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DEPARTMENT OF PHYSICS
Master Degree in Theoretical Physics

Off-equilibrium scaling driven by time-dependent external fields in $\mathbf{O}(N)$ vector-models.

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Abstract

We investigate the off-equilibrium dynamics of spin systems with O(N) symmetry arising by the presence of slowly varying time-dependent external fields. We show the general theory and then focus on two different cases: a time-dependent magnetic field $h(t, t_s) \approx t/t_s$, t_s is a time scale, at the critical temperature and the temperature deviations $T(t, t_s)/T_c - 1 \approx -t/t_s$ in the absence of magnetic fields. We demonstrate the off-equilibrium scaling behaviours and formally compute the correlation functions in the limit of large N. We study the first deviations from the equilibrium in the correlation functions and prove that the matching occurs exponentially fast. We also consider analogous phenomena at the first-order transition which occurs in the ordered phase $T < T_c$ along the line of zero magnetic field.

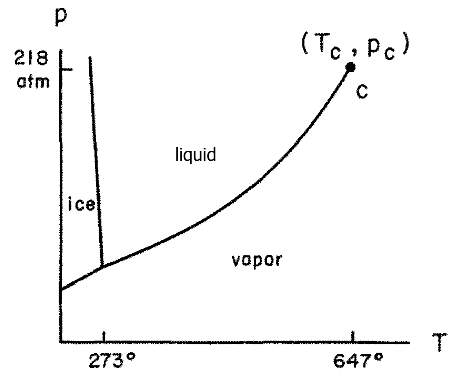

Figure 1: The liquid-vapor phase diagram of the water. This picture has been taken from the book [1].

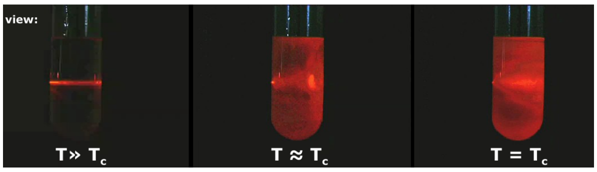

Figure 2: The phenomenon of the critical opalescence in a liquid-vapor phase transition: the fluctuation of the density in the fluid becomes of the same order of the light wavelength. The photons are therefore scattered and the sample appears opalescent.

Introduction.

Statistical systems are described in terms of macroscopic variables such as the mass density, the energy density, the magnetization and many others. The value of these quantities is generally fixed by the external fields. These ones characterize the reservoir with which the system is in contact. Some example of external fields are the temperature, the magnetic field or the pressure. However, there are some special values of the external fields such that the observables are not uniquely determined and can take different values. Let us give a concrete example: a sample of water at fixed pressure to the atmosferic value. We know that it presents different phases for different values of the temperature. The macroscopic phase (and therefore the macroscopic variables) of this system is the result of essentially two competiting effects: the interactions among the water molecules which tend to bring them closely and the temperature which increases the kinetic energy of the particles and consequently tends to decrease the mass density. There is a particular value of the temperature called *critical temperature* T_c where these two effects are of the same strenght. At the critical temperature the mass density can assume a lower value (vapor) or an higher value (liquid); thus, there is a coexistence of the two phases of the water. Transitions like this are called continuous because the two phases can be continuously connected crossing the critical temperature. The situation is illustrated in the phase diagram of fig.2.
We define the *correlation length* as the length-scale above which the correlations between two particles are suppressed. Continuous phase transitions are characterized by a divergent correlation length. This causes peculiar phenomena such as the *critical opalescence* in the case of the liquid-vapor phase transition: the fluid has a cloudy appearance due to density fluctuations at all orders of magnitude [see fig.2]. Since the correlations occur to all length scales close to a continuous phase transition, the macroscopic behaviour of the system reflects its microscopic structure. This feature is commonly called *scale invariance*. The scale invariance implies the existence of scaling relations for the statistical observables: they obey to power-law relations in terms of the external fields with certain non-integer exponents called *critical exponents*. These ones are *universal* and do not depend on the details of the

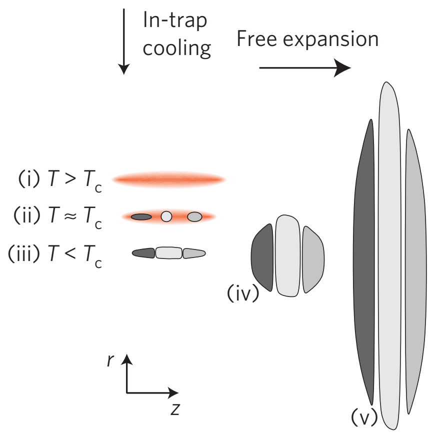

Figure 3: An example of cold-atoms experiment: a sample of atoms of sodium with elonged shape is driven to the BEC by opportune thermal variations. After crossing the transition, BEC is locally achieved forming several isles each with own phase. Further cooling makes the isles grow towards each other forming solitons. The number of solitons counted in the system is in agreement with the theoretical precitions. The figure and the example have been taken from the ref.[2].

microscopic interactions. The existence of such scaling relations was firstly supported by strong phenomenological evidences. Then, it was deeply understood in the renormalization group framework[1].
Real statistical systems have a dynamical evolution which can be thought as a stochastic process. The thermodynamic observables presents fluctuations over their equilibrium values. Close to the critical point, the divergence of the correlation length causes a divergence in the relaxation times of these fluctuations. If we consider the water example, there are a lot of minor processes such as interactions with the recipient or with possible impurities which give rise to a macroscopic noise. The value of the mass density presents random fluctuations whose relaxation time become macroscopically measurable close to the transition. This phenomenon is called *critical slowing down*.
Phase transitions generally occur in nature by varying the external fields across their critical values. When statistical systems are driven through a critical point by time-dependent external fields, they show off-equilibrium behaviours. The emergences of these behaviours are related to the phenomenon of critical slowing down i.e. to the presence of large-scale modes which cannot adapt themselves to the changes of the external parameters, even in the limit of slow passage $t_s \to \infty$.
The study of phase transitions induced by slow variations of the external fields is generally called *Kibble-Zurek* (KZ) problem. One of the most important prediction of this theory is the *Kibble-Zurek mechanism* (KZM) which explains the formation and the density of topological defects in an off-equilibrium phase transition slowly driven by temperature [see the appendix A]. The Kibble-Zurek approach well describes the off-equilibrium dynamics near the transition and leads to a non-trivial scaling theory of the observables in terms of appropriate length and time scales, different from those at the equilibrium. The scaling relations depend on the equilibrium critical exponents and also on some general features of the time-dependence of the external fields. In the limit of quasi-adiabatic time-variations $t_s \to \infty$, the results are universal.
Several experiments have investigated these off-equilibrium phenomena, in particular checking the predictions for the abundance of topological defects arising from the off-equilibrium conditions across the critical temperature, as predicted by the KZM. The first experiments

[1] The simple postulate of the existence of a fixed point for the renormalization flows is sufficient to explain the emergence of the scaling relations.

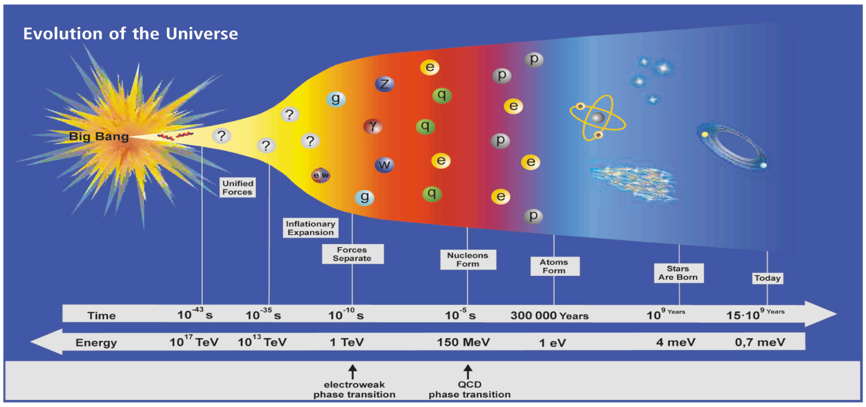

Figure 4: The evolution of the Universe in the Big Bang theory. It undergoes to a series of phase transition with time-dependent temperature. In this scenario, the Kibble-Zurek mechanism gives a possible explanation to the formation of the early structures in the Universe as results of off-equilibrium phase transitions.

meanly involved superfluids and superconductors. Modern proves of these behaviours principally come from cold-atoms experiments, ion crystals and from improved experiments still based on superfluids or superconductors [see for example [2], [3] and [4]]. In particular, Bose–Einstein condensates in trapped cold gases are extremely controllable systems and therefore an ideal platform to check the KZ mechanisms. An example of such experiments is shown in fig.3.
The study of the off-equilibrium behaviours at continuous phase transitions has also cosmological implications. Nowdays there are theoretical models and observations[2] that support the Universe expansion and its consequent cooling. In this expansion, the Universe may have undergone a series of phase transitions very early in its history. The electroweak transition, at about 100 GeV, where the W and Z particles acquire a mass through the Higgs mechanism, occurred when the age of the Universe was around 10^{-10}s. There was probably a later transition, the quark-hadron transition at which the soup of quarks and gluons separated into individual hadrons. More interesting from a cosmological point of view, however, are the hypothetical transitions at even earlier times. If the idea of grand unification is correct, there would have been a phase transition of some kind at an energy scale of around 10^{15} GeV, corresponding to a time about 10^{-36}s after the Big Bang.
These phase transitions happened in the off-equilibrium regime because the temperature changes in time as effects of the expansion[3]. Naturally, it is impossible to perform cosmological experiments in order to corroborate these speculations. However, one of the clearest signatures of these transitions would be the formation of stable topological defects which the Kibble-Zurek mechanism predicts and several condensed matter experiments have been checked, as we have seen above.
There exist also phase transitions in which the macroscopic properties of the system abruptly vary across the critical point and present discontinuity. These are called first-order phase transitions. Off-equilibrium behaviours characterize also the first-order phase transitions. In particular, one of the early off-equilibrium phenomena observed was the *hysteresis*. Hysteresis arises in ferromagnetic systems when there is an external magnetic field with a time-dependence such as $h(t, t_s) \approx t/t_s$ at fixed temperature. The magnetic field changes its direction crossing the transition at $h = 0$; it follows that also the magnetization has to change direction according with h. However, it has been observed that the system reacts in late to the external perturbation developing metastable states for a certain interval of time.

[2]For example, the measurement of the properties of the cosmic microwave background radiation or the nucleosinthesis of the light atoms support the Big Bang theory.
[3]In the radiation domain i.e. before $t \sim 300000$y the temperature varies as $T(t) \sim 1/\sqrt{t}$.

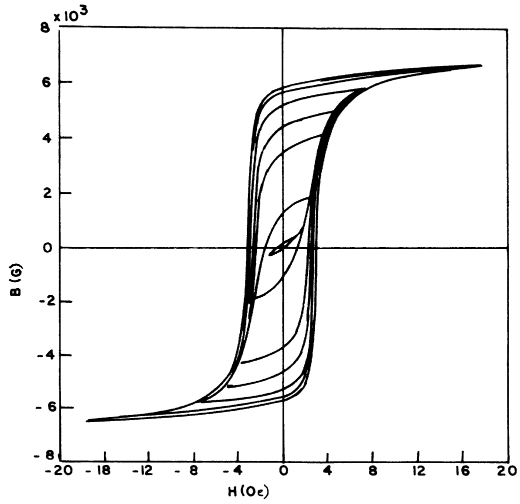

Figure 5: The characteristic curve of response of a system which presents hysteresis. The picture shows a family of hysteresis loops for various value of the external field in Permalloy. It has been taken from the ref.[5].

If we vary the external magnetic field from $t_i < 0$ to $t_f > 0$ across the first-order transition and then back from t_f to t_i, these memory effects lead the system to dissipate a non-zero value of energy. The magnetic work done by the system can be valuated as $\oint dh(t,t_s)\cdot\Sigma(t,t_s)$ where Σ is the magnetization. This integral is called *hysteresis loop area*. An example of hysteresis loop area is reported in Fig.5. It is clear that the hysteresis is related to the off-equilibrium: the equilibrium behaviour of the magnetization at a first-order transition is a jump at $h = 0$ from its initial direction to the reversed one. If the magnetization presents its equilibrium behaviour, the hysteresis curve is shrinked to a single line and the system does not spend energy in the cycle.

Overview.

In this thesis we study the off-equilibrium behaviours of the O(N) vector-models coupled to time-dependent external fields, arising near the critical point and below it. The analysis of such phenomena is made in the limit of large N which allows analytical computations. The dynamics given to the system reproduces the effects of an heat-bath and satisfies a purely dissipative Langevin equation. The external parameters are slowly varied and in such a way that they might drive the system across the critical point. In particular, two types of passage are eximined:

-The case in which the system is at the critical temperature and is coupled to a time-dependent magnetic field. The magnetic field is slowly quenched and then is turned on again in the opposite direction.

-The case in which the system starts above the critical temperature and is slowly cooled in the absence of magnetic fields.

For both these cases, scaling relations are derived and scaling functions for the correlators formally computed. The value of these scaling functions depend on the value of the effective mass of the system. The knowledge of this mass term at all times is sufficient to describe the time evolution of the correlation functions in the enteire off-equilibrium regime. However, an explicit expression cannot be easly achieved and is known only for the case of thermal variations at zero magnetic field. Therefore, we investigate the first deviations from the equilibrium proposing an ansatz for the leading off-equilibrium corrections to the scaling behaviour in terms of the effective mass. Under the assumption of exponential approach to the equilibrium, we show that the correlation functions present deviations from their equilibrium behaviour in agreement with the numerical evidences [see ref.[6]]. Two interesting phenomena related to the off-equilibrium physics are briefly discussed for the underlying model:

-The coarsening regime which occurs in the case of thermal variation without magnetic fields after crossing the critical point. It describes the development and the growth of domains characterized by different realizations of the ordered phase.

-The hysteresis phenomenon which occurs when the magnetic field is varied along a closed path.

Similiar off-equilibrium phenomena are shown by the system below the critical temperature, where it undergoes a first-order phase transition along the line of zero magnetic field. We also study the passage through this phase transition in the presence of a time-dependent magnetic field. In this case, the equations predict a rigid rotation of the magnetization in the off-equilibrium region.

The work is organized as follows. In the Sec.1 critical phenomena are presented and well known features are derived using the machinery of the field theories. The Sec.2 is dedicated to the dynamics. After a briefly introduction to the stochastic equations and their implementation in a field theory, we assume a relaxational dynamics for the model. In Sec.3 we present the O(N) vector-model in the limit of large N. Saddle point equations and critical exponents are derived. The Sec.4 is an introduction to the off-equilibrium behaviours. We formalize the passage through the transition with time-dependent external field introducing the protocols. The off-equilibrium scaling limit is defined, the finite-size effects are briefly discussed and the thermodynamic infinite-volume limit justified. In this section we also discuss the asymptotic behaviours i.e. what happens when the system approaches the off-equilibrium regime. If the system reaches the equilibrium asymptotically, it is expected that the first deviations from the equilibrium background are exponentially damped. A general ansatz in terms of the effective mass of the fields is formulated and reproduces the exponential decay of the off-equilibrium physics in the correlation functions asympotically. Within this ansatz, quantitative prediction on the leading corrections to the equilibrium behaviours of the correlators can be performed. In particular, in the Sec.5 we study the emergence of such phenomena in the O(N) vector model at large N. We first solve the Langevin equation with time-dependent external fields and compute the correlation functions. Then, we consider two different protocols separately: the thermal protocol in the absence of magnetic fields $r(t, t_s) - r_c \approx -t/t_s$ where r is the thermal coupling and r_c is its value at the critical temperature, and the magnetic field protocol which occur at the critical temperature varying the magnetic field $h(t, t_s) \approx t/t_s$. The Sec.6 and Sec.7 concern the study of the first-order transition which occurs in the O(N) vector model below the critical temperature. In Sec.6 we report the general scaling theory appropriate to describe the off-equilibrium and we consider the effects of a relaxational dynamics below the critical temperature. In the Sec.7 a magnetic field protocol at $T < T_c$ is considered and the off-equilibrium behaviour in O(N) vector models investigated. Finally, in Sec.8 we draw some conclusion.

The details of the analytical computations for Sec.3, 5 and 7 can be found in the appendices B, D and F respectively.

Contents

1 Critical Phenomena.

1.1 Introduction.

In the first section we discuss the continuous phase transitions. The general features of such transitions are shown with the concrete example of a spin system. Let us introduce the Ising model:

$$\mathcal{Z} = \sum_{\{S_i=\pm 1\}} \exp\Big(- H_{\text{Ising}}/T \Big) \tag{1}$$

where

$$H_{\text{Ising}} = J \sum_{i,j \text{ are n.n.}} S_i \cdot S_j \tag{2}$$

It describes a lattice of spins with ferromagnetic ($J < 0$) interactions among the nearest neighbour (n.n.) sites S_i. At high temperatures the directions of such spins are randomly distribuited and the system presents a macroscopic paramagnetic behaviour. When it is cooled, the macroscopic behaviour of the material changes continuously; below a certain value of the temperature that we call the *critical temperature*, the directions of the spins are no longer casual and the system presents a tendency to create large domains of aligned spins: it becomes a ferromagnet. The change in the behaviour of the system at the continuous phase transition can be related to a change in the symmetry of the model. In the Ising model, above the critical temperature if we flip the spins, all the observables are not modified and the system appears macroscopically the same. Below the critical temperature this is not true: if we flip a domain of aligned spins, the value of the magnetization of the system is flipped too. The system loses the $\mathbb{Z}_2$ symmetry crossing the transition.
More interesting magnetic systems with continuous symmetry can be obtained by a straightfoward extension of the Ising model: let us consider a lattice where in each site there is an N-component vector spin of unit length which interacts through a short-range ferromagnetic hamiltonian, qualitatively equivalent to the Ising one:

$$\mathcal{Z} = \int \prod_i dS_i \cdot \delta(S_i^2 - 1) \cdot e^{-H_N/T}, \tag{3}$$

where

$$H_N = -\sum_{i,j} V_{i,j}(S_i \cdot S_j) \tag{4}$$

and $V_{i,j}$ is a short range ferromagnetic $\mathrm{O}(N)$ symmetric two-body interaction. The direction of such spins is again random above the critical temperature and this means that the vector posses an $\mathrm{O}(N)$ symmetry. Below the critical temperature this symmetry has to change because the direction of the spins is now fixed and no more arbitrary: the group $\mathrm{O}(N)$ is reduced to $\mathrm{O}(N-1)$ which is the group of the rotation around a fixed axis.
Since the phase transition occurs continously, the state of the system at the critical point is characterized by the coexistence of the two phases. Thus, it must respect the symmetry of both the phases at the critical point. It follows[4] that the change in symmetry which occurs at a continuous phase transition reduces a certain group of symmetry to one of its subgroups. Generally, the phase of higher symmetry is related to higher temperatures and when the system is cooled loses some generators undergoing a state with reduced symmetry[5].
The change in the symmetry at a continuous phase transition can be related macroscopically to a change of the order in the material. One may consider a lattice where each site is characterized by a probability to find the microscopic variable in a certain realization of the symmetry. Above the critical temperature the probabilities for all the realization of the

[4] If we assume that the system has a certain symmetry at the critical point it can be shown that one of the two phases has the same symmetry and the other a lower symmetry. This is the only way in which two phases with different symmetries can be connected continuously.

[5] This can be shown more formally by using a group-theoretical approach. See ref.[8].

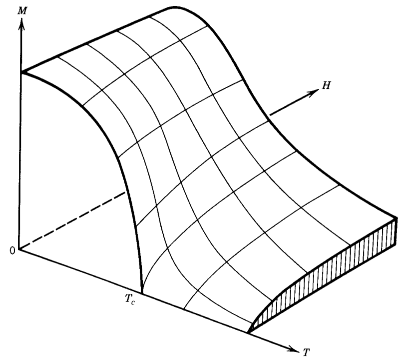

Figure 6: The behaviour of the order parameter (magnetization) in a spin system. A continuous phase transition occurs at $T = T_c$ and $H = 0$. This picture has been taken from the book [7].

symmetry are the same. Thus, the phase of the material is *disordered*. Below the critical temperature these probabilities changes and the system presents a type of *order*. To describe quantitatively the change of order of the system when it passes through a critical point, an *order parameter* can be defined such that it takes non-zero values only if the system is in the ordered phase. The physical meaning of the order parameter is very different in different systems. In the magnetic language the order parameter can be identified with the spontaneous magnetization.

It is well-known that the thermodynamic observables can be obtained from a thermodynamic potential which is a function of the external variables and of the order parameter[6]. The continuous behaviour of the system near the transition can be mathematically translated in the analiticy of the thermodynamic potential close to the critical point. It follows that this thermodynamic potential $V(\Phi)$ can be expressed as a power series in terms of the order parameter Φ

$$V(\Phi) = V_0 + a\Phi^2 + b\Phi^4 + \cdots \tag{5}$$

in which the odd terms have been neglected assuming $\mathbb{Z}_2$ symmetry. The external variables are fixed and determines the coefficients of the expansion. In contrast, the value of the order parameter is determined by the minimum of the potential, so it depends on the external variables. The expression (5) for the thermodynamic potential is a regular series and does not consider the singular behaviour of the system at the critical point. The conditions of applicability for such expansion are discussed in the ref.[8][7]. Above the critical point the shape of the thermodynamical potential is like a well whose global minimum is in zero [see fig.7]. Thus, the equilibrium value of the order parameter is zero and macroscopically the system presents a disordered phase. In the spin system example, even if a group of spins align themselves because of their ferromagnetic tendency, this microscopic order is distructed by the thermal fluctuations. When the system approaches to the transition, the shape of the well becomes weaker, i.e. growing local domains of aligned spins born but then are distructed by the thermal fluctuations. At the transition, thermal fluctuations and ordering tendency have the same strenght: a new stable configuration with ordered spins is allowed. The new phase is characterized by a non-zero value of the order parameter. Below the critical point, the thermodynamic potential presents a modified shape where only the ordered phase is admitted as stable configuration. In other words, the thermal fluctuations are no longer able to disorder the spins. We connect these concepts with the *correlation length*

[6]Crossing the transition, the macroscopic behaviour of the system changes. Thus, the order parameter reflects the changing in the structure of the system at the transition and must be taken into account in the expression of the thermodynamic potential.

[7]It is required that $\delta T = |T - T_c| \ll T_c$. However when $\delta T \to 0$ there must be a value $\delta T'$ such that for $\delta T < \delta T'$ the expansion (5) breaks down. The singularity in the specific heat can be reproduced by the Landau theory only considering the difference of the values of the specific heat at $\delta T'$ before and after the transition.

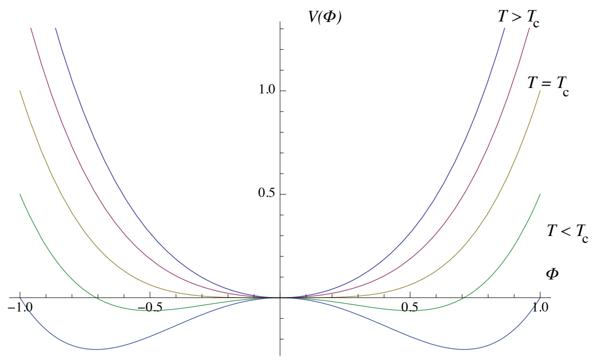

Figure 7: The shape of the thermodynamic potential $V(\Phi)$ as function of the order parameter Φ when the temperature is descreased at zero magnetic field.

of the material. We know that statistical systems have macroscopic properties determined by the weighted averages of all the possible local configurations. Local configuration means a region in which there are correlations among the sites. We call the size of this region correlation length. Away from the continuous phase transitions, the correlation length is of the order of some lattice spacings. Thus, a local configuration consists in a collection of a few spins. But when the system undergoes a continuous transition, the amount of the local fluctuations of the order parameter i.e. the correlation length of the system diverges causing a macroscopic variation in the properties of the material. Since the spins are correlated over an infinite length-scale, the system presents a *scale invariance*.
Note that an infinite value of the correlation length is found only at the critical point. In the neighbourhood of the critical point i.e. when the external variables are very close to their critical values (we call this regime *critical domain*), it can be shown that the correlation length has a power law behaviour as $\propto |T - T_c|^{-\nu}$ where ν is a certain positive exponent determined by the specific critical point under consideration. From the divergence of the correlation length, the divergences of some statistical observables follow. This means that the divergences of the statistical observables lie behind the scale invariance and therefore scaling relations in terms of the correlation length can be derived.
The continuous phase transitions are characterized by diverging correlation length and by the following divergences in some statistical observables. For this reasons the behaviours of systems near a continuous phase transitions are called *critical phenomena*.

The introduction above reports the Landau approach to the critical phenomena. Landau was the first one proposing a general framework that provided a unified explanation of several different phenomena (such as the Andrews critical opalescence or the Curie ferromagnetism) in terms of changes in the symmetry of the system. His model corresponds to the mean-field approximation and gives a good qualitative description of these phenomena. Even if it is not quantitative correct, the Landau ideas resume in a direct and simple way all the features of critical phenomena. Nowdays the modern theory of the critical phenomena is based on the use of the renormalization group.

1.2 Field Theories and renormalization approach.

A way to understand the critical phenomena is through a *renormalization group* analysis. A connection between statistical systems with short-range interactions and regularized local euclidean field theory can be established. Infact, these two branches of theoretical physics are both essentially based on the symmetry group of the model and on the use of renormalization[8]. Our purpose is to find all the empirical stuff about critical phenomena by using the formalism of field theory. We follow the discussion of the ref.[9] and [10].

Let us consider an hamiltonian functional $\mathcal{H}$ which is a function of the fundamental degrees of fredoom of the system. The correspondent field theory has to be chosen by respecting all the symmetry of the microscopic hamiltonian. Then, one considers the continuous limit for the space and for the degrees of freedom. The discrete lattice becomes an euclidean space in which the variables are fields $\phi(x)$, $x \in \mathbb{R}^d$. The resulting hamiltonian (local field theory) $\mathcal{H}(\phi)$ plays the role of an euclidean action. Suppose that we can expand this hamiltonian in powers of the field,

$$\mathcal{H}(\phi) = \sum_{n=0} \frac{1}{n!} \int d^d x_1 \cdots d^d x_n \cdot \mathcal{H}_n(x_1, \cdots, x_n); \tag{6}$$

In general this expansion contains an infinite number of terms. To the hamiltonian corresponds a set of (connected) correlation functions

$$< \phi(x_1) \cdots \phi(x_n) >_{\text{conn.}} = W^{(n)}(x_1, \cdots, x_n) = \int \mathcal{D}\phi \cdot \phi(x_1) \cdots \phi(x_n) \cdot e^{-\mathcal{H}(\phi)} \Big|_{\text{conn.}} \tag{7}$$

The correlation length ξ can be defined as the inverse of the smallest decay rate of the correlation functions (the smallest physical mass of the system). From the two-point correlation function one can define

$$\xi^2 \propto W^{(2)}(p)|_{p \to 0} \tag{8}$$

where $W^{(2)}$ is the connected two-point correlation function in the Fourier representation. We want to study the long distance properties of these correlations: let us introduce a dilatation parameter λ, the behaviour of correlators at large distances is therefore given by $W^{(n)}(\lambda x_1, \cdots, \lambda x_n)$ when $\lambda \to \infty$. This is the basic idea of the renormalization group: we consider correlation functions as functions of the dilatation parameter λ acting on the space variables x. In this way one constructs a set of a scale-dependent effective hamiltonians $\mathcal{H}_\lambda$; each of these has the same correlation functions at fixed space. Thus, we can interpretate $\mathcal{H}_\lambda$ as a flow in terms of the dilatation parameter which connect the same model viewed at different scales. Let us construct $\mathcal{H}_\lambda$ such that its correlation functions satisfy

$$W_\lambda^{(n)}(x_1, \cdots, x_n) - Z^{-n/2}(\lambda) W^{(n)}(\lambda x_1, \cdots, \lambda x_n) = R_\lambda^{(n)}(x_1, \cdots, x_n) \tag{9}$$

where $R_\lambda^{(n)}$ is a function that decrease faster of any powes of λ in the limit $\lambda \to \infty$ and therefore can be set to zero in the following lines. $Z(\lambda)$ is the scale-factor of the fields which has to be fixed consistently.
If we assume that the model has invariance under spatial translations, the previous relation can be written in the Fourier space too and becomes

$$W_\lambda^{(n)}(p_1, \cdots, p_{n-1}) = Z^{-n/2}(\lambda) \lambda^{-(n-1)d} \cdot W^{(n)}(p_1/\lambda, \cdots, p_{n-1}/\lambda). \tag{10}$$

We define as *RG-transformation* the mapping $\mathcal{H} \mapsto \mathcal{H}_\lambda$ and we fix $\mathcal{H}_{\lambda=1} = \mathcal{H}$ the microscopic "bare" hamiltonian. The renormalization group can be constructed in several forms and these differ only for the definition of $R_\lambda^{(n)}$ and $Z(\lambda)$.
We are interested to the limit $\lambda \to \infty$. We wonder if $\mathcal{H}_\lambda$ have a limit for large λ. Suppose that the answer is yes. In other case this discussion becomes meaningless.
Let us assume that exist a configuration $\mathcal{H}_*$ such that

$$\lim_{\lambda \to \infty} \mathcal{H}_\lambda = \mathcal{H}_*. \tag{11}$$

[8]Which leads to different conclusions depending on the spatial dimension of the theory.

We call this configuration *fixed point.* Even the correlation functions admit as limit their value at fixed point

$$W^{(n)}(\lambda x_1, \cdots, \lambda x_n) \overset{\lambda\to\infty}{\sim} Z^{n/2}(\lambda) \cdot W_*^{(n)}(x_1, \cdots, x_n). \tag{12}$$

If we consider another scale of dilatation μ and compute the rescaled correlation functions $W^{(n)}(\mu\lambda x_i)$, we obtain

$$W^{(n)}(\mu\lambda x_1, \cdots, \mu\lambda x_n) \sim Z^{n/2}(\mu\lambda) \cdot W_*^{(n)}(x_1, \cdots, x_n) \tag{13}$$

but also

$$W_*^{(n)}(\mu x_1, \cdots, \mu x_n) \sim Z_*^{n/2}(\mu) \cdot W_*^{(n)}(x_1, \cdots, x_n) \tag{14}$$

which implies that the scale-factor of the fields at fixed point can be defined as

$$\lim_{\lambda\to\infty} Z(\mu\lambda)/Z(\lambda) = Z_*(\mu)\ \forall\mu. \tag{15}$$

the previous relation states that Z_* has a semi-group representation

$$Z_*(\lambda_1) \cdot Z_*(\lambda_2) = Z_*(\lambda_1\lambda_2). \tag{16}$$

it is satisfied if we assume that

$$Z_*(\lambda) = \lambda^{-2d_\phi} \tag{17}$$

which is resonable: it tells to us that each field ϕ is scaled by λ at a certain exponent which we call the *scaling dimension of the field* and which is fixed by the specific properties of the fixed point under consideration.
Note that (12) implies a power-law behaviour of the correlation functions

$$W^{(n)}(\lambda x_1, \cdots, \lambda x_n) \sim \lambda^{-nd_\phi} W_*^{(n)}(x_1, \cdots, x_n). \tag{18}$$

The long-distance behaviour of the system is characterized by the fixed point of the RG flow in which the effective hamiltonian is driven to. The connection with critical phenomena lies behind the identification of the RG fixed points with the critical points. The correlation functions of different microscopic hamiltonians which flows into the same fixed point have the same critical properties. *Universality* is recovered and relies upon the exsistence of an infrared fixed points in the space of the hamiltonians. This space can be divided into subspaces and each subspace can be defined by the hamiltonians which have the same fixed point. The subspaces are the *universality classes*: the critical theory does not depend on the specific bare model we choose into a certain universality class.
From the equation (18), by considering the two-point correlation function in the Fourier space, we obtain

$$W^{(2)}(p/\lambda) \sim \lambda^{-2d_\phi}\lambda^d \cdot W_*^{(2)}(p); \tag{19}$$

thus, the correlation length is infinite whenever the fixed point is characterized by $d_\phi < d/2$ because,

$$\xi^2 \propto W^{(2)}(p=0) \sim \lambda^{-2d_\phi+d} \cdot W_*^{(2)}(0) \to \infty \tag{20}$$

The field theory at the fixed point (with $d_\phi < d/2$) is therefore critical because it is characterized by an infinite correlation length.
Let us understand a little bit more the situation. We now consider the dilatation parameter λ as a continuous variable in order to write equations which describes the flow of renormalization. These ones can be written as follows:

$$\lambda\frac{d}{d\lambda}\mathcal{H}_\lambda = \mathcal{T}[\mathcal{H}_\lambda], \tag{21}$$

$$\lambda\frac{d}{d\lambda}\log Z(\lambda) = 2 - d - \eta[\mathcal{H}_\lambda], \tag{22}$$

where $\mathcal{T}$ is a map from the space of the hamiltonians into itself and η is a real function defined on the space of the hamiltonians. It is quite obvious to define a critical point $\mathcal{H}_*$ as a solution of (21) such that

$$\mathcal{T}[\mathcal{H}_*] = 0.$$

From the other relation (22), with (17) we obtain the scaling dimension of the field

$$d_\phi = \frac{1}{2}(d - 2 + \eta[\mathcal{H}_*]). \tag{23}$$

1.2.1 Linearized renormalization group.

We consider the hamiltonian $\mathcal{H}_\lambda$ very close to a fixed point, where we can write $\mathcal{H}_\lambda \approx \mathcal{H}_* + \Delta\mathcal{H}_\lambda$. The equation (21) becomes

$$\lambda \frac{d}{d\lambda}(\Delta\mathcal{H}_\lambda) = L_*[\Delta\mathcal{H}_\lambda] \tag{24}$$

where L_* is a linear operator independent on λ. This version of renormalization group which works only in the neighbourhood of a fixed point is sometime called *linearized RG*. Since L_* is linear, it can be decomposed into some eigenoperators y_i (we sometimes call them *scaling fields*) with specific eigenvalues ℓ_i (we often refer to them as *scaling dimensions* of the operators). $\Delta\mathcal{H}_\lambda$ can be expanded in this base

$$\Delta\mathcal{H}_\lambda = \sum_i h_i(\lambda) \cdot y_i \tag{25}$$

and it follows that (24) becomes

$$\lambda \frac{d}{d\lambda} h_i(\lambda) = \ell_i \cdot h_i(\lambda) \tag{26}$$

it admits as solution $h_i(\lambda) = \lambda^{\ell_i} \cdot h_i(1)$. This means that an hamiltonian $\mathcal{H}_\lambda$ close to a fixed point can be described through the decomposition into the scaling fields. These fields can be classified as follows:

-if $\ell_i > 0$ then the scaling fields are *relevant* and thus if $\mathcal{H}_\lambda$ contains these operators, they grow with λ driving the system away from the critical point;

-if $\ell_i < 0$ then the operator is *irrelevant*: its effects goes to zero when the dilatation parameter increases;

-if $\ell_i = 0$ the scaling operator is *marginal* and remains finite in the RG flow.

The classification of fixed points is related to their local stability properties. A fixed point is locally stable if all the relevant operator are fixed to their critical values. The number of relevant parameters classifies a critical point and gives us the dimension of the *critical surface*. The critical surface of a critical point is a surface in the space of hamiltonians characterized by having all the relevant scaling fields fixed to their critical values. Thus, an hamiltonian which lies on the critical surface, presents a critical behaviour (or in other word it converges to $\mathcal{H}_*$ if we perform a scale transformation with $\lambda \to \infty$). The dimension of the critical surface is given by the co-dimension of the number of the relevant parameters.

As we have seen, the hamiltonian can be parametrized through its couplings $\{h(\lambda)\}$ close to the fixed point. The rescaled correlation functions can be written as function of the rescaled couplings

$$W_\lambda^{(n)}(x_1, \cdots, x_n) = W^{(n)}(\{h(\lambda)\}, x_1, \cdots, x_n); \tag{27}$$

We differentiate the relation (9) with respect to λ

$$\lambda\frac{d}{d\lambda}\Big(Z^{n/2}(\lambda)W^{(n)}(\{h(\lambda)\},x_1/\lambda,\cdots,x_n/\lambda)\Big)=0 \tag{28}$$

and expliciting the action of the derivative we obtain

$$\Big(-\sum_l x_l\frac{\partial}{\partial x_l}+\sum_i\beta(h_i)\frac{\partial}{\partial h_i}+\sum_i\frac{n}{2}(2-d-\eta(h_i))\Big)W^{(n)}(\{h(\lambda)\},x_1,\cdots,x_n)=0 \tag{29}$$

where we have introduced the *beta-function*

$$\beta(h_i)=\lambda\frac{d}{d\lambda}h_i(\lambda) \tag{30}$$

and $\eta(h_i)$ is given by

$$2-d-\eta(h_i)=\lambda\frac{d}{d\lambda}\log Z(\lambda). \tag{31}$$

A fixed point, in this notation, is the configuration $\{h^*\}$ such that $\beta(h_i^*)=0\ \forall i$.

1.2.2 Gaussian fixed point.

We have seen how universality and critical behaviour follows if we assume that an IR fixed point exist and it is reached by the RG flow. In practise is not obvious how to find a fixed point starting from a specific bare model. Let us start with a free theory which is quadratic in fields. We want to construct a scale transformation such that a fixed point is allowed.
A generic quadratic hamiltonian is of the form

$$\mathcal{H}_G(\phi)=\frac{1}{2}\int d^dx\cdot\sum_{r=0}\phi(x)\Box_r\phi(x) \tag{32}$$

where the box operator $\Box_r$ is a two-fields coupling constant containing $2r$ derivatives; the $\Box_0=m^2$ is the mass term. We perform a scale transformation

$$\phi\mapsto\sqrt{Z(\lambda)}\phi,$$
$$x\mapsto\lambda x.$$

The effective hamiltonian becomes

$$\mathcal{H}_{G,\lambda}=\frac{1}{2}\int\lambda^d d^dx\cdot Z(\lambda)\sum_{r=0}\phi(x)\Box_r\lambda^{-2r}\phi(x) \tag{33}$$

therefore it can be viewed as a quadratic coupling which is transformed as

$$\Box_r\mapsto\Box_r(\lambda)=Z(\lambda)\lambda^{d-2r}\Box_r(1),$$

where $\Box_r(1)=\Box_r$. When the dilatation parameters becomes large $\lambda\to\infty$, the term which contains the smallest number of derivative becomes the most important. A critical theory requires that $\Box_r^*(\lambda)=\Box_r^*=Z(\lambda)\lambda^{d-2r}\Box_r^*$: it is satisfied if,

• We choose $Z(\lambda)=\lambda^{-d}$ if $m^2\neq0$ and $\Box_{r>0}=0$. This is commonly called *trivial fixed point*,

$$\mathcal{H}_G^*=\frac{1}{2}m^2\int d^dx\cdot\phi^2(x). \tag{34}$$

The scaling dimension of the field ϕ is $d_\phi=d/2$: the theory is not critical because the correlation length ξ tends to zero. The two-point correlation function becomes a δ-propagator.

• We choose $Z(\lambda) = \lambda^{-(d-2)}$ if $m^2 = 0$ and $\Box_1 \neq 0$ (all the other terms with higher derivatives are no relevant even in $d > 4$). The scaling dimension of the field ϕ is $d_\phi = (d-2)/2$ which is often called *canonical scaling dimension* of the fields. Infact, looking at (22), the η function is defined as a correction to this canonical dimension and therefore is zero in this case. The underlying case is the *gaussian fixed point.* The theory is critical and characterized by an infinite value of the correlation length. The critical hamiltonian is given by

$$\mathcal{H}_G^* = \frac{1}{2}\int d^dx \cdot \phi(x)\Box_1\phi(x). \tag{35}$$

Let us perform a linearization of the RG near the gaussian fixed point: the effective hamiltonian can be decomposed into scaling fields

$$\Delta\mathcal{H}_\lambda = \mathcal{H}_\lambda - \mathcal{H}_G^* = \sum_{n=2}^{\infty}\sum_{r=0}\sum_i y_i^{n,r}(\lambda,\phi) \tag{36}$$

where

$$y_i^{n,r}(\lambda,\phi) = Z^{n/2}(\lambda)\lambda^{d-r}y_i^{n,r}(\phi) \tag{37}$$

Thus, we learn by the previous relation that scaling fields have scaling dimensions $\ell_{n,r} = d - \frac{n}{2}(d-2) - r$. The gaussian fixed point can be classified and its local stability can be related to the spatial dimension d of the critical theory. In particular,

-the operator $n = 2$, $r = 0$ that is a mass term is always relevant and correspond to a deviation from the critical temperature.

-the odd operator $n = 1$, $r = 0$ which breaks explicitly the invariance under $\mathrm{O}(N)$ symmetry, is always relevant and has dimension $l_{1,0} = (d+2)/2$. It can be interpreted, in spin systems, as an external magnetic field.

-if the spatial dimension $d > 4$ all the other operator are irrelevant and the gaussian fixed point is stable.

-If $d = 4$ only $n = 4$, $r = 0$ becomes marginal and logarithmic corrections are expected.

-if $d < 4$ the gaussian fixed point is certainly unstable because many operators are no longer irrelevant and move the system away from the gaussian critical point.

The gaussian fixed point has been obtaneid starting from free theories and it has no a particular physical interest. We want to find another fixed point which allows interactions. As we can see in the last few lines, this can be done only below four spatial dimensions. The impossibility to construct a theory which flows to an interacting fixed point above dimension four is called *triviality problem.*

1.2.3 Wilson-Fisher fixed point and the ϵ-expansion.

Below dimension four, the main difficult in RG approach is to construct an explicit scale transformation which drives $\mathcal{H}_\lambda$ to a fixed point. The question of the existence of another non-trivial fixed point is non-perturbative and cannot be easily answered. Some results were obtained thanks to Wilson and Fisher. They assume that the spatial dimension and the scaling dimensions of the operators are continuous functions and developed a theory in the neighbourhood of four dimension, $d = 4 - \epsilon$. If ϵ is small, the critical hamiltonian gains only the quartic coupling constant u and all the other operators still remain irrelevant. The constant u is small and remains small under the dilation trasformation because its evolution is very slow. Under these assumption one can perform a double perturbation expansion in terms of the coupling constant u and of $\epsilon = 4 - d$. The flow of $u(\lambda)$ is determined only by

$u(\lambda)$ itself and it is well-described by the function beta

$$\lambda \frac{d}{d\lambda} u(\lambda) = \beta(u(\lambda)) = -\epsilon u(\lambda) + \beta_2 u^2(\lambda) + O(u^3(\lambda)). \tag{38}$$

β_2 is a constant which depends on the spatial dimension $4-\epsilon$ but, at leading order, can be computed in $d=4$. The previous line tells to us that another non-trivial fixed point is emerged

$$\beta(u_*) = 0 \Rightarrow u_* = \epsilon/\beta_2 + O(u^3); \tag{39}$$

The sign of β_2 plays a crucial role and gives a direction to the RG flow

-if $\beta_2 < 0$ and $d < 4$ then $u(\lambda)$ increases: the ϵ-expansion loses its meaning. In $d > 4$ the previous fixed point becomes repulsive. The destiny of the system depends on the bare value of u: if $u(1) = u < u_*$ the system comes again to the gaussian fixed point; if $u = u_*$ it still remains at the critical point and finally in the case of $u > u_*$, the running coupling constant $u(\lambda)$ will increase with λ and any fixed point is reached.

-if $\beta_2 = 0$ one needs the u^3 term of the perturbation expansion to make conclusions.

-if $\beta_2 > 0$ and $d < 4$ the Wilson-Fisher fixed point is stable thus if $u < (>) u_*$ then $u(\lambda)$ decreases (increase) and goes to the fixed point value; $u(\lambda)$ remains equal to the critical value if $u = u_*$. Above dimension four is already known that the gaussian fixed point is the only stable fixed point which exists.

The explicit computation of the beta function can be performed only by fixing a specific model.

