Università degli Studi di Pisa

Dipartimento di Matematica

Corso di Laurea in Matematica

Convergence of random graphs

16 Settembre 2016

Tesi di Laurea Magistrale

Relatore
Prof. Franco Flandoli
Università di Pisa

Controrelatore
Prof. Marco Romito
Università di Pisa

Candidato
Alessandro Montagnani
Università di Pisa

Anno Accademico 2015/2016

Introduction

Recently, being natural models for a random “discretised” surface, random planar maps become relevant to theoretical physics, and in particular theories of quantum gravity. The aim of this work is to study the convergence of random graphs in local topology and scale limits, with a special look to planar maps. The major result we present, from papers of Le Galle and Miermont, is that scale limit of certain classes of planar maps is the Brownian Map.

The work is divided in three sections. In the first section we introduce the discrete objects and graph properties. We then introduce a metric in the space of pointed graph called the local topology and a notion of uniformly pointed graph through the mass transport principle. We show some examples of local limits in the topic of uniformly pointed graph with a particular attention to the local limit of Galton Watson trees. Then we briefly conclude with some connection to ergodic theory and percolation problems.

In the second section we introduce the fundamental tools for the main result of the work. We start proving the Cori-Vanquelin-Shaeffer bijection between well labelled trees with n vertices and rooted quadrangulations with n faces. The proof of the CVS bijection follows the work of Shaeffer in his PhD thesis. We spend some efforts to study the metric properties preserved by the CVS bijection. After that we discuss the bijection with a well labelled embedded tree and his contour process. Inspired by the convergence of scaled random walks to the Brownian Motion (Donsker Theorem), we briefly introduce the theory of Brownian excursions and remarking the contour process method we connect large random plane trees and Brownian excursions. This leads to a study of the objects called the Brownian Continuum Random Tree and the Browian Snake. These objects are the building blocks for the convergence result of the next section.

In the third section we prove the result of Le Gall-Miermont, the Brownian Map is the limit of class of quadrangulations, in the Gromov-Hausdorff distance. To do this we use the result that the family of laws of random walks that take values uniformly over the quadrangulations with k faces (with the distance suitably rescaled) is tight. If (X, D) is the weak limit of a subsequence, and (S, D^*) is the Brownian map, we first set an enviroment where $X = S$ almost surely, so the proof is reducted to show that also almost surely $D = D^*$. Actually, instead of considering directly quadrangulations in the convergence proof we work with well labelled trees thank to the CVS bijection, since the labelling of the resulting tree carries crucial informations about the graph distance from the root.

Contents

1 Random graphs in local topology **3**
1.1 Basic tools . . . 3
1.2 Local topology . . . 4
1.3 Unimodularity . . . 7
1.4 Random walks invariance . . . 10
1.5 Galton-Watson tree . . . 17

2 Planar Maps **22**
2.1 Circle packing . . . 22
2.2 CVS bijection . . . 24
2.3 Brownian excursions . . . 31
2.4 Convergence results . . . 35

3 The Brownian Map **46**
3.1 Basic definitions . . . 46
3.2 Considerations on the metrics . . . 52
3.3 Proof of the theorem . . . 56

Bibliography **60**

1 Random graphs in local topology

This section is mainly dedicated to give to the reader a general view on random pointed graphs. After introducing the local topology, we show some results on random pointed graphs from the local distance point of view, with a particular look to those graphs whose structure is good enough to give no relevance on the choice of the origin. At the ending of the section we briefly introduce the well known Galton-Watson tree, that turns out to be a key object for the convergence results in the next section.

1.1 Basic tools

In literature there are sligh differences on the definition of graph, so we start from that.

Definition 1.1 (Graph)**.** *Let $V = V(g)$ be a set and $E = E(g)$ a subset over $V \times V$. The pair $g = (V(g), E(g))$ is called graph, $V(g)$ set of vertices and $E(g)$ set of edges of the graph g.*

Remark that in the previous definition loops are admitted, so it is possibile to have in the set of $E(g)$ a pair of the form (x, x). Such an edge is called a loop.

Definition 1.2 (Degree)**.** *Le g be a graph, and $x \in V(g)$. The degree of x is defined as*

$$deg(x) = \#\{ \text{ edges adjacent to } x\}$$

where loops are counted twice.

From now on we consider only graphs for which holds that if $(x, y) \in E$ also $(y, x) \in E$.
So we can introduce the graph distance, which consists on the shortest path from a vertex to another.

Definition 1.3 (Graph distance)**.** *Let $g = (V, E)$ be a graph, we define the graph distance,*

$$d_{gr}(x, y) = \min_{k}\{e_1, \dots, e_k\}$$

such that if $e_i = (x_i, y_i)$ we have $y_{i-1} = x_i$, $x_1 = x$ and $y_k = y$.

We put $d_{gr}(x, y) = \infty$ if there not exist such a sequence for all $k \in \mathbb{N}$.
So we can speak about connected components of g, i.e. the equivalence classes of the relation $\sim$ defined as $x \sim y \Leftrightarrow d_{gr}(x, y) < \infty$.

Remark that we will start to speak about random variables that take values uniformly in some class of graphs, so we need a notion of equivalence between graphs, at least to ensure that such variables have finite possible values.

Definition 1.4 (Graphs equivalence)**.** *If g and g' are two graphs, g and g' are equivalent if there exist a bijection*

$$\phi : V(g) \to V(g')$$

such that $\phi(E(g)) = E(g')$. Such a ϕ is called homomorphism of graphs.

From now on, we can speak about equivalence classes of graphs under the relation $g \sim g'$ if exists ϕ homomorphism such that $\phi(g) = g'$, and we consider only graphs g such that $E(g)$ is countable, connected and locally finite (for all $x \in V(g)$ we have $deg(x) < \infty$).

1.2 Local topology

Now we start to discuss graphs where an origin point is fixed.

Definition 1.5 (Pointed graph)**.** *A pointed graph $g^\bullet$ is a pair (g, ρ) where g is a graph and $\rho \in g$.*

We call $\mathcal{G}^\bullet$ the set of all equivalence classes of pointed graphs.

Now we briefly discuss the notion of local topology for a general metric space. Let $(\mathcal{E}, \delta)$ be a metric space such that for any $x \in \mathcal{E}$ and for any $r \in \mathbb{N}$ there exist a fixed choice of the elements $[x]_r \in \mathcal{E}$ with the following properties:

- $[[x]_{r'}]_r = [x]_r$ for any $r' \geq r$,
- for any $r \geq 0$ the set $\{[x]_r : x \in \mathcal{E}\}$ is separable,
- any sequence $x_0, x_1, \ldots$ satisfying $[x_i]_j = x_j$ for all $0 \leq j \leq i$ admits a unique element $x \in \mathcal{E}$ such that $[x]_r = x_r$ for all $r \geq 0$.

Now, to help the reader, we give two examples of metric spaces satisfying the previous hypothesis:

- The space $(\mathcal{C}(\mathbb{R}, \mathbb{R}), \|\|_\infty)$, where $[f]_r(x) = f(x)\mathbf{1}_{\{x \leq r\}}$.
- The space of all the words made by letters (eventually infinite) of a given alphabet with the trivial distance, where $[w]_r$ is the word made by the first r letters of w.

Hence on $(\mathcal{E}, \delta)$ we put the local distance defined as

$$d_{loc}(x, y) = \sum_{r \geq 0} 2^{-r} \min(1, \delta([x]_r, [y]_r)).$$

From the definition should be clear why such a distance is called the local distance: two elements x and y are "close" if they coincide on the first k element of the orbits $[x]_r$ and $[y]_r$, and as r grows the importance of the difference in the orbits decrease in terms of the distance.

Theorem 1.6. *The space $(\mathcal{E}, d_{loc})$ is a Polish space (separable and complete) and a subset of $\mathcal{A} \in \mathcal{E}$ is pre-compact if and only if for every $r \geq 0$ we have $\{[x]_r : x \in \mathcal{A}\}$ is pre-compact.*

Proof. It's easy to check that d_{loc} is a distance: the only non trivial check is to show that if for all r we have $[x]_r = [y]_r$ then $x = y$, but this comes from the assumption of the unique element for a sequence.
We prove the separation, for any $x \in \mathcal{E}$ we have $d_{loc}(x, [x]_r) \leq 2^{-r}$ and we supposed the set $\{[x]_r : x \in \mathcal{E}\}$ to be separable. So we find a countable dense for all r and the union in r is still countable.
For the completeness, if (x_n) is a Cauchy sequence for d_{loc} then for every r the ball $[x_n]_r$ converges for the distance δ to an element $x_r \in \mathcal{E}$. Remembering that if $r' \geq r$ it holds $[x_{r'}]_r = x_r$ we can define a unique element $x \in \mathcal{E}$ such that $x_r = [x]_r$. So $x_n \to x$ for d_{loc}.
It remains only to show the previous characterization of compacts. If there exists $r_0 \geq 0$ and a sequence x_n in $\mathcal{A}$ whose elements $[x_n]_{r_0}$ are all at distance ε for eachother the subset couldn,t be precompact, so the condition of the theorem is necessary (such a sequence cannot admit a converging subsequence). It is also sufficient because if $\mathcal{A}$ satisfies that we can just cover $\mathcal{A}$ with a net of ball of radius 2^{-r} for δ and this will be a net for d_{loc}.

□

From now on we always consider pointed graph endowed with local distance in the following way: consider $(\mathcal{G}^\bullet, \delta)$ where δ is the trivial distance between graphs and $[g^\bullet]_r$ is the ball of radius r around the origin ρ of $g^\bullet$, where now the ball is considered using the graph distance d_{gr}. So $(\mathcal{G}^\bullet, d_{loc})$ is a polish space.

With the following proposition we give a characterization of precompacts in $(\mathcal{G}^\bullet, d_{loc})$.

Proposition 1.7. *Precompacts in $(\mathcal{G}^\bullet, d_{loc})$ are the sets $\mathcal{A}$ satisfying, for all $r \geq 0$*

$$\sup_{g^\bullet \in \mathcal{A}} \sup_{x \in V([g^\bullet]_r)} deg(x) < \infty.$$

Proof. Let $\mathcal{A}$ be in the form of the proposition. If $g_n^\bullet$ is a sequence, we can construct a subsequence with a limit in the following way: at first step we considere the neighbourhood of the origin ρ. The degree of ρ is finite, so using the pidgenhole principle there is a subsequence $g_k^\bullet(n)$ of graphs that has the same edges and vertices adjacent to ρ. At the second step we continue with the same idea taking a subsequence with the same edges and vertices adjacent to the vertices at graph distance 1 from ρ and so on. It is clear that such a subsequence admit limit (eventually on the closure).
Conversely, suppose x_n are vertices from $g_n^\bullet$ such that $\deg(x_n)$ tend to ∞, this means that we can construct a subsequence of graphs $g_{k(n)}^\bullet$ in a way that at fixed graph distance from the origin ρ we can take a different edge. □

Corollary 1.7.1. *For all $M \geq 0$, a subset of $\mathcal{G}$ with degree of vertices bounded by M is precompact.*

Now we introduce the concept of random (pointed) graph.

Definition 1.8. *A random pointed graph is a random variable*

$$G^\bullet : (\Omega, \mathcal{A}, \mathbb{P}) \to (\mathcal{G}^\bullet, d_{loc}),$$

where $(\mathcal{G}^\bullet, d_{loc})$ *is endowed with the Borel* σ*-field.*

Definition 1.9. *We say that* $(G_n^\bullet)_{n\in\mathbb{N}}$ *converges in distribution toward* $G_\infty^\bullet$ *if for any bounded continuous function* $F : \mathcal{G}^\bullet \to \mathbb{R}$ *we have*

$$\mathbb{E}[F(G_n^\bullet)] \to \mathbb{E}[F(G_\infty^\bullet)].$$

Proposition 1.10. *A family of random pointed graphs* $(G_i^\bullet)_{i\in I}$ *is tight if and only if for all* $r \geq 0$ *the family of random variables*

$$\max_{x\in V([G_i^\bullet]_r)} deg(x), i \in I$$

is tight as real valued random variables.

Proof. We first show that the condition is necessary. From definition of tight we have that for any $\varepsilon > 0$ there exist a compact subset of $\mathcal{G}^\bullet$ $\mathcal{A}_\varepsilon$ such that $\mathbb{P}(G_i^\bullet \in \mathcal{A}_\varepsilon) \geq 1 - \varepsilon$. Suppose that $\max_{x\in V([G_i^\bullet]_r)} \deg(x)$ is not tight, so for any $M > 0$ there exists an $i \in I$ such that

$$\mathbb{P}(\max_{x\in V([G_i^\bullet]_r)} \deg(x) < M) > \varepsilon,$$

hence taking the sup on M means that $\mathbb{P}(x \in \text{compact}) > \varepsilon$.
The condition of the proposition it is also sufficient, it comes directly from the fact that the set $\{g^\bullet : sup_{x\in g^\bullet} < M\}$ is a precompact. □

Proposition 1.11. *Let be* $G_1^\bullet$ *and* $G_2^\bullet$ *two random graphs such that for any* $g_0^\bullet \in \mathcal{G}^\bullet$ *and any* $r \geq 0$ *we have*

$$\mathbb{P}([G_1^\bullet]_r = g_0^\bullet) = \mathbb{P}([G_2^\bullet]_r = g_0^\bullet),$$

then $G_1^\bullet = G_2^\bullet$ *in distribution.*

Proof. The two variables agree on $\mathcal{M}$, where

$$\mathcal{M} = \{\{g^\bullet \in \mathcal{G}^\bullet : [g^\bullet]_r = g_0^\bullet\} : g_0^\bullet \in \mathcal{G}^\bullet, r \geq 0\}.$$

So the proof follows directly from the monotone class theorem. □

1.3 Unimodularity

In the previous subsection, to introduce the local topology on graphs, we do need to fix an origin, so we start to speak about pointed graphs. In this subsection we study graphs where the origin plays no special role. As the reader could expect, in the finite vertices case there is no problem in giving such a notion, while the infinite one deserves a more elaborate discussion.

Definition 1.12. *Let $G^\bullet$ be a finite (connected) random pointed graph. So $G^\bullet$ is uniformly pointed if for any measurable function $f : \mathcal{G}^\bullet \to \mathbb{R}_+$ we have*

$$\mathbb{E}[f(G^\bullet)] = \mathbb{E}[\frac{1}{\#V(G)} \sum_{x \in V(G)} f(G, x)].$$

To extend the last definition to (eventually) infinite graphs we introduce the space of (equivalence classes of) doubly pointed graphs.
We say that two graphs with two fixed ordered origin vertices (g, x, y) and (g', x', y') are equivalent if there exists an homomorphism $\phi : g \to g'$ such that $\phi(x) = x'$ and $\phi(y) = y'$.

Definition 1.13. *We call $\mathcal{G}^{\bullet\bullet}$ the set of equivalence classes of doubly pointed graphs.*

We endow $\mathcal{G}^{\bullet\bullet}$ with the local topology, where δ is the trivial distance and $[(g, x, y)]_r$ is the graph made by all vertices and edges at graph distance d_{gr} less than r, empty if not connected.

Proposition 1.14. *The projection $\pi : \mathcal{G}^{\bullet\bullet} \to \mathcal{G}^\bullet$ is continuous.*

Proof. Lets take and open set $\mathcal{A}$ in $\mathcal{G}^\bullet$, we must show that $\pi^{-1}\mathcal{A}$ is open. Without loss of generality we can consider $\mathcal{A}$ of the form

$$\mathcal{A} = \{g^\bullet : d_{loc}(g^\bullet, g_0^\bullet) < \varepsilon\}.$$

so we have for any $g^\bullet \in \mathcal{A}$ and for any $r < r(\varepsilon)$ $[g^\bullet]_r = [g_0^\bullet]_r$. Hence

$$\pi^{-1}\mathcal{A} = \bigcup_{y | d_{gr}(\rho, y) = 1} \{g^\bullet : d_{loc}(g^\bullet, (g_0, \rho, y) < \frac{\varepsilon}{2}\},$$

so, being a union of open subsets, $\pi^{-1}\mathcal{A}$ is open. □

Definition 1.15. *A Borel function $f : \mathcal{G}^{\bullet\bullet} \to \mathbb{R}^+$ (so to be well defined it is invariant for homomorphism of doubly pointed graphs) is called a transport function.*

Informally, f behaves giving the mass quantity that the vertex x sends to the vertex y.

Definition 1.16 (Unimodular)**.** *A random pointed graph (G,ρ) is unimodular if it obeys the Mass-Transport-Principle (MTP), i.e. for any transport function f we have*

$$\mathbb{E}[\sum_{x\in V(G)} f(G,\rho,x)] = \mathbb{E}[\sum_{x\in V(G)} f(G,x,\rho)].$$

Roughly speaking, obeying the MTP means that the average mass the origin receives in total is equal to the average that it sends.
Now we show that on the finite case the notions of unimodular and uniformly pointed coincide.

Proposition 1.17. *A random finite pointed graph $G^\bullet$ is uniformly pointed if and only if it is unimodular.*

Proof. Suppose that $G^\bullet$ is uniformly pointed and let f be a transport function. Then, observing that

$$\sum_{x\in V(g)} f(g,\rho,x) = F(g,\rho)$$

and

$$\sum_{x\in V(g)} f(g,x,\rho) = F'(g,\rho)$$

are measurable function for the local topology (on the single point) we have

$$\mathbb{E}[\sum_{x\in V(G)} f(G,\rho,x)] = \mathbb{E}[F(G,\rho)] = \mathbb{E}[\frac{1}{\#V(G)}\sum_{x\in V(G)} F(G,x)] =$$

$$= \mathbb{E}[\frac{1}{\#V(G)}\sum_{x\in V(G)}\sum_{y\in V(G)} f(G,x,y)] = \mathbb{E}[\frac{1}{\#V(G)}\sum_{x\in V(G)}\sum_{y\in V(G)} f(G,y,x)] =$$

$$= \mathbb{E}[\frac{1}{\#V(G)}\sum_{x\in V(G)} F'(G,x)] = \mathbb{E}[F'(G,\rho)] = \mathbb{E}[\sum_{x\in V(G)} f(G,x,\rho)].$$

So $G^\bullet$ obeys the MTP. Conversely, if $G^\bullet$ is unimodular, choosing the transport function

$$f(g,x,y) = \frac{1}{\#V(G)}h(g,x)$$

where $h : \mathcal{G}^\bullet \to \mathbb{R}^+$ is measurable function and combining with the MTP we get

$$\mathbb{E}[h(G,\rho)] = \mathbb{E}[\sum_{x\in V(G)} f(G,\rho,x)] = \mathbb{E}[\sum_{x\in V(G)} f(G,x,\rho)] = \mathbb{E}[\frac{1}{\#V(G)}\sum_{x\in V(G)} h(G,x)].$$

□

In the following theorem we show the most important result of this section, local limits preserve unimodularity.

