l(X_3^{(l)}) \circ M_1^{-1} &= q^{-2}\, \rho_l(X_3^{(l)}) \end{aligned}$$

$$\begin{aligned} M_2 \circ \rho_l(X_1^{(l)}) \circ M_2^{-1} &= q^{-2}\, \rho_l(X_1^{(l)}) \\ M_2 \circ \rho_l(X_2^{(l)}) \circ M_2^{-1} &= \rho_l(X_2^{(l)}) \\ M_2 \circ \rho_l(X_3^{(l)}) \circ M_2^{-1} &= q^{2}\, \rho_l(X_3^{(l)}) \end{aligned}$$

$$\begin{aligned} M_3 \circ \rho_l(X_1^{(l)}) \circ M_3^{-1} &= q^{2}\, \rho_l(X_1^{(l)}) \\ M_3 \circ \rho_l(X_2^{(l)}) \circ M_3^{-1} &= q^{-2}\, \rho_l(X_2^{(l)}) \\ M_3 \circ \rho_l(X_3^{(l)}) \circ M_3^{-1} &= \rho_l(X_3^{(l)}) \end{aligned}$$

The isomorphisms M_i are unique up to scalar multiplication, because of the irreducibility of the considered representations. Moreover, by means of compositions of these applications, we can obtain every change of parameters of the form

$$\begin{aligned} y_1^{(l)} &\longmapsto q^{2k_1} y_1^{(l)} \\ y_2^{(l)} &\longmapsto q^{2k_2} y_2^{(l)} \\ y_3^{(l)} &\longmapsto q^{2k_3} y_3^{(l)} \end{aligned}$$

for every $k_1, k_2, k_3 \in \mathbb{Z}_N$ such that $k_1 + k_2 + k_3 = 0$.

We observed that, given $\rho_1 \otimes \cdots \otimes \rho_m$ and $\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_m$ two representatives of ρ_λ such that ρ_j is individually isomorphic to $\bar{\rho}_j$ for every $j = 1, \ldots, m$, then there exists $c \in H_1(S; \mathbb{Z}_N)$ such that they are related as follows

$$\begin{aligned} \bar{\rho}_l(X_1^{(l)}) &= q^{2c_i \varepsilon(i,l)}\, \rho_l(X_1^{(l)}) \\ \bar{\rho}_l(X_2^{(l)}) &= q^{2c_j \varepsilon(j,l)}\, \rho_l(X_2^{(l)}) \\ \bar{\rho}_l(X_3^{(l)}) &= q^{2c_k \varepsilon(k,l)}\, \rho_l(X_3^{(l)}) \end{aligned}$$

for every triangle T_l of the ideal triangulation $\lambda \in \Lambda(S)$ (if the edges of T_l are distinct, otherwise see relation 4.3). Fixed $l = 1, \ldots, m$, there exists a linear isomorphism $M^{(l)}\colon V_l \to V_l$ such that

$$M^{(l)} \circ \rho_l(X) \circ (M^{(l)})^{-1} = \bar{\rho}_l(X)$$

for every $X \in \mathcal{T}_\lambda^q$, and this map can be expressed, up to scalar multiplication, as composition of the elementary applications M_i described above, because the elements $k_1 = c_i\varepsilon(i,l)$, $k_2 = c_j\varepsilon(j,l)$ and $k_3 = c_k\varepsilon(k,l)$ verify $k_1 + k_2 + k_3 = 0$, by virtue of the relation 4.4.

What just noticed shows that every $\rho_1 \otimes \cdots \otimes \rho_m$ and $\bar{\rho}_1 \otimes \cdots \otimes \bar{\rho}_m$ isomorphic representatives of a local representation ρ are isomorphic to each other through a tensor split linear isomorphism $M^{(1)} \otimes \cdots \otimes M^{(m)}$, in which every $M^{(l)}$ is a certain composition of the applications $M_1^{\pm 1}, M_2^{\pm 1}, M_3^{\pm 1}$ described above.

4.2 The elementary cases

The first part of our work is devoted to give the definitions of the sets $\mathscr{L}_{\lambda\lambda'}^{\rho\rho'}$ and their actions $\psi_{\lambda\lambda'}^{\rho\rho'}$ in the simplest cases, namely when λ and λ' differ by an elementary move. In particular the discussion will be divided in the following cases

- when λ and λ' are equal;
- when λ and λ' differ by a reindexing;
- when λ and λ' differ by a diagonal exchange.

In all this Section we will assume that the elements

$$\rho = \{\rho_\lambda\colon \mathcal{T}_\lambda^q \to \mathrm{End}(V_\lambda)\}_{\lambda \in \Lambda(S)} \qquad \rho' = \{\rho'_\lambda\colon \mathcal{T}_\lambda^q \to \mathrm{End}(V'_\lambda)\}_{\lambda \in \Lambda(S)}$$

are isomorphic local representations of the quantum Teichmüller space of S.

4.2.1 Same triangulation

Fixed $\lambda \in \Lambda(S)$, the maps $\rho_\lambda\colon \mathcal{T}_\lambda^q \to \mathrm{End}(V_\lambda)$ and $\rho'_\lambda\colon \mathcal{T}_\lambda^q \to \mathrm{End}(V'_\lambda)$, part of ρ and ρ' respectively, are two isomorphic local representations of the Chekhov-Fock algebra $\mathcal{T}_\lambda^q$. Let S_0 be the surface obtained by splitting S along λ and let λ_0 be its ideal triangulation. Define

$$A_{\lambda\lambda}^{\rho\rho'} := \{(\zeta,\zeta') \in \mathscr{F}_{S_0}(\rho_\lambda) \times \mathscr{F}_{S_0}(\rho'_\lambda) \mid \zeta \textit{ and } \zeta' \textit{ are isomorphic}\}$$

where $\zeta = \bigotimes_j \rho_j$ and $\zeta' = \bigotimes_j \rho'_j$ are thought as local (and irreducible) representations of $\mathcal{T}_{\lambda_0}^q$. For every $(\zeta,\zeta') \in A_{\lambda\lambda}^{\rho\rho'}$ there exists a tensor-split linear isomorphism $L^{\zeta\zeta'} = L_1 \otimes \cdots \otimes L_m\colon V'_\lambda \to V_\lambda$, unique up to multiplicative constant, such that

$$L^{\zeta\zeta'} \circ \zeta'(X) \circ (L^{\zeta\zeta'})^{-1} = \zeta(X) \in \mathrm{End}(V_\lambda)$$

for every $X \in \mathcal{T}_{S_0}^q$ (observe that each L_i is an isomorphism between ρ_j and ρ'_j). The uniqueness follows from the irreducibility of local representations when the surface is a disjoint union of ideal polygons (see Proposition 2.2.8). Now, label as $\mathscr{L}^{\rho\rho'}_{\lambda\lambda}$ the set of operators $L^{\zeta\zeta'}\colon V'_\lambda \to V_\lambda$, for varying (ζ,ζ') in $A^{\rho\rho'}_{\lambda\lambda}$. The elements $L^{\zeta\zeta'}$ are defined up to multiplicative constant, but we will always omit the equivalence class to simplify the notation. There is an obvious surjective map $p\colon A^{\rho\rho'}_{\lambda\lambda} \to \mathscr{L}^{\rho\rho'}_{\lambda\lambda}$ that associates with a couple (ζ,ζ') the corresponding isomorphism $L^{\zeta\zeta'}$.

Suppose that (ζ,ζ') and $(\bar\zeta,\bar\zeta')$ go under p in the same isomorphism

$$L = L^{\zeta\zeta'} = L^{\bar\zeta\bar\zeta'} = L_1 \otimes \cdots \otimes L_m : V'_\lambda \longrightarrow V_\lambda$$

In particular, $\zeta = \bigotimes_j \rho_j$ and $\bar\zeta = \bigotimes_j \bar\rho_j$ have to be equivalent, belonging both to $\mathscr{F}_{S_0}(\rho_\lambda)$. Therefore, there exist transition constants $\alpha_i \in \mathbb{C}^*$ such that $\bar\zeta \overset{\alpha_i}{\to} \zeta$. On the other hand, by hypothesis, the following hold

$$\begin{aligned} L_j \circ \rho'_j(X) \circ L_j^{-1} &= \rho_j(X) \\ L_j \circ \bar\rho'_j(X) \circ L_j^{-1} &= \bar\rho_j(X) \end{aligned} \tag{4.5}$$

where $\zeta' = \bigotimes_j \rho'_j$ and $\bar\zeta' = \bigotimes_j \bar\rho'_j$. By using $\bar\zeta \overset{\alpha_i}{\to} \zeta$ and the relations 4.5, we deduce that

- for every edge λ_i lying in the boundary of S, if T_{k_i} is the triangle on its side and a_i is the index of the side of T_{k_i} identified in λ to λ_i, then

$$\begin{aligned} \rho'_{k_i}(X^{(k_i)}_{a_i}) &= L^{-1}_{k_i} \circ \rho_{k_i}(X^{(k_i)}_{a_i}) \circ L_{k_i} \\ &= L^{-1}_{k_i} \circ \bar\rho_{k_i}(X^{(k_i)}_{a_i}) \circ L_{k_i} \\ &= \bar\rho'_{k_i}(X^{(k_i)}_{a_i}) \end{aligned}$$

- for every internal edge λ_i with different triangles on its sides, in the notations of Definition 4.1.1, we have

$$\begin{aligned} \rho'_{l_i}(X^{(l_i)}_{a_i}) &= L^{-1}_{l_i} \circ \rho_{l_i}(X^{(l_i)}_{a_i}) \circ L_{l_i} \\ &= \alpha_i\, L^{-1}_{l_i} \circ \bar\rho_{l_i}(X^{(l_i)}_{a_i}) \circ L_{l_i} \\ &= \alpha_i\, \bar\rho'_{l_i}(X^{(l_i)}_{a_i}) \end{aligned}$$

$$\begin{aligned} \rho'_{r_i}(X^{(r_i)}_{b_i}) &= L^{-1}_{r_i} \circ \rho_{r_i}(X^{(r_i)}_{b_i}) \circ L_{r_i} \\ &= \alpha_i^{-1}\, L^{-1}_{r_i} \circ \bar\rho_{r_i}(X^{(r_i)}_{b_i}) \circ L_{r_i} \\ &= \alpha_i^{-1}\, \bar\rho'_{r_i}(X^{(r_i)}_{b_i}) \end{aligned}$$

  and analogously when λ_i is an internal edge with the same triangle on its sides.

Therefore, we have shown that $p(\zeta,\zeta') = p(\bar\zeta,\bar\zeta')$ implies that the transition constants α_i from $\bar\zeta$ to ζ are exactly the same as those from $\bar\zeta'$ and ζ', that is

$$\begin{aligned} \bar\zeta &\xrightarrow{\alpha_i} \zeta \\ \bar\zeta' &\xrightarrow{\alpha_i} \zeta' \end{aligned}$$

We will briefly denote this phenomenon between (ζ,ζ') and $(\bar\zeta,\bar\zeta')$ by $(\zeta,\zeta') \approx (\bar\zeta,\bar\zeta')$.

Vice versa, suppose that two couples (ζ, ζ') and $(\bar\zeta, \bar\zeta')$ in $A^{\rho\rho'}_{\lambda\lambda}$ are in $\approx$-relation, with $\bar\zeta \xrightarrow{\alpha_i} \zeta$ and $\bar\zeta' \xrightarrow{\alpha_i} \zeta'$, and label $\zeta = \bigotimes_j \rho_j$, $\zeta' = \bigotimes_j \rho'_j$, $\bar\zeta = \bigotimes_j \bar\rho_j$, $\bar\zeta' = \bigotimes_j \bar\rho'_j$. Because (ζ, ζ') is in $A^{\rho\rho'}_{\lambda\lambda}$, there exists an isomorphism $L^{\zeta\zeta'} : V'_\lambda \to V_\lambda$ between ζ and ζ', with $L^{\zeta\zeta'} = L_1 \otimes \cdots \otimes L_m$. Then the following hold

- for every edge λ_i lying in the boundary of S, if T_{k_i} is the triangle on its side and a_i is the index of the side of T_{k_i} identified in λ to λ_i, then
$$\begin{aligned}\bar\rho_{k_i}(X^{(k_i)}_{a_i}) &= \rho_{k_i}(X^{(k_i)}_{a_i}) \\ &= L_{k_i} \circ \rho'_{k_i}(X^{(k_i)}_{a_i}) \circ L^{-1}_{k_i} \\ &= L_{k_i} \circ \bar\rho'_{k_i}(X^{(k_i)}_{a_i}) \circ L^{-1}_{k_i}\end{aligned}$$

- for every internal edge λ_i with different triangles on its sides, in the notations of Definition 4.1.1, we have
$$\begin{aligned}\bar\rho_{l_i}(X^{(l_i)}_{a_i}) &= \alpha_i^{-1}\, \rho_{l_i}(X^{(l_i)}_{a_i}) = \alpha_i^{-1}\, L_{l_i} \circ \rho'_{l_i}(X^{(l_i)}_{a_i}) \circ L^{-1}_{l_i} \\ &= L_{l_i} \circ \bar\rho'_{l_i}(X^{(l_i)}_{a_i}) \circ L^{-1}_{l_i} \\ \bar\rho_{r_i}(X^{(r_i)}_{b_i}) &= \alpha_i\, \rho_{r_i}(X^{(r_i)}_{b_i}) = \alpha_i\, L_{r_i} \circ \rho'_{r_i}(X^{(r_i)}_{b_i}) \circ L^{-1}_{r_i} \\ &= L_{r_i} \circ \bar\rho'_{r_i}(X^{(r_i)}_{b_i}) \circ L^{-1}_{r_i}\end{aligned}$$
  and analogously when λ_i has the same triangle on its sides.

Since these hold for every i varying from 1 to n, we have shown that $L^{\zeta\zeta'}$ is an isomorphism between $\bar\zeta$ and $\bar\zeta'$, so by irreducibility $L^{\zeta\zeta'} \doteq L^{\bar\zeta\bar\zeta'}$. The equivalence relation $\approx$ on $A^{\rho\rho'}_{\lambda\lambda}$ is therefore compatible with the map p and the corresponding application on the quotient $\mathscr{A}^{\rho\rho'}_{\lambda\lambda} := A^{\rho\rho'}_{\lambda\lambda} / \approx$, which we denote by
$$\widetilde{p} : \mathscr{A}^{\rho\rho'}_{\lambda\lambda} \longrightarrow \mathscr{L}^{\rho\rho'}_{\lambda\lambda}$$
is a bijection. Moreover, we can let $H_1(S, \mathbb{Z}_N)$ act on $A^{\rho\rho'}_{\lambda\lambda}$ as follows
$$c \cdot (\zeta, \zeta') := (\zeta, c \cdot \zeta') \tag{4.6}$$
where $c \cdot \zeta' = c \cdot (\rho'_1 \otimes \cdots \otimes \rho'_m)$ is the action defined previously, in this case on $\mathscr{F}_{S_0}(\rho'_\lambda)$. Now we want to show that the definition in 4.6 is compatible with the relation $\approx$ and then it leads to an action
$$\begin{array}{rccc}\psi^{\rho\rho'}_{\lambda\lambda} : & H_1(S; \mathbb{Z}_N) \times \mathscr{A}^{\rho\rho'}_{\lambda\lambda} & \longrightarrow & \mathscr{A}^{\rho\rho'}_{\lambda\lambda} \\ & (c, [\zeta, \zeta']) & \longmapsto & [\zeta, c \cdot \zeta']\end{array}$$
of $H_1(S; \mathbb{Z}_N)$ on the quotient $\mathscr{A}^{\rho\rho'}_{\lambda\lambda}$ and equivalently, through $\widetilde{p}$, on $\mathscr{L}^{\rho\rho'}_{\lambda\lambda}$.

Theorem 4.2.1. *The action of $H_1(S; \mathbb{Z}_N)$ on $\mathscr{A}^{\rho\rho'}_{\lambda\lambda}$, and equivalently on $\mathscr{L}^{\rho\rho'}_{\lambda\lambda}$, is well defined, transitive and free. Moreover, for every $[\zeta, \zeta'] \in \mathscr{A}^{\rho\rho'}_{\lambda\lambda}$ and for every $c \in H_1(S; \mathbb{Z}_N)$ we have*
$$c \cdot [\zeta, \zeta'] = [(-c) \cdot \zeta, \zeta'] \tag{4.7}$$

Proof. In order to show the good definition, suppose that

$$\zeta = \bigotimes_i \rho_i \xrightarrow{\alpha_i} \bigotimes_i \bar{\rho}_i = \bar{\zeta}$$
$$\zeta' = \bigotimes_i \rho'_i \xrightarrow{\alpha_i} \bigotimes_i \bar{\rho}'_i = \bar{\zeta}'$$

By definition of $c\cdot$, we have

$$\zeta' \xrightarrow{q^{2c_i}} c \cdot \zeta'$$
$$\bar{\zeta}' \xrightarrow{q^{2c_i}} c \cdot \bar{\zeta}'$$

Then

$$c \cdot \zeta' \xrightarrow{q^{-2c_i}} \zeta' \xrightarrow{\alpha_i} \bar{\zeta}' \xrightarrow{q^{2c_i}} c \cdot \bar{\zeta}'$$

so $c \cdot \zeta' \overset{\alpha_i}{\to} c \cdot \bar{\zeta}'$. On the other hand, we have $\zeta \overset{\alpha_i}{\to} \bar{\zeta}$, and these facts together tell us that $c \cdot (\zeta, \zeta') := (\zeta, c \cdot \zeta') \approx (\bar{\zeta}, c \cdot \bar{\zeta}') =: c \cdot (\bar{\zeta}, \bar{\zeta}')$, as desired.

Now we will prove that the action is transitive. Let $[\zeta, \zeta']$, $[\bar{\zeta}, \bar{\zeta}']$ be two elements of $\mathscr{A}^{\rho\rho'}_{\lambda\lambda}$ and (ζ, ζ'), $(\bar{\zeta}, \bar{\zeta}')$ two representatives of them, with $\zeta = \bigotimes_j \rho_j$, $\zeta' = \bigotimes_j \rho'_j$, $\bar{\zeta} = \bigotimes_j \bar{\rho}_j$ and $\bar{\zeta}' = \bigotimes_j \bar{\rho}'_j$. Both ζ and $\bar{\zeta}$ belong to $\mathscr{F}_{S_0}(\rho_\lambda)$, then there exists a family $(\alpha_i)_i$ of transition constants such that $\zeta \overset{\alpha_i}{\to} \bar{\zeta}$. Because (ζ, ζ') is an element of $A^{\rho\rho'}_{\lambda\lambda}$, there exist isomorphisms $L_j \colon V'_j \to V_j$ such that $L_j \circ \rho'_j \circ L_j^{-1} = \rho_j$ for every $j = 1, \dots, m$. Now we can construct a representation $\widetilde{\zeta}' = \bigotimes_j \widetilde{\rho}'_j$ defined by

$$\widetilde{\rho}'_j(X) := L_j^{-1} \circ \bar{\rho}_j(X) \circ L_j$$

for every $j = 1, \dots, m$. The representation $\widetilde{\zeta}'$ belongs to $\mathscr{F}_{S_0}(\rho'_\lambda)$ because by construction it can be obtained from ζ', which is an element of $\mathscr{F}_{S_0}(\rho'_\lambda)$, as

$$\zeta' \xrightarrow{\alpha_i} \widetilde{\zeta}'$$

So $(\bar{\zeta}, \widetilde{\zeta}')$ belongs to $A^{\rho\rho'}_{\lambda\lambda}$ and ζ', $\widetilde{\zeta}'$ are related by the transition constants $(\alpha_i)_i$, just like ζ and $\bar{\zeta}$. This means that the couples (ζ, ζ') and $(\bar{\zeta}, \widetilde{\zeta}')$ are $\approx$-equivalent, i. e. $[\zeta, \zeta'] = [\bar{\zeta}, \widetilde{\zeta}']$. Moreover, both $\widetilde{\zeta}'$ and $\bar{\zeta}'$ are isomorphic to $\bar{\zeta}$ and then they are isomorphic to each other. By Proposition 4.1.5, there exists a unique $c \in H_1(S; \mathbb{Z}_N)$ such that $c \cdot \widetilde{\zeta}' = \bar{\zeta}'$, so

$$c \cdot [\zeta, \zeta'] = c \cdot [\bar{\zeta}, \widetilde{\zeta}'] = [\bar{\zeta}, c \cdot \widetilde{\zeta}'] = [\bar{\zeta}, \bar{\zeta}']$$

and this proves that the action is transitive.

Now suppose that there exist a $c \in H_1(S; \mathbb{Z}_N)$ and an element $[\zeta, \zeta'] \in \mathscr{A}^{\rho\rho'}_{\lambda\lambda}$ such that $[\zeta, \zeta'] = c \cdot [\zeta, \zeta']$. This means that, passing on representatives, the couples (ζ, ζ') and $(\zeta, c \cdot \zeta')$ are $\approx$-equivalent. Because the first terms of the couples are exactly the same, they are in particular related by transition constants all equal to 1, and the same must hold for ζ' and $c \cdot \zeta'$. This means that $\zeta' = c \cdot \zeta'$ and so, by the second assertion of Proposition 4.1.5, we conclude.

For what concerns the equation 4.7, we firstly note that $[\zeta, \zeta'] = [c \cdot \zeta, c \cdot \zeta']$. Indeed, we have $\zeta \overset{q^{2c_i}}{\to} c \cdot \zeta$ and $\zeta' \overset{q^{2c_i}}{\to} c \cdot \zeta'$, which means that $(\zeta, \zeta') \approx (c \cdot \zeta, c \cdot \zeta')$. Now it is immediate to prove the relation:

$$[\zeta, c \cdot \zeta'] = [(-c) \cdot \zeta, (-c + c) \cdot \zeta'] = [(-c) \cdot \zeta, \zeta']$$

□

We will denote the action of $H_1(S; \mathbb{Z}_N)$ on $\mathscr{L}^{\rho\rho'}_{\lambda\lambda}$ by $\psi^{\rho\rho'}_{\lambda\lambda}$.

Remark 4.2.2. An important consequence of this fact concerns the definition of the intertwining operators exposed in [BBL07]. Recall the following assertion:

Theorem ([BBL07, Theorem 20]). *For every surface S there exists a unique family of intertwining operators $\widehat{L}^{\rho\rho'}_{\lambda\lambda'}$, indexed by couples of isomorphic local representations of $\mathcal{T}^q_S$ and by couples of ideal triangulations $\lambda, \lambda' \in \Lambda(S)$, individually defined up to scalar multiplication, such that:*

COMPOSITION RELATION: *for every $\lambda, \lambda', \lambda'' \in \Lambda(S)$ and for every triple of isomorphic local representations ρ, ρ', ρ'', we have $\widehat{L}^{\rho\rho''}_{\lambda\lambda''} \doteq \widehat{L}^{\rho\rho'}_{\lambda\lambda'} \circ \widehat{L}^{\rho'\rho''}_{\lambda'\lambda''}$;*

FUSION RELATION: *let S be a surface obtained from another surface R by fusion, and let λ, λ' be two triangulations of S obtained by fusion of two triangulations μ, μ' of R. If η, η' are two isomorphic local representations of $\mathcal{T}^q_R$ and ρ, ρ' are two isomorphic local representations of $\mathcal{T}^q_S$ obtained by fusion from η, η' respectively, then we have $\widehat{L}^{\eta\eta'}_{\mu\mu'} \doteq \widehat{L}^{\rho\rho'}_{\lambda\lambda'}$.*

In what follows we want to describe how the facts observed in Theorem 4.2.1 show a problem in the definition of the intertwining operators $\widehat{L}^{\rho\rho'}_{\lambda\lambda'}$ of [BBL07, Theorem 20], in particular in the case in which $\lambda = \lambda'$.

Fix $\rho = \{\rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)\}_{\lambda \in \Lambda(S)}$ and $\rho' = \{\rho'_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)\}_{\lambda \in \Lambda(S)}$ two isomorphic local representations of $\mathcal{T}^q_S$, where S is a certain surface with non-trivial $H_1(S; \mathbb{Z}_N)$. Since ρ and ρ' are isomorphic, we can choose representatives $\zeta = \rho_1 \otimes \cdots \otimes \rho_m$ and $\zeta' = \rho'_1 \otimes \cdots \otimes \rho'_m$ of ρ_λ and ρ'_λ, respectively, such that $\rho_j \colon \mathcal{T}^q_{T_j} \to \mathrm{End}(V_j)$ is individually isomorphic to $\rho'_j \colon \mathcal{T}^q_{T_j} \to \mathrm{End}(V'_j)$ by a linear isomorphism $L_j \colon V_j \to V'_j$. Denoting as before by S_0 the surface obtained by splitting S along λ and by λ_0 its triangulation, we have that ζ and ζ' are two local representations of $\mathcal{T}^q_{S_0}$ and by construction they are isomorphic by the linear transformation

$$L_1 \otimes \cdots \otimes L_m : V'_\lambda \longrightarrow V_\lambda$$

Moreover, this application is unique, up to scalar multiplication, because ζ and ζ' are irreducible. S_0 is a disjoint union of triangles, then it admits a unique ideal triangulation λ_0. This means that ζ and ζ' can be thought as local representations of the whole $\mathcal{T}^q_{S_0}$ and that $\widehat{L}^{\zeta\zeta'}_{\lambda_0\lambda_0} \doteq L_1 \otimes \cdots \otimes L_m$. Assuming that [BBL07, Theorem 20] holds, the element $\widehat{L}^{\rho\rho'}_{\lambda\lambda}$ must be equal to $\widehat{L}^{\zeta\zeta'}_{\lambda_0\lambda_0}$ up to scalar multiplication by virtue of the Fusion Property, so

$$\widehat{L}^{\rho\rho'}_{\lambda\lambda} \doteq L_1 \otimes \cdots \otimes L_m \tag{4.8}$$

Now we are going to show that different choices of representatives for ρ_λ and ρ'_λ produce a contradiction in relation 4.8. Take the same representative ζ for ρ_λ and replace ζ' with $c \cdot \zeta'$, for a certain non-trivial $c \in H_1(\Gamma; \mathbb{Z}_N) = H_1(S; \mathbb{Z}_N)$. Thanks to Proposition 4.1.5, the representations ζ' and $c \cdot \zeta'$ are isomorphic via an automorphism $M^{(1)} \otimes \cdots \otimes M^{(m)}$ of $V'_1 \otimes \cdots \otimes V'_m$, which is non-trivial up to scalar multiplication because $c \neq 0$. Define $\bar{\zeta}' := c \cdot \zeta'$. This is a local and irreducible representation of $\mathcal{T}^q_{\lambda_0}$ and it represents ρ'_λ on S, just like ζ'. Moreover, the representations ζ and $\bar{\zeta}'$ are isomorphic via

$$(L_1 \circ M^{(1)}) \otimes \cdots \otimes (L_m \circ M^{(m)})$$

Applying the Fusion property as before, but on $\bar{\zeta}'$ instead of ζ', we obtain

$$\begin{aligned} \widehat{L}^{\rho\rho'}_{\lambda\lambda} &\doteq \widehat{L}^{\zeta\bar{\zeta}'}_{\lambda_0\lambda_0} \\ &\doteq (L_1 \circ M^{(1)}) \otimes \cdots \otimes (L_m \circ M^{(m)}) \end{aligned}$$

but this is in contradiction with 4.8, because $M^{(1)} \otimes \cdots \otimes M^{(m)}$ is not equal to the identity up to scalar multiplication.

4.2.2 Reindexing

In the investigation of local representations we have intentionally ignored the problems concerning the case in which the ideal triangulations λ and λ' differ by reindexing, i. e. $\lambda' = \gamma(\lambda)$ with $\gamma \in \mathfrak{S}_n$. We did not focus on that because all the properties of representations are basically intrinsic and does not really depend on the ordering of the edges, but only on the structure of the triangulation. Indeed, the coordinate change isomorphisms $\Phi^q_{\lambda\lambda'}$ in this case are just the maps on the fraction rings induced by the reordering applications from $\mathcal{T}^q_\lambda$ to $\mathcal{T}^q_{\gamma(\lambda)}$. Moreover, the described action of $H_1(S; \mathbb{Z}_N)$ clearly does not depend on the fixed order of the edges. We will continue to be vague on that, we want just to enunciate the fact we will use later, analogous to Theorem 4.2.1.