1.2.4 Effective ϕ^4 model.

Let us consider a ϕ^4 model. In the next sections we focus our attention on spin systems with O(N) symmetry: this symmetry does not allow terms with odd power of the fields. Therefore the microscopic short-range statistical system can be reduced to a local regularized ϕ^4-field theory in $4-\epsilon$ spatial dimension. The microscopic hamiltonian is

$$\mathcal{H}(\phi) = \int d^d x \cdot \Big[\frac{1}{2} c(\nabla\phi)^2 + \frac{1}{2} a\phi^2(x) + \frac{b}{4!}(\phi^2(x))^2\Big], \tag{40}$$

where a, b and c are regular functions of the temperature T close to the critical temperature T_c. One can consider a more general hamiltonian, that can be expanded in powers of the field ϕ and derivatives

$$\mathcal{H}(\phi) = \int d^d x \cdot \Big[\frac{1}{2} c(\nabla\phi)^2 + \sum_j \mathcal{H}_j(\phi)\Big], \tag{41}$$

where $\mathcal{H}_j(\phi)$ is an O(N) symmetric monomial in ϕ of degree n_j and containing r_j derivative. However, by the previous discussion we know that the effective hamiltonian near dimension 4 cointain almost the quartic term and all the others cannot survive in the flow. Thus, we consider as bare field theory a local ϕ^4-model. Let us perform the rescaling of the field $\phi(x)$ in such a way that the coefficient of $(\nabla\phi)^2$ becomes the standard 1/2:

$$x \mapsto \Lambda x, \tag{42}$$

$$\phi(x) \mapsto \zeta\phi(x); \tag{43}$$

After this rescaling all quantities have a dimension in units of Λ. Our choice of normalization for the gradient term implies,

$$\zeta = c^{-1/2} \cdot \Lambda^{(2-d)/2}, \tag{44}$$

which shows that ϕ now has in terms of Λ its canonical dimension $d_\phi = (d-2)/2$. A term $\mathcal{H}_j(\phi)$ then is multiplied by

$$\mathcal{H}_j(\phi) \mapsto \Lambda^{d-n_j(d-2)/2-r_j}\mathcal{H}_j(\phi). \tag{45}$$

For large Λ all operators except $(\phi^2)^2$ should remain irrelevant. After the rescaling the hamiltnian $\mathcal{H}(\phi)$ then becomes

$$\mathcal{H}(\phi) = \int d^dx \cdot \left[\frac{1}{2}(\nabla\phi(x))^2 + \frac{1}{2}r\phi^2(x) + \frac{u}{4!}\cdot\Lambda^{4-d}(\phi^2(x))^2\right] \tag{46}$$

with $r = a\Lambda^2/c$, $u = b/c^2$.

We choose (46) as the bare model of the critical theory. This hamiltonian generates a perturbative expansion of field theory type which can be described in terms of Feynman diagrams. These have to be calculated with a momentum cut-off of order Λ, reflection of the initial microscopic structure. The corresponding theory is thus analogous to regularized quantum field theory. However, in constrast with the conventional quantum field theory, where the bare coupling constant are adjustable parameters and have not well-defined physical meaning, here the quartic coupling constant has a dependence in Λ given a priori. In terms of the microscopic length scale Λ^{-1}, the critical domain is given at large distances $\gg \Lambda^{-1}$ in which $\xi^{-1} \sim$ physical mass $\ll \Lambda$.

1.2.5 Renormalization near four dimension.

Let us find a non-trivial fixed point in perturbation theory: we perform a series expansion in powers of the coupling constant u and of the distance ϵ from dimension four. We want to study the large cut-off limit: we introduce a new scale of energy $\mu \ll \Lambda$ and define renormalized correlation functions. In order to give a meaning to the renormalization procedure, one has to declare what are the values of the physical parameters in the renormalized field theory. In the case of the hamiltonian (46) we impose for the proper vertices:

$$\Gamma^{(2)}_{\text{ren.}}(p, u_{\text{ren.}}, \mu, \Lambda)|_{p^2=0} = 0, \tag{47}$$

which states that the theory is critical. For normalization of the fields

$$\frac{\partial}{\partial p^2}\Gamma^{(2)}_{\text{ren.}}(p, u_{\text{ren.}}, \mu, \Lambda)|_{p^2=\mu} = 1 \tag{48}$$

and we declare what is the renormalized quartic coupling constant:

$$\Gamma^{(4)}_{\text{ren.}}(p, u_{\text{ren.}}, \mu, \Lambda)|_{\{p_i=\mu\theta_i,\theta_i\in\mathbb{R}\}} = \mu^\epsilon u_{\text{ren.}} \tag{49}$$

These renormalized correlation functions are related to the original ones (bare) by the relation

$$\Gamma^{(n)}_{\text{ren.}}(p, u_{\text{ren.}}, \mu, \Lambda) = Z^{n/2}(u_{\text{ren.}}, \Lambda/\mu)\Gamma^{(n)}(p, u, \Lambda) \tag{50}$$

The scale-factor of the fields Z contain a gaussian normalization which is already performed. From this relation one can write down the bare RG equation

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(u, \Lambda/\mu)\frac{\partial}{\partial u} - \frac{n}{2}\eta(u, \Lambda/\mu)\right]\Gamma^{(n)}(p, u, \Lambda) = 0; \tag{51}$$

where the function[9]

$$\beta(u, \Lambda/\mu) = \beta(u) = \Lambda\frac{d}{d\Lambda}\Big|_{u_{\text{ren}},\mu\text{ fix.}} u \tag{52}$$

[9] We have neglected the dependence on the ratio Λ/μ because these function can be obtained also by solving the equation (51) in terms of correlation functions which do not know the new scale μ.

$$\eta(u,\Lambda/\mu) = \eta(u) = -\Lambda \frac{d}{d\Lambda}\Big|_{u_{\text{ren.}},\mu \text{ fix.}} \log Z(g,\Lambda/\mu) \tag{53}$$

The equation (51) is satisfied in the limit of large Λ by the bare correlation functions of the microscopic model. We want to describe the long distance properties of the theory when the dilatation parameter $\mu/\Lambda = \lambda$ becomes very small. We are looking for $u(\lambda)$, $Z(\lambda)$ which satisfy

$$\lambda \frac{d}{d\lambda}\Big[Z^{-n/2}(\lambda)\Gamma^{(n)}(p,u(\lambda),\lambda\Lambda)\Big] = \Big(\lambda\frac{\partial}{\partial\lambda} + \beta(u(\lambda))\frac{\partial}{\partial u(\lambda)} - \frac{n}{2}\eta(u(\lambda))\Big)\Gamma^{(n)}(p,u(\lambda),\lambda\Lambda) = 0 \tag{54}$$

with

$$\beta(u(\lambda)) = \lambda \frac{d}{d\lambda} u(\lambda) \tag{55}$$

$u(\lambda)$ is a running coupling constant which changes as the scale of the theory is dilatated $\Lambda \mapsto \lambda\Lambda$. It follows that $u(1) = u$. Note that this relation has the same meaning of (21) because the running coupling constant $u(\lambda)$ full characterizes the hamiltonian $\mathcal{H}_\lambda$ in $4-\epsilon$ dimensions. The scale-factor of the fields also becomes a running coupling with $Z(1) = 1$ and satisfies the flow equation

$$\eta(u(\lambda)) = \lambda \frac{d}{d\lambda} \log Z(\lambda) \tag{56}$$

which is equivalent to the relation (22) with a gaussian normalization of Z. The correlation functions after the dilatation are related to the original ones by the equation

$$\Gamma^{(n)}(p,u,\Lambda) = Z^{-n/2}(\lambda)\Gamma^{(n)}(p,u(\lambda),\lambda\Lambda), \tag{57}$$

which has the same meaning of (50) but considering running couplings. Formally the construction (54) has solution

$$\int_u^{u(\lambda)} \frac{dg}{\beta(g)} = \log\lambda \tag{58}$$

$$\int_1^\lambda \frac{ds}{s}\eta(u(s)) = \log Z(\lambda) \tag{59}$$

These relations can be solved in perturbation theory. In particular, the beta-function follows from the relation (49) which relates the bare coupling constant to the renormalized one and then by its definition (52). If we perform the computation [see ref.[10]]

$$\beta(u,\epsilon) = -\epsilon u + \frac{N+8}{48\pi^2}u^2 + O(u^3, u^2\epsilon). \tag{60}$$

where N is the number of components of the field ϕ in the O(N) symmetric hamiltonian (46). Looking to the beta-function, we note that $\beta_2 > 0$ so the new fixed point is IR stable below four dimension. We are interested to the behaviour near the fixed point u_* where we can linearize the beta-function $\beta(u) = \omega(u-u_*)$, $\omega = \beta'(u_*) = \epsilon + O(\epsilon^2)$. By substituting the linearized beta-function in (58) we obtain

$$|u(\lambda) - u_*| \overset{\lambda\to 0}{\sim} O(\lambda^\omega) \tag{61}$$

Thus, the effect of renormalization is to bring the quartic coupling constant more and more close to the critical value.

1.2.6 Scaling behaviours at the critical point.

Let us assume that the function η and the correlation functions are finite at the critical point. This is consistently with the results which one can obtain order by order in ϵ-expansion. The equation (59) can be approximately solved near the fixed point (in the limit of small λ)

$$\log Z(\lambda) \overset{\lambda\to 0}{\sim} \eta(u_*)\log\lambda \tag{62}$$

and we call as *critical exponent* η the number $\eta(u_*) = \eta$. It is very important because describes the scaling behaviour of the correlation function at the critical point

$$\Gamma^{(n)}(p, u(\lambda), \lambda\Lambda) \sim \lambda^{n\eta/2}\Gamma^{(n)}(p, u_*, \Lambda) \tag{63}$$

but the gaussian normalization states that a rescaling performed over the momenta obey to the scaling relation

$$\Gamma^{(n)}(\lambda p, u, \lambda\Lambda) \sim \lambda^{d-n/2(d-2)}\Gamma^{(n)}(p, u, \Lambda) \tag{64}$$

Thus, the *scaling behaviour of correlation functions* close to the Wilson-Fisher critical point is

$$\Gamma^{(n)}(\lambda p, u(\lambda), \lambda\Lambda) \sim \lambda^{d-n/2(d-2+\eta)}\Gamma^{(n)}(p, u_*, \Lambda) \tag{65}$$

We conclude that d_ϕ is now different from the canonical value (at the gaussian fixed point). It has aquired anoumalus dimension η

$$d_\phi = \frac{1}{2}(d - 2 + \eta) \tag{66}$$

Finally, we derive the *weak-scaling statement*: the two-point correlation function at the critical point diverges with a specific power-law when the momenta go to zero. From the general relation (65) we consider the case $n = 2$ and use the momentum $\lambda p = k$ as scale of dilatation

$$\Gamma^{(2)}(\lambda p) = \Gamma^{(2)}(k) \underset{p\text{ fix.}}{\overset{k\to 0}{\sim}} k^{2-\eta} \tag{67}$$

$$W^{(2)}(k) \overset{k\to 0}{\sim} 1/k^{2-\eta} \tag{68}$$

We have found a scaling-behaviour for the correlation functions looking at long distances below four dimension by using the ϵ-expansion technique. This behaviour is universal in the sense that it is governed only by the properties of the fixed point and, for small ϵ, it does not depend on the value of the bare coupling constant. The critical phenomena are universal depending only on the symmetry of the field theory and on the spatial dimension (which characterize the fixed point).

1.2.7 Scaling behaviours above the critical point.

We are interested to the scaling properties of the system where the correlation length is large with respect to the microscopic scale Λ^{-1} but finite. This region is called *critical domain.* Let us define this region formally.
Consider a variation in the temperature T from the critical value T_c. It moves all the relevant couplings away from the critical point. However, in $4 - \epsilon$ spatial dimensions, the most important contribution is given by the ϕ^2 operator. Thus, we can consieder into (46) the coupling $r = r(T)$ and close to the critical point

$$r(T) \sim \frac{T - T_c}{T_c} + r_c = \tau + r_c, \tag{69}$$

$r_c = r(T_c)$ a constant. By dimensional analysis, the bare correlation functions have a scale relation

$$\Gamma^{(n)}(p, \tau, u, \Lambda) = \Lambda^{d-\frac{n}{2}(d-2)}\Gamma^{(n)}(p/\Lambda, \tau/\Lambda^2, u, 1) \tag{70}$$

We define the critical domain as the region in which the system presents small thermal variations with respect to the critical temperature

$$|\tau| \ll \Lambda^2 \tag{71}$$

Since in the critical domain $\tau \neq 0$, to the renormalization conditions given before one has to add the conditions at zero momentum

$$\Gamma^{(2)}_{\text{ren.}}(p = 0, \tau_{\text{ren.}}, u_{\text{ren.}}) = \tau_{\text{ren.}}, \tag{72}$$

$$\Gamma^{(4)}_{\text{ren.}}(p=0,\tau_{\text{ren.}},u_{\text{ren.}}) = \tau^{\epsilon}_{\text{ren}} u_{\text{ren.}} \tag{73}$$

The renormalized correlation functions are related to the bare correlations by

$$\Gamma^{(n)}_{\text{ren.}}(p,\tau_{\text{ren.}},u_{\text{ren.}}) \overset{\Lambda\to\infty}{\sim} Z^{n/2}(\tau_{\text{ren.}}/\Lambda,u_{\text{ren.}})\Gamma^{(n)}(p,\tau,u,\Lambda) \tag{74}$$

Note that the renormalized quadratic coupling has dimension of a mass $\tau_{\text{ren.}} = m_{\text{ren.}}$. So, if we perform a rescaling in the renormalized correlation functions through $m_{\text{ren.}}$

$$\Gamma^{(n)}_{\text{ren.}}(p,m_{\text{ren.}},u_{\text{ren.}}) \sim m_{\text{ren.}}^{d-\frac{n}{2}(d-2)}\Gamma^{(n)}_{\text{ren.}}(p/m_{\text{ren.}},1,u_{\text{ren.}}) \tag{75}$$

This rescaling, applied to the case $n=2$ tells to us that the two-point correlation function decays with a power-law of a characteristic length-scale $\xi = m_{\text{ren.}}^{-1}$ which is the correlation length. Let us perform a dilatation transformation of the correlation functions with a parameter $\lambda = m_{\text{ren.}}/\Lambda$: from (74) we obtain the flow equation

$$\lambda\frac{d}{d\lambda}\Big[Z^{-n/2}(\lambda)\Gamma^{(2)}(p,\tau(\lambda),u(\lambda),\lambda\Lambda)\Big] = 0 \tag{76}$$

We introduce the functions

$$-\lambda\frac{d}{d\lambda}u(\lambda) = \beta(u(\lambda)) \tag{77}$$

$$\lambda\frac{d}{d\lambda}\log Z(\lambda) = \eta(u(\lambda)) \tag{78}$$

$$\lambda\frac{d}{d\lambda}\log\tau(\lambda) = \eta_2(u(\lambda)) \tag{79}$$

which describe the flow of the running coupling constant. We set $u(1)=u$, $Z(1)=1$, $\tau(1)=\tau$. The RG equation becomes

$$\Big[\lambda\frac{\partial}{\partial\lambda} - \beta(u(\lambda))\frac{\partial}{\partial u(\lambda)} + \frac{n}{2}\eta(u(\lambda)) - \eta_2(u(\lambda))\tau(\lambda)\frac{\partial}{\partial\tau(\lambda)}\Big]\Gamma^{(n)}(p,\tau(\lambda),u(\lambda),\lambda\Lambda) = 0; \tag{80}$$

The critical region is defined by (71), so we can impose as upper limit that the running coupling constant $\tau(\lambda) = \lambda^2\Lambda^2$ after the rescaling. From (79) we obtain

$$\tau(\lambda) = \tau\cdot\exp\Big(-\int_1^\lambda\frac{ds}{s}\cdot\frac{1}{\nu(u(s))}\Big) \tag{81}$$

where ν is a real function of u. For $\lambda\to 0$ we can substitute $\nu(u(\lambda))\sim\nu(u_*)=\nu$ and the upper limit becomes

$$\log(\tau/\Lambda^2) \simeq \frac{1}{\nu}\log\lambda \tag{82}$$

It follows that

$$\tau \propto \lambda^{1/\nu} \propto m_{\text{ren.}}^{1/\nu} \Rightarrow \xi\sim\tau^{-\nu} \tag{83}$$

ν is the critical exponent that drives the scaling behaviour of the correlation length into the critical domain. The correlation functions are related in the way we know

$$\Gamma^{(n)}(\lambda p,\lambda^2\tau,\lambda\Lambda) \sim \lambda^{d-\frac{n}{2}(d-2+\eta)}\Gamma^{(n)}(p,\tau,u,\Lambda) \tag{84}$$

but, from this discussion, a new property arises: since $\tau\neq 0$ the correlation functions at zero momenta are finite and present a scaling behaviour given by

$$\Gamma^{(n)}(p=0,\lambda^2\tau,u,\Lambda) \propto \tau^{\nu(d-\frac{n}{2}(d-2+\eta))} \tag{85}$$

this is commonly called *strong scaling law*. In particular the two-point correlation function at zero momenta, often called *susceptibility*, has a behaviour

$$\chi = W^{(2)}(p=0,\tau,u,\Lambda) \propto \tau^{-\nu(2-\eta)} \tag{86}$$

we define as critical exponent $\gamma = \nu(2-\eta)$ which describes the power-law behaviour of the susceptibility. Other thermodinamic critical exponents can be defined but they are all reducible to a certain combination of ν and η. These last two exponents directly follow from scaling properties of the critical system and therefore lead to the scaling relations in the thermodinamic observables.

1.2.8 Symmetry breaking and scaling behaviours below the critical point.

The system (46) is invariant under global O(N) transformations. The infinitesimal variation of the field ϕ is

$$\delta\phi(x)_\alpha = T^i_{\alpha\beta}\phi_\beta(x)\cdot v_i \tag{87}$$

where $T_{\alpha\beta}$ are the generators of the group and v_i are the space-independent parameters of the specific transformation $g = 1 + v \in \mathrm{O}(N)$ close to the identity. In order to connect continuosly the state of the system above and below the critical point, we need a spontaneus symmetry breaking. A standard way to proceed is to couple the hamiltonian with an external magnetic field which explicitly brokes O(N) symmetry into O($N-1$). Then, if the system is not physically coupled to a magnetic field, one takes the limit in which this field goes to zero. The hamiltonian (46) becomes

$$\mathcal{H}(\phi) = \int d^dx\cdot\Big[\frac{1}{2}(\nabla\phi(x))^2 - \frac{1}{2}|\tau|\phi^2(x) + \frac{1}{2}r_c\phi^2(x) + \frac{u}{4!}\cdot\Lambda^{4-d}(\phi^2(x))^2 \; - h_\alpha\cdot\phi_\alpha(x)\Big] \tag{88}$$

$$= \mathcal{H}_{\mathrm{sym}}(\phi) - \int d^dx\cdot h_\alpha\cdot\phi_\alpha(x).$$

where the magnetic field has a fixed direction

$$h_\alpha = \delta_{1,\alpha}\cdot h. \tag{89}$$

The field ϕ has a non-trivial expectation value σ that we call *order parameter* or, more physically, the magnetization. We parametrize the field as a fluctuation over the background $\phi(x) = \sigma + \varphi(x)$. To the hamiltonian (88) are associated correlation functions which are no longer O(N)-symmetric. These satisfiy the Ward identities, that state

$$\int d^dx\cdot T^i_{\alpha\beta}\Big[\frac{\delta\Gamma(\varphi(x)+\sigma)}{\delta\varphi_\alpha(x)} + h_\alpha\Big](\varphi_\beta(x)+\sigma_\beta) = 0; \tag{90}$$

where $\Gamma(\phi)$ is the effective action, the functional generator of the proper vertices of the theory. In particular, two important relations follow from the Ward identities. The first is obtained by consider $\varphi = 0$

$$T^i_{\alpha\beta}h_\alpha\sigma_\beta = 0 \tag{91}$$

The magnetization has the same direction of the external magnetic field and are both left invariant under O($N-1$) rotations around the direction of σ. The second important relation involves the two-point correlation function: if we differentiate once (90) with respect to $\varphi_\gamma(y)$ and then set $\varphi = 0$ we obtain

$$\int d^dx\cdot\Big[\sigma_\beta T^i_{\alpha\beta}\cdot\frac{\delta^2\Gamma(\varphi(x)+\sigma)}{\delta\varphi_\alpha(x)\delta\varphi_\gamma(y)}\Big|_{\varphi=0} + T^i_{\alpha\beta}\delta_{\beta\gamma}\delta^d(x-y)\sigma_\alpha\Big] =$$

$$\int d^dx\cdot\Big[\sigma_\beta T^i_{\alpha\beta}\Gamma^{(2)}_{\alpha\gamma}(x,y) + T^i_{\alpha\gamma}\delta^d(x-y)\sigma_\alpha\Big] = 0;$$

In the Fourier space,

$$\sigma_\beta T^i_{\beta\alpha}\Gamma^{(2)}_{\alpha\gamma}(p=0) + T^i_{\gamma\alpha}h_\alpha = 0 \tag{92}$$

If we consider the limit in which $h \to 0$ this relation tells to us that the matrix $\Gamma^{(2)}_{\alpha\gamma}(0)$ has $N-1$ zero-modes: they are commonly called *Nambu-Goldstone bosons* and can be interpreted as massless particles (because the eigenvalue of the operator $\Gamma^{(2)}(0)$ is a square mass).

Since the magnetic field and the magnetization define a direction in the space we can distinguish the correlation functions as transverse and longitudinal to σ. The $N-1$ transverse two-point correlation functions at zero momenta are proportional to the magnetic field and goes to zero (Nambu-Goldstone bosons formation) when it is removed

$$\Gamma^{(2)}_T(0) = \chi^{-1} = h/\sigma \tag{93}$$

The longitudinal two-point function $\Gamma_L^{(2)}(p)$ is dressed by the transverse propagators too as one can easly see by computing the one-loop order contribution in perturbation theory. Thus, when the magnetic field is turned-off, the $W_L^{(2)}(p=0)$ operator has IR divergences. These are due to the Goldstone-waves and not to the critical fluctuations. Therefore, below the critical point and when the magnetic field goes to zero, it is expected a cross-over behaviour in the momentum: there is a value of momentum $p^\diamond \sim \xi^{-1}$ which divedes the critical behaviour from the Goldstone-waves behaviour at large distances. Below the critical point (where the correlation length is finite) the long distance behaviour is dominated by Goldstone waves. When $\tau \to 0$, $\tau < 0$ or, in other words, when we are into the critical domain below the critical point, the correlation length becomes large with respect to the microscopic scale $\xi^{-1} \ll \Lambda^{-1}$ but it is still finite. Thus, there is a regime of universal critical behaviour at length scales of order $\sim \xi$ and at larger length scales a regime dominated by Goldstone modes.

The correlation functions present a scaling behaviour also in the low temperature phase. Here we can construct a thermodinamic potential density functional $\mathcal{F}$ as function of the magnetization as follows

$$\mathcal{F}(\sigma,\tau,u,\Lambda) = \sum_{n=0} \frac{\sigma^n}{n!} \cdot \Gamma^{(n)}(p=0,\tau,u,\Lambda) \tag{94}$$

and it follows that $\mathcal{F}$ generates the proper vertices at zero momenta. The magnetic field can be obtained by diffentiate once $\mathcal{F}$ with respect to σ

$$h(\sigma,\tau,u,\Lambda) = \frac{\delta\mathcal{F}(\sigma,\tau,u,\Lambda)}{\delta\sigma} = \sum_{n=1} \frac{\sigma^n}{n!} \cdot \Gamma^{(n+1)}(p=0,\tau,u,\Lambda) \tag{95}$$

By dimensional analysis we observe

$$h(\sigma,\tau,u,\Lambda) \sim \Lambda^{(d+2)/2} h(\sigma/\Lambda^{(d-2)/2}, \tau/\Lambda^2, u, 1) \tag{96}$$

Let us perform a dilatation transformation $\Lambda \mapsto \lambda\Lambda$ on the magnetic field

$$h(\sigma,\tau,u,\Lambda) = Z^{-1/2}(\lambda) h(\sigma(\lambda),\tau(\lambda),u(\lambda),\lambda\Lambda) \tag{97}$$

By this relation follows the RG equation

$$\lambda\frac{d}{d\lambda}\Big[Z^{-1/2}(\lambda) h(\sigma(\lambda),\tau(\lambda),u(\lambda),\lambda\Lambda)\Big] = 0 \tag{98}$$

it can be written as

$$\Big[\lambda\frac{\partial}{\partial\lambda} - \beta(u(\lambda))\frac{\partial}{\partial u(\lambda)} + \frac{1}{2}\eta(u(\lambda))\Big(1+\sigma(\lambda)\frac{\partial}{\partial\sigma(\lambda)}\Big) - \eta_2(u(\lambda))\tau(\lambda)\frac{\partial}{\partial\tau(\lambda)}\Big] h(\sigma(\lambda),\tau(\lambda),u(\lambda),\lambda\Lambda) = 0 \tag{99}$$

where we have introduced the function β, η and η_2 as in (77), (78) and (79). The running coupling constant $\sigma(\lambda)$ satisfies the flow equation

$$\lambda\frac{d}{d\lambda}\log\sigma(\lambda) = -\frac{1}{2}\eta(u(\lambda)) \tag{100}$$

with $\sigma(1) = \sigma$. Comparing (78) and (100) we read that

$$\sigma(\lambda) = \sigma Z^{-1/2}(\lambda); \tag{101}$$

The choice of λ is arbitrary and its value is set to

$$\sigma(\lambda) = (\lambda\Lambda)^{(d-2)/2} \tag{102}$$

The equation (100) has solution

$$\log(\sigma(\lambda)/\sigma) = -\frac{1}{2}\int_1^\lambda \frac{ds}{s}\cdot\eta(u(s)) \tag{103}$$

Using (102) one finds

$$\log(\sigma/(\Lambda)^{(d-2)/2} = \frac{1}{2}\int_1^\lambda \frac{ds}{s}[d-2+\eta(u(s))] \tag{104}$$

in the critical domain the magnetization has very small values

$$\sigma \ll \Lambda^{(d-2)/2} \tag{105}$$

Thus, in (104) we can consider, for small values of λ, $u(\lambda)$ close to u_*. In this limit the magnetization presents a scaling law

$$\sigma\Lambda^{(d-2)/2} \sim \lambda^{(d-2+\eta)/2} \tag{106}$$

We have already seen that in the critical domain $\tau(\lambda)/\lambda^2 \sim \tau\lambda^{-1/\nu}$ and $Z(\lambda) \sim \lambda^\eta$. We can replace in (97) the asymptotic forms for the running coupling constants in order to eliminate the parameter λ

$$h(\sigma,\tau,u,1) \sim \sigma^\delta f(\tau\sigma^{-1/\beta}) \tag{107}$$

and for $h = 0$, $\tau < 0$

$$\sigma \propto (-\tau)^\beta \tag{108}$$

This is the scaling behaviour of the equation of state of the system. We have introduced the critical exponent $\delta = (d+2-\eta)/(d-2+\eta)$ and $\beta = \nu/2(d-2+\eta)$. As we can see they are a combination of the critical exponents ν and η related to renormalization. The function f is an example of *scaling function*. A scaling function depends on a specific combination of its variables and is universal.

Finally we note that the strong scaling statement is still valid below the critical point into the critical domain $p \ll \Lambda$, $|\tau| \ll \Lambda^2$, $\sigma \ll \Lambda^{(d-2)/2}$. The correlation functions stastisfy the RG equation

$$\Big[\Lambda\frac{\partial}{\partial\Lambda} + \beta(u)\frac{\partial}{\partial u} - \frac{1}{2}\Big(n+\sigma\frac{\partial}{\partial\sigma}\Big) - \eta_2(u)\tau\frac{\partial}{\partial\tau}\Big]\Gamma^{(n)}(p,\tau,\sigma,u,\Lambda) = 0 \tag{109}$$

where we can note that here the correlation functions depends on σ too. Using the same arguments of before (one looks to (99) and its solution (107)), we can find a solution to this equation

$$\Gamma^{(n)}(p,\tau,\sigma,u,\Lambda) \sim m^{d-n(d-2+\eta)/2}F^{(n)}(p/m, \tau m^{-1/\nu}) \tag{110}$$

where $m = \sigma^{\nu/\beta}$ and $f^{(n)}$ is a scaling function. If we set the magnetic field $h = 0$, below the critical point, the magnetization does not vanish and becomes a spontaneus magnetization. The spontaneus magnetization can be expressed in terms of τ by (108). Thus, the previous solution becomes

$$\Gamma^{(n)}(p,\tau,u,\Lambda) \sim m^{d-n(d-2+\eta)/2}F_-^{(n)}(p/m) \tag{111}$$

$$\Gamma^{(n)}(p=0,\tau,u,\Lambda) \propto m^{d-n(d-2+\eta)/2} \tag{112}$$

which is equivalent to (85) with the same critical exponents but with different amplitudes[10].

1.3 Static equilibrium scaling behaviours.

The model does not know time: all the stuff are derived for time-independent fields and parameters. The only variation of the coupling constants of the system is due to change of the scale λ. The system, at a fixed scale, remains itself at all times. We refer to this situation as *static equilibrium* and call the previous relations *static equilibrium scaling forms*. By the previous discussion, we define the *static equilibrum scaling behaviour* for a generic operator O with scaling dimension Δ to be the limit in which the correlation length ξ goes to infinity keeping x/ξ fixed such that

$$< O(x_1,x_2,\cdots) >= G_O(x_1,x_2,\cdots) \sim \xi^{-\Delta}\cdot\mathcal{G}_O^{\text{eq.}}(\frac{x_1}{\xi},\frac{x_2}{\xi},\cdots) \tag{113}$$

where $\mathcal{G}_O^{\text{eq.}}$ is a scaling function. The expectation value of the operator O is a generic observable. For instance, one can consider O as the product of several fields $O(x_1,\cdots,x_n) \sim \phi(x_1)\cdots\phi(x_n)$ having scaling dimension $\Delta = nd_\phi$.

[10] Or if $p \neq 0$ with different scaling functions $F_+ \neq F_-$. The amplitude ratio is a universal quantity.

2 Critical dynamics.

Up to now, we considered only the static equilibrium properties of critical systems. We shall now study the time evolution of the fields. The dynamics of a statistical system is the effect of several processes and complicated interactions which can be summarized in a *stochastic equation.* Close to the critical point, the divergence of the correlation length causes the emergence of modes with large relaxation times. This phenomenon is called *critical slowing down* and leads to universal scaling behaviour for time-dependent quantities. The scaling behaviour of the system depends on the type of dynamics: we assume that the time evolution of the fields satisfies a purely dissipative Langevin equation. In order to use the machinery of field theory, we implement the stochastic equation in the field theory itself. Then, we extract the dynamical exponent z from the Langevin equation and define the *dynamical equilibrium scaling behaviour.*

2.1 Stochastic field equations.

We report a briefly discussion on the stochastic field equation following the ref. [11], [12] and [13]. A *stochastic partial differential field equation* is

$$D\phi(x,t) = F[\phi(x,t)] + \varsigma(x,t) \tag{114}$$

where ς is a random function of its argument which describes the macroscopic noise affecting the system. We consider a field-independent type of noise. D is a differential operator which does not involve fields explicitly and $F[\vec{\phi}]$ is a forcing term. Some standard example are:

$D = \partial/\partial t - w\nabla^2$ diffusion operator,

$D = \partial^2/\partial t^2 - \nabla^2$ wave operator,

$D = \partial/\partial t$ Langevin operator;

and for the forcing term,

$F[\phi]_\alpha = -\frac{\Omega}{2}\delta\mathcal{H}(\phi)/\delta\phi_\alpha$ purely dissipative equation,

$F[\phi] = \frac{c}{2}(\vec{\nabla}\cdot\vec{\phi})^2$ Kadar-Parisi equation, and so on.

We assume that the stochastic equation admits a unique solution $\phi_{\text{sol}}(x,t|\varsigma)$. For any function of the field $Q(\phi)$ we can define the averages on noise distribution as

$$< Q(\phi) >_\varsigma = \int \mathcal{D}\varsigma \cdot P(\varsigma) \cdot Q(\phi_{\text{sol}}) \tag{115}$$

where $P(\varsigma)$ is the noise distribution, normalized to 1. Even if the solution is not known, the expectation values can be computed by using a trick. We start from the identity

$$\phi_{\text{sol}}(x,t|\varsigma) = \int \mathcal{D}\phi\cdot\phi(x,t)\cdot\delta\Big(\phi(x,t)-\phi_{\text{sol}}(x,t|\varsigma)\Big) = \int \mathcal{D}\phi\cdot\delta\Big(D\phi(x,t)-F[\phi(x,t)]-\varsigma(x,t)\Big)\cdot\det\mathbb{M}. \tag{116}$$

where $\det\mathbb{M} = \det\left(D - \frac{\delta F[\phi]}{\delta\phi}\right)$ is the jacobian associated to the transformation. Using the relation above in (115), we obtain:

$$< Q(\phi) >_\varsigma = \int \mathcal{D}\varsigma \cdot \mathcal{D}\phi \cdot P(\varsigma) \cdot Q(\phi) \cdot \delta\Big(D\phi(x,t) - F[\phi(x,t)] - \varsigma(x,t)\Big) \cdot \det\mathbb{M}. \tag{117}$$

At this point we define the generating functional of correlation functions as $Q(\phi) = \exp\Big(\int d^dx \cdot \int dt \cdot J(x,t)\cdot\phi(x,t)\Big)$. More precisely,

$$\mathcal{Z}[J] =< \exp\Big(\int d^dx \cdot \int dt \cdot J(x,t)\cdot\phi(x,t)\Big) >_\varsigma = \tag{118}$$

$$\int \mathcal{D}\varsigma \cdot \mathcal{D}\phi \cdot P(\varsigma)\cdot\delta\Big(D\phi(x,t) - F[\phi(x,t)] - \varsigma(x,t)\Big)\cdot \det\mathbb{M}\cdot\exp\Big(\int d^dx \cdot \int dt \cdot J(x,t)\cdot\phi(x,t)\Big).$$

All the correlation functions are given by

$$< \phi(1)\cdots\phi(n) >_\varsigma = \frac{1}{\mathcal{Z}}\cdot\frac{\delta^n \mathcal{Z}[J]}{\delta J(1)\cdots\delta J(n)}\Big|_{J=0} \tag{119}$$

$$= \frac{\delta^n}{\delta J(1)\cdot\delta J(n)}\cdot\frac{\int \mathcal{D}\varsigma\cdot\mathcal{D}\phi\cdot P(\varsigma)\cdot\exp\Big(J\cdot\phi\Big)\cdot\delta\Big(\phi-\phi_{\rm sol}\Big)}{\int \mathcal{D}\varsigma\cdot\mathcal{D}\phi\cdot P(\varsigma)\cdot\delta\Big(\phi-\phi_{\rm sol}\Big)}\Big|_{J=0}$$

where we have introduced a compact notation[11]. If the solution of the stochastic field equation is known, the relation is simplified because we can integrate over the field ϕ

$$< \phi_{\rm sol}(1)\cdots\phi_{\rm sol}(n) >_\varsigma = \frac{\delta^n}{\delta J(1)\cdots\delta J(n)}\cdot\frac{\int \mathcal{D}\varsigma\cdot P(\varsigma)\cdot\exp\Big(J\cdot\phi_{\rm sol}\Big)}{\int \mathcal{D}\varsigma\cdot P(\varsigma)}\Big|_{J=0}$$

$$= \frac{\delta^n}{\delta J(1)\cdots\delta J(n)}\int \mathcal{D}\varsigma\cdot P(\varsigma)\cdot\exp\Big(J\cdot\phi_{\rm sol}\Big)\Big|_{J=0} \tag{120}$$

Gaussian noise distribution.

Let us assume that the noise distribution is gaussian. Without loss of generality we can take the noise to have zero mean: if the mean is not zero, we can redefine the forcing term $F[\phi]$ in such a way to have zero mean. This implies that the only non-zero cumulant is the variance. Therefore, the distribution of the noise can be written as

$$P(\varsigma) = \frac{1}{\sqrt{\det(2\pi\Omega)}}\cdot\int\mathcal{D}\varsigma\cdot\exp\Big(-\frac{1}{2}\int dt\cdot\int d^dx\cdot\int d^dy\cdot\varsigma(x,t)\cdot\Omega(x,y)^{-1}\cdot\varsigma(y,t)\Big). \tag{121}$$

If we look to (118), the integration over the noise becomes gaussian and can be performed. The result is:

$$\mathcal{Z}[J] = \int\mathcal{D}\phi\cdot\exp\Big(-\frac{1}{2}\int dt\cdot\int d^dx\cdot\int d^dy\cdot(D\phi(x,t)-F[\phi(x,t)])\cdot\Omega(x,y)^{-1}\cdot(D\phi(y,t)-F[\phi(y,t)])$$

$$+\int dt\cdot\int d^dx\cdot J(x,t)\cdot\phi(x,t)\Big)\cdot\det\mathbb{M}. \tag{122}$$

The previous relation can be viewed as a dynamical action

$$S_0^{\rm dyn.}[\phi] = \frac{1}{2}\int dt\cdot\int d^dx\cdot\int d^dy\cdot\Big(D\phi(x,t)-F[\phi(x,t)]\Big)\Omega^{-1}(x,y)\Big(D\phi(y,t)-F[\phi(y,t)]\Big). \tag{123}$$

Only for completeness, we conclude this discussion introducing the *ghosts fields*. These are not related to physical quantities. They implement a constraint in the field theory, in this case, the stochastic equation. We can write the determinant as a gaussian integral of Grassmann anticommuting scalar particles

$$\det\mathbb{M} = \det\Big(D - \frac{\delta F[\phi]}{\delta\phi}\Big)$$

[11] For instance, $\phi(i) = \phi(x_i, t)$ and so on and so forth.

$$= \frac{1}{\det(2\pi\mathbb{I})} \cdot \int \mathcal{D}\bar{c} \cdot \mathcal{D}c \cdot \exp\Big(-\frac{1}{2} \int dt \cdot \int d^dx \cdot \bar{c}(x,t)(D - \delta F[\phi]/\delta\phi)^{-1} c(x,t)\Big);$$

and the partition function becomes

$$\mathcal{Z} = \int \mathcal{D}\phi \cdot \mathcal{D}\bar{c} \cdot \mathcal{D}c \cdot \exp\Big\{ -\frac{1}{2} \int dt \cdot \int d^dx \cdot \int d^dy \cdot$$

$$(D\phi(x,t) - F[\phi(x,t)])\Omega^{-1}(x,y)(D\phi(y,t) + F[\phi(y,t)])$$

$$-\frac{1}{2} \int dt \cdot \int d^dx \cdot \bar{c}(x,t)(D - \frac{\delta F[\phi]}{\delta\phi})c(x,t) + \int dt \cdot \int d^dx \cdot J(x,t) \cdot \phi(x,t)\Big\} \qquad (124)$$

The dynamical action can be written as the previous term plus a term which is commonly called *Fadeev Popov action* and it expresses the constraint in our theory

$$S_{\text{dyn.}}[\phi] = S_0^{\text{dyn.}}[\phi] + S_{\text{fix}}[\phi] =$$

$$\frac{1}{2} \int dt \cdot \int d^dx \cdot \int d^dy \cdot \Big(D\phi(x,t) - F[\phi(x,t)]\Big)\Omega^{-1}(x,y)\Big(D\phi(y,t) - F[\phi(y,t)]\Big)$$

$$+\frac{1}{2} \int dt \cdot \int d^dx \cdot \bar{c}(x,t)(D - \frac{\delta F[\phi]}{\delta\phi})c(x,t) =$$

$$\Big\{\frac{1}{2} \int dt \cdot \int d^dx \cdot \int d^y \cdot D\phi(x,t)\Omega^{-1}(x,y)D\phi(y,t) + \frac{1}{2} \int dt \cdot \int d^dx \cdot \bar{c}(x,t)(D^{-1})c(x,t)\Big\} +$$

$$\Big\{ \int dt \cdot \int d^dx \cdot \int d^y \cdot (-D\phi(x,t)\Omega^{-1}(x,y)F[\phi(y,t)]) + \frac{1}{2} F[\phi(x,t)]\Omega^{-1}(x,y)F[\phi(y,t)]$$

$$-\frac{1}{2} \int dt \cdot \int d^dx \cdot \bar{c}(x,t)(\frac{\delta F[\phi]}{\delta\phi})c(x,t)\Big\} = S^{\text{free}}_{\text{dyn.}}[\phi] + S^{\text{int}}_{\text{dyn.}}[\phi].$$

From this action, one can derive the Feynman rules for propagators and vertices [see ref.[11]].

Note.
The dynamical action has a BRS symmetry. It is related to the invariance of the measure:

$$\prod_\alpha [d\phi_\alpha] \cdot \delta(E_\alpha[\phi]) \cdot \det(E), \qquad (125)$$

under traslation that takes $E_\alpha[\phi] \mapsto E_\alpha[\phi] + \varsigma_\alpha$, where E is a general constraint applied to fields. The infinitesimal transformation of fields is

$$\delta\phi_\alpha(x,t) = [E[\phi(x,t)]]^{-1}_{\alpha\beta} \cdot \varsigma_\beta(x,t). \qquad (126)$$

Introducing auxiliary fields and ghosts fields, it's simple to demonstrate that this symmetry of the action is BRS[12].

2.2 Langevin dynamics with gaussian noise.

We assume that dynamical evolution of the system is driven by a purely dissipative Langevin equation

$$\frac{\partial\phi_\alpha(x,t)}{\partial t} = -\frac{1}{2} \int d^dx \cdot \int d^dy \cdot \frac{\delta S[\phi]}{\delta\phi_\alpha(y,t)} \cdot \Omega(x,y) + \varsigma_\alpha(x,t). \qquad (127)$$

We also assume gaussian noise distribution with cumulants

$$< \varsigma_\alpha(x,t) >_\varsigma = 0; \qquad (128)$$

$$< \varsigma_\alpha(x,t) \cdot \varsigma_\beta(x',t') >_\varsigma = \Omega(x,x') \cdot \delta_{\alpha\beta} \cdot \delta(t-t'); \qquad (129)$$

[12]The BRS transformation and the proof of the statement above can be found in [9].

The previous equation of motion, written in Fourier space, becomes

$$\frac{\partial\phi_\alpha(k,t)}{\partial t} = -\frac{\Omega(k)}{2}\frac{\delta S[\phi(k,t)]}{\delta\phi_\alpha(k,t)} + \varsigma_\alpha(k,t) \tag{130}$$

We are interested to the long-distances properties of the system i.e. to the small-k behaviour of $\Omega(k)$. One may note that, since $\Omega(x,y)$ is a short-range function, $\Omega(k)$ is an analytic function. At this point there are two possibilities: the underlying dynamics may converve or not the order parameter of the transition. We follows the discussion of the ref.[14].

2.2.1 Gaussian model.

We consider first the simplest case: the free theory, described by the euclidean action

$$S[\phi] = \int d^dx \cdot \Big[\frac{1}{2}(\partial_\mu\phi(x,t))^2 + \frac{1}{2}m^2\phi^2(x,t)\Big]. \tag{131}$$

The equation (130) becomes

$$\frac{\partial\phi_\alpha(k,t)}{\partial t} = -\frac{\Omega(k)}{2}(k^2+m^2)\phi_\alpha(k,t) + \varsigma_\alpha(k,t) \tag{132}$$

We consider expectation value of the field:

$$\frac{\partial <\phi_\alpha(k,t)>_\varsigma}{\partial t} = -\frac{\Omega(k)}{2}(k^2+m^2)\cdot <\phi_\alpha(k,t)>_\varsigma \tag{133}$$

• If the order parameter is conserved, $\frac{\partial}{\partial t} <\phi_\alpha(0,t)>_\varsigma = 0$, this implies $\Omega(k=0)=0$. The expansion for $k\to 0$ of $\Omega(k)$ starts as $\Omega(k)\sim k^2\cdot\Omega$, where Ω is a constant. It corresponds to fields which can have only self-interactions. This type of dynamcs is called *model B*. The equation (133), in the limit of small momenta $\Omega(k) \overset{k\to 0}{\sim} k^2\Omega$, becomes

$$\frac{\partial <\phi_\alpha(k,t)>_\varsigma}{\partial t} \overset{k\sim 0}{\sim} -\frac{k^2\Omega}{2}(k^2+\xi^{-2}) <\phi_\alpha(k,t)>_\varsigma \tag{134}$$

Each mode has a relaxation time given by $\tau_k \sim \Omega^{-1}k^{-2}(k^2+\xi^{-2})^{-1}$. When $k\to 0$, the relaxation times diverge like ξ^z. This phenomenon is called *critical slowing down.* From the previous relation we fix the dynamical exponent $z=4$ for the gaussian model.

• If the order parameter is not conserved, fields can have interactions also with external fields. In this case the expansion of $\Omega(k)$ starts with a constant $\Omega(k=0)=\Omega$. This type of dynamics sometimes is called *relaxational* because reproduces the effects of an heat-bath which absorbs the fluctuations. The underlying situation is known as *model A*. As $\Omega(k) \overset{k\to 0}{\sim} \Omega$, the equation (133) becomes

$$\frac{\partial <\phi_\alpha(k,t)>_\varsigma}{\partial t} \overset{k\sim 0}{\sim} -\frac{\Omega}{2}(k^2+\xi^{-2}) <\phi_\alpha(k,t)>_\varsigma \tag{135}$$

The relaxation time is $\tau_k \sim \Omega(k^2+\xi^{-2})^{-1}$. Thus, we read a value of the dynamical exponent $z=2$ for the gaussian model.