Theorem 1.18. *Let $G_n^\bullet = (G_n,\rho_n)$ be a sequence of unimodular random graphs converging in distribution for d_{loc} towards $G_\infty^\bullet$. Then $G_\infty^\bullet$ is unimodular.*

Proof. If f is a transport function with finite range, i.e. such that $f(g,x,y)$ is zero as soon as x and y are at least at distance r_0 and that $f(g,x,y)$ only depends on $[(g,x,y)]_{r_0}$ then it follows that for every $k \geq 0$

$$F_k(g,\rho) = \sum_{x \in V(g)} (k \wedge f(g,\rho,x))\mathbf{1}_{\#V([g,\rho,x]_{r_0}) \leq k}$$

and

$$F'_k(g,\rho) = \sum_{x \in V(g)} (k \wedge f(g,x,\rho))\mathbf{1}_{\#V([g,x,\rho]_{r_0}) \leq k}$$

are both bounded continuous functions for the local topology. Hence, applying th MTP on $G_n^\bullet$ we have

$$\mathbb{E}[F_k(G_n^\bullet)] = \mathbb{E}[F'_k(G_n^\bullet)].$$

By the local convergence of $G_n^\bullet$ to $G_\infty^\bullet$ we get

$$\mathbb{E}[F_k(G_\infty^\bullet)] = \mathbb{E}[F'_k(G_\infty^\bullet)].$$

For $k \to \infty$ we have by monotone convergence

$$\mathbb{E}[\sum_{x \in V(G_\infty)} f(G_\infty, \rho_\infty, x)] = \mathbb{E}[\sum_{x \in V(G_\infty)} f(G_\infty, x, \rho_\infty)],$$

so the MTP is satisfied for all transport functions depending only on finite range around the origin.
Unluckly we can't conclude briefly from this, because there are transport functions that are not increasing limits of finite range functions.
Let $r_0, k \geq 0$ and denote by

$$\mathcal{D}_{r_0,k} = \{(g,x,y) : d_{gr}(x,y) \leq r_0 \text{ and } \#V([g,x,y]_{r_0}) \leq k\} \subset \mathcal{G}^{\bullet\bullet}.$$

So we define the family of measurable sets

$$\mathcal{M}_{r_0,k} = \{\mathcal{A} \subset \mathcal{G}^{\bullet\bullet} \text{ measurable } : \mathbb{E}[\sum_{x \in V(G_\infty)} \mathbf{1}_{(G_\infty,\rho,x) \in \mathcal{A} \cap \mathcal{D}_{r_0,k}}] =$$

$$= \mathbb{E}[\sum_{x \in V(G_\infty)} \mathbf{1}_{(G_\infty,x,\rho) \in \mathcal{A} \cap \mathcal{D}_{r_0,k}}]\}.$$

All elementary sets $A = \{(g,x,y) : [(g,x,y)]_r = g_0^{\bullet\bullet}\}$ when $g_0^{\bullet\bullet} \in \mathcal{G}^{\bullet\bullet}$ is a finite bi-pointed graph are in $\mathcal{M}_{r_0,k}$ and those sets generate the Borel σ-field of $\mathcal{G}^{\bullet\bullet}$ and are stable under finite intersection. We show that $\mathcal{M}_{r_0,k}$ is a monotone class: the stability under monotone union comes from monotone convergence theorem, the stability under difference follows from

$$\mathbb{E}[\sum_{x \in V(G_\infty)} \mathbf{1}_{(G_\infty,\rho,x) \in \mathcal{A}^c \cap \mathcal{D}_{r_0,k}}] = \mathbb{E}[\sum_{x \in V(G_\infty)} \mathbf{1}_{(G_\infty,\rho,x) \in \mathcal{D}_{r_0,k}}] -$$

$$-\mathbb{E}[\sum_{x \in V(G_\infty)} \mathbf{1}_{(G_\infty,\rho,x) \in \mathcal{A} \cap \mathcal{D}_{r_0,k}}],$$

and it is anologous when the role of ρ and x are exchanged. Now observe that the first expectation in the right side is finite. It follows that $\mathcal{M}_{r_0,k}$ is the Borel σ-field of $\mathcal{G}^{\bullet\bullet}$.
So, sending $r_0 \to \infty$ and $k \to \infty$, from the monotone convergence theorem we have that $G^{\bullet}_{\infty}$ obeys the MTP for any indicator functions. Approximating positive functions with increasing simple functions we can conclude.

□

We conclude this subsection showing that Cayley graphs are unimodular.

Definition 1.19 (Cayley graph)**.** *Let G be a group with a given choice of finite symmetric generating set*

$$S = \{s_1, s_1^{-1}, \dots, s_k, s_k^{-1}\},$$

his Cayley graph is a graph with vertices the elements of G, and (x, y) is an edge if and only if there is an s inS such that $x = sy$.

Proposition 1.20. *Any Cayley graph is unimodular.*

Proof. Let g be the Caylet graph of G (with a fixed S) pointed at the identity e. So, for any $x, y \in G$ there is an homomorphism of doubly pointed graphs between

$$\phi : (g, x, y) \to (g, e, yx^{-1})$$

such that

$$\phi(z) = \phi(zx^{-1}).$$

So, being a transport function f invariant for homorphism of doubly pointed graphs, we have $f(g, x, y) = \tilde{\mathrm{f}}(yx^{-1})$ for some function $\tilde{\mathrm{f}} : G \to \mathbb{R}^+$. Hence, being $x \to x^{-1}$ an involution of the group,

$$\mathbb{E}[\sum_{x\in G} f(g, e, x)] = \sum_{x\in G} f(g, e, x) = \sum_{x\in G} \tilde{\mathrm{f}}(x) = \sum_{x\in G} \tilde{\mathrm{f}}(x^{-1}) = \mathbb{E}[\sum_{x\in G} f(g, x, e)].$$

□

1.4 Random walks invariance

In this subsection we see how to connect unimodularity and law invariance for random walks. To do this, we start introducing rooted graph.

Definition 1.21. *A rooted graph is a pair $\overrightarrow{g} = (g, \overrightarrow{e})$ where $\overrightarrow{e}$ is an oriented edge. Remark that in this case we distinguish (x, y) from (y, x), differently from what seen since now.*

Definition 1.22. *We call $\overrightarrow{\mathcal{G}}$ the class of equivalence of homomorphism of rooted graphs, where an homomorphism between two rooted graphs $(g, \overrightarrow{e})$ and $(g', \overrightarrow{e}')$ is a graph homomorphism*

$$\phi : g \to g'$$

such that $\phi(\overrightarrow{e}) = \overrightarrow{e}'$.

As aspected we endow $\overrightarrow{\mathcal{G}}$ with the local topology, where δ is the trivial distance between rooted graphs and $[\overrightarrow{g}]_r$ is obtained by keeping vertices and edges at graph distance $d_{gr} \leq r$ from the origin of the root edge $\overrightarrow{e}$ (and keeping $\overrightarrow{e}$ as root edge in $[\overrightarrow{g}]_r$).

Definition 1.23. *If $\overrightarrow{g} = (g, \overrightarrow{e})$ is a rooted graph, denote by $\pi_\bullet(\overrightarrow{g})$ the pointed graph obtained by keeping distinguished in g the origin of the root edge $\overrightarrow{e}$.*

Definition 1.24. *If $G^\bullet = (g, \rho)$ is a (eventually random) graph, denote by $\pi_\to(G^\bullet)$ the random rooted graph obtained keeping the graph g and choosing the oriented edge starting from the origin vertex ρ and ending at random uniformly in an adjacent vertex.*

Note that it holds $\pi_\bullet \circ \pi_\to = id_{\mathcal{G}^\bullet}$.

Proposition 1.25. *The operator $\pi_\bullet : \overrightarrow{\mathcal{G}} \to \mathcal{G}^\bullet$ is continuous in the local topology for the two spaces.*

Proof. Let $\mathcal{A}$ be an open subset of $(\mathcal{G}^\bullet, d_{loc})$. Without loss of generality we can consider $\mathcal{A}$ of the form

$$\mathcal{A} = \{g^\bullet \in \mathcal{G}^\bullet \text{such that } d_{loc}(g^\bullet, g_0^\bullet) < \frac{1}{2^{r+1}}\}$$

for a given $g_0^\bullet \in \mathcal{G}^\bullet$. So $g^\bullet \in \mathcal{A}$ if and only if $[g^\bullet]_k = [g_0^\bullet]_k$ for all $k \leq r$. So

$$\pi_\bullet^{-1}(\mathcal{A}) = \{\overrightarrow{g} \in \overrightarrow{\mathcal{G}} \text{such that } d_{loc}(\overrightarrow{g}, \pi_\bullet^{-1}(g_0^\bullet)) < \frac{1}{2^{r+1}}\},$$

so it is an open subset of $\overrightarrow{\mathcal{G}}$.

□

As for unimodularity, we start the discussion in the a.s. finite case.

Definition 1.26. *Let $\overrightarrow{G} = (G, \overrightarrow{e})$ an a.s. finite random rooted graph. We say that $\overrightarrow{G}$ is uniformly rooted if for all Borel function $f : \overrightarrow{\mathcal{G}} \to \mathbb{R}_+$ we have*

$$\mathbb{E}[f(\overrightarrow{G})] = \mathbb{E}[\frac{1}{\#\overrightarrow{E}(G)} \sum_{\overrightarrow{\sigma} \in \overrightarrow{E}(G)} f(G, \overrightarrow{\sigma})].$$

Definition 1.27. *Given two random variables defined on the same space*

$$X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathcal{E}, d)$$
$$Y : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$$

such that Y has a finite non zero expectation, we define

$$\tilde{X} : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathcal{E}, d)$$

such that

$$\mathbb{E}[f(\tilde{X})] = \frac{1}{\mathbb{E}[Y]}\mathbb{E}[f(x)y]$$

for every Borel function $f : \mathcal{E} \to \mathbb{R}^+$. The law of $\tilde{X}$ is said the law of the random variable X biased by Y.

Definition 1.28. *Let $\overrightarrow{G}$ be a random uniformly rooted graph, we define $\underline{G}^\bullet$ as the random graph obtained from $\pi_\bullet(\overrightarrow{G})$ by biasing by deg$(\overrightarrow{e_*})^{-1}$, where for $\overrightarrow{e_*}$ we intend the origin vertex of the root edge $\overrightarrow{e}$.*

Proposition 1.29. *With the notation of the previous definition, $\underline{G}^\bullet$ is a random uniformly pointed graph.*

Proof. Let f be a positive Borel function on $\mathcal{G}^\bullet$, we have

$$\mathbb{E}[f(\underline{G}^\bullet)] = \frac{1}{\mathbb{E}[\deg^{-1}(\overrightarrow{e_*})]}\mathbb{E}[\deg^{-1}(\overrightarrow{e_*})f(\pi_\bullet(\overrightarrow{G}))] =$$

here we use the definition of random uniformly rooted for $\overrightarrow{G}$

$$= \frac{1}{\mathbb{E}[\deg^{-1}(\overrightarrow{e_*})]}\mathbb{E}[\frac{1}{\#\overrightarrow{E}(G)}\sum_{\overrightarrow{\sigma}\in\overrightarrow{E}(G)} \deg^{-1}(\overrightarrow{\sigma_*})f(G, \overrightarrow{\sigma_*})] =$$

$$= \frac{1}{\mathbb{E}[\deg^{-1}(\overrightarrow{e_*})]}\mathbb{E}[\frac{1}{\#\overrightarrow{E}(G)}\sum_{x\in V(G)} f(G, x)] =$$

$$= \frac{1}{\mathbb{E}[\deg^{-1}(\overrightarrow{e_*})]}\mathbb{E}[\frac{1}{\#\overrightarrow{E}(G)}\sum_{\overrightarrow{\sigma}\in\overrightarrow{E}(G)} \deg^{-1}(\overrightarrow{\sigma_*})\frac{1}{\#V(G)}\sum_{x\in V(G)} f(G, x)]$$

we then apply again the hypothesis of uniformly rooted for the borel function $\deg^{-1}(\overrightarrow{\sigma_*})\frac{1}{\#V(G)}\sum_{x\in V(G)} f(G, x)$ and we get

$$\frac{\mathbb{E}[F(G, \overrightarrow{e})]}{\mathbb{E}[\deg^{-1}(\overrightarrow{e_*})]} = \frac{1}{\mathbb{E}[\deg^{-1}(\overrightarrow{e_*})]}\mathbb{E}[\deg^{-1}(\overrightarrow{\sigma_*})\frac{1}{\#V(G)}\sum_{x\in V(G)} f(G, x)] =$$

$$= \mathbb{E}[\frac{1}{\#V(\underline{G})}\sum_{x\in V(\underline{G})} f(\underline{G}, x)].$$

□

Conversely, if $G^\bullet$ is a random uniformly pointed graph, $\pi_{\rightarrow}(G^\bullet)$ biased by $\deg(\rho)$ is uniformly rooted.

Now again we have to adapt the definition in the infinite graph case. The relation is given biasing by the degree of the origin as above.

Definition 1.30 (Deterministic case)**.** *If $\overrightarrow{g} = (g, \overrightarrow{e})$ is a fixed rooted graph, we denote by $\boldsymbol{P}_{\overrightarrow{g}}$ the law of a simple random walk on g starting from $\overrightarrow{E}$. So there is a sequence $\overrightarrow{E_0}, \overrightarrow{E_1}, \dots$ where $\overrightarrow{E_0} = \overrightarrow{e}$ and we choose at step i independently of the past the next oriented edge $\overrightarrow{E}_{i+1}$ with origin the end point of $\overrightarrow{E}_i$.*

Definition 1.31 (Random case)**.** *If $\overrightarrow{G} = (G, \overrightarrow{e})$ is a random rooted graph, a random walk on $\overrightarrow{G}$ is the law of $(G, \overrightarrow{E}_i)$ under*

$$\int d\mathbb{P}(\overrightarrow{G}) \int d\boldsymbol{P}_{\overrightarrow{G}}(\overrightarrow{E}_i)_{i\geq 0}.$$

Finally we can introduce the concepts of stationary and reversible, that should give to the reader a sense of “invariance” for random walk.

Definition 1.32. *Let $\overrightarrow{G} = (G, \overrightarrow{e})$ be a random rooted graph, denote by $(\overrightarrow{E}_i)_{i\geq 0}$ the edges visited by a simple random walk on it. The random graph $\overrightarrow{G}$ is said to be stationary if for any $k \geq 0$ the law of $(G, \overrightarrow{E}_k)$ is the same of $\overrightarrow{G}$.*

Definition 1.33. *With the notation of the previous definition, we say that $\overrightarrow{G}$ is reversible if $(G, \overrightarrow{E_0}) = (G, \overleftarrow{E_0})$ in law.*

Proposition 1.34. *$\overrightarrow{G}$ is stationary if and only if $(G, \overrightarrow{E_1}) = (G, \overrightarrow{E_0})$ in distribution.*

Proof. The condition of the proposition is clearly necessary.
Let $\overrightarrow{E_0}, \overrightarrow{E_1}, \dots$ be the visit of a simple random walk. We have that $(G, \overrightarrow{E_2})$ has the law of $(G, \overrightarrow{E_1})$ with root one step after the root origin, so for hypotesis has the law of $(G, \overrightarrow{E_0})$ with root one step after the origin, so has the same law of $(G, \overrightarrow{E_1})$, so the same of $(G, \overrightarrow{E_0})$. Repeating this inductively gives the proof. $\square$

As expected, stationarity and uniformly rooted coincide in the a.s. finite case.

Proposition 1.35. *Let $\overrightarrow{G}$ be an a.s. finite random rooted graph. We have that $\overrightarrow{G}$ is uniformly rooted if and only if $\overrightarrow{G}$ is stationary.*

Proof. If $\overrightarrow{G}$ is a.s. finite and uniformly rooted, keep an oriented edge $\overrightarrow{E_0}$ and perform a k step random walk. It is known that on finite (connected) graph the law on a random walk on it is the uniform measure on the edges, so $(G, \overrightarrow{E_k})$ is equal in distribution to $(G, \overrightarrow{E_0})$, since the graph is uniformly rooted.
For the converse we use a classical ergodic theorem, if $\overrightarrow{g}$ is finite connected rooted graph and $(\overrightarrow{E_i})$ has law $\mathbf{P}_{\overrightarrow{g}}$, for any $f : \overrightarrow{\mathcal{G}} \to \mathbb{R}_+$ we have

$$S_f(\overrightarrow{g}, n) = \frac{1}{n} \sum_{k=0}^{n-1} f(g, \overrightarrow{E_k}) \longrightarrow_{n\to\infty} \frac{1}{\#\overrightarrow{E}(g)} \sum_{\overrightarrow{\sigma}\in\overrightarrow{E}(g)} f(g, \overrightarrow{\sigma}) = U_f(\overrightarrow{g}).$$

If $\overrightarrow{G}$ is a finite stationary random graph it holds

$$\int d\mathbb{P}(\overrightarrow{G}) \int d\mathbf{P}_{\overrightarrow{G}}(\overrightarrow{E_k}) f(G, \overrightarrow{E_k}) = \mathbb{E}[f(\overrightarrow{G})]$$

hence for stationarity

$$\mathbb{E}[f(\overrightarrow{G})] = \mathbb{E}[\int d\mathbf{P}_{\overrightarrow{G}} S_f(\overrightarrow{G}, n)]$$

and the last term as n tends to ∞ converges to $\mathbb{E}[U_f(\overrightarrow{G})]$ for dominate convergence.

□

We then can show the connection between stationary and reversible random rooted graphs and unimodular random graph.

Proposition 1.36. *Let $G^\bullet = (G, \rho)$ be an unimodular random pointed graph. Let $\overline{G}^\bullet = (\overline{G}, \overline{\rho})$ obtained from (G, ρ) after biasing by the degree of its origin and consider $\overrightarrow{\overline{G}} = \pi_\to(\overline{G}^\bullet)$. Then $\overrightarrow{\overline{G}}$ is stationary and reversible.*

Proof. We start showing that $\overrightarrow{\overline{G}} = (\overline{G}, \overrightarrow{E})$ has the same law of $(\overline{G}, \overleftarrow{E})$. So, let $h(g, \overrightarrow{e})$ be a function from $\overrightarrow{\mathcal{G}}$ to $\mathbb{R}_+$ and

$$f(g, x, y) = \mathbf{1}_{\{x\sim y\}} \sum_{x\to y} h(g, \overrightarrow{e}),$$

the associated transport function obtained summing over all possible choice of a link from x to y. Then for MTP

$$\mathbb{E}[\deg\rho] \cdot \mathbb{E}[h(\overline{G}, \overrightarrow{E})] = \mathbb{E}[\deg(\rho) \frac{1}{\deg(\rho)} \sum h(G, \overrightarrow{e})] =$$
$$= \mathbb{E}[\sum_{x\in V(G)} f(G, \rho, x)] = \mathbb{E}[\sum_{x\in V(G)} f(G, x, \rho)] =$$
$$= \mathbb{E}[\deg(\rho) \frac{1}{\deg(\rho)} \sum h(G, \overleftarrow{e})] = \mathbb{E}[\deg\rho] \cdot \mathbb{E}[h(\overline{G}, \overleftarrow{E})].$$

The previous also proves that the graph is stationary: if $(\overrightarrow{E_0} = \overleftarrow{E}, \overrightarrow{E_1})$ are the first two step of a random walk on $(\overline{G}, \overleftarrow{E})$ then $(\overline{G}, \overrightarrow{E_1})$ has the same distribution of $(\overline{G}, \overrightarrow{E})$. □

Lemma 1.37. *Let (G, ρ) be a random pointed graph satisfying the MTP for transport functions $f(g, x, y)$ which are null as soon as x and y are not neighbors in g. Then G, ρ is unimodular.*

Proof. Suppose that $f(g, x, y)$ is a transport function that is zero unless $d_{gr}(x, y) = k$ for some $k \geq 1$. We denote by $\mathcal{P}(g, x, y)$ the number of geodesics path from x to y and $\mathcal{P}_j(g, x, y, u, v)$ the number of such paths such that the jth step links u to v, where $1 \leq j \leq k$. Consider the transport functions

$$f_j(g, u, v) = \sum_{x,y \in V(g)} f(g, x, y) \frac{\mathcal{P}_j(g, x, y, u, v)}{\mathcal{P}(g, x, y)}$$

which are null except if u and v are neighbors in g. Then we can apply MTP for those functions:

$$\mathbb{E}[\sum_{x \in V(G)} f_j(G, \rho, x)] = \mathbb{E}[\sum_{x \in V(G)} f_j(G, x, \rho)].$$

And we have:

1.
$$\sum_{x \in V(G)} f(G, \rho, x) = \sum_{x \in V(G)} f(G, \rho, x) \sum_{y: d_{gr}(x,y)=1} \frac{\mathcal{P}_j(g, \rho, x, \rho, y)}{\mathcal{P}(g, \rho, x)} = \sum_{y \in V(G)} f_1(G, \rho, y),$$

2.
$$\sum_{x \in V(G)} f(G, x, \rho) = \sum_{x \in V(G)} f(G, x, \rho) \sum_{y: d_{gr}(x,y)=k-1} \frac{\mathcal{P}_j(g, x, \rho, y, \rho)}{\mathcal{P}(g, x, \rho)} = \sum_{y \in V(G)} f_k(G, y, \rho),$$

3. for $1 \leq j < k$ we have
$$\sum_{x \in V(G)} f_j(G, x, \rho) = \sum_{u,v \in V(G)} f(G, u, v) \sum_{x \in V(G)} \frac{\mathcal{P}_j(G, u, v, x, \rho)}{\mathcal{P}(G, u, v)} =$$
$$= \sum_{u,v \in V(G)} f(G, v, u) \sum_{x \in V(G)} \frac{\mathcal{P}_{j+1}(G, u, v, \rho, x)}{\mathcal{P}(G, u, v)} = \sum_{x \in V(G)} f_{j+1}(G, \rho, x).$$

So the proof is complete.