Let $\lambda, \lambda' \in \Lambda(S)$ be two ideal triangulations differing by a reindexing of the edges, with $\lambda' = \gamma(\lambda)$. Define $A^{\rho\rho'}_{\lambda\lambda'}$ as the set of couples $(\zeta_{\lambda_0}, \zeta'_{\lambda'_0})$, where ζ_{λ_0} is an element of $\mathscr{F}_{S_0}(\rho_\lambda)$, $\zeta_{\lambda'_0}$ is an element of $\mathscr{F}_{S_0}(\rho'_{\lambda'})$ and $\zeta_{\lambda_0} \circ \Phi^q_{\lambda_0\lambda'_0}$ is isomorphic to $\zeta'_{\lambda'_0}$ (λ and λ' clearly induce the same splitted surface S_0, we should give details of indexing of triangulations λ_0 and λ'_0 in order give a sense to $\Phi^q_{\lambda_0\lambda'_0}$, but we omit this boring procedure). We say that $(\zeta_{\lambda_0}, \zeta'_{\lambda'_0}), (\bar{\zeta}_{\lambda_0}, \bar{\zeta}'_{\lambda'_0}) \in A^{\rho\rho'}_{\lambda\lambda'}$ are $\approx$-equivalent if

$$\zeta_{\lambda_0} \xrightarrow{\alpha_i} \bar{\zeta}_{\lambda_0}$$
$$\zeta'_{\lambda'_0} \xrightarrow{\beta_i} \bar{\zeta}'_{\lambda'_0}$$

with $\beta_i = \alpha_{\gamma(i)}$ for every i. As before $\mathscr{A}^{\rho\rho'}_{\lambda\lambda'}$ denotes the quotient of $A^{\rho\rho'}_{\lambda\lambda'}$ by the relation $\approx$. On $A^{\rho\rho'}_{\lambda\lambda'}$ we consider a natural map $\widetilde{p}\colon \mathscr{A}^{\rho\rho'}_{\lambda\lambda'} \to \mathrm{Hom}(V'_{\lambda'}, V_\lambda)$, sending an element $[\zeta_{\lambda_0}, \zeta'_{\lambda'_0}]$ in the linear isomorphism between ζ_{λ_0} and $\zeta'_{\lambda'_0}$. The map is injective and we designate its image as $\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}$. We can define on $\mathscr{A}^{\rho\rho'}_{\lambda\lambda'}$ an action $\psi^{\rho\rho'}_{\lambda\lambda'}$ of $H_1(S; \mathbb{Z}_N)$ setting $c \cdot [\zeta_{\lambda_0}, \zeta'_{\lambda'_0}] := [\zeta_{\lambda_0}, c \cdot \zeta'_{\lambda'_0}]$. In light of Theorem 4.2.1, it is straightforward to prove that the following holds

Theorem 4.2.3. *The action of $H_1(S;\mathbb{Z}_N)$ on $\mathscr{A}^{\rho\rho'}_{\lambda\lambda'}$, and equivalently on $\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}$, is well defined, transitive and free. Moreover, for every $[\zeta_{\lambda_0},\zeta'_{\lambda'_0}]\in\mathscr{A}^{\rho\rho'}_{\lambda\lambda'}$ and for every $c\in H_1(S;\mathbb{Z}_N)$ we have*

$$c\cdot[\zeta_{\lambda_0},\zeta'_{\lambda'_0}]=[(-c)\cdot\zeta_{\lambda_0},\zeta'_{\lambda'_0}]$$

4.2.3 Diagonal exchange

Assume that λ and λ' are ideal triangulations of S that differ by a diagonal exchange along λ_i. Designate as R the surface obtained from S by splitting it along all the edges of λ except for λ_i. R is the disjoint union of an ideal square Q and $m-2$ ideal triangles. In order to simplify the notation, we will assume that the triangles composing Q are labelled as T_1 and T_2 and the others as $T_3,\dots,T_m$. The triangulations λ and λ' induce on R two ideal triangulations μ,μ'. μ is just the disjoint union of an ideal triangulation μ_Q of Q and the only possible triangulation μ_0 on the disjoint union of the triangles T_j for $j\geq 3$. Analogously $\mu'=\mu'_Q\sqcup\mu_0$ where μ'_Q is the only ideal triangulation on Q different from μ_Q. Observe that the Chekhov-Fock algebras associated with the triangulation μ and μ' on R are canonically isomorphic to the tensor products

$$\mathcal{T}^q_{\mu_Q}\otimes\mathcal{T}^q_{T_3}\otimes\cdots\otimes\mathcal{T}^q_{T_m},\quad \mathcal{T}^q_{\mu'_Q}\otimes\mathcal{T}^q_{T_3}\otimes\cdots\otimes\mathcal{T}^q_{T_m}$$

We will denote by S_0 the surface obtained by splitting S along λ, by S'_0 the surface obtained by splitting S along λ' and by λ_0 and λ'_0 the respective triangulations on these surfaces.

Fix $\rho=\{\rho_\eta\colon\mathcal{T}^q_\eta\to\mathrm{End}(V_\eta)\}_{\eta\in\Lambda(S)}$ and $\rho'=\{\rho'_\eta\colon\mathcal{T}^q_\eta\to\mathrm{End}(V_\eta)\}_{\eta\in\Lambda(S)}$ two local representations of $\mathcal{T}^q_S$. We introduce the following notations

$$V_\eta=V_{\eta,1}\otimes\cdots\otimes V_{\eta,m}$$
$$V'_\eta=V'_{\eta,1}\otimes\cdots\otimes V'_{\eta,m}$$

Denote by $\mathscr{F}_{S_0}(\rho_\lambda)$ the set of local representations of $\mathcal{T}^q_{\lambda_0}$ that represent ρ_λ on S and analogously label as $\mathscr{F}_{S'_0}(\rho'_{\lambda'})$ the set of local representations of $\mathcal{T}^q_{\lambda'_0}$ that represent $\rho'_{\lambda'}$ on S. Given ζ_{λ_0} an element of $\mathscr{F}_{S_0}(\rho_\lambda)$, ζ_{λ_0} represents a local representation ζ_μ of the Chekhov-Fock algebra $\mathcal{T}^q_\mu$, and in the same way a representation $\zeta'_{\lambda'_0}\in\mathscr{F}_{S'_0}(\rho'_{\lambda'})$ induces a representation $\zeta'_{\mu'}$ of $\mathcal{T}^q_{\mu'}$.

Now we define

$$A^{\rho\rho'}_{\lambda\lambda'}:=\{(\zeta_{\lambda_0},\zeta'_{\lambda'_0})\in\mathscr{F}_{S_0}(\rho_\lambda)\times\mathscr{F}_{S'_0}(\rho'_{\lambda'})\mid\zeta_\mu\circ\Phi^q_{\mu\mu'}\ \textit{is isomorphic to}\ \zeta'_\mu\}$$

It is easy to verify that the composition $\zeta_\mu\circ\Phi^q_{\mu\mu'}$ makes sense because $\rho_\lambda\circ\Phi^q_{\lambda\lambda'}$ does (the key ingredient is that the invariant of the representation ζ_μ associated with μ_i coincides with the one of ρ_λ for λ_i, which is not equal to -1 because $\rho_\lambda\circ\Phi^q_{\lambda\lambda'}$ makes sense, being ρ_λ part of a global representation of the quantum Teichmüller space). Given ζ_{λ_0} in $\mathscr{F}_{S_0}(\rho_\lambda)$, ζ_μ is equal to the tensor product of a representation ζ_{μ_Q} of $\mathcal{T}^q_{\mu_Q}$ and a representation ζ_{μ_0} of $\mathcal{T}^q_{\mu_0}=\mathcal{T}^q_{T_3}\otimes\cdots\otimes\mathcal{T}^q_{T_m}$. In the same way, given $\zeta'_{\lambda'_0}\in\mathscr{F}_{S'_0}(\rho'_{\lambda'})$, $\zeta'_{\mu'}$ is the tensor product of a representation $\zeta'_{\mu'_Q}$ of $\mathcal{T}^q_{\mu'_Q}$ and a representation ζ'_{μ_0} of $\mathcal{T}^q_{\mu_0}$. Recalling the Disjoint union property of $\Phi^q_{\lambda\lambda'}$ exposed in Theorem 3.1.1, the restriction of $\Phi^q_{\mu\mu'}$ on $\mathcal{T}^q_\mu = \mathcal{T}^q_{\mu_Q} \otimes \mathcal{T}^q_{\mu_0}$ coincides with $\Phi^q_{\mu_Q\mu'_Q} \otimes id$. Thus the representation $\zeta_\mu \circ \Phi^q_{\mu\mu'}$ is equal to

$$(\zeta_{\mu_Q} \circ \Phi^q_{\mu_Q\mu'_Q}) \otimes \zeta_{\mu_0} \tag{4.9}$$

By virtue of the irreducibility of ζ_μ and $\zeta'_{\mu'}$ (observe that R is a disjoint union of ideal polygons) there exists an isomorphism $L^{\zeta_\mu\zeta'_{\mu'}}$, unique up to scalar multiplication, such that

$$L^{\zeta_\mu\zeta'_{\mu'}} \circ \zeta'_{\mu'}(X') \circ (L^{\zeta_\mu\zeta'_{\mu'}})^{-1} = (\zeta_\mu \circ \Phi^q_{\mu\mu'})(X')$$

for every $X' \in \mathcal{T}^q_{\mu'}$. In analogy with the case $\lambda = \lambda'$, we designate as $\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}$ the set of operators $L^{\zeta_\mu\zeta'_{\mu'}}$, for varying $(\zeta_\mu, \zeta'_{\mu'})$ in $A^{\rho\rho'}_{\lambda\lambda'}$. Because of relation 4.9, every $L^{\zeta_\mu\zeta'_{\mu'}}$ is the tensor product of an isomorphism

$$L^{\zeta_{\mu_Q}\zeta'_{\mu'_Q}} : V'_{\lambda',1} \otimes V'_{\lambda',2} \longrightarrow V_{\lambda,1} \otimes V_{\lambda,2}$$

between $\zeta_{\mu_Q} \circ \Phi^q_{\mu_Q\mu'_Q}$ and $\zeta'_{\mu'_Q}$, and of an isomorphism

$$L^{\zeta_{\mu_0}\zeta'_{\mu_0}} : V'_{\lambda',3} \otimes \cdots \otimes V'_{\lambda',m} \longrightarrow V_{\lambda,3} \otimes \cdots \otimes V_{\lambda,m}$$

between ζ_{μ_0} and ζ'_{μ_0}, which is tensor-split.

As before, we define the map

$$\begin{array}{rccc} p: & A^{\rho\rho'}_{\lambda\lambda'} & \longrightarrow & \mathscr{L}^{\rho\rho'}_{\lambda\lambda'} \\ & (\zeta_{\lambda_0}, \zeta'_{\lambda'_0}) & \longmapsto & L^{\zeta_\mu\zeta'_{\mu'}} \end{array}$$

The map p is tautologically surjective, we want to characterize its injective quotient. If ζ_{λ_0} and $\bar{\zeta}_{\lambda_0}$ belong to $\mathscr{F}_{S_0}(\rho_\lambda)$, then they both represent ρ_λ. Fixed arbitrary orientations on the edges of λ, there exist transition constants $(\alpha_j)_j$ such that $\zeta_{\lambda_0} \stackrel{\alpha_j}{\to} \bar{\zeta}_{\lambda_0}$. If $\zeta_\mu = \zeta_{\mu_Q} \otimes \zeta_3 \otimes \cdots \otimes \zeta_m$ and $\bar{\zeta}_{\mu_Q} \otimes \bar{\zeta}_3 \otimes \cdots \otimes \bar{\zeta}_m$ are the induced representations on $\mathcal{T}^q_\mu$, then, for every λ_j with $j \neq i$, the following hold

- if λ_j is on the boundary of S, then the two representations must coincide on the only variable in $\mathcal{T}^q_\mu$ corresponding to λ_j;
- if λ_j is internal and it is side of two triangles T_{l_j} and T_{r_j}, with $l_j, r_j \geq 3$, on the left and on the right respectively of λ_j, then

$$\bar{\zeta}_{l_j}(X^{(l_j)}_{a_j}) = \alpha_j\, \zeta_{l_j}(X^{(l_j)}_{a_j})$$
$$\bar{\zeta}_{r_j}(X^{(r_j)}_{b_j}) = \alpha_j^{-1}\, \zeta_{r_j}(X^{(r_j)}_{b_j})$$

  where a_j and b_j are the indices of the sides in T_{l_j} and T_{r_j}, respectively, identified to λ_j in S (analogously if $T_{l_j} = T_{r_j}$);
- if λ_j is internal and it is side of a triangle T_{k_j} and of the square Q, then

$$\bar{\zeta}_{\mu_Q}(X^{(Q)}_{a_j}) = \alpha_j^{\varepsilon(j,Q)}\, \zeta_{\mu_Q}(X^{(Q)}_{a_j})$$
$$\bar{\zeta}_{k_j}(X^{(k_j)}_{b_j}) = \alpha_j^{\varepsilon(j,k_j)}\, \zeta_{k_j}(X^{(k_j)}_{b_j})$$

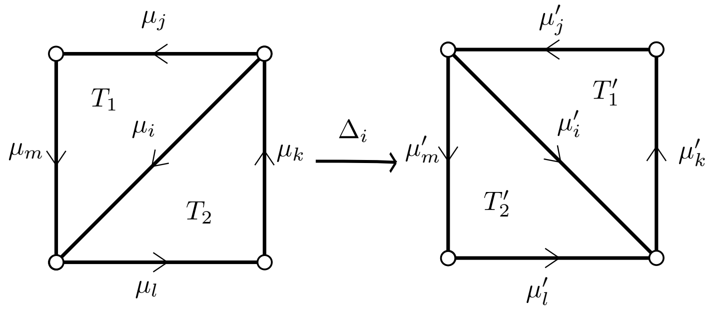

Figure 4.1: The ideal triangulations $\mu_Q, \mu'_Q \in \Lambda(Q)$

where a_j and b_j are the indices of the sides in Q and T_{k_j}, respectively, identified to λ_j in S, $\varepsilon(j, Q)$ is equal to $+1$ if the orientation of λ_j coincides with the boundary orientation of Q, -1 otherwise, and $\varepsilon(j, k_j)$ is equal to $+1$ if the orientation of λ_j coincides with the boundary orientation of T_{k_j}, -1 otherwise;

and analogously in the case in which λ_j has on both sides the square Q. Observe that the constant α_i does not appear in the discussion because we are considering the equivalence classes ζ_μ, $\bar{\zeta}_\mu$ of local representation of $\mathcal{T}^q_\mu$, instead of the representations ζ_{λ_0} and $\bar{\zeta}'_{\lambda'_0}$. We will say that the constants α_j, for $j \neq i$, are the *transition constants* from ζ_μ to $\zeta'_{\mu'}$, which can be thought as supported on $\lambda \setminus \lambda_i$. The same can be noted for a couple of $\zeta'_{\lambda'_0}$, $\bar{\zeta}'_{\lambda_0}$ in $\mathscr{F}_{S_0}(\rho'_{\lambda'})$, providing a collection of transition constants β_j from $\zeta'_{\mu'}$ to $\bar{\zeta}'_{\mu'}$. The two collections (α_j) and (β_j) can be compared in a natural way because there is a canonical correspondence between $\lambda \setminus \lambda_i$ and $\lambda' \setminus \lambda'_i$.

Given two couples $(\zeta_{\lambda_0}, \zeta'_{\lambda'_0}), (\bar{\zeta}_{\lambda_0}, \bar{\zeta}'_{\lambda'_0}) \in A^{\rho\rho'}_{\lambda\lambda'}$, they are in $\approx$-relation if the transition constants from ζ_μ to $\bar{\zeta}_\mu$ are the same of those from $\zeta'_{\mu'}$ to $\bar{\zeta}'_{\mu'}$. Observe that the relation $\approx$ can be expressed in terms of the transition constants $\zeta_{\lambda_0} \overset{\alpha_j}{\to} \bar{\zeta}_{\lambda_0}$ and $\zeta'_{\lambda'_0} \overset{\beta_j}{\to} \bar{\zeta}'_{\lambda'_0}$ by asking $\alpha_j = \beta_j$ for every $j \neq i$, and by not requiring any restriction on α_i and β_i.

We want to show that, if $(\zeta_{\lambda_0}, \zeta'_{\lambda'_0}) \approx (\bar{\zeta}_{\lambda_0}, \bar{\zeta}'_{\lambda'_0})$, then $L^{\zeta_\mu \zeta'_{\mu'}} \doteq L^{\bar{\zeta}_\mu \bar{\zeta}'_{\mu'}}$. It is sufficient to prove that the following hold

$$L^{\zeta_{\mu_Q} \zeta'_{\mu'_Q}} \doteq L^{\bar{\zeta}_{\mu_Q} \bar{\zeta}'_{\mu_Q}}$$
$$L^{\zeta_{\mu_0} \zeta'_{\mu_0}} \doteq L^{\bar{\zeta}_{\mu_0} \bar{\zeta}'_{\mu_0}}$$

The second equality can be obtained with exactly the same observations of the case $\lambda = \lambda'$ done above. Therefore, we will concentrate only on the first one, for which we must pay a little more attention because we have to manage the composition $\zeta_{\mu_Q} \circ \Phi^q_{\mu_Q \mu'_Q}$.

Assume that the edges of the square are labelled as in Figure 4.1 and, in order to simplify the notation, that they are oriented counter-clockwise with respect to the orientation of Q. Then there exist $\alpha_h \in \mathbb{C}^*$ for $h \in \{j, k, l, m\}$ such that

$$\zeta_{\mu_Q}(X_h^{(Q)}) = \alpha_h \, \bar{\zeta}_{\mu_Q}(X_h^{(Q)})$$

where we are denoting by $X_h^{(Q)}$ the element of the Chekhov-Fock algebra $\mathcal{T}^q_{\mu_Q}$ associated with the edge μ_h. Using the fact that $\zeta_{\mu_Q}(X_i^{(Q)}) = \bar{\zeta}_{\mu_Q}(X_i^{(Q)})$ (this equality holds because both ζ_μ and $\bar{\zeta}_\mu$ represent ρ_λ and the edge μ_i corresponding to λ_i is already fused in μ) and the explicit formulas for $\Phi^q_{\mu_Q \mu'_Q}$, we obtain that

$$\zeta_{\mu_Q}(X_h^{(Q)}) = \alpha_h \, \bar{\zeta}_{\mu_Q}(X_h^{(Q)}) \Leftrightarrow (\zeta_{\mu_Q} \circ \Phi^q_{\mu_Q \mu'_Q})(Y_h^{(Q)}) = \alpha_h \, (\bar{\zeta}_{\mu_Q} \circ \Phi^q_{\mu_Q \mu'_Q})(Y_h^{(Q)}) \tag{4.10}$$

Now the same argument of the previous case can be applied in order to conclude the desired equality, the only difference is that in this case one of the representations is on the square Q instead of a triangle.

The map p induces on the quotient

$$\mathscr{A}^{\rho\rho'}_{\lambda\lambda'} := A^{\rho\rho'}_{\lambda\lambda'} / \approx$$

an application $\widetilde{p}$. With the same argument done in the case $\mathscr{A}^{\rho\rho'}_{\lambda\lambda}$ and using relation 4.10, we can prove that $\widetilde{p}$ is injective, and so bijective, since the surjectivity is obvious. Moreover, we can describe an action of $H_1(S;\mathbb{Z}_N)$ on $\mathscr{A}^{\rho\rho'}_{\lambda\lambda'}$ when λ and λ' differ by a diagonal exchange along λ_i, by defining

$$\begin{array}{rccc} \psi^{\rho\rho'}_{\lambda\lambda'}: & H_1(S;\mathbb{Z}_N) \times \mathscr{A}^{\rho\rho'}_{\lambda\lambda'} & \longrightarrow & \mathscr{A}^{\rho\rho'}_{\lambda\lambda'} \\ & (c, [\zeta_{\lambda_0}, \zeta'_{\lambda'_0}]) & \longmapsto & [\zeta_{\lambda_0}, c \cdot \zeta'_{\lambda'_0}] \end{array}$$

where $c \cdot \zeta'_{\lambda'_0}$ is the action of c on $\zeta'_{\lambda'_0}$ in $\mathscr{F}_{S'_0}(\rho'_{\lambda'})$ as in Proposition 4.1.5.

Theorem 4.2.4. *The action of $H_1(S;\mathbb{Z}_N)$ on $\mathscr{A}^{\rho\rho'}_{\lambda\lambda'}$, and equivalently on $\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}$, is well defined, transitive and free. Moreover, for every $[\zeta_{\lambda_0}, \zeta'_{\lambda'_0}] \in \mathscr{A}^{\rho\rho'}_{\lambda\lambda'}$ and for every $c \in H_1(S;\mathbb{Z}_N)$ we have*

$$c \cdot [\zeta_{\lambda_0}, \zeta'_{\lambda'_0}] = [(-c) \cdot \zeta_{\lambda_0}, \zeta'_{\lambda'_0}] \tag{4.11}$$

Proof. The proof will be very similar to the one of Theorem 4.2.1. Take two couples $(\zeta_{\lambda_0}, \zeta'_{\lambda'_0}), (\bar{\zeta}_{\lambda_0}, \bar{\zeta}'_{\lambda'_0})$ in $A^{\rho\rho'}_{\lambda\lambda'}$ that are $\approx$-equivalent. Then

$$\zeta_{\lambda_0} \xrightarrow{\alpha_j} \bar{\zeta}_{\lambda_0}$$
$$\zeta'_{\lambda'_0} \xrightarrow{\beta_j} \bar{\zeta}'_{\lambda'_0}$$

with $\alpha_j = \beta_j$ for every $j \neq i$. By definition of the action $\psi^{\rho\rho'}_{\lambda\lambda'}$ we have

$$\zeta'_{\lambda'_0} \xrightarrow{q^{2c_j}} c \cdot \zeta'_{\lambda'_0}$$
$$\bar{\zeta}'_{\lambda'_0} \xrightarrow{q^{2c_j}} c \cdot \bar{\zeta}'_{\lambda'_0}$$

Then

$$c \cdot \zeta'_{\lambda'_0} \xrightarrow{q^{-2c_j}} \zeta'_{\lambda'_0} \xrightarrow{\beta_j} \bar{\zeta}'_{\lambda'_0} \xrightarrow{q^{2c_j}} c \cdot \bar{\zeta}'_{\lambda'_0}$$

We conclude that $\zeta_{\lambda_0} \xrightarrow{\alpha_j} \bar{\zeta}_{\lambda_0}$ and $c \cdot \zeta'_{\lambda'_0} \xrightarrow{\beta_j} c \cdot \bar{\zeta}'_{\lambda'_0}$, with $\alpha_j = \beta_j$ for every $j \neq i$, hence $(\zeta_{\lambda_0}, c \cdot \zeta'_{\lambda'_0}) \approx (\bar{\zeta}_{\lambda_0}, c \cdot \bar{\zeta}'_{\lambda'_0})$, which proves the good definition of the action.

In order to prove the transitivity, fix two elements $[\zeta_{\lambda_0}, \zeta'_{\lambda'_0}], [\bar{\zeta}_{\lambda_0}, \bar{\zeta}'_{\lambda'_0}]$ in $\mathscr{A}^{\rho\rho'}_{\lambda\lambda'}$ and two respective representatives $(\zeta_{\lambda_0}, \zeta'_{\lambda'_0}), (\bar{\zeta}_{\lambda_0}, \bar{\zeta}'_{\lambda'_0})$. Denote by $(\alpha_j)_j$ the transition constants from ζ_{λ_0} to $\bar{\zeta}_{\lambda_0}$. The element $(\zeta_{\lambda_0}, \zeta'_{\lambda'_0})$ belongs to $A^{\rho\rho'}_{\lambda\lambda'}$, hence the induced local representations $\zeta_\mu \circ \Phi^q_{\mu\mu'}$ and $\zeta'_{\mu'}$ are isomorphic via an isomorphism

$$L^{\zeta_\mu \zeta'_{\mu'}} = L^{\zeta_{\mu_Q} \zeta'_{\mu'_Q}} \otimes L^{\zeta_{\mu_0} \zeta'_{\mu'_0}} : V'_{\lambda'} \longrightarrow V_\lambda$$

We can construct a representation $\widetilde{\zeta}'_{\lambda'_0}$ such that $(\bar{\zeta}_{\lambda_0}, \widetilde{\zeta}'_{\lambda'_0})$ belongs to $A^{\rho\rho'}_{\lambda\lambda'}$ and $(\zeta_{\lambda_0}, \zeta'_{\lambda'_0}) \approx (\bar{\zeta}_{\lambda_0}, \widetilde{\zeta}'_{\lambda'_0})$, simply by defining $\widetilde{\zeta}'_{\lambda'_0}$ as the representation verifying

$$\zeta'_{\lambda'_0} \xrightarrow{\alpha_j} \widetilde{\zeta}'_{\lambda'_0}$$

where the α_j are the transition constants from ζ_{λ_0} to $\bar{\zeta}_{\lambda_0}$ (it is not important which is the transition constant in the edge λ'_i). Because ζ_μ and $\zeta'_{\mu'}$ verify

$$L^{\zeta_\mu \zeta'_{\mu'}} \circ \zeta'_{\mu'}(X') \circ (L^{\zeta_\mu \zeta'_{\mu'}})^{-1} = (\zeta_\mu \circ \Phi^q_{\mu\mu'})(X')$$

for every $X' \in \mathcal{T}^q_{\mu'}$ and because $\zeta_{\lambda_0} \overset{\alpha_j}{\to} \bar{\zeta}_{\lambda_0}$, then we have also

$$L^{\zeta_\mu \zeta'_{\mu'}} \circ \widetilde{\zeta}'_{\mu'}(X') \circ (L^{\zeta_\mu \zeta'_{\mu'}})^{-1} = (\bar{\zeta}_\mu \circ \Phi^q_{\mu\mu'})(X')$$

The proof can be done using relation 4.10 and the irreducibility of the considered representations. This justifies the fact that $(\bar{\zeta}_{\lambda_0}, \widetilde{\zeta}'_{\lambda'_0})$ belongs to $A^{\rho\rho'}_{\lambda\lambda'}$ and $(\zeta_{\lambda_0}, \zeta'_{\lambda'_0}) \approx (\bar{\zeta}_{\lambda_0}, \widetilde{\zeta}'_{\lambda'_0})$. Both the representations $\widetilde{\zeta}'_{\mu'}$ and $\bar{\zeta}'_{\mu'}$ are isomorphic to $\bar{\zeta}_\mu \circ \Phi^q_{\mu\mu'}$, so they are isomorphic to each other. Possibly by changing $\widetilde{\zeta}'_{\lambda'_0}$ in its class of local representation of $\mathcal{T}^q_\mu$, we can assume that $\bar{\zeta}'_{\lambda'_0}$ and $\widetilde{\zeta}'_{\lambda'_0}$ are isomorphic. Indeed, change $\widetilde{\zeta}'_{\lambda'_0}$ in its class of local representation of $\mathcal{T}^q_\mu$ is equivalent to take a representation $\widetilde{\eta}'_{\lambda'_0}$ defined by $\widetilde{\zeta}'_{\lambda'_0} \overset{\gamma_j}{\to} \widetilde{\eta}'_{\lambda'_0}$ with $\gamma_j = 1$ for every $j \neq i$. We assert that γ_i can be chosen so that $\widetilde{\eta}'_{\lambda'_0}$ is isomorphic to $\bar{\zeta}'_{\lambda'_0}$. We know that the central loads $\widetilde{h}$ and $\bar{h}$, associated with Q, of $\widetilde{\zeta}'_{\mu'}$ and $\bar{\zeta}'_{\mu'}$ are the product of the central loads of $\widetilde{\zeta}'_{\lambda'_0}$ and $\bar{\zeta}'_{\lambda'_0}$ on the triangles composing Q. Moreover, being $\widetilde{\zeta}'_{\mu'}$ and $\bar{\zeta}'_{\mu'}$ isomorphic, we have $\tilde{h} = \bar{h}$. Denoting by $\widetilde{h}^1$ and $\widetilde{h}^2$ the central loads of $\widetilde{\zeta}'_{\lambda'_0}$ on the triangles in Q and by $\bar{h}^1$, $\bar{h}^2$ the ones of $\bar{\zeta}'_{\lambda'_0}$, we observe that the relative central loads of $\widetilde{\eta}'_{\lambda'_0}$ change like $\gamma_i \widetilde{h}^1$ and $\gamma_i^{-1} \widetilde{h}^2$. Then there exists a unique γ_i such that $\bar{h}^1 = \gamma_i \widetilde{h}^1$. Doing this choice, the following holds

$$\bar{h}^2 = \frac{\bar{h}}{\bar{h}^1} = \frac{\widetilde{h}}{\gamma_i \widetilde{h}^1} = \gamma_i^{-1} \widetilde{h}^2$$

Now the central loads on the triangles are equal. It remains to prove that the invariants on the couple of edges in λ'_0 corresponding to λ_i are equal, but this is clear because we already know that, for both the triangles in Q, their representations have the same central loads and two invariants of edges coinciding. Recalling that $h^N = x_1 x_2 x_3$ the assertion follows.

Hence we can assume that $\bar{\zeta}'_{\lambda'_0}$ and $\widetilde{\zeta}'_{\lambda'_0}$ are isomorphic. Via Proposition 4.1.5, there exists a $c \in H_1(S;\mathbb{Z}_N)$ such that $c \cdot \widetilde{\zeta}'_{\lambda'_0} = \bar{\zeta}'_{\lambda'_0}$, hence

$$c \cdot [\zeta_{\lambda_0}, \zeta'_{\lambda'_0}] = c \cdot [\bar{\zeta}_{\lambda_0}, \widetilde{\zeta}'_{\lambda'_0}] = [\bar{\zeta}_{\lambda_0}, c \cdot \widetilde{\zeta}'_{\lambda'_0}] = [\bar{\zeta}_{\lambda_0}, \bar{\zeta}'_{\lambda'_0}]$$

and so the transitivity is proved.

Now suppose that there exist a $c \in H_1(S;\mathbb{Z}_N)$ and an element $[\zeta_{\lambda_0}, \zeta'_{\lambda'_0}] \in \mathscr{A}^{\rho\rho'}_{\lambda\lambda'}$ such that $[\zeta_{\lambda_0}, \zeta'_{\lambda'_0}] = c \cdot [\zeta_{\lambda_0}, \zeta'_{\lambda'_0}]$. This means that, passing on representatives, the couples $(\zeta_{\lambda_0}, \zeta'_{\lambda'_0})$ and $(\zeta_{\lambda_0}, c \cdot \zeta'_{\lambda'_0})$ are $\approx$-equivalent. Because the first terms of the couples are exactly the same, they are in particular related by transition constants all equal to 1, and the same must hold for $\zeta'_{\lambda'_0}$ and $c \cdot \zeta'_{\lambda'_0}$, except possibly along the edge λ_i, where there are not restrictions. But $(\lambda'_i)^*$ has distinct vertices, so there are not elements of $H_1(S;\mathbb{Z}_N)$ that act only in the edge λ_i, hence c must be trivial. Therefore the action is free.