2.2.2 ϕ^4 model.

We want to extract the value of the dynamical critical exponent z in the case of an interacting theory. In particular we consider an interacting O(N) symmetric model, which can be reduced near dimension four to the $(\phi^2)^2$ theory without loss of generality. Thus, we consider the action:

$$S[\phi] = \int d^dx \cdot \Big[\frac{1}{2}(\partial_\mu\phi(x))^2 + \frac{1}{2}r^2\phi^2(x) + \frac{u}{4!}(\phi^2(x))^2\Big]. \tag{136}$$

The equation (130) describes the dynamics of the system:

$$\frac{\partial\phi_\alpha(k,t)}{\partial t} = -\frac{\Omega(k)}{2} L[\phi(k,t)] + \varsigma_\alpha(k,t) \tag{137}$$

where $\delta S[\phi]/\delta\phi = L[\phi] = (k^2 + r + u/6(\phi^2(k,t)))\phi_\alpha(k,t)$. To the dynamical model ((136) plus (137)) is associated a set of dynamical correlation functions $W^{(n)}(x_i, t)$.
In particular, the connected two-point correlation function, at the critical point and in the limit of small momenta has a scaling behaviour

$$W^{(2)}(k,t) \overset{k\to 0}{\sim} k^{-2+\eta+z}\Psi(tk^z) \tag{138}$$

where we have taken into account the scaling also for the time variable through the dynamic critical exponent z and Ψ is a scaling function. But if we compute the connected two-point correlation function from (137), we expect that:

$$W^{(2)}(k,t) \overset{k\to 0}{\sim} \Omega \tag{139}$$

for the model A and

$$W^{(2)}(k,t) \overset{k\to 0}{\sim} \Omega\cdot k^2 \tag{140}$$

for the model B. Comparing the relations above, we conclude that the critical exponent $z = 2-\eta$ for the model A and $z = 4-\eta$ for the model B.

2.3 Relaxational dynamics.

We assume that the dynamics of the fields occurs through a *relaxational dynamics* which means to consider a purely dissapative Langevin equation with a white gaussian noise distrubution:

$$\int \mathcal{D}\varsigma\cdot P(\varsigma) = \frac{1}{\sqrt{\det(2\pi\Omega)}}\cdot\int \exp\Big(-\frac{1}{2}\int d^dx\cdot\int d^dy\cdot\varsigma(x)\cdot\Omega^{-1}\delta^d(x-y)\cdot\varsigma(y)\Big) = 1. \tag{141}$$

The cumulants of the noise distribution are fixed to

$$< \varsigma_\alpha(x,t) >_\varsigma = 0; \tag{142}$$

$$< \varsigma_\alpha(x,t)\cdot\varsigma_\beta(x',t' >_\varsigma = \Omega\cdot\delta_{\alpha\beta}\delta^d(x-x')\delta(t-t'); \tag{143}$$

The dynamical action (123) becomes

$$S_0^{\text{dyn.}}[\phi] = \frac{1}{2}\Omega^{-1}\int dt\cdot\int d^dx\cdot\Big[\Big(\partial_t + \frac{\Omega}{2}\ell[\phi(x,t)]\Big)\phi(x,t)\Big]^2 \tag{144}$$

such that

$$\frac{\delta S_0^{\text{dyn.}}[\phi]}{\delta\phi_\alpha(x,t)} = 0 = \Big(\partial_t + \frac{\Omega}{2}\ell[\phi(x,t)]\Big)\phi_\alpha(x,t). \tag{145}$$

where $\delta S[\phi]/\delta\phi = L[\phi] = \ell[\phi]\cdot\phi$.
Since we consider a white gaussian type of noise, $\Omega(k) = \Omega$ thus we have a model A type of dynamics. Model A is the dynamical universality class is characterized by the dynamic scaling exponent $z = 2-\eta$.
The equilibrium scaling behaviour (113) now takes into account also the time-variable. We define therefore the *dynamical equilibrium scaling behaviour* to be the limit in which $\xi\to\infty$ keeping x/ξ and t/ξ^z fixed such that an operator O with scaling dimension Δ sastisfies

$$< O(x_1, x_2, \cdots, t) >= G_O(x_1, x_2, \cdots, t) \sim \xi^{-\Delta}\cdot\mathcal{G}_O^{\text{eq.}}(\frac{x_1}{\xi}, \frac{x_2}{\xi}, \cdots, \frac{t}{\xi^z}). \tag{146}$$

This statement completely defines the framework in which the off-equilibrium scaling theory will be developed.

Finally, there is one thing that must be point out. Why a relaxational dynamics might be a good dynamical model for our system? Let us consider the probability distribution of the system

$$P(\phi,t) =< \delta\Big(\phi(x,t)-\phi(x)\Big) >_\varsigma \tag{147}$$

It satisfies the *Fokker-Planck equation* [see, for example, ref. [9] and [12]]

$$\dot{P}(\phi,t) = -\Omega\mathcal{H}_{\rm FP}P(\phi,t); \tag{148}$$

where $\mathcal{H}_{\rm FP}$ is called *Fokker-Planck Hamiltonian*

$$\mathcal{H}_{\rm FP}\Big(\phi,\frac{\delta}{\delta\phi}\Big) = -\frac{1}{2}\int d^dx\cdot\frac{\delta}{\delta\phi(x)}\Big[\frac{\delta}{\delta\phi(x)}+L[\phi(x)]\Big]. \tag{149}$$

We write the Fokker-Planck equation as an *equation of continuity* for the distribution of probabibility

$$\frac{\partial P(\phi,t)}{\partial t} = \frac{\Omega}{2}\int d^dx\cdot\frac{\delta}{\delta\phi(x)}\cdot J(x),$$

where the current is defined as

$$J(x) = \frac{\delta P(\phi,t)}{\delta\phi(x)} + L[\phi(x)]P(\phi,t); \tag{150}$$

The first term of the current reproduces the diffusion process and the second one the dissipations. After long times we expect that the system does not depend on time and all the dynamical effects are relaxed through the damping mechanisms: in other words we want that the probability distribution tends to a steady state

$$\frac{\partial P(\phi,t)}{\partial t} = 0 = \frac{\Omega}{2}\int d^dx\cdot\frac{\delta}{\delta\phi(x)}\cdot J(x);$$

and this implies

$$\frac{\delta P(\phi,t)}{\delta\phi(x)} + \frac{\delta S[\phi]}{\delta\phi(x)}P(\phi,t) = 0;$$

The last equation is satisfied if the distribution of probability is like

$$P(\phi,t) \propto e^{-S[\phi]} \tag{151}$$

that is the Boltzmann distribution of probability in our notations. Real statistical systems, even in the presence of off-equilibrium phenomena at the transition, approaches the equilibrium after long times. This physical requirement justifies the assumption of relaxational dynamics.

3 O(N) vector model in the large N limit.

In this section we present the O(N) vector model in the limit of large N. The limit $N \to \infty$ may be rather unphysical but it turns out to be exactly solvable.

3.1 O(N) vector model.

The N-vector model is a lattice model described in terms of N-vector spin variables S_i of unit length on each lattice site i, interacting through a short range ferromagnetic O(N) symmetric two-body interaction $V_{i,j}$. The partition function of such a model can be written as

$$\mathcal{Z} = \int \prod_i dS_i \cdot \delta(S_i^2 - 1) \cdot e^{-\varepsilon(S)/T}, \tag{152}$$

in which the configuration energy ε is

$$\varepsilon(S) = -\sum_{i,j} V_{i,j}(S_i \cdot S_j) \tag{153}$$

This model has a second order phase transition between a disordered phase at high temperature, and a low temperature ordered phase where the O(N) symmetry is spontaneously broken, and the order parameter S_i has a non-vanishing expectation value. One can add to $\varepsilon(S)$ a linear coupling

$$\varepsilon(S) = -\sum_{i,j} V_{i,j}(S_i \cdot S_j) + \sum_i h \cdot S_i \tag{154}$$

which can be interpreted as a uniform external magnetic field. The presence of a non-zero magnetic field leads to a first-order transition in the low-temperature phase along the line of $h = 0$. At the continuous transition the correlation length diverges and therefore a non-trivial long distance physics emerges. As we have already seen in the first section, the long distance physics of (154) can be described through an effective ϕ^4 model

$$S[\phi] = \int [\frac{1}{2}(\partial_\mu \phi(x))^2 + \frac{1}{2} r\phi^2(x) + \frac{u}{4!}(\phi^2(x))^2 - h_\alpha \phi_\alpha(x)] \cdot d^d x. \tag{155}$$

In the following sections we consider this model when the number of components N of the vector become very large.

3.2 Statics of O(N) vector model for large N.

We present the static model and derive its critical properties [see ref. [9], [10] and [15]]. The system is described by the partition function

$$\mathcal{Z} = \int \mathcal{D}\phi \cdot \exp\Big(- S[\phi]\Big) \tag{156}$$

where the action functional is

$$S[\phi] = \int d^d x \cdot \Big[\frac{1}{2}(\partial_\mu \phi(x))^2 + N \cdot U(\phi^2(x)/N) - h_\alpha \phi_\alpha(x)\Big]. \tag{157}$$

where the magnetic field has a fixed direction $h_\alpha = \delta_{1,\alpha} \cdot h$. The potential U is a general polynomial. However, if we look to the long-distance properties, we know that a general spin system with O(N) symmetry can be reduced to a ϕ^4 model below dimension four. Thus, we fix the shape of the potential to:

$$U(\phi^2(x)/N) = \frac{1}{2N} r\phi^2(x) + \frac{u}{4!N}(\phi^2(x))^2 \tag{158}$$

where $u \sim O(1/N)$ is a coupling constant with dimensions Λ^ϵ. The large N limit makes the system exactly solvable. The main simplification lies behind the following idea. One may observe that, by the central limit theorem, the square field $\phi^2(x) = \sum_{\alpha=1}^N \phi_\alpha^2(x)$ should have a normal distribution as $N \to \infty$. Thus, it is expected that the averages of several fields can be reduced to

$$< \phi^2(x) \cdot \phi^2(y) > \overset{N\to\infty}{\sim} < \phi^2(x) > \cdot < \phi^2(y) > \tag{159}$$

This idea suggest us to introduce a new density field $N\rho(x) = \phi^2(x)$ which satisfies the condition

$$1 = N \int \mathcal{D}\rho \cdot \delta(N\rho(x) - \phi^2(x)); \tag{160}$$

This constraint can be implemented in our model by introducing another field $\lambda(x)$

$$1 = N \int \mathcal{D}\rho \cdot \delta(N\rho(x) - \phi^2(x)) = \frac{N}{4i\pi} \int \mathcal{D}\rho \cdot \mathcal{D}\lambda \cdot \exp\Big(- \frac{\lambda}{2}(\phi^2(x) - N\rho(x))\Big) \tag{161}$$

λ field integration runs over imaginary axis in the complex plane because the field λ is not related to any physical quantity; it tells us which is the relation between ϕ^2 and ρ, nothing more. By inserting the identity (161), the partition function can be written as

$$\mathcal{Z} = \int \mathcal{D}\phi \cdot \mathcal{D}\rho \cdot \mathcal{D}\lambda \cdot \exp\Big(- S[\phi, \rho, \lambda]\Big) \tag{162}$$

with

$$S[\phi, \rho, \lambda] = \int d^d x \cdot \Big[\frac{1}{2}(\partial_\mu \phi(x))^2 + \frac{1}{2} N \cdot r\rho + \frac{uN^2}{4!} \cdot \rho^2 + \frac{1}{2}\lambda(x) \cdot (\phi^2(x) - N\rho(x)) - h \cdot \phi_1(x)\Big]. \tag{163}$$

We note that the new action is quadratic in ϕ integration, thus we are able to perform the integral (162). The field has N components, $\phi = (\phi_1 = \sigma \ , \ \phi_{\alpha>1} = \vec{\pi})$. We perform the integration over the $N-1$ transverse components

$$\mathcal{Z} = \int \mathcal{D}\sigma \cdot \mathcal{D}\rho \cdot \mathcal{D}\lambda \cdot \exp\Big(- S_N[\sigma, \rho, \lambda]\Big) \tag{164}$$

The action is

$$S_N[\sigma, \rho, \lambda] = \int d^d x \cdot \Big[\frac{1}{2}(\partial_\mu \sigma(x))^2 + \frac{1}{2} N \cdot r\rho + \frac{uN^2}{4!}\rho^2 + \frac{1}{2}\lambda(x) \cdot (\sigma^2(x) - N\rho(x)) - h \cdot \sigma(x)\Big] \tag{165}$$

$$+ \frac{1}{2}(N-1) \mathrm{Tr} \log(-\partial_\mu \partial_\mu + \lambda(\cdot))$$

It may be useful to rescale all the quantities with N,

$$\phi(x) \mapsto \sqrt{N}\phi(x), \tag{166}$$

$$u \mapsto u/N, \tag{167}$$

$$h_\alpha \mapsto \sqrt{N} h_\alpha. \tag{168}$$

After such rescaling, the action above becomes explicitly of order N:

$$\begin{aligned} S_N[\phi] = \int [\frac{1}{2}(\partial_\mu \sqrt{N}\phi(x))^2 + \frac{1}{2} r N \phi^2(x) + \frac{u}{4!N}((\sqrt{N}\phi(x))^2)^2 - \sqrt{N} h_\alpha \sqrt{N}\phi_\alpha(x)] \cdot d^d x \mapsto \\ N \cdot \int [\frac{1}{2}(\partial_\mu \phi(x))^2 + \frac{1}{2} r \phi^2(x) + \frac{u}{4!}((\phi(x))^2)^2 - h_\alpha \phi_\alpha(x)] \cdot d^d x = N \cdot S[\phi]. \end{aligned} \tag{169}$$

and therefore the expression (165) becomes:

$$S_N[\sigma, \rho, \lambda] = N \cdot \int d^d x \cdot \Big[\frac{1}{2}(\partial_\mu \sigma(x))^2 + \frac{1}{2} r\rho + \frac{u}{4!}\rho^2 + \frac{1}{2}\lambda(x) \cdot (\sigma^2(x) - N\rho(x)) - h \cdot \sigma(x)\Big] \tag{170}$$

$$+\frac{1}{2}\mathrm{Tr}\log(-\partial_\mu\partial_\mu+\lambda(\cdot))=N\cdot S[\sigma,\rho,\lambda].$$

If we consider the limit of the large N, the partition function (164) can be computed by the steepest descend method. We perform a saddle point expansion of the action S in terms of $1/N$ and we keep only the zero order[13]. Further developments are made by using the N scaled action $S\sim O(1)$ valuated at the leading order in $1/N$.

3.2.1 Saddle Point Equations.

Defining the *action density* $\mathcal{E}$ as

$$S[\sigma,\rho,\lambda]=\int d^dx\cdot\mathcal{E}[\sigma,\rho,\lambda], \tag{175}$$

we look for an uniform space-independent saddle point $X=\{\sigma(x),\rho(x),\lambda(x)\}=\{\sigma,\rho,m^2\}$ in which $\mathcal{E}$ has a minima:

$$\frac{\delta\mathcal{E}}{\delta\sigma(x)}\Big|_{\sigma(x)=\sigma}=0=\frac{\delta\mathcal{E}}{\delta\rho(x)}\Big|_{\rho(x)=\rho}=\frac{\delta\mathcal{E}}{\delta\lambda(x)}\Big|_{\lambda(x)=m^2}.$$

At saddle point the action density is

$$\mathcal{E}=\frac{1}{2}r\rho+\frac{u}{4!}\rho^2+\frac{1}{2}m^2\Big(\sigma^2-\rho\Big)-h\cdot\sigma+\frac{1}{2}\int^\Lambda\frac{d^dk}{(2\pi)^d}\log(k^2+m^2). \tag{176}$$

From the last relations, we extract the saddle point equations:

$$m^2\sigma-h=0 \tag{177}$$

this equation implies that the magnetic field breaks O(N) symmetry explicitly and gives to the system an induced magnetization, that is

$$\sigma=h/m^2 \tag{178}$$

The second saddle point equation tells to us

$$U'(\rho)-\frac{1}{2}m^2=0;$$

By fixing the potential as (158) we have

$$m^2-r-\frac{u}{6}\rho=0 \tag{179}$$

this leads to the results

$$\rho=\frac{6(m^2-r)}{u} \tag{180}$$

[13] Let us parametrize the deviation from the saddle point value of the field as

$$\phi(x)=X+\widetilde{\phi}(x)/\sqrt{N} \tag{171}$$

the action can be written as

$$S[\phi]=S[X]+\frac{1}{2!}\widetilde{\phi}(x)\frac{\delta^2S[\phi]}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi=X}\widetilde{\phi}(y)+R_N(\widetilde{\phi}); \tag{172}$$

where

$$R_N(\widetilde{\phi})=\sum_{k=3}^{\infty}\frac{(1/N)^{k/2}-1}{k!}\cdot\frac{\delta^kS[\phi]}{\delta\phi(x_1)\cdots\delta\phi(x_k)}\Big|_{\phi=X}\cdot\widetilde{\phi}(x_1)\cdots\widetilde{\phi}(x_k). \tag{173}$$

The corrections to the system valuated at its saddle point generate an $1/N$ expansion and they are negligible because

$$\lim_{N\to\infty}R_N(\widetilde{\phi})=0. \tag{174}$$

The third saddle point equation is

$$\sigma^2 - \rho - \Omega(m) = 0$$

which can be written, using (180), as

$$r + \frac{u}{6}(\sigma^2 + \Omega(m)) = m^2 \tag{181}$$

where Ω is defined as

$$\Omega(m) = \int^{\Lambda} \frac{d^d p}{(2\pi)^d} \cdot \frac{1}{p^2 + m^2}. \tag{182}$$

If we turn off the magnetic field $h = 0$ the saddle point equations becomes:

$$m^2 \cdot \sigma = 0; \tag{183}$$

$$r + \frac{u}{6}\rho = m^2; \tag{184}$$

$$\sigma^2 - \rho - \Omega(m) = 0. \tag{185}$$

Note.
The choice of the quartic potential (158) makes the system quadratic in the ρ field. After its integration the action (170) becomes:

$$S[\sigma,\lambda] = \int d^d x \cdot \Big[\frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}\lambda(x)\sigma^2(x) - \frac{3}{2u}\lambda^2(x) + \frac{3}{u} r\lambda(x) - h\cdot\sigma(x)\Big] + \frac{1}{2}\mathrm{Tr}\log(-\partial_\mu\partial_\mu + \lambda(\cdot)) \tag{186}$$

The result of the integration is equivalent to the substitution in (170) of the ρ field with the solution of the classical equation

$$\lambda(x) = r + \frac{u}{6}\rho(x). \tag{187}$$

It can be shown that the square field can be replaced by its expectation value $\phi^2(x) \sim \rho$ at the leading order in $1/N$[14]. It follows that $\lambda(x) \sim r + u/6\rho = m^2$. The action above becomes:

$$S[\sigma,\lambda] = \int d^d x \cdot \Big[\frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2} m^2 \sigma^2(x) - h\cdot\sigma(x) + \text{const.}\Big] + \frac{1}{2}\mathrm{Tr}\log(-\partial_\mu\partial_\mu + \lambda(\cdot)) + O(1/N). \tag{188}$$

The theory seems to be gaussian at order $O(1)$. The connected two-point correlation function is given by:

$$W^{(2)}(p) = \frac{1}{p^2 + m^2} \tag{189}$$

Let us derive the critical properties of the system. We investigate the two phases and compute the specific values of some critical exponents.

3.2.2 Low Temperature Phase.

In the absence of magnetic fields, the low-temperature phase presents the spontaneus symmetry breaking $\mathrm{O}(N) \mapsto \mathrm{O}(N-1)$. The expectation value of the field σ is not zero, thus the saddle point equation (183) states that $m^2 = 0$: the $\vec{\pi}$ components of ϕ become Goldstone bosons. The full symmetry loses $N-1$ generators. From the second saddle point equation (184) we obtain

$$\rho = \frac{6(m^2 - r)}{u} = \frac{-6r}{u}; \tag{190}$$

[14] At leading order, it is reasonable to consider subleading the variation of the square field $\phi^2(x)$ with respect to those of $\phi(x)$. Thus, one can replace $\phi^2(x) \sim <\phi^2(x)>= \rho$. A formal proof of this statement can be found in [15]. This property still remain true in critical dynamics where the Langevin equation can be linearized by the substution $\phi^2(x,t) \sim <\phi^2(x,t)>_\varsigma$.

it follows that

$$\sigma^2 = \rho - \rho_c = \frac{6}{u}(r_c - r), \tag{191}$$

with the definitions $\rho_c = \Omega(0)$ and $r_c = -(u/6)\rho_c < 0$. We can study the departure from critical temperature by defining:

$$\tau = \frac{6}{u}(r - r_c) \tag{192}$$

that implies

$$\sigma^2 = \frac{6}{u}(r_c - r) = -\tau \tag{193}$$

Using the last expression we are able to extract the critical exponent β for the behaviour of order parameter close to the critical point

$$\sigma \sim \sqrt{-\tau} = (-\tau)^\beta \Rightarrow \beta = 1/2 \tag{194}$$

We find for O(N) vector model in the large N limit a gaussian exponent β, in all spatial dimensions.

3.2.3 High Temperature Phase.

We consider again the case $h = 0$. The expectation value $\sigma = 0$ because of the full O(N) symmetry. The saddle point equation (183) now leads to $m^2 \neq 0$: the system has a typical mass and therefore a finite length scale of correlation. The value of the physical mass can be also read in the poles of the two-point correlation function (189). From the saddle point equation (185) with $\sigma = 0$ we can write $\rho = \Omega(m)$ that implies

$$\frac{\partial \rho}{\partial m} = \frac{(-2m)}{(2\pi)^d} \int^\Lambda \frac{d^d p}{(p^2 + m^2)^2} < 0 \tag{195}$$

The thermal coupling r is an increasing function of m^2 and it has minimum in r_c where $m^2 = 0$. From (184) using definitions $\rho_c = \Omega(0)$, $r_c = -u/6\rho_c$, $\tau = 6/u(r - r_c)$, we obtain

$$m^2 = \frac{u}{6}(\rho - \rho_c + \tau). \tag{196}$$

Let us introduce the notation

$$\rho - \rho_c = -m^2 D_d(m) \tag{197}$$

where

$$D_d(m) = \frac{1}{m^2}[\Omega(m) - \Omega(0)] = \frac{1}{(2\pi)^d} \int^\Lambda \frac{d^d p}{p^2(p^2 + m^2)}. \tag{198}$$

The value of this integral depends on the spatial dimension of the system.
We develop our results in $d = 4 - \epsilon$ where, moving to the critical point we can expand the function $D_d(m)$ for $m << \Lambda$ as

$$D_d(m) \simeq C(d) \cdot m^{d-4} - a(d) \cdot \Lambda^{d-4} + O(m^2 \cdot \Lambda^{d-6}) \tag{199}$$

where $C(d) = (1/(4\pi)^{d/2})\Gamma(1 - d/2)$ and $a(d)$ is a constant that depends on regularization method choosen[15] At the leading order of this expansion we find

$$\nu = 1/(2 - \epsilon) = 1/(d - 2) \tag{200}$$

Below dimension four the theory presents a critical exponent ν different from those at the gaussian fixed point. This reflects the presence of a non-trivial interaction behind the definition of m^2.

[15] For instance, using dimensional regularization:

$$D_d(m) = \frac{m^{-2}}{(2\pi)^d}\Big(\int \frac{d^d p}{(p^2 + m^2)} - \int \frac{d^d p}{p^2}\Big) = m^{d-4}\frac{1}{(2\pi)^d}\int \frac{d^d p}{(p^2 + 1)} = m^{d-4}\frac{1}{(4\pi)^{d/2}}\Gamma(1 - d/2).$$

3.2.4 Critical Point.

We have two relevant variables that characterize the critical point now: the thermal coupling and the magnetic field. The general critical surface for a ferromagnetic system is described by

∘ $\tau = 0$: The temperature is at its critical value so the thermal coupling is fixed to

$$r = r_c = -(u/6)\Omega(0).$$

∘ $h_\alpha = 0$: The continuous transition occurs at null magnetic field.

We can extract the critical exponent η recalling the weak scaling statement (68): since the theory is massless we read from (189) that

$$W^{(2)}(p) \sim 1/p^2 \Rightarrow \eta = 0 \tag{201}$$

This is a very remarkable result: for all spatial dimensions the anomalous dimensions of the fields are zero in the large N limit. It follows that the scaling dimension of the fields is set to the canonical value

$$d_\phi = \frac{1}{2}(d - 2 + \eta) = (d-2)/2 \tag{202}$$

For an external source which linearly couples the field (as will be magnetic field), the scaling dimension is

$$d_h = d - d_\phi = \frac{1}{2}(d + 2 - \eta) = (d+2)/2 \tag{203}$$

It is possible to verify that the scaling exponents found before satisfy the relation $\beta = \nu d_\phi$.

3.3 Equilibrium relaxational dynamics of O(N) vector model for large N.

We consider the action (169) when the fields becomes time-dependent:

$$S[\phi] = \int d^d x \cdot \Big[\frac{1}{2}(\partial_\mu \phi(x,t))^2 + \frac{1}{2} r\phi^2(x,t) + \frac{u}{4!}(\phi^2(x,t))^2 - h_\alpha \phi_\alpha(x,t)\Big]. \tag{204}$$

where the magnetic field has a fixed direction $h_\alpha = \delta_{1,\alpha} \cdot h$. We assume that the dynamical evolution of the system is driven by a purely dissipative Langevin equation:

$$\frac{\partial \phi_\alpha(x,t)}{\partial t} = -\frac{\Omega}{2}\frac{\delta S[\phi]}{\delta \phi_\alpha(x,t)} + \varsigma_\alpha(x,t) \tag{205}$$

The parameters of the noise distribution are fixed to:

$$< \varsigma_\alpha(x,t) >_\varsigma = 0; \tag{206}$$

$$< \varsigma_\alpha(x,t) \cdot \varsigma_\beta(x',t' >_\varsigma = \Omega \cdot \delta_{\alpha\beta}\delta^d(x-x')\delta(t-t'); \tag{207}$$

We write explicitly the stochastic field equation

$$\dot{\phi}_\alpha(x,t) = -\frac{\Omega}{2}\Big(-\partial_\mu\partial_\mu + r + \frac{u}{6}(\phi^2(x,t))\Big)\phi_\alpha(x,t) + \frac{\Omega}{2}h_\alpha + \varsigma_\alpha(x,t); \tag{208}$$

In the large N limit the Langevin equation can be linearized replacing the square field with its expectation value

$$\phi^2(x,t) \overset{N\to\infty}{\sim} < \phi^2(x,t) >_\varsigma \tag{209}$$

It follows that the mass term, defined by the saddle point equation (179) is a constant

$$r + \frac{u}{6}\phi^2(x,t) \sim r + \frac{u}{6} < \phi^2(x,t) >_\varsigma = m^2 \tag{210}$$

Since we consider the leading order of the saddle point expansion, this approximation is consistent. The Langevin equation becomes:

$$\dot{\phi}_\alpha(x,t) = -\frac{\Omega}{2}(-\partial_\mu\partial_\mu + m^2)\phi_\alpha(x,t) + \frac{\Omega}{2}h_\alpha + \varsigma_\alpha(x,t); \tag{211}$$

In the Fourier space this equation becomes:

$$\dot{\phi}_\alpha(k,t) = -\frac{\Omega}{2}(k^2+m^2)\phi_\alpha(k,t) + \frac{\Omega}{2}\cdot(2\pi)^d\delta^d(k)\cdot h_\alpha + \varsigma_\alpha(k,t); \tag{212}$$

The cumulants of noise distribution, written in Fourier space, are:

$$< \varsigma_\alpha(k,t) >_\varsigma = 0 \tag{213}$$

$$< \varsigma_\alpha(k,t)\varsigma_\beta(k',t') >_\varsigma = \Omega \cdot \delta_{\alpha\beta}(2\pi)^d\delta^d(k+k')\delta(t-t') \tag{214}$$

The solution of the equation (212) is [see app.B.1]:

$$\phi_\alpha(k,t) = \phi^0_\alpha(k,t) + (2\pi)^d\delta^d(k)\Big(1 - \exp\Big(-\frac{\Omega}{2}(k^2+m^2)(t-t_0)\Big)\Big)\frac{h_\alpha}{k^2+m^2}$$
$$+ \int_{t_0}^{t} dt' \cdot \exp\Big(-\frac{\Omega}{2}(k^2+m^2)(t-t')\Big)\cdot\varsigma_\alpha(k,t'). \tag{215}$$

with $\phi^0_\alpha(k,t) = \delta_{1,\alpha}(2\pi)^d\delta^d(k)\cdot\sigma\exp\Big(-\frac{\Omega}{2}(k^2+m^2)\cdot(t-t_0)\Big)$.

3.4 Correlation functions.

We conclude this section by computing the dynamical correlation functions. We remember that, in the large N limit, the $\mathrm{O}(N)$ vector-model is exactly solvable in terms of one and two-point correlation functions. All higher-order correlators can be computed by using Wick's theorem. The solution (215) permits us to make easly averages on noise distribution by using relation (120).

3.4.1 One-point Correlator.

From solution (215), exploiting the noise characteristics (213), we obtain the one-point correlation function:

$$< \phi_\alpha(k,t) >_\varsigma = \phi_{\alpha,0}(k,t) + (2\pi)^d \delta^d(k) \Big(1 - \exp\Big(-\frac{\Omega}{2}(k^2+m^2)(t-t_0)\Big)\Big) \frac{h_\alpha}{k^2+m^2}.$$

We define the magnetization Σ to be

$$(2\pi)^d \delta^d(k) \Sigma_\alpha(t) =< \phi_\alpha(k,t) >_\varsigma \tag{216}$$

By integrating over the momenta in the previous relation we obtain

$$\Sigma_\alpha(t) = \delta_{1,\alpha}\sigma \exp\Big(-\frac{\Omega}{2} m^2 \cdot (t-t_0)\Big) + \Big(1 - \exp\Big(-\frac{\Omega}{2} m^2 (t-t_0)\Big)\Big) \frac{\delta_{1,\alpha} \cdot h}{m^2} = \delta_{1,\alpha}\sigma.$$

Thus, we learn that the transverse magnetization $\Sigma_T = \Sigma_{\alpha>1}$ is equal to zero and the longitudinal component $\Sigma_L = \Sigma_1$ is

$$\Sigma_L(t) = \sigma. \tag{217}$$

The longitudinal magnetization is fixed to its equilibrium value given by (177). Since the transverse components are zero, we call the longitudinal magnetization $\Sigma_L(t) = \Sigma(t)$.

3.4.2 Two-point Correlator.

At zero magnetic field the two-point correlation function $G_{\phi\phi}$ is defined to be:

$$< \phi(k,t)_\alpha \phi(k',t)_\beta >_\varsigma = \delta_{\alpha\beta} (2\pi)^d \delta^d(k+k') \cdot G_{\phi\phi}(k,t) \tag{218}$$

where the $\delta_{\alpha\beta}$ makes equivalents all the spatial directions. In the presence of a symmetry breaking the space is no longer isotropic. Therefore we define a transverse two point function

$$< \phi(k,t)_\alpha \phi(k',t)_\beta >_\varsigma = \delta_{\alpha\beta} (2\pi)^d \delta^d(k+k') \cdot G_T(k,t) \tag{219}$$

with α, $\beta > 1$ and a longitudinal two point function

$$< \phi(k,t)_1 \phi(k',t)_1 >_\varsigma = (2\pi)^d \delta^d(k+k') \cdot G_L(k,t) \tag{220}$$

Thus, we consider the expectation value of two fields over the noise distribution [see app. B.2]

$$< \phi(k,t)_\alpha \phi(k',t)_\beta >_\varsigma =$$

$$(2\pi)^d \delta^d(k+k') \delta_{\alpha,\beta} \Big\{ (2\pi)^d \delta^d(k) \sigma^2 + \frac{\Omega}{2} \cdot \frac{1}{k^2+m^2} \Big(1 - \exp\Big(-\Omega(k^2+m^2) \cdot (t-t_0)\Big)\Big)\Big\} \tag{221}$$

It follows that the two-point correlation functions are:

$$G_L(k,t) = \frac{\Omega}{2} \cdot \frac{1}{k^2+m^2} \Big(1 - \exp\Big(-\Omega(k^2+m^2) \cdot (t-t_0)\Big)\Big) + (2\pi)^d \delta^d(k) \sigma^2 \tag{222}$$

and

$$G_T(k,t) = \frac{\Omega}{2} \cdot \frac{1}{k^2+m^2} \Big(1 - \exp\Big(-\Omega(k^2+m^2) \cdot (t-t_0)\Big)\Big) \tag{223}$$

after long times $t \gg t_0$

$$G_L(k,t) = \frac{\Omega}{2} \cdot \frac{1}{k^2+m^2} + (2\pi)^d \delta^d(k)\sigma^2 \tag{224}$$

and

$$G_T(k,t) = \frac{\Omega}{2} \cdot \frac{1}{k^2+m^2} = W_T^{(2)}(k) \tag{225}$$

We recover the static two-point correlation functions if we set $\Omega = 2$.
Finally we note that the equilibrium dynamical correlation functions satisfy the constraint relation (181) which becomes:

$$r + \frac{u}{6}\Big(\Sigma^2(t) + \int^{\Lambda} \frac{d^dk}{(2\pi)^d} \cdot G_T(k,t)\Big) = m^2. \tag{226}$$

4 Introduction to the scaling behaviours out of equilibrium.

This section provides an introduction to the off-equilibrium scenario. The results of the previous sections apply in the equilibrium framework. Now we want to investigate the off-equilibrium behaviours emerging when the system approaches the transition with time-dependent parameters.
Thus, we start with a system at equilibrium and we change the parameters in such a way that the system is driven to the transition at $t = 0$. At the critical point, the fluctuations of the system have diverging relaxation times, therefore it is expected that there is a time in which the system goes out of equilibrium. This remains true even in the limit of slow time variations $t_s \to \infty$: even if the parameters slowly change in time, the system cannot adapt itself to the external variations.
Very close to the transition, the dynamics is described in terms of new length and time scales. These ones can be defined through the equilibrium correlation length and time and are called *Kibble-Zurek scales*. The observables present a non-trivial rescaling in terms of these scales.

4.1 Protocols.

We define the protocols $\delta(t, t_s)$ as paths in the parameters space of the system that cross the critical point at $t = 0$ [see ref.[16]]. Protocols are essentially characterized by their symmetry and by the leading order behaviour a near the transition. For $t \simeq 0$ the protocol can be written as

$$\delta(t, t_s) \overset{t \sim 0}{\sim} \delta_0 \cdot (t/t_s)^a \tag{227}$$

where δ_0 is a constant and t_s is the scale of time-variations. Since the protocol takes the system across the critical point, the leading order behaviour around the transition is an odd number. The most general protocol for a ferromagnetic critical point involve both temperature and magnetic field and can be expressed as

$$\vec{\delta}(t, t_s) = (\vec{h}(t, t_s), r(t, t_s) - r_c) \tag{228}$$

If we consider a protocol in which both the parameters have a proper time-evolution, the system undergoes a multicritical phase-transition and therefore the previous results should be reviewed. We restrict our attention to the following two protocols:

4.1.1 Magnetic field protocol.

$$\vec{\delta}(t, t_s) = (h_\alpha(t, t_s), 0). \tag{229}$$

It is a protocol along the magnetic field at the critical temperature (the thermal coupling is fixed to its critical value $r = r_c$). The magnetic field has a fixed direction

$$h_\alpha(t, t_s) = \delta_{1,\alpha} \cdot h(t, t_s). \tag{230}$$

This is a symmetry-breaking protocol because there is an explicit symmetry-breaking term in the system due to external magnetic field. The dependence of the magnetic field on time is $\delta(t, t_s) = h(t, t_s) = \tanh(t/t_s)$ and we will investigate the area near the transition in the limit of slow-variations $t_s \to \infty$ where we write $\delta(t, t_s) = h(t, t_s) \approx t/t_s$ as linear ramp protocol.

4.1.2 Thermal protocol.

$$\vec{\delta}(t,t_s) = (0, r(t,t_s) - r_c) \tag{231}$$

It is a protocol at zero magnetic field $h_\alpha = 0$ moving along the thermal direction with a law $\delta(t,t_s) = r(t,t_s) - r_c$. Even if we do not specify the time-dependence of the thermal coupling at all times, we assume that near the transition the protocol can be written as in (227). Thus, we assume that near the transition $\delta(t,t_s) \approx -t/t_s$. This protocol is non-symmetry breaking because the most relevant operator along this path respect all the symmetries of the model.

4.1.3 Round-trip protocols.

We define the *round-trip protocol* associated to $\delta(t,t_s)$ to be the protocol such that it has the same time-dependence of the external fields of δ but occurs along a closed path: it starts at a time $t_i = -\infty$, goes to $t_f = +\infty$ crossing the transition at $t = 0$ and then comes back from t_f to t_i. It follows that the initial and the final values of the external fields are the same along a round-trip protocol.

4.1.4 Strong scaling behaviour.

We define $\xi(t,t_s)$ and $\xi_t(t,t_s)$ to be the instantaneous correlation length and time if the system is at the equilibrium at $\delta(t,t_s)$. We also define for all kind of protocols the critical exponent ν_g as follows:

$$\nu_g = 1/(d - \Delta) \tag{232}$$

where Δ is the scaling dimension of the most relevant operator along the path near $\delta = 0$. For small value of the protocol the correlation length diverges as

$$\xi(t,t_s) \sim |\delta(t,t_s)|^{-\nu_g}. \tag{233}$$

The quantity ν_g plays the role of a generalized critical exponent ν[16] Let us consider in particular the two protocols (229) and (231):

-Magnetic field protocol.

The magnetic field has scaling dimension $d_h = d + 2/2$ and couples the field ϕ having scaling dimension $d_\phi = d - 2/2 = \Delta$. It follows that

$$\nu_g = \nu_h = \frac{1}{d - \frac{(d-2)}{2}} = \frac{2}{d+2} = \frac{1}{d_h}; \tag{234}$$

-Thermal protocol.

In Sec.3 we have already derived the exponent ν related to the thermal coupling constant (200). Thus,

$$\nu = \frac{1}{d-2}; \tag{235}$$

in $2 < d < 4$. This result can be viewed (looking at relation (196)) as a coupling

$$r - r_c = m^2 - \frac{u}{6}(\rho - \rho_c) = m^2 + \frac{u}{6}(m^2 D_d(m)) = m^2 + \frac{u}{6}(C(d)m^{2-\epsilon}) \sim m^{2-\epsilon}, \tag{236}$$

at leading order. This term couples an operator having scaling dimension $\Delta = d - 2 + \epsilon = d - 2 + (4 - d) = 2$. It follows that

$$\nu_g = \nu = \frac{1}{d - \Delta} = \frac{1}{d-2} = \frac{1}{2d_\phi}. \tag{237}$$

[16]As we have seen in Sec.1, the critical exponent ν is used to describe the power-law with which the correlation length diverges along the thermal protocol. Here we extend this idea by defining the an exponent ν_g for generic type of protocols [see ref.[16].

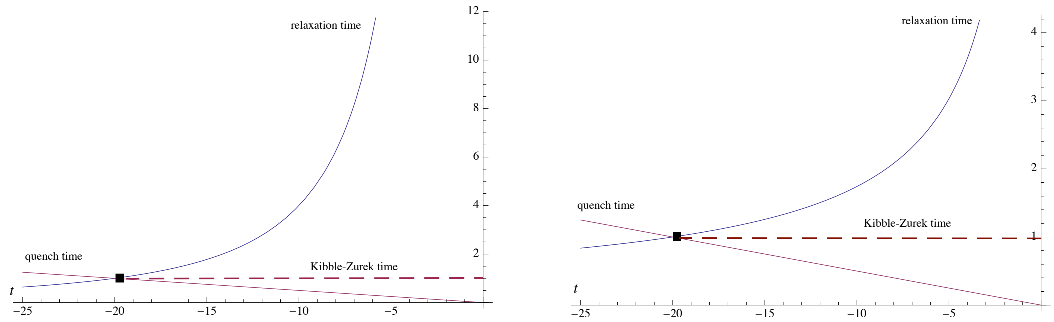

Figure 8: The Kibble-Zurek time for the thermal (left) and the magnetic field (right) protocol at fixed $t_s = 20$ in three spatial dimensions.

4.2 Kibble-Zurek scales.

There are two competiting effects which characterize the system while it approaches the critical point: the divergence of the correlation length and the phenomenon of the critical slowing down. One considers an instantaneous value of the protocol $\delta(t, t_s)$: if the system is at the equilibrium, the correlation length diverges as

$$\xi(t, t_s) \sim |\delta(t, t_s)|^{-\nu_g} \sim |\delta_0 \cdot (t/t_s)^a|^{-\nu_g}. \tag{238}$$

We define the *quench-time* as the time which the system needs to adapt itself to the variations of the external fields:

$$\tau_{\text{quench}}(t, t_s) = \xi(t, t_s)/\dot{\xi}(t, t_s). \tag{239}$$

Since the instantaneous relaxation time of the system $\xi_t(t, t_s) < \tau_{\text{quench}}(t, t_s)$ the system remains at the equilibrium.
But the relaxation time diverges at the transition as

$$\xi_t(t, t_s) \sim \xi^z(t, t_s) \sim |\delta(t, t_s)|^{-\nu_g z} \sim |\delta_0 \cdot (t/t_s)^a|^{-\nu_g z} \tag{240}$$

Thus, there must be a time in which $\xi_t(t, t_s) > \tau_{\text{quench}}(t, t_s)$ and therefore the system goes out of the equilibrium. We consider the instant of time t_Q such that

$$\xi_t(t_Q, t_s) = \tau_{\text{quench}}(t_Q, t_s); \tag{241}$$

For times $|t| < t_Q$ the correlation length and time differ from their equilibrium value and are no longer able to describe the scaling behaviour of the system. By the previous relations follow that

$$t_Q = (t_s/|\delta_0|^{1/a})^{(a\nu_g z)/(a\nu_g z+1)} \tag{242}$$

We call this quantity *Kibble-Zurek time*. One may note that t_Q depends on the protocol through the leading order behaviour a and the exponent ν_g, which is related to the most relevant scaling field along the path. We define also a length-scale describing the off-equilibrium dynamics as

$$l_Q = t_Q^{1/z} = (t_s/|\delta_0|^{1/a})^{(a\nu_g)/(a\nu_g z+1)} \tag{243}$$

that is generally called *Kibble-Zurek length*. We compute the specific values of the Kibble-Zurek length and time along the protocols (229) and (231):

-Magnetic field protocol.

For a linear ramp protocol $a = 1$, $\delta_0 = 1$ with $\nu_g = \nu_h = 1/d_h$, the KZ time is:

$$t_Q = (t_s)^{4/(6+d)} \tag{244}$$

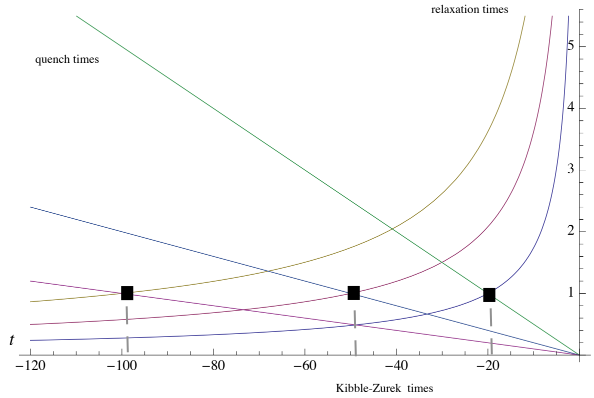

Figure 9: The figure show how the KZ time increase with t_s. It has been made in $d = 3$ and for the magnetic field protocol. The thermal protocol shows the same feature.

and the KZ length

$$l_Q = t_Q^{1/z} = (t_s)^{2/(6+d)} \tag{245}$$

where we have used a value $z = 2$ of the dynamic critical exponent.
For instance, in three spatial dimension $d = 3$ we have: $d_h = 5/2$, $d_\phi = 1/2$, $\nu = 2/5$ and $t_Q = (t_s)^{4/9}$, $l_Q = (t_s)^{2/9}$.