□

Theorem 1.38. *Let $\overrightarrow{G} = (G, \overrightarrow{E})$ be a stationary and reversible random graph. Let $\underline{\overrightarrow{G}} = (\underline{G}, \underline{\overrightarrow{E}})$ the graph obtained biasing $\overrightarrow{G}$ by the inverse of the degree of the origin of $\overrightarrow{E}$. Then $\pi_\bullet(\underline{\overrightarrow{G}})$ is a unimodular random graph.*

Proof. We do need to verify the MTP. For the previous lemma we can ask the transport function $f(g, x, y)$ to be zero as soon as x and y are not neighbors. We then construct $h(g, \overrightarrow{e})$ as in the previous proposition. So:

$$\mathbb{E}[\frac{1}{\deg(\overrightarrow{E_*})}]\mathbb{E}[\sum_{x\in V(\underline{G})} f(\underline{G},\rho,x)] = \mathbb{E}[\frac{1}{\deg(\overrightarrow{E_*})}\sum_{x\in V(G)} f(G,\overrightarrow{E}_*,x)] =$$

$$= \mathbb{E}[\frac{1}{\deg(\overrightarrow{E_*})}\sum_{\overrightarrow{\sigma}=\overrightarrow{E_*}} h(G,\overrightarrow{\sigma})] = \mathbb{E}[h(G,\overrightarrow{E})] =$$

$$= \mathbb{E}[h(G,\overleftarrow{E})] = \mathbb{E}[\frac{1}{\deg(\overrightarrow{E_*})}\sum_{\overrightarrow{\sigma}=\overrightarrow{E_*}} h(G,\overleftarrow{\sigma})] =$$

$$= \mathbb{E}[\frac{1}{\deg(\overrightarrow{E_*})}\sum_{x\in V(G)} f(G,x,\overrightarrow{E}_*)] = \mathbb{E}[\frac{1}{\deg(\overrightarrow{E_*})}]\mathbb{E}[\sum_{x\in V(\underline{G})} f(\underline{G},x,\rho)].$$

□

We conclude this subsection showing a connection between stationarity and ergodic theory. Let $G = (g,(\overrightarrow{e_i}))$ be a graph with a labelled path of oriented edges $\overrightarrow{e_i}$. Let θ be a shift on said space:

$$\theta(g,(\overrightarrow{e_i})) \to (g,(\overrightarrow{e_{i+1}})).$$

Let μ be the distribution of a simple random walk on the random rooted graph $(G\overrightarrow{E})$:

$$\mu = \int d\mathbb{P}(G,\overrightarrow{E})\int d\mathrm{P}_{(G,\overrightarrow{E})}(\overrightarrow{E_i}),$$

where $\mathrm{P}_{(g,\overrightarrow{e})}$ is the law of a simple random walk on the fixed graph g starting in $\overrightarrow{e}$. We have that the random graph $(G,\overrightarrow{E})$ is stationary if and only if μ is θ-invariant.

Recall Kingman subadditive ergodic theorem, that is a generalization of the classical Birkhoff result.

Theorem 1.39 (Kingman)**.** *If θ is a measure preserving transformation on a probability space $(E,\mathcal{A},\mu)$ and $(h_n)_{n\geq 1}$ is a sequence of integrable functions satisfying for $n,m\geq 1$*

$$h_{n+m}(x) \leq h_n(x) + h_m(\theta^n x)$$

so it holds

$$\frac{(h_n)(x)}{n} \to h(x)$$

where the convergence is both a.s. and in L^1, and $h(x)$ is θ-invariant.

Definition 1.40. *Let $\theta : (E,\mathcal{A},\mu) \to (E,\mathcal{A})$ measurable and mu-invariant. We say that θ is ergodic (respect to mu) if for any A measurable ,*

$$\mu(A\Delta\theta^{-1}(A)) = 0 \Rightarrow \mu(A) \in \{0,1\}.$$

Proposition 1.41. *Let $\theta : (E, \mathcal{A}, \mu) \to (E, \mathcal{A})$ be ergodic respect to μ. Let f be a function $f : E \to \mathbb{R}$ θ-invariant. Then f is equal to a costant μ almost surely.*

Proof. Suppose there exist A such that $\mu(A) = c \neq 0, 1$ such that $f(A) < C_1$ and $f(A^c) > C_2$. We then have $f(\theta(A)) < C_1$, so $\mu(A\Delta\theta^{-1}(A)) = 0$ for maximality and μ-invariance , but this is a contradiction.

□

As corollary of this proposition we have that if θ is ergodic, in Kingman theorem, then $\frac{h_n}{n}$ converge to a costant.

Definition 1.42. *An ergodic random graph is a random graph where the shift θ is ergodic.*

Now we want to use Kingman theorem for ergodic random graph. Consider the following function

$$h_n(g, (\overrightarrow{e_i})) = d_{gr}((\overrightarrow{e_0})_*, (\overrightarrow{e_n})_*)$$

that satisfies the hypothesis of Kingman theorem:

$$h_{m+n}(g, (\overrightarrow{e_i})) = d_{gr}((\overrightarrow{e_0})_*, (\overrightarrow{e}_{m+n})_*) \leq d_{gr}((\overrightarrow{e_0})_*, (\overrightarrow{e_n})_*) + d_{gr}((\overrightarrow{e_n})_*, (\overrightarrow{e}_{m+n})_*) =$$

$$= h_n(g, (\overrightarrow{e_i})) + h_m(g, (\overrightarrow{e_i})_{i \geq n}) = h_n(g, (\overrightarrow{e_i})) + h_m(\theta^n(g, (\overrightarrow{e_i}))).$$

Hence there exist a costant s such that

$$\frac{d_{gr}((\overrightarrow{E_o})(\overrightarrow{E_n}))}{n} \to s.$$

Such an s could be interpretated as the speed of the random walk .

1.5 Galton-Watson tree

In this subsection we study some aspects of a well know random graph, the Galton-Watson tree (GWT). After a brief introduction on the basic properties of this random tree we give to the reader results involving GWT and the applications of the tools introduced in the previous subsections.

Definition 1.43 (Galton-Watson tree)**.** *Let $p = (p_k)_{k \geq 0}$ a distribution over $\mathbb{N}$. A Galton-Watson tree (with offspring distribution p) is a random tree obtained by starting from an ancestor particle, and then each particles reproduce independently with offspring distribution p.*

Proposition 1.44. *The extinction probability is the smallest solution in* $[0, 1]$ *of*

$$F_p(z) = z,$$

where

$$F_p(z) = \sum_{k \geq 0} z^k p_k.$$

Proof. If z is the extinction probability of the ancestor, this should be exactly equal to the product of extinction probability of his sons (for independence), then z solves $z = \sum_{k \geq 0} z^k p_k$. □

Proposition 1.45. *Except the trivial case of offspring distribution* $p = \delta_1$*, the GWT is almost surely finite if and only if* $\mathbb{E}[p] \leq 1$.

Proof. It follows directly from the fact that $\mathbb{E}[p] = F_p'(1)$. □

Definition 1.46. *Given a graph* g*, a Bernoulli bond percolation on* g *of parameter* $p \in (0, 1)$ *is the random graph obtained by keeping each edge, and relative vertices, independently with probability* p.

Definition 1.47. *Let* $\lambda > 0$*,* $T_\lambda^\bullet$ *is the Galton-Watson tree with offspring distribution Poisson*$(\lambda) = (e^{-\lambda} \frac{\lambda^n}{n!})_{n \in \mathbb{N}}$*, pointed at the ancestor vertex.*

Let k^n be the complete graph on n vertices and let $G^\bullet(n, p)$ be the random graph made by performing a Bernoulli bond percolation of parameter p on k^n and keeping the connected component containing the vertex 1, pointed at this vertex.

Theorem 1.48. *With the notation as above, we have the following convergence in distribution for* d_{loc} *of the two spaces*

$$G^\bullet(n, \frac{\lambda}{n}) \to T_\lambda^\bullet.$$

Proof. Fix a pointed tree

$$t^\bullet = (t, \rho)$$

of heigh (max graph distance from the origin) at most r. We first want to show that

$$\mathbb{P}([G^\bullet(n, \frac{\lambda}{n})]_r = t^\bullet) \to \mathbb{P}([T_\lambda^\bullet = t^\bullet])$$

as $n \to \infty$. To make things more clear, we take an order on the vertices of $t^\bullet$, so we obtain $\tilde{\mathrm{t}}^\bullet$, that is $t^\bullet$ embedded in the plane and giving to each vertex and order from left to right.

We also consider $\tilde{\mathrm{T}}^{\bullet}_{\lambda} = (T^{\bullet}_{\lambda}, \prec)$ the previous GWT equipped with such vertices order. Then

$$\mathbb{P}([\tilde{\mathrm{T}}^{\bullet}_{\lambda}]_r = \tilde{\mathrm{t}}^{\bullet}) = \prod_{x \in V(\tilde{\mathrm{t}}^{\bullet}) : d_{gr}(\rho, x) < r} e^{-\lambda} \frac{\lambda^{\#\mathrm{Children}(x)}}{\#\mathrm{Children}(x)!}.$$

On the other hand we have

$$\mathbb{P}([G^{\bullet}(n, \frac{\lambda}{n})]_r = \tilde{\mathrm{t}}^{\bullet}) = N_{n,\tilde{\mathrm{t}}^{\bullet}} \prod_{x \in V(\tilde{\mathrm{t}}^{\bullet}) : d_{gr}(\rho, x) < r} (\frac{\lambda}{n})^{\#\mathrm{Children}(x)} (1 - \frac{\lambda}{n})^{n-1-\#\mathrm{Children}(x)},$$

where $N_{n,\tilde{\mathrm{t}}^{\bullet}}$ is equal to the number of ways to assign labels from $\{1, 2, \dots, n\}$ to the vertices of $\tilde{\mathrm{t}}^{\bullet}$ so that the ancestor gets label 1 and the numbers assigned to the children of a given vertex are increasing from left to right.
Asintotically we have

$$N_{n,\tilde{\mathrm{t}}^{\bullet}} = \prod_{x \in V(\tilde{\mathrm{t}}^{\bullet}) : d_{gr}(\rho, x) < r} \frac{n^{\#\mathrm{Children}(x)}}{\#\mathrm{Children}(x)!}.$$

So, recalling that $\mathrm{Bin}(n-2, \frac{\lambda}{n}) \to \mathrm{Poisson}(\lambda)$ we have:

$$\mathbb{P}([G^{\bullet}(n, \frac{\lambda}{n})]_r = \tilde{\mathrm{t}}^{\bullet}) \to \mathbb{P}([\tilde{\mathrm{T}}^{\bullet}_{\lambda}]_r = \tilde{\mathrm{t}}^{\bullet}),$$

hence, cause ordering was only to make things clear

$$\mathbb{P}([G^{\bullet}(n, \frac{\lambda}{n})]_r = t^{\bullet}) \to \mathbb{P}([T^{\bullet}_{\lambda}]_r = t^{\bullet}).$$

It lasts only to check that for any $g^{\bullet}$ pointed graph that is not a tree we have

$$\mathbb{P}([G^{\bullet}(n, \frac{\lambda}{n})]_r = g^{\bullet}) \to 0.$$

This follows from

$$\limsup_{n\to\infty} \mathbb{P}([G^{\bullet}(n, \frac{\lambda}{n})]_r \text{is not a tree}) = 1 - \liminf_{n\to\infty} \sum_{t^{\bullet} \text{tree}} \mathbb{P}([G^{\bullet}(n, \frac{\lambda}{n})]_r = t^{\bullet}) \leq$$

here we use Fatou lemma

$$\leq 1 - \sum_{t^{\bullet} \text{tree}} \mathbb{P}([T^{\bullet}_{\lambda}]_r = t^{\bullet}) = 0.$$

□

Note that a *GWT* is not stationary and reversible, the origin of the tree has expectation of degree 1 less than the other vertices. Here we show how to construct a similar object to avoid this problem.

Definition 1.49. *A plane tree τ is a subset of*

$$\mathcal{U} = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n$$

such that:

1. *$\emptyset \in \tau$ (that is the ancestor of the root),*
2. *if $v \in \tau$, all its ancestor belong to τ,*
3. *for every $u \in \mathcal{U}$ there exist $c_u(\tau) \geq 0$ (number of children of τ) such that $uj \in \tau$ if and only if $j \leq c_u(\tau)$,*

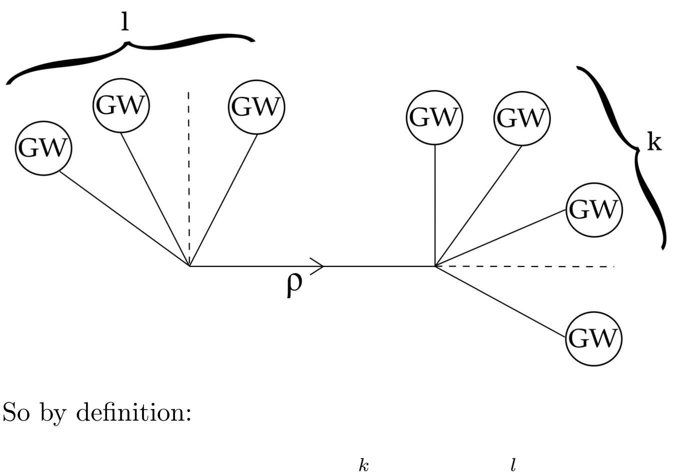
where if $v = uj$ we say that v is descendant of u and u is ancestor of u.

Definition 1.50. *A rooted tree (τ) can be embedded in the space of plane trees keeping by keeping as root the edge $(\emptyset, 1)$. We denote by $\pi_{\circlearrowleft}(\tau)$ this immersion.*

Definition 1.51. *An augmented GWT with distribution p is the random tree obtained by grafting two independent GWT of distribution p such that the origin of the root is the origin of the first GWT and the endpoint of the root is the origin of the second GWT.*

Theorem 1.52. *Let τ be an augmented GWT of distribution p. So $\pi_{\circlearrowleft}(\tau)$ is a stationary and reversible random rooted tree.*

Proof. We fix $k, l \geq 0$ and measurable subset of trees $A_1, \ldots, A_k, B_1, \ldots, B_l$. The event $\mathcal{E}$ is the one where from the root endpoint start the trees $A_1, \ldots, A_k$ and from the root origin star the trees $B_1, \ldots, B_l$, and no other subtrees are part of the tree.



So by definition:

$$\mathbb{P}_\tau(\mathcal{E}) = p_k p_l \prod_{i=1}^{k} \mathbb{P}_{GW}(A_i) \prod_{i=1}^{l} \mathbb{P}_{GW}(B_i).$$

Then we compute the probability that with a new root E_1 the tree is $\mathcal{E}$. This probability is equal to the sum of tha case $\overrightarrow{E_1} \neq \overleftarrow{E_0}$ and his complement $\overrightarrow{E_1} = \overleftarrow{E_0}$, so it is:

$$p_l \frac{l}{l+1} \prod_{i=1}^{l} \mathbb{P}_{GW}(B_i) p_k \prod_{i=1}^{k} \mathbb{P}_{GW}(A_i) + pl \frac{1}{l+1} p_k \prod_{i=1}^{l} \mathbb{P}_{GW}(B_i) \prod_{i=1}^{k} \mathbb{P}_{GW}(A_i).$$

So the law coincide on event of the form of $\mathcal{E}$ and for the monotone class theorem the proof is complete.

$\square$

2 Planar Maps

This section is the core of the work. After giving the notion of planar maps, we introduce the contour process of a rooted tree and we prove the so called CVS-bijection between well labelled trees and rooted quadrangulations. The bijection is the key step for the main result of this work, the Brownian map is the scaling limit of random variables that take values uniformly in certain class of quadrangulations.
The second key step for the main proof presented in this section is that the contour functions of certain random trees converges in law (after being properly rescaled) to a normalized Brownian excursion.

2.1 Circle packing

We start this section with a warmup, introducing the basic definitions of planar maps and discussing circle packing method.

Definition 2.1. *A planar graph is a graph such that there exist an embedding from the graph to the* 2*-sphere* $\mathbb{S}^2$.

Definition 2.2. *A finite planar map is a finite connected planar graph properly embedded in the sphere, viewed up to homeomorphism that preserve the orientation.*

Definition 2.3. *The faces of a planar map are the connected component of the complement of the embedding.*

Definition 2.4. *The degree of a face correspond to the number of edges incident to the face, where edges that are incident to only one face are to be counted twice.*

Remark that two planar maps are identified if there exist an orientation-preserving homeomorphism that sends one map to the other.
So a finite planar map can be seen as a finite graph with a system of coherent orientation around each vertex in the map.
Hence should be clear that the number of planar maps of given number of vertices is finite.

Definition 2.5. *Given a planar map* m*, we denote with* $V(m)$*,* $E(m)$ *and* $F(m)$ *the sets of vertices, edges and faces of the map.*

Theorem 2.6 (Euler's formula)**.** *For any finite planar map* m *we have*

$$\#V(m) + \#F(m) - \#E(m) = 2.$$

Proof. With homology theory this proof would be an instant kill. Anyway, there is no need to introduce homology in our work, so we prove the formula by induction on the number of edges.
A map with 0 edge has 1 vertex and 1 face, then satisfies Euler's formula.
Suppose now $E(m) \geq 1$ and erase one edge of m, then we have two possible scenarios:

1. either the map is still connected, so we have one less edge but also one less face, and by induction hypothesis the formula is good,

2. or we have that now the map is not connected and we get two maps m_1 and m_2, so we can apply the formula to each (that holds for induction hypothesis)

$$\#V(m_1) + \#F(m_1) - \#E(m_1) = 2$$

$$\#V(m_2) + \#F(m_2) - \#E(m_2) = 2$$

   and noting that $\#V(m) = \#V(m_1) + \#V(m_2)$, $\#E(m) = \#E(m_1) + \#E(m_2) + 1$ and $\#F(m) = \#F(m_1) + \#F(m_2) - 1$ (the external face is counted twice in the splitting), we have the thesis.