Finally, the relation 4.11 can be proved exactly in the same way of relation 4.7 in Theorem 4.2.1. □

An explicit calculation

The previous discussion shows us that the elements in $\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}$ are tensor products of a tensor split isomorphism $L^{\zeta_{\mu_0}\zeta'_{\mu_0}}$ and an isomorphism $L^{\zeta_{\mu_Q}\zeta'_{\mu'_Q}}$ between two irreducible representations $\zeta_{\mu_Q} \circ \Phi^q_{\mu_Q \mu'_{Q'}}$ and $\zeta_{\mu'_{Q'}}$ on the square Q. In what follows we want to give an explicit description of this linear isomorphism $L^{\zeta_{\mu_Q}\zeta'_{\mu'_Q}}$.

Redefine the notations: let Q be an ideal square and let λ, λ' be its ideal triangulations, with edges labelled as in Figure 4.2. Given $\rho = \{\rho_\lambda, \rho_{\lambda'}\}$ a local representation of $\mathcal{T}^q_Q$, we know that there exists a linear isomorphism $L^{\rho\rho}_{\lambda\lambda'} \colon V_{\lambda'} \to V_\lambda$, unique up to scalar multiplication, such that

$$L^{\rho\rho}_{\lambda\lambda'} \circ \rho_{\lambda'}(X') \circ (L^{\rho\rho}_{\lambda\lambda'})^{-1} = (\rho_\lambda \circ \Phi^q_{\lambda\lambda'})(X')$$

Let us describe in a explicit way this linear isomorphism. We firstly reduce to the standard situation, which means that ρ_λ and $\rho_{\lambda'}$ are represented by the tensor product of standard irreducible representations of the triangle algebras (here standard means that the representation sends each generator X_s of $\mathcal{T}^q_T$ in a $N \times N$ matrix, which is a multiple of the B_i described in Proposition 2.1.4). In order to identify a standard representation of the triangle algebra, we need the following data: a clockwise indexing of the edges of each triangle and the choice of N-th roots of the invariants on the edges of the square. We will order the edges of each triangle as described in Figure 4.2 by red numbers (the square on the left represents the ideal triangulation λ and the indexing on its triangles T_1 and T_2, the square on the right represents the ideal triangulation λ' and the indexing on its triangles T'_1 and T'_2). The representations $\rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(\mathbb{C}^N \otimes \mathbb{C}^N)$ and

$\rho_{\lambda'}\colon \mathcal{T}^q_{\lambda'} \to \mathrm{End}(\mathbb{C}^N \otimes \mathbb{C}^N)$ have the following form

$$\begin{aligned}
\rho_\lambda(X_i) &= y_i\, B_2 \otimes B_1\\
\rho_\lambda(X_j) &= y_j\, B_1 \otimes I\\
\rho_\lambda(X_k) &= y_k\, I \otimes B_2\\
\rho_\lambda(X_l) &= y_l\, I \otimes B_3\\
\rho_\lambda(X_m) &= y_m\, B_3 \otimes I
\end{aligned}$$

$$\begin{aligned}
\rho_{\lambda'}(X'_i) &= v_i\, B_3 \otimes B_2\\
\rho_{\lambda'}(X'_j) &= v_j\, B_1 \otimes I\\
\rho_{\lambda'}(X'_k) &= v_k\, B_2 \otimes I\\
\rho_{\lambda'}(X'_l) &= v_l\, I \otimes B_3\\
\rho_{\lambda'}(X'_m) &= v_m\, I \otimes B_1
\end{aligned}$$

where the numbers y_i, y_j, y_k, y_l, y_m are N-th roots of x_i, x_j, x_k, x_l, x_m and the numbers v_i, v_j, v_k, v_l, v_m are N-th roots of $x_i^{-1}, (1+x_i)x_j, (1+x_i^{-1})^{-1}x_k, (1+x_i)x_l, (1+x_i^{-1})^{-1}x_m$. Moreover, the product of the y_s and the product of the v_s are equal and coincide with the central load of the representations ρ_λ and $\rho_{\lambda'}$. Denote by z_i the number in $\mathbb{Z}_N$ such that $v_i = y_i^{-1} q^{2z_i}$.

Because of the expression of $\Phi^q_{\lambda\lambda'}$, the representation $\rho_\lambda \circ \Phi^q_{\lambda\lambda'}$ has the following behaviour

$$\begin{aligned}
(\rho_\lambda \circ \Phi^q_{\lambda\lambda'})(X'_i) &= y_i^{-1}\, B_2^{-1} \otimes B_1^{-1}\\
(\rho_\lambda \circ \Phi^q_{\lambda\lambda'})(X'_j) &= y_j\, (I \otimes I + q y_i\, B_2 \otimes B_1) B_1 \otimes I\\
(\rho_\lambda \circ \Phi^q_{\lambda\lambda'})(X'_k) &= y_k\, (I \otimes I + q y_i^{-1}\, B_2^{-1} \otimes B_1^{-1})^{-1} I \otimes B_2\\
(\rho_\lambda \circ \Phi^q_{\lambda\lambda'})(X'_l) &= y_l\, (I \otimes I + q y_i\, B_2 \otimes B_1) I \otimes B_3\\
(\rho_\lambda \circ \Phi^q_{\lambda\lambda'})(X'_m) &= y_m\, (I \otimes I + q y_i^{-1}\, B_2^{-1} \otimes B_1^{-1})^{-1} B_3 \otimes I
\end{aligned}$$

Now we define, for $a \in \mathbb{N}$, the following function

$$f(a) := \begin{cases} \left(\frac{y_j}{v_j}\right)^a \prod_{u=1}^{a} (1 + y_i q^{1-2(u+z_i)}) & \text{if } a \neq 0\\ 1 & \text{if } a = 0 \end{cases}$$

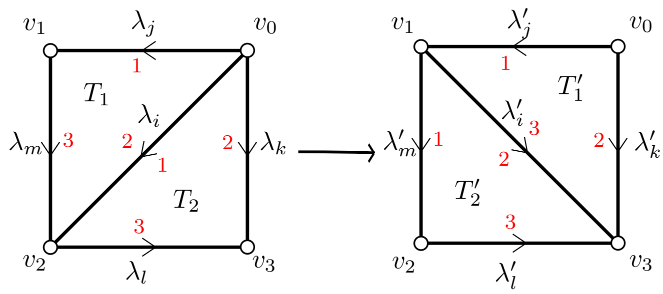

Figure 4.2: Useful notations

Observe that, because q is a primitive N-th root of $(-1)^{N+1}$, we have

$$\prod_{u=1}^{N}(1+y_i q^{1-2(u+z_i)}) = 1+y_i^N = 1+x_i$$

On the other hand, $\frac{y_j}{v_j}$ is an N-th root of $(1+x_i)^{-1}$, indeed

$$\left(\frac{y_j}{v_j}\right)^N = \frac{x_j}{(1+x_i)x_j} = (1+x_i)^{-1}$$

These facts imply immediately that, for every $a \in \mathbb{N}$, the following relation holds

$$f(a+N) = f(a)$$

Moreover, we define the following polynomial

$$p(x) := \sum_{d=0}^{N-1}\left(\frac{y_l y_k y_i}{v_l v_k}\right)^d x^d$$

Now we have all the tools required for the description of the isomorphism $L^{\rho\rho}_{\lambda\lambda'}$, up to scalar multiplication.

Proposition 4.2.5. *The map* $L^{\rho\rho}_{\lambda\lambda'} \colon \mathbb{C}^N \otimes \mathbb{C}^N \to \mathbb{C}^N \otimes \mathbb{C}^N$ *has components* $(L^{\rho\rho}_{\lambda\lambda'})^{b,c}_{s,t}$ *equal to*

$$q^{-s^2+2z_i(b-c-z_i)+2bc}\left(\frac{y_j y_k y_i}{v_j v_k}\right)^{c+z_i} p(q^{2(s+t-c-z_i)})\left(\sum_{a=0}^{N-1} q^{2a(b-s)} f(a)\right) \tag{4.12}$$

for varying $s,t,b,c \in \{0,\dots,N-1\}$ *and the indices of* $e_{-r,r-(s+t+z_i)}$ *are thought as elements of* $\mathbb{Z}_N = \{[0],\dots,[N-1]\}$.

Recall that $L^{\rho\rho}_{\lambda\lambda'}$ is defined up to scalar multiplication, so by multiplying the relations above, for varying k and l, by a common scalar, we obtain another linear isomorphism verifying the property

$$L^{\rho\rho}_{\lambda\lambda'} \circ \rho_{\lambda'}(X') \circ (L^{\rho\rho}_{\lambda\lambda'})^{-1} = (\rho_\lambda \circ \Phi^q_{\lambda\lambda'})(X') \qquad \forall X' \in \mathcal{T}^q_{\lambda'}$$

Proof of Proposition 4.2.5. In what follows we will describe the strategy conducing to the relation 4.12. Firstly, we choose an indexing on the edges of the triangles different from the one of Figure 4.2, but more appropriate in order to do an explicit calculus. In particular, we have chosen the indexing in Figure 4.3. Denote by $\bar\rho_\lambda$ and $\bar\rho_{\lambda'}$ the standard representations determined by this choice of indexing and by the N-th roots y_s and v_s respectively, the same that we have chosen above. In particular, in this case we have the following relations

$$\begin{aligned}
\bar\rho_\lambda(X_i) &= y_i\, B_1 \otimes B_1\\
\bar\rho_\lambda(X_j) &= y_j\, B_3 \otimes I\\
\bar\rho_\lambda(X_k) &= y_k\, I \otimes B_2\\
\bar\rho_\lambda(X_l) &= y_j\, I \otimes B_3\\
\bar\rho_\lambda(X_m) &= y_m\, B_2 \otimes I
\end{aligned}$$

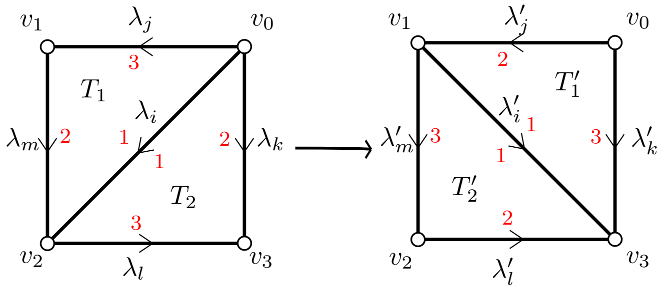

Figure 4.3: Another indexing

$$\begin{aligned}
\bar\rho_{\lambda'}(X'_i) &= v_i\, B_1 \otimes B_1 \\
\bar\rho_{\lambda'}(X'_j) &= v_j\, B_2 \otimes I \\
\bar\rho_{\lambda'}(X'_k) &= v_k\, B_3 \otimes I \\
\bar\rho_{\lambda'}(X'_l) &= v_l\, I \otimes B_2 \\
\bar\rho_{\lambda'}(X'_m) &= v_m\, I \otimes B_3
\end{aligned}$$

As previously done, we will denote by $z_i \in \mathbb{Z}_N$ the number verifying $v_i = q^{2z_i} y_i^{-1}$. The representation $\bar\rho_\lambda \circ \Phi^q_{\lambda\lambda'}$ has the following behaviour

$$\begin{aligned}
(\bar\rho_\lambda \circ \Phi^q_{\lambda\lambda'})(X'_i) &= y_i^{-1}\, B_1^{-1} \otimes B_1^{-1} \\
(\bar\rho_\lambda \circ \Phi^q_{\lambda\lambda'})(X'_j) &= y_j\, (I \otimes I + q y_i\, B_1 \otimes B_1) B_3 \otimes I \\
(\bar\rho_\lambda \circ \Phi^q_{\lambda\lambda'})(X'_k) &= y_k\, (I \otimes I + q y_i^{-1}\, B_1^{-1} \otimes B_1^{-1})^{-1} I \otimes B_2 \\
(\bar\rho_\lambda \circ \Phi^q_{\lambda\lambda'})(X'_l) &= y_l\, (I \otimes I + q y_i\, B_1 \otimes B_1) I \otimes B_3 \\
(\bar\rho_\lambda \circ \Phi^q_{\lambda\lambda'})(X'_m) &= y_m\, (I \otimes I + q y_i^{-1}\, B_1^{-1} \otimes B_1^{-1})^{-1} B_2 \otimes I
\end{aligned}$$

The main benefits of this choice are

- the matrices $I \otimes I + q y_i\, B_1 \otimes B_1$ and $(I \otimes I + q y_i^{-1}\, B_1^{-1} \otimes B_1^{-1})^{-1}$ are diagonal;

- it is very simple to find the family of isomorphisms ψ verifying

$$\psi^{-1} \circ (\bar\rho_\lambda \circ \Phi^q_{\lambda\lambda'})(X'_i) \circ \psi = \bar\rho_{\lambda'}(X'_i)$$

  Indeed, they both are diagonal matrices, so it is sufficient to ask that ψ carries the α-eigenspace of $\bar\rho_{\lambda'}(X_i)$ in the α-eigenspace of $(\bar\rho_\lambda \circ \Phi^q_{\lambda\lambda'})(X'_i)$ for every α. In particular, such a ψ has the following behaviour

$$e_{s,t} \longmapsto \sum_{r=0}^{N-1} a_{r,s,t}\, e_{-r,r-(s+t+z_i)}$$

  where the indices of $e_{-r,r-(s+t+z_i)}$ must be thought as elements of $\mathbb{Z}_N = \{[0], \dots, [N-1]\}$.

Now we require that ψ carries the whole representation $\bar\rho_\lambda \circ \Phi^q_{\lambda\lambda'}$ in $\bar\rho_{\lambda'}$ by conjugation and we find equations in the constants $a_{r,s,t}$, which determine these elements $a_{r,s,t}$ up to a common multiplicative scalar. In particular, we obtain the following equations

$$\begin{cases} y_j(1+y_i q^{-2(s+t+z_i+1)+1})q^{1+2(r-1)}\, a_{r-1,s,t} = v_j\, a_{r,s+1,t} \\ y_k(1+y_i^{-1} q^{2(s+t+z_i-1)+1})^{-1}\, a_{r,s,t} = v_k q^{1-2s}\, a_{r,s-1,t} \\ y_l(1+y_i q^{-2(s+t+z_i+1)+1})q^{1+2(s+t+z_i-r)}\, a_{r,s,t} = v_l\, a_{r,s,t+1} \\ y_m(1+y_i^{-1} q^{2(s+t+z_i-1)+1})^{-1}\, a_{r+1,s,t} = v_m\, a_{r,s,t-1} \end{cases}$$

conducing to the following expression

$$a_{r,s,t} = \left(\frac{v_k}{y_k y_i}\right)^s \left(\frac{y_j y_k y_i}{v_j v_k}\right)^r \left(\frac{y_l}{v_l}\right)^t q^{(t-r)^2+2(s+t-r)z_i+2st} \prod_{u=1}^{s+t} (1+y_i q^{1-2(u+z_i)})\, a_{0,0,0}$$

Now, defining $\xi\colon \mathbb{C}^N \to \mathbb{C}^N$ the isomorphism

$$\begin{array}{cccc} \xi: & \mathbb{C}^N & \longrightarrow & \mathbb{C}^N \\ & e_k & \longmapsto & \frac{1}{\sqrt{N}} \sum_{h=0}^{N-1} q^{2hk+h^2}\, e_h \end{array}$$

we observe that the following relations hold

$$(\xi \otimes \xi^{-1}) \circ \bar{\rho}_{\lambda'} \circ (\xi^{-1} \otimes \xi) = \rho_{\lambda'}$$
$$(\xi^{-1} \otimes I) \circ \bar{\rho}_{\lambda} \circ (\xi \otimes I) = \rho_{\lambda}$$

The point is that ξ is the linear isomorphism that change a standard triangular representation in another standard triangular representation simply by rotate the indexing. More precisely, if X_1, X_2, X_3 are generators of $\mathcal{T}_T^{\mathrm{II}}$, corresponding to edges $\lambda_1, \lambda_2, \lambda_3$, ordered clockwise, and if η is a representation of $\mathcal{T}_T^q$ defined by

$$\eta(X_i) = u_i\, B_i$$

for every $i \in \{1,2,3\}$, then $\xi^{-1} \circ \eta(\cdot) \circ \xi$ verifies

$$\xi^{-1} \circ \eta(X_i) \circ \xi = u_i\, B_{i+1}$$

where the indices are in $\mathbb{Z}_3 = \{[1],[2],[3]\}$. So we obtain that the composition

$$(\xi^{-1} \otimes I) \circ \psi \circ (\xi^{-1} \otimes \xi)$$

verifies the property defining $L_{\lambda\lambda'}^{\rho\rho}$. The relation 4.12 can be found by develop the composition, where we have chosen $a_{0,0,0} = N\sqrt{N}$. □

Remark 4.2.6. In the notations used in Proposition 4.2.5, we define

$$w_0 := \frac{y_j}{v_j}$$
$$w_1 := \frac{y_k}{v_k}$$
$$w_2 := -y_i$$
$$\zeta := q^2$$

Moreover, we assume that the following hold

$$\begin{cases} w_0 = \frac{y_j}{v_j} = \frac{y_l}{v_l} \\ w_1 = \frac{y_k}{v_k} = \frac{y_m}{v_m} \\ w_2 = -y_i = -v_i^{-1} q^{-2} \\ w_0 w_1 w_2 = -q^{-1} \end{cases}$$

It is not difficult to see that these conditions are coherent with the relations verified by the y_s and v_s by virtue of the expression of $\Phi^1_{\lambda\lambda'}$. In this situation, the formula 4.12 for the numbers $(L^{\rho\rho}_{\lambda\lambda'})^{b,c}_{s,t}$ (with ρ_λ and $\rho_{\lambda'}$ defined by the indexing in Figure 4.2 and the N-roots y_s and v_s) becomes the following

$$N\zeta^{-(m+1)s^2-b+(m+1)c+bc-(m+1)}\delta(2s+2t-2c+1)\left(\sum_{a=1}^{N}\zeta^{a(b-s)}w_0^a\prod_{u=1}^{a}(1-w_2\zeta^{3(m+1)-u})\right)$$

where $\delta(k)$ is equal to 1 if $k \equiv 0(N)$ and is equal to 0 otherwise and $N = 2m+1$.

4.3 The elementary properties

In this Section we will focus on the properties verified by the objects $(\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}, \psi^{\rho\rho'}_{\lambda\lambda'})$ that we have just defined in the previous Section. In particular, we will investigate on the relations that will conduce to the Fusion and Composition properties in the general case. The first part is dedicated to the "baby" version of the Fusion property. The second Subsection will request some efforts and will conduce us to the proof of a technical lemma that will be useful in the last Subsection, where we will explicit the elementary version of the Composition property.

4.3.1 Elementary Fusion property

If S is obtained by fusion from R along some boundary components, there is a natural map of projection $\pi\colon R \to S$. Given $\mu \in \Lambda(R)$ an ideal triangulation and $\lambda \in \Lambda(S)$ the induced ideal triangulation on S, the map π induced an identification of $\Gamma_{R,\mu}$ with a subgraph of $\Gamma_{S,\lambda}$. Moreover, thinking to $\Gamma_{R,\mu}$ as a deformation retract of R, the map $\pi_*\colon H_1(R;\mathbb{Z}_N) \to H_1(S;\mathbb{Z}_N)$ is injective, because the map obtained from the inclusion of $\Gamma_{R,\mu}$ in $\Gamma_{S,\lambda}$ on $H_1(\,\cdot\,;\mathbb{Z}_N)$ is.

Lemma 4.3.1. *Let R be a surface as above and S be obtained by fusion from R. Fix $\eta = \{\eta_\mu\colon \mathcal{T}^q_\mu \to \mathrm{End}(W_\mu)\}_{\mu\in\Lambda(R)}$, $\eta' = \{\eta'_\mu\colon \mathcal{T}^q_\mu \to \mathrm{End}(W'_\mu)\}_{\mu\in\Lambda(R)}$ two isomorphic local representations of $\mathcal{T}^q_R$ and $\rho = \{\rho_\lambda\colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)\}_{\lambda\in\Lambda(S)}$, $\rho' = \{\rho'_\lambda\colon \mathcal{T}^q_\lambda \to \mathrm{End}(V'_\lambda)\}_{\lambda\in\Lambda(S)}$ two isomorphic local representations of $\mathcal{T}^q_S$, with ρ and ρ' obtained by fusion from η and η', respectively. Then, for every $\mu, \mu' \in \Lambda(R)$ that differ by diagonal exchange or a re-indexing, if $\lambda, \lambda' \in \Lambda(S)$ are the corresponding ideal triangulations on S, there exists a natural inclusion $j\colon \mathscr{L}^{\eta\eta'}_{\mu\mu'} \to \mathscr{L}^{\rho\rho'}_{\lambda\lambda'}$ such that, for every $L \in \mathscr{L}^{\eta\eta'}_{\mu\mu'}$, the following holds*

$$j(c \cdot L) = \pi_*(c) \cdot j(L)$$

for every $c \in H_1(R;\mathbb{Z}_N)$.

Proof. We will prove only the case in which λ and λ' differ by a diagonal exchange, the other situation is analogous. Moreover, we will use the sets $\mathscr{A}_{\lambda\lambda'}^{\rho\rho'}$ instead of $\mathscr{L}_{\lambda\lambda'}^{\rho\rho'}$, in order to describe the relations with the action in a more explicit way. On the $\mathscr{L}$-level, the map j will be just the inclusion as subsets of $\mathrm{Hom}(V'_{\lambda'}, V_\lambda)$.

Fix $\mu, \mu' \in \Lambda(R)$ and $\lambda, \lambda' \in \Lambda(S)$ as in the statement, with $\lambda = \Delta_i(\lambda')$ and $\mu = \Delta_i(\mu')$. It is clear that the surfaces S_0 and R_0, obtained by splitting S and R along λ and μ respectively, can be identified and analogously for S'_0 and R'_0, obtained by splitting S and R along λ' and μ'. If η_μ represents ρ_λ, then $\mathscr{F}_{R_0}(\eta_\mu) \subseteq \mathscr{F}_{S_0}(\rho_\lambda)$, by definition. From this fact we deduce an inclusion i of $A_{\mu\mu'}^{\eta\eta'}$ in $A_{\lambda\lambda'}^{\rho\rho'}$. The map j will be the application induced by i from $\mathscr{A}_{\mu\mu'}^{\eta\eta'}$ to $\mathscr{A}_{\lambda\lambda'}^{\rho\rho'}$. We need to prove the good definition of j and the injectivity.

Let $(\zeta, \zeta') \in \mathscr{F}_{R_0}(\eta_\mu) \times \mathscr{F}_{R_0}(\eta'_{\mu'})$ be an element in $A_{\mu\mu'}^{\eta\eta'}$ and denote by $[\zeta, \zeta']_S$ its image in $\mathscr{A}_{\lambda\lambda'}^{\rho\rho'}$. Take $(\bar\zeta, \bar\zeta')$ another representative of $[\zeta, \zeta']_R \in \mathscr{A}_{\mu\mu'}^{\eta\eta'}$, the equivalence class of (ζ, ζ') in $A_{\mu\mu'}^{\eta\eta'}$. Then there exist transition constants α_j and β_j, one for each internal edge of μ, such that $\zeta \overset{\alpha_j}{\to} \bar\zeta$ and $\zeta' \overset{\beta_j}{\to} \bar\zeta'$, with $\alpha_j = \beta_j$ for every $j \neq i$. Note that the representations ζ and $\bar\zeta$ need to coincide on the variables corresponding to the boundary edges of R. In particular this means that the elements ζ and $\bar\zeta$, as representations in $\mathscr{F}_{S_0}(\rho_\lambda)$, have transition constants equal to 1 for every λ_j that is the result of the identification of a couple of boundary components in R, and equal to $\alpha_j \in \mathbb{C}^*$ otherwise. The same must hold for ζ' and $\bar\zeta'$, so in particular $[\zeta, \zeta']_S = [\bar\zeta, \bar\zeta']_S$, which proves the good definition of j.

Now take $(\zeta, \zeta'), (\bar\zeta, \bar\zeta') \in A_{\mu\mu'}^{\eta\eta'}$ and assume that $[\zeta, \zeta']_S = [\bar\zeta, \bar\zeta']_S$. This means that there exist transitions constants α_j and β_j, one for each internal edge of λ, such that $\zeta \overset{\alpha_j}{\to} \bar\zeta$ and $\zeta' \overset{\beta_j}{\to} \bar\zeta'$, with $\alpha_j = \beta_j$ if $j \neq i$. On the other hand, the representations ζ and $\bar\zeta$ belong to $\mathscr{F}_{R_0}(\eta_\mu)$, so they must coincide on the variables corresponding to the boundary edges of R. This implies that $\alpha_j = 1$ for every j such that $\lambda_j^* \notin \Gamma_{R,\mu}$ (we are identifying $\Gamma_{R,\mu}$ with its image in $\Gamma_{S,\lambda}$ under π). In the same way we can see that $\beta_j = 1$ for every j such that $(\lambda'_j)^* \notin \Gamma_{R,\mu'}$, and these observations lead to the equality $[\zeta, \zeta']_R = [\bar\zeta, \bar\zeta']_R$. Hence we have concluded the proof of the injectivity.

Finally observe that, for every $c \in H_1(R; \mathbb{Z}_N)$, we have

$$\begin{aligned} j(c \cdot [\zeta, \zeta']_R) &= j([\zeta, c \cdot \zeta']_R) \\ &= [\zeta, \pi_*(c) \cdot \zeta']_S \\ &= \pi_*(c) \cdot [\zeta, \zeta']_S \\ &= \pi_*(c) \cdot j([\zeta, \zeta']_R) \end{aligned}$$

□

4.3.2 A technical Lemma

In the previous Section we have given a presentation of the elements in $\mathscr{A}_{\lambda\lambda'}^{\rho\rho'}$ in terms of equivalence classes of representations on the surfaces S_0 or S'_0. In this Subsection we are going to prove a Lemma that will give an alternative construction of the sets $\mathscr{A}^{\rho\rho'}_{\lambda\lambda'}$ in terms of equivalence classes of local representations in a intermediate common level R' between S and the surfaces S_0, S'_0.

In other words, we want to represent local representations on S with local representations on another surface R', which can be obtained by fusion from both S_0 and S'_0 and which is a splitting of S along certain edges of an ideal triangulation of S. In addition, we will require that R' is a disjoint union of ideal polygons. The first important example of this situation is the surface R that we have introduced in the Subsection 4.2.3.

In the setting that this technical Lemma will allow us to introduce, the proofs of the Elementary Composition property in the next Section will be simpler and more expressive.

Diagonal exchange

Let $\lambda, \lambda' \in \Lambda(S)$ be two ideal triangulations that differ by a diagonal exchange along λ_i. We have denoted by S_0 and S'_0 the surfaces, obtained by splitting S along λ and λ' respectively, endowed with the triangulations λ_0 and λ'_0. Moreover, we have defined R as the surface obtained by splitting S along all the edges except for λ_i, on which we have the ideal triangulations $\mu = \mu_Q \sqcup \mu_0$ and $\mu' = \mu'_Q \sqcup \mu_0$ induced by λ and λ'.

The triangulations λ_0 and λ'_0 are the result of the splitting of μ and μ' along μ_i and μ'_i, diagonals of the square Q in R. Now we take an intermediate surface R' between R and S, that is a surface obtained by splitting S along certain λ_j, with $j \neq i$, with induced ideal triangulations ν and ν'. Furthermore, we assume that R' is the disjoint union of ideal polygons. We represent the situation of surfaces and triangulations related by fusion in the diagram on the right, where an arrow from A to B means that B is obtained by fusion from A, and on the sides there are the relative triangulations.

$$\begin{tikzcd} \lambda_0\, S_0 \arrow[dr] & & S'_0\, \lambda'_0 \arrow[dl] \\ & \mu\ R\ \mu' \arrow[d] & \\ & \nu\ R'\ \nu' \arrow[d] & \\ & \lambda\ S\ \lambda' & \end{tikzcd}$$

Fixed $\rho = \{\rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)\}_{\lambda\in\Lambda(S)}$, $\rho' = \{\rho'_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V'_\lambda)\}_{\lambda\in\Lambda(S)}$ two local representations of $\mathcal{T}^q_S$, we define the following sets

$$\mathscr{F}_{R'}(\rho_\lambda) := \{\zeta_\nu \in \mathrm{Repr}_{loc}(\mathcal{T}^q_\nu, V_\lambda) \mid \zeta_\nu \textit{ represents } \rho_\lambda\}$$
$$\mathscr{F}_{R'}(\rho'_{\lambda'}) := \{\zeta'_{\nu'} \in \mathrm{Repr}_{loc}(\mathcal{T}^q_{\nu'}, V'_{\lambda'}) \mid \zeta'_{\nu'} \textit{ represents } \rho'_{\lambda'}\}$$

Now we introduce a set $B^{\rho\rho'}_{\lambda\lambda'}$ that will perform the role of $A^{\rho\rho'}_{\lambda\lambda'}$ in the new setting:

$$B^{\rho\rho'}_{\lambda\lambda'} := \{(\zeta_\nu, \zeta'_{\nu'}) \in \mathscr{F}_{R'}(\rho_\lambda) \times \mathscr{F}_{R'}(\rho'_{\lambda'}) \mid \zeta_\nu \circ \Phi^q_{\nu\nu'} \textit{ is isomorphic to } \zeta'_{\nu'}\}$$

Fix two elements $(\zeta_\nu, \zeta'_{\nu'}), (\bar{\zeta}_\nu, \bar{\zeta}'_{\nu'})$ of $B^{\rho\rho'}_{\lambda\lambda'}$. For any choice of representatives η_{λ_0} in $\mathscr{F}_{S_0}(\zeta_\nu)$, $\bar{\eta}_{\lambda_0}$ in $\mathscr{F}_{S_0}(\bar{\zeta}_\nu)$, both η_{λ_0} and $\bar{\eta}_{\lambda_0}$ represents ρ_λ on S. This means that there exist transition constants α_j such that $\eta_{\lambda_0} \overset{\alpha_j}{\to} \bar{\eta}_{\lambda_0}$, with α_j that corresponds to the edge λ^*_j in the dual graph $\Gamma_{S,\lambda}$. In the same way, fixed $\eta'_{\lambda'_0}$ in $\mathscr{F}_{S'_0}(\zeta'_{\nu'})$ and $\bar{\eta}'_{\lambda'_0}$ in $\mathscr{F}_{S'_0}(\bar{\zeta}'_{\nu'})$, there exist transition constants β_j, indexed by the edges of $\Gamma_{S,\lambda'}$, such that $\eta'_{\lambda'_0} \overset{\beta_j}{\to} \bar{\eta}'_{\lambda'_0}$. The transition constants α_j and

β_j can be naturally compared for every $j \neq i$. We will say that $(\zeta_\nu, \zeta'_{\nu'})$ and $(\bar{\zeta}_\nu, \bar{\zeta}'_{\nu'})$ are $\approx_{R'}$-equivalent if $\alpha_j = \beta_j$ for every j such that λ_j^* does not belong to the subgraph $\Gamma_{R',\nu} \subset \Gamma_{S,\lambda}$, for any choice of the representatives η_{λ_0}, $\bar{\eta}_{\lambda_0}$, $\eta'_{\lambda'_0}$ and $\bar{\eta}'_{\lambda'_0}$. This definition does not depend on the choices of representatives, because we are not comparing the transition constants on the edges of $\Gamma_{R',\nu}$ and $\Gamma_{R',\nu'}$, which are the only ones that can be modified by a different choice of representatives. We will denote by $\mathscr{B}^{\rho\rho'}_{\lambda\lambda'}$ the quotient of $B^{\rho\rho'}_{\lambda\lambda'}$ by the equivalence relation $\approx_{R'}$.