-Thermal protocol.

the Kibble-Zurek time for a linear thermal protocol $\nu_g = \nu = 1/(d-2)$ with $|\delta_0| = 1$ is

$$t_Q = (t_s)^{2/d} \tag{246}$$

The Kibble-Zurek length for $z = 2$ is

$$l_Q = t_Q^{1/z} = (t_s)^{1/d} \tag{247}$$

In $d = 3$ we have: $d_h = 5/2$, $d_\phi = 1/2$, $\nu = 1$ and $t_Q = (t_s)^{2/3}$, $l_Q = (t_s)^{1/3}$.

The off-equilibrium physics is observed for an interval of size $t_Q \sim (t_s)^e$ around the transition, with $e = (a\nu_g z)/(a\nu_g z + 1)$. When we take the limit of very slow time-variations $t_s \to \infty$, this interval become very large:

$$t_Q \sim (t_s/|\delta_0|^{1/a})^e \overset{t_s \to \infty}{\to} \infty \tag{248}$$

Since $e < 1$, the focus on the off-equilibrium scenario is related to a very small values of the external fields: near the transition a generic protocol can be written as $\delta(t, t_s) \approx \delta_0 \cdot (t/t_s)^a$, so if we express the protocol in terms of the KZ time:

$$\delta(t, t_s) \sim \pm(t/t_Q)^a \cdot (t_s/|\delta_0|^{1/a})^{a(e-1)} \overset{t_s \to \infty}{\to} 0 \tag{249}$$

where the two signs are related to the sign of the constant δ_0. In the limit of very slow passage the system is arbitrarily close to the critical point.

4.3 Kibble-Zurek scaling limit.

We assume the existence of a non-trivial scaling behaviour for $\delta(t, t_s) \simeq 0$ in terms of the variables:

$$\bar{t} = t/t_Q.$$

$$\overline{x} = x/l_Q.$$

The protocol $\delta(t, t_s)$ can be expressed in terms of the new scaling variables near the transition:

$$\delta(t, t_s) \sim \delta_0 \cdot (t/t_s)^a \sim \pm \bar{t}^a \cdot l_Q^{-1/\nu_g} \tag{250}$$

We formally define the off-equilibrium scaling limit (also called *Kibble-Zurek scaling limit*) to be the limit $t_s \to \infty$ when time and length scales are measured in units of the diverging KZ scales, t_Q and l_Q.
Let us consider an operator O with scaling dimension Δ: the KZ scaling for its correlation functions is the limit $t_s \to \infty$ holding $\bar{t}$ and $\overline{x}$ fixed, such that:

$$< O(x, t, t_s, \delta) >_\varsigma = G_O(t, t_s, \delta) \sim \frac{1}{l_Q^\Delta} \cdot \mathcal{G}_O(\bar{t}); \tag{251}$$

for one-point correlator, and

$$< O(x, t, t_s, \delta) O(0, t, t_s, \delta) >_\varsigma = G_{OO}(x, t, t_s, \delta) \sim \frac{1}{l_Q^{2\Delta}} \cdot \mathcal{G}_{OO}(\overline{x}, \bar{t}), \tag{252}$$

for two-point correlation functions and so on and so forth.

The dynamics across the transition present a *universal* scaling behaviour in the limit of slow passage. Infact, it does not depends on the choice of the initial and final values of the external fields because the off-equilibrium scaling occurs in a range of values of the protocol that shrinks near zero when $t_s \to \infty$. Very slow-dynamics provides to universalty; the off-equilibrium scaling behaviours depend only on a few things: the static and the dynamic class of universality[17] plus the leading order behaviour a of the protocol near the transition.

4.4 Finite-size effects.

We study the off-equilibrium scaling relations arising by the presence of time-dependent parameters coupled to a system of finite-size L which approaches the critical point [see ref. [6]][18]. Thus, we consider a protocol $\delta(t, t_s)$ associated to the system and we investigate the limit of very slow passage $t_s \to \infty$. Assuming the existence of a non-trivial scaling behaviour for $\delta(t, t_s) \simeq 0$, we expect that the off-equilibrium behaviour to be controlled by the scaling variables:

$$\bar{t} = t/t_Q,$$
$$\overline{x} = x/l_Q,$$
$$\ell = l_Q/L;$$

We also assume that the dynamics across the transition present a universal scaling behaviour when t_s and L becomes large keeping the variables $\bar{t}$, $\overline{x}$ and ℓ fixed. The protocol can be written in terms of the scaling variables close to the critical point:

$$\delta(t, t_s) \cdot L^{1/\nu_g} \sim \delta_0 \cdot (t/t_s)^a \cdot L^{1/\nu_g} \sim \pm \bar{t}^a \cdot \ell^{-1/\nu_g}. \tag{253}$$

Let us derive more quantitative predicitions, considering first the equilibrium scaling behaviour of statistical observables. The two-point correlation function of an operator O with scaling dimension Δ, obey to the relation[19]:

$$< O(x, t, L, \delta) O(0, t, L, \delta) >_\varsigma = G_{OO}(x, t, L, \delta) \sim L^{-2\Delta} \cdot \mathcal{G}^{\text{eq.}}_{OO}(\frac{x}{L}, \frac{t}{L^z}, \delta \cdot L^{1/\nu_g}); \tag{254}$$

[17] Which means the symmetry of the model and the spatial dimensions plus the type of equilibrium-dynamics

[18] Up to now we have considered the scaling relation for an infinite-volume system. However, it is not obvious to understand why the infinite-volume off-equilibrium behaviour is well-defined. Thus, let us consider the effects of a finite-size L. Then, we consider again the infinite-volume limit of the system.

[19] It is an extension of the equilibrium scaling behaviour (146) with finite-size effects. See for instance ref.[19]

close to the critical point. In the infinite-volume limit[20], for $|\delta| \to 0$ at x/ξ, t/ξ^z fixed:

$$< O(x,t,\delta)O(0,t,\delta) >_\varsigma \sim \xi^{-2\Delta} \cdot \mathcal{G}^{\text{eq.}}_{OO}(\frac{x}{\xi}, \frac{t}{\xi^z}) = |\delta|^{2\nu_g \Delta} \cdot \mathcal{G}^{\text{eq.}}_{OO}(x|\delta|^{\nu_g}, t|\delta|^{z\nu_g}); \qquad (255)$$

$\mathcal{G}^{\text{eq.}}_{OO}$ is the equilibrium scaling function and the exponent ν_g is defined by (234). The protocol δ resume in a compact notation all the informations about the relevant external fields[21]. We require that the equilibrium finite size-scaling matches the infinite-volume scaling behaviour:

$$\mathcal{G}^{\text{eq.}}_{OO}(\frac{x}{L}, \frac{t}{L^z}, \delta \cdot L^{1/\nu_g}) \overset{L=\infty}{\sim} |\delta \cdot L^{1/\nu_g}|^{2\nu_g \Delta} \cdot \mathcal{G}^{\text{eq.}}_{OO}(x|\delta|^{\nu_g}, t|\delta|^{z\nu_g}). \qquad (256)$$

when $\delta \cdot L^{1/\nu_g} \to \infty$. Let us give a concrete example: the equilibrium scaling behaviour of the magnetization at the critical point for small values of the magnetic field is

$$\Sigma(t, L, h, T_c) \sim L^{-d_\phi} \cdot \Theta^{\text{eq.}}(\frac{t}{L^z}, h \cdot L^{d_h}); \qquad (257)$$

where $\Theta^{\text{eq.}}$ is the equilibrium scaling function of the magnetization. In the infinite-volume limit

$$\Sigma(t, L = \infty, h, T_c) \sim \pm a|h|^{d_\phi/d_h}; \qquad (258)$$

since $|h| \to 0$[22]. Thus, the scaling function obey to the relation

$$\Theta^{\text{eq.}}(\frac{t}{L^z}, h \cdot L^{d_h}) \overset{L=\infty}{\sim} \pm a|h \cdot L^{d_h}|^{d_\phi/d_h} \qquad (259)$$

when $h \cdot L^{d_h} \to \infty$.
In order to describe the off-equilibrium regime around the transition where $\delta(t, t_s) \approx \delta_0 \cdot (t/t_s)^a$, we generalize the relation (254):

$$G_{OO}(x, t, t_s, L, \delta) \sim L^{-2\Delta} \cdot \mathcal{G}_{OO}(\overline{x}, \overline{t}, \ell) \qquad (260)$$

$\mathcal{G}_{OO}$ is a general function of the off-equilibrium scaling variables. The off-equilibrium scaling limit in the infinite-volume limit has been already discussed in the previous subsection [see Eq. (251) and ref.[16]]. We report the result:

$$G_{OO}(x, t, t_s, \delta) \sim l_Q^{-2\Delta} \cdot \mathcal{G}_{OO}(\overline{x}, \overline{t}). \qquad (261)$$

The infinite-volume limit can be formally obtained by performing the limit $\ell \to 0$ keeping $\overline{t}$, $\overline{x}$ fixed

$$\mathcal{G}_{OO}(\overline{x}, \overline{t}, \ell) \overset{\ell \to 0}{\sim} \ell^{-2\Delta} \cdot \mathcal{G}_{OO}(\overline{x}, \overline{t}) = (l_Q/L)^{-2\Delta} \cdot \mathcal{G}_{OO}(\overline{x}, \overline{t}) \qquad (262)$$

4.5 Asymptotic behaviours.

We investigate the first deviations from the equilibrium behaviour in the correlation functions occuring at a time $|t| \sim t_Q$ before the transition. In terms of the rescaled time, the equilibrium has to be recovered in the asymptotic limit $\overline{t} \to -\infty$.

4.5.1 Matching of the scaling behaviours.

By costruction, it is possible to connect the off-equilibrium scaling with the equilibrium one, in the appropriate limit $\overline{t} \to -\infty$. Within the notation of the previous subsection, in the infinite-volume limit

$$l_Q^{-\Delta} \cdot \mathcal{G}_O(\overline{t}) \sim |\delta|^{\nu_g \Delta} \cdot \mathcal{G}^{\text{eq.}}_O(\overline{t} \cdot |\delta|^{za\nu_g}) \qquad (263)$$

[20]The volume tends to infinity at fixed density of the spins variables.
[21]At the equilibrium the protocol δ does not depends on time: it is a fixed configuration.
[22]The constant $a > 0$; the two signs reflects the fact that Σ aligns itself with the direction of the magnetic field and therefore changes sign if $h \mapsto -h$.

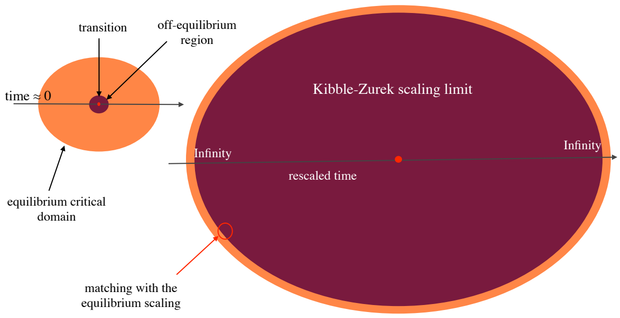

Figure 10: A qualitatively picture shows the region around the transition before and after the KZ scaling limit. In this figure, we assumed that the system approaches the equilibrium also after the transition.

Using the relation (250) one finds [23]

$$\mathcal{G}_O(\bar{t}) \sim |\bar{t}|^{a\nu_g \Delta} \cdot \mathcal{G}_O^{\text{eq.}}(\bar{t} \cdot |\bar{t}|^{za\nu_g}). \tag{264}$$

For the two-point correlation function:

$$\mathcal{G}_{OO}(\overline{x}, \bar{t}) \sim |\bar{t}|^{2a\nu_g \Delta} \cdot \mathcal{G}_{OO}^{\text{eq}}(\overline{x} \cdot |\bar{t}|^{\nu_g a}, \bar{t} \cdot |\bar{t}|^{za\nu_g}); \tag{265}$$

We consider the particular case of a spin system which crosses the ferromagnetic critical point. One can find for the one-point correlation function:

$$\mathcal{G}_O(\bar{t}) \propto |\bar{t}|^{a\nu_g \Delta} \tag{266}$$

because spin systems present a constant magnetization at the equilibrium. The equilibrium two-point correlation function decays exponentially:

$$\mathcal{G}_{OO}(\overline{x}, \bar{t}) \sim |\bar{t}|^{2a\nu_g \Delta} \cdot e^{-|\overline{x}| \cdot |\bar{t}|^{\nu_g a}}; \tag{267}$$

In the Fourier space the last relation becomes:

$$G_{OO}^{\text{eq.}}(k, t, \delta) \sim \xi^{2\Delta} \cdot \mathcal{G}_{OO}^{\text{eq.}}(k \cdot \xi, t/\xi^z) \tag{268}$$

Thus,

$$\mathcal{G}_{OO}(\overline{k}, \bar{t}) \sim |\bar{t}|^{-2a\nu_g \Delta} \cdot \mathcal{G}_{OO}^{\text{eq.}}(\overline{k} \cdot |\bar{t}|^{-a\nu_g}, \bar{t} \cdot |\bar{t}|^{za\nu_g}) \sim |\bar{t}|^{-2a\nu_g \Delta} \frac{|\bar{t}|^{-a\nu_g}}{\overline{k}^2 \cdot |\bar{t}|^{-2a\nu_g} + 1}. \tag{269}$$

Let us consider also the finite-size effects. In a finite geometry, a necessary condition to obtain equilibrium results is that $t_s \gg \tau$ i.e. $t_s \cdot \tau \to \infty$, where τ is the slowest time-scale of the system at the equilibrium given by $\tau \sim L^z$. Since $t_s \to \infty$ at fixed ℓ we have:

$$t_s \cdot L^{-z} = \ell^{z/e} \cdot L^{z(1-e)/z} \tag{270}$$

This condition is satisfied only if $L \to \infty$.
The previous relations implies that the matching occur at a time in which $\xi(t, t_s) < L$. Thus,

[23] It can be viewed also as a comparation of the two length scales:

$$\frac{x/l_Q}{x/\xi(t,t_s)} = \frac{\xi(t,t_s)}{l_Q} \sim |\delta(t,t_s)|^{-\nu_g} \cdot l_Q^{-1} \sim |\delta_0 \cdot (t/t_s)^a|^{-\nu_g} \cdot l_Q^{-1} \sim |t/t_Q|^{-a\nu_g}.$$

the limit $\bar{t} \to \infty$ at fixed ℓ is expected to lead to the infinite-volume equilibrium behaviour. Using the relation (253), we find the link between the two scalings regimes:

$$\mathcal{G}_{OO}(\overline{x}, \bar{t}, \ell) \sim |\bar{t}^{a} \cdot \ell^{-1/\nu_g}|^{2\nu_g \Delta} \cdot \mathcal{G}_{OO}^{\text{eq.}}(\overline{x} \cdot |\bar{t}|^{a\nu_g}, \bar{t} \cdot |\bar{t}|^{za\nu_g}) \tag{271}$$

The asymptotic behaviour of a finite-size system matches the infinite-volume equilibrium scaling relations because occurs in a region with a finite correlation length. Further developments are made in the infinite-volume limit.

4.5.2 Leading correction to the asymptotic equilibrium scaling.

We discuss the leading off-equilibrium corrections to the equilibrium scaling behaviour in the correlation functions. Let us consider, for instance, the two-point correlator. At the equilibrium it presents an exponential decay

$$\mathcal{G}_{OO}^{\text{eq.}}(x/\xi, t/\xi^{z}) \sim e^{-|t|/\xi^{z}(t,t_s)} \tag{272}$$

From the equation (267), we can write the last relation in terms of the KZ scales

$$\mathcal{G}_{OO}^{\text{eq.}}(\overline{x}, \bar{t}) \sim |\bar{t}|^{2a\nu_g \Delta} \cdot e^{-|\bar{t}| \cdot |\bar{t}|^{a\nu_g z}} \tag{273}$$

Approaching the equilibrium, the scaling function $\mathcal{G}_{OO}$ presents small fluctuations whose lifetime τ_o is of the order of the ratio between the two competing time-scales

$$\tau_o \sim \xi_t(t, t_s)/t_Q \sim |\delta(t, t_s)|^{-a\nu_g z} \cdot t_Q^{-1} \sim |\bar{t}|^{-a\nu_g z}; \tag{274}$$

We assume that these fluctuations are exponentially damped

$$\left\{\mathcal{G}_{OO}(\overline{x}, \bar{t}) - \mathcal{G}_{OO}^{\text{eq.}}(\overline{x} \cdot |\bar{t}|^{a\nu_g}, \bar{t} \cdot |\bar{t}|^{a\nu_g z})\right\} \overset{\bar{t}\to-\infty}{\sim} \mathcal{G}_{OO}^{\text{eq.}}(\overline{x} \cdot |\bar{t}|^{a\nu_g}, \bar{t} \cdot |\bar{t}|^{a\nu_g z}) \cdot K(\bar{t}) \cdot e^{-c|\bar{t}|/\tau_o}$$

$$\frac{\mathcal{G}_{OO}(\overline{x}, \bar{t})}{\mathcal{G}_{OO}^{\text{eq.}}(\overline{x} \cdot |\bar{t}|^{a\nu_g}, \bar{t} \cdot |\bar{t}|^{a\nu_g z})} - 1 \sim K(\bar{t}) \cdot e^{-c|\bar{t}|^{1+az\nu_g}} \tag{275}$$

where K is a regular function and c is a positive constant. This ansatz can be generalized to the other observables and has been numerically checked in the ref.[6] for the O(N) vector models.

5 Off-equilibrium scaling behaviours for O(N) vector model at large N.

We study the off-equilibrium physics arising in the O(N) vector model at large N. The dynamics of the system now depends also on the external fields: therefore we consider again the Langevin equation with variable parameters and we solve it. We find that the correlation functions depends on the external fields and on the effective mass term $m^2(t, t_s)$ of the O(N) model. A constraint-equation defines the mass term at all times.
Since the correlation functions and the constraint-equation are different for different protocols, we divide the discussion for the thermal protocol and for the magnetic field protocol. For both the protocols, the scaling relations are derived and the analysis of the asymptotic behaviours of the system computed.

5.1 Dynamics with time-dependent external fields.

Let us insert a time-dependence of the parameters in the action (204)

$$S[\phi] = \int d^dx \cdot \Big[\frac{1}{2}(\partial_\mu \phi(x,t))^2 + \frac{1}{2} r(t,t_s)\phi^2(x,t) + \frac{u}{4!}(\phi^2(x,t))^2 - h_\alpha(t,t_s)\phi_\alpha(x,t)\Big]. \quad (276)$$

The dynamical evolution of the system at large N is driven by the linearized Langevin equation (211):

$$\dot{\phi}_\alpha(x,t) = -\frac{\Omega}{2}\Big(- \partial_\mu\partial_\mu\phi_\alpha(x,t) + m^2(t,t_s)\phi_\alpha(x,t)\Big) + \frac{\Omega}{2} h_\alpha(t,t_s) + \varsigma_\alpha(x,t); \quad (277)$$

Writing the Fourier transform of the equation (277), we obtain

$$\dot{\phi}_\alpha(k,t) = -\frac{\Omega}{2}(k^2 + m^2(t,t_s))\phi_\alpha(k,t) + \frac{\Omega}{2}\cdot(2\pi)^d\delta^d(k)\cdot h_\alpha(t,t_s) + \varsigma_\alpha(k,t); \quad (278)$$

The cumulants of noise distribution in the Fourier space are (213) and (214).

The set of equation which defines the dynamics are [see ref. [20] and [21]]

$$\dot{\Sigma}(t,t_s) = -\frac{\Omega}{2} m^2(t,t_s)\cdot\Sigma(t,t_s) + \frac{\Omega}{2} h(t,t_s) \quad (279)$$

$$\dot{G}_T(k,t,t_s) = -\Omega(k^2 + m^2(t,t_s))\cdot G_T(k,t,t_s) + \Omega; \quad (280)$$

and finally the relation which defines the mass term [see ref.[16]]:

$$m^2(t,t_s) = r(t,t_s) + \frac{u}{6}\Big(\Sigma^2(t,t_s) + \int^\Lambda \frac{d^dk}{(2\pi)^d}\cdot G_T(k,t,t_s)\Big). \quad (281)$$

The first two equations directly follow from the equation of motion by considering expectation values over the noise distribution [see app. D.1]. The equation (281) is the saddle point equation (181) expressed in terms of the dynamical fields and parameters.
The dynamical evolution of the field ϕ is written in the solution of (278). The detail of the computation are reported in the appendix D.2. The final result is:

$$\phi_\alpha(k,t) = \phi^0_\alpha(k,t) + \int_{t_0}^t dt' \cdot \exp\Big(-\frac{\Omega}{2}\int_{t'}^t dt''\cdot(k^2 + m^2(t'',t_s))\Big)\cdot\Big\{\frac{\Omega}{2}\cdot(2\pi)^d\delta^d(k)h_\alpha(t',t_s) + \varsigma_\alpha(k,t')\Big\} \quad (282)$$

where we have defined the term proportional to the original condition as

$$\phi^0_\alpha(k,t) = \exp\Big(- \frac{\Omega}{2}\int_{t_0}^t dt'\cdot(k^2 + m^2(t',t_s))\Big)(2\pi)^d\delta^d(k)\delta_{1,\alpha}\cdot\sigma. \quad (283)$$

5.2 Correlation functions.

From the general solution above, using relation (120), we are able to compute the dynamical correlation functions when the system is coupled to time-dependent external fields.

5.2.1 One-point Correlator.

From solution (282), exploiting the noise characteristics (213), we get the one-point correlation function:

$$< \phi_\alpha(k,t) >_\varsigma =$$

$$\phi^0_\alpha(k,t)+\int_{t_0}^t dt'\cdot\exp\Big(-\frac{\Omega}{2}\int_{t'}^t dt''\cdot(k^2+m^2(t'',t_s))\Big)\cdot\Big\{\frac{\Omega}{2}\cdot(2\pi)^d\delta^d(k)h_\alpha(t',t_s)+ < \varsigma_\alpha(k,t') >_\varsigma \Big\} =$$

$$(2\pi)^d\delta^d(k)\delta_{1,\alpha}\cdot\Sigma^0(t,t_s)+\frac{\Omega}{2}\int_{t_0}^t dt'\cdot\exp\Big(-\frac{\Omega}{2}\int_{t'}^t dt''\cdot(k^2+m^2(t'',t_s))\Big)\cdot(2\pi)^d\delta^d(k)\delta_{1,\alpha}h(t',t_s)$$

Integrating over the delta function in momenta we obtain the magnetization

$$\Sigma(t,t_s) = \Sigma_L(t,t_s) = \Sigma^0(t,t_s) + \frac{\Omega}{2}\int_{t_0}^t dt'\cdot\exp\Big(-\frac{\Omega}{2}\int_{t'}^t dt''\cdot(m^2(t'',t_s))\Big)\cdot h(t',t_s) \quad (284)$$

where we've defined the term proportional to the original condition as

$$\Sigma^0(t) = \sigma\cdot\exp\Big(-\frac{\Omega}{2}\int_{t_0}^t dt'\cdot m^2(t',t_s)\Big). \quad (285)$$

Note that the $N-1$ transverse components of the magnetization $\Sigma_{\alpha>1} = \Sigma_T = 0$ [24]

5.2.2 Two-point Correlator.

The expectation value of two fields can be computed starting from the expression of the dynamical fields (282), using the cumulants of the noise distribution (213) and (214). The result is [see app. D.3]

$$< \phi_\alpha(k,t)\phi_\beta(k',t) >_\varsigma = \delta_{\alpha\beta}(2\pi)^d\delta^d(k+k')\cdot\Big[\delta_{1,\alpha}(2\pi)^{2d}\delta^d(k)\Big\{(\Sigma^0(t,t_s))^2$$

$$+2\Sigma^0(t,t_s)\cdot\frac{\Omega}{2}\int_{t_0}^t dt'\cdot\exp\Big(-\frac{\Omega}{2}\int_{t'}^t dt''\cdot(k^2+m^2(t'',t_s))\Big)h(t',t_s)+\Sigma^2(t,t_s)\Big\}$$

$$+\Omega\int_{t_0}^t dt'\cdot\exp\Big(-\Omega\int_{t'}^t dt''\cdot(k^2+m^2(t'',t_s))\Big)\Big]. \quad (286)$$

It follows that the transverse two-point correlation function α, $\beta > 1$ is

$$G_T(k,t,t_s) = \Omega\int_{t_0}^t dt'\cdot\exp\Big(-\Omega\int_{t'}^t dt''\cdot(k^2+m^2(t'',t_s))\Big) = W_T^{(2)}(k,t,t_s). \quad (287)$$

Note that the transverse two-point correlation function is connected because $\Sigma_T = 0$. The longitudinal two-point correlation function is:

$$G_L(k,t,t_s) = G^0(k,t,t_s)+\Omega\int_{t_0}^t dt'\cdot\exp\Big(-\Omega\int_{t'}^t dt''\cdot(k^2+m^2(t'',t_s))\Big)+(2\pi)^d\delta^d(k)\Sigma^2(t,t_s) \quad (288)$$

where we have defined the term proportional to the initial condition as

$$G^0(k,t,t_s) = (2\pi)^{2d}\delta^d(k)\Big[(\Sigma^0(t,t_s))^2 \quad (289)$$

[24] Protocols with high frequency $\delta(t,t_s) = h(t,t_s) = \sin(t/t_s)$ admit the existence of a transverse magnetization as was found by [21]. We investigate only protocols with quasi-adiabatic variations where the transverse magnetization is zero.

$$+2\Sigma^0(t,t_s)\cdot\frac{\Omega}{2}\int_{t_0}^{t}dt'\cdot\exp\Big(-\frac{\Omega}{2}\int_{t'}^{t}dt''\cdot(k^2+m^2(t'',t_s))\Big)h(t',t_s)\Big].$$

These relations for the dynamic correlation functions are valid for generic protocols. Naturally, if we consider a non symmetry-breaking protocol like the thermal protocol (231), the distinction between transverse and longitudinal directions becomes meaningless and we have $G_L = G_T = G_{\phi\phi}$.

Note.
If we consider the long-time limit $t \to \pm\infty$ in which the action (and its parameters) comes back to be time-independent, the standard form of the two-point correlation function has to be recovered. Thus, from the results of Sec.2,3.4.2 we set $\Omega = 2$.

5.3 The constraint-equation.

The correlators (284), (345) found in the last subsection are function of the external fields and of the mass term $m^2(t,t_s)$. Since the external fields are specified by the protocol $\delta(t,t_s)$, the knowledge of the mass term determines the explicit expression of the correlation functions at all times. As we have seen, it is defined by the equation (281):

$$r(t,t_s)+\frac{u}{6}\Big(\Sigma^2(t,t_s)+\int^{\Lambda}\frac{d^dk}{(2\pi)^d}\cdot G_T(k,t,t_s)\Big)=m^2(t,t_s) \tag{290}$$

The equation above is the best self-consistency relation of our model. In the next discussions we refer to this relation as *constraint-equation*.
We obtain the KZ scaling relation of the mass term using dimensional argument: for $t_s \to \infty$ at fixed $\bar{t}$

$$m^2(t,t_s)\sim l_Q^{-2}\cdot\mathcal{M}^2(\bar{t}); \tag{291}$$

where $\mathcal{M}^2$ is a scaling function. In the following, we discuss the thermal protocol (229) and magnetic field protocol (229) separately.

5.4 Thermal protocol.

Let us consider first the off-equilibrium dynamics arising by a time variations of the temperature in the absence of magnetic fields:

$$\boxed{\delta(t,t_s)=r(t,t_s)-r_c\approx -t/t_s} \tag{292}$$

For this protocol, the exponent $\nu_g = \nu = 1/2d_\phi$.

5.4.1 Correlation functions for thermal protocol.

Let us derive the specific form of the correlation functions for this protocol. The value of the magnetization is:

$$\Sigma_\alpha(t,t_s)=0; \tag{293}$$

because the value of the spontaneous magnetization at t_0 is zero, since the protocol start above the critical temperature. Since the system posses the full-symmetry before the transition, we do not have to distinguish between transverse and longitudinal directions. Thus, the two-point correlation function is:

$$G_{\phi\phi}(k,t,t_s)=2\int_{t_0}^{t}dt'\cdot\exp\Big(-2\int_{t'}^{t}dt''\cdot(k^2+m^2(t'',t_s))\Big). \tag{294}$$

5.4.2 Off-equilibrium scaling behaviours.

Let us write the constraint-equation (290) in this specific case:

$$m^2(t;t_s) = r(t,t_s) + \frac{u}{6}\int^{\Lambda} \frac{d^dk}{(2\pi)^d} \cdot G_{\phi\phi}(k,t,t_s,m^2). \tag{295}$$

In order to have the protocol explicitly, we write $r(t,t_s) = \delta(t,t_s) + r_c$. The critical value of the thermal coupling r_c can be written as[25]

$$r_c = -\frac{u}{6}\int^{\Lambda} \frac{d^dk}{(2\pi)^d} \cdot G_{\phi\phi}(k,t,t_s,m^2=0). \tag{296}$$

Thus, the constraint-equation becomes

$$m^2(t;t_s) = \delta(t,t_s) + \frac{u}{6}\int^{\Lambda} \frac{d^dk}{(2\pi)^d} \cdot \Big(G_{\phi\phi}(k,t,t_s,m^2) - G_{\phi\phi}(k,t;t_s,0)\Big). \tag{297}$$

By dimensional analysis we can check the consistence of the constraint-equation before taking scaling limit. Remember that the dimension of terms are $[m^2] = 2$, $[G_{\phi\phi}] = -2$ and $[u] = 4-d$, so, we have: $2 = 4 - d + d - 2$.
We consider the off-equilibrium scaling limit $t_s \to \infty$ keeping $\bar{t} = t/t_Q$, $\bar{k} = k \cdot l_Q$ fixed. The scaling hypotesis are:

- ∘ $m^2(t;t_s) \sim \mathcal{M}^2(\bar{t}) \cdot l_Q^{-2}$, for dimensional arguments.
- ∘ $G_{\phi\phi}(k,t,t_s) \sim \mathcal{G}_{\phi\phi}(\bar{k},\bar{t}) \cdot l_Q^2$.
- ∘ $\delta(t,t_s) \approx -(t/t_s) = -\bar{t}\cdot(t_Q/t_s) = -\bar{t}\cdot(t_s)^{\frac{2}{d}-1} = -\bar{t}/l_Q^{1/\nu} = -\bar{t}/l_Q^{d-2}$;

$\delta(t,t_s) \sim -\bar{t}\cdot l_Q^{-2d_\phi}$ as we read in the relation (250) with $\nu_g = \nu = 1/(d-2)$.

where $\mathcal{M}^2$ and $\mathcal{G}$ are scaling functions. It might be useful to define the quantity:

$$A(k,t,t') = \int_{t'}^{t} dt'' \cdot (k^2 + m^2(t'';t_s)); \tag{298}$$

In the KZ scaling limit:

$$A(k,t,t') = \int_{t'}^{t} (k^2 + m^2(t'';t_s)) \cdot dt'' \overset{(t''\mapsto t''/t_Q)}{=} \tag{299}$$

$$\int_{\bar{t}'}^{\bar{t}} (\bar{k}^2 \cdot l_Q^{-2} + \mathcal{M}^2(\bar{t}'')\cdot l_Q^{-2})\cdot d\bar{t}''\cdot t_Q = \int_{\bar{t}'}^{\bar{t}} (\bar{k}^2 + \mathcal{M}^2(\bar{t}''))\cdot d\bar{t}'' = A(\bar{k},\bar{t},\bar{t}').$$

$$(t_Q = l_Q^2)$$

We verify the scaling ansatz for the two-point correlation function starting from (294)

$$G_{\phi\phi}(k,t,t_s) = 2\int_{t_0}^{t} dt' \cdot \exp\Big(-2\int_{t'}^{t} dt''\cdot(k^2 + m^2(t'',t_s))\Big) \tag{300}$$

Taking the scaling limit and using the relation (299)

$$G_{\phi\phi}(k,t,t_s) \sim 2\int_{-\infty}^{\bar{t}} d\bar{t}'\cdot t_Q \cdot e^{-2A(\bar{k},\bar{t},\bar{t}')} = l_Q^2 \cdot \mathcal{G}_{\phi\phi}(\bar{k},\bar{t}). \tag{301}$$

The scaling function is:

$$G_{\phi\phi}(k,t,t_s) \sim l_Q^2\cdot\mathcal{G}_{\phi\phi}(\bar{k};\bar{t}) = l_Q^2\cdot 2\int_{-\infty}^{\bar{t}} d\bar{t}'\cdot\exp\Big(-2\int_{\bar{t}'}^{\bar{t}} d\bar{t}''(\bar{k}^2 + \mathcal{M}^2(\bar{t}''))\Big). \tag{302}$$

[25] see Sec.3, 3.2.2

At this point we can consider the KZ scaling limit in the constraint equation (295):

$$\mathcal{M}^2(\bar{t}) \cdot l_Q^{-2} \sim 0 = -\bar{t} \cdot l_Q^{-1/\nu} + \frac{u}{6} \int^{+\infty} \frac{d^d \overline{k}}{(2\pi)^d} \cdot l_Q^{-d} \cdot l_Q^2 \cdot \Big(\mathcal{G}_{\phi\phi}(\overline{k}, \bar{t}, \mathcal{M}^2) - \mathcal{G}_{\phi\phi}(\overline{k}, \bar{t}, 0) \Big)$$

$$\bar{t} = \frac{u}{6} \int^{+\infty} \frac{d^d \overline{k}}{(2\pi)^d} \cdot \Big(\mathcal{G}_{\phi\phi}(\overline{k}, \bar{t}, \mathcal{M}^2) - \mathcal{G}_{\phi\phi}(\overline{k}, \bar{t}, 0) \Big); \tag{303}$$

in which we have neglected the subleading terms $\mathcal{M}^2 \sim 0$. We will refer to the equation above as the *scaling constraint-equation* for the thermal protocol.
We have explicitly computed the off-equilibrium scaling function for the system: these quantities depends on the scaling function of the mass term $\mathcal{M}^2$. The scaling function of the mass term is formally defined at all times as the solution of the scaling-constraint equation:

$$\frac{u}{3} \int^{+\infty} \frac{d^d \overline{k}}{(2\pi)^d} \cdot \int_{-\infty}^{\bar{t}} d\bar{t}' \cdot e^{-2\int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot \overline{k}^2} \Big[\exp\Big(-2 \int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot \mathcal{M}^2(\bar{t}'') \Big) - 1 \Big] = \bar{t}; \tag{304}$$

In this relation the integral over momenta is gaussian and it can be easly performed

$$2 \int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \frac{(2\pi)^{d/2}}{(2\pi)^d (2|\bar{t} - \bar{t}'|)^{d/2}} \cdot \Big[\exp\Big(-2 \int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot \mathcal{M}^2(\bar{t}'') \Big) - 1 \Big] = \bar{t};$$

The scaling-constraint equation can be rewritten in terms of the function f, defined as

$$f(\bar{t}) = \exp\Big(2 \int_0^{\bar{t}} \mathcal{M}^2(\bar{t}') d\bar{t}' \Big) \Leftrightarrow \mathcal{M}^2(\bar{t}) = \frac{\dot{f}(\bar{t})}{2 f(\bar{t})} \tag{305}$$

instead of $\mathcal{M}^2$. With a little bit of algebra

$$\mathcal{G}_{\phi\phi}(\overline{k}, \bar{t}, \mathcal{M}^2) = \int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \exp\Big(-2 \int_0^{\bar{t}} (\overline{k}^2 + \mathcal{M}^2(\bar{t}'')) d\bar{t}'' \Big) \cdot \exp\Big(+2 \int_0^{\bar{t}'} (\overline{k}^2 + \mathcal{M}^2(\bar{t}'')) d\bar{t}'' \Big) =$$

$$\int_{-\infty}^{\bar{t}} \Big(\frac{e^{-2\overline{k}^2 \bar{t}}}{f(\bar{t})} \cdot e^{2\overline{k}^2 \bar{t}'} \cdot f(\bar{t}') \Big) \cdot d\bar{t}' = \frac{e^{-2\overline{k}^2 \bar{t}}}{f(\bar{t})} \int_{-\infty}^{\bar{t}} e^{2\overline{k}^2 \bar{t}'} \cdot f(\bar{t}') \cdot d\bar{t}'$$

Thus, the scaling constraint-equation is:[26]

$$\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \frac{f(\bar{t}) - f(\bar{t}')}{|\bar{t} - \bar{t}'|^{d/2}} = \bar{t} \cdot f(\bar{t}). \tag{306}$$

The theory is completely solved in this case because the scaling constraint-equation has a solution in term of the function f [see ref. [16]]. The solution in three spatial dimensions is:

$$f(\bar{t}) \overset{d=3}{=} -3^{1/3} \Gamma(1/3) \cdot e^{2\bar{t}^3/3} \cdot (\bar{t} \cdot \mathrm{Ai}(\bar{t}^2) + \mathrm{Ai}'(\bar{t}^2)). \tag{307}$$

where Ai is the convergent Airy function[27] and Γ is the Euler's Gamma function.

A comment.
In the constraint-equation (297), the cutoff Λ appears as the upper limit on the momentum integral. In the scaling limit, the upper limit $\sim \Lambda \cdot l_Q \to \infty$ so the scaling constraint-equation (303) is *cutoff independent.* This statement is true for general protocols.

[26] The value of the quartic coupling constant u is set to $1/3(4\pi)^{d/2}$ in order to have a simplified result.
[27] A brief description of the Airy function is reported into the appendix C

5.4.3 A proof of the scaling relations.

We investigate the limit $t_s \to \infty$ keeping $\bar{t} = t/(t_s)^e$ and $\bar{k} = k(t_s)^{e/z}$ fixed. We show that there is only a value of e and of z which lead to a non-trivial rescaling of all the quantities and these are in agreement with the previous definitions.
In order to be general, we consider the thermal protocol (231) with a generic leading order behaviour $\delta(t, t_s) \approx -(t/t_s)^a$ near $t \simeq 0$; then we fix $a = 1$ as before. The general rescaling of the two-point correlation function is:

$$G_{\phi\phi}(k, t, t_s) = 2 \int_{-\infty}^{\bar{t}} d\bar{t}' \cdot (t_s)^e \cdot \exp\Big(-2 \int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot (\bar{k}^2 + \mathcal{M}^2(\bar{t}''))\Big) = (t_s)^e \cdot \mathcal{G}_{\phi\phi}(\bar{k}, \bar{t}) \quad (308)$$

where we have rescaled the mass term as $m^2(t, t_s) \sim \mathcal{M}^2(\bar{t}) \cdot (t_s)^{-2e/z}$ by dimensional analysis. There is only a choice of z which make the exponential a scaling quantity:

$$-2e/z + e = 0 \Rightarrow z = 2$$

which is the dynamical exponent of the O(N) model in the large N limit under a model A dynamics. Taking the KZ scaling limit into the constraint-equation (281):

$$\bar{t} \cdot (t_s)^{a(e-1)} = \frac{u}{6}\Big(\frac{d^d\bar{k}}{(2\pi)^d} \cdot (t_s)^{-de/z} \cdot (t_s)^e \Big[\mathcal{G}_{\phi\phi}(\bar{k}, \bar{t}, \mathcal{M}^2) - \mathcal{G}_{\phi\phi}(\bar{k}, \bar{t}, 0)\Big]\Big) \quad (309)$$

The only value of e which satisfies the scaling constraint-equation is:

$$a(e-1) = -\frac{de}{z} + e \Rightarrow e = \frac{a}{a + \frac{d}{z} - 1}.$$

If we consider $a = 1$, $d = 3$ and $z = 2$ we obtain $e = 2/3$. This result is in agreement with the definition of the Kibble-Zurek time

$$t_Q = (t_s)^{a\nu_g z/(a\nu_g z+1)} \overset{?}{=} (t_s)^e = (t_s)^{a/(a+d/z-1)} \quad (310)$$

only if

$$\nu_g = \nu = \frac{2}{2(d-2)} = \frac{1}{2d_\phi} \quad (311)$$

The definition of the Kibble-Zurek scales has a physical meaning: thus, we first have introduced these scales and then performed the rescaling of the correlation function with t_Q and l_Q. We have found a non-trivial off-equilibrium scaling behaviour. One might think that off-equilibrium scaling relations can be obtained also with the definition of other scales instead of the KZ ones. The previous lines demonstrate that the KZ scaling is the only one that the system admits.

5.4.4 Asymptotic behaviours

In Sec.4.5.2, we discussed the leading off-equilibrium corrections in the correlators. These correlation functions are uniquely determined at all times by the value of the scaling function of the mass term $\mathcal{M}^2$.
We propose a general ansatz in terms of this function that is sufficient to reproduce asymptotically the exponential damping of the off-equilibrium fluctuations in the observables. In the limit $\bar{t} \to -\infty$ we can write the scaling function:

$$\mathcal{M}^2(\bar{t}) = \mathcal{M}_e^2(\bar{t}) + \mathcal{M}_o^2(\bar{t}), \quad (312)$$

as an equilibrium term plus a very small off-equilibrium deviation $\mathcal{M}_o^2 \simeq 0$. At the equilibrium, we can relate the effective mass of the system with the inverse of the instantaneous correlation length:

$$m^2(t, t_s) \sim \xi^{-2}(t, t_s) \sim |\delta(t, t_s)|^{2\nu_g}; \quad (313)$$

In the KZ scaling limit

$$\mathcal{M}_e^2(\bar{t}) \sim |\bar{t}|^{2\nu_g}. \tag{314}$$

Under the assumption of asymptotic exponential approach to the equilibrium, we write $\mathcal{M}_o^2$ as:

$$\mathcal{M}_o^2(\bar{t}) \sim |\bar{t}|^{2\nu_g}\Big(b(\bar{t}) \cdot e^{-c|\bar{t}|^{1+z\nu_g}}\Big). \tag{315}$$

Thus, the scaling function of the mass term can be written close to the equilibrium $\bar{t} \to -\infty$ as

$$\mathcal{M}^2(\bar{t}) \sim |\bar{t}|^{2\nu_g} \cdot \Big(1 + b(\bar{t}) \cdot e^{-c|\bar{t}|^{1+z\nu_g}}\Big) \tag{316}$$

where b is a regular function and c is a constant. The ansatz (316) applies, close to the equilibrium, for general protocols. In the case of thermal protocol: $\nu_g = \nu = 1/2d_\phi$.

Equilibrium asymptotic forms.

In first approximation, we consider the mass term equal to the equilibrium value i.e. we set the term $\mathcal{M}_o^2 = 0$. This means that we look at the earliest rescaled times, when the off-equilibrium physics does not appear yet. Thus, we assume that

$$\mathcal{M}^2(\bar{t}) \simeq |\bar{t}|^{1/d_\phi} \tag{317}$$

With this relation the scaling functions of the Sec.5.4.2 can be computed explicitly. In particular, we consider the *susceptibility* $\chi(t, t_s)$ of the system, defined to be the two-point correlation function at zero momenta:

$$\chi(t, t_s) = G_{\phi\phi}(k = 0, t, t_s) = 2\int_{t_0}^{t} dt' \cdot \exp\Big(-2\int_{t'}^{t} dt'' \cdot m^2(t'', t_s)\Big) \tag{318}$$

It has a scaling relation:

$$\chi(t, t_s) \sim l_Q^2 \cdot \mathcal{G}(\bar{t}) = l_Q^2 \cdot 2\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \exp\Big(-2\int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot \mathcal{M}^2(\bar{t}'')\Big); \tag{319}$$

where $\mathcal{G}$ denotes the scaling function of the susceptibility. Using (312) in the scaling function above [see the appendix D.4.1]

$$\mathcal{G}(\bar{t}) = 2\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \exp\Big(-2\int_{\bar{t}'}^{\bar{t}} \mathcal{M}^2(\bar{t}'') \cdot d\bar{t}''\Big) \underset{\mathcal{M}^2 \simeq \mathcal{M}_e^2}{\overset{\bar{t} \to -\infty}{\sim}} 2\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \exp\Big(-2\int_{\bar{t}'}^{\bar{t}} |\bar{t}''|^{2\nu} \cdot d\bar{t}''\Big) \sim |\bar{t}|^{-2\nu}; \tag{320}$$

We find that the $\mathcal{M}_e^2$ reproduces the equilibrium value of the susceptibility:

$$\mathcal{G}_{\phi\phi}(\bar{t}) \sim |\bar{t}|^{-2\nu} = |\bar{t}|^{-1/d_\phi} \tag{321}$$

The critical exponent γ which describes the power-law behaviour of the susceptibility at the equilibrium is recovered:

$$\gamma = \nu(2 - \eta) = 2\nu \tag{322}$$

because η is zero in the large N limit. This relation is also in agreement with (269) valuated at zero momentum setting $a = 1$, $\nu = 1$ and $\Delta = d_\phi$ in three spatial dimensions $d = 3$.

Leading correction to the asymptotic equilibrium scaling.