□

Now we roughly introduce the circle packing method, that answer to the question of if it is possible to have a canonical embedding from a planar map to the sphere. This method became relevant in the past due to the proof of the famous 4-colour problem.

Definition 2.7. *A simple map m is represented by a circle packing if there exist a collection $(C_v : v \in V(m))$ of non overlapping disk in $\mathbb{R}^2$ (or in the sphere) such that C_u is tangent to C_v if and only if u, v are neighbors.*

Theorem 2.8. *Any finite map admits a circle packing representation in $\mathbb{S}^2$.*

Proof. (Sketch)
At first glance, notice that if the theorem holds for triangulations (maps where all faces have degree 3) it holds also for every map, since each map can have faces cutted by new edges until they become triangulation. So now on we will consider only triangulations.
The algorithm to construct circle packing on triangulations is the following:

1. Find the radii of the circles, and then construct the packing starting from the external face and continue deploying all circles.
   Once the radii are out, to deploy properly we can use the angle between cirles using the fact that we are "packing" a triangulation, so angles should be done.

2. To find the radii we start by an arbitrary assignment, except the three vertices of a starting face that has for now on radii of 1.
   Starting from this face we just deploy the circles of the neighbors for each vertex of the face, in cycling order and going on.

There is a way to prove that this algorithm ends.

□

Definition 2.9. *A rooted map is a map with a distinguished oriented edge e_*, called root edge. The origin of e_* is called the root vertex.*

Definition 2.10. *The class of (rooted) planar maps, up to homeomorphisms that preserve orientation, is denoted by $\mathcal{M}$. The subclass of (rooted) planar maps with exactly n faces is denoted by $\mathcal{M}_n$.*

Proposition 2.11. *Let m be a planar map, so it holds*

$$\sum_{f \in F(m)} deg_m(f) = 2\#E(m).$$

Proof. Each edge in the sum contributes twice (either one for each face if it is incident to two different faces or two if it is incident to a single face).

□

Definition 2.12. *A map is called a quadrangulation if all of his faces have degree 4.*

Definition 2.13. *We will denote by $\mathcal{Q}$ the set of plane rooted quadrangulations, $\mathcal{Q}_n$ the set of plane rooted quadrangulations with exactly n faces, Q_n the random variable uniformly distributed in $\mathcal{Q}_n$.*

Note that unless otherwise stated random variables are always to be considered definited in a general probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Proposition 2.14. *Let $m \in \mathcal{M}_n$. Then $m \in \mathcal{Q}_n$ if and only if $\#E(m) = 2n$ and $\#V(m) = n + 2$.*

Proof. We know that $\#F(m) = n$, and that $\#E(m) = 2\#F(m)$ because m is a quadrangulation. Applying Euler's formula we get $\#V(m) = n + 2$.

□

2.2 CVS bijection

In this subsection we will contruct the bijection between rooted quadrangulations with n-faces and well labelled trees of n edges.

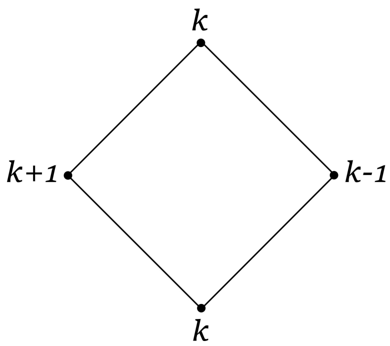
Proposition 2.15. *Let $q \in \mathcal{Q}$, let v_0 be the root vertex of $\mathcal{Q}$, u_1 , u_2 and w_1 , w_2 opposite vertices of a given face f of q. So we have either $d_q(u_1, v_0) = d_q(u_2, v_0)$ or $d_q(w_1, v_0) = d_q(w_2, v_0)$.*

Proof. It follows directly from the fact that for adjacent vertices the graph distance from a third point could be different at most by one.

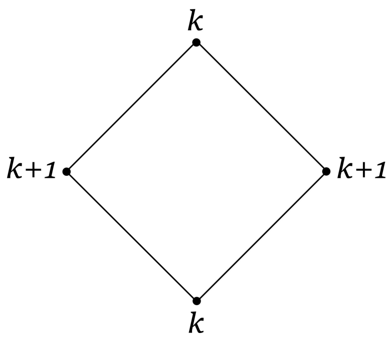
∎

Definition 2.16. *With the above notation, we say that f is simple if only one equality is satisfied, confluent if both are satisfied.*

The following is a simple face.



And the following is a confluent face.



Definition 2.17. *We denote by $\mathcal{P}$ the class of plane trees, by $\mathcal{P}_n$ the class of plane trees with n edges.*

Proposition 2.18. *$\#\mathcal{P}_n = C_n$ where C_n is the n-th Catalan number*

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Proof. (Scketch)

Let $p_n = \#\mathcal{P}_n$. Consider

$$P(z) = \sum_{n\geq 0} p_n z^n$$

the generating function of p_n. We then have

$$P(z) = \sum_{n\geq 0} (\text{rooted trees with } n \text{ nodes} = C_n) z^N,$$

if we forget the orgin and we sum on the possible degree of the root,

$$p_n = [Z^{n-1}](P(z) + P(z^2) + P(z^3) + \ldots).$$

So $C_n = \frac{P(z)}{1-P(z)}$, hence $P(z)$ is such that

$$P(z) = \frac{z}{1-P(z)}.$$

It follows from this recurrence relation using Lagrange formula or defining the binomial for all real that:

$$p_n = \frac{1}{n+1}\binom{2n}{n}.$$

□

Now we discuss the contour process of a tree.

Definition 2.19 (Contour process)**.** *Let $\tau_n \in \mathcal{P}_n$, let v_0 be the root vertex of τ_n. Let $(e_1, \ldots, e_{2n-1})$ be the sequence of (oriented) edges bounding the only face of τ_n, starting with the edge incident to v_0.*
This sequence is called the contour exploration, or contour process, of τ_n.

Denote by $u_i = e_i^-$ the i-th visited vertex in the contour exploration and set

$$D_{\tau_n}(i) = d_{\tau_n}(u_0, u_i)$$

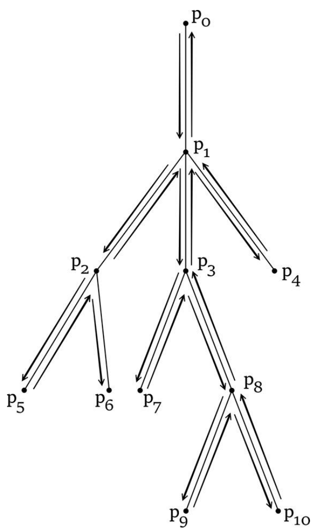
for every $i \in \{0, \ldots, 2n-1\}$. We the set $e_{2n} = e_0$ and $U_{2n} = U_0$. After doing this, we extend by linear interpolation the function D_{τ_n}

$$C_{\tau_n}(s) = (1 - \{s\})D_{\tau_n}([s]) + \{s\}D_{\tau_n}([s]+1),$$

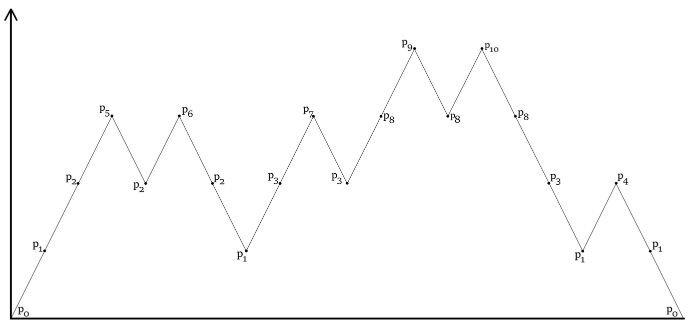
where $\{s\} = s - [s]$ is the fractional part of s.
Note that C_{τ_n} is a non negative path starting and ending at 0.

Definition 2.20 (Contour function)**.** *We call C_{τ_n} defined as above the contour function of τ_n. The set of all contour function (of size n) is $\mathcal{C}_n$.*

The following is an exploration of a tree by the contour process.



And the associated contour function.



Proposition 2.21. *The mapping $f : \mathcal{P}_n \to \mathcal{C}_n$ defined as*

$$f(\tau_n) = C_{\tau_n}$$

is a bijection.

Proof. The number of possible contour 'graph' is exactly the Catalan number. □

Now we go into the core of this section and we show the result of Schaeffer, work of his PhD thesis.

Theorem 2.22. *There exist a bijection between rooted quadrangulations with n faces and well labelled trees with n edges, such that the profile of a rooted quadrangulation is mapped onto the label distribution of the corresponding well-labelled tree.*

First of all we show how to encode quadrangulations as well labelled trees.

Definition 2.23. *Let $q \in \mathcal{Q}_n$ be a quadrangulation, $e_* = (v_0, v_1)$ the root edge of q.*
Now, label each vertex of q with the distance from the vertex to the root vertex, i.e. we contruct a labelling l

$$l : V(q) \to \mathbb{N}$$
$$l(v) = d_q(v_0, v).$$

Then, we define a map

$$\phi : \mathcal{Q}_n \to \mathcal{W}_n$$

in the following way:

- *Contruct a new map q' by dividing the confluent faces of q into two triangular faces with and edge that has for origin and endpoint the vertices with maximal label of the face.*
- *Extract a new subset of edges of q' in the following steps*
  1. *Each edge addes in a confluent face to form q' is choosen.*
  2. *If f is a simple face of q, let v be the vertex with maximal label in f. The edge $\{v, w\}$ in f such that f is on the left side of the directed edge (v, w) is choosen.*
- *The edges choosen above in 1 and 2 are the edges of $\phi(q)$. Then keep v_1 as the root vertex of $\phi(q)$ and discard v_0 and the previous root edge of q. The construction of ϕ is complete.*

Proposition 2.24. *The operator ϕ defined above sends a rooted quadrangulation with n faces to a well labelled tree with n edges.*

Proof. At first glance we show that the vertices of $\phi(q)$ are the vertices of $q \setminus v_0$.

If $v \neq v_0$ there exist $w\ inV(q)$ such that v, w are neighbors. Without loss of generality we can assume

$$l(v) = l(w) - 1,$$

remarking that $l(x) = d_q(v_0, x)$.
So the edge (v, w) is at least incident to one of the following

1. a confluent face
2. a simple face, where v has maximal label
3. two simple faces, where v has intermediate label.

We now prove that in all three cases v is incident to an edge choosen by $\phi(q)$, so v belongs to $V(\phi(q))$.

In case 1, v is the greatest vertex in terms of label of a confluent face, so it belongs to the new edge added to contruct q', hence is choosen.

In case 2, v is the vertex of maximal label in a simple face, so it belongs to an edge choosen by ϕ.

In case 3, suppose (v, w) is incident to two simple faces f' and f''. The vertex v is neighbor to two opposite vertices w', w'' of labels greater than the label of v (otherwise one of the two simple faces has v as maximal label and we are in case 2). So w' and w'' are the vertices of maximal label in f' and f'', so they are both choosen by ϕ. Since the face f' is on the left (right) to the edge (w', v), the face f'' is on the right (left) to (w'', v), so one of these two edges is choosen by ϕ, so $v \in V(\phi(q))$.

Hence, in each scenario $v \in V(\phi(q))$, so

$$V(q) \setminus \{v_0\} \subset V(\phi(q)).$$

Since each vertex of $\phi(q)$ is a former vertex of q', the inclusion above is actually and identity.

Remark that in every planar map that is a quadrangulation q we have $\#V(q) = n + 2$, so having dropped v_0 with ϕ we have

$$\#V(\phi(q)) = n + 1.$$

We also know that $\#E(\phi(q))$ is exactly n, since ϕ takes only one edge from each face of q.
With the above considerations, if $\phi(q)$ has no cycles, then $\phi(q)$ is a forest of trees, and having $\#V(\phi(q)) = n + 1$ and $\#E(\phi(q)) = n$ is exactly a tree.

Suppose now there exist a cycle C in $\phi(q)$. Consider the labels of C, they could be equal or there exists a path of the form $\{k+1, k, k+1\}$ of labels. By selection rules there exist two vertices v, w, inside and outside C, such that

$$l(v) = l(w) = \min(l(x) | x \in V(C)) - 1.$$

The shortest path from v or w to v_0 has to intersect the cycle C for the Jordan's curve theorem (applied to the cycle C). Remark that $l(x) = d_q(v_0, x)$, so this leads to a contradiction. Hence $\phi(q)$ is a tree, and the labelling (l) on $\phi(q)$ is actually a well labelling.

□

Definition 2.25 (Corner)**.** *Let $\tau_n \in \mathcal{P}_n$. Let E_{τ_n} be the contour exploration of τ_n*

$$E_{\tau_n} = (e_0, \dots, e_{2n-1}).$$

A corner is a sector between two consecutive edges of E_{τ_n} around a vertex. The label of a corner is the label of the corresponding vertex. We denote by c^- the vertex associated to the corner c.

Definition 2.26 (Successor)**.** *Let $\tau_n \in \mathcal{W}_n$ with contour exploration* $(e_0, \dots, e_{2n-1})$*. The successor s of a corner $i \in \{0, \dots 2n-1\}$ is*

$$s(i) = \inf\{j > i : l(j) = l(i) - 1\},$$

and it is denoted also by $s(e_i) = e_{s(i)}$.

Let see now how to encode trees as quadrangulations.
Our goal is to construct an operator ψ that acts like the inverse of ϕ precently defined.
Let $\tau \in \mathcal{W}_n$ and let v_0 be the root vertex of τ. Suppose also we have l as label function.

Definition 2.27. *The map $\psi(\tau)$ is defined as follows:*

1. *Introduce a new vertex v_*, labelled with 0, in the (unique) face of τ.*
2. *Introduce new edges, linking each corner e_i with the successor $s(e_i)$, for all $i \in \{0, \dots, 2n-1\}$.*
3. *Delete all edges of the original tree τ.*
4. *The new root of τ is the edge $e_* = (v_*, v_0)$, introduced in 2, since the successor of v_0 can only be v_*.*

Proposition 2.28. *The edges added in the above procedure ψ can be drawn in a way that the resulting graph τ is a planar map.*

Proof. Suppose there exist four different corners $c_1, \dots, c_4$, ordered in this way by the contour exploration, such that they violate the planarity, i.e. $c_3 = s(c_1)$ and $c_4 = s(c_2)$.
This clearly leads to a contradiction, since in this scenario $l(c_3) < l(c_4)$ and $l(c_4) < l(c_2)$, by labelling rule we also have $s(c_1) = c_3$ so $l(c_2) \geq l(c_1)$ and $s(c_2) = c_4$ implies $l(c_3) \geq l(c_2)$, so

$$l(c_2) \leq l(c_3) < l(c_1) \leq l(c_2).$$

□

Proposition 2.29. *Let $\tau_n \in \mathcal{W}_n$, then $\psi(\tau_n)$ is a rooted quadrangulation with n faces.*

Proof. The image $\psi(\tau_n)$ is connected, since for every corner there exists a finite path $(c, s(c), s(s(c)), \dots, v_*)$. So v_* is connected with every corner, so the resulting graph is connected.

The next step is to show that every face of $\psi(\tau_n)$ has degree four. To do this, we show that every face of $\psi(\tau_n)$ is simple or confluent.
Fix an edge of τ_n that has e and $\overline{e}$ in the contour exploration. So we have three possibile scenarios:

1. $l(e^+) = l(e^-) - 1$. So we fix $l(e^+) = k - 1$ and $l(e^-) = k$. Note that we have an edge from e^- to e^+ since $s(e^-) = e^+$.
   Now consider the head of the directed edge $\overline{e}$, we denote this corner by c'. We have $l(c') = l(\overline{e}) = k - 1$.
   The successor of such edge, $s(s(c'))$ is the first corner coming after c' such that has label $k - 2$, so $s(\overline{e}) = s(s(c'))$, hence this four vertices form a simple face.

2. $l(e^+) = l(e^-) + 1$. This case is equivalent to 1 once having changed $\overline{e}$ with e.

3. $l(e^+) = l(e^-)$. Denote by c' the corner of the head of e and by c'' the corner of the head of $\overline{e}$.
   We have
   $$l(e) = l(c') = l(\overline{e}) = l(c'')$$
   so $s(e) = s(c')$ and $s(\overline{e}) = s(s(c''))$, then we have a confluent face with diagonal edge (no more existing) $\{e, \overline{e}\}$.

So every face of $\psi(\tau_n)$ has degree four.
Remark that $\psi(\tau_n)$ has $2n$ edges (twice the edges of τ_n, one for each corner) and $n + 2$ vertices, so it must have n faces for Euler's formula (by the previous proposition we already know it is a planar map).

Hence $\psi(\tau_n)$ is a quadrangulation, and actually it is rooted due to step 4 of ψ definition.

□

So we have finished the proof of the CVS bijection for a well labelled tree, since ψ is the inverse of ϕ.

2.3 Brownian excursions

As we will prove in the next subsection, symmetric random walks converge in scale limit to a Brownian motion (Donsker's invariance principle).
We have seen that for each rooted tree one can associate the contour function. For a random tree uniformly distributed over a certain "regular" class of trees one should expect to the relative random contour function to act like a random walk.
Differently from symmetric random walks the contour function cannot be negative and has for endpoint the value of the starting that is 0.
So the natural object to introduce is the Brownian excursion, that informally acts like a Brownian motion conditioned to stay positive for all time except the starting and the endpoint that is 0.

Definition 2.30. *Let $\{B_t\}_{t \geq 0}$ be a standard Brownian motion, so we set*

$$d = \inf\{t \geq 1 : B_t = 0\},$$
$$g = \sup\{t \leq 1 : B_t = 0\}.$$

Remark that $\mathbb{P}\{B_0 = 0\} = 1$ *and* $\mathbb{P}\{B_1 = 0\} = 0$ *so almost surely we have* $g < 1 < d$.

A normalized Brownian excursion

$$\boldsymbol{e} = \{\boldsymbol{e}_t : t \in [0,1]\}$$

is a stochastic process such that

$$\boldsymbol{e}_t = \frac{|B_{g+t(d-g)}|}{\sqrt{d-g}}.$$

Roughly speaking, it is the process B_t *with* $t \in [g,d]$ *rescaled to be a process at values in* $[0,1]$.

Now we give the definition of compact real tree.

Definition 2.31 ($\mathbb{R}$-tree)**.** *A metric space* (τ, d) *is a real tree or* $\mathbb{R}$*-tree is it holds, for any* $x, y \in \tau$ *:*

- *There exists a unique isometric map* $\phi_{x,y} : [0, d(x,y)] \to \tau$ *such that* $\phi_{x,y}(0) = x$ *and* $\phi_{x,y}(d(x,y)) = y$.
- *If* $\phi' : [0,1] \to \tau$ *is continuous and injective such that* $\phi'_{x,y}(0) = x$ *and* $\phi'_{x,y}(d(x,y)) = y$*, then* $\phi'([0,1]) = \phi([0, d(x,y)])$.