Analogously to what previously done, we define a map

$$\bar{p} : B^{\rho\rho'}_{\lambda\lambda'} \longrightarrow \operatorname{Hom}(V'_{\lambda'}, V_\lambda)$$

carrying an element $(\zeta_\nu, \zeta'_{\nu'})$ in the isomorphism $M^{\zeta_\nu,\zeta'_{\nu'}}$ that verifies

$$(\zeta_\nu \circ \Phi^q_{\nu\nu'})(X') = M^{\zeta_\nu,\zeta'_{\nu'}} \circ \zeta'_{\nu'}(X') \circ (M^{\zeta_\nu,\zeta'_{\nu'}})^{-1}$$

Note that there is a natural map $f\colon \mathscr{A}^{\rho\rho'}_{\lambda\lambda'} \to \mathscr{B}^{\rho\rho'}_{\lambda\lambda'}$, which sends a couple $[\zeta_{\lambda_0}, \zeta'_{\lambda'_0}] \in \mathscr{A}^{\rho\rho'}_{\lambda\lambda'}$ in the couple $[\zeta_\nu, \zeta'_{\nu'}] \in \mathscr{B}^{\rho\rho'}_{\lambda\lambda'}$, where ζ_ν is represented by ζ_{λ_0} and $\zeta'_{\nu'}$ is represented by $\zeta'_{\lambda'_0}$. It is very easy to check that this map is well defined.

Now we are able to give the statement of the announced technical lemma:

Lemma 4.3.2. *In the above notations, the following hold:*

- *the map* $f\colon \mathscr{A}^{\rho\rho'}_{\lambda\lambda'} \to \mathscr{B}^{\rho\rho'}_{\lambda\lambda'}$ *is a bijection;*
- *the following diagram is commutative*

$$\begin{tikzcd}
\mathscr{A}^{\rho\rho'}_{\lambda\lambda'} \arrow[rr, "f"] \arrow[dr, "\widetilde{p}"'] & & \mathscr{B}^{\rho\rho'}_{\lambda\lambda'} \arrow[dl, "\bar{p}"] \\
& \operatorname{Hom}(V'_{\lambda'}, V_\lambda) &
\end{tikzcd}$$

In particular $\mathscr{L}^{\rho\rho'}_{\lambda\lambda'} = \operatorname{Im}\widetilde{p} = \operatorname{Im}\bar{p}$.

Before dealing with the proof, we want to remark the consequences of this fact. Thanks to this statement, we can see that it is not important to represent an element of $\mathscr{A}^{\rho\rho'}_{\lambda\lambda'}$ as an equivalence class of irreducible representations on the algebras

$$\mathcal{T}^q_{T_1} \otimes \cdots \otimes \mathcal{T}^q_{T_m} \qquad\qquad \mathcal{T}^q_{T'_1} \otimes \cdots \otimes \mathcal{T}^q_{T'_m}$$

but it is sufficient to choose a certain surface R', which is a disjoint union of polygons, and take couples of local representations on R', with a proper equivalence relation that generalizes the one defined in original construction of $\mathscr{A}^{\rho\rho'}_{\lambda\lambda'}$. In other words, in order to obtain all the intertwining operators in $\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}$ via the action of $H_1(S;\mathbb{Z}_N)$, it is not important to split S in all the ideal triangles that compose it but is sufficient to split the surface in simple connected pieces.

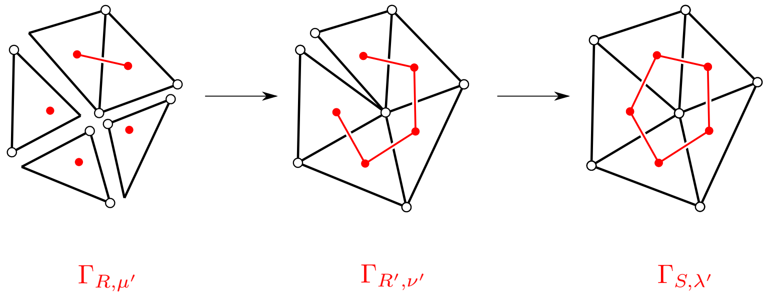

Figure 4.4: An example of $R \to R' \to S$ and their dual graphs

Proof of Lemma 4.3.2. Firstly we will prove the surjectivity of f. Fixed $[\zeta_\nu, \zeta'_{\nu'}]$ in $\mathscr{B}^{\rho\rho'}_{\lambda\lambda'}$, we want to find an element $(\eta_{\lambda_0}, \eta'_{\lambda'_0}) \in A^{\rho\rho'}_{\lambda\lambda'}$ such that the local representations η_ν and $\eta'_{\nu'}$ represented by η_{λ_0} and $\eta'_{\lambda'_0}$, on $\mathcal{T}^q_\nu$ and $\mathcal{T}^q_{\nu'}$ respectively, verify $[\eta_\nu, \eta'_{\nu'}] = [\zeta_\nu, \zeta'_{\nu'}] \in \mathscr{B}^{\rho\rho'}_{\lambda\lambda'}$. Take a representative ζ_{λ_0} of the local representation ζ_ν and analogously $\zeta'_{\lambda'_0}$ of $\zeta'_{\nu'}$. We denote by ζ_μ and $\zeta'_{\mu'}$ the corresponding representations on $\mathcal{T}^q_\mu$ and $\mathcal{T}^q_{\mu'}$. Take $\eta_{\lambda_0} := \zeta_{\lambda_0} \in \mathscr{F}_{S_0}(\rho_\lambda)$. We want to change the element $\zeta'_{\lambda'_0}$ with an $\eta'_{\lambda'_0} \in \mathscr{F}_{S'_0}(\zeta'_{\nu'})$ in such a way that the corresponding $\eta'_{\mu'}$ is isomorphic to $\eta_\mu \circ \Phi^q_{\mu\mu'} = \zeta_\mu \circ \Phi^q_{\mu\mu'}$ (observe that $\zeta_\mu \circ \Phi^q_{\mu\mu'}$ is well defined because ζ_{λ_0} is a representative of ρ_λ, and $\rho_\lambda \circ \Phi^q_{\lambda\lambda'}$ makes sense). In other words, we need to find transition constants α_j, one for every edge $(\nu'_j)^*$ in the graph $\Gamma_{R',\nu'}$, such that, if $\eta'_{\lambda'_0}$ verifies the following relation

$$\zeta'_{\lambda'_0} \xrightarrow{\alpha_j} \eta'_{\lambda'_0}$$

as elements of $\mathscr{F}_{S'_0}(\zeta'_{\nu'})$, then the local representation $\eta'_{\mu'}$, represented by $\eta'_{\lambda'_0}$ on $\mathcal{T}^q_{\mu'}$, has the same invariants of the ones of $\zeta_\mu \circ \Phi^q_{\mu\mu'}$. Exhibiting such a $\eta'_{\lambda'_0}$, we will find a couple $(\eta_{\lambda_0}, \eta'_{\lambda'_0})$ that belongs to $A^{\rho\rho'}_{\lambda\lambda'}$, because by construction $\eta'_{\mu'}$ is isomorphic to $\eta_\mu \circ \Phi^q_{\mu\mu'}$, and such that $\eta_\nu = \zeta_\nu$, $\eta'_{\nu'} = \zeta'_{\nu'}$.

Remember that $\zeta_\nu \circ \Phi^q_{\nu\nu'}$ and $\zeta_{\nu'}$ are isomorphic, so the invariants of all the edges of (R', ν') and the central loads of every component of R' must coincide. In particular, for every j such that the edge μ'_j in ∂R goes in $\partial R'$ through the fusion, the invariant $\zeta'_{\mu'}((X'_j)^N)$ is the same of $\zeta_\mu \circ \Phi^q_{\mu\mu'}$. Hence we have to find the α_s in order to make coincide the central loads of the connected components of R and the invariants of $\eta'_{\mu'}$ associated with the edges of μ' that are fused in ν'.

We will focus at the moment on the component R'_1 of R' containing the edge ν_i, along which we make a diagonal exchange. The same procedure that we are going to describe can be applied to each component and will lead to the conclusion. Take the graph $\Gamma_{R',\eta'}$ and denote by Γ_0 the component of $\Gamma_{R',\eta'}$ that corresponds to R'_1. Γ_0 is a tree because is a deformation retract of R'_1, which is simply connected by hypothesis. We are going to describe a recursive procedure, with

INPUT: a sub-tree Γ_k of Γ_0, containing $(\nu'_i)^*$, and a transition constant α_j for every $(\nu'_j)^*$ that is in $\Gamma_0^{(1)} \setminus \Gamma_k^{(1)}$ verifying: if $(\zeta'_{\lambda'_0})^{(k)}$ denotes the representation in $\mathscr{F}_{S'_0}(\zeta'_{\mu'})$ defined by

$$\zeta'_{\lambda'_0} \xrightarrow{\beta_j} (\zeta'_{\lambda'_0})^{(k)}$$

where $\beta_j = \alpha_j$ when $(\nu'_j)^* \in \Gamma_0^{(1)}$, $\beta_j = 1$ otherwise, then $(\zeta'_{\mu'})^{(k)}$, the local representation of $\mathcal{T}^q_{\mu'}$ represented by $(\zeta'_{\lambda'_0})^{(k)}$, has the same invariants of $\zeta_\mu \circ \Phi^q_{\mu\mu'}$ on all the couples of edges corresponding to $\Gamma_0^{(1)} \setminus \Gamma_k^{(1)}$ and the same central loads on all the triangles corresponding to vertices in $\Gamma_0^{(0)} \setminus \Gamma_k^{(0)}$;

OUTPUT: a sub-tree Γ_{k+1} of Γ_k, obtained by removing a certain edge $(\nu'_n)^*$, $n \neq i$ and a vertex corresponding to a triangle T'_s, and a new transition constant α_n such that the conditions in the input are verified by Γ_{k+1} instead of Γ_{k+1}.

The algorithm ends when the last Γ_k is composed of the only edge $(\nu'_i)^*$ and its ends. Before describing the procedure, we want to convince ourselves that the final transition constants $(\alpha_j)_j$ provide the desired representation. By construction, the resulting $\eta'_{\lambda'_0} := (\zeta'_{\lambda'_0})^{(k)}$ leads to a representation $\eta'_{\mu'}$ that has the proper central loads and edge invariants on every triangle. The last thing we need to check is that the invariants on the square Q are correct too.

Recall that the central load of a fusion is the product of the central loads of the glued terms. We already know that the central load on R'_1 of $\zeta_\nu \circ \Phi^q_{\nu\nu'}$ is equal to the one of $\zeta'_{\nu'}$ and we have constructed a representation $\zeta_{\mu'}$ that has the same loads of $\zeta_\mu \circ \Phi^q_{\mu\mu'}$ on all the triangles not contained in Q, so it is immediate to check that the same holds on the square Q, by the Fusion property. With analogous observations we can check that also the invariants on the boundary of Q have to be the ones of $\zeta_\mu \circ \Phi_{\mu\mu'}$. In order to conclude the proof of the surjectivity, it is sufficient to repeat the procedure on the other components of $\Gamma_{R',\nu'}$, removing from the conditions on the input the restrictions on $(\nu'_i)^*$. In these cases the procedure ends with transition constants that conduce to a representation with the proper invariants on all the triangles composing the fixed component.

Now we can describe the algorithm. Γ_k is a tree, so we can select a leaf of it, i. e. a vertex with valence equal to 1. Assume that the vertex corresponds to the triangle T'_s of the triangulation of R'_1. By hypothesis this vertex is on the side of a unique cell $(\nu'_n)^* \in \Gamma_k^{(1)}$, dual of the edge ν'_n. If $n = i$ and there are not any other leaves, then the tree Γ_k is the graph of the only square Q, and so the algorithm ends. Otherwise, replace the first leaf considered with this one.

Because $(T'_s)^*$ is a leaf, the $(\alpha_j)_j$, selected in the previous steps, lead to a representation $(\zeta'_{\lambda'_0})^{(k)}$ that has the correct invariants on two of the three sides of T'_s, the ones different from ν'_n. Now we want to select a transition constant α_n in order to make correct also the central load of T'_s and the invariant of the edge of T'_s corresponding to ν'_n. Suppose that the sides of T'_s in μ' are labelled as μ'_l, μ'_m and μ'_n, and that the invariants prescribed by $\zeta_\mu \circ \Phi^q_{\mu\mu'}$ are y_l, y_m, y_n and h. Moreover, denote by $\bar{y}_l$, $\bar{y}_m$, $\bar{y}_n$, $\bar{h}$ the invariants of the tensor term of $(\zeta'_{\lambda'_0})^{(k)}$ related to T'_s. By hypothesis, we already know that the following hold

$$
\begin{aligned}
y_l &= \bar{y}_l \\
y_m &= \bar{y}_m
\end{aligned}
$$

If α_n is the transition constant associated with ν'_n, it is immediate to check that the suitable multiplication by α_n of the tensor term of $(\zeta'_{\lambda'_0})^{(k)}$ associated with T'_s modifies the invariants $\bar{y}_u, h$ as follows

$$
\begin{aligned}
\bar{y}_l &\longrightarrow \bar{y}_l \\
\bar{y}_m &\longrightarrow \bar{y}_m \\
\bar{y}_n &\longrightarrow \alpha_n^{N\varepsilon(n,s)}\bar{y}_n \\
\bar{h} &\longrightarrow \alpha_n^{\varepsilon(n,s)}\bar{h}
\end{aligned}
$$

Now fix a certain N-th root β of $\left(\frac{y_n}{\bar{y}_n}\right)^{\varepsilon(n,s)}$ and define $\alpha_n := q^{2t}\beta$, with $t \in \mathbb{Z}_N$ to be determined. By construction

$$
\begin{aligned}
\bar{y}_n \longrightarrow (q^{2t}\beta)^{N\varepsilon(n,s)}\bar{y}_n &= q^{2Nt}\left(\frac{y_n}{\bar{y}_n}\right)^{\varepsilon(n,s)^2}\bar{y}_n \\
&= y_n
\end{aligned}
$$

Now we want to choose $t \in \mathbb{Z}_N$ in order to send $\bar{h}$ in h. Recall that $h^N = y_l y_m y_n$ and $\bar{h}^N = \bar{y}_l \bar{y}_m \bar{y}_n$, so with every choice of t we have $h^N = \bar{h}^N$. On the other hand

$$
\bar{h} \longrightarrow q^{2t\varepsilon(n,s)}\beta\bar{h}
$$

and this implies that we can realize, by changing t, all the possible N-th roots of $\bar{y}_l\bar{y}_m\bar{y}_n = y_l y_m y_n = h^N$, and then there exists a $\bar{t}$ such that $\bar{h} \to h$.

Denote by Γ_{k+1} the tree obtained by removing $(\nu'_n)^*$ and $(T'_s)^*$ from Γ_k. Let us verify that Γ_{k+1} has all the properties in order to repeat the algorithm on it: by construction the representation $(\zeta'_{\lambda'_0})^{(k+1)}$ has the same central loads of

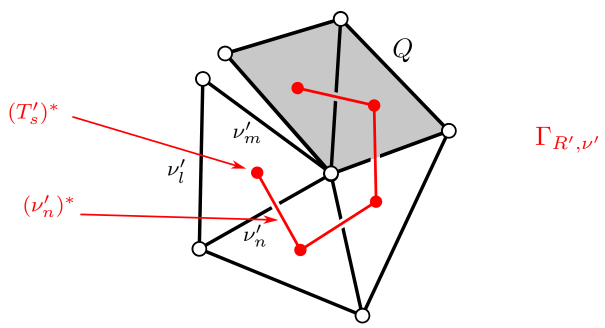

Figure 4.5: The first step of the algorithm

$\zeta_\mu \circ \Phi^q_{\mu\mu'}$ on all the triangles corresponding to vertices in $\Gamma_0^{(0)} \setminus \Gamma_{k+1}^{(0)}$. Indeed, we do not have modified $(\zeta'_{\lambda'_0})^{(k)}$ on the triangles associated with the vertices in $\Gamma_0^{(0)} \setminus \Gamma_k^{(0)}$ and we have chosen α_n in order to have the same central load also on T'_s. The only thing that we need to check is that the invariants on the couples of edges corresponding to the elements of $\Gamma_0^{(1)} \setminus \Gamma_{k+1}^{(1)}$ are correct. The edge ν'_n is the result of the fusion in μ' of the edge μ'_n of T'_s and of another edge μ'_u of a certain triangle T'_v. The last thing we need to check is that the invariant y_u of $(\zeta'_{\mu'})^{(k+1)}$ on the edge μ'_u is also correct. But this easily follows from the fact that, labelled as $\bar{y}_n, \bar{y}_u$ the invariants of $\zeta_\mu \circ \Phi^q_{\mu\mu'}$ on μ_n and μ_u, the products $y_n y_u$ and $\bar{y}_n \bar{y}_u$ are equal because $\rho'_{\lambda'}$ is isomorphic to $\rho_\lambda \circ \Phi^q_{\lambda\lambda'}$ and, thanks to the choice made for α_n, we also have $y_n = \bar{y}_n$. With this last observation we conclude the proof of the surjectivity of f.

Now we have to deal with the injectivity. Fix $[\zeta_{\lambda_0}, \zeta'_{\lambda'_0}]$, $[\bar{\zeta}_{\lambda_0}, \bar{\zeta}'_{\lambda'_0}]$ two elements of $\mathscr{A}^{\rho\rho'}_{\lambda\lambda'}$ and suppose that their images $[\zeta_\nu, \zeta'_{\nu'}]$, $[\bar{\zeta}_\nu, \bar{\zeta}'_{\nu'}]$ in $\mathscr{B}^{\rho\rho'}_{\lambda\lambda'}$ coincide. As usual we are denoting by ζ_ν, $\bar{\zeta}_\nu$ the local representations on $\mathcal{T}^q_\nu$ represented by ζ_{λ_0}, $\bar{\zeta}_{\lambda_0}$ and analogously for $\zeta'_{\nu'}$, $\bar{\zeta}'_{\nu'}$. Because $[\zeta_\nu, \zeta'_{\nu'}] = [\bar{\zeta}_\nu, \bar{\zeta}'_{\nu'}]$, we have

$$\zeta_{\lambda_0} \xrightarrow{\alpha_j} \bar{\zeta}_{\lambda_0}$$
$$\zeta'_{\lambda'_0} \xrightarrow{\beta_j} \bar{\zeta}'_{\lambda'_0}$$

with $\alpha_j = \beta_j$ for every j such that $\lambda^*_j \in \Gamma_{S,\lambda} \setminus \Gamma_{R',\nu'}$. We have to show that $\alpha_j = \beta_j$ for every $j \neq i$. Similarly to what done before, we take Γ_0 the sub-tree of $\Gamma_{R',\nu'}$ related to a connected component of R', and we prove that on all the edges $(\nu'_j)^*$ of Γ_0, except for $(\lambda'_i)^* = (\nu'_i)^*$ possibly, we have $\alpha_j = \beta_j$. Select a leaf of Γ_0, with vertex T'_s and edge $(\nu'_n)^*$ as before. If $n = i$ then we look for another leaf: if it exists we replace ν'_n with it in the following procedure; if it does not, then this component is dual of the only square Q, hence we can skip it and focus on a different connected component. Assume that $n \neq i$. Because $[\zeta_{\lambda_0}, \zeta'_{\lambda'_0}]$ belongs to $\mathscr{A}^{\rho\rho'}_{\lambda\lambda'}$, the representation $\zeta'_{\mu'}$ is isomorphic to $\zeta_\mu \circ \Phi^q_{\mu\mu'}$ and analogously $\bar{\zeta}'_{\mu'}$ is isomorphic to $\bar{\zeta}_\mu \circ \Phi^q_{\mu\mu'}$. In particular, because T'_s is not contained in Q and thanks to the Fusion property of $\Phi^q_{\mu\mu'}$, the invariants of the edges and the central load on T'_s of $\zeta'_{\lambda'_0}$ are the same of those of ζ_{λ_0}, and analogously the invariants of the edges and the central load on T'_s of $\bar{\zeta}'_{\lambda'_0}$ are the same of those of $\bar{\zeta}_{\lambda_0}$. Denoting by ν'_l, ν'_m, ν'_n the edges of T'_s, we already know that $\alpha_l = \beta_l$ and $\alpha_m = \beta_m$ because $(T'_h)^*$ is a leaf. Now we see that, if $\alpha_n \neq \beta_n$, then it can not happen in the same moment that $\zeta_\mu \circ \Phi^q_{\mu\mu'}$ is isomorphic to $\zeta'_{\mu'}$ and $\bar{\zeta}_\mu \circ \Phi^q_{\mu\mu'}$ is isomorphic to $\bar{\zeta}'_{\mu'}$, by inspection of the invariants. Indeed, if $\alpha_n^N \neq \beta_n^N$, then the invariants on the edge ν'_n can not be equal in both cases. If $\alpha_n^N = \beta_n^N$ but $\alpha_n \neq \beta_n$, then the invariants on the edges coincide in both case, but not the central loads of T'_s. This concludes the proof of the main part of the Lemma.

To see that $\bar{p} \circ f = \widetilde{p}$, it is sufficient to observe that, if L is an isomorphism between $\zeta_\mu \circ \Phi_{\mu\mu'}$ and $\zeta'_{\mu'}$, then L is an isomorphism between the fusions $\zeta_\nu \circ \Phi^q_{\nu\nu'}$ and $\zeta'_{\nu'}$ too. □

If $\zeta_\nu, \bar{\zeta}_\nu$ are two elements of $\mathscr{F}_{R'}(\rho_\lambda)$, then we can define a notion of transition constants like in the case of $R' = S_0$. Indeed, taken two representatives ζ_{λ_0}

and $\bar{\zeta}_{\lambda_0}$ of ζ_ν and $\bar{\zeta}_\nu$, then for every λ_j in λ there exists a constant α_j such that $\zeta_{\lambda_0} \xrightarrow{\alpha_j} \bar{\zeta}_{\lambda_0}$. The constants α_j depend in general on the chosen representatives, but only those α_j such that λ_j^* is an edge of $\Gamma_{R',\nu} \subset \Gamma_{S,\lambda}$. Therefore we can define the transition constants from ζ_ν and $\bar{\zeta}_\nu$ as the collection of the α_j that correspond to edges in $\Gamma_{S,\lambda} \setminus \Gamma_{R',\nu}$, fixed a certain couple of representatives. We will briefly write this as

$$\zeta_\nu \xrightarrow[R']{\alpha_j} \bar{\zeta}_\nu$$

Moreover, given ζ_{λ_0} a representative of ζ_ν and such a collection of transitions constants, we can extend it arbitrarily to a set with one α_j for each λ_j and define a new representation $\bar{\zeta}_{\lambda_0}$ of $\mathcal{T}^q_{\lambda_0}$ by taking $\zeta_{\lambda_0} \xrightarrow{\alpha_j} \bar{\zeta}_{\lambda_0}$. It is immediate to see that the local representation $\bar{\zeta}_\nu$ on $\mathcal{T}^q_\nu$ represented by $\bar{\zeta}_{\lambda_0}$ does not depend on the way we extended the set $(\alpha_j)_j$. In conclusion, given $\zeta_\nu \in \mathscr{F}_{R'}(\rho_\lambda)$ and a set of $\alpha_j \in \mathbb{C}^*$, one for each edge of $\Gamma_{S,\lambda} \setminus \Gamma_{R',\nu}$, there is a unique local representation $\bar{\zeta}_\nu \in \mathscr{F}_{R'}(\rho_\lambda)$ such that $\zeta_\nu \xrightarrow[R']{\alpha_j} \bar{\zeta}_\nu$.

The other cases

It is not difficult to deduce a Lemma for the case in which $\lambda = \alpha(\lambda')$, with $\alpha \in \mathfrak{S}_n$, analogous to Lemma 4.3.2. For the sake of simplicity, we will deal with the case $\lambda = \lambda'$, but the same holds in the case of a generic reindexing. Fix a surface R', obtained by splitting S along certain edges, which is disjoint union of polygons and endowed with an induced triangulation $\nu \in \Lambda(R')$.

Now define $B^{\rho\rho'}_{\lambda\lambda}$ as the set of couples (ζ_ν, ζ'_ν) in the product $\mathscr{F}_{R'}(\rho_\lambda) \times \mathscr{F}_{R'}(\rho'_\lambda)$ such that ζ_μ and ζ'_μ are isomorphic. The equivalence relation on $B^{\rho\rho'}_{\lambda\lambda}$ leading to $\mathscr{B}^{\rho\rho'}_{\lambda\lambda}$ is defined as follows: (ζ_ν, ζ'_ν) is $\approx$-equivalent to $(\bar{\zeta}_\nu, \bar{\zeta}'_\nu)$ if

$$\zeta_\nu \overset{\alpha_j}{[}\ R'] \longrightarrow \bar{\zeta}_\nu$$
$$\zeta'_\nu \overset{\beta_j}{[}\ R'] \longrightarrow \bar{\zeta}'_\nu$$

with $\alpha_j = \beta_j$ for every j such that $\lambda_j^* \in \Gamma_{S,\lambda} \setminus \Gamma_{R',\nu}$. As in the previous case, we have natural maps $f\colon \mathscr{A}^{\rho\rho'}_{\lambda\lambda} \to \mathscr{B}^{\rho\rho'}_{\lambda\lambda}$ and $\bar{p}\colon \mathscr{B}^{\rho\rho'}_{\lambda\lambda} \to \mathrm{Hom}(V'_\lambda, V_\lambda)$, defined in the same way. With the same procedure as in the proof of Lemma 4.3.2 the following fact can be shown:

Lemma 4.3.3. *In the above notations, the following hold:*

- *the map $f\colon \mathscr{A}^{\rho\rho'}_{\lambda\lambda} \to \mathscr{B}^{\rho\rho'}_{\lambda\lambda}$ is a bijection;*
- *the following diagram is commutative*

$$\begin{tikzcd}
\mathscr{A}^{\rho\rho'}_{\lambda\lambda} \arrow[rr, "f"] \arrow[dr, "\widetilde{p}"'] & & \mathscr{B}^{\rho\rho'}_{\lambda\lambda} \arrow[dl, "\bar{p}"] \\
& \mathrm{Hom}(V'_\lambda, V_\lambda) &
\end{tikzcd}$$

In particular $\mathscr{L}^{\rho\rho'}_{\lambda\lambda'} = \operatorname{Im}\widetilde{p} = \operatorname{Im}\bar{p}$.

Let $\lambda, \lambda' \in \Lambda(S)$ be two ideal triangulations, which differ by a diagonal exchange or a reindexing. Then we can describe an action of $H_1(S;\mathbb{Z}_N)$ on the set $\mathscr{B}^{\rho\rho'}_{\lambda\lambda'}$ just introduced. Given $c \in H_1(S;\mathbb{Z}_N)$ and $[\zeta_\nu, \zeta'_{\nu'}] \in \mathscr{B}^{\rho\rho'}_{\lambda\lambda'}$, we take a representative $\zeta'_{\lambda'_0}$ of $\zeta'_{\nu'}$ and we define

$$c \cdot [\zeta_\nu, \zeta'_{\nu'}] := [\zeta_\nu, c \cdot \zeta'_{\nu'}]$$

where we are denoting by $c \cdot \zeta'_{\nu'}$ the local representation on (R', ν') induced by $c \cdot \zeta'_{\lambda'_0}$. It is not difficult to see that $c \cdot \zeta'_{\nu'}$ does not depend on the chosen representative $\zeta'_{\lambda'_0}$ of $\zeta'_{\nu'}$ and that the action is well defined (as in the proof of Lemma 4.3.2, here is crucial the simply-connectedness of R'). Moreover, it is clear that this action corresponds via f to the usual action of $H_1(S;\mathbb{Z}_N)$ on $\mathscr{A}^{\rho\rho'}_{\lambda\lambda'}$, hence, thanks to Lemmas 4.3.2 and 4.3.3, the properties of the action that we observed on $\mathscr{A}^{\rho\rho'}_{\lambda\lambda'}$, like transitivity and freeness, hold also on $\mathscr{B}^{\rho\rho'}_{\lambda\lambda'}$.