We consider the leading off-equilibrium corrections to the equilibrium scaling behaviour given by (316):

$$\mathcal{M}^2(\bar{t}) \overset{\bar{t} \to -\infty}{\sim} |\bar{t}|^{2\nu}\Big(1 + b(|\bar{t}|) \cdot e^{-c|\bar{t}|^{1+z\nu}}\Big) \tag{323}$$

Under this assumption, we investigate the first deviations of the susceptibility from the equilibrium behaviour

$$\mathcal{G}(\bar{t}) = 2\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \exp\Big(-2\int_{\bar{t}'}^{\bar{t}} \mathcal{M}^2(\bar{t}'')\cdot d\bar{t}''\Big) = 2\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \exp\Big(-2\int_{\bar{t}'}^{\bar{t}} (\mathcal{M}_e^2(\bar{t}'')+\mathcal{M}_o^2(\bar{t}''))\cdot d\bar{t}''\Big) \overset{\bar{t}\to-\infty}{\sim}$$

$$2\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \exp\Big(-2\int_{\bar{t}'}^{\bar{t}} \mathcal{M}_e^2(\bar{t}'')\cdot d\bar{t}''\Big)\cdot\Big(1-2\int_{\bar{t}'}^{\bar{t}} \mathcal{M}_o^2(\overline{t''})\cdot d\bar{t}''\Big) \simeq$$

$$|\bar{t}|^{2-\kappa} - 4e^{2|\bar{t}|^\kappa/\kappa}\int_{-\infty}^{\bar{t}} d\bar{t}'\cdot e^{-2|\bar{t}'|^\kappa/\kappa}\cdot\Big(\int_{\bar{t}'}^{\bar{t}} d\bar{t}''\cdot|\bar{t}''|^{2\nu}\cdot b(\bar{t}'')\cdot\exp(-c|\bar{t}''|^{1+z\nu})\Big);$$

where $\kappa = 1 + z\nu = 1 + 1/d_\phi$. We assume that the function $b(|\bar{t}|)$ is in the form of a power-law:

$$b(\bar{t}) \sim b\cdot|\bar{t}|^a \tag{324}$$

Thus, we have three free parameters a, b and c (constants) into (316). The previous expression becomes

$$\mathcal{G}(\bar{t}) \sim |\bar{t}|^{2-\kappa} - 4e^{2|\bar{t}|^\kappa/\kappa}\int_{-\infty}^{\bar{t}} d\bar{t}'\cdot e^{-2|\bar{t}'|^\kappa/\kappa}\cdot\Big(\int_{\bar{t}'}^{\bar{t}} d\bar{t}''\cdot|\bar{t}''|^{2\nu}\cdot b\cdot|\bar{t}''|^a\cdot\exp(-c|\bar{t}''|^{1+z\nu})\Big); \tag{325}$$

After the computation [see app. D.5.2], the result for the transverse susceptibility is

$$\mathcal{G}(\bar{t}) \sim |\bar{t}|^{-2\nu}\Big(1 + K'|\bar{t}|^a\cdot e^{-c|\bar{t}|^{1+z\nu}}\Big). \tag{326}$$

where $K' = -2b/(2+c\kappa)$. This relation is in agreement with (275): the ansatz of exponential approach for the scaling function of the mass term $\mathcal{M}^2$ is sufficient to reproduce the same approach in the susceptibility.
We note that the mass term and the two-point correlation function are related by the relation (280) at all times[28]. In the KZ scaling limit, this relation becomes:

$$\frac{\partial}{\partial\bar{t}}\mathcal{G}(\bar{k},\bar{t}) = -2(\bar{k}^2 + \mathcal{M}^2(\bar{t}))\cdot\mathcal{G}(\bar{k},\bar{t}) + 2; \tag{327}$$

at zero momenta:

$$\mathcal{M}^2(\bar{t}) = -\frac{1}{2}\frac{d\mathcal{G}(\bar{t})/d\bar{t}}{\mathcal{G}(\bar{t})} + \frac{1}{\mathcal{G}(\bar{t})}; \tag{328}$$

The last equation gives us a constraint between the scaling functions which has to be satisfied at all times. In particular, we check the consistence of the result (380) taking the limit $\bar{t}\to-\infty$ in the this equation:

$$|\bar{t}|^{\kappa-1} + b|\bar{t}|^{\kappa-1+a}\cdot e^{-c|\bar{t}|^\kappa} \sim \frac{1}{2}(1-\kappa)|\bar{t}|^{-1} - \frac{1}{2}K'(1-\kappa)|\bar{t}|^{-1+a}\cdot e^{-c|\bar{t}|^\kappa}$$

$$-\frac{1}{2}c\kappa K'|\bar{t}|^{\kappa-1+a}\cdot e^{-c|\bar{t}|^\kappa} + |\bar{t}|^{\kappa-1} - K'|\bar{t}|^{\kappa-1+a}\cdot e^{-c|\bar{t}|^\kappa} + O(e^{-2c|\bar{t}|^\kappa});$$

The equilibrium terms satisfy the equation.

[28] The equation (280) is written for the transverse two-point correlator because it has been introduced for general protocols. In the absence of magnetic fields, we can remove the restriction to the transverse components.

The off-equilibrium leading terms are:

$$b|\bar{t}|^{\kappa-1+a}\cdot e^{-c|\bar{t}|^{\kappa}}\sim-\frac{1}{2}c\kappa K'|\bar{t}|^{\kappa-1+a}\cdot e^{-c|\bar{t}|^{\kappa}}-K'|\bar{t}|^{\kappa-1+a}\cdot e^{-c|\bar{t}|^{\kappa}};$$

These terms satisfy the equation only if $K'=-2b/(2+c\kappa)$, which is in agreement with our result.[29]

[29]The details of the computation are reported in the following table. The asymptotic contribution of the three terms into (328) in the limit $\bar{t}\to-\infty$ is:

$$\bullet\ \mathcal{M}^2(\bar{t})\sim|\bar{t}|^{\kappa-1}+b|\bar{t}|^{\kappa-1+a}\cdot e^{-c|\bar{t}|^{\kappa}};$$

$$\circ\ \mathcal{G}(\bar{t})\sim|\bar{t}|^{-2\nu}\left(1+K'|\bar{t}|^{a}\cdot e^{-c|\bar{t}|^{1+z\nu}}\right)=|\bar{t}|^{1-\kappa}+K'|\bar{t}|^{1-\kappa+a}\cdot e^{-c|\bar{t}|^{\kappa}};$$

$$\bullet\ \mathcal{G}(\bar{t})^{-1}\sim|\bar{t}|^{\kappa-1}-K'|\bar{t}|^{\kappa-1+a}\cdot e^{-c|\bar{t}|^{\kappa}};$$

$$\circ\ \dot{\mathcal{G}}(\bar{t})\sim-(1-\kappa)|\bar{t}|^{-\kappa}-K'(1-\kappa+a)|\bar{t}|^{-\kappa+a}\cdot e^{-c|\bar{t}|^{\kappa}}-K'|\bar{t}|^{1-\kappa+a}(-c\kappa)|\bar{t}|^{\kappa-1}\cdot e^{-c|\bar{t}|^{\kappa}};$$

$$\bullet\ \dot{\mathcal{G}}(\bar{t})\cdot\mathcal{G}_T(\bar{t})^{-1}\sim-(1-\kappa)|\bar{t}|^{-1}+K'(1-\kappa)|\bar{t}|^{-1+a}\cdot e^{-c|\bar{t}|^{\kappa}}-K'(1-\kappa)|\bar{t}|^{-1+a}\cdot e^{-c|\bar{t}|^{\kappa}}$$
$$+c\kappa K'|\bar{t}|^{\kappa-1+a}\cdot e^{-c|\bar{t}|^{\kappa}}+O(e^{-2c|\bar{t}|^{\kappa}});$$

The time-derivatives have been computed using the relation:

$$\frac{d}{d\bar{t}}F(|\bar{t}|)=\frac{|\bar{t}|}{\bar{t}}\cdot\frac{d}{d|\bar{t}|}F(|\bar{t}|)\overset{\bar{t}\to-\infty}{\sim}-\frac{d}{d|\bar{t}|}F(|\bar{t}|)$$

with F a generic function.

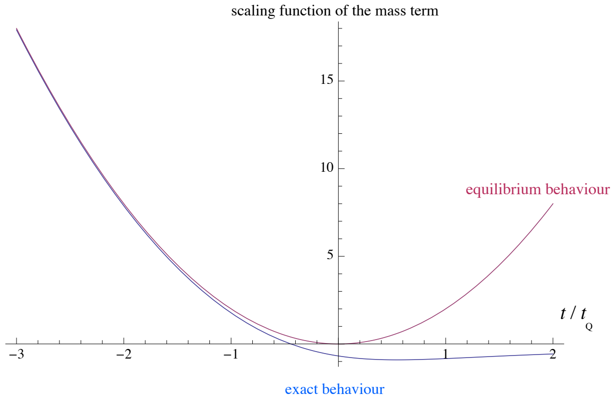

Figure 11: The asymptotic behaviour of the scaling function $\mathcal{M}^2(\bar{t})$. In the limit $\bar{t} \to -\infty$, the solution (329) approaches the equilibrium behaviour (317).

5.4.5 Analysis of the solution.

We discuss the asymptotic limits of the system starting from the solution of the scaling constraint-equation (303):

$$f(\bar{t}) \overset{d=3}{=} -3^{1/3}\Gamma(1/3) \cdot e^{2\bar{t}^3/3} \cdot (\bar{t} \cdot \mathrm{Ai}(\bar{t}^2) + \mathrm{Ai}'(\bar{t}^2));$$

From this result, using (305), we compute the scaling function of the mass term:

$$\mathcal{M}^2(\bar{t}) = \frac{\dot{f}(\bar{t})}{2f(\bar{t})} = \bar{t}^2 + \frac{\left[\mathrm{Ai}(\bar{t}^2) + 2\bar{t}^3 \cdot \mathrm{Ai}(\bar{t}^2) + 2\bar{t}^2 \cdot \mathrm{Ai}'(\bar{t}^2)\right]}{\left[\bar{t} \cdot \mathrm{Ai}(\bar{t}^2) + \mathrm{Ai}'(\bar{t}^2)\right]} \tag{329}$$

We first investigate the limit $\bar{t} \to -\infty$ using the asymptotic expansions of the Airy functions [see the appendix C][30]:

$$\mathcal{M}^2(\bar{t}) \overset{\bar{t}\to-\infty}{\sim} 2\bar{t}^2 \tag{330}$$

which is in agreement with (317) in three spatial dimensions. The exact solution for $\mathcal{M}^2$ matches the equilibrium in the appropriate limit [see fig.11].
Since we know a solution at all times, we are able to compute also the tail $\bar{t} \to +\infty$:

$$\mathcal{M}^2(\bar{t}) \sim -\frac{5}{4\bar{t}} \tag{331}$$

The asymptotic behaviour after the transition differs from the equilibrium prediction. The two-point correlation function at $\bar{t} \to +\infty$ with effective mass (331) is known in literature [see ref.[16] and [22]]:

$$\mathcal{G}_{\phi\phi}(\bar{k}, \bar{t}) \overset{\bar{t}\to+\infty}{\sim} \bar{t}^{5/2} \cdot e^{-2\bar{k}^2\bar{t}} \tag{332}$$

In the following, we argue that the asymptotic regime after the transition is the *coarsening.*

5.4.6 Coarsening physics.

When a system is quenched to the ordered phase with multiple vacua, it generally undergoes coarsening [see ref. [13], [16] and [22]]. Coarsen physics means that a system, with no explicit symmetry-breaking terms before transition, realizes different orientation of local broken symmetry. Infact, it has no reasons to prefer one vacuum to another. Each

[30] The relations (504) and (505) work in both the limit $\bar{t} \to \pm\infty$ because $z = \bar{t}^2 \in \mathbb{R}^+_{\{0\}} \leftrightarrow |z| = \bar{t}^2$ and $|\mathrm{Arg}(z)| = 0$.

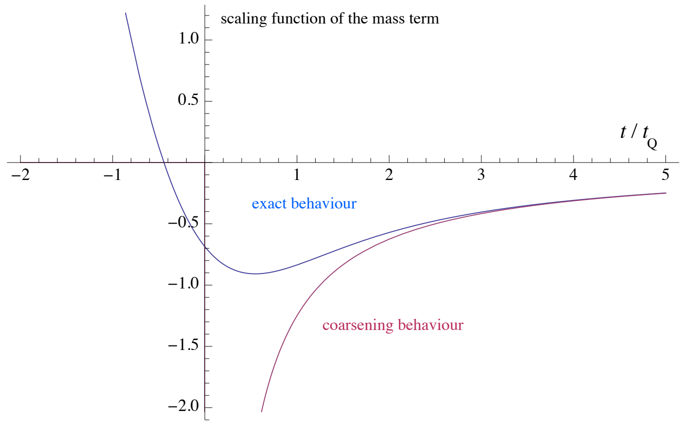

Figure 12: The asymptotic behaviour of the scaling function $\mathcal{M}^2(\bar{t})$. In the limit $\bar{t} \to +\infty$, the system approaches the coarsening regime.

local symmetry-breaking region also grows in time with a typical length scale called *coarsening length* $l_{co}(t) >> \xi$ and appears self-similiar over this scale. The system approaches the equilibrium into each single domain, over a scale ξ that is, the equilibrium correlation length but, among the domains, the equilibrium regime is exponentially suppressed. For late times $l_{co} >> \xi$ the domains grow but the correlation length does not reach to take the system to the equilibrium in whole: there are no competitor with the scale due to coarsening.

Coarsening scaling limit.

The *coarsening scaling limit* can be defined as the limit $t, t' \to \infty$ in which the two-point correlation function of an operator O, with scaling dimension Δ, has a behaviour[31]

$$< O(x,t,\delta)O(0,t',\delta) >_{\varsigma} \sim \xi^{-2\Delta} \cdot \mathcal{G}^{co}_{OO}(\frac{x}{l_{co}(t)}, \frac{x}{l_{co}(t')}) \tag{333}$$

We define the coarsening length as

$$l_{co}(t) = |t|^{\theta} \tag{334}$$

where the exponent

$$\theta = -a\nu_g + (a\nu_g z + 1)/z_d \tag{335}$$

and z_d is the specific *coarsening dynamical exponent*. For the O(N) vector-model at large N with relaxational dynamics we have $z_d = z = 2$ [see ref. [22]].
The scaling function of the two-point correlator in the coarsening regime has a behaviour [see ref.[22]]

$$\mathcal{G}^{co}_{OO}(kl_{co}(t)) \sim t^{d/2} \exp\Big(- 2(k \cdot l_{co}(t))^2 \Big). \tag{336}$$

We match the Kibble-Zurek scaling behaviour with the coarsening one [see Ref. [16]]. Comparing the two length scales:

$$\frac{l_{co}(t)}{l_Q} \sim \frac{|t|^{\theta}}{t_Q^{1/z}} \sim |\bar{t}|^{\theta} \cdot t_Q^{\theta-(1/z)}; \tag{337}$$

[31] $\mathcal{G}^{co}_{OO}$ is the scaling function in the coarsen regime.

Since $z_d = z$ follows that $\theta - 1/z = 0$. Taking the limit $\bar{t} \to +\infty$ at fixed $x/l_{co}(t)$, the scaling relation for the two-point correlation functions is

$$\mathcal{G}_{OO}(\overline{x},\overline{t}) \sim |\overline{t}|^{2a\nu_g\Delta} \cdot \mathcal{G}^{co}_{OO}(\frac{\overline{x}}{|\overline{t}|^{\theta}}); \tag{338}$$

If we consider the thermal protocol ($a = 1$, $\nu_g = \nu = 1$ in $d = 3$), the coarsening length is

$$l_{co}(t) = |t|^{\theta} \sim \sqrt{|t|}. \tag{339}$$

We compare the relation (332) with (336) using (338) in $d = 3$:

$$\mathcal{G}_{\phi\phi}(\overline{k},\overline{t}) \sim \overline{t}^{5/2} \cdot \exp\Big(- 2\overline{k}^2\overline{t}\Big) \sim \overline{t} \cdot \{\overline{t}^{3/2} \cdot \exp\Big(- 2(\overline{k}\sqrt{\overline{t}})^2\Big)\} \sim \overline{t} \cdot \mathcal{G}^{co}_{\phi\phi}(\overline{k} \cdot |\overline{t}|^{\theta}). \tag{340}$$

The expression (332) for the two-point correlator agrees with the coarsening prediction.

5.5 Magnetic field protocol.

We also investigate the slow passage across the critical point of the system driven by a time-dependent magnetic field at the critical temperature

$$\boxed{\delta(t,t_s) = \delta_{1,\alpha} \cdot h(t,t_s) \approx t/t_s} \tag{341}$$

For this protocol, the thermal coupling is fixed to the critical value $r = r_c$ and the exponent $\nu_g = \nu_h = 1/d_h$.

5.5.1 Correlation functions for magnetic field protocol.

Let us write explicitly the correlation functions along the magnetic field protocol. The value of the longitudinal magnetization is:

$$\Sigma(t,t_s) = \Sigma_L(t,t_s) = \Sigma^0(t) + \int_{t_0}^{t} dt' \cdot \exp\Big(- \int_{t'}^{t} dt'' \cdot (m^2(t'',t_s))\Big) \cdot h(t',t_s); \tag{342}$$

and the transverse components

$$\Sigma_T(t,t_s) = 0; \tag{343}$$

The longitudinal two-point correlation function is given by

$$G_L(k,t,t_s) = G^0(k,t,t_s) + 2\int_{t_0}^{t} dt' \cdot \exp\Big(-2\int_{t'}^{t} dt'' \cdot (k^2 + m^2(t'',t_s))\Big) + (2\pi)^d\delta^d(k)\Sigma^2(t,t_s) \tag{344}$$

while the transverse components

$$G_T(k,t,t_s) = 2\int_{t_0}^{t} dt' \cdot \exp\Big(- 2\int_{t'}^{t} dt'' \cdot (k^2 + m^2(t'',t_s))\Big). \tag{345}$$

There is an important thing to point out: the *role of the initial conditions* in the KZ scaling limit. Since the scaling behaviours turn out to be universal in the limit of slow variations, it is expected that the initial state of the system at t_0 does not influence the critical theory. However, the correlation above depend on ϕ^0 through Σ^0 and G^0. If we consider the expression of ϕ^0:

$$\phi^0_\alpha(k,t) = (2\pi)^d\delta^d(k) \cdot \delta_{1,\alpha}\sigma \cdot \exp\Big(- \int_{t_0}^{t} dt' \cdot (k^2 + m^2(t,t_s)\Big);$$

When we consider the off-equilibrium scaling limit $t_s \to \infty$

$$\phi^0_\alpha(k,t) \sim l_Q^d \cdot (2\pi)^d\delta^d(\overline{k}) \cdot \delta_{1,\alpha}\sigma \cdot \exp\Big(- \int_{-\infty}^{\overline{t}} (\overline{k}^2 + \mathcal{M}^2(\overline{t}'')) \cdot d\overline{t}''\Big) \to 0$$

The focus on the off-equilibrium scenario enlarge the small area $t \simeq 0$ near the critical point [see fig.10]. In the KZ scaling limit, the starting time becomes very far in time $t_0 \sim -\infty$ and the term ϕ^0 is therefore driven exponentially to zero[32]. The sensibility of the system to the initial conditions disappears in the scaling limit. This confirms that the details of the protocol are not important and the dynamics in the off-equilibrium region depends only on the leading order behaviour near the critical point.

5.5.2 Off-equilibrium scaling behaviour.

The thermal coupling is fixed to the critical value and can be written using the relation (296). The constraint-equation (290) for the magnetic protocol is:

$$m^2(t,t_s) = \frac{u}{6}\Big(\int^{\Lambda} \frac{d^dk}{(2\pi)^d} \cdot \Big\{G_T(k,t,t_s,m^2) - G_T(k,t,t_s,0)\Big\} + \Sigma^2(t;t_s)\Big); \qquad (346)$$

By dymensional analysis $2 = 4-d+d-2 = 4-d+2(d-2/2)$.
Let us consider the KZ scaling limit $t_s \to \infty$ keeping $\bar{t}$ and $\bar{k}$ fixed. We make the following scaling hypotesis:

- ∘ $m^2(t;t_s) \sim \mathcal{M}^2(\bar{t}) \cdot l_Q^{-2}$ using dimensional arguments, as before.
- ∘ $G_{\phi\phi}(k,t,t_s) \sim \mathcal{G}_{\phi\phi}(\bar{k},\bar{t}) \cdot l_Q^2$.
- ∘ $\Sigma(t,t_s) \sim \Theta(\bar{t}) \cdot l_Q^{-d_\phi}$.
- ∘ $\delta(t,t_s) \approx t/t_s \sim \bar{t} \cdot l_Q^{-d_h}$ as we read in (250) with $\nu_g = \nu_h = 1/d_h$.

Θ is the scaling function of the magnetization. We verify the scaling hypotesis for the correlation functions starting from their definitions[33]. Firstly, we consider the longitudinal two-point correlation function:

$$G_L(k,t,t_s) \sim 2\int_{-\infty}^{t} dt' \cdot e^{-2\int_{t'}^{t} dt''\cdot(k^2+m^2(t'',t_s))} + (2\pi)^d\delta^d(k)\Big(\int_{-\infty}^{t} dt' \cdot e^{-\int_{t'}^{t} dt''\cdot(m^2(t'',t_s))}\cdot h(t')\Big)^2 \qquad (347)$$

In the scaling limit, using the relation (299)

$$G_L(k,t,t_s) \sim 2\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot t_Q \cdot e^{-2A(\bar{k},\bar{t},\bar{t}')} + (2\pi)^d\delta^d(\bar{k}) \cdot l_Q^d \cdot \Big(\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot t_Q \cdot e^{-A(\bar{k},\bar{t},\bar{t}')} \cdot \bar{t}' \cdot l_Q^{-d_h}\Big)^2$$

$$= l_Q^2 \cdot \Big(2\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot e^{-2A(\bar{k},\bar{t},\bar{t}')}\Big) + l_Q^{(d+4-2d_h)} \cdot (2\pi)^d\delta^d(\bar{k})\Big(\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot e^{-A(\bar{k},\bar{t},\bar{t}')} \cdot \bar{t}'\Big)^2 = l_Q^2 \cdot \mathcal{G}_L(\bar{k},\bar{t}).$$

The result for the longitudinal two-point correlation function is

$$G_L(k,t,t_s) \sim l_Q^2 \cdot \mathcal{G}_L(\bar{k};\bar{t}) = l_Q^2\Big(\mathcal{G}_L^{\text{conn.}}(\bar{k},\bar{t}) + (2\pi)^d\delta(\bar{k})\Theta(\bar{t})\Big). \qquad (348)$$

where the first term is the scaling function of the connected part of the longitudinal two-point function

$$W_L^{(2)}(k,t,t_s) = l_Q^2 \cdot \mathcal{G}_L^{\text{conn.}}(\bar{k},\bar{t}) = l_Q^2 \cdot 2\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \exp\Big(-2\int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot (\bar{k}^2 + \mathcal{M}^2(\bar{t}''))\Big). \quad (349)$$

[32]The momenta are set to zero by the delta function and the scaling function of the mass term at $-\infty$ is equal to $|\bar{t}|^{2\nu_g}$. Thus, $\phi^0 \to 0$.

[33]We do not consider in the following the terms proportional to the initial state of the system because, as we have seen before, they do not influence the dynamics in the off-equilibrium region.

We already read, in the scaling limit of the disconnected part of the longitudinal two-point correlator, the scaling function of the magnetization. However, it can be obtained starting from (284):

$$\Sigma(t,t_s)=\int_{-\infty}^{t}dt'\cdot e^{-\int_{t'}^{t}dt''\cdot m^2(t'',t_s)}\cdot h(t')=\int_{-\infty}^{\bar{t}}d\bar{t}'\cdot t_Q\cdot e^{-A(0,\bar{t},\bar{t}')}\cdot\bar{t}'\cdot l_Q^{-d_h}=l_Q^{-d_\phi}\cdot\Theta(\bar{t}). \tag{350}$$

Thus,

$$\Theta(\bar{t})=\int_{-\infty}^{\bar{t}}d\bar{t}'\cdot\bar{t}'\cdot\exp\Big(-\int_{\bar{t}'}^{\bar{t}}d\bar{t}''\cdot\mathcal{M}^2(\bar{t}'')\Big). \tag{351}$$

The transverse two-point correlation function is equal to the connected part of the longitudinal two-point correlation function[34]. Thus, its scaling function is:

$$G_T(k,t,t_s)=l_Q^2\cdot\mathcal{G}_T(\bar{k},\bar{t})=l_Q^2\cdot 2\int_{-\infty}^{\bar{t}}d\bar{t}'\cdot\exp\Big(-2\int_{\bar{t}'}^{\bar{t}}d\bar{t}''\cdot(\bar{k}^2+\mathcal{M}^2(\bar{t}''))\Big). \tag{352}$$

Let us derive the scaling constraint-equation for the magnetic field protocol:

$$\mathcal{M}^2(\bar{t})\cdot l_Q^{-2}=\frac{u}{6}\Big(\int^{+\infty}\frac{d^d\bar{k}}{(2\pi)^d}l_Q^{-d}l_Q^2\cdot\Big[\mathcal{G}_T(\bar{k},\bar{t},\mathcal{M}^2)-\mathcal{G}_T(\bar{k},\bar{t},0)\Big]+\Theta^2(\bar{t})\cdot l_Q^{-2d_\phi}\Big); \tag{353}$$

$$\mathcal{M}^2(\bar{t})\cdot l_Q^{-2}\sim 0=l_Q^{-(d-2)}\cdot\frac{u}{6}\Big(\int^{+\infty}\frac{d^d\bar{k}}{(2\pi)^d}\Big[\mathcal{G}_T(\bar{k},\bar{t},\mathcal{M}^2)-\mathcal{G}_T(\bar{k},\bar{t},0)\Big]+\Theta^2(\bar{t})\Big);$$

We have set m^2 to zero because it is a subleading term in the scaling limit. The problem is formally solved in this case too: the scaling functions for the correlators are computed and the function $\mathcal{M}^2$ is defined by

$$\int^{+\infty}\frac{d^d\bar{k}}{(2\pi)^d}\cdot 2\int_{-\infty}^{\bar{t}}d\bar{t}'\cdot e^{-2\int_{\bar{t}'}^{\bar{t}}d\bar{t}''\cdot\bar{k}^2}\Big[\exp\Big(-2\int_{\bar{t}'}^{\bar{t}}d\bar{t}''\cdot\mathcal{M}^2(\bar{t}'')\Big)-1\Big]=$$
$$-\Big(\int_{-\infty}^{\bar{t}}d\bar{t}'\cdot\exp\Big(-\int_{\bar{t}'}^{\bar{t}}d\bar{t}''\cdot\mathcal{M}^2(\bar{t}'')\Big)\Big)^2 \tag{354}$$

This equation can be expressed in terms of the function f: the left hand side has already been computed. The magnetization, in terms of f, is:

$$\Theta^2(\bar{t},\mathcal{M}^2)=\Big(\int_{-\infty}^{\bar{t}}e^{-\int_{\bar{t}'}^{\bar{t}}(\mathcal{M}^2(\bar{t}'')d\bar{t}'')}\cdot\bar{t}'\cdot d\bar{t}'\Big)^2$$

$$=\Big(\int_{-\infty}^{\bar{t}}\Big[e^{-\int_0^{\bar{t}}(\mathcal{M}^2(\bar{t}'')d\bar{t}'')}\cdot e^{+\int_0^{\bar{t}'}(\mathcal{M}^2(\bar{t}'')d\bar{t}'')}\Big]\cdot\bar{t}'d\bar{t}'\Big)^2=\Big(\int_{-\infty}^{\bar{t}}\frac{\sqrt{f(\bar{t}')}}{\sqrt{f(\bar{t})}}\cdot\bar{t}'\cdot d\bar{t}'\Big)^2;$$

Thus, the scaling constraint equation becomes

$$\frac{2}{(4\pi)^{d/2}}\Big(\int_{-\infty}^{\bar{t}}d\bar{t}'\cdot\frac{f(\bar{t})-f(\bar{t}')}{|\bar{t}-\bar{t}'|^{d/2}}\Big)=\Big(\int_{-\infty}^{\bar{t}}\sqrt{f(\bar{t}')}\cdot\bar{t}'\cdot d\bar{t}'\Big)^2 \tag{355}$$

It might be a useful because the relation (354), in terms of f, becomes more compact. However, in this form, the scaling constraint-equation loses its physical meaning. Therefore we prefer to work with the relation

$$\int^{+\infty}\frac{d^d\bar{k}}{(2\pi)^d}\Big(\mathcal{G}_T(\bar{k},\bar{t},\mathcal{M}^2)-\mathcal{G}_T(\bar{k},\bar{t},0)\Big)=-\Theta^2(\bar{t},\mathcal{M}^2). \tag{356}$$

Unfortunately, the scaling constraint-equation for the magnetic field protocol does not have known solution.

[34] See (345) and (288).

5.5.3 A proof of the scaling relations.

We investigate the limit $t_s \to \infty$ keeping $\bar{t} = t/(t_s)^e$ and $\bar{k} = k(t_s)^{e/z}$ fixed with e, z free numbers. As in the thermal protocol, we demonstrate that a scaling behviour is allowed in the off-equilibrium regime only if e and z are fixed respectively to the KZ value and to the dynamical critical exponent of the system at the equilibrium.
We consider a generic leading order behaviour for the magnetic field $h(t, t_s) \approx (t/t_s)^a$ near $t \simeq 0$. The rescaling of magnetization is[35]

$$\Sigma(t, t_s) = \int_{-\infty}^{\bar{t}} d\bar{t}' \cdot (t_s)^e \cdot \exp\Big(- \int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot (t_s)^e \cdot \mathcal{M}^2(\bar{t}'') \cdot (t_s)^{-2e/z} \Big) \cdot \bar{t}'^a \cdot (t_s)^{a(e-1)} \quad (357)$$

The rescaling of the mass term $m^2(t, t_s) \sim \mathcal{M}^2(\bar{t}) \cdot (t_s)^{-2e/z}$ fixes the value of z as in the previous case:

$$-2e/z + e = 0 \Rightarrow z = 2$$

It follows that

$$\Sigma(t, t_s) \sim (t_s)^{(a+1)e-a} \cdot \Theta(\bar{t}) \quad (358)$$

The transverse two-point correlation function rescales as

$$G_T(k, t, t_s) = 2 \int_{-\infty}^{\bar{t}} d\bar{t}' \cdot (t_s)^e \cdot \exp\Big(-2 \int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot (\bar{k}^2 + \mathcal{M}^2(\bar{t}''))\Big) = (t_s)^e \cdot \mathcal{G}_T(\bar{k}, \bar{t}) \quad (359)$$

Performing the KZ scaling limit in the constraint-equation

$$(t_s)^{-e} \cdot \mathcal{M}^2(\bar{t}) = \frac{u}{6}\Big((t_s)^{2(a+1)e-2a} \cdot \Theta^2(\bar{t}) + \int^{+\infty} \frac{d^d\bar{k}}{(2\pi)^d} \cdot (t_s)^{-de/z} \cdot (t_s)^e \cdot \Big[\mathcal{G}_T(\bar{k}, \bar{t}, \mathcal{M}^2) - \mathcal{G}_T(\bar{k}, \bar{t}, 0)\Big]\Big) \quad (360)$$

The mass term is subleading. The scaling constraint-equation is satisfied only if

$$-\frac{d}{z}e + e = 2(a+1)e - 2a$$

Thus,

$$e = \frac{-2az}{(-d + z - 2z(a+1))} \quad (361)$$

if we set $a = 1$, $d = 3$ and we insert the previous result $z = 2$ we obtain $e = 4/9$. This result is in agreement with the definition:

$$t_Q = (t_s)^{a\nu_g z/(a\nu_g z+1)} \stackrel{?}{=} (t_s)^e = (t_s)^{-2az/(-d+z-2z(a+1))} \quad (362)$$

setting

$$\nu_g = \nu_h = \frac{1}{d_h} \quad (363)$$

5.5.4 Asymptotic behaviours.

For symmetry-breaking protocols like the magnetic field one, there are no reasons to think about coarsening phenomena asymptotically.
We assume that the system approaches the equilibrium also after the transition and therefore the ansatz (316):

$$\mathcal{M}^2(\bar{t}) \stackrel{\bar{t}\to\pm\infty}{\sim} |\bar{t}|^{2/d_h} \cdot \Big(1 + b(\bar{t}) \cdot e^{-c|\bar{t}|^{1+z/d_h}}\Big)$$

applies in both the asymptotic limit $\bar{t} \to \pm\infty$, maybe with different values of b, c. The exponent $1 + z/d_h$ is expected to be the same.
However, memory effects make us unable to investigate the limit $\bar{t} \to +\infty$ because it needs to

[35] We do not consider the term proportional to initial condition into the relations: for all values of e and z it is exponentially damped.

know a complete solution of the scaling constraint-equation (356): the analytical expressions for the correlators involve integrals which start from t_0 up to the current time t. When we try to compute the limit $\bar{t} \to +\infty$, we also consider instant of time $\bar{t}'$ around zero where the asymptotic ansatz for the scaling function $\mathcal{M}^2$ breaks down. We focus therefore on the asymptote $\bar{t} \to -\infty$ only.

Equilibrium asymptotic forms.

Firstly, we look to the equilibrium term of (316) setting $\mathcal{M}_o^2 = 0$. For the magnetic field protocol the equilibrium contribution to the mass term is given by

$$\mathcal{M}^2(\bar{t}) \simeq |\bar{t}|^{2/d_h} \tag{364}$$

The expression above give us the possibility to write explicitly the scaling functions of the correlators. In particular, using (364) in the scaling relation of the magnetization (351) [see app.D.5.1]

$$\Theta(\bar{t}) \sim \int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot \exp\left(-\int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot \mathcal{M}^2(\bar{t}'')\right) \underset{\mathcal{M}^2 \simeq \mathcal{M}_e^2}{\overset{\bar{t}\to-\infty}{\sim}} \int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot \exp\left(-\int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot |\bar{t}''|^{2/d_h}\right) \sim -|\bar{t}|^{d_\phi/d_h} \tag{365}$$

We learn that the equilibrium asymptotic contribution of magnetization for $\bar{t} \to -\infty$ is:

$$\Theta(\bar{t}) \sim -|\bar{t}|^{d_\phi/d_h} \tag{366}$$

the equilibrium scaling behaviour has been recovered. The result is in agreement with the well-known relations among the equilibrium critical exponents:

$$\beta = \nu_g \cdot d_\phi = \nu_h \cdot d_\phi = d_\phi/d_h. \tag{367}$$

The matching with the equilibrium scaling behaviour is consistent with the general predictions given by (266), where $\Delta = d_\phi$, $\nu_g = \nu_h = 1/d_h$ and $a = 1$.
If we consider the system at the equilibrium at all times, we can consider the limit $\bar{t} \to +\infty$ in the scaling function (351) using (364). After the computation, we obtain [see app. D.5.1]

$$\Theta(\bar{t}) \sim \int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot \exp\left(-\int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot \mathcal{M}^2(\bar{t}'')\right) \underset{\mathcal{M}^2 = \mathcal{M}_e^2}{\overset{\bar{t}\to+\infty}{\sim}} \int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot \exp\left(-\int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot |\bar{t}''|^{2/d_h}\right) \sim \bar{t}^{d_\phi/d_h} \tag{368}$$

In agreement with the equilibrium predictions as above. The scaling function of magnetization can be expressed in terms of the magnetic field:

$$\Sigma(t, t_s) \sim l_Q^{-d_\phi} \cdot \Theta(\bar{t}) = \Theta(\bar{t}) \cdot (l_Q^{-d_h})^{d_\phi/d_h} = \pm(\Theta(\bar{t}) \cdot |\bar{t}|^{-d_\phi/d_h}) \cdot |h|^{d_\phi/d_h} = \widetilde{\Theta}(\bar{t}) \cdot |h|^{d_\phi/d_h}. \tag{369}$$

The relation between the two scaling function is:

$$\Theta(\bar{t}) = \pm\widetilde{\Theta}(\bar{t}) \cdot |\bar{t}|^{d_\phi/d_h} \tag{370}$$

The new scaling function can be computed in the asymptotic limit $\bar{t} \to -\infty$ [see app. D.5.1]

$$\widetilde{\Theta}(\bar{t}) = -|\bar{t}|^{-d_\phi/d_h} \int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot \exp\Big(-\int_{\bar{t}'}^{\bar{t}} \mathcal{M}^2(\bar{t}'') \cdot d\bar{t}''\Big) \underset{\mathcal{M}^2 \simeq \mathcal{M}_e^2}{\overset{\bar{t}\to\pm\infty}{\sim}} -1. \tag{371}$$

and if we assume that the system is at the equilibrium at all times, $\widetilde{\Theta}(\bar{t}) \sim +1$ in the limit $\bar{t} \to +\infty$. Thus,

$$\Sigma(t, t_s) \sim \pm|h|^{d_\phi/d_h} \tag{372}$$

where we read the critical exponent[36]

$$\delta = d_h/d_\phi \tag{373}$$

[36]We never use this notation again because the symbol δ indicates the protocols.

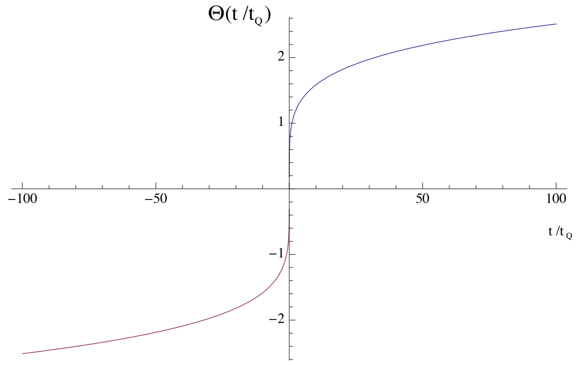

Figure 13: The equilibrium scaling function of the magnetization $\Theta(\bar{t}) \sim \pm|\bar{t}|^{d_\phi/d_h}$ as function of rescaled time $\bar{t}$ in three spatial dimensions $d = 3$.

which is the well-known scaling relation at the equilibrium. Let us consider also the transverse susceptibility:

$$\chi_T(t, t_s) = G_T(k = 0, t, t_s) \sim l_Q \cdot \mathcal{G}_T(\bar{t}). \tag{374}$$

The equilibrium result is already known: one may note that the transverse two-point correlation function (345) in the magnetic field protocol has the same analytical expression of the two-point correlation function of the thermal case (294). Thus, using another definition of the exponent ν_g, we conclude from (321) that

$$\mathcal{G}_T(\bar{t}) \sim |\bar{t}|^{-2\nu_h} = |\bar{t}|^{-2/d_h} \tag{375}$$

which is in agreement with the general predictions (269) and with the definition of the equilibrium critical exponents, as we have discussed in the thermal case.

Leading corrections to the asymptotic equilibrium scaling.

We consider the leading off-equilibrium corrections to the equilibrium scaling behaviour given by (316) plus (324):

$$\mathcal{M}^2(\bar{t}) \overset{\bar{t}\to-\infty}{\sim} |\bar{t}|^{2\nu_h}\Big(1 + b \cdot |\bar{t}|^a \cdot e^{-c|\bar{t}|^{1+z\nu_h}}\Big) \tag{376}$$

Within this ansatz, we investigate the first deviations of the correlation functions from their equilibrium behaviour. Let us start with the magnetization

$$\Theta(\bar{t}) = \int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot \exp\Big(-\int_{\bar{t}'}^{\bar{t}} \mathcal{M}^2(\bar{t}'') \cdot d\bar{t}''\Big) \overset{\bar{t}\to-\infty}{\sim}$$

$$\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot \exp\Big(-\int_{\bar{t}'}^{\bar{t}} \mathcal{M}_e^2(\bar{t}'') \cdot d\bar{t}''\Big) \cdot \Big(1 - \int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot \mathcal{M}_o^2(\bar{t}'')\Big) =$$

$$-|\bar{t}|^{d_\phi/d_h} - e^{|\bar{t}|^\kappa/\kappa}\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot e^{-|\bar{t}'|^\kappa/\kappa} \cdot \bar{t}' \cdot \Big(\int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot |\bar{t}''|^{2\nu_h} \cdot b \cdot |\bar{t}''|^a \cdot \exp(-c|\bar{t}''|^{1+z\nu_h})\Big); \tag{377}$$

where $\kappa = 1 + z \cdot \nu_h = 1 + 2/d_h$. The details of the computation are reported in the appendix D.5.2. The final result for the magnetization is

$$\Theta(\bar{t}) \sim -|\bar{t}|^{d_\phi/d_h}\Big(1 + K \cdot |\bar{t}|^a \cdot e^{-c|\bar{t}|^{1+z/d_h}}\Big). \tag{378}$$

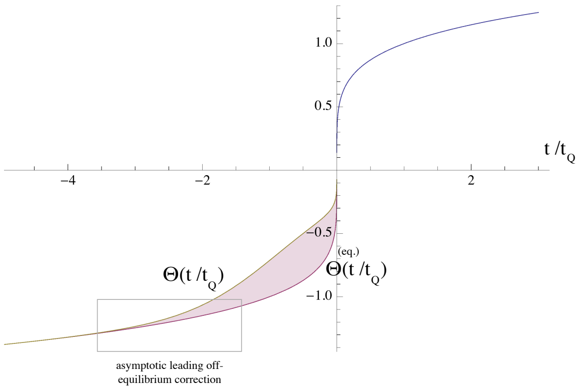

Figure 14: The leading off-equilibrium correction for the scaling function of the magnetization given by (378) with respect to the equilibrium behaviour. This plot shows qualitatively the deviations from the equilibrium: it has been made by fixing the parameters to $d = 3$, $c = \kappa^{-1} = 5/9$, $a = 0$, $b = 1$.

with $K = -b/(c\kappa + 1)$. The relation above is a remarkable result: if we assume that the scaling function of the mass term has an exponential approach to the equilibrium, the same type of approach is reproduced in the scaling function of the magnetization.
The mass term and the magnetization are linked at all times by the equation (279). If we take the KZ scaling limit of this relation:

$$\mathcal{M}^2(\bar{t}) = -\frac{d\Theta(\bar{t})/d\bar{t}}{\Theta(\bar{t})} + \frac{\bar{t}}{\Theta(\bar{t})} \tag{379}$$

We have derived a constraint between the scaling function of the mass term and the scaling function of the magnetization. In particular, in the asymptotic limit, the expression (316) and (378) must satisfies the equation above. Thus, the equation (379), in the limit $\bar{t} \to -\infty$, becomes [37]:

$$|\bar{t}|^{\kappa-1} + b|\bar{t}|^{\kappa-1+a} \cdot e^{-c|\bar{t}|^\kappa} \sim |\bar{t}|^{\kappa-1} - K|\bar{t}|^{\kappa-1+a} \cdot e^{-c|\bar{t}|^\kappa}$$
$$+(2-\kappa)|\bar{t}|^{-1} - K(2-\kappa)|\bar{t}|^{a-1} \cdot e^{-c|\bar{t}|^\kappa} + K(2-\kappa+a)|\bar{t}|^{a-1} \cdot e^{-c|\bar{t}|^\kappa} - c\kappa K|\bar{t}|^{\kappa-1+a} \cdot e^{-c|\bar{t}|^\kappa} + O(e^{-2c|\bar{t}|^\kappa});$$

The equilibrium terms satisfy the relation. The off-equilibrium leading terms are:

$$b|\bar{t}|^{\kappa-1+a} \cdot e^{-c|\bar{t}|^\kappa} \sim -K|\bar{t}|^{\kappa-1+a} \cdot e^{-c|\bar{t}|^\kappa} - c\kappa K|\bar{t}|^{\kappa-1+a} \cdot e^{-c|\bar{t}|^\kappa};$$

[37] The details of the computation are:

- $\bullet\ \mathcal{M}^2(\bar{t}) \sim |\bar{t}|^{2/d_h} \cdot \left(1 + b \cdot |\bar{t}|^a \cdot e^{-c|\bar{t}|^{1+z/d_h}}\right) = |\bar{t}|^{\kappa-1} + b|\bar{t}|^{\kappa-1+a} \cdot e^{-c|\bar{t}|^\kappa};$
- $\circ\ \Theta(\bar{t}) \sim -|\bar{t}|^{d_\phi/d_h} \cdot \left(1 + K|\bar{t}|^a \cdot e^{-c|\bar{t}|^{1+z/d_h}}\right) = -|\bar{t}|^{2-\kappa} - K|\bar{t}|^{2-\kappa+a} \cdot e^{-c|\bar{t}|^\kappa};$
- $\circ\ \Theta^{-1}(\bar{t}) \sim -|\bar{t}|^{\kappa-2} + K|\bar{t}|^{\kappa-2+a} \cdot e^{-c|\bar{t}|^\kappa};$
- $\bullet\ \bar{t}/\Theta(\bar{t}) \sim -|\bar{t}|/\Theta(\bar{t}) = |\bar{t}|^{\kappa-1} - K|\bar{t}|^{\kappa-1+a} \cdot e^{-c|\bar{t}|^\kappa};$
- $\circ\ \dot{\Theta}(\bar{t}) \sim (2-\kappa)|\bar{t}|^{1-\kappa} + K(2-\kappa+a)|\bar{t}|^{1-\kappa+a} \cdot e^{-c|\bar{t}|^\kappa} + K|\bar{t}|^{2-\kappa+a} \cdot (-c\kappa|\bar{t}|^{\kappa-1})e^{-c|\bar{t}|^\kappa};$
- $\bullet\ \dot{\Theta}(\bar{t}) \cdot \Theta^{-1}(\bar{t}) \sim -(2-\kappa)|\bar{t}|^{-1} + K(2-\kappa)|\bar{t}|^{a-1} \cdot e^{-c|\bar{t}|^\kappa} - K(2-\kappa+a)|\bar{t}|^{a-1} \cdot e^{-c|\bar{t}|^\kappa}$
  $+c\kappa K|\bar{t}|^{-c|\bar{t}|^\kappa} + O(e^{-2c|\bar{t}|^\kappa});$

The equation is satisfied if $K = -b/(c\kappa + 1)$, in agreement with the results.