Definition 2.32. *Let* $g : [0,1] \to \mathbb{R}_+$ *non negative continuous function such that* $g(0) = g(1) = 0$*. So for all* s, t *in*$[0,1]$ *we set*

$$m_g(s,t) = \inf_{r \in [s \wedge t, s \vee t]} g(r)$$

and

$$d_g(s,t) = g(s) + g(t) - 2m_g(s,t).$$

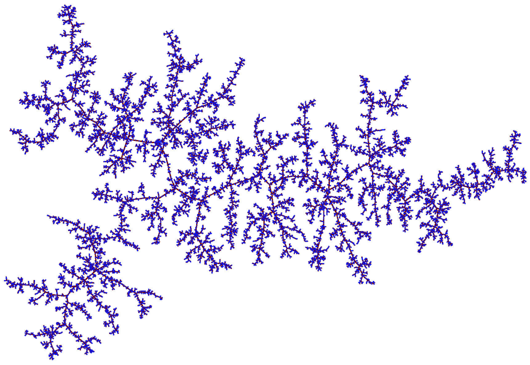
We observe that d_g defined as above is a pseudo-metric on $[0,1]$.
So, we put $s \sim_g t$ if and only if $d_g(s,t) = 0$, or equivalently if and only if $g(s) = g(t) = m_g(s,t)$.
Hence, given $\tau_g = [0,1]/\sim_g$ we have that (τ_g, d_g) is a metric space.
We observe that with this definition (τ_g, d_g) is a compact $\mathbb{R}$-tree. So we can speak about $\mathbb{R}$-tree coded by g.

Definition 2.33 (CRT)**.** *The CRT is the random* $\mathbb{R}$*-tree*

$$(\tau_{\boldsymbol{e}}, d_{\boldsymbol{e}})$$

coded by a Brownian excursion $\boldsymbol{e}$.

The following is a picture from the web of the CRT.



At the moment we have introduced planar maps but we don't get yet a topology in the planar map space that permits us to speak about convergence and limits. So we introduce the Gromov-Hausdorff topology, that informally express how two maps differ by checking in all the space where they are both embeddable minimizing on the possible embedding the Hausdorff distance between the images.

Definition 2.34. *Let (X, d) be a metric space, A, B non empty subsets. We denote by $d_H(A, B)$ the Hausdorff distance between the two subsets, defined as follows*

$$d_H(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} d(a, b), \sup_{b\in B} \inf_{a\in A} d(a, b)\}.$$

Definition 2.35. *Let (X, d_x), (Y, d_y) be two compact metric spaces. We denote by $d_{GH}(X, Y)$ the Gromov-Hausdorff distance defined as follows*

$$d_{GH}(X, Y) = \inf d_H(\phi_X(X), \phi_Y(Y)),$$

where the infimum is taken over all metric spaces (Z, d_z) and all isometric embeddings

$$\phi_X : X \to Z,$$
$$\phi_Y : Y \to Z.$$

Definition 2.36 (Correspondence)**.** *Let X, Y be two sets, we say that $\mathcal{R} \subset X \times Y$ is a correspondence between X and Y if for every $x \in X$ it exists at least one $y \in Y$ such that $(x, y) \in \mathcal{R}$, and the same for y.*
We denote by $Cor(X, Y)$ the set of all the correspondences between X and Y.

Definition 2.37 (Distortion)**.** *Given a correspondence $\mathcal{R}$ between metric spaces (X, d_x) and (Y, d_y), the distortion ($dis(\mathcal{R})$) is defined as*

$$dis(\mathcal{R}) = \sup\{|d_x(x, y) - d_y(x', y')| : (x, x'), (y, y') \in \mathcal{R}\}.$$

Observe that the Gromov-Hausdorff distance can be seen in terms of distortion:

$$d_{GH}(X,Y) = \frac{1}{2}\inf\{\mathrm{dis}(\mathcal{R}) : \mathcal{R} \in \mathrm{Cor}(X,Y)\}.$$

It also holds that (τ, d_{GH}) is a complete and separable metric space.

In the following definition we introduce the Brownian snake, that is used to contruct the Brownian map. Actually not all of the Brownian snake structure is need, for our work it only sufficient to consider the "head" of such process.

Definition 2.38 (Brownian snake)**.** *Let $\boldsymbol{e}$ be a normalized Brownian excursion. We call Brownian snake the following path valued stochastic process*

$$W_s = \{W_s(t) : t \in [0, \boldsymbol{e}_s]\}$$

such that:

- *For all $s \in [0,1]$, $t \to W_s(t)$ is a standard Brownian motion defined for $t \in [0, \boldsymbol{e}_s]$.*
- *$\{W_s : s \in [0,1]\}$ is a continuous Markov process satisfying for any $s_1 < s_2$*
  1. *$\{W_{s_1}(t) : t \in [0, m_{\boldsymbol{e}}(s_1,s_2)]\} = \{W_{s_2}(t) : t \in [0, m_{\boldsymbol{e}}(s_1,s_2)]\}$*
  2. *$\{W_{s_2}(m_{\boldsymbol{e}}(s_1,s_2)+t) : t \in [0, \mathbf{e} - m_{\boldsymbol{e}}(s_1,s_2)]\}$ is a standar Brownian motion starting from $W_{s_2}(m_{\boldsymbol{e}}(s_1,s_2))$ and independent of W_{s_1}.*

Definition 2.39. *Let $(W_s)_{s\in[0,1]}$ be the Brownian snake and $W'_s = W_s(\boldsymbol{e}_s)$. The contour pair*

$$X_s = (\boldsymbol{e}_s, W'_s)$$

such that $s \in [0,1]$ is called head of the Brownian snake (driven by $\boldsymbol{e}$).

The following is a picture from the web of the Brownian snake.

2.4 Convergence results

In this subsection we show the preliminar convergence results that make the building blocks for the proof of the main result in the next section.

Definition 2.40. *We say that a martingale $\{X_n\}_{n\in\mathbb{N}}$ is a binary splitting if for any $\{x_0,\dots,x_n\}\in\mathbb{R}$ the event*

$$A(x_0,\dots,x_n)=\{X_0=x_0,\dots,X_n=x_n\}$$

of positive probability, the random variable X_{n+1} conditioned on $A(x_0,\dots,x_n)$ is supported in at most two values.

Proposition 2.41. *Let X be a random variable with $\mathbb{E}[X^2]<\infty$. Then there exists a binary splitting martingale $\{X_n\}_{n\in\mathbb{N}}$ such that*

$$X_n\to X$$

almost surely and in L^2.

Proof. We define the martingale $\{X_n\}_{n\in\mathbb{N}}$ and the associate filtration $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ rescursively. Let

- $\mathcal{G}_0$ be the trivial σ-algebra,
- $X_0=\mathbb{E}[X]$,
- $\xi_0=\begin{cases}1, & \text{if } X\geq X_0\\ -1, & \text{if } X<X_0\end{cases}.$

And for any $n>0$ let

- $\mathcal{G}_n=\sigma(\xi_0,\dots\xi_{n-1})$,
- $X_n=\mathbb{E}[X|\mathcal{G}_n]$,
- $\xi_n=\begin{cases}1, & \text{if } X\geq X_n\\ -1, & \text{if } X<X_n\end{cases}.$

Note that $\mathcal{G}_n$ is generated by a partition P_n of the probability space into 2^n sets of the form $A(x_0,\dots,x_n)$.
Each element of P_n is the union of two elements of P_{n+1}, so the martingale $\{X_n\}_{n\in\mathbb{N}}$ is a binary splitting.
We also have

$$\mathbb{E}[X^2]=\mathbb{E}[(X-X_n)^2]+\mathbb{E}[X_n^2]\geq\mathbb{E}[X_n^2].$$

So $\{X_n\}_{n\in\mathbb{N}}$ is bounded in L^2, hence

$$X_n\to X_\infty=\mathbb{E}[X|\mathcal{G}_\infty]\text{almost surely and in } L^2,$$

where $\mathcal{G}_\infty = \sigma(\cup_{i=0}^\infty \mathcal{G}_i)$.
To conclude we have to show that $X = X_\infty$ almost surely. Note that

$$\lim_{n\to\infty} \xi_n(X - X_{n+1}) = |X - X_\infty|$$

since:

- If $X(\omega) = X_\infty(\omega)$ it is trivial.
- If $X(\omega) < X_\infty(\omega)$ there exists N such that $X(\omega) < X_n(\omega)$ for any $n > N$, so $\xi_n = -1$.
- If $X(\omega) > X_\infty(\omega)$ there exists N such that $X(\omega) > X_n(\omega)$ for any $n > N$, so $\xi_n = 1$.

Now, ξ_n is $\mathcal{G}_{n+1}$ measurable, so

$$\mathbb{E}[\xi_n(X - X_{n+1})] = \mathbb{E}[\mathbb{E}[\xi_n(X - X_{n+1})|\mathcal{G}_{n+1}]] = 0$$

hence we can conclude that $\mathbb{E}[|X - X_\infty|] = 0$.

□

Theorem 2.42 (Skorokhod embedding theorem)**.** *Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion. Let X be a real valued random variable such that $\mathbb{E}[X] = 0$ and $\mathbb{E}[X^2] < \infty$.*
So there exist a stopping time T, with respect to the natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ of the Brownian motion such that B_T has the law of X and

$$\mathbb{E}[T] = \mathbb{E}[X^2].$$

Proof. Using the above proposition, we construct a binary splitting martingale $\{X_n\}_{n\in\mathbb{N}}$ such that $X_n \to X$ almost surely and in L^2.
As X_n conditioned on $A(x_0, \ldots, x_n)$ is supported on at most two values, we construct a sequence of stopping time $T_0, T_1, \ldots$ such that B_{T_n} is distributed as X_n and

$$\mathbb{E}[T_n] = \mathbb{E}[X_n^2],$$

to do this if for example a, b are such values, take $T = \inf\{t : B_t \in \{a, b\}\}$.
So we have $T_n \uparrow T$, so $T_n \uparrow T$ almost surely, and by monotone convergence theorem

$$\mathbb{E}[T] = \lim_{n\uparrow\infty} \mathbb{E}[T_n] = \lim_{n\uparrow\infty} \mathbb{E}[X_0^2] = \mathbb{E}[X^2].$$

Now note that B_{T_n} converges in distribution to X and almost surely to B_T (by continuity), so B_T is distributed as X.

□

Proposition 2.43. *Suppose $\{B_t\}_{t\geq 0}$ is a linear Brownian motion. Then, for any random variable X such that*

- $\mathbb{E}[X] = 0$

- *Var*$[X] = 1$

there exists a sequence of stopping times

$$0 = T_0 \leq T_1 \leq T_2 \leq T_3 \leq \dots$$

with respect to the natural filtration of the Brownian motion, such that

1. $\{B_{T_n}\}_{n\geq 0}$ *has the distribution of the random walk with increment given by the law of* X*,*
2. *the sequence of functions* $\{S_n^*\}_{n\geq 0}$ *construct from this random walk is such that*
$$\lim_{n\to\infty} \mathbb{P}\{\sup_{0\leq t\leq 1} |\frac{B_{nt}}{\sqrt{n}} - S_{nt}^*| > \varepsilon\} = 0.$$

Proof. We use Skorokhod embedding, so we define T_1 stopping time such that

$$\mathbb{E}[T_1] = 1$$

$$B_{T_1} = X \text{ in distribution.}$$

Then

$$\{B_t^2\}_{t\geq 0} = \{B_{T_1+t} - B_{T_1}\}_{t\geq 0}$$

is a Brownian motion independent of $\mathcal{F}_{T_1}^*$, in particular independent of (T_1, B_{T_1}). So we can define a stopping time T_2' with respect to the natural filtration of B_t^2 such that

$$\mathbb{E}[T_2'] = 1$$

$$B_{T_2'}^2 = X \text{ in distribution.}$$

Then $T_2 = T_1 + T_2'$ is a stopping time for B_t with $\mathbb{E}[T_2] = 2$, such that B_{T_2} is the second value in a random walk with increments given by the law of X.
So we can contruct recursively

$$0 = T_0 \leq T_1 \leq T_2 \leq T_3 \leq \dots$$

such that $S_n = B_{T_n}$ is the embedded random walk, and $\mathbb{E}[T_n] = n$.
Now let $W_t^n = \frac{B_{nt}}{\sqrt{n}}$ and let

$$A_n = \{\text{exists } t \in [0,1) \text{ such that } |S_n^* - W_t^n| > \varepsilon\}.$$

We want to prove that $\mathbb{P}(A_n) \to 0$. Let $k = k(t)$ the (unique) integer with

$$\frac{k-1}{n} \leq t \leq \frac{k}{n}.$$

Since S_n^* is linear we have

$$A_n \subset \{\text{exists } t \in [0,1) \text{ such that } |\frac{S_k}{\sqrt{n}} - W_t^n| > \varepsilon\} \cup$$

$$\cup \{\text{exists } t \in [0,1) \text{ such that } |\frac{S_{k-1}}{\sqrt{n}} - W_t^n| > \varepsilon\},$$

And since $S_k = B_{T_k} = \sqrt{n} W^n_{\frac{T_k}{n}}$ we have

$$A_n \subset A^*_n = \{\text{exists } t \in [0,1) \text{ such that } |W^n_{\frac{T_k}{n}} - W^n_t| > \varepsilon\} \cup$$

$$\cup \{\text{exists } t \in [0,1) \text{ such that } |W^n_{\frac{T_{k-1}}{n}} - W^n_t| > \varepsilon\}.$$

So given $0 < \delta < 1$, the event $A^*_n \subset (D^*_n \cup C^*_n)$ where

$$D^*_n = \{\text{exist } s,t \in [0,2] \text{ such that for } |s-t| < \delta \text{ we have } |W^n_s - W^n_t| > \varepsilon\}$$

$$C^*_n = \{\text{exists } t \in [0,1] \text{ such that } |T_{\frac{k}{n}-t}| \vee |T_{\frac{k-1}{n}-t}| \geq \delta\}.$$

Now we note that $\mathbb{P}(D^*_n)$ does not depend on n, and by the uniform continuity of the Brownian motion in $[0,2]$ as $\delta \to 0$ we have $\mathbb{P}(D^*_n) \to 0$.
So fixed $\delta > 0$, we have to show that

$$\mathbb{P}(C^*_n) \to 0$$

as $n \to \infty$. It holds, for large numbers on the sequence $T_k - T_{k-1}$ of i.i.d. random variables with $\mathbb{E}[T_k - T_{k-1}] = 1$,

$$\lim_{n\to\infty} \frac{T_n}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (T_k - T_{k-1}) = 1 \text{ almost surely.}$$

Note also that, for any sequence $\{a_n\}_{n\in\mathbb{N}} \in \mathbb{R}$ (so deterministic) it holds

$$\lim_{n\to\infty} \frac{a_n}{n} = 1 \Rightarrow \lim_{n\to\infty} \sup_{0\leq k\leq n} \frac{|a_k - k|}{n} = 0.$$

So we have

$$\lim_{n\to\infty} \mathbb{P}\{\sup_{0\leq k\leq n} \frac{|T_k - k|}{n} \geq \delta\} = 0.$$

Recall that $t \in [\frac{k-1}{n}, \frac{k}{n}]$ and let $n > \frac{2}{\delta}$. Then

$$\mathbb{P}(C^*_n) \leq \mathbb{P}(\{\sup_{1\leq k\leq n} \frac{(T_k - (k-1)) \vee (k - T_{k-1})}{n}\}) \leq$$

$$\leq \mathbb{P}(\{\sup_{1\leq k\leq n} \frac{T_k - k}{n} \geq \frac{\delta}{2}\}) + \mathbb{P}(\{\sup_{1\leq k\leq n} \frac{k-1-T_{k-1}}{n} \geq \frac{\delta}{2}\}),$$

and they both converge to 0 for what proved above.

□

We are now able to prove Donsker's invariance principle, that gives to us the convergence in scale limit of a random walk to a Brownian motion.

Theorem 2.44 (Donsker's invariance principle)**.** *Let $\{S_n\}_{n\geq 0}$ be a symmetric simple random walk, i.e.*

$$S_n = \sum_{k=1}^{n} X_k$$

where $\mathbb{E}[X_k] = 1$ and $Var[X_k] = 1$.
Let S be the continuous linear interpolation of S_n. Let $\tilde{S}_n(t) = \frac{S(nt)}{\sqrt{n}}$ and $B = \{B_t\}_{t\in[0,1]}$ a standard Brownian motion. So $\tilde{S}_n \to B$ as $n \to \infty$, where the convergence is intended to be in distribution on the space of $(\mathcal{C}[0,1], \|\|_\infty)$.

Proof. Let $\{B_t\}_{t\in[0,1]}$ be a standard Brownian motion, so $\tilde{\mathrm{B}}^n = \frac{B_{nt}}{\sqrt{n}}$ is also a standard Brownian motion.
Suppose now $K \subset \mathcal{C}[0,1]$ is closed, and for any $\varepsilon > 0$ let

$$K_\varepsilon = \{f \in \mathcal{C}[0,1] : \exists g \in K : \|f - g\|_\infty \leq \varepsilon\},$$

then

$$\mathbb{P}\{\tilde{\mathrm{S}}_n \in K\} \leq \mathbb{P}\{\tilde{\mathrm{B}}_n \in K_\varepsilon\} + \mathbb{P}\{\|\tilde{\mathrm{S}}_n - \tilde{\mathrm{B}}_n\|_\infty > \varepsilon\},$$

so

$$\mathbb{P}\{\tilde{\mathrm{S}}_n \in K\} \leq \mathbb{P}\{B \in K_\varepsilon\} + \mathbb{P}\{\|\tilde{\mathrm{S}}_n - \tilde{\mathrm{B}}_n\|_\infty > \varepsilon\},$$

and by the previous proposition we have $\lim_{n\to\infty} \mathbb{P}\{\|\tilde{\mathrm{S}}_n - \tilde{\mathrm{B}}_n\|_\infty > \varepsilon\} = 0$. Now, since K is closed

$$\lim_{\varepsilon\to 0} \mathbb{P}\{B \in K_\varepsilon\} = \mathbb{P}\{B \in \bigcap_{\varepsilon\to 0} K_\varepsilon\} = \mathbb{P}\{B \in K\}.$$

So we get that for any K closed

$$\lim_{n\to\infty} \sup \mathbb{P}\{\tilde{\mathrm{S}}_n \in K\} \leq \mathbb{P}\{B \in K\},$$

so $\tilde{\mathrm{S}}_n \to B$ in distribution for a well known criterium of convergence in distribution.

□

Now we discuss the convergence of the (suitably rescaled) contour functions of random tree uniformly distributed on $\mathcal{T}_n$ to the normalized Brownian excursion. These contour functions ends up to be random walks conditioned to stay positive. So in some sense we study a conditional version of Donsker's theorem.

Definition 2.45. *Let θ be a Galton-Watson tree with distribution μ_0, the critical geometric offspring distribution*

$$\mu_0(k) = 2^{-k-1}.$$

We write Π_{μ_0} for the distribution of θ on $\mathcal{T}$

Observe that $\Pi_{\mu_0}(\tau)$ only depends on $|\tau|$. So, for every integer $k \geq 0$ the conditional probability distribution $\Pi_{\mu_0}(\cdot\,||\tau| = k)$ is the uniform probability measure on $\mathcal{T}_k$.