4.3.3 Elementary Composition property

Let λ be an ideal triangulation of a surface S and assume that there exist in S three triangles that compose a pentagon with diagonals λ_i and λ_j, possibly not embedded in S. We enumerate the sequence of triangulations appearing in the Pentagon relation as follows

$$\begin{aligned}
\lambda^{(0)} &:= \lambda \\
\lambda^{(1)} &:= \Delta_i(\lambda) \\
&\;\;\vdots \\
\lambda^{(4)} &:= (\Delta_j \circ \Delta_i \circ \Delta_j \circ \Delta_i)(\lambda) \\
\lambda^{(5)} &:= \alpha_{ij}(\lambda) \\
\lambda^{(6)} &:= \lambda
\end{aligned}$$

Designate as R the surface obtained from S by splitting it along all the edges except for λ_i and λ_j. Then R is the disjoint union of an ideal pentagon P and $m-3$ triangles $T_4, \ldots, T_m$. The ideal triangulations $\lambda^{(k)}$ lift to a sequence $\mu^{(k)}$ of triangulations on R, which are related by diagonal exchanges along μ_i and μ_j, the diagonals of P. Each ideal triangulation $\mu^{(k)}$ can be naturally presented as the disjoint union of a triangulation $\mu^{(k)}_P$ of the pentagon and the only possible triangulation $\bar{\mu}$ on $T_4 \sqcup \cdots \sqcup T_m$.

By definition of the sets $\mathscr{L}^{\rho\rho}_{\lambda^{(k)}\lambda^{(k+1)}}$, it makes sense to compose the intertwining operators as follows

$$\begin{array}{ccc}
\prod_{k=0}^{5} \mathscr{L}^{\rho\rho}_{\lambda^{(k)}\lambda^{(k+1)}} & \longrightarrow & \operatorname{End}(V_\lambda) \\
(L_0, \ldots, L_5) & \longmapsto & L_0 \circ \cdots \circ L_5
\end{array}$$

Lemma 4.3.4. *Let* $\rho = \{\rho_\lambda \colon \mathcal{T}^q_\lambda \to \operatorname{End}(V_\lambda)\}_{\lambda \in \Lambda(S)}$ *be a local representation of* $\mathcal{T}^q_S$ *and* $\lambda^{(k)}$ *a sequence of triangulations as described above. Then the composition*

$$\begin{array}{ccc}
\prod_{k=0}^{5} \mathscr{L}^{\rho\rho}_{\lambda^{(k)}\lambda^{(k+1)}} & \longrightarrow & \mathscr{L}^{\rho\rho}_{\lambda\lambda} \\
(L_0, \ldots, L_5) & \longmapsto & L_0 \circ \cdots \circ L_5
\end{array}$$

is well defined and it verifies

$$(c_0 \cdot L_0) \circ \cdots \circ (c_5 \cdot L_5) = \left(\sum_{k=0}^{5} c_k \right) \cdot (L_0 \circ \cdots \circ L_5)$$

Proof. By virtue of Lemma 4.3.2, each $\mathscr{L}^{\rho\rho}_{\lambda^{(k)}\lambda^{(k+1)}}$ is in bijection with the set $\mathscr{B}^{\rho\rho}_{\lambda^{(k)}\lambda^{(k+1)}}$, defined as

$$\{(\zeta^{(k)}, \zeta^{(k+1)}) \in \mathscr{F}_R(\rho_{\lambda^{(k)}}) \times \mathscr{F}_R(\rho_{\lambda^{(k+1)}}) \mid \zeta^{(k)} \circ \Phi^q_{\lambda^{(k)}\lambda^{(k+1)}} \text{ isom to } \zeta^{(k+1)}\}\big/\approx$$

where R is the surface described above, which is a disjoint union of an ideal pentagon and triangles, in particular a disjoint union of polygons. The first step of the proof will be the following: we want to translate the composition map on the $\mathscr{L}^{\rho\rho}_{\lambda^{(k)}\lambda^{(k+1)}}$ in an application defined on the product of the sets $\mathscr{B}^{\rho\rho}_{\lambda^{(k)}\lambda^{(k+1)}}$, in order to have a better control of the behaviour of the actions of $H_1(S; \mathbb{Z}_N)$. All the efforts spent to prove Lemma 4.3.2 allow us to manage local representations $\zeta^{(k)}$ defined on Chekhov-Fock algebras of the same surface R associated with the ideal triangulations $\mu^{(k)}$.

We will denote an element of $\prod_{k=0}^{5} \mathscr{B}^{\rho\rho}_{\lambda^{(k)}\lambda^{(k+1)}}$ by $([\zeta_0^{(k)}, \zeta_1^{(k)}])_{k=0}^5$, where $\zeta_0^{(k)}$ is a local representation of $\mathcal{T}^q_{\mu^{(k)}}$ that represents $\rho_{\lambda^{(k)}}$ on S and $\zeta_1^{(k)}$ is a local representation of $\mathcal{T}^q_{\mu^{(k+1)}}$ that represents $\rho_{\lambda^{(k+1)}}$ on S. Suppose that the 6-tuple corresponding to $([\zeta_0^{(k)}, \zeta_1^{(k)}])_{k=0}^5$ in $\prod_{k=0}^{5} \mathscr{L}^{\rho\rho}_{\lambda^{(k)}\lambda^{(k+1)}}$ is $(L_0, \ldots, L_5)$. We would like to understand if there exists an element in $\mathscr{B}^{\rho\rho}_{\lambda\lambda}$ corresponding to $L_0 \circ \cdots \circ L_5$ and how can be described. The elements $[\zeta_0^{(k)}, \zeta_1^{(k)}]$ belong to $\mathscr{B}^{\rho\rho}_{\lambda^{(k)}\lambda^{(k+1)}}$, so for every $k \in \{0, \ldots, 5\}$ and $i \in \{0, 1\}$ the local representation $\zeta_i^{(k)}$ represents $\rho_{\lambda^{(k+i)}}$, i. e. it is an element of $\mathscr{F}_R(\rho_{\lambda^{(k+i)}})$. Then, for every $k = 1, \ldots, 5$ there exist transition constants $\alpha_h^{(k)}$ such that

$$\zeta_0^{(k)} \xrightarrow[R]{\alpha_h^{(k)}} \zeta_1^{(k-1)}$$

where $\alpha_h^{(k)}$ is associated with an edge $\lambda_h^* \in \Gamma_{S,\lambda^{(k)}} \setminus \Gamma_{R,\mu^{(k)}}$. Observe that there is a natural bijection

$$\Gamma_{S,\lambda^{(k)}} \setminus \Gamma_{R,\mu^{(k)}} \longleftrightarrow \Gamma_{S,\lambda^{(k+1)}} \setminus \Gamma_{R,\mu^{(k+1)}}$$

for every $k = 0, \ldots, 5$. Indeed, $\lambda^{(k)}$ and $\lambda^{(k+1)}$ differ by a diagonal exchange, so we have a canonical correspondence between all the edges of them except for the ones on which we do diagonal exchange; in particular on all the edges different from $\lambda_i^{(k)}$ and $\lambda_j^{(k)}$ that compose $\Gamma_{R,\mu^{(k)}}$.

We can change representative of $[\zeta_0^{(1)}, \zeta_1^{(1)}]$ by taking $(\zeta_1^{(0)}, \bar{\zeta}_1^{(1)})$, where

$$\zeta_0^{(1)} \xrightarrow[R]{\alpha_h^{(1)}} \zeta_1^{(0)}$$

$$\zeta_1^{(1)} \xrightarrow[R]{\alpha_h^{(1)}} \bar{\zeta}_1^{(1)}$$

Analogously we can construct a representative $(\bar{\zeta}_1^{(1)}, \bar{\zeta}_1^{(2)})$ of $[\zeta_0^{(2)}, \zeta_1^{(2)}]$ defined by

$$\zeta_0^{(2)} \xrightarrow[R]{\alpha_h^{(2)}} \zeta_1^{(1)} \xrightarrow[R]{\alpha_h^{(1)}} \bar{\zeta}_1^{(1)}$$
$$\zeta_1^{(2)} \xrightarrow[R]{\alpha_h^{(2)}} \tilde{\zeta}_1^{(2)} \xrightarrow[R]{\alpha_h^{(1)}} \bar{\zeta}_1^{(2)}$$

In the same way, for every $k = 3, 4, 5$ we choose the representative $(\bar{\zeta}_1^{(k-1)}, \bar{\zeta}_1^{(k)})$ of $[\zeta_0^{(k)}, \zeta_1^{(k)}]$ obtained as follows

$$\zeta_0^{(k)} \xrightarrow[R]{\alpha_h^{(k)}} \zeta_1^{(k-1)} \xrightarrow[R]{\alpha_h^{(k-1)}} \cdots \xrightarrow[R]{\alpha_h^{(1)}} \bar{\zeta}_1^{(k-1)}$$
$$\zeta_1^{(k)} \xrightarrow[R]{\alpha_h^{(k)}} \tilde{\zeta}_1^{(k)} \xrightarrow[R]{\alpha_h^{(k-1)}} \cdots \xrightarrow[R]{\alpha_h^{(1)}} \bar{\zeta}_1^{(k)}$$

In this way we obtain a 6-tuple of representatives of $([\zeta_0^{(k)}, \zeta_1^{(k)}])_{k=0}^5$ that looks like

$$([\zeta_0^{(0)}, \zeta_1^{(0)}], [\zeta_1^{(0)}, \bar{\zeta}_1^{(1)}], [\bar{\zeta}_1^{(1)}, \bar{\zeta}_1^{(2)}], [\bar{\zeta}_1^{(2)}, \bar{\zeta}_1^{(3)}], [\bar{\zeta}_1^{(3)}, \bar{\zeta}_1^{(4)}], [\bar{\zeta}_1^{(4)}, \bar{\zeta}_1^{(5)}])$$

Despite their terrible appearance, these representatives have the good property that the elements appearing that belong to $\mathscr{F}_R(\rho_{\lambda^{(k)}})$ are both equal to $\bar{\zeta}_1^{(k-1)}$ for $k = 2, 3, 4$, or to $\zeta_1^{(0)}$ when $k = 1$. We claim that the local representations $\zeta_0^{(0)}$ and $\bar{\zeta}_1^{(5)}$, both elements of $\mathscr{F}_R(\rho_\lambda)$, are isomorphic via $L_0 \circ \cdots \circ L_5$, so $[\zeta_0^{(0)}, \bar{\zeta}_1^{(5)}]$ is an element of $\mathscr{B}_{\lambda\lambda}^{\rho\rho}$ whose image is $L_0 \circ \cdots \circ L_5$.

Because we have chosen representatives of the classes $[\zeta_0^{(i)}, \zeta_1^{(i)}]$ corresponding to the linear isomorphisms L_i, we have that

$$\begin{aligned}
(\zeta_0^{(0)} \circ \Phi^q_{\mu^{(0)}\mu^{(1)}})(X^{(1)}) &= L_0 \circ \zeta_1^{(0)}(X^{(1)}) \circ L_0^{-1} & \forall X^{(1)} \in \mathcal{T}^q_{\mu^{(1)}} \\
(\zeta_1^{(0)} \circ \Phi^q_{\mu^{(1)}\mu^{(2)}})(X^{(2)}) &= L_1 \circ \bar{\zeta}_1^{(1)}(X^{(2)}) \circ L_1^{-1} & \forall X^{(2)} \in \mathcal{T}^q_{\mu^{(1)}} \\
(\bar{\zeta}_1^{(k-1)} \circ \Phi^q_{\mu^{(k)}\mu^{(k+1)}})(X^{(k+1)}) &= L_k \circ \bar{\zeta}_1^{(k)}(X^{(k+1)}) \circ L_k^{-1} & \forall X^{(k+1)} \in \mathcal{T}^q_{\mu^{(k+1)}}
\end{aligned}$$

for $k = 2, \ldots, 5$. Hence we deduce that, for every $X \in \mathcal{T}^q_\lambda$

$$\begin{aligned}
\zeta_0^{(0)}(X) &= (\zeta_0^{(0)} \circ \Phi^q_{\mu^{(0)}\mu^{(6)}})(X) \\
&= (\zeta_0^{(0)} \circ \Phi^q_{\mu^{(0)}\mu^{(1)}} \circ \cdots \circ \Phi^q_{\mu^{(5)}\mu^{(6)}})(X) \\
&= L_0 \circ (\zeta_1^{(0)} \circ \Phi^q_{\mu^{(1)}\mu^{(2)}} \circ \cdots \circ \Phi^q_{\mu^{(5)}\mu^{(6)}})(X) \circ L_0^{-1} \\
&\vdots \\
&= (L_0 \circ \cdots \circ L_5) \circ \bar{\zeta}_1^{(5)}(X) \circ (L_0 \circ \cdots \circ L_5)^{-1}
\end{aligned}$$

where we have used the Pentagon relation of $\Phi^q_{\lambda\lambda'}$. This proves the claim.

Denote by $\odot$ the operation from $\prod_{k=0}^{5} \mathscr{B}^{\rho\rho}_{\lambda^{(k)}\lambda^{(k+1)}}$ to $\mathscr{B}^{\rho\rho}_{\lambda\lambda}$ corresponding to the composition of $\mathscr{L}^{\rho\rho}_{\lambda^{(k)}\lambda^{(k+1)}}$ just described. Then we have proved the following equality

$$[\zeta_0^{(0)}, \zeta_1^{(0)}] \odot [\zeta_1^{(0)}, \bar{\zeta}_1^{(1)}] \odot [\bar{\zeta}_1^{(1)}, \bar{\zeta}_1^{(2)}] \odot \cdots \odot [\bar{\zeta}_1^{(4)}, \bar{\zeta}_1^{(5)}] = [\zeta_0^{(0)}, \bar{\zeta}_1^{(5)}] \tag{4.13}$$

Observe that we do not have to verify any dependence on the chosen representatives, because the composition map is obviously well defined and we have just rewrite it on the sets $\mathscr{B}^{\rho\rho}_{\lambda^{(k)}\lambda^{(k+1)}}$.

Now the conclusion is just a verification. Define

$$s_k := \sum_{i=0}^{k} c_i \in H_1(S; \mathbb{Z}_N)$$

for every $k = 0, \dots, 5$. If $\bar{p}\colon \mathscr{B}^{\rho\rho}_{\lambda\lambda} \to \mathscr{L}^{\rho\rho}_{\lambda\lambda}$ is the bijection described in Lemma 4.3.3, we observe that

$$\begin{aligned}
&\bar{p}^{-1}((c_0 \cdot L_0) \circ \cdots \circ (c_5 \cdot L_5)) = \\
&\quad = (c_0 \cdot [\zeta_0^{(0)}, \zeta_1^{(0)}]) \odot (c_1 \cdot [\zeta_1^{(0)}, \bar{\zeta}_1^{(1)}]) \odot \cdots \odot (c_5 \cdot [\bar{\zeta}_1^{(4)}, \bar{\zeta}_1^{(5)}]) \\
&\quad = [\zeta_0^{(0)}, c_0 \cdot \zeta_1^{(0)}] \odot [\zeta_1^{(0)}, c_1 \cdot \bar{\zeta}_1^{(1)}] \odot \cdots \odot [\bar{\zeta}_1^{(4)}, c_5 \cdot \bar{\zeta}_1^{(5)}] \\
&\quad = [\zeta_0^{(0)}, s_0 \cdot \zeta_1^{(0)}] \odot [\zeta_1^{(0)}, (s_1 - s_0) \cdot \bar{\zeta}_1^{(1)}] \odot \cdots \odot [\bar{\zeta}_1^{(4)}, (s_5 - s_4) \cdot \bar{\zeta}_1^{(5)}] \\
&\quad = [\zeta_0^{(0)}, s_0 \cdot \zeta_1^{(0)}] \odot [s_0 \cdot \zeta_1^{(0)}, s_1 \cdot \bar{\zeta}_1^{(1)}] \odot \cdots \odot [s_4 \cdot \bar{\zeta}_1^{(4)}, s_5 \cdot \bar{\zeta}_1^{(5)}] \\
&\quad = [\zeta_0^{(0)}, s_5 \cdot \bar{\zeta}_1^{(5)}] \\
&\quad = s_5 \cdot [\zeta_0^{(0)}, \bar{\zeta}_1^{(5)}] \\
&\quad = s_5 \cdot ([\zeta_0^{(0)}, \zeta_1^{(0)}] \odot [\zeta_1^{(0)}, \bar{\zeta}_1^{(1)}] \odot \cdots \odot [\bar{\zeta}_1^{(4)}, \bar{\zeta}_1^{(5)}]) \\
&\quad = s_5 \cdot \bar{p}^{-1}(L_0 \circ \cdots \circ L_5) \\
&\quad = \bar{p}^{-1}(s_5 \cdot (L_0 \circ \cdots \circ L_5))
\end{aligned}$$

where we are using the relation 4.13 and the equality $[\zeta, \zeta'] = [c \cdot \zeta, c \cdot \zeta'] \in \mathscr{B}^{\rho\rho}_{\lambda^{(k)}\lambda^{(k+1)}}$. Finally, by applying $\bar{p}$ to the first and the last members we obtain

$$(c_0 \cdot L_0) \circ \cdots \circ (c_5 \cdot L_5) = s_5 \cdot (L_0 \circ \cdots \circ L_5) = \left(\sum_{k=0}^{5} c_k \right) \cdot (L_0 \circ \cdots \circ L_5)$$

as desired. □

In the same way, we can prove analogous results with respect to the other relations that hold between the elementary operations on the ideal triangulations. We limit ourselves to the enunciations of these properties, their proof can be obtained with procedures analogous to the Pentagon relation case, by considering the surface R, result of the splitting of S along all the edges except for the ones along we do diagonal exchange, if there is any.

Lemma 4.3.5. *Let $\rho = \{\rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)\}_{\lambda \in \Lambda(S)}$ be a local representation of $\mathcal{T}^q_S$ and let $\lambda \in \Lambda(S)$ be an ideal triangulation.*

COMPOSITION RELATION: *given $\alpha, \beta \in \mathfrak{S}_n$, consider the following path of ideal triangulations*

$$\lambda^{(0)} := \lambda, \quad \lambda^{(1)} := \alpha(\lambda), \quad \lambda^{(2)} := (\beta \circ \alpha)(\lambda)$$

Then the composition

$$\begin{array}{ccc} \mathscr{L}^{\rho\rho}_{\lambda^{(0)}\lambda^{(1)}} \times \mathscr{L}^{\rho\rho}_{\lambda^{(1)}\lambda^{(2)}} & \longrightarrow & \mathscr{L}^{\rho\rho}_{\lambda^{(0)}\lambda^{(2)}} \\ (L_0, L_1) & \longmapsto & L_0 \circ L_1 \end{array}$$

is well defined and it verifies

$$(c_0 \cdot L_0) \circ (c_1 \cdot L_1) = (c_0 + c_1) \cdot (L_0 \circ L_1)$$

for every $c_i \in H_1(S; \mathbb{Z}_N)$;

REFLEXIVITY RELATION: *given λ_i a diagonal of a certain square in λ, consider the following path of ideal triangulations*

$$\lambda^{(0)} := \lambda, \quad \lambda^{(1)} := \Delta_i(\lambda), \quad \lambda^{(2)} := \lambda$$

Then the composition

$$\begin{array}{ccc} \mathscr{L}^{\rho\rho}_{\lambda^{(0)}\lambda^{(1)}} \times \mathscr{L}^{\rho\rho}_{\lambda^{(1)}\lambda^{(2)}} & \longrightarrow & \mathscr{L}^{\rho\rho}_{\lambda\lambda} \\ (L_0, L_1) & \longmapsto & L_0 \circ L_1 \end{array}$$

is well defined and it verifies

$$(c_0 \cdot L_0) \circ (c_1 \cdot L_1) = (c_0 + c_1) \cdot (L_0 \circ L_1)$$

for every $c_i \in H_1(S; \mathbb{Z}_N)$;

RE-INDEXING RELATION: *given λ_i a diagonal of a certain square in λ and $\alpha \in \mathfrak{S}_n$, consider the following path of ideal triangulations*

$$\begin{aligned} \lambda^{(0)} &:= \lambda \\ \lambda^{(1)} &:= \alpha(\lambda) \\ \lambda^{(2)} &:= \Delta_i(\alpha(\lambda)) = \alpha(\Delta_{\alpha(i)}(\lambda)) \\ \lambda^{(3)} &:= \Delta_{\alpha(i)}(\lambda) \\ \lambda^{(4)} &:= \lambda \end{aligned}$$

Then the composition

$$\begin{array}{ccc} \prod_{i=0}^{3} \mathscr{L}^{\rho\rho}_{\lambda^{(k)}\lambda^{(k+1)}} & \longrightarrow & \mathscr{L}^{\rho\rho}_{\lambda\lambda} \\ (L_0, L_1, L_2, L_3) & \longmapsto & L_0 \circ L_1 \circ L_2 \circ L_3 \end{array}$$

is well defined and it verifies

$$(c_0 \cdot L_0) \circ (c_1 \cdot L_1) \circ (c_2 \cdot L_2) \circ (c_3 \cdot L_3) = \left(\sum_{k=0}^{3} c_k \right) \cdot (L_0 \circ L_1 \circ L_2 \circ L_3)$$

for every $c_i \in H_1(S; \mathbb{Z}_N)$. The same holds for the inverse sequence $\overline{\lambda}^{(i)}$, with $\overline{\lambda}^{(i)} := \lambda^{(4-i)}$ for $i = 0, \ldots, 4$;

Distant Commutativity relation: *given λ_i and λ_j diagonals in λ that do not belong to a common triangle, consider the following path of ideal triangulations*

$$\begin{aligned}
\lambda^{(0)} &:= \lambda \\
\lambda^{(1)} &:= \Delta_i(\lambda) \\
\lambda^{(2)} &:= (\Delta_j \circ \Delta_i)(\lambda) = (\Delta_i \circ \Delta_j)(\lambda) \\
\lambda^{(3)} &:= \Delta_j(\lambda) \\
\lambda^{(4)} &:= \lambda
\end{aligned}$$

Then the composition

$$\begin{array}{ccc}
\prod_{i=0}^{3} \mathscr{L}^{\rho\rho}_{\lambda^{(k)}\lambda^{(k+1)}} & \longrightarrow & \mathscr{L}^{\rho\rho}_{\lambda\lambda} \\
(L_0, L_1, L_2, L_3) & \longmapsto & L_0 \circ L_1 \circ L_2 \circ L_3
\end{array}$$

is well defined and it verifies

$$(c_0 \cdot L_0) \circ (c_1 \cdot L_1) \circ (c_2 \cdot L_2) \circ (c_3 \cdot L_3) = \left(\sum_{k=0}^{3} c_k \right) \cdot (L_0 \circ L_1 \circ L_2 \circ L_3)$$

for every $c_i \in H_1(S; \mathbb{Z}_N)$;

Pentagon relation: *given λ_i and λ_j diagonals of a common pentagon in λ, consider the following path of ideal triangulations*

$$\begin{aligned}
\lambda^{(0)} &:= \lambda \\
\lambda^{(1)} &:= \Delta_i(\lambda) \\
&\vdots \\
\lambda^{(4)} &:= (\Delta_j \circ \Delta_i \circ \Delta_j \circ \Delta_i)(\lambda) \\
\lambda^{(5)} &:= \alpha_{ij}(\lambda) \\
\lambda^{(6)} &:= \lambda
\end{aligned}$$

Then the composition

$$\begin{array}{ccc}
\prod_{k=0}^{5} \mathscr{L}^{\rho\rho}_{\lambda^{(k)}\lambda^{(k+1)}} & \longrightarrow & \mathscr{L}^{\rho\rho}_{\lambda\lambda} \\
(L_0, \ldots, L_5) & \longmapsto & L_0 \circ \cdots \circ L_5
\end{array}$$

is well defined and it verifies

$$(c_0 \cdot L_0) \circ \cdots \circ (c_5 \cdot L_5) = \left(\sum_{k=0}^{5} c_k \right) \cdot (L_0 \circ \cdots \circ L_5)$$

for every $c_i \in H_1(S; \mathbb{Z}_N)$.

The relations between the actions exposed in Lemma 4.3.5 and the transitivity of the actions proved in Theorem 4.2.1 imply that the compositions maps are surjective in every case exposed in Lemma 4.3.5.

Lemma 4.3.6. *Let*

$$\begin{aligned} \rho &= \{\rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)\}_{\lambda\in\Lambda(S)} \\ \rho' &= \{\rho'_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V'_\lambda)\}_{\lambda\in\Lambda(S)} \\ \rho'' &= \{\rho''_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V''_\lambda)\}_{\lambda\in\Lambda(S)} \end{aligned}$$

be three isomorphic local representations of $\mathcal{T}^q_S$, and let λ, λ' be two ideal triangulations of S that differ by a diagonal exchange or a re-indexing. Then the compositions

$$\begin{array}{cccc} \circ : & \mathscr{L}^{\rho\rho'}_{\lambda\lambda'} \times \mathscr{L}^{\rho'\rho''}_{\lambda'\lambda'} & \longrightarrow & \mathscr{L}^{\rho\rho''}_{\lambda\lambda'} \\ & (L, M) & \longmapsto & L \circ M \end{array}$$

$$\begin{array}{cccc} \circ : & \mathscr{L}^{\rho\rho'}_{\lambda\lambda} \times \mathscr{L}^{\rho'\rho''}_{\lambda\lambda'} & \longrightarrow & \mathscr{L}^{\rho\rho''}_{\lambda\lambda'} \\ & (L, M) & \longmapsto & L \circ M \end{array}$$

are well defined and they verify

$$(c \cdot L) \circ (d \cdot M) = (c + d) \cdot (L \circ M)$$

for every $c, d \in H_1(S; \mathbb{Z}_N)$.

Proof. Assume that the triangulations differ by diagonal exchange along the edge λ_i. Now take R the surface obtained by splitting S along all the edges except for λ_i. Then we can represent, as in the proof of Lemma 4.3.2, the compositions on the sets $\mathscr{B}^{\rho\rho'}_{\lambda\lambda'} \times \mathscr{B}^{\rho'\rho''}_{\lambda'\lambda'} \longrightarrow \mathscr{B}^{\rho\rho''}_{\lambda\lambda'}$ and $\mathscr{B}^{\rho\rho'}_{\lambda\lambda} \times \mathscr{B}^{\rho'\rho''}_{\lambda\lambda'} \longrightarrow \mathscr{B}^{\rho\rho''}_{\lambda\lambda'}$ respectively, where the set $\mathscr{B}$ are defined as quotients of sets of local representations on R. Now the proof can be achieved with the same ideas of Lemma 4.3.2. □

Lemma 4.3.7. *Given $\lambda, \lambda' \in \Lambda(S)$ that differ by an elementary move, i. e. a diagonal exchange or a re-indexing of the edges, the map*

$$\begin{array}{cccc} (\cdot)^{-1} : & \mathscr{L}^{\rho\rho}_{\lambda\lambda'} & \longrightarrow & \mathscr{L}^{\rho\rho}_{\lambda'\lambda} \\ & L & \longmapsto & L^{-1} \end{array}$$

verifies

$$(c \cdot L)^{-1} = (-c) \cdot L^{-1}$$

where c is an element of $H_1(S; \mathbb{Z}_N)$, the action in the first member is on $\mathscr{L}^{\rho\rho}_{\lambda\lambda'}$ and the action in the second member is on $\mathscr{L}^{\rho\rho}_{\lambda'\lambda}$.

Proof. Thanks to Lemma 4.3.5, the proof is immediate:

$$((-c) \cdot L^{-1}) \circ (c \cdot L) = (-c + c) \cdot (L^{-1} \circ L) = id$$

□

4.4 The complete definition

In the previous paragraphs we have studied the elementary properties of the sets $\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}$ endowed with certain actions $\psi^{\rho\rho'}_{\lambda\lambda'}$ of $H_1(S; \mathbb{Z}_N)$, but we have defined these objects only in the case in which λ and λ' differ by an elementary move. Now we have achieved all the elements to deal with the general construction.

Let $\lambda, \lambda' \in \Lambda(S)$ be two ideal triangulations of S. Thanks to Theorem 0.3 there exists a sequence of elementary moves that leads from λ to λ'. Label the ideal triangulations we pass through as follows

$$\lambda = \lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(h)}, \lambda^{(h+1)} = \lambda'$$

Then we define $\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}$ as the image of the composition map

$$\begin{array}{ccc} \prod_{k=0}^{h} \mathscr{L}^{\rho\rho}_{\lambda^{(k)}\lambda^{(k+1)}} \times \mathscr{L}^{\rho\rho'}_{\lambda'\lambda'} & \longrightarrow & \mathrm{Hom}(V'_{\lambda'}, V_\lambda) \\ (L_0, \dots, L_h, L_{h+1}) & \longmapsto & L_0 \circ \cdots \circ L_{h+1} \end{array}$$

Moreover, denoting by $\pi\colon H_1(S;\mathbb{Z}_N)^{h+2} \to H_1(S;\mathbb{Z}_N)$ the map

$$\pi(c_0, \dots, c_{h+1}) := \sum_{i=0}^{h+1} c_i$$

we can fix a section $s\colon H_1(S;\mathbb{Z}_N)^{h+2} \to H_1(S;\mathbb{Z}_N)$ of π, in other words a map that verifies $\pi \circ s = id$, not necessarily a homomorphism, and we define the action of $c \in H_1(S;\mathbb{Z}_N)$ on an element $L = L_0 \circ \cdots \circ L_{h+1}$ in $\mathscr{L}^{\rho\rho}_{\lambda\lambda'}$ as follows

$$c \cdot L := (c_0 \cdot L_0) \circ \cdots \circ (c_{h+1} \cdot L_{h+1})$$

where $s(c) = (c_0, \dots, c_{h+1})$ and we have chosen an element $(L_0, \dots, L_{h+1})$ in the fibre of π under the composition map.