We consider also the leading off-equilibrium correction of the asymptotic equilibrium behaviour in the transverse susceptibility. In this case we can read the final result in the relation (326) using $\nu_g = \nu_h = 1/d_h$

$$\mathcal{G}_T(\bar{t}) \sim |\bar{t}|^{-2/d_h}\Big(1 + K'|\bar{t}|^a \cdot e^{-c|\bar{t}|^{1+z/d_h}}\Big). \tag{380}$$

where $K' = -2b/(2 + c\kappa)$ and $\kappa = 1 + z\nu_h = 1 + z/d_h$. The exponential approach is found also in the transverse susceptibility. The mass term and the transverse two-point correlation function are linked by the relation (280) at all times. In the KZ scaling limit:

$$\mathcal{M}^2(\bar{t}) = -\frac{1}{2}\frac{d\mathcal{G}_T(\bar{t})/d\bar{t}}{\mathcal{G}_T(\bar{t})} + \frac{1}{\mathcal{G}_T(\bar{t})}; \tag{381}$$

The consistence of (380) in the asymptotic limit follows from the result of Sec.5.4.4 using $\nu_g = \nu_h = 1/d_h$, instead of ν.

A comment.

The previous results for the magnetization and for the transverse susceptibility cannot be used as a demonstration of the ansatz (316) because (379) and (381) are equivalent to the definition of the correlators (351) and (352). Thus, they give only a check of consistence. The ansatz has to be validated in the scaling constraint-equation. We recall the scaling constraint-equation (356) for the magnetic field protocol

$$\int^{+\infty} \frac{d^d\bar{k}}{(2\pi)^d}\Big(\mathcal{G}_T(\bar{k},\bar{t},\mathcal{M}^2) - \mathcal{G}_T(\bar{k},\bar{t},0)\Big) = -\Theta^2(\bar{t},\mathcal{M}^2).$$

The integration over the momenta makes the relation above not writable in terms of special functions[38]. Thus, we are not able to proceed further. However, a different ansatz for $\mathcal{M}^2$ does not lead to consistence among the relations. Let us consider, for instance

$$\mathcal{M}^2(\bar{t}) \overset{\bar{t}\to\pm\infty}{\sim} |\bar{t}|^{2\nu_h}\Big(1 + b/|\bar{t}|^{\alpha}\Big) \tag{382}$$

with $\alpha > 0$ big enough. The asymptotic scaling function of the magnetization, in the limit $\bar{t} \to -\infty$, using (382), is [see app.D.5.3]

$$\Theta(\bar{t}) \sim |\bar{t}|^{d_\phi/d_h}; \tag{383}$$

The off-equilibrium leading corrections cancels themselves. The equation (379) becomes:

$$|\bar{t}|^{\kappa-1} + b|\bar{t}|^{\kappa-1-\alpha} \sim (2-\kappa)|\bar{t}|^{-1} + |\bar{t}|^{\kappa-1};$$

The equilibrium term is satisfied but the off-equilibrium leading terms do not satisfy the relation above.

The assumption of the exponential approach to the equilibrium for the mass term ensures consistence. In addition, the ansatz (316) has the remarkable property to reproduces itself in the correlation functions.
The approach to the equilibrium for the scaling function of the correlators, has been numerically studied by [6] in the O(3) vector-model of finite size L. The numerical simulation shows that the behaviour of the correlators, and in particular of the magnetization, tends to the equilibrium exponentially. It is expected that qualitatively similiar behaviours are shown by O(N) vector-models at finite $N = 3$. Thus, a scaling relation for the magnetization can be

[38] For more details, see the appendix D.5.3

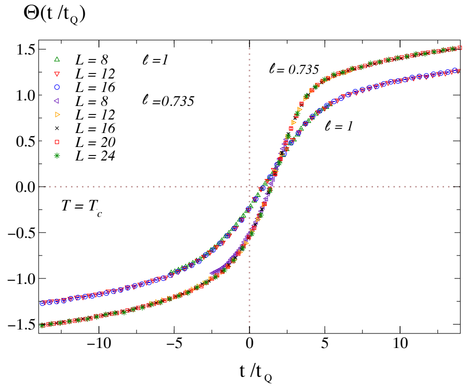

Figure 15: The off-equilibrium dynamics of the scaling function of the magnetization $\Theta(\bar{t})$ driven by the magnetic field protocol (229) for different values of $\ell = L/l_Q$. The numerical simulations have been made for an Heisenberg ferromagnet $N = 3$ in three spatial dimensions. The system has a cubic finite-size $V = L^3$. This picture has been taken from the ref. [6].

estimated starting from (351) and using the exponents of the Heinsenberg universality class [39]

$$\Sigma^{(N=3)}(t, t_s, L) \sim l_Q^{-d_\phi} \cdot \Theta^{(N=3)}(\bar{t}, \ell) \simeq (t_s)^{-0.11} \cdot \Theta^{(N=3)}(\bar{t}, \ell) \tag{384}$$

The behaviour of the scaling funtion $\Theta^{(N=3)}(\bar{t}, \ell)$ is shown in Fig.15. One may note that the asymptotic behaviour of the scaling function does not depend on the size of the system and matches the infinite-volume equilibrium scaling, as it has been discussed in Sec.4.5.

[39]For an Heisenberg ferromagnet $d_\phi = (1+\eta)/2$, $d_h = (5-\eta)/2$, in three spatial dimensions. The value of the critical exponent is $\eta \simeq 0.03(7)$ from the ref. [18],[19].
It follows that $d_\phi \simeq 0.52$, $d_h \simeq 2.3$, and $z \simeq 2.02$ [see ref. [6]]. Thus the exponent $e = (z/d_h)/((z/d_h)+1) \simeq 0.45$: the Kibble-Zurek scales are $t_Q \simeq (t_s)^{0.45}$ and $l_Q \simeq (t_s)^{0.22}$.

5.5.5 Hysteresis phenomena.

We define the *hysteresis loop area* $\mathcal{A}$ as the area beetween the two cuves described by the magnetization $\Sigma(t,t_s)$ going from $t_i = -\infty$ to $t_f = +\infty$ and coming back (round-trip protocol γ) when the dynamics of the system is driven by the magnetic field protocol $\delta(t,t_s) = h(t,t_s) \approx t/t_s$ [see ref. [6]]

$$\mathcal{A} = \oint_\gamma dt \cdot \Sigma(t,t_s) \tag{385}$$

We note that the magnetization (342) has a symmetry with respect to a reflection of the magnetific field: if we reverse the direction of the magnetic field $h \mapsto -h$, it follows

$$\Sigma^{\text{inv.}}(t,t_s) = -\Sigma(-t,t_s); \tag{386}$$

The vaue of the magnetization with reversed time is:

$$\Sigma(-t,t_s) = \int_{+\infty}^{-t} dt' \cdot \exp\Big(-\int_{t'}^{-t} dt'' \cdot m^2(t'',t_s)\Big) \cdot t'/t_s = \int_{-\infty}^{t} dt' \cdot \exp\Big(+\int_{t'}^{t} dt'' \cdot m^2(t'';t_s)\Big) \cdot t'/t_s. \tag{387}$$

Thus, the hysteresis loop area can be also written as

$$\mathcal{A} = \oint_\gamma dt \cdot \Sigma(t,t_s) = \int_{t_i=-\infty}^{t_f=+\infty} dt \cdot \Big(\Sigma(t,t_s) + \Sigma(-t,t_s)\Big) \tag{388}$$

because the integral of the magnetization over γ is equivalent to an integral over a unique trip from $t_i = -\infty$ to $t_f = +\infty$ considering the magnetization, given by (342), plus the magnetization with reversed time.

The hysteresis loop area can be easly connected with the magnetic work $\mathcal{W}$ which the system performs over γ

$$\mathcal{W} = \oint_\gamma dh(t,t_s) \cdot \Sigma(t,t_s) = t_s^{-1} \cdot \mathcal{A}. \tag{389}$$

Therefore the hysteresis loop area has a direct physical meaning. Let us explicitly compute its expression:

$$\mathcal{A} = \int_{-\infty}^{+\infty} dt \cdot \int_{-\infty}^{t} dt' \cdot (t'/t_s) \cdot \Big[\exp\Big(-\int_{t'}^{t} m^2(t'';t_s) \cdot dt''\Big) + \exp\Big(+\int_{t'}^{t} m^2(t'';t_s) \cdot dt''\Big)\Big] =$$

$$= \int_{-\infty}^{+\infty} dt \cdot \int_{-\infty}^{t} dt' \cdot (t'/t_s) \cdot 2\cosh\Big(\int_{t'}^{t} m^2(t'',t_s) \cdot dt''\Big).$$

In the KZ scaling limit

$$\mathcal{A} = \int_{-\infty}^{+\infty} d\bar{t} \cdot t_Q \cdot \int_{-\infty}^{\bar{t}} d\bar{t}' \cdot t_Q \cdot \bar{t}' \cdot l_Q^{-d_h} \cdot 2\cosh\Big(\int_{\bar{t}'}^{\bar{t}} \mathcal{M}^2(\bar{t}'') \cdot l_Q^{-2} \cdot d\bar{t}'' \cdot t_Q\Big) \tag{390}$$

Thus, the scaling relation of the hysteresis loop area is

$$\mathcal{A} \sim l_Q^{4-d_h} \cdot \Xi = l_Q^{2-d_\phi} \cdot \Xi. \tag{391}$$

The amplitude Ξ of hysteresis loop area is a constant which depends on the scaling function $\mathcal{M}^2$:

$$\Xi = 2\int_{-\infty}^{+\infty} d\bar{t} \cdot \int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot \cosh\Big(\int_{\bar{t}'}^{\bar{t}} \mathcal{M}^2(\bar{t}'') \cdot d\bar{t}''\Big); \tag{392}$$

It follows that the magnetic work has a scaling relation:

$$\mathcal{W} \sim (t_s)^{-2/3} \cdot \Xi \tag{393}$$

in three spatial dimensions. This means that the energy spent by the system in a cycle decrease if the time-scale of the variation of the magnetic field occur very-slow. In the limit

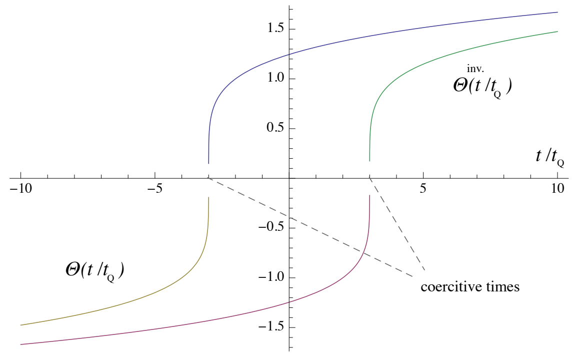

Figure 16: Qualitatively picture of the hysteresis loop area which the spin system presents at $T = T_c$ as effects of an external perturbation $h(t, t_s) \approx t/t_s$.

of quasi-adiabatic protocol $t_s \to \infty$, the magnetic work becomes zero. We note that the hysteresis is an off-equilibrium phenomenon: since the system is at the equilibrium[40]:

$$\mathcal{A} = \oint dt \cdot \Sigma^{\text{eq.}}(t, t_s) \sim l_Q^{2-d_\phi} \int_{-\infty}^{+\infty} d\bar{t} \cdot \Big(|\bar{t}|^{d_\phi/d_h} - |\bar{t}|^{d_\phi/d_h} \Big) = 0; \tag{394}$$

The central area is shrinked to a single line as fig.13 shows. The system does not spend energy over the cycle.

The hysteresis loop area differs from zero when the time-evolution of the system is characterized also by off-equilibrium regimes: when the system is coupled to a time-dependent magnetic field, it needs a finite time to readjust itself and adapt to the external perturbation. Therefore the magnetization crosses the zero at a certain time $\bar{t}_c \neq 0$ that we call *coercitive time* and it corresponds to a value of h called *coercitive magnetic field*. A qualitatively picture of the hysteresis loop area is shown in fig.16.

Since we do not know the behaviour of $\mathcal{M}^2$ at all times, we are not able to compute the coercitive times and the integral of the hysteresis. However, one may note that the asymptotic exponential approach to the equilibrium of the scaling function of the magnetization (378) makes the amplitude Ξ finite. Since the tails of the magnetization are exponentially damped and since the $\mathbb{Z}_2$ symmetry of the magnetization makes the hysteresis loop area a closed curve, we can conclude from the ansatz (316) that

$$\Xi = \int_{-\infty}^{+\infty} d\bar{t} \cdot \Big(\Theta(\bar{t}) + \Theta(-\bar{t}) \Big) < \infty \tag{395}$$

Physically speaking, the variations of the external magnetic field tend to modify the value of the magnetization of the system. Because of the memory effects, the value of the magnetization is not instantaneously modified and the system develops a metastable state for a time $\bar{t}_c$ before that it agrees with the reservoir. When the external pertubation becomes large, the system presents a saturation and does not respond to the magnetic field anymore.

Hysteresis phenomena have been numerically studied in O(3) vector-model with a finite size L by [6]. Using the Heisenberg universality class critical exponents in $d = 3$, we find a scaling relation:

$$\mathcal{A} \sim l_Q^{z-d_\phi} \cdot \Xi^{(N=3)} \sim (t_s)^{0.33} \cdot \Xi^{(N=3)} \tag{396}$$

[40] We use the expression for the scaling function of the magnetization at the equilibrium (366)

$$\Theta^{\text{eq.}}(\bar{t}) = \pm|\bar{t}|^{d_\phi/d_h}$$

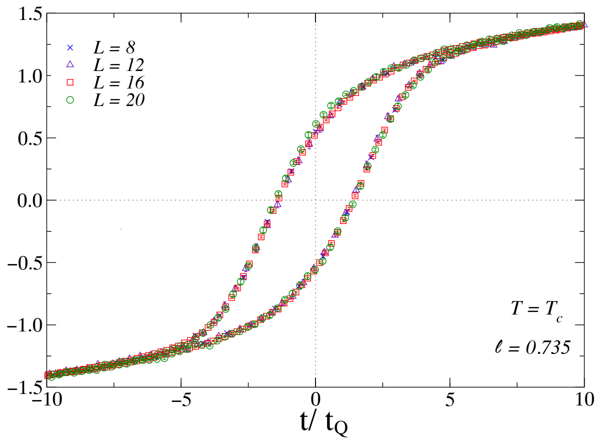

Figure 17: The hysteresis loop area for an Heisenberg ferromagnet $N = 3$ of finite-size system with cubic shape in three spatial dimension. The numerical simulations has been done with $\ell = L/l_Q$ fixed. The round-trip protocol moves from $t_i < 0$ to $t_f > 0$ at the critical temperature along the magnetic field $\delta(t, t_s) = h(t, t_s) = t/t_s$. Then, it comes back from t_f to t_i. This picture has been taken from the ref.[6].

The rescaled hysteresis loop area $\Xi^{(N=3)}$ is reported in fig.17. It is expected that the hysteresis loop area at large N is qualitatively similiar to the case $N = 3$ and that the finite-size does not modify the asymptotic tails because the matching with the equilibrium occur always with the infinite-volume behaviour.
The discussion above predicts that the magnetic work done by the system in a round-trip protocol for an O(3) Heisenberg ferromagnet in $d = 3$ scales as

$$\mathcal{W} \sim (t_s)^{-0.66} \cdot \Xi^{(N=3)}. \tag{397}$$

6 First-order transition in the low-temperature phase.

In the last sections we want to show how off-equilibrium behaviours emerge crossing the first-order transitions. We investigate the low-temperature phase at fixed temperature $T < T_c$ in which a transition is driven by magnetic field around $h = 0$. Firstly, we construct a scaling theory for the first-order transition using the renormalization group. Then, we investigate the effects of a relaxational dynamics in a finite-size spin system below the critical temperature and extract the dynamical exponents.

6.1 Renormalization approach to first-order transitions.

In the RG approach to statistical physics, the singularities of thermodynamic functions associated with phase transitions are located on critical surfaces determined by the fixed points of the renormalization transformation. The properties of the fixed points which give rise to continuous phase transitions and critical phenomena have been already discussed in Sec.1. The corresponding properties associated with first-order phase transitions, i.e. discontinuous change in an order parameter are shown in the following [see ref. [23], [24], [25] and [26]].

Let us consider the hamiltionian functional $\mathcal{H}(\phi)$ of O(N) symmetric spin-sytems. It depends on a set of external fields $\{h_\alpha\} = \{h, K_\alpha\}$ and h is the field conjugate to the order parameter σ. A first-order phase transition for temperatures T below the critical temperature T_c, can be described by a discontinuity in σ as a function of h which can be taken to occur at $h = 0$.
We assume that exsists an RG transformation for $\mathcal{H}$ such that the fields $\{h, K_\alpha\}$ are mapped into new fields $h'(h, K)$ and $K'_\alpha(h, K)$ which are analytic functions of the original fields $\{h, K_\alpha\}$. Note that this transformation has to preserve the origin and the sign near the origin of the field h[41]. Thus, the new fields must satisfy $h'(h = 0, K) = 0$ while $\partial h'(0, K)/\partial h \neq 0$. As we have seen in Sec.1 the RG approach to the critical behavior of spin-systems is determined by a fixed point $\{h, K_\alpha\} = \{0, K^*_\alpha\}$, with relevant eigenvalues associated to the ordering field h, $\ell_h = \partial h'(0, K^*)/\partial h$ and to the other fields[42] which can be obtained by diagonalizing the matix $T_{\alpha\beta} = \partial K'_\alpha(0, K^*)/\partial K_\beta$. From these eigenvalues of the RG flow near the fixed point all the stuff about the critical behaviour (critical exponents, scaling behaviour) can be determined.

We want to state some conditions which are sufficient for the occurence of a discontinuity in the order parameter across the first-order transition:

• There exists another fixed point for the spin-system at $h = 0$ and $K_\alpha = K^{**}_\alpha$ such that the configurations $\{0, K_\alpha\}$ of the system which belong to a domain $\mathcal{D}$ (corresponding to $T < T_c$) bounded on one the side of the critical surface, are mapped by many iterations of the RG transformation into this fixed point. We refer to this fixed point as *discontinuous fixed point.*

• At this fixed point, the eigenvalue $\ell_h = \partial h'(0, K^{**})/\partial h$ associated with the ordering field h is given by L^d where L^d is the change in the scale of volume under the renormalization transformations[43].

• The discontinuity of the order parameter $\Delta\sigma(K)$[44] does not vanish when we consider the limit $K \mapsto K^{**}$ i.e. $\Delta\sigma^{**} = \Delta\sigma(K^{**}) \neq 0$.

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[41] It determines the discontinuous fixed point.

[42] For the underlying model the relevant operator are magnetic field and temperature only.

[43] In the first-order transitions the size of the system is very important. We will recover the infinite-volume limit but we begin with a finite-size spin-system.

[44] It can be defined by considering the equilibrium value in the low temperature phase given by (177). Since the magnetic field reverse the direction of the order parameter, $\Delta\sigma(T) = \sigma^{(+)}(T) - \sigma^{(-)}(T) = 2\sigma(T)$.

We prove that the second condition is necessary for a discontinuity $\Delta\sigma$ at $h = 0$ $K_\alpha \in \mathcal{D}$ Let us consider the thermodynamic potential $\mathcal{F}(\sigma, h, K)$[45]: it is well known that it scales with the volume of the system [see for instance [14]]:

$$\mathcal{F}(\sigma, h, K) \sim g_L(\sigma, h, K) + \frac{1}{L^d} \cdot \mathcal{F}(\sigma, h', K'). \tag{398}$$

The order parameter can be obtained by

$$\sigma(K) = \frac{\partial \mathcal{F}(\sigma, h, K)}{\partial h} \tag{399}$$

and

$$\sigma^{(\pm)}(K) = \lim_{h \to 0^\pm} \frac{\partial \mathcal{F}(\sigma, h, K)}{\partial h} \tag{400}$$

Thus,

$$\Delta\sigma(K) = R(K) \cdot \Delta\sigma(K') \tag{401}$$

where $R(K) = L^{-1} \partial h'(h, K)/\partial h\Big|_{h=0}$. Since the second condition is satisfied, $R(K^{**}) = 1$ and therefore $\Delta\sigma^{**} \neq 0$ after many iterations. The second condition also implies that the non-scaling part of the thermodynamic potential g_L has a behaviour $\partial g_L(0, K^{**})/\partial h = 0$ and this makes $\sigma^{(\pm)}$ not logharitmically divergent at the discontinuous fixed point.
We consider the effects of many iterations of the RG transformation. From the last equation (401) we obtain:

$$\Delta\sigma(K) = \prod_{n=0}^{\infty} R(K^{(n)}) \cdot \Delta\sigma^{**} \tag{402}$$

where $K_\alpha^{(n)}$ is the mapping of the field K_α after n iterations of the RG transformations and $K_\alpha^{(0)} = K_\alpha$. A sufficient condition to have a finite product of the $R(K^{(n)})$ is:

$$\rho = \lim_{n \to \infty} \frac{R(K^{(n+1)}) - 1}{R(K^{(n)}) - 1} < 1;$$

Since the critical value of the fields K_α^{**} are finite[46], ρ is equal to the largest eigenvalue ℓ' associated to the external fields close to the discontinuous fixed point. If the system $\in \mathcal{D}$, we expect that successive RG mappings drive the system closer to the discontinuous fixed point i.e. it is a stable fixed point. Thus, we have necessary $\ell' < 1$. If the system $\notin \mathcal{D}$ i.e. $T > T_c$, it is mapped toward a different fixed point. In general, it is expected that $R < 1$ there so there is no discontinuity in the order parameter.

6.2 Scaling hypotesis for first-order transitions.

From the previous discussion about RG approach to first-order transitions, a scaling behaviour of the observables follows. Our purpose is to extract the scaling dimensions of the fields and the power-law exponents[47] related to the discontinuous fixed point. In the following we proceed making a scaling hypotesis on the behaviour of the order parameter. Then, we check the consistence with the RG formalism above.
Let us consider a first-order transition in which the system switches from one non-critical phase, with finite correlation length, to another non-critical phase. This is the case of a spin-system below the critical temperature [see fig. 18].

We postulate that the phenomenological power-law used for the critical phase transitions is still valid for the discontinuous fixed point [see ref.[25]]:

$$\sigma \sim \pm a|h|^{1/\delta} \tag{403}$$

[45] see Sec. 1.2.8.
[46] A similiar proof for K_α^{**} infinite can be found in the work [24].
[47] They are the equivalent of the critical exponent for a first-order phase transition,

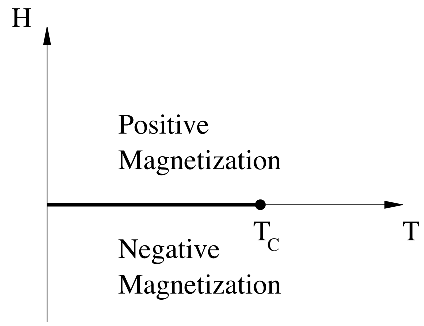

Figure 18: The phase diagram of a ferromagnetic spin-system. This picture has been taken from the ref. [19].

A first-order transition is described by $\delta \to \infty$ [48] which leads to a discontinuity in the behaviour of σ at the fixed point. It follows that $a = |\sigma|$.
Let us connect the scaling hypotesis with the RG formalism: a generic RG transformation maps the ordering field [49]:

$$h \mapsto h' = \ell_h \cdot h = \lambda^{d_h} \cdot h; \tag{404}$$

and for the scaling part of the thermodynamic potential:

$$\mathcal{F}(h) \mapsto \mathcal{F}(h') = \lambda^{-d} \cdot \mathcal{F}(h) \tag{405}$$

where we have neglected the dependece of $\mathcal{F}$ on other relevant operators K_α because we are considering a spin-system at fixed temperature $T < T_c$ [50]. We fix the scale of the RG transformation as

$$\lambda = |h|^{-1/d_h} \tag{406}$$

and therefore the thermodynamic potential scales as

$$\mathcal{F}(h') = F_\pm \cdot |h|^{d/d_h} \tag{407}$$

where $F_\pm = \mathcal{F}(\pm 1)$. Since the magnetization is given by (399), we obtain from the scaling hypotesis (403) the relation:

$$d/d_h = 1 - (1/\delta); \tag{408}$$

but since a first-order transition is characterized by $\delta \to \infty$, we conclude that the scaling dimension of the magntic field is $d_h = d$. This result is in agreement with the assumption made before to construct the discontinuous fixed point. From the general result $\delta = d_h/d_\phi = \infty$ we can conclude, since $d_h = d$, that $d_\phi = 0$ [see ref.[25]].

Let us consider the equilibrium correlation length ξ for the system below the critical temperature. Since the temperature is fixed, ξ is finite and the theory is not critical. When we make an RG transformation in the neighbourhood of a discontinuous fixed point we expect that

$$\xi \mapsto \xi' = \lambda^{-1} \cdot \xi(h') \tag{409}$$

Using the relation (406) we obtain

$$\xi' = A_\pm \cdot |h|^{-1/d} \tag{410}$$

with $A_\pm = \xi(\pm 1)$. Thus, when the system undergoes a first-order transition $h \to 0$, the correlation length diverges like

$$\xi \sim |h|^{-\nu_h} \Rightarrow \nu_h = \frac{1}{d_h} = \frac{1}{d}; \tag{411}$$

[48] $\delta = d_\phi/d_h$ is the critical exponent extended to the first-order transition. Then, we consider the protocols.
[49] Note that $\lambda^{d_h} = \ell_h$. Thus, we already know from the previous subsection that $d_h = d$. We check this result in order to check the scaling hypotesis too.
[50] In addition to the ordering field, only the temperature is relevant but we keep it fixed

However, the relation above must be interpreted. We know that the ordered phases connected through the first-order transition are both non-critical, so what does the last result mean?
Physically speaking, the system below the critical temperature is in the ordered phase where the spins are aligned with the external field. When $h \to 0$, we consider the phase transition between two different ordered phases which coexists at the discontinuous fixed point. Let us consider the longitudinal two-point correlation function: it is composed by a connected part $W^{(2)}(p)$ plus the square of the magnetization σ^2 (non-connected). The connected part defines the correlation length of the system and remains finite. Infact, when we consider $< \phi(0)_1 \phi(x)_1 >$ with $|x| \to \infty$, the two-point correlation function approaches the value of the square of the spontaneous magnetization. Thus, $W^{(2)} \to 0$ which means that spins are connected over a finite length.
But, when the system undergoes the first-order transition, a long-range order arises because the system does not distinguish the ordered phases anymore. Even if the correlation length is still finite, ξ can be interpreted as *coherence* or *persistence length* for the coexisting phases. This interpretation is also quite consistent with the standard hyperscaling relations [see for instance ref.[1]] $2-\eta = d(\delta-1)/(\delta+1) = 2/\nu_h - d$ which, when $\delta \to \infty$ yield the decay exponent $d_\phi = d - 2 + \eta = 0$ corresponding to no decay, i.e. to long-range order.

6.3 Relaxational dynamics at $T < T_c$.

We derive the dynamical exponents of the O(N) vector model below the critical temperature by following the discussion of the ref. [6]. We assume that the dynamics of the fields occurs through a purely dissipative Langevin equation. It is useful to consider a system of finite-size L with a cubic shape $V = L^d$[51] in order to extract the dynamical critical exponent z. For simplicity, we investigate the case $N = 2$. However, as will be more clear later[52], the results hold for any $N \geq 2$. The equation of motion, in the absence of external magnetic fields, is:

$$\frac{\partial \phi(x,t)}{\partial t} = -(-\nabla^2 + r + \frac{u}{6}|\phi|^2) \cdot \phi(x,t) + \varsigma(x,t); \tag{412}$$

where $\phi(x,t)$ is a two-real-components field and $\varsigma(x,t)$ is a random variable with white gaussian distribution. In the low-temperature phase, the spontaneous magnetization is given by [see eq. (190) of Sec.3.2.2.]:

$$\sigma = \sqrt{-\frac{6r}{u}} > 0;$$

Thus, the field can be represented as a one-component complex field:

$$\phi(x,t) = \sigma \cdot (1 + \varrho(x,t)) \cdot e^{i\theta(x,t)}, \tag{413}$$

where ϱ is a real field which takes into account the possibility of radial fluctuation of $|\phi|$. Let us insert the parametrization into the equation of motion:

$$i\frac{\partial}{\partial t}\theta(x,t) + \frac{\partial}{\partial t}\varrho(x,t) = \tag{414}$$

$$\Big(e^{-i\theta}\nabla^2\{e^{i\theta(x,t)}(1+\varrho(x,t))\} - |r|\varrho(x,t)(1+\varrho(x,t))\cdot(2+\varrho(x,t))\Big) + e^{-i\theta(x,t)}\varsigma(x,t)/\sigma;$$

Separating real and imaginary parts we obtain:

$$\frac{\partial}{\partial t}\theta(x,t)(1+\varrho(x,t)) = \Big(\nabla^2\theta(x,t)(1+\varrho(x,t)) + 2\nabla\theta(x,t)\cdot\nabla\varrho(x,t)\Big) + \varsigma_\theta(x,t); \tag{415}$$

[51] The shape of the system determines the dynamic scaling of the system at first-order transitions, See Ref.[6]. We consider a cubic shape in order to take, then, the infinite-volume limit.

[52] The mainly dynamics occurs into a plane for every $N \geq 2$. Thus, the O(N) vector-model, even in the limit of large N, can be reduced to a planar model. See 462.

and

$$\frac{\partial}{\partial t}\varrho(x,t) = -\Big((\nabla\theta(x,t))^2(1+\varrho(x,t))-\nabla^2\varrho(x,t)+|r|\varrho(x,t)(1+\varrho(x,t))\cdot(2+\varrho(x,t))\Big)+\varsigma_\varrho(x,t); \tag{416}$$

We have ridefined the distribution of the noise for the phase degree of fredoom ς_θ according to

$$P(\varsigma_\theta) \propto \int \mathcal{D}\varsigma_\theta \cdot \exp\Big(-\frac{\sigma^2}{2\Omega}\int dt \int d^dx \cdot |\varsigma_\theta(x,t)|^2\Big). \tag{417}$$

The variance of the new noise distribution is rescaled: $\Omega = 2 \mapsto 2/\sigma^2$, the mean value is zero. The radial fluctuations are affected by the noise ς_ϱ which has the same distribution of ς_θ. We focus on the phase degree of fredoom θ under the following approximations:

-The radial fluctuations are very small and therefore we neglect them: $(1+\varrho) \approx 1$.

-ϱ and θ spatial variations are uncorrelated i.e. $<\nabla\theta\cdot\nabla\varrho>_\varsigma = 0$.

These approximations decouple the radial fluctuations of the modulus from its angular precession. Thus, the previous equation results simplified:

$$\frac{\partial}{\partial t}\theta(x,t) = \nabla^2\theta(x,t) + \varsigma_\theta(x,t); \tag{418}$$

in the Fourier space it becomes:

$$\frac{\partial}{\partial t}\theta(k,t) = -k^2\theta(k,t) + \varsigma_\theta(k,t); \tag{419}$$

Note that in a finite-size system the Fourier transform of the field is given by:

$$\theta(k,t) = \int_{L^d} d^dx \cdot e^{ik\cdot x} \cdot \theta(x,t) \tag{420}$$

and the inverse

$$\theta(x,t) = \frac{1}{L^d}\sum_k e^{-ik\cdot x} \cdot \theta(k,t). \tag{421}$$

The distribution of the noise in the Fourier space is

$$P(\varsigma_\theta) \propto \int \mathcal{D}\varsigma_\theta \cdot \exp\Big(-\frac{\sigma^2}{2V\Omega}\int dt \cdot \sum_k |\varsigma_\theta(k,t)|^2\Big). \tag{422}$$

The solution of the equation of motion (419) is:

$$\theta(k,t) = e^{-k^2t}\int_0^t dt' \cdot e^{k^2t'} \cdot \varsigma_\theta(k,t'). \tag{423}$$

where we have assumed $t_0 = 0$[53], $\theta(k,t_0) = 0$ without loss of generality. Let us consider the spatial average angle of the system:

$$\widetilde{\theta}(t) = \frac{1}{L^d}\int d^dx \cdot \theta(x,t); \tag{424}$$

It follows that $<\widetilde{\theta}(t)>_{\varsigma_\theta} = 0$ and the two-point correlation function is [see ref.[6]]:

$$<\widetilde{\theta}(t)\widetilde{\theta}(s)>_{\varsigma_\theta} = \frac{1}{\sigma^2 L^d}(t+s-|t-s|); \tag{425}$$

[53]Since the system is at the equilibrium, the initial time can be chosen arbitrarily.

The average angle changes over a time-scale of the order of the volume $\sim L^d$.
Let us now consider the two-point correlation function of single fields not spatially averaged:

$$< \theta(k,t)\theta(k,s) >_{\varsigma_\theta} = \frac{L^d}{\sigma^2 k^2}\Big(e^{-k^2|t-s|} - e^{-k^2(t+s)}\Big) \tag{426}$$

We consider a "wall quantity" in three spatial dimensions $x = (x_1, x_2, x_3)$ defined as

$$\delta\theta_T(x_3,t) = \frac{1}{L^2}\int_{L^2} dx_1 \cdot dx_2 \cdot \Big[\theta(x,t) - \widetilde{\theta}(t)\Big] \tag{427}$$

The two-point function of the previous quantity is:

$$< \delta\theta_T(x_3,t)\delta\theta_T(x_3',s) >_{\varsigma} = \tag{428}$$

$$\frac{1}{\sigma^2 L^d}\sum_{k_3\neq 0}\frac{e^{-k_3^2|t-s|}}{k_3^2}\cdot e^{ik_3(x_3-x_3')} = \frac{1}{4\pi^2\sigma^2 L^2}\sum_{n\neq 0}\frac{e^{-4\pi^2|t-s|/L^2}}{n^2}\cdot e^{i2\pi n(x_3-x_3')/L}$$

where in the last computation the two-point correlation function for the wall quantity has been written in the limit $t, s \to \infty$ helding $|t-s|$ fixed, where the system is at the equilibrium. This fluctuations are small and have a typical time scale of the order of L^2.

We compute the autocorrelation function for the magnetization:

$$\frac{1}{L^{2d}}\int d^dx \cdot d^dx' \cdot < \phi(x,t)\phi(x',s) >_{\varsigma} = \frac{\sigma^2}{L^{2d}}\int d^dx \cdot d^dx' \cdot < e^{i(\theta(x,t)-\theta(x',s))} >_{\varsigma}; \tag{429}$$

the angle $\theta(x,t) \sim \widetilde{\theta}(t) + O(1/\sqrt{L})$, thus we approximate the last relation:

$$\approx \sigma^2 \cdot < \exp\Big(i\widetilde{\theta}(t) - i\widetilde{\theta}(s)\Big) >_{\varsigma} = \sigma^2 e^{-\frac{1}{2}<(\widetilde{\theta}(t)-\widetilde{\theta}(s))^2>_{\varsigma}} \tag{430}$$

$$= \sigma^2 \cdot \exp\Big(-\frac{1}{L^d\sigma^2}|t-s|\Big);$$

The time-scale of the autocorrelation is $\tau \sim L^d$. We have considered a spin system without magnetic fields: in this case the magnetization has fixed modulus but there are no constraint on its direction. The random orientation of the vector magnetization into the space is expected to be the slowest dynamics of the system and has a dynamical exponent $z = d$ because requires a variation in the enteire volume of the system. Note that there is also a motion in the transverse planes due to the spin-waves with a time-scale $z = 2$. However, it is faster and thermalizes over larger time-scales $\sim L^d$.
In the presence of a magnetic field, the magnetization has a fixed direction. The only degrees of freedom of the system are the spin-waves propagating along the transverse directions. Since the system is non-critical, the spin-waves is expected to be non-interacting, so the dynamic exponent of the system is gaussian $z = 2$.

7 Off-equilibrium scaling behaviour at the first-order transition in O(N) vector models.

In this section we focus on the off-equilibrium dynamics arising in the O(N) vector-model at large N, when this system is at fixed temperature $T < T_c$ and is coupled to a time-dependent magnetic field. Thus, we consider a magnetic field protocol

$$\boxed{\delta(t,t_s) = \delta_{1,\alpha} \cdot h(t,t_s) \approx \delta_{1,\alpha} \cdot t/t_s} \tag{431}$$

with $T < T_c$ fixed which means that the thermal coupling r is a constant. Note that the general constraint-equation (290) is still valid below the critical temperature because related to the large N limit and not to the specific protocol. Since we assume a relaxational dynamics of the fields, we can use the relations (279), (280) and the results for the correlation functions with time-dependent parameters (342)-(345). These are function of the mass term which is defined at all times by (290). In particular, the constraint-equation (290) for the protocol (431) is

$$r + \frac{u}{6}\Big(\Sigma^2(t,t_s) + \int^{\Lambda} \frac{d^dk}{(2\pi)^d} \cdot G_T(k,t,t_s)\Big) = m^2(t,t_s). \tag{432}$$

Let us consider the off-equilibrium scaling limit in the constraint-equation above: we take the limit $t_s \to \infty$ keeping $\bar{t} = t/(t_s)^e$ and $\bar{k} = k(t_s)^{e/z}$ fixed. Since the thermal coupling r is a constant, it scales as $r \sim r \cdot (t_s)^0$. Thus, from the rescaling of the magnetization (358):

$$2(a+1)e - 2a = 0 \Rightarrow e = \frac{a}{(a+1)},\ \forall d.$$

The exponent e in the low-temperature phase does not depend on the spatial dimension of the system. If we consider a linear ramp protocol $a = 1$, it follows that $e = 1/2$. Thus, we can define the *off-equilibrium time-scale* t_Q as

$$t_Q = (t_s)^e = \sqrt{t_s}. \tag{433}$$

The off-equilibrium scaling limit of the constraint-equation requires that the magnetization scales has $(t_s)^0$ i.e. its scaling dimension is $d_\phi = 0$ and therefore the scaling dimension of the magnetic field is $d_h = d - d_\phi = d$, in agreement with the general results shown before. Since $e < 1$, the off-equilibrium dynamics arises for very small values of the magnetic field:

$$h(t,t_s) \approx t/t_s = (t/\sqrt{t_s}) \cdot (t_s)^{-1/2} \overset{t_s \to \infty}{\to} 0. \tag{434}$$

Therefore the slowest dynamics of the system in the off-equilibrium region is expected to be a change in direction of the vector-magnetization whose time-scale is given by a dynamical exponent $z = d$. Other types of dynamics occur with faster time-scales and can be neglected. It follows that the *off-equilibrium length scale* can be defined as:

$$l_Q = t_Q^{1/z} = t_Q^{1/d} = (t_s)^{1/2d}. \tag{435}$$

For the transverse two-point correlation function, we read into (359) that

$$\int^{\Lambda} \frac{d^dk}{(2\pi)^d} \cdot G_T(k,t,t_s) \sim ((t_s)^e)^{-(d-z)/z} \cdot \int \frac{d^d\bar{k}}{(2\pi)^d} \cdot \mathcal{G}_T(\bar{k},\bar{t}); \tag{436}$$

$$-(d-z)/z = 0 \Rightarrow z = d$$

The transverse two-point correlation function is not subleading in the off-equilibrium scaling limit only if we consider the slowest dynamics of the system.

Let us show that the definition of the off-equilibrium length and time scales are also in agreement with the definition of the Kibble-Zurek scales

$$t_Q = (t_s)^{a\nu_h z/(a\nu_h z+1)} \overset{?}{=} (t_s)^e = (t_s)^{a/(a+1)} \tag{437}$$

the definitions are equivalent only if

$$\nu_h = 1/d_h = 1/d. \tag{438}$$

The name "Kibble-Zurek" is generally related to continuous phase transitions. We will call t_Q KZ time in order to emphasize the analogies between this case and the continuous phase transistion treated in the first part.
The mass term satisfies the equation (279) also below the critical temperature:

$$m^2(t,t_s) = -\frac{1}{\Sigma(t,t_s)}\Big(\dot{\Sigma}(t,t_s) - h(t,t_s)\Big) \tag{439}$$

When we take the off-equilibrium scaling limit

$$\mathcal{M}^2(\bar{t})\cdot(l_Q)^{X=?} = -\frac{\dot{\Theta}(\bar{t})}{\Theta(\bar{t})}\cdot t_Q^{-1} + \frac{\bar{t}}{\Theta(\bar{t})}\cdot l_Q^{-d_h} = -\frac{\dot{\Theta}(\bar{t})}{\Theta(\bar{t})}\cdot t_Q^{-1} + \frac{\bar{t}}{\Theta(\bar{t})}\cdot t_Q^{-d/z}$$

Therefore the scaling relation of the mass term is given by

$$m^2(t,t_s) \sim \mathcal{M}^2(\bar{t})\cdot l_Q^{-d}. \tag{440}$$

Note that the transverse two-point correlation function generally has a critical behaviour in the limit of zero momenta. In this case the transverse correalation function remains finite at all times because the mass term is:

$$m^2(t,t_s) \sim 0\cdot l_Q^{-2} + \mathcal{M}^2(\bar{t})\cdot l_Q^{-d} \tag{441}$$

the presence of a small magnetic field makes the transverse susceptibility not IR-divergent.
Let us discuss the rescaling of the momenta. Below the critical temperature the degrees of freedom of the system are essentially given by the spin-waves. The short-wavelength modes can be considered at the equilibrium for all times and therefore the off-equilibrium behaviour depends only on the long-wavelength modes. In the KZ scaling limit, it can be shown that only the zero-momentum modes are relevant. A more detailed discussion is remanded to the appendix E.
We define the quantity:

$$S(t,t_s) = \int^{\Lambda} \frac{d^dk}{(2\pi)^d}\cdot G_T(k\to 0,t,t_s) \sim l_Q^0\cdot\mathcal{S}(\bar{t}) \tag{442}$$

The *scaling constraint-equation* below the critical temperature is:

$$r\cdot l_Q^0 + \frac{u}{6}\Big(\Theta^2(\bar{t})\cdot l_Q^0 + \mathcal{S}(\bar{t})\cdot l_Q^0\Big) = 0; \tag{443}$$

The mass term is subleading also below the critical temperature. Thus, the result is simplified:

$$r + \frac{u}{6}\Big(\Theta^2(\bar{t}) + \mathcal{S}(\bar{t})\Big) = 0; \tag{444}$$

The previous result can be written also as

$$\Theta^2(\bar{t}) + \mathcal{S}(\bar{t}) = \sigma^2. \tag{445}$$

where $\sigma = \sqrt{-6r/u}$. The scaling constraint-equation (445) states that the vector magnetization performs a rigid rotation with fixed length equal to σ. Infact, the equilibrium behaviour of the longitudinal magnetization has to be recovered in the appropriate limits: it makes a jump from $-\sigma$ to $+\sigma$, crossing the transition. We expect therefore that the longitudinal component of the magnetization, in the off-equilibrium region, first decreases dissipating into the transverse modes and then increases going to $+\sigma$ at $+\infty$. Note that the transverse magnetization is zero because of the $O(N-1)$ symmetry[54] but the correlations among the transverse components (resumed in the function $\mathcal{S}$) rotate the vector in one of the $N-1$ planes transverse to the longitudinal direction.