Definition 2.46. *Let $(S_n)_{n\geq 0}$ be a simple random walk, i.e.*

$$S_n = X_1 + X_2 + \ldots + X_n$$

Where $X_1, X_2, \ldots$ are i.i.d. random variables with distribution $\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = \frac{1}{2}$.
Let $T = \inf\{n \geq 0 : S_n = -1\}$. Remark that $T < \infty$ almost surely. So the random finite path

$$(S_0, S_1, \ldots, S_{T-1})$$

is called an excursion of simple random walk.

Proposition 2.47. *Let θ be a μ_0 Galton-Watson tree. The the contour function of θ is an excursion of simple random walk.*

Proof. The statement of the proposition is equivalent to saying that if θ is coded by an excursion of simple random walk, then is a μ_0 Galton Watson tree. We introduce the upcrossing times of the random walk from 0 to 1

$$U_1 = \inf\{n \geq 0 : S_n = 1\}, V_1 = \inf\{n \geq U_1 : S_n = 0\}$$
$$U_{j+1} = \inf\{n \geq V_j : S_n = 1\}, V_{j+1} = \inf\{n \geq U_{j+1} : S_n = 0\}.$$

Let $K = \sup\{j : U_j \leq T\}$, where $\sup \emptyset = 0$. By contruction we have that the number of children of the origin $\emptyset$ of θ is exactly K. Moreover, for every $1 \leq i \leq K$ the contour function associated with the subtree $T_i\theta = \{u \in \mathcal{T} : iu \in \theta\}$ is the path ω_i, with

$$\omega_i(n) = S_{(U_i+n)\wedge(V_i-1)} - 1, 0 \leq n \leq V_i - U_i - 1.$$

So K is distributed according to μ_0 and the paths $\omega_1, \ldots, \omega_k$ are, conditionally on $K = k$, k independent excursions of simple random walk. So θ is a μ_0 Galton-Watson tree.

□

Let now $\beta_t = |B_t|$ where B_t is a Brownian motion. We set

$$L_t^0 = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t ds \mathbf{1}_{[0,\varepsilon]}(\beta_s),$$

and $\sigma_l = \inf\{t \geq 0 : L_t^0 > l\}$, for every $l \geq 0$. For any $l \in D$, where D is the set of discontinuity times of the mapping $l \to \sigma_l$, we define the excursion $e_l = (e_l(t))_{t\geq 0}$ with the following

$$e_l(t) = \begin{cases} \beta_{\sigma_{l-}+t}, & \text{if } 0 \leq t \leq \sigma_l - \sigma_{l-} \\ 0, & \text{if } t > \sigma_l - \sigma_{l-} \end{cases}.$$

Definition 2.48. *Let E be the set of (non identically zero) excursion functions. We also set*

$$\zeta(e) = \sup\{s > 0 : e(s) > 0\}.$$

The space E is equipped with the metric

$$d(e, e') = \sup_{t \geq 0} |e(t) - e'(t)| + |\zeta(e) - \zeta(e')|.$$

Theorem 2.49. *The point measure*

$$\sum_{l \in D} \delta_{(l, e_l)}(dsde)$$

is a Poisson measure on $\mathbb{R}_+ \times E$ with intensity

$$2ds \otimes \boldsymbol{n}(de)$$

where $\boldsymbol{n}(de)$ is a σ-finite measure on E.

The measure $\mathbf{n}(de)$ is called the Itô measure of positive excursion of linear Brownian motion. By a consequence of this theorem we have

- $\mathbf{n}(\max_{t \geq 0} e(t) > \varepsilon) = \frac{1}{2\varepsilon}$,
- $\mathbf{n}(\zeta(e) > \varepsilon) = \frac{1}{\sqrt{2\pi\varepsilon}}$.

Definition 2.50. *We write $\boldsymbol{n}_{(s)}$ for $\boldsymbol{n}_{(s)} = \boldsymbol{n}(\cdot\,|\zeta = s)$. With this notation $\boldsymbol{n}_{(1)}$ is the law of the normalized Brownian excursion.*

Definition 2.51. *For every $t > 0$ and $x > 0$, we set*

$$q_t(x) = \frac{x}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}}$$

and for every $t > 0$, $x, y \in \mathbb{R}$

$$p_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}}.$$

Proposition 2.52. *The measure $\boldsymbol{n}$ is the only σ-finite measure on E that satisfies:*

1. *For every $t > 0$, $f \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$*

$$\boldsymbol{n}(f(e(t)) \boldsymbol{1}_{\zeta > t}) = \int_0^\infty f(x) q_t(x) dx.$$

2. *Let $t > 0$, under the conditional probability measure $\boldsymbol{n}(\cdot\,|\zeta > t)$ the process $(e(t+r))_{r\geq 0}$ is Markov with the transition kernels of Brownian motion stopped upon hitting* 0.

If $\mathcal{F}_t$ denote the σ-field on E generated by $r \to e(r)$ for $0 \leq r \leq t$, we have

$$\frac{d\mathbf{n}_{(1)}}{d\mathbf{n}}|_{\mathcal{F}_t}(e) = 2\sqrt{2\pi} q_{1-t}(e(t)).$$

So for every $p \geq 1$, $t_1, \ldots, t_p < 1$ the distribution of $(e(t_1), \ldots, e(t_n))$ under $\mathbf{n}_{(1)}(de)$ has density

$$2\sqrt{2\pi} q_{t_1}(x_1) p^*_{t_2-t_1}(x_1, x_2) p^*_{t_3-t_2}(x_2, x_3) \ldots p^*_{t_p-t_1}(x_{p-1}, x_p) q_{1-t_p}(x_p)$$

where

$$p^*_t(x,y) = p_t(x,y) - p_t(x,-y), t > 0, x, y > 0$$

is the transition density of Brownian motion stopped when hitting 0.

As expected, we show the convergence of contour functions of T_n to a normalized Brownian excursion.

Theorem 2.53. *Let $k \geq 1$, let T_k uniformly distributed over $\mathcal{T}_k$, and let $(C_k(t))_{t\geq 0}$ be its contour function. Then*

$$(\frac{1}{\sqrt{2k}} C_k(2kt)_{0\leq t\leq 1}) \to (\boldsymbol{e}_t)_{0\leq t\leq 1},$$

where $\boldsymbol{e}$ is distributed according to $\boldsymbol{n}_{(1)}$ and the convergence holds in the sense of weak convergence on the space $\mathcal{C}([0,1], \mathbb{R}_+)$ with the topology of uniform convergence.

Proof. We know that $\Pi_{\mu_0}(\cdot\,||\tau| = k)$ coincides with the uniform distribution over $\mathcal{T}_k$, so by a previous proposition we get that $(C_k(0), C_k(1), \ldots, C_k(2k))$ is distributed as an excursion of simple random walk conditioned to have lenght $2k$. With the notation already used, $(S_n)_{n\geq 0}$ is the simple random walk and $T = \inf\{n \geq 0 S_n = -1\}$. So our goal is to verify that the law of

$$(\frac{1}{\sqrt{2k}} S_{[2kt]})_{0\leq t\leq 1}$$

under $\mathbb{P}(\cdot\,|T = 2k+1)$ converges to $\mathbf{n}_{(1)}$ as $k \to \infty$.
The proof is divided in two parts, the first to establish the convergence of finite-dimension marginals and the second to prove the tightness of the sequence of laws.

Finite-dimensional marginals. Consider first the one dimensional marginals, so for a fixed $t \in (0,1)$ we have to verify

$$\lim_{k\to\infty} \sqrt{2k}\mathbb{P}(S_{[skt]} = [x\sqrt{2k}] \text{ or } [x\sqrt{2k}] + 1 | T = 2k+1) = 4\sqrt{2\pi} q_t(x) q_{1-t}(x)$$

uniformly when x varies over a compact subset of $(0,\infty)$. Using the formula above for marginals we see that the law of $(2k)^{-\frac{1}{2}}S_{[2kt]}$ under $\mathbb{P}(\cdot\,|T=2k+1)$ converges to the law of $e(t)$ under $\mathbf{n}_{(1)}(de)$.
Recall a case of classical local limit theorems, for every $\varepsilon>0$

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}\sup_{s\geq\varepsilon}|\sqrt{n}\mathbb{P}(S_{[ns]}=[x\sqrt{n}]\text{ or }[x\sqrt{n}]+1)-2p_s(0,x)|=0.$$

The second step to prove the statement for the one dimensional marginals is the following, for every $l\in\mathbb{Z}_+$ and every integer $n\geq 1$,

$$P_l(T=n)=\frac{l+1}{n}P_l(S_n=-1),$$

where P_l is a probability measure under which the simple random walk S starts from $l\in\mathbb{Z}$. To prove this we note that

$$P_l(T=n)=\frac{1}{2}P_l(S_{n-1}=0,T>n-1),$$

and on the other hand

$$P_l(S_{n-1}=0,T>n-1)=P_l(S_{n-1}=0)-P_l(S_{n-1}=0,T\leq n-1)=$$

$$=P_l(S_{n-1}=0)-P_l(S_{n-1}=-2,T\leq n-1)=P_l(S_{n-1}=0)-P_l(S_{n-1}=2).$$

So we have

$$P_l(T=n)=\frac{1}{2}(P_l(S_{n-1}=0)-P_l(S_{n-1}=-2)).$$

So to prove

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}\sup_{s\geq\varepsilon}|\sqrt{n}\mathbb{P}(S_{[ns]}=[x\sqrt{n}]\text{ or }[x\sqrt{n}]+1)-2p_s(0,x)|=0$$

we first let for $i\in\{1,\dots,2k\}$ and $l\in\mathbb{Z}_+$,

$$\mathbb{P}(S_i=l|T=2k+1)=\frac{\mathbb{P}(\{S_i=l\}\cap\{T=2k+1\})}{\mathbb{P}(T=2k+1)},$$

and applying Markov property of S

$$\mathbb{P}(\{S_i=l\}\cap\{T=2k+1\})=\mathbb{P}(S_i=l,T>i)P_l(T=2k+1-i).$$

By time reversal argument it also holds

$$\mathbb{P}(S_i=l,T>i)=2P_l(T=i+1).$$

So we obtain

$$\mathbb{P}(S_i=l|T=2k+1)=\frac{2P_l(T=i+1)P_l(T=2k+1-i)}{\mathbb{P}(T=2k+1)}=$$

$$=\frac{2(2k+1)(l+1)^2}{(i+1)(2k+1-i)}\frac{P_l(S_{i+1}=-1)P_l(S_{2k+1-i}=-1)}{\mathbb{P}(S_{2k+1}=-1)}.$$

In the equality above let $i=[2kt]$ and $l=[x\sqrt{2k}]$ or $l=[x\sqrt{2k}]+1$. By the local limit result mentioned above

$$\frac{2(2k+1)([x\sqrt{2k}]+1)^2}{([2kt]+1)(2k+1-[2kt])}\times\frac{1}{\mathbb{P}(S_{2k+1}=-1)}\approx 2\sqrt{2\pi}(\frac{k}{2})^{\frac{1}{2}}\frac{x^2}{t(1-t)}.$$

Now we apply again the second step result above to get

$$P_{[x\sqrt{2k}]}(S_{[2kt]+1}=-1)P_{[x\sqrt{2k}]}(S_{2k+1-[2kt]}=-1)+$$

$$+P_{[x\sqrt{2k}]+1}(S_{[2kt]+1}=-1)P_{[x\sqrt{2k}]+1}(S_{2k+1-[2kt]}=-1)\approx 2k^{-1}p_t(0,x)p_{1-t}(0,x).$$

Since $q_t(x)=\frac{x}{t}p_t(0,x)$ we have proved the one dimensional marginals case. Higher order marginals follow a similar argument, we just sketch the argument in case of two marginals. If $0<i<j<2k$ and if $l,m\in\mathbb{N}$ we have

$$\mathbb{P}(S_i=l,S_j=m,T=2k+1)=2P_l(T=i+1)P_l(S_{j-i}=m,T>j-i)P_m(T=k+1-j).$$

The only term different from the case of one dimension marginals is

$$P_l(S_{j-i}=m,T>j-i)=P_l(S_{j-i}=m)-P_l(S_{j-i}=-m-2).$$

Using again the same tools we obtain

$$P_{[x\sqrt{2k}]}(S_{[2kt]-[2ks]}=[y\sqrt{2k}])+P_{[x\sqrt{2k}]+1}(S_{[2kt]-[2ks]}=[y\sqrt{2k}])\approx(2k)^{-\frac{1}{2}}p^*_{t-s}(x,y).$$

Tightness. Let $x_0,\dots,x_{2k}$ be a contour exploration for a fixed k, let $i\in\{0,1,\dots,2k-1\}$. So for every $j\in\{0,1,\dots,2k\}$ we set

$$x_j^{(i)}=x_i+x_{i\oplus j}-2\min_{i\wedge(i\oplus j)\le n\le i\vee(i\oplus j)}x_n$$

where $i\oplus j=i+j \bmod(2k)$. Observe that the mapping

$$\Phi_i:(x_0,\dots,x_{2k})\to(x_0^{(i)},\dots,x_{2k}^{(i)})$$

is a bijection from the set of contour process of lenght $2k$ onto itself. Now we set

$$\overline{C}_k^{i,j}=\min_{i\wedge j\le n\le i\vee j}C_k(n).$$

It holds from the bijection the identity in distribution

$$(C_k(i)+C_k(i\oplus j)-2\overline{C}_k^{i,i\oplus j})_{0\le j\le 2k}=(C_k(j))_{0\le j\le 2k}.$$

If we would have that for any integer $p\ge 1$ there exists a constant K_p such that for every $k\ge 1$ and every $i\in\{0,\dots,2k\}$

$$\mathbb{E}[C_k(i)^{2p}]\le K_p i^p$$

this would imply that for $0\le i\le j\le 2k$

$$\mathbb{E}[(C_k(j)-C_k(i))^{2p}]\le\mathbb{E}[(C_k(i)+C_k(j)-2\overline{C}_k^{i,j})^{2p}]=\mathbb{E}[C_k(j-i)^{2p}]\le K_p(j-i)^p.$$

So to complete the tightness we have to proof the above bound. We restric without loss of generality to the case $1 \leq i \leq k$. Since $C_k(i)$ has the same distribution of S_i under $\mathbb{P}(\cdot\,|T = 2k+1)$ we have

$$\mathbb{P}(C_k(i) = l) = \frac{2(2k+1)(l+1)^2}{(i+1)(2k+1-i)} \frac{P_l(S_{i+1} = -1)P_l(S_{2k+1-i} = -1)}{\mathbb{P}(S_{2k+1} = -1)}.$$

By the local limit theorem result mentioned above we have the existence of two positive costant c_1 and c_2 such that

- $\mathbb{P}(S_{2k+1} = -1) \geq c_0(2k)^{-\frac{1}{2}}$
- $P_l(S_{2k+1-i} = -1) \leq c_1(2k)^{-\frac{1}{2}}$

then

$$\mathbb{P}(C_k(i) = l) \leq 4c_1(c_0)^{-1}\frac{(l+1)^2}{i+1}P_l(S_{i+1} = -1) = 4c_1(c_0)^{-1}\frac{(l+1)^2}{i+1}\mathbb{P}(S_{i+1} = l+1).$$

This leads to

$$\mathbb{E}[C_k(i)^{2p}] = \sum_{l=0}^{\infty} l^{2p}\mathbb{P}(C_k(i) = l) \leq \frac{4c_1(c_0)^{-1}}{i+1}\sum_{l=0}^{\infty} l^{2p}(l+1)^2\mathbb{P}(S_{i+1} = l+1) \leq$$

$$\leq \frac{4c_1(c_0)^{-1}}{i+1}\mathbb{E}[(S_{i+1})^{2p+2}].$$

Since $\mathbb{E}[(S_{i+1})^{2p+2}] \leq K'_p(i+1)^{p+1}$ for some constant K'_p independent of i, this completes the proof of the tightness and the proof of the theorem. $\square$

3 The Brownian Map

In the previous section we studied how rooted quadrangulations can be encoded by well labelled trees. We have also discussed the convergence of uniformly distributed contour functions to the Brownian excursion. Using the CVS bijection and the contour process a rooted quadrangulation is hence identified by a pair made by the contour function of the tree and the labelling of the tree.
In this section we then show the convergence in distribution of this pair, suitably rescaled, to the pair made by a normalized Brownian excursion $\mathbf{e}$ and the head of the Brownian snake driven by $\mathbf{e}$.
We also introduce the Brownian map as a quotient space of the CRT, the continuum random tree, and we prove that the Brownian map is the limit of rescaled random quadrangulations.
The strategy of the proof is the following, using the result from Le Gall that shows the tightness of the laws of random quadrangulations, we see along a converging subsequence the limit of (C_n, L_n, D_n) resulting from the contour process, the labelling and the graph distance. We have the convergence of the first two element to $(\mathbf{e}, Z)$, so defining the Brownian map as $(\mathbf{e}, Z, D^*)$, where D^* is a pseudo distance, we show that if D is the limit of D_n along the subsequence then $D = D^*$ almost surely. So the limit do not depend on the subsequence and we have that the Brownian map is the global limit.

3.1 Basic definitions

In this section we identify Q_n with the metric space $(V(Q_n), d_{Q_n})$.
Remark that Q_n is the random variable uniformly distributed in $\mathcal{Q}_n$, the set of rooted quadrangulations with n faces.
A result from a paper of Chassing and Schaeffer (see **[4]** for details) shows that the (typical) graph distances d_{Q_n} are of order $n^{\frac{1}{4}}$ as $n \to \infty$.
Let $\mathbb{M}$ be the set of compact metric spaces considered up to isometry and endowed with the Gromov-Hausdorff distance.
We are interested to study any weak limits of

$$(n^{-\frac{1}{4}} Q_n) = (V(Q_n), n^{-\frac{1}{4}} d_{Q_n}).$$

The first result by Le Gall (see **[8]**) is that this family of probability distributions on $\mathbb{M}$ is relatively compact. The aim of this section is to show that any weak limit of $(V(Q_n), n^{-\frac{1}{4}} d_{Q_n})$ has the same law, so that we can speak about a global limit that does not depends on the choice of the subsequence, and this weak limit is the Brownian Map.

Let $(\mathbf{e}, Z)$ be a pair made of a normalized Brownian excursion

$$\mathbf{e} = \{\mathbf{e}_t\}_{t \in [0,1]}$$

and

$$Z = \{Z_t\}_{t \in [0,1]}$$

the head of the Brownian snake driven by $\mathbf{e}$. Informally, one may think about this pair as the limit of contour functions with relative labels: thank to the CVS bijection we know that any quadrangulation is encoded by a well labelled tree, and each well labelled tree can be seen as a pair (C_τ, l_τ) where C_τ is the contour function and l_τ the labelling.

Definition 3.1. *We set*

$$D^0(s,t) = Z_s + Z_t - 2\max(\inf_{s \le u \le t} Z_u, \inf_{t \le u \le s} Z_u), s, t \in [0,1]$$

where $s \le u \le t$ means $u \in [s,1] \cup [0,t]$ when $t < s$.
Let $a, b \in \tau_{\boldsymbol{e}}$,

$$D^0(a,b) = \inf\{D^0(s,t) : s, t \in [0,1], p_{\boldsymbol{e}}(s) = a, p_{\boldsymbol{e}}(t) = b\},$$

where $p_{\boldsymbol{e}} : [0,1] \to \tau_{\boldsymbol{e}}$ is the canonical projection.

Unluckly, D^0 on $\tau_{\mathbf{e}}$ does not satisfy the triangle inequality. This leads to the next definition.