In this definition of $(\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}, \psi^{\rho\rho'}_{\lambda\lambda'})$ we have done some arbitrary choices:

- the path of triangulations between λ and λ';
- the section s of π;
- the decomposition of $L \in \mathscr{L}^{\rho\rho'}_{\lambda\lambda'}$ as image under the composition map of a certain $(h+2)$-tuple $(L_0, \dots, L_{h+1})$.

We need to prove that the object $(\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}, \psi^{\rho\rho'}_{\lambda\lambda'})$ does not depend on the choices made. We start from the last one: take $(L_i)_i$ and $(L'_i)_i$ such that

$$L_0 \circ \cdots \circ L_{h+1} = L'_0 \circ \cdots \circ L'_{h+1}$$

We want to show that, for every $(c_0, \dots, c_{h+1}) \in H_1(S;\mathbb{Z}_N)^N$, the following holds

$$(c_0 \cdot L_0) \circ \cdots \circ (c_{h+1} \cdot L_{h+1}) = (c_0 \cdot L'_0) \circ \cdots \circ (c_{h+1} \cdot L'_{h+1})$$

Firstly observe that, because of the transitivity of the action of $H_1(S;\mathbb{Z}_N)$, for every $i = 0, \dots, h+1$ there exists a $d_i \in H_1(S;\mathbb{Z}_N)$ such that $d_i \cdot L_i = L'_i$. By hypothesis, we have

$$(L_0 \circ \cdots \circ L_{h+1}) \circ (L'_0 \circ \cdots \circ L'_{h+1})^{-1} = id \in \mathrm{End}(V_\lambda)$$

On the other hand

$$
\begin{aligned}
&(L_0 \circ \cdots \circ L_{h+1}) \circ (L'_0 \circ \cdots \circ L'_{h+1})^{-1} = \\
&\quad = L_0 \circ \cdots \circ L_{h+1} \circ (d_{h+1} \cdot L_{h+1})^{-1} \circ \cdots \circ (d_0 \cdot L_0)^{-1} \\
&\quad = L_0 \circ \cdots \circ L_{h+1} \circ ((-d_{h+1}) \cdot L_{h+1}^{-1}) \circ \cdots \circ ((-d_0) \cdot L_0^{-1}) \qquad \text{Lemma 4.3.7} \\
&\quad = L_0 \circ \cdots \circ L_h \circ ((-d_{h+1}) \cdot id) \circ ((-d_h) \cdot L_h^{-1}) \circ \cdots \circ ((-d_0) \cdot L_0^{-1}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{Lemma 4.3.5} \\
&\quad = L_0 \circ \cdots \circ L_h \circ ((-d_{h+1} - d_h) \cdot L_h^{-1}) \circ \cdots \circ ((-d_0) \cdot L_0^{-1}) \quad \text{Lemma 4.3.6} \\
&\quad \vdots \\
&\quad = \left(- \sum_{i=0}^{h+1} d_i \right) \cdot id
\end{aligned}
$$

We observed that the action of $H_1(S;\mathbb{Z}_N)$ on $\mathscr{L}_{\lambda\lambda}^{\rho\rho}$ is free, so $\sum_{i=0}^{h+1} d_i$ must be equal to 0. It is simple to see that the steps of the relation above prove also that the following relation holds

$$
\begin{aligned}
((c_0 \cdot L_0) \circ \cdots \circ (c_{h+1} \cdot L_{h+1})) \circ ((c'_0 \cdot L_0) \circ \cdots \circ (c'_{h+1} \cdot L_{h+1}))^{-1} = \\
= \left(\sum_{i=0}^{h+1} (c_i - c'_i) \right) \cdot id
\end{aligned}
\tag{4.14}
$$

In particular, we have

$$
\begin{aligned}
&(c_0 \cdot L_0) \circ \cdots \circ (c_{h+1} \cdot L_{h+1}) \circ ((c_0 \cdot L'_0) \circ \cdots \circ (c_{h+1} \cdot L'_{h+1}))^{-1} = \\
&= (c_0 \cdot L_0) \circ \cdots \circ (c_{h+1} \cdot L_{h+1}) \circ (((c_0 + d_0) \cdot L_0) \circ \cdots \circ ((c_{h+1} + d_{h+1}) \cdot L_{h+1}))^{-1} \\
&= \left(\sum_{i=0}^{h+1} c_i - \sum_{i=0}^{h+1} (d_i + c_i) \right) \cdot id \\
&= id
\end{aligned}
$$

and this concludes the proof of the independence of $(c_0 \cdot L_0) \circ \cdots \circ (c_{h+1} \cdot L_{h+1})$ from the $(h+2)$-tuple $(L_i)_i$.

Fixed s, s' two sections of π and c an element of $H_1(S;\mathbb{Z}_N)$, we have

$$
\sum_{i=0}^{h+1} c_i = \sum_{i=0}^{h+1} c'_i
$$

where $s(c) = (c_1, \ldots, c_{h+1})$ and $s'(c) = (c'_1, \ldots, c'_{h+1})$. Hence relation 4.14 proves the independence of $c \cdot L$ from the fixed section too. In addition, by selecting an homomorphism as section, which clearly exists, we conclude that $(c, L) \mapsto c \cdot L$ is indeed an action. The last step in order to obtain the good definition of $(\mathscr{L}_{\lambda\lambda'}^{\rho\rho'}, \psi_{\lambda\lambda'}^{\rho\rho'})$ is the independence from the choice of the path of ideal triangulations between λ and λ'.

Consider two sequences of ideal triangulations from λ to λ'. By Theorem 0.4 we know that these sequences are connected by a chain of certain moves. It is sufficient to prove that, starting from a sequence

$$
\lambda = \lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(k)}, \ldots, \lambda^{(h)}, \lambda^{(h+1)} = \lambda'
$$

and by applying any move described in Theorem 0.4, the sets $(\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}, \psi^{\rho\rho'}_{\lambda\lambda'})$ defined through the different sequences are the same. In what follows, we will show that this fact is a simple consequence of Lemma 4.3.5. Take a sequence obtained by modifying the original one, which will look like

$$\lambda = \lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(k)} = \overline{\lambda}^{(0)}, \overline{\lambda}^{(1)}, \ldots, \overline{\lambda}^{(n+1)} = \lambda^{(k+1)}, \ldots, \lambda^{(h)}, \lambda^{(h+1)} = \lambda'$$

where $\lambda^{(k)} = \overline{\lambda}^{(0)}, \overline{\lambda}^{(1)}, \ldots, \overline{\lambda}^{(n)}$ is one of the sequences appearing in the assertion of Lemma 4.3.5. Then we need to compare the images of the following composition maps, labelled as φ_1 and φ_2 respectively:

$$\prod_{i=0}^{k-1} \mathscr{L}^{\rho\rho}_{\lambda^{(i)}\lambda^{(i+1)}} \times \prod_{j=0}^{n} \mathscr{L}^{\rho\rho}_{\overline{\lambda}^{(j)}\overline{\lambda}^{(j+1)}} \times \prod_{l=k+1}^{h} \mathscr{L}^{\rho\rho}_{\lambda^{(l)}\lambda^{(l+1)}} \times \mathscr{L}^{\rho\rho'}_{\lambda'\lambda'} \xrightarrow{\varphi_1} \mathrm{Hom}(V'_{\lambda'}, V_\lambda)$$

$$\prod_{i=0}^{h} \mathscr{L}^{\rho\rho}_{\lambda^{(i)}\lambda^{(i+1)}} \times \mathscr{L}^{\rho\rho'}_{\lambda'\lambda'} \xrightarrow{\varphi_2} \mathrm{Hom}(V'_{\lambda'}, V_\lambda)$$

Because of Lemma 4.3.5, the composition of an $(n+1)$-tuple in $\prod_{j=0}^{n} \mathscr{L}^{\rho\rho}_{\overline{\lambda}^{(j)}\overline{\lambda}^{(j+1)}}$ provide an element in $\mathscr{L}^{\rho\rho}_{\lambda^{(k)}\lambda^{(k+1)}}$, so the inclusion $\mathrm{Im}\,\varphi_1 \subseteq \mathrm{Im}\,\varphi_2$ is obvious. Moreover, the composition is surjective, so also the inverse inclusion holds and therefore $\mathrm{Im}\,\varphi_1 = \mathrm{Im}\,\varphi_2$. Hence the sets $\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}$ are well defined, it remains to prove the good definition of the action $\psi^{\rho\rho'}_{\lambda\lambda'}$.

Fixed $L \in \mathscr{L}^{\rho\rho'}_{\lambda\lambda'}$, we can write it as an element in the image of φ_1 and φ_2, respectively, as follows

$$\begin{aligned} L &= L_0 \circ \cdots \circ L_{k-1} \circ \overline{L}_0 \circ \cdots \circ \overline{L}_n \circ L_{k+1} \circ \cdots \circ L_h \circ L_{h+1} \\ &= L'_0 \circ \cdots \circ L'_{k-1} \circ L'_k \circ L'_{k+1} \circ \cdots \circ L'_h \circ L'_{h+1} \end{aligned} \tag{4.15}$$

By virtue of the transitivity of the action on the terms $\mathscr{L}^{\rho\rho}_{\lambda^{(i)}\lambda^{(i+1)}}$ for every $i \neq k$, there exist $c_i \in H_1(S; \mathbb{Z}_N)$ such that $c_i \cdot L'_i = L_i$. Moreover, we can find an element $c_k \in H_1(S; \mathbb{Z}_N)$ such that $c_k \cdot L'_k = \overline{L}_0 \circ \cdots \circ \overline{L}_n$, thanks to Lemma 4.3.5. Denoting by $\prod_{i=0}^{n} M_i$ the composition $M_0 \circ \cdots \circ M_n$, from relation 4.15 we can deduce

$$\begin{aligned} id &= \prod_{i=0}^{k-1} L_i \circ \prod_{j=0}^{n} \overline{L}_j \circ \prod_{l=k+1}^{h+1} L_l \circ \prod_{l=0}^{h-k} (L'_{h+1-l})^{-1} \circ (L'_k)^{-1} \circ \prod_{i=0}^{k-1} (L'_{k-1-i})^{-1} \\ &= \prod_{i=0}^{k-1} (c_i \cdot L'_i) \circ (c_k \cdot L'_k) \circ \prod_{l=k+1}^{h+1} (c_l \cdot L'_l) \circ \prod_{l=0}^{h-k} (L'_{h+1-l})^{-1} \circ (L'_k)^{-1} \circ \prod_{i=0}^{k-1} (L'_{k-1-i})^{-1} \\ &= \left(\sum_{i=0}^{h+1} c_i \right) \cdot id \qquad \text{Relation 4.14} \end{aligned}$$

Because the action of $H_1(S; \mathbb{Z}_N)$ on $\mathscr{L}^{\rho\rho}_{\lambda\lambda}$ is free, we must have $\sum_{i=0}^{h+1} c_h = 0$. We have shown that the actions are independent from the choices of the sections, so given

$$\begin{array}{rccc} \pi_1: & H_1(S; \mathbb{Z}_N)^{h+n+2} & \longrightarrow & H_1(S; \mathbb{Z}_N) \\ & (d_i)_i & \longmapsto & \sum_{i=0}^{h+n+1} d_i \end{array}$$

$$\begin{array}{rccc} \pi_2: & H_1(S; \mathbb{Z}_N)^{h+2} & \longrightarrow & H_1(S; \mathbb{Z}_N) \\ & (d_i)_i & \longmapsto & \sum_{i=0}^{h+1} d_i \end{array}$$

the maps through which the actions are defined, we can choose

$$s_1(d)_i = (d, 0, \dots, 0) \in H_1(S; \mathbb{Z}_N)^{h+n+2}$$
$$s_2(d)_i = (d, 0, \dots, 0) \in H_1(S; \mathbb{Z}_N)^{h+2}$$

as sections. Then the actions of d, defined through these two sections, give respectively

$$(d \cdot L)_1 = (d \cdot L_0) \circ \cdots \circ L_{k-1} \circ \overline{L}_0 \circ \cdots \circ \overline{L}_n \circ L_{k+1} \circ \cdots \circ L_h \circ L_{h+1}$$
$$(d \cdot L)_2 = (d \cdot L'_0) \circ \cdots \circ L'_{k-1} \circ L'_k \circ L'_{k+1} \circ \cdots \circ L'_h \circ L'_{h+1}$$

where $(c \cdot L)_1$ denotes the action of d on L defined with the first sequence and the section s_1, and $(c \cdot L)_1$ denotes the action of d on L defined with the second sequence and the section s_2.

Now, by virtue of the presentations of $(d \cdot L)_1$ and $(d \cdot L)_2$ just given and of the relation 4.14, we can rewrite the isomorphism $(d \cdot L)_1 \circ (d \cdot L)_2^{-1}$ as the following composition:

$$\begin{aligned}
&= (d \cdot L_0) \circ \prod_{i=1}^{k-1} L_i \circ \prod_{j=0}^{n} \overline{L}_j \circ \prod_{l=k+1}^{h+1} L_l \circ \prod_{l=0}^{h-k} (L'_{h+1-l})^{-1} \circ (L'_k)^{-1} \circ \prod_{i=0}^{k-2} (L'_{k-1-i})^{-1} \circ \\
&\qquad \circ (d \cdot L'_0)^{-1} \\
&= ((d + c_0) \cdot L'_0) \circ \prod_{i=1}^{k-1} (c_i \cdot L'_i) \circ (c_k \cdot L'_k) \circ \prod_{l=k+1}^{h+1} (c_l \cdot L'_l) \circ \prod_{l=0}^{h-k} (L'_{h+1-l})^{-1} \circ (L'_k)^{-1} \circ \\
&\qquad \circ \prod_{i=0}^{k-2} (L'_{k-1-i})^{-1} \circ (d \cdot L'_0)^{-1} \\
&= \left(d + \sum_{i=0}^{h+1} c_i - d \right) \cdot id \\
&= id
\end{aligned}$$

and this finally proves the independence of the action from the chosen path of ideal triangulations. Hence the objects $(\mathscr{L}_{\lambda\lambda'}^{\rho\rho'})$ are well defined.

In order to prove the transitivity of the action, fix a certain path of ideal triangulation and two elements $L = L_0 \circ \cdots \circ L_{h+1}$ and $L' = L'_0 \circ \cdots \circ L'_{h+1}$ in $\mathscr{L}_{\lambda\lambda'}^{\rho\rho'}$. Because the actions are transitive in the elementary cases, for every $i = 0, \dots, h+1$ there exists a $c_i \in H_1(S; \mathbb{Z}_N)$ such that $c_i \cdot L_i = L'_i$. So, choosing a section s such that $s(\sum_i c_i) = (c_0, \dots, c_n)$ (because s is not required to be a homomorphism, we can always do so), we observe that

$$c \cdot (L_0 \circ \cdots \circ L_{h+1}) = L'_0 \circ \cdots \circ L'_{h+1}$$

hence the action is transitive. Now suppose that $c \cdot L = L$, then the following relation must hold

$$((c_0 \cdot L_0) \circ \cdots \circ (c_{h+1} \cdot L_{h+1})) \circ (L_0 \circ \cdots \circ L_{h+1})^{-1} = id$$

Now, applying the relation 4.14 we obtain

$$\left(\sum_{i=0}^{h+1} c_i \right) \cdot id = id$$

By freeness of the action of $H_1(S;\mathbb{Z}_N)$ on $\mathscr{L}_{\lambda\lambda}^{\rho\rho}$, we deduce $\sum_i c_i = 0$ and then $c = 0$, which proves the freeness of the action in the generic case.

Now we have defined all the elements needed to deal with the proof of the Existence Theorem:

Theorem 4.4.1 (Existence Theorem)**.** *For every surface S (see Chapter 0 for details) there exists a collection $\{(\mathscr{L}_{\lambda\lambda'}^{\rho\rho'}, \psi_{\lambda\lambda'}^{\rho\rho'})\}$, indexed by couples of isomorphic local representations ρ, ρ' of the quantum Teichmüller space $\mathcal{T}_S^q$ and by couples of ideal triangulations $\lambda, \lambda' \in \Lambda(S)$ such that*

Intertwining: *for every couple of isomorphic local representations*

$$\rho = \{\rho_\lambda \colon \mathcal{T}_\lambda^q \to \mathrm{End}(V_\lambda)\}_{\lambda\in\Lambda(S)} \qquad \rho' = \{\rho'_\lambda \colon \mathcal{T}_\lambda^q \to \mathrm{End}(V'_\lambda)\}_{\lambda\in\Lambda(S)}$$

and for every $\lambda, \lambda' \in \Lambda(S)$, $\mathscr{L}_{\lambda\lambda'}^{\rho\rho'}$ is a set of linear isomorphisms $L_{\lambda\lambda'}^{\rho\rho'}$ from $V'_{\lambda'}$ to V_λ such that

$$(\rho_\lambda \circ \Phi_{\lambda\lambda'}^q)(X') = L_{\lambda\lambda'}^{\rho\rho'} \circ \rho'_{\lambda'}(X') \circ (L_{\lambda\lambda'}^{\rho\rho'})^{-1}$$

for every $X' \in \mathcal{T}_{\lambda'}^q$;

Action: *every set $\mathscr{L}_{\lambda\lambda'}^{\rho\rho'}$ is endowed with a transitive and free action $\psi_{\lambda\lambda'}^{\rho\rho'}$ of $H_1(S;\mathbb{Z}_N)$;*

Fusion property: *let R be a surface and S obtained by fusion from R. Fix*

$$\eta = \{\eta_\mu \colon \mathcal{T}_\mu^q \to \mathrm{End}(W_\mu)\}_{\mu\in\Lambda(R)} \qquad \eta' = \{\eta'_\mu \colon \mathcal{T}_\mu^q \to \mathrm{End}(W'_\mu)\}_{\mu\in\Lambda(R)}$$

two isomorphic local representations of $\mathcal{T}_R^q$ and

$$\rho = \{\rho_\lambda \colon \mathcal{T}_\lambda^q \to \mathrm{End}(V_\lambda)\}_{\lambda\in\Lambda(S)} \qquad \rho' = \{\rho'_\lambda \colon \mathcal{T}_\lambda^q \to \mathrm{End}(V'_\lambda)\}_{\lambda\in\Lambda(S)}$$

two isomorphic local representations of $\mathcal{T}_S^q$, with ρ and ρ' obtained by fusion from η and η', respectively. Then for every $\mu, \mu' \in \Lambda(R)$, if $\lambda, \lambda' \in \Lambda(S)$ are the corresponding ideal triangulations on S, there exists a natural inclusion $j \colon \mathscr{L}_{\mu\mu'}^{\eta\eta'} \to \mathscr{L}_{\lambda\lambda'}^{\rho\rho'}$ (the inclusion as subsets of $\mathrm{Hom}(V'_{\lambda'}, V_\lambda)$) such that, for every $L \in \mathscr{L}_{\mu\mu'}^{\eta\eta'}$, the following holds

$$(j \circ \psi_{\mu\mu'}^{\eta\eta'})(c, L) = \psi_{\lambda\lambda'}^{\rho\rho'}(\pi_*(c), j(L))$$

for every $c \in H_1(R;\mathbb{Z}_N)$, where $\pi \colon R \to S$ is the projection map;

Composition property: *for every ρ, ρ', ρ'' isomorphic local representations of $\mathcal{T}_S^q$ and for every $\lambda, \lambda', \lambda'' \in \Lambda(S)$ the composition map*

$$\begin{array}{ccc} \mathscr{L}_{\lambda\lambda'}^{\rho\rho'} \times \mathscr{L}_{\lambda'\lambda''}^{\rho'\rho''} & \longrightarrow & \mathscr{L}_{\lambda\lambda''}^{\rho\rho''} \\ (L, M) & \longmapsto & L \circ M \end{array}$$

is well defined and it verifies

$$(c \cdot L) \circ (d \cdot M) = (c + d) \cdot (L \circ M)$$

Proof. We need to verify that Fusion and Composition properties hold.

$$\begin{array}{c} \nu\,\nu' \ R \ \mu\,\mu' \\ \big\downarrow \\ \lambda\,\lambda' \ S \ \rho\,\rho' \end{array}$$

FUSION PROPERTY: Let R and S be surfaces like in the assertion, each one endowed with a couple of ideal triangulations, $\mu, \mu' \in \Lambda(R)$ and $\lambda, \lambda' \in \Lambda(S)$ respectively, where the fusions of the firsts are the seconds, and with a couple of isomorphic local representations, η, η' of $\mathcal{T}_R^q$ and ρ, ρ' of $\mathcal{T}_S^q$, where the fusions of the firsts are the seconds. Firstly observe that $W_\mu = V_\lambda$ and $W'_{\mu'} = V'_{\lambda'}$ because ρ is fusion of η and ρ' is fusion of η', so the sets $\mathscr{L}_{\lambda\lambda'}^{\rho\rho'}$ and $\mathscr{L}_{\mu\mu'}^{\eta\eta'}$ are both subsets of $\mathrm{Hom}(V'_{\lambda'}, V_\lambda)$.

Now we choose a sequence of ideal triangulations in R from μ to μ'

$$\mu = \mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(k)}, \mu^{(k+1)} = \mu'$$

which induces a corresponding sequence in S from λ to λ' by fusion

$$\lambda = \lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(k)}, \lambda^{(k+1)} = \lambda'$$

The set $\mathscr{L}_{\mu\mu'}^{\eta\eta'}$ can be realized as the image under the composition map of the set

$$\mathscr{L}_{\mu\mu^{(1)}}^{\eta\eta} \times \cdots \times \mathscr{L}_{\mu^{(k)}\mu'}^{\eta\eta} \times \mathscr{L}_{\mu'\mu'}^{\eta\eta'}$$

Lemma 4.3.1 tells us that this product is contained, through a natural map that we still denote by j, in

$$\mathscr{L}_{\lambda\lambda^{(1)}}^{\rho\rho} \times \cdots \times \mathscr{L}_{\lambda^{(k)}\lambda'}^{\rho\rho} \times \mathscr{L}_{\lambda'\lambda'}^{\rho\rho'}$$

The image of this last set, under the composition map, is equal to $\mathscr{L}_{\lambda\lambda'}^{\rho\rho'}$. This clearly shows $\mathscr{L}_{\mu\mu'}^{\eta\eta'} \subseteq \mathscr{L}_{\lambda\lambda'}^{\rho\rho'}$ and the map is just the inclusion as subsets of $\mathrm{Hom}(V'_{\lambda'}, V_\lambda)$, so it does not depend on the chosen sequence of ideal triangulations. Now we have to prove the relation between the actions: we fix $c \in H_1(R; \mathbb{Z}_N)$ and an element $L \in \mathscr{L}_{\mu\mu'}^{\eta\eta'}$, we take a presentation $L = L_0 \circ \cdots \circ L_{k+1}$ as compositions of a $(k+2)$-tuple in

$$\mathscr{L}_{\mu\mu^{(1)}}^{\eta\eta} \times \cdots \times \mathscr{L}_{\mu^{(k)}\mu'}^{\eta\eta} \times \mathscr{L}_{\mu'\mu'}^{\eta\eta'}$$

Then, recalling Lemma 4.3.1 and what just seen, we have

$$\begin{aligned} j(c \cdot (L_0 \circ \cdots \circ L_{k+1})) &= j((c_0 \cdot L_0) \circ \cdots \circ (c_{k+1} \cdot L_{k+1})) \\ &= j(c_0 \cdot L_0) \circ \cdots \circ j(c_{k+1} \cdot L_{k+1}) \\ &= (\pi_*(c_0) \cdot j(L_0)) \circ \cdots \circ (\pi_*(c_{k+1}) \cdot j(L_{k+1})) \\ &= \pi_*(c) \cdot (j(L_0) \circ \cdots \circ j(L_{k+1})) \\ &= \pi_*(c) \cdot j(L_0 \circ \cdots \circ L_{k+1}) \end{aligned}$$

where $\pi \colon R \to S$ is the quotient map. So the Fusion property holds.

COMPOSITION PROPERTY: Given $\lambda, \lambda'\lambda'' \in \Lambda(S)$, we choose paths of elementary moves on ideal triangulations

$$\lambda = \lambda^{(0)}, \ldots, \lambda^{(k)}, \lambda^{(k+1)} = \lambda', \lambda^{(k+2)}, \ldots, \lambda^{(h)}, \lambda^{(h+1)} = \lambda''$$

Then the sets $\mathscr{L}_{\lambda\lambda'}^{\rho\rho'}$ and $\mathscr{L}_{\lambda'\lambda''}^{\rho'\rho''}$ can be constructed as images of the composition maps as follows

$$\begin{aligned} \mathscr{L}_{\lambda\lambda'}^{\rho\rho'} &= \mathrm{Im}(\mathscr{L}_{\lambda\lambda^{(1)}}^{\rho\rho} \times \cdots \times \mathscr{L}_{\lambda^{(k)}\lambda'}^{\rho\rho} \times \mathscr{L}_{\lambda'\lambda'}^{\rho\rho'} \longrightarrow \mathrm{Hom}(V'_{\lambda'}, V_\lambda)) \\ \mathscr{L}_{\lambda'\lambda''}^{\rho'\rho''} &= \mathrm{Im}(\mathscr{L}_{\lambda'\lambda^{(k+2)}}^{\rho'\rho'} \times \cdots \times \mathscr{L}_{\lambda^{(h)}\lambda''}^{\rho'\rho'} \times \mathscr{L}_{\lambda''\lambda''}^{\rho'\rho''} \longrightarrow \mathrm{Hom}(V''_{\lambda''}, V'_{\lambda'})) \end{aligned}$$

On the other hand, the set $\mathscr{L}^{\rho\rho''}_{\lambda\lambda''}$ can be presented as

$$\mathscr{L}^{\rho\rho''}_{\lambda\lambda''} = \mathrm{Im}(\mathscr{L}^{\rho\rho}_{\lambda\lambda^{(1)}} \times \cdots \times \mathscr{L}^{\rho\rho}_{\lambda^{(h)}\lambda''} \times \mathscr{L}^{\rho\rho''}_{\lambda''\lambda''} \longrightarrow \mathrm{Hom}(V''_{\lambda''}, V_\lambda))$$

Firstly we need to show that this set is equal to

$$\mathrm{Im}\left(\prod_{i=0}^{k} \mathscr{L}^{\rho\rho}_{\lambda^{(i)}\lambda^{(i+1)}} \times \mathscr{L}^{\rho\rho'}_{\lambda'\lambda'} \times \prod_{i=k+1}^{h} \mathscr{L}^{\rho'\rho'}_{\lambda^{(i)}\lambda^{(i+1)}} \times \mathscr{L}^{\rho'\rho''}_{\lambda''\lambda''} \to \mathrm{Hom}(V''_{\lambda''}, V_\lambda)\right) \tag{4.16}$$

and then to explain how the actions are related. The first fact is an easy consequence of Lemma 4.3.6. Indeed, by virtue of that result, the following holds

$$\begin{aligned}\mathrm{Im}(\mathscr{L}^{\rho\rho'}_{\lambda'\lambda'} \times \mathscr{L}^{\rho'\rho'}_{\lambda'\lambda^{(k+2)}} &\longrightarrow \mathrm{Hom}(V'_{\lambda^{(k+2)}}, V_{\lambda'})) = \\ &= \mathrm{Im}(\mathscr{L}^{\rho\rho}_{\lambda'\lambda^{(k+2)}} \times \mathscr{L}^{\rho\rho'}_{\lambda^{(k+2)}\lambda^{(k+2)}} \longrightarrow \mathrm{Hom}(V'_{\lambda^{(k+2)}}, V_{\lambda'}))\end{aligned}$$

where the maps are the obvious compositions. This implies that the set in 4.16 is equal to the image of the composition on the set

$$\prod_{i=0}^{k+1} \mathscr{L}^{\rho\rho}_{\lambda^{(i)}\lambda^{(i+1)}} \times \mathscr{L}^{\rho\rho'}_{\lambda^{(k+2)}\lambda^{(k+2)}} \times \prod_{i=k+2}^{h} \mathscr{L}^{\rho'\rho'}_{\lambda^{(i)}\lambda^{(i+1)}} \times \mathscr{L}^{\rho'\rho''}_{\lambda''\lambda''}$$

Now, iterating this process we conclude that the set in 4.16 is equal to the image of the composition on

$$\prod_{i=0}^{h} \mathscr{L}^{\rho\rho}_{\lambda^{(i)}\lambda^{(i+1)}} \times \mathscr{L}^{\rho\rho'}_{\lambda''\lambda''} \times \mathscr{L}^{\rho'\rho''}_{\lambda''\lambda''}$$

and by applying one last time Lemma 4.3.6 on $\mathscr{L}^{\rho\rho'}_{\lambda''\lambda''} \times \mathscr{L}^{\rho'\rho''}_{\lambda''\lambda''}$ we obtain the equality we are looking for. Now it remains to prove the relation between the actions. Given $L \in \mathscr{L}^{\rho\rho'}_{\lambda\lambda'}$ and $M \in \mathscr{L}^{\rho'\rho''}_{\lambda'\lambda''}$, we write L as composition of a certain

$$(L_0, \ldots, L_{k+1}) \in \mathscr{L}^{\rho\rho}_{\lambda\lambda^{(1)}} \times \cdots \times \mathscr{L}^{\rho\rho}_{\lambda^{(k)}\lambda'} \times \mathscr{L}^{\rho\rho'}_{\lambda'\lambda'}$$

and analogously M as composition of

$$(M_{k+2}, \ldots, M_{h+1}) \in \mathscr{L}^{\rho'\rho'}_{\lambda'\lambda^{(k+2)}} \times \cdots \times \mathscr{L}^{\rho'\rho'}_{\lambda^{(h)}\lambda''} \times \mathscr{L}^{\rho'\rho''}_{\lambda''\lambda''}$$