It is useful to write the longitudinal magnetization $\Sigma_1 = \Sigma$ as

$$\Sigma(t, t_s) = \sigma \cdot \cos(\vartheta(t, t_s)) \tag{446}$$

where the angle $\vartheta \in [0, \pi]$ is the azimut with respect to the longitudinal direction into the N-dimensional sphere. It follows from (342) that

$$\cos(\vartheta(t, t_s)) = \frac{1}{\sigma} \int_{t_0}^{t} dt' \cdot h(t', t_s) \cdot \exp\Big(- \int_{t'}^{t} dt'' \cdot m^2(t'', t_s) \Big). \tag{447}$$

In the KZ scaling limit

$$\cos(\vartheta(t, t_s)) \sim l_Q^0 \cdot \cos(\vartheta(\bar{t})) = \frac{1}{\sigma} \int_{-\infty}^{\bar{t}} d\bar{t}' \cdot t_Q \cdot \bar{t}' \cdot t_Q^{-1} \cdot \exp\Big(- \int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot t_Q \cdot \mathcal{M}^2(\bar{t}'') \cdot t_Q^{-1} \Big). \tag{448}$$

where $\vartheta(\bar{t})$ is the scaling function of $\vartheta(t, t_s)$[55]. In the following, we want to demonstrate the conjecture about the off-equilibrium dynamics: we first check the asymptotic equilibrium behaviour and then demonstrate that the longitudinal projection decreases.

Finite-size effects.

It has been considered the system in the infinite-volume limit. The infinite-volume limit is well defined also below the critical temperature and can be obtained starting from a system with a finite size L taking the limit $\ell = l_Q/L \to 0$ at fixed $\bar{k}$, $\bar{t}$ in the correlation functions. In addition, the matching with the equilibrium behaviour in the limit $\bar{t} \to \pm\infty$ requires that $t_s \cdot L^{-z} \to \infty$ i.e. $L \to \infty$. Since we are able to investigate only the firstly deviations from the asymptotic equilibrium, the presence of a finite size does not modify the discussion.

7.1 Asymptotic behaviours.

At the equilibrium, the mass term of the system can be related to the inverse of the coherence length of the system. In particular, since the scaling relation of the mass term is (441), it follows that:

$$m^2(t, t_s) \sim \xi^{-d}(t, t_s) \tag{449}$$

and in the KZ scaling limit

$$\mathcal{M}_e^2(\bar{t}) \sim |\bar{t}| \tag{450}$$

in the asymptotic limits $\bar{t} \to \pm\infty$. Since the scaling functions of the correlators (342)-(345) depend on the value of the scaling function of the mass term $\mathcal{M}^2$, this relation permits to recover their equilibrium behaviour.

We investigate the first deviations from the equilibrium assuming that the exponential ansatz (316), with appropriate exponents, applies in the case of first-order phase transition too [see ref.[6]]:

$$\mathcal{M}^2(\bar{t}) \overset{\bar{t} \to \pm\infty}{\sim} |\bar{t}| \cdot \Big(1 + b|\bar{t}|^a \cdot e^{-c|\bar{t}|^{1+z\nu_h}} \Big) \tag{451}$$

where a, b, c are free parameters and $\nu_h = 1/d$, $z = d$. In the following, we show that this assumption leads to an exponential approach to the equilibrium in the correlation functions.

[54]It is an average quantity: all the transverse directions are equivalent and their average value is zero.
[55]Since $d_\phi = 0$, the rescaling is only a change of variables.

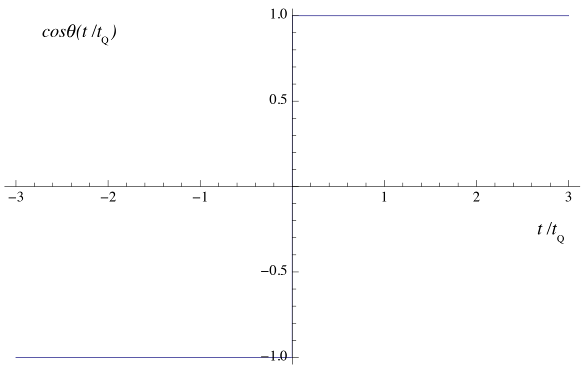

Figure 19: The equilibrium behaviour of the magnetization below the critical temperature.

We expect that the ansatz (451) is correct for both the asymptotic limits, maybe with different values of the free parameters. However, since we do not know the general expression for $\mathcal{M}^2$, [56] we are not able to investigate the limit $\bar{t} \to +\infty$ because of memory effects. Thus, we will investigate the off-equilibrium leading correction for $\bar{t} \to -\infty$ only.

7.1.1 Equilibrium asymptotic forms.

We want to recover the equilibrium behaviour of the correlators in the asymptotic limits $\bar{t} \to \pm\infty$. At the equilibrium the magnetization has fixed direction and lies in the longitudinal direction. Thus, we can assume that

$$\Sigma(t, t_s) \overset{\bar{t}\to\pm\infty}{\sim} \pm\sigma. \tag{452}$$

From the equation (279), using the last relation, follows that

$$m^2(t, t_s) = \pm\frac{h(t, t_s)}{\sigma} \tag{453}$$

that is the equation (177) valid at the equilibrium. In the KZ scaling limit, it becomes

$$m^2(t, t_s) \sim \pm\frac{\bar{t}}{\sigma} \cdot l_Q^{-d} \tag{454}$$

In other words, the equilibrium contribution to the scaling function $\mathcal{M}^2$ is

$$\mathcal{M}^2(\bar{t}) \overset{\bar{t}\to\pm\infty}{\sim} \pm\frac{\bar{t}}{\sigma} = \frac{|\bar{t}|}{\sigma} = \mathcal{M}_e^2(\bar{t}). \tag{455}$$

which is in agreement with (450). Note that this relation makes us able to compute the equilibrium contribution of the correlation function (342) and (345).
We start from the magnetization: the general expression for the correlator has to reduce to the relation (452) when $\bar{t} \to -\infty$. We check the consistence in terms of the azimut angle ϑ [see app.F.1.1]

$$\cos(\vartheta(\bar{t})) = \frac{1}{\sigma}\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot \exp\Big(-\int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot \mathcal{M}^2(\bar{t}'')\Big) \underset{\mathcal{M}^2\simeq\mathcal{M}_e^2}{\overset{\bar{t}\to-\infty}{\sim}} -1. \tag{456}$$

That is in agreement with $\theta(t_0) = \pi$ such that $\Sigma(t_0) = \sigma\cos(\theta(t_0)) = -\sigma$. If we consider the system at the equilibrium for all times, similiar computations leads to $\cos(\vartheta(\bar{t} \to +\infty)) \sim +1$.

[56] Formally, it is the solution of the scaling constrain-equation (445) by expressing the correlators as functions of $\mathcal{M}^2$.

The behaviour of the magnetization at the equilibrium is shown in fig.19.
We compute the equilibrium contribution for the transverse susceptibility using the relation (455) in (345) at zero momenta [see app.F.1.2]

$$\mathcal{G}_T(\bar{t}) = 2\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \exp\Big(-2\int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot \mathcal{M}^2(\bar{t}'')\Big) \underset{\mathcal{M}^2 \simeq \mathcal{M}_e^2}{\overset{\bar{t}\to -\infty}{\sim}} \sigma/|\bar{t}| = (\mathcal{M}_e^2(\bar{t}))^{-1}. \qquad (457)$$

In other words,

$$\chi_T^{-1}(t,t_s) \sim -\frac{h(t,t_s)}{\sigma} \qquad (458)$$

in agreement with the general prediction for the transverse susceptibility below the critical temperature at the equilibrium [see Sec.1.2.8].
We note that the scaling constraint-equation is satisfied at the equilibrium: since we assume that the vector magnetization has a fixed direction i.e. $\Sigma \sim -\sigma$, the transverse correlation function is negligible[57]. Thus, from (432):

$$\sigma^2 = \sigma^2\Big(\cos(\vartheta(\bar{t}))\Big)^2 \sim \sigma^2 \qquad (459)$$

7.1.2 Phase dynamics.

We want to understand the effects of the transverse correlations on the value of the vector magnetization at early times. From the scaling constraint-equation (432) we learn that the longitudinal component of the vector magnetization rotates into one of the $N-1$ transverse planes. Thus, the O(N) vector-model can be reduced to a planar X-Y model by considering the longitudinal direction plus one of the $N-1$ transverse ones[58].The N-dimensional vector ϕ obey to the equation of motion:

$$\dot{\phi}_\alpha(x,t) = -(\nabla^2 + m^2(t,t_s))\phi_\alpha(x,t) + \delta_{1,\alpha}\cdot h(t,t_s) + \varsigma_\alpha(x,t) \qquad (460)$$

for $\alpha = 1,\cdots,N$. Let us consider a polar coordinates for the components of ϕ:

$\phi_1(x,t) = \sigma(1+\varrho(x,t))\cos(\theta_1(x,t));$

$\phi_2(x,t) = \sigma(1+\varrho(x,t))\cos(\theta_2(x,t))\cdot\sin(\theta_1(x,t));$

$\cdots$

$\phi_{N-1}(x,t) = \sigma(1+\varrho(x,t))\cos(\theta_{N-1}(x,t))\cdots\sin(\theta_1(x,t));$

$\phi_N(x,t) = \sigma(1+\varrho(x,t))\sin(\theta_{N-1}(x,t))\cdot\sin(\theta_1(x,t));$

We have parametrized the vector on a N-dimensional sphere[59]. Since the O(N) vector-model can be reduced to a planar one [see ref.[28] and [29]], we integrate $N-2$ transverse degrees of freedom and we consider the 2-dimensional vector ϕ in the complex plane:

$$\phi(x,t) = \sigma(1+\varrho(x,t))\cdot e^{i\theta_1(x,t)} \qquad (461)$$

In the following we refer to the phase $\theta_1 = \theta \in [0,2\pi]$[60]. We can obtain an equation for the phase dynamics using the parametrization (461) in (460) and then separating the real and

[57] If the magnetization has a fixed direction, the dynamics of the spin-waves turns out to be gaussian $z=2$.
[58] This justifies the computation of the subsection 6.3, where we have considered the $N=2$ case.
[59] We have inserted also a radial fluctuation ϱ to be general. Then, we neglect the radial fluctuations.
[60] Since we are on a N-dimensional sphere, θ is the azimut and thus $0 \le \theta \le \pi$. When we consider a planar model the phase takes values in the enteire unitary circle.

the imaginay part. The detail of the computation are reported in the appendix F.2.1. The final result is:

$$\begin{aligned}\dot{\theta}(x,t)(1+\varrho(x,t)) &= \nabla^2\theta(x,t)(1+\varrho(x,t)) + 2\nabla\theta(x,t)\cdot\nabla\varrho(x,t) \\ &-\frac{h(t,t_s)}{\sigma}\sin\Big(\theta(x,t)\Big) - \frac{\varsigma_1(x,t)}{\sigma}\sin(\theta(x,t)) + \frac{\varsigma_2(x,t)}{\sigma}\cos(\theta(x,t))\end{aligned} \tag{462}$$

for the phase dynamics, and

$$\dot{\varrho}(x,t) = (\nabla\theta(x,t))^2(1+\varrho(x,t)) - m^2(t,t_s)(1+\varrho) + \frac{h(t,t_s)}{\sigma}\cos\Big(\theta(x,t)\Big) \tag{463}$$

$$+\frac{\varsigma_1(x,t)}{\sigma}\cos(\theta(x,t)) + \frac{\varsigma_2(x,t)}{\sigma}\sin(\theta(x,t)).$$

for the dynamics of the radial fluctuations. This system of equations full describes the time-evolution of the field but it is very hard to solve analitically. Thus, we solve the equation for the phase dynamics under some approximations:

-The radial fluctuations of the field are very small so $(1+\varrho(x,t)) \approx 1$. The off-equilibrium physics is related to very small values of the magnetic field which do not lead to significative length-variations of the vector magnetization.

-The spatial variations of angular and radial fluctuations are uncorrelated: $< \nabla\theta(x,t)\cdot\nabla\varrho(x,t) >_\varsigma = 0$.

We want to study the time-evolution of the phase by solving the equation of motion (462). Since we are interested to the off-equilibrium behaviour we have seen that the momenta are subleading terms [see app.E and ref. [28]].
Thus, we write the equation of motion (462) in the Fourier space

$$\dot{\theta}(k,t) = -k^2\theta(k,t) - h_\alpha(t,t_s)/\sigma\cdot\sin(\theta(k;t)) + \varsigma_\theta(k,t)$$

and we consider the zero-momentum case [61]:

$$\dot{\theta}(t) = -h_\alpha(t,t_s)/\sigma\cdot\sin(\theta(t)) + \varsigma_\theta(t) \tag{464}$$

where we have called $\varsigma_\theta = -\varsigma_1\sin(\theta) + \varsigma_2\cos(\theta)$. We have started with a white gaussian noise distribution proportional to

$$P(\varsigma) \propto \int\mathcal{D}\varsigma\cdot\exp\Big(-\frac{1}{2\Omega}\int dt\cdot\int d^dx\cdot|\varsigma(x,t)|^2\Big) = \int\mathcal{D}\varsigma\cdot\exp\Big(-\frac{1}{2\Omega}\int dt\cdot\int d^dx\cdot\Big\{\varsigma_1^2(x,t)+\varsigma_2^2(x,t)\Big\}\Big) \tag{465}$$

with zero mean and variance (207). The new noise variable ς_θ has distribution

$$P(\varsigma_\theta) \propto \int\mathcal{D}\varsigma_\theta\cdot\exp\Big(-\frac{\sigma^2}{2\Omega}\int dt\cdot\int d^dx\cdot|\varsigma_\theta(x,t)|^2\Big) \tag{466}$$

The variance of the new noise distribution is rescaled: $\Omega = 2 \mapsto 2/\sigma^2$, the mean value is zero. The solution at zero-momenta for the phase $\theta(t)$ is [see app.F.2.2]

$$\theta(t) = 2\arctan\Big\{\tan(\theta_0/2)\cdot\exp\Big(-\int_{t_0}^t dt'\cdot h(t',t_s)/\sigma\Big) + \int_{t_0}^t dt'\cdot\exp\Big(-\int_{t'}^t dt''\cdot h(t'',t_s)/\sigma\Big)\cdot\varsigma'_\theta(t')\Big\} \tag{467}$$

where we have ridefined the distribution of the noise $\varsigma'_\theta = \varsigma_\theta/2\cos^2(\theta/2)$ having the same cumulants of ς_θ. We investigate the dynamics for small angles:

$$\theta(t) = \pi - \theta'(t) \tag{468}$$

[61] We denote $\theta(k=0,t) = \theta(t)$.

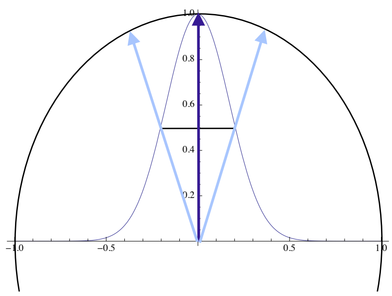

Figure 20: The effects of the transverse correlation functions on the value of the phase θ at early times.

with $\theta'(t)$ very small. It follows that

$$\sin(\theta(t)) = \sin(\pi - \theta'(t)) = -\sin(\theta'(t)) \simeq -\theta'(t).$$

The initial condition for θ' is $\theta'_0 = 0$. In this approximation we can linearize the equation of motion (464):

$$\dot{\theta}'(t) = h(t, t_s)/\sigma \cdot \theta'(t) + \varsigma'_{\theta'}(t) \tag{469}$$

and the solution (467) becomes[62]:

$$\theta'(t) = \int_{t_0}^{t} dt' \cdot \exp\Big(\int_{t'}^{t} dt'' \cdot h(t'', t_s)/\sigma\Big) \cdot \varsigma'_{\theta'}(t') \tag{470}$$

Let us now compute the expectation values of the phase. We remember that the cumulants of the noise distribution $\varsigma'_{\theta'}$ are:

$$< \varsigma'_{\theta'}(t) >_{\varsigma'_{\theta'}} = 0; \tag{471}$$

and

$$< \varsigma'_{\theta'}(t) \cdot \varsigma'_{\theta'}(t') >_{\varsigma'_{\theta'}} = \frac{2}{\sigma^2}\delta(t - t'); \tag{472}$$

Thus, the average value of the phase $\hat{\theta}$ at small angles follows from (470)

$$\hat{\theta}(t, t_s) =< \theta'(t)) >_{\varsigma'_{\theta'}} = 0 \tag{473}$$

The variance of the phase distribution $\gamma^2(t, t_s)$ is [see app.F.2.3]

$$\gamma^2(t, t_s) =< \Big(\theta'(t) - \hat{\theta}(t, t_s)\Big)^2 >_{\varsigma'_{\theta'}} =< (\theta'(t))^2 >_{\varsigma'_{\theta'}} \sim \frac{2}{\sigma^2} e^{t^2/t_s\sigma} \cdot \Big\{\frac{\sqrt{\pi}}{2} \cdot \sqrt{\sigma t_s} \cdot \mathrm{Erfc}\Big(|t|/\sqrt{\sigma t_s}\Big)\Big\} \tag{474}$$

In the KZ scaling limit and looking at early times $\bar{t} \to -\infty$, where the approximation for small angles hold, we obtain:

$$\gamma^2(t, t_s) \sim t_Q \cdot \gamma'^2(\bar{t}) \overset{\bar{t}\to-\infty}{\sim} \frac{1}{\sigma|\bar{t}|} \sim \frac{\mathcal{G}_T^{\mathrm{eq.}}(\bar{t})}{\sigma^2}. \tag{475}$$

[62] The term proportional to the initial state

$$\theta'_0(t) = \theta'_0 \cdot \exp\Big(\int_{t_0}^{t} dt' \cdot h(t', t_s)/\sigma\Big) = 0,$$

because $\theta'_0 = 0$.

The Fokker-Planck equation for the phase distribution (near θ_0) $P(\theta',t)$ can be obtained directly from (469)[63][see ref.[28]]:

$$\frac{\partial}{\partial t}P(\theta',t) = -2\mathcal{H}_{\rm FP}\cdot P(\theta',t);$$

The Fokker-Planck hamiltonian is:

$$\mathcal{H}_{\rm FP} = -\frac{1}{2}\int d^dx\cdot\frac{\delta}{\delta\theta'(x)}\Big(\frac{\delta}{\delta\theta'(x)}+L[\theta'(x)]\Big) = -\frac{1}{2}\Big(\frac{\delta^2}{\delta\theta'^2}+(-\theta'\cdot\frac{h(t,t_s)}{\sigma})\Big).$$

It follows that the Fokker-Planck equation is

$$\frac{\partial}{\partial t}P(\theta',t) = -\frac{\delta}{\delta\theta'}\Big(\theta'\frac{h(t,t_s)}{\sigma}\cdot P(\theta',t)\Big)+\frac{\delta^2}{\delta\theta^2}P(\theta',t); \tag{476}$$

with $P(\theta',t_0)=\delta(\theta')$. This equation admits a gaussian solution in the form:

$$P(\theta',t) = \frac{e^{-\theta'^2/2\gamma^2(t,t_s)}}{\sqrt{2\pi}\gamma(t,t_s)}; \tag{477}$$

We learn that the effect of the spin-waves fluctuations along the transverse direction give rise to a variance in the distribution of the phase close to the equilibrium. However, the vector magnetization still remains fixed along the longitudinal direction at early times.

7.1.3 Leading correction to the asymptotic equilibrium scaling.

Our purpose is to understand how the magnetization starts its departure from the initial position. We consider the leading order correction to the asymptotic equilibrium behaviour in the scaling function of the mass term:

$$\mathcal{M}^2(\bar{t}) \overset{\bar{t}\to-\infty}{\sim} \frac{|\bar{t}|}{\sigma}\Big(1+b|\bar{t}|^a\cdot e^{-c|\bar{t}|^{1+z\nu}}\Big) = \frac{|\bar{t}|}{\sigma}\Big(1+b|\bar{t}|^a\cdot e^{-c|\bar{t}|^2}\Big) \tag{478}$$

where a, $b>0$, $c>0$ free parameters. Let us insert this ansatz in the equation (279):

$$\mathcal{M}^2(\bar{t}) = -\frac{\dot{\Theta}(\bar{t})}{\Theta(\bar{t})}+\frac{\bar{t}}{\Theta} = -\dot{\vartheta}(\bar{t})\cdot\tan\Big(\vartheta(\bar{t})\Big)+\frac{\bar{t}}{\sigma\cos(\vartheta(\bar{t}))} \overset{\bar{t}\to-\infty}{\sim} \frac{|\bar{t}|}{\sigma}\Big(1+b|\bar{t}|^a\cdot e^{-c|\bar{t}|^2}\Big). \tag{479}$$

The equilibrium behaviour of the mass term consists in the zero-order of the Taylor expansion of ϑ around π. If we consider the first-order expansion:

$$\mathcal{M}^2(\bar{t}) \simeq -\dot{\vartheta}(\bar{t})\cdot\vartheta(\bar{t})+\frac{|\bar{t}|}{\sigma}\sim\frac{|\bar{t}|}{\sigma}\Big(1+b|\bar{t}|^a\cdot e^{-c|\bar{t}|^2}\Big); \tag{480}$$

We can find an approximate solution for the time-evolution of ϑ by solving the previous equation:

$$\int_\pi^{\vartheta(\bar{t})}\vartheta\cdot d\vartheta = -\int_{-\infty}^{\bar{t}}\frac{b}{\sigma}|\bar{t}'|^{1+a}\cdot e^{-c|\bar{t}'|^2}\cdot d\bar{t}';$$

Thus,

$$\frac{1}{2}\vartheta^2(\bar{t}) = \frac{\pi^2}{2}+\Big(-\frac{b}{2\sigma}c^{-a/2-1}\cdot\Gamma[(2+a)/2,c|\bar{t}'|^2]\Big|_{-\infty}^{\bar{t}}\Big)$$

Using the asymptotic expansion (502) we obtain

$$\vartheta(\bar{t}) \simeq \pi\Big(1-\frac{b}{2\pi^2c\sigma}|\bar{t}|^a\cdot e^{-c|\bar{t}|^2}\Big). \tag{481}$$

[63]A more detailed computation has to start with the equation of motion (464) and then has to consider the Kramers-Moyal expansion. However, in the approximation scheme (468), it is sufficient to keep the Fokker-Planck truncation.

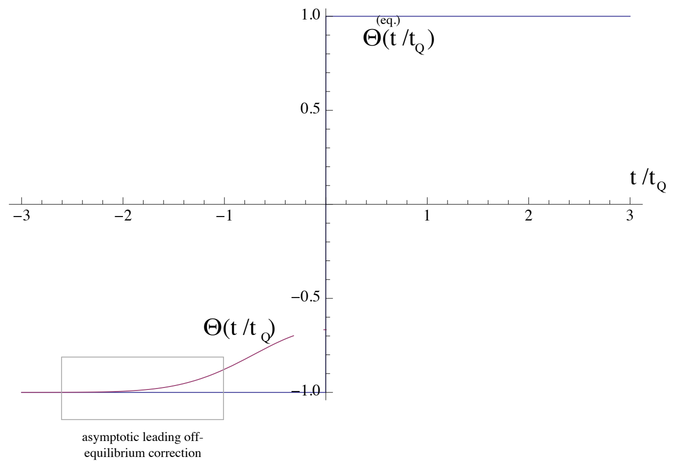

Figure 21: The leading off-equilibrium correction for the scaling function of the magnetization $\Sigma(\bar{t})/\sigma = \cos(\vartheta(\bar{t}))$ given by (482) with respect to the equilibrium behaviour. This plot has been made with $d = 3$, $c = \sigma^{-1} = 1$, $a = 0$.

This solution tells to us that the average azimut angle decreases under the ansatz (478). The computation above neglects the quadratic terms $O(\vartheta^2)$. Thus, it cannot be used to estimate correctly the leading off-equilibrium correction into the longitudinal magnetization (448)[64].
The correct result can be obtained using the ansatz (478) directly in (448) [see app. F.3.1]

$$\cos\Big(\vartheta(\bar{t})\Big) \sim -1\Big(1 + K|\bar{t}|^a \cdot e^{-c|\bar{t}|^2}\Big) \tag{482}$$

with $K = (-b/(1 + 2c\sigma))$. The exponential approach to the equilibrium has been found also for the longitudinal magnetization. Note that the relation (482) tells to us that the projection of the vector magnetization $\vec{\Sigma}$ along the longitudinal direction decreases.
We can check the consistence of this result using the relation (279) in the scaling limit. It has to be satisfied at all times also below the critical temperature; in particular, in the asymptotic limit $\bar{t} \to -\infty$, we obtain[65]:

$$\frac{|\bar{t}|}{\sigma} + \frac{b}{\sigma}|\bar{t}|^{1+a} \cdot e^{-c|\bar{t}|^2} \sim -aK|\bar{t}|^{a-1} \cdot e^{-c|\bar{t}|^2} - 2cK|\bar{t}|^{a+1} \cdot e^{-c|\bar{t}|^2}\, \frac{|\bar{t}|}{\sigma} - \frac{K}{\sigma}|\bar{t}|^{1+a} \cdot e^{-c|\bar{t}|^2} + O(e^{-2c|\bar{t}|^2});$$

[64] At the same order of the approximation $\cos(\vartheta) = -1$ and it gives the equilibrium results.
[65] The detail of the computation are reported in the following table:

- $\mathcal{M}^2(\bar{t}) \sim |\bar{t}|/\sigma \cdot \Big(1 + b \cdot |\bar{t}|^a \cdot e^{-c|\bar{t}|^2}\Big) = |\bar{t}|/\sigma + \frac{b}{\sigma}|\bar{t}|^{1+a} \cdot e^{-c|\bar{t}|^2}$;
  - $\cos(\vartheta(\bar{t})) \sim -1\Big(1 + K|\bar{t}|^a \cdot e^{-c|\bar{t}|^2}\Big)$;
  - $(\cos(\vartheta(\bar{t})))^{-1} \sim -1\Big(1 - K|\bar{t}|^a \cdot e^{-c|\bar{t}|^2}\Big)$;
- $\bar{t}/\cos(\vartheta(\bar{t})) \sim -|\bar{t}|/\cos\Big(\vartheta(\bar{t})\Big) = |\bar{t}|/\sigma - \frac{K}{\sigma}|\bar{t}|^{1+a} \cdot e^{-c|\bar{t}|^2}$;
  - $\frac{d}{d\bar{t}}\cos(\vartheta(\bar{t})) \sim -\frac{d}{d|\bar{t}|}\cos(\vartheta(\bar{t})) \sim -aK|\bar{t}|^{a-1} \cdot e^{-c|\bar{t}|^2} - 2cK|\bar{t}|^{a+1} \cdot e^{-c|\bar{t}|^2}$;
- $\frac{d}{d\bar{t}}\cos(\vartheta(\bar{t})) \cdot (\cos(\vartheta(\bar{t})))^{-1} \sim aK|\bar{t}|^{a-1} \cdot e^{-c|\bar{t}|^2} + 2cK|\bar{t}|^{a+1} \cdot e^{-c|\bar{t}|^2} + O(e^{-2c|\bar{t}|^2})$;

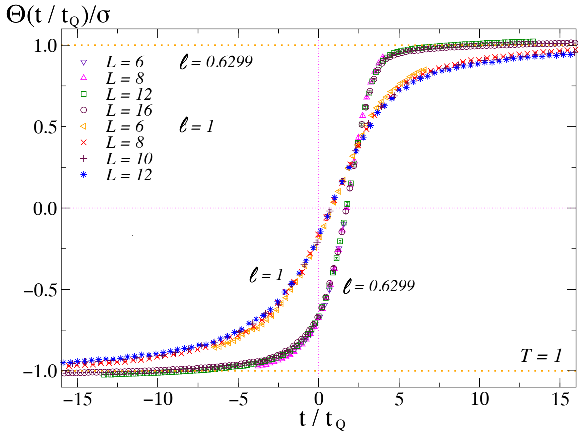

Figure 22: The behaviour of the scaling function of the relative magnetization Θ/σ as function of the rescaled time along protocol (431), below the critical temperature. The numerical simulations are made for an Heisenberg ferromagnet $N = 3$ in three spatial dimensions. The system has a cubic finite-size $V = L^3$. This picture has been taken from the ref.[6].

.

The equilibrium terms satisfy the relation. The leading off-equilibrium terms are:

$$\frac{b}{\sigma}|\bar{t}|^{1+a}\cdot e^{-c|\bar{t}|^2} \sim -2cK|\bar{t}|^{a+1}\cdot e^{-c|\bar{t}|^2} - \frac{K}{\sigma}|\bar{t}|^{1+a}\cdot e^{-c|\bar{t}|^2};$$

The last equation is satisfied if $K = (-b/(1+2c\sigma))$, in agreement with the result.
We investigate the first correction to the asymptotic equilibrium behaviour also in the transverse susceptibility. Under the assumption (478), the result is an exponential deviation [see app.F.3.2]:

$$\mathcal{G}_T(\bar{t}) \sim \frac{\sigma}{|\bar{t}|}\Big(1 + K'|\bar{t}|^a\cdot e^{-c|\bar{t}|^2}\Big) \tag{483}$$

with $K' = (-b/(1+c\sigma))$. We can check the consistence of this result using the relation (381) in the limit $\bar{t} \to -\infty$[66]:

$$\frac{|\bar{t}|}{\sigma}+\frac{b}{\sigma}|\bar{t}|^{1+a}\cdot e^{-c|\bar{t}|^2} \sim -\frac{1}{2}\Big[|\bar{t}|^{-1}-K'|\bar{t}|^{a-1}\cdot e^{-c|\bar{t}|^2}-(a-1)K'|\bar{t}|^{a-1}\cdot e^{-c|\bar{t}|^2}+2cK'|\bar{t}|^{a+1}\cdot e^{-c|\bar{t}|^2}\Big]$$
$$+\frac{|\bar{t}|}{\sigma}-\frac{K'}{\sigma}|\bar{t}|^{1+a}\cdot e^{-c|\bar{t}|^2}+O(e^{-2c|\bar{t}|^2});$$

The equilibrium terms satisfy the relation. The off-equilibrium leading terms are:

$$\frac{b}{\sigma}|\bar{t}|^{1+a}\cdot e^{-c|\bar{t}|^2} \sim -cK'|\bar{t}|^{a+1}\cdot e^{-c|\bar{t}|^2} - \frac{K'}{\sigma}|\bar{t}|^{1+a}\cdot e^{-c|\bar{t}|^2};$$

[66] The detail of the computation are given in the following table:

- $\mathcal{M}^2(\bar{t}) \sim |\bar{t}|/\sigma\cdot\Big(1+b\cdot|\bar{t}|^a\cdot e^{-c|\bar{t}|^2}\Big) = |\bar{t}|/\sigma + \frac{b}{\sigma}|\bar{t}|^{1+a}\cdot e^{-c|\bar{t}|^2}$;
- ∘ $\mathcal{G}_T(\bar{t}) \sim \sigma/|\bar{t}|\Big(1+K'|\bar{t}|^a\cdot e^{-c|\bar{t}|^2}\Big)$;
- $\mathcal{G}_T(\bar{t})^{-1} \sim |\bar{t}|/\sigma\Big(1-K'|\bar{t}|^a\cdot e^{-c|\bar{t}|^2}\Big)$;
- ∘ $\frac{d}{d\bar{t}}\mathcal{G}_T(\bar{t}) \sim -\frac{d}{d|\bar{t}|}\mathcal{G}_T(\bar{t}) \sim \sigma|\bar{t}|^{-2} - \sigma(a-1)K'|\bar{t}|^{a-2}\cdot e^{-c|\bar{t}|^2} - \sigma K'|\bar{t}|^a\cdot(-2c|\bar{t}|)\cdot e^{-c|\bar{t}|^2}$;
- $\dot{\mathcal{G}}_T(\bar{t})\cdot\mathcal{G}_T(\bar{t})^{-1} \sim |\bar{t}|^{-1} - K'|\bar{t}|^{a-1}\cdot e^{-c|\bar{t}|^2} - (a-1)K'|\bar{t}|^{a-1}\cdot e^{-c|\bar{t}|^2} + 2cK'|\bar{t}|^{a+1}\cdot e^{-c|\bar{t}|^2} + O(e^{-2c|\bar{t}|^2})$;

The expression is satisfied only if $K' = (-b/(1+c\sigma))$, in agreement with the previous result.

We have found that the ansatz (478) for the leading off-equilibrium correction of the scaling function $\mathcal{M}^2$ is sufficient to have an asymptotic exponential approach to the equilibrium in the correlation functions. In particular, the longitudinal component of the magnetization slowly decreases as exponential according to the rigid rotation conjecture. In order to demonstrate the ansatz (478) one should insert the asymptotic expression of the correlators in the scaling constraint-equation (445). The results above are only check of consistence. We are not able to compute analytically the function $S(t, t_s)$ because of the integration over the momenta. Even in the asymptotic limit $\bar{t} \to -\infty$ and keeping $k \to 0$ is not obvious how to make approximation to arise a qualitatively correct result. However, as in the case $T = T_c$, it can be demonstrated that different ansatzs do not lead to consistence.
The approach to the equilibrium in finite-size O(3) vector-models at the first-order transition has been numerically studied by [6] in three spatial dimensions.
We have shown that the off-equilibrium dynamics occur into one of the $N-1$ transverse planes, so we expect that our results are valid for each $N \geq 2$. Thus,

$$\sigma^{-1} \cdot \Sigma^{(N=3)}(t, t_s, L) \sim (t_s)^0 \cdot \cos^{(N=3)}(\bar{t}, \ell). \tag{484}$$

The presence of a finite size does not modify the asymptotic matching which always occur with the infinite-volume equilibrium scaling behaviour. The numerical result for $\cos^{(N=3)}(\bar{t}, \ell)$ is shown in the fig.22.

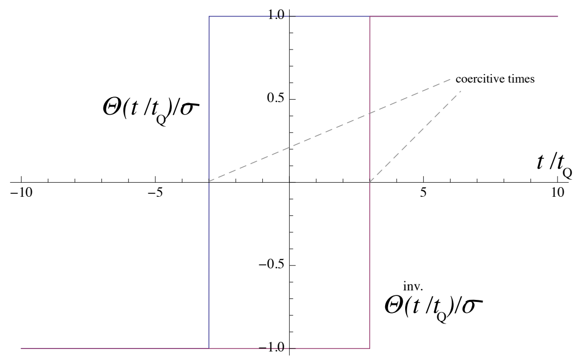

Figure 23: A qualitatively picture of the hysteresis loop area below the critical temperature.

7.2 Hysteresis phenomena.

When we consider a round-trip protocol i.e. when the magnetic field $h(t, t_s) \approx t/t_s$ is varied along a closed path, the system shows hysteresis also below the critical temperature. The hysteresis loop area can be defined in the same way of the case $T = T_c$ [see eq. (385)]:

$$\mathcal{A} = \int_{-\infty}^{+\infty} dt \cdot \Big(\Sigma(t, t_s) + \Sigma(-t, t_s)\Big) = \sigma \int_{-\infty}^{+\infty} dt \cdot \Big(\cos(\vartheta(t)) - \cos(\vartheta(-t))\Big). \tag{485}$$

The scaling relation of the hysteresis loop area is:

$$\mathcal{A} \sim \sigma \int_{-\infty}^{+\infty} d\bar{t} \cdot t_Q \cdot \Big(\cos(\vartheta(\bar{t})) + \cos(\vartheta(-\bar{t}))\Big) \sim t_Q \cdot \Xi. \tag{486}$$

where

$$\Xi = 2 \int_{-\infty}^{+\infty} d\bar{t} \cdot \int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot \cosh\Big(\int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot \mathcal{M}^2(\bar{t}'')\Big). \tag{487}$$

The energy spent by the system in a cycle is:

$$\mathcal{W} = \oint dh(t, t_s) \cdot \Sigma(t, t_s) = t_s \cdot \mathcal{A} \sim (t_s)^{-1/2} \cdot \Xi. \tag{488}$$

Hysteresis phenomena are related to the off-equilibrium: since the system is at the equilibrium, the magnetic work is zero as fig.19 shown. In contrast, when the system is coupled to a time-dependent magnetic field, it develops metastable because it cannot reach to adapt instantaneously to the external variations. A qualitatively picture of the hysteresis loop area below the critical temperature is shown in fig.23.
Since we do not have an explicit solution for $\mathcal{M}^2$, we cannot compute the amplitude Ξ. Neverthless, the ansatz (478) ensures the finitess of this amplitude Ξ.
A roughly approximation of the value of the hysteresis loop area can be obtained assuming that the system is at the equilibrium up to the coercitive time. Then, the magnetization jumps from -1 to $+1$ and the same for the return. Thus, the coercitive time t_c can be estimated as the time in which the contribute of the transverse correlations are of the same order of the magnetization. These can be valueted through the function:

$$S(t, t_s) \approx \text{width} \cdot \text{peak}$$

The peak of the distribution $S(t, t_s)$ is given by the zero momenta contribution [see Sec.462]:

$$\gamma^2(t, t_s) \sim \frac{2}{\sigma^2} e^{t^2/t_s\sigma} \cdot \Big\{\frac{\sqrt{\pi}}{2} \cdot \sqrt{\sigma t_s} \cdot \Big(1 + \text{Erf}(|t|/\sqrt{\sigma t_s})\Big)\Big\} \tag{489}$$

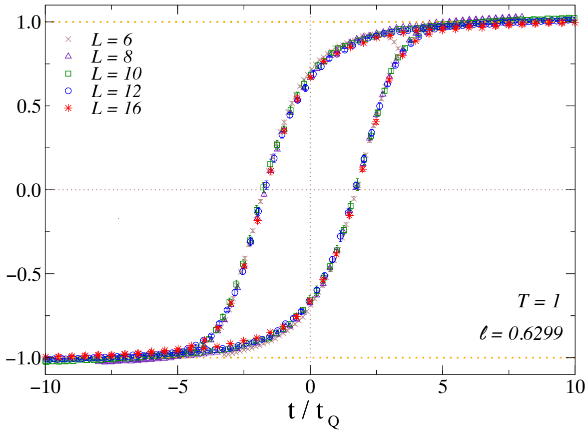

Figure 24: The hysteresis loop area for an Heisenberg ferromagnet $N = 3$ of finite-size system with cubic shape in three spatial dimension. The numerical simulations has been done with $\ell = L/l_Q$ fixed. The round-trip protocol moves from $t_i < 0$ to $t_f > 0$ below the critical temperature along the magnetic field $h(t, t_s) = t/t_s$. Then, it comes back from t_f to t_i. This picture has been taken from the ref.[6].

the width of the distribution $S(t, t_s)$ can be approximated with the value $k^\diamond = (1/t_s\sigma)^{1/4}$ such that $k < k^\diamond$ are relevant [see the appendix E]. Thus,

$$S(t_c, t_s) \approx \sqrt{t_s} \cdot \frac{\sqrt{\pi}}{\sigma^{3/2}} e^{t_c^2/t_s\sigma} \cdot \Big(1 + \mathrm{Erf}(|t|/\sqrt{\sigma t_s})\Big) \cdot (1/t_s\sigma)^{d/4} \sim O(1) \tag{490}$$

It follows that

$$t_c \approx \sqrt{t_s} \cdot \sqrt{\sigma} \cdot \sqrt{\log\Big(\sigma^{(6+d)/4} \cdot (t_s)^{(d-2)/4}\Big) + A}; \tag{491}$$

where A is a constant. In the squarish approximation, the hysteresis loop area is given by

$$\mathcal{A} \approx t_c \cdot \sigma \approx \sqrt{t_s} \cdot \sigma^{3/2} \cdot \sqrt{\log\Big(\sigma^{(6+d)/4} \cdot (t_s)^{(d-2)/4}\Big) + A}; \tag{492}$$

and the magnetic work

$$\mathcal{W} \approx (t_s)^{-1/2} \cdot \sigma^{3/2} \cdot \sqrt{\log\Big(\sigma^{(6+d)/4} \cdot (t_s)^{(d-2)/4}\Big) + A}; \tag{493}$$

this results are in agreement with [21] and [28].
Hysteresis phenomena at the first-order transition have been numerically studied in finite-size O(3) vector-models by [6]. It has been shown that the off-equilibrium dynamics occurs in one of the $N - 1$ transverse planes: we therefore expect that the hysteresis loop area is almost similiar for any $N \geq 2$. In the fig.24 is shown the numerical result for the case $N = 3$.

8 Conclusions.

We study the slow passage through the critical point of a statistical system in the presence of time-dependent external fields, focusing on a spin system with $O(N)$ symmetry. This model shows a continuous phase transition occuring at the critical temperature $T = T_c$ and at zero magnetic field $h = 0$. Two different protocols $\delta(t, t_s)$ are investigated for such transition: a magnetic field with linear time dependence at $T = T_c$, $\delta(t, t_s) \sim h(t, t_s) \approx t/t_s$, where t_s is a time scale, and the time-variations of the temperature $\delta(t, t_s) \sim T(t, t_s)/T_c - 1 \approx -t/t_s$ in the absence of magnetic fields [see Sec.4.1]. Very close to the critical point $\delta(t, t_s) \simeq 0$, the system goes out of the equilibrium because it develops large scale modes which cannot adapt themselves to the variations of the external parameters, even in the limit of slow passage $t_s \to \infty$.
The dynamics of the system shows universal scaling behaviours, which are controlled by the time t and the time scale t_s. In this regime the time dependence of the correlations can be expressed in terms of universal scaling functions that depend on the scaling variables $t/(t_s)^e = t/t_Q$ and $x/(t_s)^{e/z} = x/l_Q$, where z is the dynamical critical exponent and $0 < e < 1$ is a universal exponent depending on the static universality class of the model, on the type of dynamics and on the behaviour of the specific protocol near the transition [see Sec. 4.2].
The magnetic field protocol was numerically studied [see ref.[6]]. We provide to analytical computations in the limit of large N for the magnetic field protocol and we extend the analytical results in the thermal case. For both the protocols we demonstrate the existence of a non-trivial rescaling very close to the critical point $\delta(t, t_s) \approx 0$ and check that the relations for the $O(N)$ vector model at large N are satisfied only if the rescaling is made with l_Q and t_Q [see Sec. 5.4.2, 5.4.3 for the thermal protocol and Sec. 5.5.2, 5.5.3 for the magnetic field protocol].
The prediction for the scaling relations at large N are in agreement with the numerical result for the case $N = 3$ [see ref.[6]]. The large N limit does not modify the qualitatively off-equilibrium behaviour of the system and the scaling relations apply for finite N with appropriate exponents.
We also perform the study of the first deviations from the equilibrium scaling behaviour [see Sec.4.5] occuring at a time $\sim t_Q$ before the transition. We expect that the fluctuations over the equilibrium background in the correlation functions, decay exponentially with a lifetime of the order of the ratio between the equilibrium relaxation time and the off-equilibrium time scale. The same behaviour is shown by the system also after the transition if it approaches again the equilibrium.
In particular, we demonstrate that is sufficient to formulate an ansatz [see eq.(316)] in terms of the scaling function of the effective mass term $\mathcal{M}^2$ of the $O(N)$ vector model at large N to have an exponential decay in the correlation functions. We verify the consistence of this ansatz in the both the cases [see Sec. 5.4.4 and Sec.5.5.4].
We also investigate the off-equilibrium behaviour arising when the system is coupled to a magnetic field which varies in time from $t_i < 0$ to $t_f > 0$ and then back from t_f to t_i. The system presents hysteresis phenomena related to the off-equilibrium. The area of the hysteresis obeys to a scaling relation and can be easly connected to the magnetic work done by the system over a round-trip protocol [see Sec. 5.5.5]. We obtained a scaling relation for the magnetic work in three spatial dimensions:

$$\mathcal{W} \sim (t_s)^{-2/3} \cdot \Xi,$$

where Ξ is an amplitude constant which is finite under the assumption of exponential decay abovementioned. Extending this result to the case $N = 3$ using the critical exponents of the Heisenberg universality class, we found that the magnetic work scales as

$$\mathcal{W}^{(N=3)} \sim (t_s)^{-0.66} \cdot \Xi^{(N=3)}$$

which is in agreement with the numerical results [6].
The general features of the off-equilibrium and the ansatz for the asymptotic behaviours are not specific of the continuous phase transition. Thus, the same scaling theory, with appropriate exponents, is applied also below the critical temperature, where the $\mathrm{O}(N)$ vector model undergoes a first-order phase transition at $h = 0$. For a magnetic field protocol $h(t, t_s) \approx t/t_s$ at $T < T_c$, we derive the scaling relations and the leading corrections to the equilibrium scaling, pointing out the analogies with the case $T = T_c$ [see Sec. 7]. In this case, a constraint equation of the $\mathrm{O}(N)$ vector model at large N predicts that the magnetization of the system behaves as a rigid spin under the effects of a time-dependent magnetic fields and makes a slowly rotation in the off-equilibrium region. Hysteresis phenomena are shown by the system also in the ordered phase when we consider a round-trip protocol. The scaling relation of the magnetic work is:

$$\mathcal{W} \sim (t_s)^{-1/2} \cdot \Xi.$$

where Ξ is a finite constant under the assumption of exponential damping for $\mathcal{M}^2$. We also compute the explicit value of the hysteresis loop area in the squarish approximation: if we assume that the system is at the equilibrium up to the coercitive time and then jump to the other realization of the ordered phase, the amount of the magnetic work over a round trip protocol can be estimated as:

$$\mathcal{W} \approx (t_s)^{-1/2} \cdot \sigma^{3/2} \cdot \sqrt{\log\left(\sigma^{(6+d)/4} \cdot (t_s)^{(d-2)/4}\right) + A}$$

where A is a constant and σ is the equilibrium value of the magnetization.
The study of the off-equilibrium phase transitions is very interesting because permits to investigate peculiar phenomena such as the hysteresis or the defects formation and it is also object of several experiments in condensed matter physics [e.g. [2],[4],[3]]. Furthermore, they might be important in the study of the early Universe.