Definition 3.2. *Let $a, b \in \tau_{\boldsymbol{e}}$. We set*

$$D^*(a,b) = \inf\{\sum_{i=1}^{k-1} D^0(a_i, a_{i+1}) \text{ such that } k \ge 1, a = a_1, a_k = b\}.$$

Definition 3.3. *Since D^* on $\tau_{\mathbf{e}}$ is a pseudo-distance, we define*

$$S = \tau_{\boldsymbol{e}} / \{D^* = 0\}.$$

Note that if we set, for any $s, t \in [0,1]$

$$D^*(s,t) = \inf\{\sum_{i=1}^{k} D^0(s_i, t_i) : k \ge 1, s = s_1, t = t_k, d_{\mathbf{e}}(t_i, s_{i+1}) = 0 \text{ for any } 1 \le i \le k-1\},$$

it holds

$$S = [0,1]/\{D^* = 0\}.$$

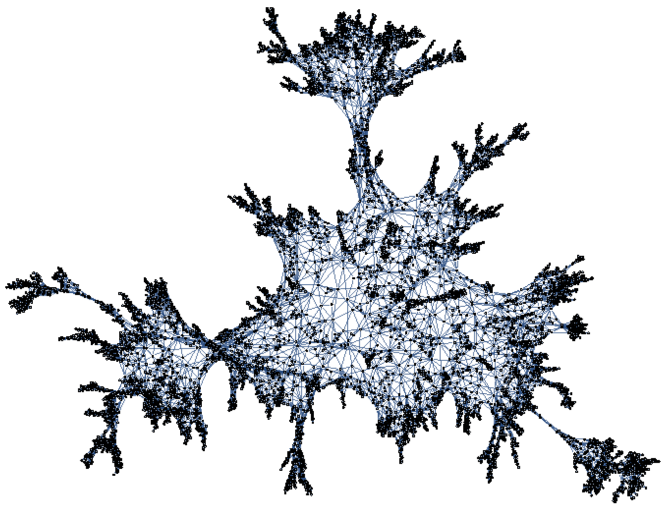
Observe also that S is a geodesic metric space, i.e. for any $x, y \in S$ there exists an isometry

$$\gamma : [0, D^*(x,y)] \to S$$

such that $\gamma(0) = x$, $\gamma(D^*(x,y)) = y$. Such a γ is called geodesic from x to y.

Definition 3.4 (Brownian Map)**.** *The Brownian map is the metric space (S, D^*).*

The following is a picture from the web of the Brownian Map.



Definition 3.5. *Set*

$$\underline{Z} = \inf_{t\in[0,1]} Z_t.$$

Let s_* be the almost surely unique time in $[0,1]$ such that $Z_{s_*} = \underline{Z}$.
Let $s \oplus t = s + t$ mod(1). Then

- $\overline{\boldsymbol{e}_t} = \boldsymbol{e}_{s_*} + \boldsymbol{e}_{s_*\oplus t} - 2m_{\boldsymbol{e}}(s_*, s_* \oplus t)$,
- $\overline{Z_t} = Z_{s_*\oplus t} - Z_{s_*}$.

Let $\theta_n = (\tau_n, (l_u^n)_{u\in\tau_n^0})$ be uniformly distributed over the set $\mathcal{T}_n$.
Let $C^n = (C_t^n)_{0\le t\le 2n}$ be the contour function of τ_n^0 and $V^n = (V_t^n)_{0\le t\le 2n}$ the labels of the contour exploration. From CVS bijection, the pair (C^n, V^n) determines θ_n.
As expected, the head of the Brownian snake driven by $\mathbf{e}$ is the weak limit of the labelling of random trees as well $\mathbf{e}$ is the limit of the contour functions of those trees.

Theorem 3.6. *We have*

$$(\frac{1}{\sqrt{2}} n^{-\frac{1}{2}} C_{2nt}^n, (\frac{9}{8})^{\frac{1}{4}} n^{-\frac{1}{4}} V_{2nt}^n)_{0\le t\le 1} \to (\overline{\boldsymbol{e}_t}, \overline{Z_t})_{0\le t\le 1},$$

where the convergence holds in distribution in the space of probability measure on $\mathcal{C}([0,1],\mathbb{R}^2)$.

Proof. From the last section result, using Skorokhod representation theorem, without loss of generality we can assume that

$$\sup_{0\le t\le 1} |(2k)^{-\frac{1}{2}} C_{2kt}^k - \mathbf{e}_t| \to 0$$

almost surely when $k \to \infty$. As usual we first show the convergence for finite dimensional marginals, so we show that for every choice of $0 \le t_1, < \ldots < t_p \le 1$ we have the convergence in distributions

$$(\frac{1}{\sqrt{2k}} C^k_{2kt_i}, (\frac{9}{8k})^{\frac{1}{4}} V^k_{2kt_i}) \to (\mathbf{e}_{t_i}, Z_{t_i}).$$

Since for every $i \in \{1, \ldots, n\}$ it holds $|C^k_{2kt_i} - C^k_{[2kt_i]}| \le 1$, $|V^k_{2kt_i} - V^k_{[2kt_i]}| \le 1$ we can replace in the thesis $2kt_i$ with $[2kt_i]$.
Consider now $p = 1$ and assume $0 < t_1 < 1$. Conditionally on θ_k, the label increments $l^k(v) - l^k(\pi(v)), v \in \theta_k \setminus \{\emptyset\}$ are independet and uniformly distributed over $\{-1, 0, 1\}$. So we let

$$(C^k_{[2kt_1]}, V^k_{[2kt_1]}) = (C^k_{[2kt_1]}, \sum_{i=1}^{C^k_{[2kt_1]}} \eta_i)$$

where the identity holds in distribution and the variables $\eta_1, \eta_2, \ldots$ are independent, independent from θ_k and uniformly distributed over $\{-1, 0, 1\}$. So, by the central limit theorem

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \eta_i \to (\frac{2}{3})^{\frac{1}{2}} N$$

where the convergence holds in distribution as $n \to \infty$ and N is a standard normal variable. Let now $\lambda \in \mathbb{R}$ and

$$\Phi(n, \lambda) = \mathbb{E}[\exp(i \frac{\lambda}{\sqrt{n}} \sum_{i=1}^{n} \eta_i)],$$

so

$$\Phi(n, \lambda) \to \exp(\frac{-\lambda^2}{3})$$

as $n \to \infty$. Hence for every $\lambda, \lambda' \in \mathbb{R}$ and conditioning on θ_k we get

$$\mathbb{E}[\exp(i \frac{\lambda}{\sqrt{2k}} C^k_{[2kt_1]} + i \frac{\lambda'}{\sqrt{C^k_{[2kt_1]}}} \sum_{i=1}^{C^k_{[2kt_1]}} \eta_i)] =$$

$$= \mathbb{E}[\exp(i \frac{\lambda}{\sqrt{2k}} C^k_{[2kt_1]}) \times \Phi(C^k_{[2kt_1]}, \lambda')] \to \mathbb{E}[\exp(i \lambda \mathbf{e}_{t_1})] \times \exp(\frac{-\lambda'^2}{3}).$$

So we have

$$(\frac{C^k_{[2kt_1]}}{\sqrt{2k}}, \frac{1}{\sqrt{C^k_{[2kt_1]}}} \sum_{i=1}^{C^k_{[2kt_1]}} \eta_i) \to (\mathbf{e}_{t_1}, (\frac{2}{3})^{\frac{1}{2}} N)$$

where the convergence holds in distributions and N is a normal variable independent of $\mathbf{e}$. Gluing the two results we first have the following convergence in distribution

$$(\frac{C^k_{[2kt_1]}}{\sqrt{2k}}, (\frac{9}{8k})^{\frac{1}{4}} V^k_{[2kt_1]}) = (\frac{C^k_{[2kt_1]}}{\sqrt{2k}}, (\frac{3}{2})^{\frac{1}{2}} (\frac{C^k_{[2kt_1]}}{\sqrt{2k}})^{\frac{1}{2}} \frac{1}{\sqrt{C^k_{[2kt_1]}}} \sum_{i=1}^{C^k_{[2kt_i]}} \eta_i)$$

so from the result above we have the following convergence in distribution

$$(\frac{C^k_{[2kt_1]}}{\sqrt{2k}}, (\frac{9}{8k})^{\frac{1}{4}} V^k_{[2kt_1]}) \to (\mathbf{e}_{t_1}, \sqrt{\mathbf{e}_{t_1}} N).$$

So the proof when $p = 1$ is completed. Proof for higher dimension follows similar, we show how to go with $p = 2$. Let $0 < t_1 < t_2 < 1$ and recall, for $i, j \in \{0, 1, \dots, 2k\}$

$$\overline{C}^{i,j}_k = \min_{i \wedge j \le n \le i \vee j} C^k_n.$$

We know that

- $C^k_{[2kt_1]} = |v^k_{[2kt_1]}|$
- $C^k_{[2kt_2]} = |v^k_{[2kt_2]}|$
- $V^k_{[2kt_1]} = l^k(v^k_{[2kt_1]})$
- $V^k_{[2kt_2]} = l^k(v^k_{[2kt_2]})$

and that $\overline{C}^{[2kt_1],[2kt_2]}_k$ is the generation in θ_k of the last common ancestor to $v^k_{[2kt_1]}$ and $v^k_{[2kt_2]}$. Using the properties of the labelling on θ_k, we see conditionally on θ_k

$$(V^k_{[2kt_1]}, V^k_{[2kt_2]}) = (\sum_{i=1}^{\overline{C}^{[2kt_1],[2kt_2]}_k} \eta_i + \sum_{i=\overline{C}^{[2kt_1],[2kt_2]}_k+1}^{C^k_{[2kt_1]}} \eta'_i, \sum_{i=1}^{\overline{C}^{[2kt_1],[2kt_2]}_k} \eta_i + \sum_{i=\overline{C}^{[2kt_1],[2kt_2]}_k+1}^{C^k_{[2kt_2]}} \eta''_i)$$

where the identity holds in distribution and η_i , η'_i and η''_i are independent and uniformly distributed over $\{-1, 0, 1\}$. From the result of the previous section we have

$$((2k)^{-\frac{1}{2}} C^k_{[2kt_1]}, (2k)^{-\frac{1}{2}} C^k_{[2kt_2]}, (2k)^{-\frac{1}{2}} \overline{C}^{[2kt_1],[2kt_2]}_k) \to (\mathbf{e}_{t_1}, \mathbf{e}_{t_2}, m_{\mathbf{e}}(t_1, t_2)).$$

In a similar way of the case $p = 1$ we have

$$(\frac{C^k_{[2kt_1]}}{\sqrt{2k}}, \frac{C^k_{[2kt_2]}}{\sqrt{2k}}, (\frac{9}{8k})^{\frac{1}{4}} V^k_{[2kt_1]}, (\frac{9}{8k})^{\frac{1}{4}} V^k_{[2kt_2]})$$

$$\to (\mathbf{e}_{t_1}, \mathbf{e}_{t_2}, \sqrt{m_{\mathbf{e}}(t_1, t_2)} N + \sqrt{e_{t_1} - m_{\mathbf{e}}(t_1, t_2)} N', \sqrt{m_{\mathbf{e}}(t_1, t_2)} N + \sqrt{e_{t_2} - m_{\mathbf{e}}(t_1, t_2)} N'')$$

where N, N', N'' are independent standard normal variables, also independent of $\mathbf{e}$. The last limit in distribution is $(\mathbf{e}_{t_1}, \mathbf{e}_{t_2}, Z_{t_1}, Z_{t_2})$.

To complete the proof we have to show the tightness of the processes

$$((\frac{9}{8k})^{\frac{1}{4}} V^k_{2kt})_{t \in [0,1]}.$$

Actually it is sufficient to prove that for every integer $p \ge 1$ there exists a costant $K_p < \infty$ such that, for every $k \ge 1$ and every $s, t \in [0, 1]$

$$\mathbb{E}[(\frac{V^k_{2kt} - V^k_{2ks}}{k^{\frac{1}{4}}})^{4p}] \le K_p |t - s|^p.$$

Without loss of generality we restrict to $s = \frac{i}{2k}$, $t = \frac{j}{2k}$ with $i, j \in \{0, 1, \ldots, 2k\}$. Using the same decomposition as above we get

$$V_j^k - V_i^k = \sum_{n=1}^{d_{gr}(v_i^k, v_j^k)} \eta_n.$$

Recalling that

$$d_{gr}(v_i^k, v_j^k) = C_i^k + C_j^k - 2\overline{C}_k^{i,j}$$

and conditioning on θ_k we get

$$\mathbb{E}[(V_i^k - V_j^k)^{4p}] \leq K_p' \mathbb{E}[(d_{gr}(v_i^k, v_j^k))^{2p}].$$

And from what seen in the last convergence result of the previous section the proof is complete since

$$\mathbb{E}[(C_i^k + C_j^k - 2\overline{C}_k^{i,j})^{2p}] \leq K_p'' |j - i|^p.$$

□

As already said, another result from Le Gall states that the laws of $(n^{-\frac{1}{4}} Q_n)$ form a relative compact subset of probability distributions on $\mathbb{M}$.
Let now state the main theorem of this work.

Theorem 3.7. *The metric space*

$$(V(Q_n), (\frac{9}{8})^{\frac{1}{4}} n^{-\frac{1}{4}} d_{Q_n})$$

converges in distribution for the Gromov-Hausdorff topology on $\mathbb{M}$ to the space (S, D^*).

To prove this, we first reformulate the statement. Since $(\frac{8n}{9})^{-\frac{1}{4}} Q_n$ is relative compact in law, we consider a random variable (S', D') that is the weak limit along a given subsequence. In order to compare properly the two spaces our strategy is to use an enviroment in which holds $S = S'$ almost surely, so after that we need only to prove $D = D'$ almost surely.

If $q = Q_n$ is a uniform random variable in $\mathcal{Q}_n$ and v_* is uniform among the $n+2$ vertices of Q_n, the resulting labelled tree (T_n, l_n) has contour and label function (C_n, L_n), such that C_n is a simple random walk on $\mathbb{Z}$ starting at 0 and conditioned to ending at 0 at time $2n$ and to stay non negative for all i such that $0 \leq i \leq 2n$.
Let u_i^n be the vertex of T^n visited at step i by the contour process, and let

$$D_n(\frac{i}{2n}, \frac{j}{2n}) = (\frac{9}{8n})^{\frac{1}{4}} d_{Q_n}(u_i^n, u_j^n), 0 \leq i, j \leq 2n.$$

Extend D_n by interpolation on $[0, 1]^2$. Then

$$((\frac{C_{2ns}^n}{\sqrt{2n}})_{0 \leq s \leq 1}, (\frac{L_{2ns}^n}{(\frac{8n}{9})^{\frac{1}{4}}})_{0 \leq s \leq 1}, (D_n(s, t))_{0 \leq s, t \leq 1})$$

form a relative compact family of probability distributions. So for a given subsequence we can extract a limit

$$(\mathbf{e}, Z, D),$$

where D is a random pseudo metric on $[0,1]$. The function D induce a pseudo distance on $\tau_{\mathbf{e}}/\{d_{\mathbf{e}} = 0\}$. We can view D as a pseudo metric distance on $[0,1]$, so let

$$S' = [0,1]/\{D = 0\} = \tau_{\mathbf{e}}/\{D = 0\}$$

endowed with the distance induced by D that we still call D.
Note that (S', D) is a random geodesic space. So, using a result by Le Gall (see **[8]**) it holds the following.

Proposition 3.8. • *The subsets of $\tau_{\mathbf{e}}$ of the form $\{D = 0\}$, $\{D^0 = 0\}$, $\{D^* = 0\}$ are equivalent.*

- *Along the subsequence (n_k),*

$$(Q_n, (\frac{9}{8n})^{\frac{1}{4}} d_{Q_n}) \to (S, D)$$

in the Gromov-Hausdorff sense.

Remark that we can assume that this convergence holds almost surely, up to changing the underlying probability space, thank to the Skorokhod representation theorem. So the main theorem can be reformulated as follows.

Theorem 3.9. *Almost surely, it holds that $D = D^*$.*

3.2 Considerations on the metrics

Recall that D^0 is defined on $[0,1]^2$ and $\tau_{\mathbf{e}}^2$, so it can be seen as a function of S^2 with the following:

$$D^0(x, y) = \inf\{D^0(a, b) : a, b \in \tau_{\mathbf{e}}, p_z(a) = x, p_z(b) = y\}$$

where $p_z : \tau_{\mathbf{e}} \to S$ is the canonical projection. D^0 defined above on S is not yet a pseudo distance (it does not satisfies the triangle inequality), however it holds

$$D(x, y) \leq D^*(x, y) \leq D^0(x, y)$$

for every $x, y \in S$.

Definition 3.10. *Let $p = p_z \circ p_e$. We denote by λ the push forward on S of the Lebasgue measure $\mathcal{L}$ on $[0,1]$ by p, i.e.*

$$\lambda(A) = \mathcal{L}(p^{-1}(A)).$$

The process Z has almost surely one unique minimum at a point $s_* \in [0,1]$, the projection of s_*, $\rho = p(s_*)$ is called the root of the space (S, D). By the labelling rules we have

$$D(\rho, x) = D^0(\rho, x) = D^*(\rho, x) = Z_x - \inf Z$$

for any $x \in S$.
We also have $D(s,t) \geq |Z_s - Z_t|$ for any $s, t \in [0,1]$, since it is just the limit of

$$D_n(\frac{i}{2n}, \frac{j}{2n}) \geq (\frac{9}{8n})^{\frac{1}{4}} |L_n(i) - L_n(j)|,$$

where the lattest comes from the labelling rule and CVS bijection properties. For a formal proof we refer to Le Gall **[7]**.

Definition 3.11. *In a geodesic metric space (X, d) we say that (x, y, z) are aligned if*

$$d(x,y) + d(y,z) = d(x,z).$$

Proposition 3.12. *Almost surely for any $x, y \in S$ it holds that (ρ, x, y) are aligned in (S, D) if and only if they are aligned in (S, D^*).*

Proof. Let (ρ, x, y) be aligned in (S, D). This means $D(x,y) = D^0(x,y)$. So $D^*(x,y) = D(x,y)$ since $D \leq D^* \leq D^0$.
Conversely, if (ρ, x, y) are aligned in (S, D^*) we have

$$D^*(\rho, y) = D(\rho, y) \leq D(\rho, x) + D(x,y) \leq D^*(\rho, x) + D^*(x,y) \leq D^*(\rho, y).$$

□

Proposition 3.13. *Let $s, t \in [0,1]$. Let $\gamma^{(s)}$ and $\gamma^{(t)}$ be the geodesics from ρ to s and t. So the images of $\gamma^{(s)}$ and $\gamma^{(t)}$ coincide in the complement of $B_D(p(s), D^0(s,t))$.*

Proof. $D^0(s,t)$ is the lenght of the path in the tree τ_Z from $p(s)$ to $p(t)$. So the two geodesics from two given points to the roots will merge before the root.

□

Proposition 3.14. *Let (X, d) be an arcwise connected metric space, let x, y be two distinct points in X. Let γ be a continuous path from x to y. Then for any $\eta > 0$ there exists at least*

$$k = [\frac{d(x,y)}{2\eta}] + 1$$

points $y_1, \ldots, y_k$ in the image of γ such that $d(y_i, y_j) \geq 2\eta$ for any $i, j \in \{1, 2, \ldots k\}$ with $i \neq j$.