The element $L_{k+1} \circ M_{k+2}$ belongs to $\mathscr{L}^{\rho\rho'}_{\lambda'\lambda^{(k+2)}}$, which is equal to the image of compositions of isomorphisms in

$$\mathscr{L}^{\rho\rho}_{\lambda'\lambda^{(k+2)}} \times \mathscr{L}^{\rho\rho'}_{\lambda^{(k+2)}\lambda^{(k+2)}}$$

by virtue of Lemma 4.3.6. Then there exists a couple $(L'_{k+1}, \overline{M}_{k+2})$ in the set $\mathscr{L}^{\rho\rho}_{\lambda'\lambda^{(k+2)}} \times \mathscr{L}^{\rho\rho'}_{\lambda^{(k+2)}\lambda^{(k+2)}}$ such that $L_{k+1} \circ M_{k+2} = L'_{k+1} \circ \overline{M}_{k+2}$. In this way we have written $L \circ M$ as composition of an element in

$$\prod_{i=0}^{k+1} \mathscr{L}^{\rho\rho}_{\lambda^{(i)}\lambda^{(i+1)}} \times \mathscr{L}^{\rho\rho'}_{\lambda^{(k+2)}\lambda^{(k+2)}} \times \prod_{i=k+2}^{h} \mathscr{L}^{\rho'\rho'}_{\lambda^{(i)}\lambda^{(i+1)}} \times \mathscr{L}^{\rho'\rho''}_{\lambda''\lambda''}$$

Iterating this process, we rewrite $L \circ M$ as follows

$$\begin{aligned} L \circ M &= L_0 \circ \cdots \circ L_k \circ L_{k+1} \circ M_{k+1} \circ M_{k+2} \circ \cdots \circ M_{h+1} \\ &= L_0 \circ \cdots \circ L_k \circ L'_{k+1} \circ \overline{M}_{k+2} \circ M_{k+3} \circ \cdots \circ M_{h+1} \\ &= L_0 \circ \cdots \circ L_k \circ L'_{k+1} \circ M'_{k+2} \circ \overline{M}_{k+3} \circ \cdots \circ M_{h+1} \\ &\;\vdots \\ &= L_0 \circ \cdots \circ L_k \circ L'_{k+1} \circ M'_{k+2} \circ M'_{k+3} \circ \cdots \circ M'_{h+1} \end{aligned}$$

where

$$(L_0, \ldots, L_k, L'_{k+1}, M'_{k+2}, \ldots, M'_{h+1}) \in \mathscr{L}^{\rho\rho}_{\lambda\lambda^{(1)}} \times \cdots \times \mathscr{L}^{\rho\rho}_{\lambda^{(h)}\lambda''} \times \mathscr{L}^{\rho\rho''}_{\lambda''\lambda''}$$

so we have found a decomposition of $L \circ M$ as element of $\mathscr{L}^{\rho\rho''}_{\lambda\lambda''}$, described by the path $\lambda = \lambda^{(0)}, \ldots, \lambda^{(h+1)} = \lambda''$. The image of L under the action of an element $c \in H_1(S; \mathbb{Z}_N)$ will have the form

$$c \cdot L = (c_0 \cdot L_0) \circ \cdots \circ (c_{k+1} \cdot L_{k+1})$$

with $\sum_i c_i = c$, and analogously the image of M under the action of $d \in H_1(S; \mathbb{Z}_N)$ will appear like

$$d \cdot M = (d_{k+2} \cdot M_{k+2}) \circ \cdots \circ (d_{h+1} \cdot M_{h+1})$$

Recalling the relation between the actions in Lemma 4.3.6 we see that

$$\begin{aligned} (c_{k+1} \cdot L_{k+1}) \circ (d_{k+2} \circ M_{k+2}) &= (c_{k+1} + d_{k+2}) \cdot (L_{k+1} \circ M_{k+2}) \\ &= (c_{k+1} + d_{k+2}) \cdot (L'_{k+1} \circ \overline{M}_{k+2}) \\ &= (c_{k+1} \cdot L'_{k+1}) \circ (d_{k+2} \circ \overline{M}_{k+2}) \end{aligned}$$

This implies the following relation

$$\begin{aligned} (c \cdot L) \circ (d \cdot M) &= (c_0 \cdot L_0) \circ \cdots \circ (c_{k+1} \cdot L_{k+1}) \circ (d_{k+2} \cdot M_{k+2}) \circ \cdots \circ M_{h+1} \\ &= (c_0 \cdot L_0) \circ \cdots \circ (c_{k+1} \cdot L'_{k+1}) \circ (d_{k+2} \cdot \overline{M}_{k+2}) \circ \cdots \circ (d_{h+1} \circ M_{h+1}) \end{aligned}$$

By iterating this process as before we obtain that

$$(c \cdot L) \circ (d \cdot M) = (c_0 \cdot L_0) \circ \cdots \circ (c_{k+1} \cdot L'_{k+1}) \circ (d_{k+2} \cdot M'_{k+2}) \circ \cdots \circ (d_{h+1} \cdot M'_{h+1})$$

Now observe that the second member is equal to $(c + d) \cdot (L \circ M) \in \mathscr{L}^{\rho\rho''}_{\lambda\lambda''}$ by definition of $(\mathscr{L}^{\rho\rho''}_{\lambda\lambda''}, \psi^{\rho\rho''}_{\lambda\lambda''})$ and because

$$\sum_{i=0}^{k+1} c_i + \sum_{j=k+2}^{h+1} d_j = c + d \in H_1(S; \mathbb{Z}_N)$$

This concludes the proof. □

Theorem 4.4.2 (Uniqueness Theorem). *Suppose that* $\{\mathscr{M}^{\rho\rho'}_{\lambda\lambda'}\}$ *is a collection indexed by couples of isomorphic local representations* ρ, ρ' *of the quantum Teichmüller space* $\mathcal{T}^q_S$ *and by couples of ideal triangulations* $\lambda, \lambda' \in \Lambda(S)$ *such that*

Intertwining: *for every couple of isomorphic local representations*

$$\rho = \{\rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)\}_{\lambda\in\Lambda(S)} \qquad \rho' = \{\rho'_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V'_\lambda)\}_{\lambda\in\Lambda(S)}$$

and for every $\lambda, \lambda' \in \Lambda(S)$, $\mathscr{M}^{\rho\rho'}_{\lambda\lambda'}$ is a non-empty set of linear isomorphisms $M^{\rho\rho'}_{\lambda\lambda'}$ from $V'_{\lambda'}$ to V_λ such that

$$(\rho_\lambda \circ \Phi^q_{\lambda\lambda'})(X') = M^{\rho\rho'}_{\lambda\lambda'} \circ \rho'_{\lambda'}(X') \circ (M^{\rho\rho'}_{\lambda\lambda'})^{-1}$$

for every $X' \in \mathcal{T}^q_{\lambda'}$;

Weak Fusion property: *let R be a surface and S obtained by fusion from R. Fix*

$$\eta = \{\eta_\mu \colon \mathcal{T}^q_\mu \to \mathrm{End}(W_\mu)\}_{\mu\in\Lambda(R)} \qquad \eta' = \{\eta'_\mu \colon \mathcal{T}^q_\mu \to \mathrm{End}(W'_\mu)\}_{\mu\in\Lambda(R)}$$

two isomorphic local representations of $\mathcal{T}^q_R$ and

$$\rho = \{\rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)\}_{\lambda\in\Lambda(S)} \qquad \rho' = \{\rho'_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V'_\lambda)\}_{\lambda\in\Lambda(S)}$$

two isomorphic local representations of $\mathcal{T}^q_S$, with ρ and ρ' obtained by fusion from η and η', respectively. Then for every $\mu, \mu' \in \Lambda(R)$, if $\lambda, \lambda' \in \Lambda(S)$ are the corresponding ideal triangulations on S, the inclusion as subset of $\mathrm{Hom}(V'_{\lambda'}, V_\lambda)$ $j \colon \mathscr{M}^{\eta\eta'}_{\mu\mu'} \to \mathscr{M}^{\rho\rho'}_{\lambda\lambda'}$ is well defined;

Weak Composition property: *for every ρ, ρ', ρ'' isomorphic local representations of $\mathcal{T}^q_S$ and for every $\lambda, \lambda', \lambda'' \in \Lambda(S)$ the composition map*

$$\begin{array}{ccc} \mathscr{M}^{\rho\rho'}_{\lambda\lambda'} \times \mathscr{M}^{\rho'\rho''}_{\lambda'\lambda''} & \longrightarrow & \mathscr{M}^{\rho\rho''}_{\lambda\lambda''} \\ (M, N) & \longmapsto & M \circ N \end{array}$$

is well defined.

Then, for every ρ and ρ' isomorphic local representations and for every $\lambda, \lambda' \in \Lambda(S)$ we have

$$\mathscr{L}^{\rho\rho'}_{\lambda\lambda'} \subseteq \mathscr{M}^{\rho\rho'}_{\lambda\lambda'}$$

where $\{\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}\}$ is the family previously constructed.

Proof. Thanks to the Weak Composition property and to the surjectivity of the composition maps for the $(\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}, \psi^{\rho\rho'}_{\lambda\lambda'})$, it is sufficient to show the inclusion $\mathscr{L}^{\rho\rho'}_{\lambda\lambda'} \subseteq \mathscr{M}^{\rho\rho'}_{\lambda\lambda'}$ in the elementary cases, in which the triangulations differ by a diagonal exchange or a re-indexing. Let S be a surface and take $\lambda = \lambda' \in \Lambda(S)$, the other situation is analogous. Denote by S_0 the surface obtained by splitting S along all the edges of λ and by λ_0 the ideal triangulation induced on S_0. Moreover, we fix ρ and ρ' two isomorphic local representations of $\mathcal{T}^q_S$ and we choose two isomorphic representatives ζ_{λ_0} and ζ'_{λ_0} of ρ_λ and ρ'_λ respectively. The representations ζ_{λ_0} and ζ'_{λ_0} can be thought as local representations ζ and ζ' of the whole quantum Teichmüller space of $\mathcal{T}^q_{S_0}$, because S_0 admits the only triangulation λ_0, being a disjoint union of ideal triangles. The element ζ'_{λ_0} belongs by construction to the set $\mathscr{F}_{S_0}(\rho'_\lambda)$, so we can consider $c\cdot\zeta'_{\lambda_0}$ for every $c \in H_1(S; \mathbb{Z}_N)$. In this way, for every $c \in H_1(S; \mathbb{Z}_N)$ we obtain a local representation

$c \cdot \zeta'$ of $\mathcal{T}^q_{S_0}$ isomorphic to ζ' and that still leads by fusion to ρ'_λ. Because ζ and $c \cdot \zeta'$ are isomorphic irreducible representations of $\mathcal{T}^q_{\lambda_0}$, there exists a linear isomorphism $L^{\zeta\, c\cdot\zeta'}_{\lambda\lambda'} : V'_{\lambda'} \to V_\lambda$, unique up to multiplicative scalar, verifying

$$L^{\zeta\, c\cdot\zeta'}_{\lambda\lambda} \circ \zeta'_{\lambda_0}(X) \circ (L^{\zeta\, c\cdot\zeta'}_{\lambda\lambda})^{-1} = \zeta_{\lambda_0}(X)$$

for every $X \in \mathcal{T}^q_{\lambda_0}$, by virtue of Proposition 2.2.8. Then, because $\mathscr{M}^{\zeta\zeta'}_{\lambda_0\lambda_0}$ must be non-empty, the isomorphism $L^{\zeta\, c\cdot\zeta'}_{\lambda_0\lambda_0}$ necessarily belongs to $\mathscr{M}^{\zeta\, c\cdot\zeta'}_{\lambda_0\lambda_0}$. The weak Fusion property tells us that $\mathscr{M}^{\zeta\, c\cdot\zeta'}_{\lambda_0\lambda_0}$ is contained in $\mathscr{M}^{\rho\rho'}_{\lambda\lambda}$, so the isomorphism $L^{\zeta\, c\cdot\zeta'}_{\lambda_0\lambda_0}$ must belong to $\mathscr{M}^{\rho\rho'}_{\lambda\lambda}$ for every $c \in H_1(S;\mathbb{Z}_N)$. By definition $L^{\zeta\zeta'}_{\lambda_0\lambda_0}$ is an element of $\mathscr{L}^{\rho\rho'}_{\lambda\lambda}$, and $L^{\zeta\, c\cdot\zeta'}_{\lambda_0\lambda_0}$ coincide with $c \cdot L^{\zeta\zeta'}_{\lambda_0\lambda_0}$, where $c \in H_1(S;\mathbb{Z}_N)$ is acting on $\mathscr{L}^{\rho\rho'}_{\lambda\lambda}$. This means that, by transitivity of the action $\psi^{\rho\rho'}_{\lambda\lambda}$, the whole set $\mathscr{L}^{\rho\rho'}_{\lambda\lambda}$ is contained in $\mathscr{M}^{\rho\rho'}_{\lambda\lambda}$, which is what we were looking for. □

Lemma 4.4.3. *Let $\lambda, \lambda' \in \Lambda(S)$ be two ideal triangulations and*

$$\rho = \{\rho_\lambda : \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)\}_{\lambda\in\Lambda(S)} \qquad \rho' = \{\rho'_\lambda : \mathcal{T}^q_\lambda \to \mathrm{End}(V'_\lambda)\}_{\lambda\in\Lambda(S)}$$

two isomorphic local representations of the quantum Teichmüller space $\mathcal{T}^q_S$ of S. Then for every $c \in H_1(S;\mathbb{Z}_N)$ there exists an automorphism $B(c)$ of $V'_{\lambda'}$ with $\det B(c) = 1$, *uniquely determined up to scalar multiplication by an N-th root of unity, such that*

$$c \cdot L \doteq L \circ B(c)^{-1}$$

for every $L \in \mathscr{L}^{\rho\rho'}_{\lambda\lambda'}$.

Proof. Firstly we observe that we can assume $\lambda = \lambda'$. Indeed, every $L \in \mathscr{L}^{\rho\rho'}_{\lambda\lambda'}$ can be written as $L_0 \circ L_1$, with

$$(L_0, L_1) \in \mathscr{L}^{\rho\rho}_{\lambda\lambda'} \times \mathscr{L}^{\rho\rho'}_{\lambda'\lambda'}$$

and the element $c \cdot L$ is equal to $L_0 \circ (c \cdot L_1)$. So, by showing that the condition holds for $\mathscr{L}^{\rho\rho'}_{\lambda'\lambda'}$, we will conclude the general case.

An element $L \in \mathscr{L}^{\rho\rho'}_{\lambda'\lambda'}$ corresponds to a certain class $[\zeta, \zeta'] \in \mathscr{A}^{\rho\rho'}_{\lambda'\lambda'}$, where ζ and ζ' are representatives of $\rho_{\lambda'}$ and $\rho'_{\lambda'}$, respectively, and the following holds

$$L \circ \zeta'(X) \circ L^{-1} = \zeta(X)$$

for every $X \in \mathcal{T}^q_{\lambda'_0}$, where λ'_0 is the triangulation on the surface S'_0, obtained by splitting S along the triangulation λ'. On the other hand, for every $c \in H_1(S;\mathbb{Z}_N)$ the element $c \cdot L$ corresponds to the class $[\zeta, c \cdot \zeta']$, where $c \cdot \zeta'$ is the action of c on $\zeta' \in \mathscr{F}_{S'_0}(\rho'_{\lambda'})$. In Remark 4.1.6 we observed that ζ' and $c \cdot \zeta'$ are isomorphic and that there exists a linear isomorphism $D(c)$, described explicitly, such that

$$D(c) \circ \zeta'(X) \circ D(c)^{-1} = (c \cdot \zeta')(X)$$

for every $X \in \mathcal{T}^q_{\lambda'_0}$. These two relations imply immediately that

$$(L \circ D(c)^{-1}) \circ (c \cdot \zeta')(X) \circ (L \circ D(c)^{-1})^{-1} = \zeta(X)$$

for every $X \in \mathcal{T}^q_{\lambda_0}$, and this proves that $c \cdot L \doteq L \circ D(c)^{-1}$. By asking that $\det D(c) = 1$, we obtain a linear isomorphism $B(c)$, uniquely determined up to scalar multiplication by an N-th root of unity, verifying the requests.

More precisely, in Remark 4.1.6 we have found $D(c)$ as conjugated to linear isomorphisms that are tensor products in $\mathrm{GL}(\bigotimes_i (\mathbb{C}^N)_i)$ of isomorphisms of $\mathbb{C}^N$ obtained as compositions of the applications B_i for $i = 1, 2, 3$ and their inverses. It is immediate to see that every B_i verifies $\det(B_i) = (-1)^{N+1}$, where we are using the fact that $q^N = (-1)^{N+1}$. Furthermore, the following relation holds

$$\det(L_1 \otimes \cdots \otimes L_m) = \det(L_1)^{N^{m-1}} \cdots \det(L_1)^{N^{m-1}} \tag{4.17}$$

for every $L_1, \ldots, L_k \in \mathrm{GL}_n(\mathbb{C})$, where m is the number of triangles in an ideal triangulation of S. So, if $D(c)$ is equal to $A_1 \otimes \cdots \otimes A_m$, with A_i composition of the B_i, then $\det(A_i)$ is a certain power of $(-1)^{N+1}$ and consequently $\det(A_i)^{N^{m-1}}$ is a power of $(-1)^{N^{m-1}(N+1)}$. If $m > 1$, then $N^{m-1}(N+1)$ is even, and consequently $\det(A_i)^{N^{m-1}} = 1$, which proves that $\det(D(c)) = 1$ thanks to the relation 4.17. So actually we do not need to rescale $D(c)$ in order to obtain the additional property $\det = 1$ if $m > 1$. □

4.5 Invariants of pseudo-Anosov diffeomorphisms

We firstly need to define actions of the mapping class group $\mathcal{MCG}(S)$ of S on the sets $\mathscr{E}\mathscr{H}(S)$ and $\mathrm{Repr}_{loc}(\mathcal{T}^q_S)$, which will be very useful in what follows.

Let $[r, \{\xi_\pi\}_{\pi\in\Pi}]$ be a conjugation class of a peripherally generic enhanced homomorphism and let $\varphi\colon S \to S$ be a diffeomorphism. We define

$$[r, \{\xi_\pi\}_{\pi\in\Pi}] \cdot \varphi$$

as the conjugation class of a peripherally generic enhanced homomorphism $(s, \{\eta_\pi\}_{\pi\in\Pi})$ defined as follows:

- s is equal to the composition $r \circ \varphi_*$, where $\varphi_*\colon \pi_1(S) \to \pi_1(S)$ is the isomorphism induced by φ for an arbitrary choice of a path joining the base point of S to its image under φ;
- for every $\pi \in \Pi$ η_π is equal to $\xi_{\varphi_*(\pi)}$.

The conjugation class $[s, \{\eta_\pi\}_{\pi\in\Pi}]$ does not depend on the choices of the representative $(r, \{\xi_\pi\}_{\pi\in\Pi})$ and the path joining the base point of S to its image under φ, so this construction defines a right action of $\mathcal{MCG}(S)$ on the set $\mathscr{E}\mathscr{H}(S)$

$$\begin{array}{ccc} \mathscr{E}\mathscr{H}(S) \times \mathcal{MCG}(S) & \longrightarrow & \mathscr{E}\mathscr{H}(S) \\ ([r, \{\xi_\pi\}_\pi], [\varphi]) & \longmapsto & [r, \{\xi_\pi\}_\pi] \cdot \varphi \end{array}$$

This concludes the definition of the action on $\mathscr{E}\mathscr{H}(S)$.

If φ is a diffeomorphism of S, then for every $\lambda \in \Lambda(S)$ we denote by $\varphi(\lambda) \in \Lambda(S)$ the ideal triangulation defined by $\varphi(\lambda)_i := \varphi(\lambda_i)$ for every $i = 1, \ldots, n$. If S_0 and S_0' are the surfaces obtained from S by splitting it along the triangulations λ and $\varphi(\lambda)$ respectively, then φ induces also a diffeomorphism from S_0

to S_0'. S_0 and S_0' can be endowed with unique ideal triangulations λ_0 and λ_0', which clearly verify $\varphi(\lambda_0) = \lambda_0'$. Consequently there is a natural algebra isomorphism $\bar{\varphi}^q_{\lambda_0} \colon \mathcal{T}^q_{\lambda_0} \to \mathcal{T}^q_{\lambda_0'}$ that sends, for every i, the variable in $\mathcal{T}^q_{\lambda_0}$ corresponding to $(\lambda_0)_j$ in the variable in $\mathcal{T}^q_{\lambda_0}$ corresponding to $(\lambda_0')_j$. This implies that every local representation $\rho_{\varphi(\lambda)} \colon \mathcal{T}^q_{\varphi(\lambda)} \to \mathrm{End}(W)$ induces a local representation $\rho'_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(W)$ defined as follows: fixed a representation $\zeta_0 \in \mathscr{F}_{S_0'}(\rho_{\varphi(\lambda)})$, we take the local representation ρ'_λ represented by

$$\zeta_0 \circ \bar{\varphi}^q_\lambda : \mathcal{T}^q_{\lambda_0} \longrightarrow \mathrm{End}(W)$$

The local representation ρ'_λ just constructed does not depend on the choice of the representative ζ_0 and we will label it as $\rho_{\varphi(\lambda)} \cdot \varphi$.

Moreover, the diffeomorphism φ induces also, for every $\lambda \in \Lambda(S)$, an algebra isomorphism $\varphi^q_\lambda \colon \mathcal{T}^q_\lambda \to \mathcal{T}^q_{\varphi(\lambda)}$ which sends, for every i, the variable in $\mathcal{T}^q_\lambda$ corresponding to λ_i in the variable in $\mathcal{T}^q_{\varphi(\lambda)}$ corresponding to $\varphi(\lambda_i)$. It is clear that these isomorphisms have a good behaviour with respect to the coordinate changes $\Phi^q_{\lambda\lambda'}$. More precisely, they induce isomorphisms also on the fraction rings $\hat{\varphi}^q_\lambda \colon \widehat{\mathcal{T}}^q_\lambda \to \widehat{\mathcal{T}}^q_{\varphi(\lambda)}$ in the obvious way and the following relation holds

$$\hat{\varphi}^q_\lambda \circ \Phi^q_{\lambda\lambda'} = \Phi^q_{\varphi(\lambda)\varphi(\lambda')} \circ \hat{\varphi}^q_{\lambda'} \tag{4.18}$$

for every $\lambda, \lambda' \in \Lambda(S)$.

Given $\rho = \{\rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)\}_{\lambda \in \Lambda(S)}$ a local representation of the quantum Teichmüller space $\mathcal{T}^q_S$, we define a collection of local representations

$$\rho \cdot \varphi = \{(\rho \cdot \varphi)_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_{\varphi(\lambda)})\}_{\lambda \in \Lambda(S)}$$

by setting $(\rho \cdot \varphi)_\lambda := \rho_{\varphi(\lambda)} \cdot \varphi$ for every $\lambda \in \Lambda(S)$, where $\rho_{\varphi(\lambda)} \cdot \varphi$ is constructed with the procedure described above. It is simple to see that the relation 4.18 implies that the collection $\rho \cdot \varphi$ is indeed a local representation of the quantum Teichmüller space $\mathcal{T}^q_S$. Hence we have described also a right action

$$\begin{array}{ccc} \mathrm{Repr}_{loc}(\mathcal{T}^q_S) \times \mathcal{MCG}(S) & \longrightarrow & \mathrm{Repr}_{loc}(\mathcal{T}^q_S) \\ (\rho, [\varphi]) & \longmapsto & \rho \cdot \varphi \end{array}$$

In addition, observe that the isomorphisms φ^q_λ send the central element H_λ of $\mathcal{T}^q_\lambda$ in the central element $H_{\varphi(\lambda)}$ of $\mathcal{T}^q_{\varphi(\lambda)}$. Therefore, the central load of the representation $\rho \cdot \varphi$ is the same of the one of ρ.

Recall that in Theorem 3.3.13 we have shown the existence of a surjective map

$$\begin{array}{cccc} \Theta : & \mathscr{R}_{loc}(\mathcal{T}^q_S) & \longrightarrow & \mathscr{E}\mathscr{H}(S) \\ & [\rho] & \longmapsto & [r, \{z_\pi\}_\pi] \end{array}$$

that sends each isomorphism class $[\rho]$ in the conjugation class of its hyperbolic shadow $[r, \{z_\pi\}_\pi]$and that has the fibre on $[r, \{z_\pi\}_\pi]$ composed of the N isomorphisms classes of representations, one for each possibile central loads, which are the N-th roots of $x_1 \cdots x_n$.

Lemma 4.5.1. *The following relation holds*

$$\Theta([\rho] \cdot \varphi) = \Theta([\rho]) \cdot \varphi$$

where φ is acting on $\mathscr{R}_{loc}(\mathcal{T}^q_S)$ in the first member and on $\mathscr{E}\mathscr{H}(S)$ in the second.

Proof. Let $\widetilde{\varphi}\colon \widetilde{S} \to \widetilde{S}$ a certain lift of φ on the universal covering. Recalling Proposition 3.3.9 and Theorem 3.3.13, if ρ is equal to a collection $\{\rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)\}_{\lambda \in \Lambda(S)}$ and $[r, \{\xi_\pi\}_{\pi_\Pi}]$ is its hyperbolic shadow, then for every $\lambda \in \Lambda(S)$ we can find a pleated surface with bending locus λ associated with ρ_λ that looks like $(\tilde{f}_\lambda, r)$. Then, by inspection of the shear-bend coordinates, the pleated surface $(\tilde{f}_{\varphi(\lambda)} \circ \widetilde{\varphi}, r \circ \varphi_*)$ is a pleated surface with bending locus λ associated with the representation $\rho_{\varphi(\lambda)} \cdot \varphi$. Following the definitions, this fact implies the assertion. □

Let $\varphi\colon S \to S$ be a diffeomorphism of the surface S. Denote by M_φ the mapping torus of φ, which is the 3-manifold obtained as quotient of $S \times \mathbb{R}$ by the group of diffeomorphisms generated by

$$\begin{array}{rccc} \tau_\varphi : & S \times \mathbb{R} & \longrightarrow & S \times \mathbb{R} \\ & (p,t) & \longmapsto & (\varphi(p), t+1) \end{array}$$

We define also the inclusion

$$\begin{array}{rccc} i : & S & \longrightarrow & M_\varphi \\ & p & \longmapsto & [p,0] \end{array}$$

Observe that the homomorphism $i_*\colon \pi_1(S) \to \pi_1(M_\varphi)$ induced by i is injective. Indeed, assume that $\gamma\colon S^1 \to S$ is a closed path such that $i \circ \gamma$ it homotopically trivial in M_φ. Then there exists a map $f\colon D^2 \to M_\varphi$ such that $f|_{S^1} = i \circ \gamma$. Because D^2 is simply connected, we can lift f to an application $\tilde{f}\colon D^2 \to S \times \mathbb{R}$ such that

- $\pi \circ \tilde{f} = f$, where $\pi\colon S \times \mathbb{R} \to M_\varphi$ is the projection map;
- $\tilde{f}(S^1) \subset S \times \{0\}$.

By construction we have $\tilde{f}|_{S^1} = \gamma \times \{0\}\colon S^1 \to S \times \{0\}$. If $p\colon S \times \mathbb{R} \to S \times \{0\}$ is the obvious projection on the first component, then $p \circ \tilde{f}$ is a continuous map from D^2 to $S \times \{0\}$ whose restriction to S^1 is equal to $\gamma \times \{0\}$. This proves that γ was homotopically trivial also in S, hence that i_* is injective. Moreover, by construction of the mapping torus, the maps i and $i \circ \varphi$ are homotopic, hence $i_* = i_* \circ \varphi_*$.

By virtue of the Thurston's Hyperbolization Theorem, the mapping torus M_φ admits a complete finite-volume hyperbolic structure if and only if the diffeomorphism φ is isotopic to a pseudo-Anosov diffeomorphism (see [Thu83] and [Ota96]). Assume that φ is pseudo-Anosov and let $r\colon \pi_1(M_\varphi) \to \mathbb{P}\mathrm{SL}_2(\mathbb{C})$ be the holonomy of the complete structure on M_φ, which is unique up to conjugation in $\mathbb{P}\mathrm{SL}_2(\mathbb{C})$ because of the Mostow Rigidity Theorem. From r we obtain the homomorphism $r_\varphi\colon \pi_1(S) \to \mathbb{P}\mathrm{SL}_2(\mathbb{C})$, defined as the composition $r \circ i_*$. Thanks to the injectivity of i_*, we have that r_φ is also injective. Moreover, because r is the holonomy of a complete finite-volume structure, r_φ leads every peripheral subgroup π of $\pi_1(S)$ in a parabolic subgroup of $\mathbb{P}\mathrm{SL}_2(\mathbb{C})$. This means that r_φ admits a unique enhancement $\{\xi_\pi\}_{\pi \in \Pi}$. Observe also that $(r_\varphi, \{\xi_\pi\}_{\pi \in \Pi})$ is peripherally generic, because of Lemma 3.3.6. Therefore a pseudo-Anosov diffeomorphism φ produces a unique conjugation class of enhanced homomorphisms from $\pi_1(S)$ to $\mathbb{P}\mathrm{SL}_2(\mathbb{C})$ peripherally generic $[r_\varphi, \{\xi_\pi\}_{\pi \in \Pi}] \in \mathscr{E}\mathscr{H}(S)$. By virtue of Proposition 3.3.10, there exists a unique non-quantum representation ρ^1_φ of $\mathcal{T}^1_S$ related to $[r_\varphi, \{\xi_\pi\}_{\pi \in \Pi}]$.