Proof. Assume that γ is parametrized by $[0,1]$ and $\gamma(0) = x$, $\gamma(1) = y$, $d(x,y) \geq 2\eta$, so we have $k \geq 2$ ($k = 1$ is trivial).
Let $s_0 = 0$ and let

$$s_{i+1} = \sup\{t \leq 1 : d(\gamma(t), \gamma(s_i)) \leq 2\eta\}, i \geq 0.$$

Observe that $(s_i)_{i \geq 0}$ is non-decreasing and it holds $d(\gamma(s_i), \gamma(s_{i+1})) \leq 2\eta$ for any $i \leq 0$.
Let $i \in \{0, 1, \ldots k-2\}$,

$$d(x, \gamma(s_i)) \leq \sum_{j=0}^{i-1} d(\gamma(s_j), \gamma(s_{j+1})) \leq 2\eta i \leq 2\eta(k-2) \leq d(x,y) - 2\eta.$$

So $d(\gamma(s_i), \gamma(s_j)) \geq 2\eta$ for any i, j. Then we have the thesis with $y_i = \gamma(s_{i-1})$. □

We now give an upper extimate by Le Gall (see **[7]** for details) for the volume of D-balls in S,

Proposition 3.15. *Let $\eta \in (0,1)$. Then almost surely there exists a (random) costant $c \in (0,\infty)$ such that for every $r \geq 0$ and every $x \in S$ one has*

$$\lambda(B_D(x,r)) \leq cr^{4-\eta}.$$

The following is a lower extimate for the volume of D^* -balls.

Proposition 3.16. *Let $\eta \in (0,1)$. Almost surely there exists a random $c \in (0,\infty)$ and $r_0 > 0$ such that for every $r \in [0, r_0]$ and every $x \in S$*

$$\lambda(B_{D^*}(x,r)) \geq cr^{4+\eta}.$$

Proof. Remark $B_{D^0}(x,r) \subset B_{D^*}(x,r)$ for any $x \in S$ and $r \geq 0$.
Since $D^0(x,y) = \inf_{s,t} D^0(s,t)$, where $s, t \in [0,1]$ such that $p(s) = x$, $p(t) = y$, we have for any $s \in [0,1]$ such that $p(s) = x$

$$p(\{t \in [0,1] : D^0(s,t) < t\}) \in B_{D^0}(x,r).$$

So, by definition of λ,

$$\lambda(B_{D^*}(x,r)) \geq \mathcal{L}(\{t \in [0,1] : D^0(s,t) \leq \frac{r}{2}\}).$$

Now we use that Z is almost surely Hölder-continuous with exponent $\frac{1}{4+\eta}$, which gives that almost surely exists a random $c \in (0,\infty)$ such that for any $h \geq 0$

$$w(Z,h) \leq ch^{\frac{1}{4+\eta}},$$

where

$$w(Z,h) = \sup\{|Z_t - Z_s| : s,t \in [0,1], |t-s| \leq h\}.$$

Now, from

$$D^0(s,t) \leq Z_s + Z_t - 2 \min_{u \in [s \wedge t, s \vee t]} Z_u \leq 2e(Z, |t-s|)$$

we get that for any $s \in [0,1]$, $h > 0$ and $t \in [(s-h) \vee 0, (s+h) \wedge 1]$,

$$D^0(s,t) \leq 2ch^{\frac{1}{4+\eta}}.$$

With $h = (\frac{r}{4c})^{4+\eta}$ it gives

$$\mathcal{L}(\{t \in [0,1] : D^0(s,t) \leq \frac{r}{2}\}) \geq 2h \wedge 1,$$

and combining with the inequality above we have

$$\lambda(B_{D^*}(x,r)) \geq \mathcal{L}(\{t \in [0,1] : D^0(s,t) \leq \frac{r}{2}\}) \geq 2h \wedge 1 = (\frac{2}{(4c)^{4+\eta}} r^{4+\eta}) \wedge 1,$$

so we get the thesis letting $r_0 = \frac{4c}{2^{\frac{1}{4+\eta}}}$ and $c = \frac{2}{(4c)^{4+\eta}}$.

□

Now we state a key proposition for the proof of the main theorem.

Proposition 3.17. *Key Proposition 1*

Let $\alpha \in (0,1)$ fixed. Then almost surely there exists a random $\varepsilon_1 > 0$ such that for every $x, y \in S$ with $D(x,y) \leq \varepsilon_1$ it holds

$$D^*(x,y) \leq D(x,y)^\alpha.$$

Proof. Let $\alpha \in (0,1)$, suppose we can find with positive probability μ two sequence $(x_n)_{n\geq 0}$, $(y_n)_{n \geq 0}$ in S such that $D(x_n, y_n) \to 0$ and

$$D^*(x_n, y_n) > D(x_n, y_n)^\alpha,$$

for any $n \geq 0$. From now on almost surely is intended restriced to this event with positive probability μ.
Let γ_n be a geodesic path from x_n to y_n for the distance D. We set

$$V_\beta^D(\gamma_n) = \{x \in S : \text{ there exists } t \in [0, D(x_n, y_n)], D(\gamma(t), x) < \beta\}.$$

For a previous proposition in general metric spaces, $V_\beta^D(\gamma_n)$ is contained in a union of at most $[\frac{D(x_n,y_n)}{2\beta}] + 1$ D-balls of radius 2β. So

$$\lambda(V_\beta^D(\gamma_n)) \leq (\frac{D(x_n, y_n)}{2\beta} + 1) \sup_{x \in S} \lambda(B_D(x, 2\beta)).$$

Hence, for any $\eta \in (0,1)$ almost surely there exists a $c \in (0,\infty)$ such that for any $n \geq 0$, $\beta \in [0, D(x_n, y_n)]$

$$\lambda(V_\beta^D(\gamma_n)) \leq c\beta^{3-\eta} D(x_n, y_n).$$

Let $V_\beta^{D^*}(\gamma_n)$ defined as $V_\beta^D(\gamma_n)$ changing D with D^*, γ_n is a continuous path in the arcwise connected space (S, D^*). So for every $\beta > 0$ we can find $y_1, \ldots, y_k$,

$k = [\frac{D^*(x_n,y_n)}{2\beta}] + 1$, such that $D^*(y_i, y_j) \geq 2\beta$ for any $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$.
Hence $B_{D^*}(y_i, \beta)$ are pairwise disjoint and included in $V_\beta^{D^*}(\gamma_n)$. So we have

$$\lambda(V_\beta^{D^*}(\gamma_n)) \geq \sum_{i=1}^{k} \lambda(B_{D^*}(y_i, \beta)) \geq k \inf_{s \in S} \lambda(B_{D^*}(x, \beta)) \geq \frac{D^*(x_n, y_n)}{2\beta} \inf_{x \in S} \lambda(B_{D^*}(x, \beta)).$$

So for the same $\eta \in (0, 1)$ as before we use the lower extimate for the volume of D^* balls of the proposition above and we get that almost surely there exists a $c \in (0, \infty)$, $r_0 > 0$ such that for any $\beta \in [0, r_0]$

$$\lambda(V_\beta^{D^*}(\gamma_n)) \geq c\beta^{3+\eta} D^*(x_n, y_n) \geq c\beta^{3+\eta} D(x_n, y_n)^\alpha.$$

Recall that $D^* \geq D$, so for every $\beta \in [0, D(x_n, y_n) \wedge r_0]$ we have

$$\beta^{2n} \leq cD(x_n, y_n)^{1-\alpha}.$$

Letting

- $\eta = \frac{1-\alpha}{4}$
- $\beta = D_n(x_n, y_n) \wedge r_0$

for $D(x_n, y_n) \to 0$ we have $D(x_n, y_n)^{\frac{1-\alpha}{2}} = O(D(x_n, y_n)^{1-\alpha})$ and this is clearly a contradiction.

$\square$

3.3 Proof of the theorem

Definition 3.18. *Let (X, d) be a geodesic metric space and $x_1, \ldots, x_k, x$ be $k + 1$ points in X. We say that x is a k-star point respect to $x_1, \ldots, x_k$ if for every geodesic paths $\gamma_1, \ldots, \gamma_k$ from x to $x_1, \ldots, x_k$ it holds that with $i \neq j$ $\gamma_i \cap \gamma_j = \{x\}$.*

Definition 3.19. *We set $\mathcal{G}(X; x_1, \ldots, x_k)$ for the set of points $x \in X$ that are k-star points with respect to $x_1, \ldots, x_k$.*

If x_1, x_2 are choosen at random, i.e. there are two random variable U_1, U_2 uniform in $[0, 1]$ independent of $(\mathbf{e}, Z, D)$ such that $x_1 = p(U_1)$ and $x_2 = p(U_2)$, with probability 1 there is a unique D-geodesic γ from x_1 to x_2.
The geodesics γ_1, γ_2 from ρ to x_1 and x_2 are also uniques. Recall that γ_1 and γ_2 almost surely share an initial segment, so almost surely ρ is not in γ. So

$$D(x_1, \rho) + D(x_2, \rho) > D(x_1, x_2),$$

so the ordered points (x_1, ρ, x_2) are not aligned.

Definition 3.20. *We set*

$$\Gamma = \gamma([0, D(x_1, x_2)]) \cap \mathcal{G}(S; x_1, x_2, \rho).$$

It also holds with probability 1 that $y \in \Gamma$ if and only if any geodesic γ_y from y to ρ is such that $\gamma_y \cap \gamma = \{y\}$.

Proposition 3.21. *Key Proposition 2*

It exists $\delta \in (0,1)$ such that almost surely there exists a random $\varepsilon_2 > 0$ such that for every $\varepsilon \in (0, \varepsilon_2)$ the set Γ can be covered with at most $\varepsilon^{-(1-\delta)}$ balls of radius ε in (S, D).

For the proof of this we refer to Miermont **[10]**.

Proposition 3.22. *Let (s,t) be a (non empty) interval of $[0, D(x_1, x_2)]$ such that $\gamma(v) \neq \Gamma$ for every $v \in (s,t)$. Then there exists a unique $u \in [s,t]$ such that $(\rho, \gamma(s), \gamma(u))$ and $(\rho, \gamma(t), \gamma(u))$ are aligned.*

Proof. Let $v \in (s,t)$. By assumption $\gamma(v) \neq \Gamma$, so there exists a geodesic from $\gamma(v)$ to ρ that intersects $\mathrm{Imm}(\gamma)$ in $\gamma(v')$ with $v' \neq v$. So $(\rho, \gamma(v'), \gamma(v))$ are aligned. Assume first $v' < v$ and set

$$w = \inf\{v'' \in [s,v] : (\rho, \gamma(v''), \gamma(v)) \text{ are aligned}\}.$$

So $w \in [s,v)$, since v' is in the above set. We want to show $w = s$, since if it was true that $w > s$ then $\gamma(w) \neq \Gamma$ and some geodesic from $\gamma(w)$ to ρ would intersect $\mathrm{Imm}(\gamma)$ at some point w' with $w' \neq w$, and this would be a contradiction since for minimality of w $w \leq w'$ and $w' \leq w$ since otherwise $(\rho, \gamma(w), \gamma(w'))$ and $(\rho, \gamma(w'), \gamma(w))$ would both be aligned.
So $w = s$ and $(\rho, \gamma(s), \gamma(v))$ are aligned. Doing the same reasoning, in the case $v' > v$ the ordered points $(\rho, \gamma(t), \gamma(v))$ are aligned. So for every $v \in (s,t)$ either $(\rho, \gamma(s), \gamma(v))$ or $(\rho, \gamma(t), \gamma(v))$ are aligned. So we set

$$u = \sup\{u' \in [s,t] : (\rho, \gamma(s), \gamma(u')) \text{ are aligned}\}.$$

□

By the Key Proposition 2 there exists δ, $\varepsilon > 0$ such that

$$\Gamma \subset \bigcup_{i=1}^{k} B_D(x_{(i)}, \varepsilon),$$

where $k = [\varepsilon^{-(1-\delta)}]$. We assume without loss of generality $x_1, x_2 \in \{x_{(1)}, \ldots, x_{(k)}\}$ and that the covering is minimal, discarding eventually the balls that does not intersect Γ.

Definition 3.23. *We set*

$$r_i = \inf\{t \geq 0 : \gamma(t) \in B_D(x_{(i)}, \varepsilon)\},$$

$$r_i' = \sup\{t \le D(x_1, x_2) : \gamma(t) \in B_D(x_{(i)}, \varepsilon)\},$$

$$A = \bigcup_{i=1}^{k} [r_i, r_i'].$$

Observe that $\Gamma \subset \gamma(A)$.

Proposition 3.24. *For every $i \in \{1, 2, \dots k\}$ and $r \in [r_i, r_i']$ we have $D(\gamma(r), x_{(i)}) \le 2\varepsilon$.*

Proof. We argue by contradiction. Suppose γ is a D-geodesic that pass through $\gamma(r_i), \gamma(t)$ and $\gamma(r_i')$ in this order, so

$$D(\gamma(r_i), \gamma(r_i')) = D(\gamma(r_i), x_{(i)}) + D(x_{(i)}, \gamma(r_i')) > 2\varepsilon.$$

We get a contradiction since it also holds

$$D(\gamma(r_i), \gamma(r_i')) \le D(\gamma(r_i), x_{(i)}) + D(x_{(i)}, \gamma(r_i')) < 2\varepsilon.$$

□

Let now set $I = \{j \in 1, 2, \dots, k\}$ such that $[r_j, r_j']$ is maximal for the inclusion, i.e. in a way that it still holds

$$A = \bigcup_{i \in I} [r_i, r_i'] = \bigcup_{i=0}^{k'-1} [t_i, s_{i+1}].$$

Let $x_{(i)} = \gamma(s_i)$ and $y_{(i)} = \gamma(t_i)$

Proposition 3.25. *Almost surely it holds, for any ε small enough*

$$\sum_{i=0}^{k'-1} D^*(y_{(i)}, x_{(i+1)}) \le 4k\varepsilon^{\frac{2-\delta}{2}}.$$

Proof. We set

$$J_i = \{j \in I : [r_j, r_j'] \subset [t_i, s_{i+1}]\}$$

so that $\sum_{0 \le i \le k'-1} \#J_i = \#I \le k$. Our aim is to show that

$$D^*(y_{(i)}, x_{(i+1)}) \le 2\#J_i \varepsilon^{\frac{2-\delta}{2}}$$

for any ε small enough. Observe that if $[r_j, r_j'] \cap [r_k, r_k'] \ne \emptyset$ the two intervals overlap, this means $r_j \le r_k \le r_j' \le r_k'$ or viceversa.
Let now reorder $\{r_j\}_{j \in J_i}$ in $\{r_{j_k}\}_{1 \le k \le \#J_i}$ in non decreasing order. Then

- $\gamma(r_{j_1}) = y_{(i)}$,
- $\gamma(r'_{j_{\#J_i}}) = x_{(i+1)}$,

- $D^*(y_{(i)}, x_{(i+1)}) \leq \sum_{k=1}^{\#J_i-1} D^*(\gamma(r_{j_k}), \gamma(r_{j_{k+1}})) + D^*(\gamma(r_{j_{\#J_i}}), x_{(i+1)})$.

For the previous proposition and the overlapping property of $[r_j, r_j']$ we have

$$D(\gamma(r_{j_k}), \gamma(r_{j_{k+1}})) \leq 4\varepsilon,$$

$$D(\gamma(r_{j_{\#J_i}}), x_{(i+1)}) \leq 4\varepsilon.$$

Now we apply the Key Proposition 1 with $\alpha = \frac{2-\delta}{2}$ to get that almost surely for every ε small enough

$$D^*(y_{(i)}, x_{(i+1)}) \leq \#J_i(4\varepsilon)^{\frac{2-\delta}{2}} \leq 4\#J_i\varepsilon^{\frac{2-\delta}{2}}.$$

□

We are able now to prove the main theorem. Remark that we have to show that holds $D = D^*$ almost surely.
Let $A^c \cap [0, D(x_1, x_2)]$ be the union of intervals (s_i, t_i) for $1 \leq i \leq k' - 1$. For any i we have $\gamma((s_i, t_i)) \cap \Gamma = \emptyset$ since $\Gamma \subset \gamma(A)$ and γ is injective.
By a result above we know that for every $i \in \{1, 2, \ldots, k' - 1\}$ we can find $u_i \in [s_i, t_i]$ such that $(\rho, \gamma(s_i), \gamma(u_i))$ are aligned, and the same holds for $(\rho, \gamma(t_i), \gamma(u_i))$.
We set $x_{(i)} = \gamma(s_i)$, $y_{(i)} = \gamma(t_i)$, $z_{(i)} = \gamma(u_i)$. Since if (ρ, x, y) are aligned in (S, D) then they are aligned in (S, D^*) and $D(x, y) = D^*(x, y)$ we get

$$D^*(x_{(i)}, z_{(i)}) = D(x_{(i)}, z_{(i)}),$$

$$D^*(y_{(i)}, z_{(i)}) = D(y_{(i)}, z_{(i)}).$$

Applying the triangle inequality and the fact that γ is a D-geodesic we have almost surely

$$D^*(x_1, x_2) \leq \sum_{i=1}^{k'-1} (D^*(x_{(i)}, z_{(i)}) + D^*(z_{(i)}, y_{(i)})) + \sum_{i=0}^{k'-1} D^*(y_{(i)}, x_{(i+1)}) =$$

$$= \sum_{i=1}^{k'-1} (D(x_{(i)}, z_{(i)}) + D(z_{(i)}, y_{(i)})) + \sum_{i=0}^{k'-1} D^*(y_{(i)}, x_{(i+1)}) \leq D(x_1, x_2) + \sum_{i=0}^{k'-1} D^*(y_{(i)}, x_{(i+1)})$$

and for the proposition above we can extimate the last term for an ε small enough

$$D(x_1, x_2) + \sum_{i=0}^{k'-1} D^*(y_{(i)}, x_{(i+1)}) \leq D(x_1, x_2) + 4k\varepsilon^{\frac{2-\delta}{2}}.$$

Observe that $k \leq \varepsilon^{-(1-\delta)}$ so we can let $\varepsilon \to 0$ to get

$$D^*(x_1, x_2) \leq D(x_1, x_2)$$

and since $D^* \geq D$ this implies $D^*(x_1, x_2) = D^*(x_1, x_2)$ almost surely for $x_1, x_2 \in S$.

References

[1] D. Aldous, *The continuum random tree I*, 1991.

[2] D. Aldous, *The continuum random tree III*, 1993.

[3] Itai Benjamini, Russell Lyons, Oded Schramm, *Unimodular Random Trees*, 2013.

[4] P.Chassaing, G. Schaeffer, *Random planar lattices and integrated super-Brownian excursion*, 2004.

[5] K.L. Chung, *Excursion in Brownian motion*, 1976.

[6] J.F. Le Gall, F. Paulin, *Scaling limits of bipartite planar maps are homeomorphic to the 2-sphere*, 2008.

[7] J.F. Le Gall, *Geodesics in large planar maps and in the Brownian map*, 2010.

[8] J.F. Le Gall, *The topological structure of scaling limits of large planar maps*, 2007.

[9] J.F. Le Gall, *Random geometry on the sphere*, 2014.

[10] G. Miermont, *The Brownian map is the scaling limit of uniform random plane quadrangulations*, 2013.

[11] D. Revuz, M. Yor, *Continuous Martingales and Brownian Motion*, 1991.

[12] G. Schaeffer, *Conjugaison d'arbres et cartes combinatoires aleatoires*, 1998.