Lemma 4.5.2. *The non-quantum representation ρ^1_φ verifies*

$$(\rho^1_\varphi)_\lambda(H) = (\rho^1_\varphi)_\lambda(P_i) = id_{\mathbb{C}}$$

for every $\lambda \in \Lambda(S)$ and for every $i = 1, \ldots, p$.

Proof. See [BL07, Lemma 39] □

As consequence of this lemma, the isomorphism classes of local representations in $\Theta^{-1}([r_\varphi, \{\xi_\pi\}_{\pi\in\Pi}])$ are classified by their central loads, which are all the possible N-th roots of unity. So, for every $k \in \mathbb{Z}_N$, we have found a unique isomorphism class of local representations $[\rho^k_\varphi]$ having $[r_\varphi, \{\xi_\pi\}_{\pi\in\Pi}]$ as hyperbolic shadow and q^{2k} as central load. Because $i_* \circ \varphi_* = i_*$ and because r_φ admits a unique enhancement, we have

$$[r_\varphi, \{\xi_\pi\}_{\pi\in\Pi}] \cdot \varphi = [r_\varphi, \{\xi_\pi\}_{\pi\in\Pi}]$$

In other words, the element $[r_\varphi, \{\xi_\pi\}_{\pi\in\Pi}]$ in $\mathscr{E}\mathscr{H}(S)$ is fixed by the right action of φ. Hence we deduce, recalling Lemma 4.5.1, that the action of φ on $\mathscr{R}_{loc}(S)$ preserves the set $\Theta^{-1}([r_\varphi, \{\xi_\pi\}_{\pi\in\Pi}])$. Furthermore, we have seen that the isomorphisms φ^q_λ preserve the central element H, so the action of φ on $\Theta^{-1}([r_\varphi, \{\xi_\pi\}_{\pi\in\Pi}])$ is necessarily the trivial one. We have proved

Theorem 4.5.3. *Let S be a closed surface with punctures. Fix q a N-th primitive root of $(-1)^{N+1}$, $k \in \mathbb{Z}_N$ and $\varphi\colon S \to S$ a pseudo-Anosov diffeomorphism. Then there exists a conjugation class of peripherally generic enhanced homomorphism $[r_\varphi, \{\xi_\pi\}_{\pi\in\Pi}]$, uniquely determined by φ, which is fixed by the action of φ on $\mathscr{E}\mathscr{H}(S)$. Moreover, the set $\Theta^{-1}([r_\varphi, \{\xi_\pi\}_{\pi\in\Pi}])$ is composed of exactly such isomorphisms classes $[\rho^k_\varphi]$ of local representations of $\mathcal{T}^q_S$ having $[r_\varphi, \{\xi_\pi\}_{\pi\in\Pi}]$ as hyperbolic shadow and q^{2k} as central load, and each $[\rho^k_\varphi]$ is fixed by the action of φ on $\mathscr{R}_{loc}(S)$.*

Henceforth, the numbers $q \in \mathbb{C}^*$ and $k \in \mathbb{Z}_N$ will be fixed. Choosing a representative ρ of the class $[\rho^k_\varphi]$, Theorem 4.5.3 implies that the representations ρ and $\rho \cdot \varphi$ are isomorphic as local representations of $\mathcal{T}^q_S$. Therefore, we can consider the family $\{(\mathscr{L}^{\rho\cdot\varphi\,\rho}_{\lambda\lambda'}, \psi^{\rho\cdot\varphi\,\rho}_{\lambda\lambda'})\}$ of intertwining operators between $\rho \cdot \varphi$ and ρ.

In particular, fixing $\lambda \in \Lambda(S)$, we can consider the element

$$(\mathscr{L}^{\rho\cdot\varphi\,\rho}_{\lambda\,\varphi(\lambda)}, \psi^{\rho\cdot\varphi\,\rho}_{\lambda\,\varphi(\lambda)})$$

Observe that both the representations $(\rho \cdot \varphi)_\lambda$ and $\rho_{\varphi(\lambda)}$ go in $\mathrm{End}(V_{\varphi(\lambda)})$, so the set $\mathscr{L}^{\rho\cdot\varphi\,\rho}_{\lambda\,\varphi(\lambda)}$ is contained in $\mathrm{GL}(V_{\varphi(\lambda)})$.

The element $(\mathscr{L}^{\rho\cdot\varphi\,\rho}_{\lambda\,\varphi(\lambda)}, \psi^{\rho\cdot\varphi\,\rho}_{\lambda\,\varphi(\lambda)})$ depends on the chosen representative ρ of the isomorphism class $[\rho^k_\varphi]$ and on the ideal triangulation λ. We want to produce a more intrinsic object.

Definition 4.5.4. Let $\rho = \{\rho_\lambda\colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)\}_{\lambda\in\Lambda(S)}$ be a local representation of $\mathcal{T}^q_S$. Fix $\mu \in \Lambda(S)$ an ideal triangulation and a tensor-split linear isomorphism

$$M = M_1 \otimes \cdots \otimes M_m : \bigotimes_{j=1}^m V_{\mu,j} = V_\mu \longrightarrow W = \bigotimes_{j=1}^m W_j$$

Then we denote by $M \bullet_\mu \rho$ the local representation defined as follows

$$(M \bullet_\mu \rho)_\lambda := \begin{cases} \rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda) & \textit{if } \lambda \neq \mu \\ M \circ \rho_\mu(\cdot) \circ M^{-1} \colon \mathcal{T}^q_\mu \to \mathrm{End}(W) & \textit{if } \lambda = \mu \end{cases}$$

Observe that $M \circ \rho_\mu(\cdot) \circ M^{-1} \colon \mathcal{T}^q_\mu \to \mathrm{End}(W)$ is a local representation because M is tensor splitting.

Lemma 4.5.5. *Let $\rho = \{\rho_\lambda \colon \mathcal{T}^q_\lambda \to \mathrm{End}(V_\lambda)\}_{\lambda \in \Lambda(S)}$ be a local representation of the quantum Teichmüller space $\mathcal{T}^q_S$. The following hold:*

1. *let $\mu \in \Lambda(S)$ be an ideal triangulation and $M \colon V_\mu \to W$ a tensor-split isomorphism. Then M belongs to $\mathscr{L}^{M \bullet_\mu \rho\ \rho}_{\mu\mu}$;*
2. *let $\varphi \colon S \to S$ be a diffeomorphism, $\lambda \in \Lambda(S)$ be an ideal triangulation and $M \colon V_{\varphi(\lambda)} \to W$ a tensor-split isomorphism. Then*

$$M \bullet_\lambda (\rho \cdot \varphi) = (M \bullet_{\varphi(\lambda)} \rho) \cdot \varphi$$

Proof. We will focus on one point at time.
1. As usual, we denote by S_0 the surface obtained by splitting S along μ and by μ_0 the ideal triangulation on S_0 induced by μ. Fixed an element $\zeta_\mu \in \mathscr{F}_{S_0}(\rho_\mu)$, a representative of $(M \bullet_\mu \rho)_\mu$ is given by $\zeta'_\mu = M \circ \zeta_\mu(\cdot) \circ M^{-1}$ (it is a well defined representation of $\mathcal{T}^q_{\mu_0}$ because M is tensor-split). Clearly M sends ζ_μ in ζ'_μ. Hence we have that the couple (ζ'_μ, ζ_μ) belongs to $\mathscr{F}_{S_0}((M \bullet_\mu \rho)_\mu) \times \mathscr{F}_{S_0}(\rho_\mu)$, so we have found an element $[\zeta'_\mu, \zeta_\mu] \in \mathscr{A}^{M \bullet_\mu \rho\ \rho}_{\mu\mu}$ that corresponds to M, which clearly implies $M \in \mathscr{L}^{M \bullet_\mu \rho\ \rho}_{\mu\mu}$.
2. Denote by S'_0 and S''_0 the surfaces obtained by splitting S along the ideal triangulations λ and $\varphi(\lambda)$ respectively, endowed with the ideal triangulations λ_0 and $\varphi(\lambda_0)$. Recall that the diffeomorphism φ induces an isomorphism $\overline{\varphi}^q_{\lambda_0}$ from $\mathcal{T}^q_{\lambda_0}$ to $\mathcal{T}^q_{\varphi(\lambda_0)}$. Fixed $\zeta \in \mathscr{F}_{S''_0}(\rho_{\varphi(\lambda)})$, a representative of the representation $(M \bullet_\lambda (\rho \cdot \varphi))_\lambda$ is given by

$$\begin{aligned} M \circ (\zeta \cdot \varphi)(\cdot) \circ M^{-1} &= M \circ (\zeta \circ \overline{\varphi}^q_{\lambda_0})(\cdot) \circ M^{-1} \\ &= M \circ (\zeta(\overline{\varphi}^q_{\lambda_0}(\cdot)) \circ M^{-1} \\ &= (M \circ \zeta(\cdot) \circ M^{-1}) \cdot \varphi \end{aligned}$$

Now observe that $M \circ \zeta(\cdot) \circ M^{-1}$ is a representative of $(M \bullet_{\varphi(\lambda)} \rho)_{\varphi(\lambda)}$ and consequently $(M \circ \zeta(\cdot) \circ M^{-1}) \cdot \varphi$ is a representative of

$$(M \bullet_{\varphi(\lambda)} \rho)_{\varphi(\lambda)} \cdot \varphi = ((M \bullet_{\varphi(\lambda)} \rho) \cdot \varphi)_\lambda$$

Hence we have proved

$$(M \bullet_\lambda (\rho \cdot \varphi))_\lambda = ((M \bullet_{\varphi(\lambda)} \rho) \cdot \varphi)_\lambda$$

It is simple to see that on all the other triangulations these representations are obviously equal, because they coincide with $\rho \cdot \varphi$ on them, so the proof of the second assertion is done. □

Recall that we are interested in the construction of an intrinsic object starting from

$$(\mathscr{L}_{\lambda\,\varphi(\lambda)}^{\rho\cdot\varphi\,\rho}, \psi_{\lambda\,\varphi(\lambda)}^{\rho\cdot\varphi\,\rho})$$

In order to do so, we choose a tensor-split isomorphism M from $V_{\varphi(\lambda)}$ to a fixed vector space $W = \bigotimes_i W_i$. For example, a natural choice of W is

$$W := \bigotimes_{i=1}^{m} (\mathbb{C}^N)_i$$

Now we modify the representations $\rho \cdot \varphi$ and ρ by taking $M \bullet_\lambda (\rho \cdot \varphi)$ and $M \bullet_{\varphi(\lambda)} \rho$. Considering now the object

$$i(\varphi; \lambda, \rho, M) := (\mathscr{L}_{\lambda\,\varphi(\lambda)}^{M\bullet_\lambda(\rho\cdot\varphi)\ M\bullet_{\varphi(\lambda)}\rho}, \psi_{\lambda\,\varphi(\lambda)}^{M\bullet_\lambda(\rho\cdot\varphi)\ M\bullet_{\varphi(\lambda)}\rho})$$

we obtain a set of isomorphisms in $\mathrm{GL}(W)$ and so automorphisms of a vector space that is independent from the choices done. The following study will be focused on the investigation of the dependence of $i(\varphi; \lambda, \rho, M)$ on

- the representative ρ of the isomorphism class $[\rho_\varphi^k]$;
- the tensor split isomorphism $M\colon V_{\varphi(\lambda)} \to W$;
- the ideal triangulation $\lambda \in \Lambda(S)$.

Observe that the set $\mathscr{L}_{\lambda\lambda'}^{\rho\rho'}$ does depend only on the local representations ρ_λ and $\rho'_{\lambda'}$. In particular, in the case of $i(\varphi; \lambda, \rho, M)$, the only local representations involved are $M \circ \rho_{\varphi(\lambda)}(\cdot) \circ M^{-1}$ and $M \circ (\rho_{\varphi(\lambda)} \circ \varphi_\lambda^q)(\cdot) \circ M^{-1}$. Therefore, fixed $\lambda \in \Lambda(S)$, $i(\varphi; \lambda, \rho, M)$ depends only on M and $\rho_{\varphi(\lambda)}$. By virtue of the first point of Lemma 4.5.5, the representation $M \bullet_{\varphi(\lambda)} \rho$ is isomorphic to ρ, in particular it belongs to $[\rho_\varphi^k]$, and it has the property that $(M \bullet_{\varphi(\lambda)} \rho)_{\varphi(\lambda)}$ has values in $\mathrm{End}(W)$. Moreover, if ρ' is a representation of $\mathcal{T}_S^q$ isomorphic to ρ such that $\rho'_{\varphi(\lambda)}$ has values in $\mathrm{End}(W)$, then $\rho'_{\varphi(\lambda)}$ is equal to $\rho_{\varphi(\lambda)} \bullet_{\varphi(\lambda)} M'$ for some $M'\colon V_{\varphi(\lambda)} \to W$ (it is sufficient to take $M' \in \mathscr{L}_{\varphi(\lambda)\,\varphi(\lambda)}^{\rho'\rho}$). The second point of Lemma 4.5.5 tells us that the representation $M \bullet_\lambda (\rho \cdot \varphi)$ is just the image under the action of φ of $M \bullet_{\varphi(\lambda)} \rho$. Putting together these observations, we have proved that every $i(\varphi; \lambda, \rho, M)$ is equal to $i(\varphi; \lambda, M \bullet_{\varphi(\lambda)} \rho, id)$. Moreover, for every ρ' and ρ'' representations in $[\rho_\varphi^k]$ such that $\rho'_{\varphi(\lambda)}$ and $\rho''_{\varphi(\lambda)}$ are equal and have values in $\mathrm{End}(W)$, we have

$$i(\varphi; \lambda, \rho', id) = i(\varphi; \lambda, \rho'', id)$$

Henceforth, fixed $\lambda \in \Lambda(S)$ and W, the study of the objects $i(\varphi; \lambda, \rho, M)$ can be reduced to the investigations of the elements

$$j(\varphi; \lambda, \rho) := i(\varphi; \lambda, \rho, id) = (\mathscr{L}_{\lambda\,\varphi(\lambda)}^{\rho\cdot\varphi\,\rho}, \psi_{\lambda\,\varphi(\lambda)}^{\rho\cdot\varphi\,\rho})$$

where ρ is a local representation of $\mathcal{T}_S^q$ having $\rho_{\varphi(\lambda)}$ with values in $\mathrm{End}(W)$.

Lemma 4.5.6. *Let $\lambda, \lambda' \in \Lambda(S)$ be two ideal triangulations of S and ρ, ρ' two local representations of $\mathcal{T}_S^q$. Then for every $L \in \mathscr{L}_{\varphi(\lambda')\,\varphi(\lambda)}^{\rho'\rho}$ the application*

$$\begin{array}{rccc} f_L : & \mathscr{L}_{\lambda\,\varphi(\lambda)}^{\rho\cdot\varphi\,\rho} & \longrightarrow & \mathscr{L}_{\lambda'\,\varphi(\lambda')}^{\rho'\cdot\varphi\,\rho'} \\ & A & \longmapsto & L \circ A \circ L^{-1} \end{array}$$

is well defined and bijective. Moreover, it respects the actions, i.e. $f_L(c \cdot A) = c \cdot f_L(A)$ *for every* $A \in \mathscr{L}^{\rho\cdot\varphi\,\rho}_{\lambda\,\varphi(\lambda)}$ *and* $c \in H_1(S;\mathbb{Z}_N)$.

Proof. Because L is an element of $\mathscr{L}^{\rho'\rho}_{\varphi(\lambda')\,\varphi(\lambda)}$, the composition $A \circ L^{-1}$ belongs to $\mathscr{L}^{\rho\cdot\varphi\,\rho'}_{\lambda\,\varphi(\lambda')}$. If L belongs also $\mathscr{L}^{\rho'\cdot\varphi\ \rho\cdot\varphi}_{\lambda'\lambda}$, then the assertion is an obvious corollary of the Composition property of the family $\{(\mathscr{L}^{\rho\rho'}_{\lambda\lambda'}, \psi^{\rho\rho'}_{\lambda\lambda'})\}$. It is sufficient to show that $L \in \mathscr{L}^{\rho'\cdot\varphi\ \rho\cdot\varphi}_{\lambda'\lambda}$ when λ, λ' differ by an elementary move. We will investigate the diagonal exchange case $\lambda' = \Delta_i(\lambda)$, the other possibilities are simpler to study.

Firstly we need to introduce the notations: we denote by S_0 the surface obtained by splitting S along all the edges of λ except for λ_i and we label its ideal triangulations as λ_0 and λ_0', the first corresponding to $\lambda \in \Lambda(S)$ and the second to $\lambda' \in \Lambda(S)$. Moreover, $\varphi(S_0)$ will be the surface obtained by splitting S along all the edges of $\varphi(\lambda)$ except for $\varphi(\lambda_i)$ and $\varphi(\lambda_0)$, $\varphi(\lambda_0')$ its ideal triangulations induced by $\varphi(\lambda)$ and $\varphi(\lambda')$ respectively.

Because L is in $\mathscr{L}^{\rho'\rho}_{\varphi(\lambda')\,\varphi(\lambda)}$, there exists a couple

$$(\zeta', \zeta) \in \mathscr{F}_{\varphi(S_0)}(\rho'_{\varphi(\lambda')}) \times \mathscr{F}_{\varphi(S_0)}(\rho_{\varphi(\lambda)})$$

such that

$$L \circ \zeta(X) \circ L^{-1} = (\zeta' \circ \Phi^q_{\varphi(\lambda_0')\varphi(\lambda_0)})(X)$$

for every $X \in \mathcal{T}^q_{\varphi(\lambda_0)}$. Because ζ represents the local representation $\rho_{\varphi(\lambda)}$, by composing it with the natural algebra isomorphism $\overline{\varphi}^q_{\lambda_0} : \mathcal{T}^q_{\lambda_0} \to \mathcal{T}^q_{\varphi(\lambda_0)}$ we obtain a representative of the local representation $(\rho \cdot \varphi)_\lambda$. The same argument shows that $\zeta' \circ \overline{\varphi}^q_{\lambda_0'}$ is a representative of $(\rho' \cdot \varphi)_{\lambda'}$. This means that the couple $(\zeta' \circ \overline{\varphi}^q_{\lambda_0'}, \zeta \circ \overline{\varphi}^q_{\lambda_0})$ belongs to

$$\mathscr{F}_{S_0}((\rho' \cdot \varphi)_{\lambda'}) \times \mathscr{F}_{S_0}((\rho \cdot \varphi)_\lambda)$$

Moreover, the following holds

$$\begin{aligned} L \circ (\zeta \circ \overline{\varphi}^q_{\lambda_0})(Y) \circ L^{-1} &= L \circ \zeta(\overline{\varphi}^q_{\lambda_0}(Y)) \circ L^{-1} \\ &= (\zeta' \circ \Phi^q_{\varphi(\lambda_0')\varphi(\lambda_0)})(\overline{\varphi}^q_{\lambda_0}(Y)) \\ &= (\zeta' \circ \overline{\varphi}^q_{\lambda_0'} \circ \Phi^q_{\lambda_0'\lambda_0})(Y) \end{aligned}$$

So the equivalence class $[\zeta' \circ \overline{\varphi}^q_{\lambda_0'}, \zeta \circ \overline{\varphi}^q_{\lambda_0}]$ belongs to $\mathscr{A}^{\rho'\cdot\varphi\ \rho\cdot\varphi}_{\lambda'\lambda}$ and corresponds in $\mathscr{L}^{\rho'\cdot\varphi\ \rho\cdot\varphi}_{\lambda'\lambda}$ exactly to the isomorphism L. This shows that $L \in \mathscr{L}^{\rho'\cdot\varphi\ \rho\cdot\varphi}_{\lambda'\lambda}$ and so the assertion. □

For every $\lambda \in \Lambda(S)$ and for every local representation ρ such that $\rho_{\varphi(\lambda)}$ has values in $\mathrm{End}(W)$, the couples $(\mathscr{L}^{\rho\cdot\varphi\,\rho}_{\lambda\,\varphi(\lambda)}, \psi^{\rho\cdot\varphi\,\rho}_{\lambda\,\varphi(\lambda)})$ verifies

- the set $\mathscr{L}^{\rho\cdot\varphi\,\rho}_{\lambda\,\varphi(\lambda)}$ in contained in $\mathrm{GL}(W)$;
- $H_1(S;\mathbb{Z}_N)$ acts on $\mathscr{L}^{\rho\cdot\varphi\,\rho}_{\lambda\,\varphi(\lambda)}$ by right multiplication via certain tensor-split isomorphisms of W, uniquely determined up to multiplication by an N-th root of unity.

We will say that two such couples are *conjugate* if there exists an element L in $\mathrm{GL}(W)$ such that

- the conjugation map $f_L\colon \mathrm{GL}(W) \to \mathrm{GL}(W)$ that sends A in LAL^{-1} is a bijection between the sets;
- f_L commutes with the actions.

Two couples are tensor-split conjugate if there exists an automorphism L like above that is tensor-split.

Theorem 4.5.7. *The conjugacy class of a set $i(\varphi; \lambda, \rho, M)$ does not depend on the tensor-split isomorphism $M\colon V_{\varphi(\lambda)} \to W$, on the representation $\rho \in [\rho^k_\varphi]$ and on $\lambda \in \Lambda(S)$.*

Proof. We have seen that it is sufficient to study the sets $j(\varphi; \lambda, \rho)$ with $\rho_{\varphi(\lambda)}$ with values in $\mathrm{End}(W)$. Fixing λ and varying ρ in the sets of local representations in $[\rho^k_\varphi]$ with values in $\mathrm{End}(W)$, Lemma 4.5.6 with $\lambda = \lambda'$ shows that the tensor-split conjugacy class does not change (when $\lambda = \lambda'$ the isomorphism $L \in \mathscr{L}^{\rho'\rho}_{\varphi(\lambda)\,\varphi(\lambda)}$ is tensor-split). So the tensor-split conjugacy class of $j(\varphi; \lambda, \rho)$ depends only on λ.

Now choose two different ideal triangulations and two representations ρ, ρ' such that $\rho_{\varphi(\lambda)}$ and $\rho'_{\varphi(\lambda')}$ have values in $\mathrm{End}(W)$. Because $i(\varphi; \lambda, \rho)$ does not depend on $\rho_{\varphi(\lambda')}$, we can assume that also $\rho_{\varphi(\lambda')}$ has values in $\mathrm{End}(W)$, by replacing ρ with $M \bullet_{\varphi(\lambda')} \rho$ for a certain M. Analogously we can assume that $\rho'_{\varphi(\lambda)}$ has values in $\mathrm{End}(W)$. Now we have that each element L of $\mathscr{L}^{\rho'\rho}_{\varphi(\lambda')\,\varphi(\lambda)}$ belongs to $\mathrm{GL}(W)$. Hence, by applying Lemma 4.5.6 on a fixed L in $\mathscr{L}^{\rho'\rho}_{\varphi(\lambda')\,\varphi(\lambda)}$, we have that $j(\varphi; \lambda, \rho)$ and $j(\varphi; \lambda', \rho')$ are conjugated (not necessarily tensor-split conjugated) and this finally proves the announced result. $\square$

We will denote by $I(q, k, \varphi)$ this conjugacy class, depending only on the primitive N-th root of unity q, the number $k \in \mathbb{Z}_N$ and the pseudo-Anosov diffeomorphism φ.

Explicitly, in order to obtain the invariant of the diffeomorphism φ with q and k fixed, we can proceed as follows

1. we fix an ideal triangulation $\lambda \in \Lambda(S)$ and a local representation $\rho \in [\rho^k_\varphi]$;
2. we possibly replace ρ with a representation ρ' such that $\rho_{\varphi(\lambda)}$ has values in $\mathrm{End}(W)$. We can also assume that $\rho_{\varphi(\lambda)}$ is in a standard situation. More precisely, we can choose ρ such that every representative of $\rho_{\varphi(\lambda)}$ is the tensor product of triangle representations that are in the standard form described in the proof of Proposition 2.1.4. Observe that, if $\rho_{\varphi(\lambda)}$ is in standard position, the same holds for the representation $(\rho \cdot \varphi)_\lambda$;
3. we fix a sequence of ideal triangulations $\lambda = \lambda^{(0)}, \ldots, \lambda^{(k)} = \varphi(\lambda)$ leading from λ to $\varphi(\lambda)$ and we find through it an element L of $\mathscr{L}^{\rho\cdot\varphi\,\rho}_{\lambda\,\varphi(\lambda)}$. The other elements of $\mathscr{L}^{\rho\cdot\varphi\,\rho}_{\lambda\,\varphi(\lambda)}$ can be produced as $L \circ B(c)^{-1}$ for varying $c \in H_1(S; \mathbb{Z}_N)$, where $B(c)$ is an element of $\mathrm{GL}(W)$ as described in Lemma 4.4.3;

4. we take the conjugacy class of this set, which will have the following form
$$\{C \circ (H_1(S;\mathbb{Z}_N) \cdot L) \circ C^{-1} \mid C \in \mathrm{GL}(W)\}$$
where
$$C \circ (H_1(S;\mathbb{Z}_N) \cdot L) \circ C^{-1} = \{C \circ L \circ B(c)^{-1} \circ C^{-1} \mid c \in H_1(S;\mathbb{Z}_N)\}$$

This is the resulting invariant for the pseudo-Anosov diffeomorphism, having chosen q a primitive N-th root of unity and k a certain element of $\mathbb{Z}_N$.

APPENDIX A

Algebraic notions

In what follows, we will always assume a ring is has an identity and it is not necessarily commutative.

Definition A.1. Let R be a ring. R is an *integral domain* if $R \setminus \{0\}$ is multiplicatively closed.

An element s of R is said to be a *unit* if there exists a $s' \in R$ such that $ss' = s's = 1$. We denote with R^* the set of all units of R.

A *regular element* in R is any non-zero divisor, i.e. any element $a \in R$ such that, for every $b \in B \setminus \{0\}$, both ab and ba are non-zero.

Theorem A.2 ([GWJ04, Theorem 1.9])**.** *Let $R[X]$ be a polynomial ring in one indeterminate. If the coefficient ring R is right (left) Noetherian, the so is $R[X]$.*

Corollary A.3. *For every field $\mathbb{K}$ and for every $n \in \mathbb{N}$, the ring $\mathbb{K}[X_1, \dots, X_n]$ is a bilateral Noetherian ring.*

Definition A.4. Let R be a ring and X a subset of R. We will say that X is a *multiplicative set* of R if X is multiplicative closed and $1 \in X$.

Furthermore, X satisfies the *right Ore condition* if, for each $x \in X$ and $r \in R$, there exist $y \in X$ and $s \in R$ such that $ry = xs$. A multiplicative set X satisfying the right Ore condition is called a *right Ore set*.

Definition A.5. Let R be a ring and $X \subset R$ a multiplicative set. A *right ring of fractions* for R with respect to X is any overring $S \supseteq R$ such that:

- every element of X is a unit in S;
- every element of S can be written as ax^{-1}, for some $a \in R$ and $x \in X$.

Remark A.6. Let X be a right Ore set of regular elements in a ring R. We define a relation on $R \times X$ as follows: we say that two couples (a, x), $(b, y) \in R \times X$ are $\sim$-equivalent if there exist $c, d \in R$ such that $ac = bd$ and $xc = yd \in X$. Then $\sim$ is an equivalence relation. We denote with S the set of equivalence classes of $R \times X$ with respect to $\sim$.

Given $[a, x]$, $[b, y]$ elements of S, choose $c, d \in R$ such that $xc = yd$ and set

$$[a, x] + [b, y] := [ac + bd, xc]$$

Given $[a, x]$, $[b, y]$ elements of S, choose $c \in R$ and $z \in X$ such that $bz = xc$ and set

$$[a, x] \cdot [b, y] := [ac, yz]$$

The above operations are well defined and they give S a ring structure. The map $R \ni r \mapsto [r, 1] \in S$ is an injective ring homomorphism. Also, if we identify R with $i(R)$ in S, then S becomes a right ring of fractions for R with respect to X.

Theorem A.7 ([GWJ04, Theorem 6.2])**.** *Let R be a ring and $X \subset R$ a multiplicative set of regular elements. Then there exists a right ring of fractions for R with respect to X if and only if X is a right Ore set.*

Proposition A.8 ([GWJ04, Proposition 6.3])**.** *Let R be a ring, $X \subset R$ a right Ore set of regular elements and S a right ring of fractions for R with respect to X. Suppose that $\phi\colon R \to T$ is a ring homomorphism such that $\phi(X) \subseteq T^*$, then there exists a unique extension of ϕ to a ring homomorphism $\widetilde{\phi}\colon S \to T$.*

Corollary A.9 ([GWJ04, Corollary 6.4])**.** *Let R be a ring, $X \subset R$ a right Ore set of regular elements and S, S' right rings of fractions for R with respect to X. Then the identity map on R extends uniquely to an isomorphism of S onto S'.*

Definition A.10. If R is a ring and $X \subseteq R$ a right Ore set of regular elements, we shall write RX^{-1} to denote any right ring of fractions for R with respect to X. Similarly, we shall write $Y^{-1}R$ for a left ring of fractions.

Proposition A.11 ([GWJ04, Theorem 1.9])**.** *Let R be a ring and $X \subseteq R$ a right and left Ore set of regular elements. Then $RX^{-1} = X^{-1}R$, that is, any right ring of fractions for R with respect to X is also a left ring of fractions for R with respect to X and vice versa.*

Definition A.12. A *classical right quotient ring* for a ring R is a right ring of fractions for R with respect to the set of all regular elements in R.

Definition A.13. A *right Ore integral domain* is any domain R in which the non-zero elements form a Ore set, i. e. for each $x, y \in R\backslash\{0\}$ there exist $r, s \in R$ such that $xr = ys \neq 0$.

Proposition A.14 ([Coh95, Proposition 1.3.6])**.** *Let R be a integral domain, then either R is a right Ore integral domain or it contains a right ideal which is free of infinite rank as R-module.*

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