Some further developments of this thesis are shown in the following.

• It is possible to investigate a quantum version of our model. Let us consider a quantum system characterized by a transition rate ϵ and by the energy gap Δ between the ground state and the first excited level. The transition rate is the quantum analogue of a temperature and $\hbar/\Delta$ plays the role of a relaxation time. Thus, if we consider $\epsilon(t, t_s) \approx -t/t_s$ near the critical point, it follows that $\Delta(t, t_s) \sim |\epsilon|^{z\nu} \sim |t/t_s|^{z\nu}$. Comparing the instantaneous transition rate $|\dot{\epsilon}|/|\epsilon| \sim 1/|t|$ with $\Delta(t, t_s)$ we can extract the Kibble-Zurek time $t_Q \sim (t_s)^{z\nu/(z\nu+1)}$. The validity of the Kibble-Zurek approach to the off-equilibrium dynamics across quantum phase transitions has been verified in different models [e.g. [30], [31]].

• A particular interest has the study of the spatially inhomogeneities arising as a result of an external trapping potential [see ref.[32]]. In other words, the protocol now becomes space-dependent $\delta = \delta(x, t, t_s)$. Since the external fields have a specific spatial profile, the system crosses the transition at a given position x_F at a time t_F satisfying $\delta(x_F, t_F, t_s) = 0$. In the ref.[32], spatially inhomogeneous thermal protocol in BEC are studied. They found a prediction on the density of the defects in the condensate dependent on the interplay between causality and geometry of the trap: vortices are formed only when the causality limits the formation to a small fraction of the cloud. We want to investigate the case of inhomogeneous time-dependent magnetic field coupled to the spin system looking, in particular, to the hysteresis phenomena.

• A straightfoward extension of our discussion can be done considering the interplay between time-dependent magnetic field and temperature in a spin system. This means to consider a protocol $\vec{\delta}(t, t_s) = (\vec{h}(t, t_s), T(t, t_s)/T_c - 1)$. Multicritical Kibble-Zurek mechanism has been studied in quantum spin system [see ref.[33]]. We want to examine similiar behaviours in the context of a classical $\mathrm{O}(N)$ vector model in the limit of large N.

A Appendix: The Kibble-Zurek mechanism.

When a system is cooled through a critical point, the order parameter acquires a non-zero expectation value. This value is randomly chosen and therefore widely separeted regions of the system can present different realization of the broken-symmetry phase. We known that in a continuous phase transition the equilibrium correlation length varies rapidly and diverges at the critical point. However, this is not the real world: since a system approaches the critical point at a finite-rate, the velocity with which the information about the orientation of each domain of local broken-symmetry cannot grow faster than the speed of light, or in condensed matter, than the speed of sound. Thus, there is a time in which the correlation length of the system ceases to be equal to the equilibrium one and the system comes out of the equilibrium. If we estimate the velocity with which the information about the ordered phase is transmitted into the system [see ref. [17], [34]]

$$\text{velocity} \sim \frac{\text{space}}{\text{time}}$$

The equilibrium correlation length diverges like $\xi(T) \sim |T - T_c|^{-\nu}$ for the strong scaling statement, and the relaxational time $\tau(T) \sim \xi^z(t) \sim |T - T_c|^{-\nu z}$ for the critical slowing down phenomena. It follows that

$$c(T) = \frac{\xi(T)}{\tau(T)} \sim |T - T_c|^{\nu(z-1)}; \tag{494}$$

The informations about the order parameter cannot propagate faster than this velocity. Thus, at the transition, the largest distance over which informations can propagate is given by the sonic horizon:

$$h(t) = \int_0^t dt' \cdot c(T(t')) \propto |T(t) - T_c|^{1+\nu(z-1)}; \tag{495}$$

Widely separated regions means that their distance is more than the upper limit in which informations can be propagated into the system. The sonic horizon becomes equal to the equilibrium correlation length at a time which we call Kibble-Zurek time t_Q

$$\xi(T(t_Q)) = h(T(t_Q)) \tag{496}$$

If we assume that the variations in the temperature are linear in time $|T(t)/T_c - 1| = |t|/t_s$ where t_s is the time-scale of the thermal variations or "quench-rate" of the transition, it follows that the KZ time is:

$$t_Q \sim t_s^{\nu z/(1+z\nu)} \tag{497}$$

A Kibble-Zurek length can be defined as $\xi(T(t_Q)) \sim l_Q \sim t_s^{\nu/(1+\nu z)}$.
Thus, systems driven to a phase transition with a finite rate are well-described by the equilibrium correlation length up to the Kibble-Zurek time. It marks the boundary of a region around the transition in which these systems are no longer at the equilibrium.
By these arguments, the Kibble-Zurek mechanism predicts the density per volume of topological defects[67]:

$$\text{density of defects} \sim (1/l_Q)^2 \tag{498}$$

where we have considered a system in three spatial dimension and topological defects as strings[68].
Several experiments have been tested the validity of the Kibble-Zurek mechanism. In the following we consider one of the early experiments made with superfluid helium.

[67]The topological defects formation is a common feature of symmetry-breaking phase transitions. For instance, one may think about the broken $U(1)$ group and the formation of abelian vortices in a superfluid. Non symmetry-breaking phase transition i.e. system which undergoes to the transition with explicit terms of symmetry-breaking does not realize topological defects.

[68]The topological defects separations scale as $(1/l_Q)^{d-p}$ where d are the spatial dimensions of the underlying system and p is the dimensions of these defects in the coarsening regime. See ref.[16].

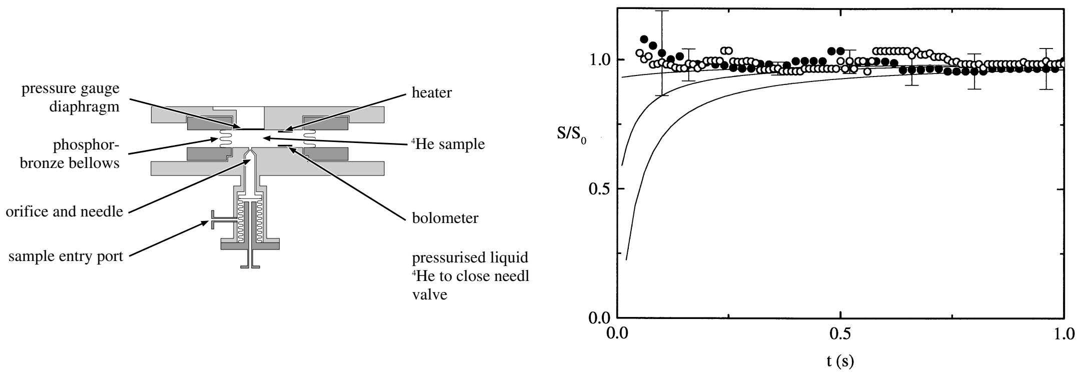

Figure 25: On the left: The expansion cell of the ^{4}He improved experiment. On the right: Time-evolution of the second sound amplitude S. S_0 is the signal amplitude in the absence of vortices. The curves refer to calculated signal evolutions for different initial vortex line densities, from the bottom, of 10^{12}, 10^{11}, $10^{10} \cdot \mathrm{m}^{-2}$. The density expected is $4 \cdot 10^{12} \cdot \mathrm{m}^{-2}$. These pictures have been taken from the ref. [34].

A.1 Liquid helium-4.

A sample of He I is contained in a small chamber that could be rapidly expanded to lower the pressure, driving the system through the lambda transition into the superfluid phase. The number of vortices produced can be found by measuring the attenuation of a second sound signal.
The first results were approximatively in agreement with the theoretical predictions. However, vorticity might have been produced by hydrodinamical effects at the walls. Another problem was that it was not possible to measure the second-sound attenuation during the first 50 ms after the transition, so that later readings had to be extrapolated back to the relevant time. Further improved experiments minimize the hydrodinamical production of vortices in the sample but leads to a trivial result: no vorticity was detected in the improved apparatus.
Why the Kibble-Zurek mechanism predictions work in several systems such as helium-3, liquid crystals or Josephson junction and not in helium-4 is still unknown[69].
This experiment borns as a gedankment experiment [see ref. [35]], later it becomes a real experiment. Even if the real experiment does not have a particular interest, it has been reported for its historical relevance.

[69]There are some explanations of this phenomenon such as the possibility that vortices decay to fast to be observed or that the thermal fluctuations tends to unwind the order parameter close to the transition. The situation is unclear.

B Algebraic computations of Sec. 3

B.1 Solution of the dynamics equation.

The dynamics of the fields satisfies the equation:

$$\dot{\phi}_\alpha(k,t) = -\frac{\Omega}{2}(k^2+m^2)\phi_\alpha(k,t) + \frac{\Omega}{2}\cdot(2\pi)^d\delta^d(k)\cdot h_\alpha + \varsigma_\alpha(k,t);$$

The cumulants of noise distribution, written in Fourier space, are:

$$< \varsigma_\alpha(k,t) >_\varsigma = 0$$

$$< \varsigma_\alpha(k,t)\varsigma_\beta(k',t') >_\varsigma = \Omega\cdot\delta_{\alpha\beta}(2\pi)^d\delta^d(k+k')\delta(t-t')$$

Let us solve the equation starting from the homogeneous case. Its solution is

$$\phi_\alpha(k,t) = \phi^0_\alpha(k)\exp\Big(-\frac{\Omega}{2}(k^2+m^2)\cdot(t-t_0)\Big).$$

with $\phi_\alpha(k,t_0) = \phi^0_\alpha(k)$ is the value of the field at the time t_0 where the system is at the equilibrium. From the equilibrium value of the magnetization (183), (177) follows:

$$\phi_\alpha(k,t_0) = \phi^0_\alpha(k) = \delta_{1,\alpha}(2\pi)^d\delta^d(k)\cdot\sigma$$

If $h=0$, the previous term is zero above the critical temperature. A particular solution can be found in the form $\phi_\alpha(x,t) = \phi^0_\alpha\exp(-A(t-t_0))\cdot w(t)$ with $A=(k^2+m^2)$. If we insert this guess into the complete equation:

$$\dot{\phi}_\alpha(x,t) = -Ae^{-A(t-t_0)}\phi^0_\alpha(k)\cdot w(t) + \dot{w}(t)\cdot\phi^0_\alpha(k)e^{-A(t-t_0)} =$$

$$-A\phi^0_\alpha(k)e^{-A(t-t_0)}\cdot w(t) + \frac{\Omega}{2}(2\pi)^d\delta^d(k)h_\alpha + \varsigma_\alpha(k,t);$$

It follows that

$$w(t) = \frac{1}{\phi^0_\alpha(k)}\int_{t_0}^t dt'\cdot e^{-A(t-t')}\cdot\Big\{\frac{\Omega}{2}(2\pi)^d\delta^d(k)h_\alpha + \varsigma_\alpha(k,t')\Big\} + w_0;$$

Therefore the solution is:

$$\phi_\alpha(k,t) = \phi^0_\alpha(k)\exp\Big(-\frac{\Omega}{2}(k^2+m^2)\cdot(t-t_0)\Big)$$

$$+\int_{t_0}^t dt'\cdot\exp\Big(-\frac{\Omega}{2}(k^2+m^2)(t-t')\Big)\cdot\Big\{\frac{\Omega}{2}(2\pi)^d\delta^d(k)h_\alpha + \varsigma_\alpha(k,t')\Big\}$$

The previous solution can be written as

$$\phi_\alpha(k,t) = \phi^0_\alpha(k,t) + (2\pi)^d\delta^d(k)\Big(1-\exp\Big(-\frac{\Omega}{2}(k^2+m^2)(t-t_0)\Big)\Big)\frac{h_\alpha}{k^2+m^2}$$

$$+\int_{t_0}^t dt'\cdot\exp\Big(-\frac{\Omega}{2}(k^2+m^2)(t-t')\Big)\cdot\varsigma_\alpha(k,t').$$

with $\phi^0_\alpha(k,t) = \delta_{1,\alpha}(2\pi)^d\delta^d(k)\cdot\sigma\exp\Big(-\frac{\Omega}{2}(k^2+m^2)\cdot(t-t_0)\Big)$.

B.2 Expectation value of two fields.

We consider the expectation value of two fields over the noise distribution:

$$< \phi(k,t)_\alpha \phi(k',t)_\beta >_\varsigma = \phi_{\alpha,0}(k,t) \cdot \phi_{\beta,0}(k',t)$$

$$+(2\pi)^{2d}\delta^d(k)\delta^d(k')h_\alpha h_\beta \Big(1 - \exp\Big(-\frac{\Omega}{2}(k^2+m^2)(t-t_0)\Big)\Big)^2 (k^2+m^2)^{-2}$$

$$+\delta_{\alpha\beta}(2\pi)^d\delta^d(k+k')\Omega \int_{t_0}^t dt' \cdot \exp\Big(-\Omega(k^2+m^2)\cdot(t-t')\Big);$$

Since the magnetic field has a fixed direction we write:

$$< \phi(k,t)_\alpha \phi(k',t)_\beta >_\varsigma = \delta_{\alpha,1}\delta_{\beta,1}(2\pi)^{2d}\delta^d(k)\delta^d(k')\sigma^2$$

$$+\delta_{\alpha\beta}(2\pi)^d\delta^d(k+k')\frac{\Omega}{2}\cdot\frac{1}{k^2+m^2}\Big(1-\exp\Big(-\Omega(k^2+m^2)\cdot(t-t_0)\Big)\Big)$$

The term $(2\pi)^{2d}\delta^d(k)\delta^d(k')$ tell to us that, after integration, both the momenta must be zero. This condition can

be written also as $(2\pi)^{2d}\delta^d(k+k')\delta^d(k)$. Thus,

$$< \phi(k,t)_\alpha \phi(k',t)_\beta >_\varsigma = (2\pi)^d\delta^d(k+k')\delta_{\alpha,\beta}\Big\{(2\pi)^d\delta^d(k)\sigma^2 + \frac{\Omega}{2}\cdot\frac{1}{k^2+m^2}\Big(1-\exp\Big(-\Omega(k^2+m^2)\cdot(t-t_0)\Big)\Big)\Big\}$$

C Appendix: Asymptotic behaviours of special functions.

The definitions and the relations for the functions are taken from *functions.wolfram.com.*

C.1 Incomplete Gamma function.

The *Incomplete Gamma function* is defined as

$$\Gamma[w,z] = \int_z^\infty dt \cdot t^{w-1} \cdot e^{-t} \tag{499}$$

where $w,\ z \in \mathbb{C}$. The Incomplete Gamma function's value at infinities is zero

$$\Gamma[w, z = \infty] = 0; \tag{500}$$

and if $\mathrm{Re}(w) > 0$

$$\Gamma[w, z = 0] = \Gamma(w) \tag{501}$$

where $\Gamma(w)$ is the Euler's Gamma function. It admits a known asymptotic expansion:

$$\Gamma[w,z] \sim e^{-z} \cdot z^{w-1}\Big(1 - \frac{1-w}{z} + \frac{(2-w)(1-w)}{z^2} + O(z^{-3})\Big); |z| \to \infty. \tag{502}$$

C.2 Airy functions.

The *Airy function* Ai or Airy function of the first type is defined by

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\Big(\frac{y^3}{3} + xy\Big)dy \tag{503}$$

where $x \in \mathbb{R}$. It is a well-known convergent solution of differential equation $y''(x) - x \cdot y(x) = 0$.
The asymptotic behaviour of Airy functions can be written in the complex plane as

$$\mathrm{Ai}(z) \sim \frac{e^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}z^{1/4}}\Big(1 - \frac{5}{48z^{3/2}} + \frac{385}{4608z^3} + O(\frac{1}{z^{9/2}})\Big); z \in \mathbb{C}, |z| \to \infty, |\mathrm{Arg}(z)| < \pi. \tag{504}$$

The derivative of Airy function Ai′ (called sometimes also as *Airy Prime function*) has a known asymptotic behaviour too

$$\mathrm{Ai}'(z) \sim -\frac{e^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}} \cdot z^{1/4} \cdot \Big(1 + \frac{7}{48z^{3/2}} - \frac{455}{4608z^3} + O(\frac{1}{z^{9/2}})\Big), z \in \mathbb{C}, |z| \to \infty, |\mathrm{Arg}(z)| < \pi. \tag{505}$$

C.3 Erf functions.

The Error function Erf is defined to be

$$\mathrm{Erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z dw \cdot e^{-w^2}; \tag{506}$$

For small values of the argument, the Erf function admit a Taylor expansion

$$\mathrm{Erf}(z) \sim \frac{2}{\sqrt{\pi}}\Big(z - \frac{z^3}{3} + \frac{z^5}{10} - O(z^7)\Big) \tag{507}$$

The asymptotic expansion of the Erf function is

$$\mathrm{Erf}(z) \sim \frac{\sqrt{z^2}}{z} - \frac{e^{-z^2}}{z\sqrt{\pi}} + O\Big(\frac{1}{z^2}\Big);\ z \in \mathbb{C},\ |z| \to \infty \tag{508}$$

The *complementary error function* Erfc is defined as

$$\mathrm{Erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty dw \cdot e^{-w^2} = 1 - \mathrm{Erf}(z). \tag{509}$$

Its values at infinities are

$$\mathrm{Erfc}(z = \infty) = 0; \tag{510}$$

and from the expansion of the Erf function we read

$$\mathrm{Erfc}(z) \sim 1 - \frac{2}{\sqrt{\pi}}\Big(z - \frac{z^3}{3} + \frac{z^5}{10} - O(z^7)\Big) \tag{511}$$

for small values of its argument. The asymptotic expansion is:

$$\mathrm{Erfc}(z) \sim 1 - \Big(\frac{\sqrt{z^2}}{z} - \frac{e^{-z^2}}{z\sqrt{\pi}}\Big) - O\Big(\frac{1}{z^2}\Big);\ z \in \mathbb{C},\ |z| \to \infty \tag{512}$$

and since the argument is real

$$\mathrm{Erfc}(x) \sim \frac{e^{-x^2}}{x\sqrt{\pi}} - O\Big(\frac{1}{x^2}\Big);\ x \in \mathbb{R},\ x \to \infty \tag{513}$$

D Algebraic computations of Sec. 5

D.1 About the equations (279) and (280) .

The equations (279) and (280) follow from the equation of motion (278). For (279) we consider the expectation value of one field:

$$\frac{d}{dt} < \phi_\alpha(k,t) >_\varsigma = -\frac{\Omega}{2}(k^2 + m^2(t,t_s)) < \phi_\alpha(k,t) >_\varsigma + \frac{\Omega}{2} \cdot (2\pi)^d \delta^d(k) \cdot h_\alpha(t,t_s);$$

Thus,

$$(2\pi)^d \delta_{1,\alpha} \delta^d(k) \dot{\Sigma}(t,t_s) = -\frac{\Omega}{2}(k^2 + m^2(t,t_s))(2\pi)^d \delta_{1,\alpha} \delta^d(k) \Sigma(t,t_s) + \frac{\Omega}{2} \cdot (2\pi)^d \delta^d(k) \cdot \delta_{1,\alpha} h(t,t_s).$$

For the relation (280) we consider the expectation value of two transverse components α, $\beta > 1$ in (278):

$$\frac{d}{dt} < \phi_\alpha(k,t)\phi_\beta(k',t) >_\varsigma = < \dot{\phi}_\alpha(k,t)\phi_\beta(k',t) >_\varsigma + < \phi_\alpha(k,t)\dot{\phi}_\beta(k',t) >_\varsigma =$$

$$-\frac{\Omega}{2}(k^2 + m^2(t,t_s)) < \phi_\alpha(k,t)\phi_\beta(k',t) >_\varsigma - \frac{\Omega}{2}(k'^2 + m^2(t,t_s)) < \phi_\alpha(k,t)\phi_\beta(k',t) >_\varsigma$$
$$+ < \varsigma_\alpha(k,t)\varsigma_\beta(k',t) >_\varsigma .$$

Thus,

$$(2\pi)\delta^d(k+k')\delta_{\alpha\beta}\dot{G}_T(k,t,t_s) =$$
$$-\Omega(k^2 + m^2(t,t_s))(2\pi)\delta^d(k+k')\delta_{\alpha\beta}G_T(k,t,s) + \Omega(2\pi)^d\delta^d(k+k')\delta_{\alpha\beta}.$$

D.2 Solution of the dynamics equation with time-dependent parameters.

The equation of the dynamics is:

$$\dot{\phi}_\alpha(k,t) = -\frac{\Omega}{2}(k^2 + m^2(t,t_s))\phi_\alpha(k,t) + \frac{\Omega}{2} \cdot (2\pi)^d \delta^d(k) \cdot h_\alpha(t,t_s) + \varsigma_\alpha(k,t);$$

It is a linear inhomogeneus differential equation with non-constant coefficients like $y'(t) = -a(t) \cdot y + f(t)$. Firstly, we consider the homogeneous case $y'(t) + a(t)y = 0$. Its solution is

$$\int_{y_0}^{y} \frac{dy'}{y'} = -\int_{t_0}^{t} a(t')dt' = -A(t) \Rightarrow y(t) = y_0 \cdot e^{-A(t)}.$$

Now we try to find a particular solution for the complete equation by making an ansatz $y(t) = w(t) \cdot e^{-A(t)}$;
If we substitute our ansatz inside the equation, we find

$$w'(t)e^{-A(t)} - a(t)w(t)e^{-A(t)} + a(t)w(t)e^{-A(t)} = f(t);$$

$$w'(t) = f(t) \cdot e^{A(t)} \Rightarrow w(t) = \int_{t_0}^{t} f(t')e^{A(t')}dt' + w_0.$$

Therefore we can conclude that the general solution of our equation is

$$y(t) = (y_0 + w_0) \cdot e^{-A(t)} + e^{-A(t)} \int_{t_0}^{t} f(t') \cdot e^{A(t')} \cdot dt'.$$

Now we come back to our system: if we translate the above soltution as

$y(t) = \phi_\alpha(k,t);$

$a(t) = \frac{\Omega}{2}(k^2 + m^2(t,t_s));$

$f(t) = \frac{\Omega}{2} \cdot (2\pi)^d \delta^d(k) h_\alpha(t,t_s) + \varsigma_\alpha(k,t);$

The boundary conditions on the equation (278) are fixed by the initial conditions of the system. At t_0 the system is assumed to be at the equilibrium. Thus,

$$y_0 = \phi_\alpha(k,t_0) = \phi^0_\alpha(k) = (2\pi)^d \delta^d(k) \cdot \delta_{1,\alpha} \cdot \sigma$$

We choose a particular solution with the costant $w_0 = 0$, or equivently ridefining the term $y_0 + w_0$ as original magnetization. It follows that the general solution of the equation above is:

$$\phi_\alpha(k,t) = \phi^0_\alpha(k,t) + \int_{t_0}^t dt' \cdot \exp\Big(-\frac{\Omega}{2}\int_{t'}^t dt'' \cdot (k^2 + m^2(t'',t_s))\Big) \cdot \Big\{\frac{\Omega}{2} \cdot (2\pi)^d \delta^d(k) h_\alpha(t',t_s) + \varsigma_\alpha(k,t')\Big\}$$

where we have defined the term proportional to the original condition as

$$\phi^0_\alpha(k,t) = \exp\Big(-\frac{\Omega}{2}\int_{t_0}^t dt' \cdot (k^2 + m^2(t',t_s))\Big)(2\pi)^d \delta^d(k) \delta_{1,\alpha} \cdot \sigma.$$

D.3 Expectation value of two fields.

The expectation value of two fields can be computed starting from the solution (282) for dynamical fields and using the cumulants of the noise distribution (213), (214):

$$< \phi_\alpha(k,t)\phi_\beta(k',t) >_\varsigma =$$

$$\Big[< \Big(\phi^0_\alpha(k,t) + \int_{t_0}^t dt' \cdot e^{-\frac{\Omega}{2}\int_{t'}^t dt'' \cdot (k^2+m^2(t'',t_s))}\Big\{\frac{\Omega}{2} \cdot (2\pi)^d \delta^d(k) \cdot h_\alpha(t',t_s) + \varsigma_\alpha(k,t')\Big\}\Big) \cdot$$

$$\Big(\phi^0_\beta(k',t) + \int_{t_0}^t d\tau' \cdot e^{-\frac{\Omega}{2}\int_{\tau'}^t dt'' \cdot (k'^2+m^2(t'',t_s))}\Big\{\frac{\Omega}{2} \cdot (2\pi)^d \delta^d(k') \cdot h_\beta(\tau',t_s) + \varsigma_\alpha(k',\tau')\Big\}\Big) >_\varsigma \Big]$$

$$= \phi^0_\alpha(k,t) \int_{t_0}^t d\tau' \cdot e^{-\frac{\Omega}{2}\int_{\tau'}^t dt'' \cdot (k'^2+m^2(t'',t_s))}\Big\{\frac{\Omega}{2} \cdot (2\pi)^d \delta^d(k') \cdot h_\beta(\tau',t_s) + < \varsigma_\beta(k',\tau') >_\varsigma \Big\}$$

$$+ \phi^0_\beta(k',t) \int_{t_0}^t dt' \cdot e^{-\frac{\Omega}{2}\int_{t'}^t dt'' \cdot (k^2+m^2(t'',t_s))}\Big\{\frac{\Omega}{2} \cdot (2\pi)^d \delta^d(k) \cdot h_\alpha(t',t_s) + < \varsigma_\alpha(k,t') >_\varsigma \Big\}$$

$$+ \int_{t_0}^t dt' \cdot \int_{t_0}^t d\tau' \cdot e^{-\frac{\Omega}{2}\int_{t'}^t dt'' \cdot (k^2+m^2(t'',t_s))} \cdot e^{-\frac{\Omega}{2}\int_{\tau'}^t dt'' \cdot (k^2+m^2(t'',t_s))} \cdot$$

$$\Big\{\frac{\Omega^2}{4} \cdot (2\pi)^{2d} \delta^d(k)\delta^d(k') \cdot h_\alpha(t',t_s) \cdot h_\beta(\tau',t_s) + < \varsigma_\alpha(k,t')\varsigma_\beta(k',\tau') >_\varsigma \Big\} + \phi^0_\alpha(k,t)\phi^0_\beta(k',t)$$

Since the magnetic field has a fixed direction

$$= \delta_{1,\alpha}\delta_{1,\beta}(2\pi)^{2d}\delta^d(k)\delta^d(k')(\Sigma^0(t,t_s))^2 +$$

$$2\delta_{1,\alpha}\delta_{1,\beta}(2\pi)^d \delta^d(k')\Sigma^0(t,t_s)\frac{\Omega}{2}\int_{t_0}^t dt' \cdot e^{-\frac{\Omega}{2}\int_{t'}^t dt'' \cdot (k^2+m^2(t'',t_s))}(2\pi)^d \delta^d(k) h(t',t_s)$$

$$+ \Omega \delta_{\alpha\beta}(2\pi)^d \delta^d(k+k') \int_{t_0}^t dt' \cdot e^{-\Omega\int_{t'}^t dt'' \cdot (k^2+m^2(t'',t_s))}$$

$$+ (2\pi)^{2d}\delta^d(k)\delta^d(k')\delta_{1,\alpha}\delta_{1,\beta}\Big(\int_{t_0}^t dt' \cdot e^{-\frac{\Omega}{2}\int_{t'}^t dt'' \cdot (k^2+m^2(t'',t_s))} \cdot \frac{\Omega}{2} \cdot h(t',t_s)\Big)^2.$$

The term $(2\pi)^{2d}\delta^d(k)\delta^d(k')$ tell to us that, after integration, both the momenta must be zero. This condition can be written also as $(2\pi)^{2d}\delta^d(k+k')\delta^d(k)$. Thus,

$$= \delta_{\alpha\beta}(2\pi)^d\delta^d(k+k')\cdot\Big[\delta_{1,\alpha}(2\pi)^{2d}\delta^d(k)\Big\{(\Sigma^0(t,t_s))^2$$

$$+2\Sigma^0(t,t_s)\cdot\frac{\Omega}{2}\int_{t_0}^{t}dt'\cdot\exp\Big(-\frac{\Omega}{2}\int_{t'}^{t}dt''\cdot(k^2+m^2(t'',t_s))\Big)h(t',t_s)+\Sigma^2(t,t_s)\Big\}$$

$$+\Omega\int_{t_0}^{t}dt'\cdot\exp\Big(-\Omega\int_{t'}^{t}dt''\cdot(k^2+m^2(t'',t_s))\Big)\Big].$$

where Σ^0 is given by (285).

D.4 Thermal protocol:

D.4.1 Equilibrium contribution of the susceptibility.

We insert the equilibrium part of the functon $\mathcal{M}^2$, (312) in the general expression of the scaling function of the susceptibility (319):

$$\mathcal{G}(\bar{t}) = 2\int_{-\infty}^{\bar{t}}d\bar{t}'\cdot\exp\Big(-2\int_{\bar{t}'}^{\bar{t}}\mathcal{M}^2(\bar{t}'')\cdot d\bar{t}''\Big)\underset{\mathcal{M}^2\simeq\mathcal{M}_e^2}{\overset{\bar{t}\to-\infty}{\sim}}$$

$$2\int_{-\infty}^{\bar{t}}d\bar{t}'\cdot\exp\Big(-2\int_{\bar{t}'}^{\bar{t}}|\bar{t}''|^{2\nu}\cdot d\bar{t}''\Big)$$

Since all the variables of integration have a defined sign in the asymptotic limit $\bar{t}\to-\infty$, we can remove the absolute values into the expression above:[70][71]

$$\mathcal{G}(\overline{t}) \simeq 2\int_{-\infty}^{\bar{t}}d\bar{t}'\cdot\exp\Big(+2\int_{\bar{t}'}^{\bar{t}}|\bar{t}''|^{2\nu}\cdot d|\bar{t}''|\Big) = -2e^{2|\bar{t}|^{\kappa}/\kappa}\int_{-\infty}^{\bar{t}}d|\bar{t}'|\cdot e^{-2|\bar{t}'|^{\kappa}/\kappa}$$

$$= -2e^{|\bar{t}|^{\kappa}/\kappa}\Big(-2^{-1/\kappa}\cdot(\kappa)^{1/\kappa-1}\cdot\Gamma[1/\kappa,2|\bar{t}'|^{\kappa}/\kappa]\Big|_{-\infty}^{\bar{t}}\Big)$$

where $\kappa = 1+2\nu$. The function $\Gamma[w,z]$ is called *Incomplete Gamma* and its features are reported into appendixC. Because of (500), the previous results becomes

$$\mathcal{G}(\bar{t}) \simeq 2e^{|\bar{t}|^{\kappa}/\kappa}\Big(-2^{-1/\kappa}\cdot(\kappa)^{1/\kappa-1}\cdot\Gamma[1/\kappa,2|\bar{t}|^{\kappa}/\kappa]\Big)$$

By taking the leading part of the asymptotic expansion (502)

$$\mathcal{G}(\bar{t}) \sim -2e^{|\bar{t}|^{\kappa}/\kappa}\cdot 2^{-1/\kappa}\cdot(\kappa)^{1/\kappa-1}\cdot e^{-2|\bar{t}|^{\kappa}/\kappa}\cdot(2|\bar{t}|^{\kappa}/\kappa)^{1/\kappa-1} = |\bar{t}|^{1-\kappa} = |\bar{t}|^{-2\nu};$$

D.4.2 Leading off-equilibrium correction to the asymptotic equilibrium behaviour of the susceptibility.

We start from the expression (325) and compute the leaging off-equilibrium correction to the asymptotic equilibrium behaviour under the assumption (316):

$$\mathcal{G}(\bar{t}) \sim 2\int_{-\infty}^{\bar{t}}d\bar{t}'\cdot\exp\Big(-2\int_{\bar{t}'}^{\bar{t}}d\bar{t}''\cdot|\bar{t}|^{2\nu}(1+b|\bar{t}|^{a}\cdot e^{-c|\bar{t}|^{1+z\nu}}\Big)$$

[70] The variables of time integration are all ≤ 0 in the limit $\bar{t}\to-\infty$ because $\bar{t}''\in[\bar{t}',\bar{t}]$ and $\bar{t}'\in(-\infty,\bar{t}]$

[71] We have used the following relation into the integral:

$$\int F(|x|)\cdot dx = \int F(|x|)\cdot\frac{dx}{d|x|}\cdot d|x| = \int F(|x|)\cdot\frac{x}{|x|}\cdot d|x| = \int \mathrm{sgn}(x)\cdot F(|x|)\cdot d|x|$$

where $F(|x|)$ is a generic function.

$$= 2e^{2|\bar{t}|^\kappa/\kappa}\int_{-\infty}^{\bar{t}} d\bar{t}'\cdot e^{-2|\bar{t}'|^\kappa/\kappa}\cdot\Big\{1+2\int_{\bar{t}'}^{\bar{t}} d|\bar{t}''|\cdot b|\bar{t}|^{\kappa-1+a}\cdot e^{-c|\bar{t}|^\kappa}\Big\}$$

$$= |\bar{t}|^{1-\kappa}+4e^{2|\bar{t}|^\kappa/\kappa}\int_{-\infty}^{\bar{t}} d\bar{t}'\cdot e^{-2|\bar{t}'|^\kappa/\kappa}\Big(-b\cdot\kappa^{-1}c^{-(a+\kappa)/\kappa-1}\cdot\Gamma[(a+\kappa)/\kappa, c|\bar{t}''|^\kappa]\Big|_{\bar{t}'}^{\bar{t}}\Big)$$

We take the leading order of the expansion (502)

$$|\bar{t}|^{1-\kappa}+4e^{2|\bar{t}|^\kappa/\kappa}\int_{-\infty}^{\bar{t}} d\bar{t}'\cdot e^{-2|\bar{t}'|^\kappa/\kappa}\Big(-b\cdot\kappa^{-1}c^{-(a+\kappa)/\kappa-1}\cdot(c|\bar{t}''|^\kappa)^{(a+\kappa)/\kappa-1}\cdot e^{-c|\bar{t}''|^\kappa}\Big|_{\bar{t}'}^{\bar{t}}\Big)$$

$$= |\bar{t}|^{1-\kappa}+\Big\{-4\frac{b\cdot e^{2|\bar{t}|^\kappa/\kappa}}{c\kappa}\cdot e^{-c|\bar{t}|^\kappa}\cdot|\bar{t}|^a\int_{-\infty}^{\bar{t}}(-d|\bar{t}'|)\cdot e^{-2|\bar{t}'|^\kappa}\Big\}$$

$$-\Big\{-4\frac{b\cdot e^{2|\bar{t}|^\kappa/\kappa}}{c\kappa}\int_{-\infty}^{\bar{t}}(-d|\bar{t}'|)\cdot e^{-2|\bar{t}'|^\kappa/\kappa}\cdot|\bar{t}|^a\cdot e^{-c|\bar{t}'|^\kappa}\Big\}$$

$$= |\bar{t}|^{1-\kappa}+4\frac{b\cdot e^{2|\bar{t}|^\kappa/\kappa}}{c\kappa}\cdot e^{-c|\bar{t}|^\kappa}\cdot|\bar{t}|^a\Big(-2^{-1/\kappa}\cdot\kappa^{1/\kappa-1}\cdot\Gamma[1/\kappa, 2|\bar{t}'|^\kappa/\kappa]\Big|_{-\infty}^{\bar{t}}\Big)$$

$$-4\frac{b\cdot e^{2|\bar{t}|^\kappa/\kappa}}{c\kappa}\Big(-(2+c\kappa)^{-(1+a)/\kappa}\cdot\kappa^{(1+a)/\kappa-1}\cdot\Gamma[(1+a)/\kappa,(2+c\kappa)|\bar{t}'|^\kappa/\kappa]\Big|_{-\infty}^{\bar{t}}\Big)$$

We consider only the asymptotic leading term of the off-equilibrium:

$$\mathcal{G}(\bar{t})\sim|\bar{t}|^{1-\kappa}+4\frac{b\cdot e^{2|\bar{t}|^\kappa/\kappa}}{c\kappa}\cdot e^{-c|\bar{t}|^\kappa}\cdot|\bar{t}|^a\Big(-2^{-1/\kappa}\cdot\kappa^{1/\kappa-1}\cdot(2|\bar{t}|^\kappa/\kappa)^{1/\kappa-1}\cdot e^{-2|\bar{t}|^\kappa/\kappa}\Big)$$

$$-4\frac{b\cdot e^{2|\bar{t}|^\kappa/\kappa}}{c\kappa}\Big(-(2+c\kappa)^{-(1+a)/\kappa}\cdot\kappa^{(1+a)/\kappa-1}\cdot((2+c\kappa)|\bar{t}|^\kappa/\kappa)^{(1+a)/\kappa-1}\cdot e^{-c|\bar{t}|^\kappa}\cdot e^{-2|\bar{t}|^\kappa/\kappa}\Big)$$

Thus,

$$\mathcal{G}(\bar{t})\sim|\bar{t}|^{1-\kappa}-2\frac{b\cdot e^{-c|\bar{t}|^\kappa}}{c\kappa}|\bar{t}|^a\cdot|\bar{t}|^{1-\kappa}+4\frac{b\cdot e^{-c|\bar{t}|^\kappa}}{c\kappa(c\kappa+2)}\cdot|\bar{t}|^{1-\kappa+a}$$

$$= |\bar{t}|^{1-\kappa}\Big(1+b\cdot\Big(\frac{-2}{(2+c\kappa)}\Big)|\bar{t}|^a\cdot e^{-c|\bar{t}|^\kappa}\Big).$$

D.5 Magnetic field protocol:

D.5.1 Equilibrium contribution of the magnetization.

We insert the equilibrium term of $\mathcal{M}^2$ given by (364) in the scaling function of the magnetization (351):

$$\Theta(\bar{t})\sim\int_{-\infty}^{\bar{t}} d\bar{t}'\cdot\bar{t}'\cdot\exp\Big(-\int_{\bar{t}'}^{\bar{t}} d\bar{t}''\cdot\mathcal{M}^2(\bar{t}'')\Big)\underset{\mathcal{M}^2\simeq\mathcal{M}_e^2}{\overset{\bar{t}\to-\infty}{\sim}}$$

$$\int_{-\infty}^{\bar{t}} d\bar{t}'\cdot\bar{t}'\cdot\exp\Big(-\int_{\bar{t}'}^{\bar{t}} d\bar{t}''\cdot|\bar{t}''|^{2\nu}\Big)=\int_{-\infty}^{\bar{t}} d\bar{t}'\cdot\bar{t}'\cdot\exp\Big(+\int_{\bar{t}'}^{\bar{t}}|\bar{t}''|^{2\nu}\cdot d|\bar{t}''|\Big)$$

$$= e^{|\bar{t}|^\kappa/\kappa}\int_{-\infty}^{\bar{t}} d|\bar{t}'|\cdot|\bar{t}'|\cdot e^{-|\bar{t}'|^\kappa/\kappa}=e^{|\bar{t}|^\kappa/\kappa}\Big(-(\kappa)^{2/\kappa-1}\cdot\Gamma[2/\kappa,|\bar{t}'|^\kappa/\kappa]\Big|_{-\infty}^{\bar{t}}\Big)$$

$$= e^{|\bar{t}|^\kappa/\kappa}\Big(-(\kappa)^{2/\kappa-1}\cdot\Gamma[2/\kappa,|\bar{t}|^\kappa/\kappa]\Big);$$

where $\kappa=2\nu_h+1=2/d_h+1$. By taking the leading part of the asymptotic expansion (502),

$$\Theta(\bar{t})\sim e^{|\bar{t}|^\kappa/\kappa}\Big(-(\kappa)^{2/\kappa-1}\cdot(|\bar{t}|^\kappa/\kappa)^{(2/\kappa)-1}\cdot e^{-|\bar{t}|^\kappa/\kappa}\Big)=-|\bar{t}|^{2-\kappa}=-|\bar{t}|^{d_\phi/d_h}.$$

If we assume that the system is at the equilibrium for all times:

$$\Theta(\bar{t}) \sim \int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot \exp\Big(- \int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot \mathcal{M}^2(\bar{t}'')\Big) \underset{\mathcal{M}^2=\mathcal{M}_e^2}{\overset{\bar{t}\to+\infty}{\sim}} \int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot \exp\Big(- \int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot |\bar{t}''|^{2\nu}\Big)$$

We divide the integral into two terms in order to have a defined sign of integral variables:

$$\Theta(\bar{t}) \sim \int_{-\infty}^{0} d|\bar{t}'| \cdot |\bar{t}'| \cdot \exp\Big(+ \int_{\bar{t}'}^{\bar{t}} d|\bar{t}''| \cdot |\bar{t}''|^{2\nu}\Big) + \int_{0}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot \exp\Big(- \int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot \bar{t}''^{2\nu}\Big)$$

$$= \int_{-\infty}^{0} d|\bar{t}'| \cdot |\bar{t}'| \cdot e^{-|\bar{t}'|^{\kappa}/\kappa} + e^{-\bar{t}^{\kappa}/\kappa} \int_{0}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot e^{\bar{t}'^{\kappa}/\kappa}$$

$$= \Big(- (\kappa)^{(2/\kappa)-1} \cdot \Gamma[2/\kappa, |\bar{t}'|^{\kappa}/\kappa]\Big|_{-\infty}^{0}\Big) + e^{-\bar{t}^{\kappa}/\kappa}\Big(- (\kappa)^{(2/\kappa)-1} \cdot \Gamma[2/\kappa, -\bar{t}'^{\kappa}/\kappa]\Big|_{0}^{\bar{t}}\Big)$$

Using the relation (500) and (501)

$$\Theta(\bar{t}) \sim -(\kappa)^{(2/\kappa)-1} \cdot \Gamma(2/\kappa) - e^{-\bar{t}^{\kappa}/\kappa} \cdot (\kappa)^{(2/\kappa)-1} \cdot \Gamma[2/\kappa, -\bar{t}^{\kappa}/\kappa] + e^{-\bar{t}^{\kappa}/\kappa} \cdot (\kappa)^{(2/\kappa)-1} \cdot \Gamma(2/\kappa)$$

Thus, by keeping the leading part of the asymptotic expansion (502):

$$\Theta(\bar{t}) \sim (\kappa)^{(2/\kappa)-1} \cdot \Gamma(2/\kappa)\Big(1 - e^{-\bar{t}^{\kappa}/\kappa}\Big) + \bar{t}^{2-\kappa} \sim +\bar{t}^{2-\kappa} = \bar{t}^{d_\phi/d_h}.$$

In terms of the scaling function $\widetilde{\Theta}$, the asymptotic equilibrium contribution at $\bar{t} \to -\infty$ can be computed using heuristic arguments:

$$\widetilde{\Theta}(\bar{t}) = \pm|\bar{t}|^{-d_\phi/d_h} \int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot \exp\Big(- \int_{\bar{t}'}^{\bar{t}} \mathcal{M}^2(\bar{t}'') \cdot d\bar{t}''\Big) \underset{\mathcal{M}^2\simeq\mathcal{M}_e^2}{\overset{\bar{t}\to\pm\infty}{\sim}}$$

$$\sim \pm \int_{-\infty}^{\bar{t}} d\bar{t}' \cdot (|\bar{t}|/|\bar{t}'|)^{-d_\phi/d_h} |\bar{t}'|^{1-d_\phi/d_h} \cdot \exp\Big(- \int_{\bar{t}'}^{\bar{t}} |\bar{t}''|^{2/d_h} \cdot d\bar{t}''\Big)$$

The ratio $|\bar{t}'|/|\bar{t}|$ tends to zero for every finite value of $\bar{t}'$. Thus, this integral has a non-trivial contribution only on the tails where we can write $|\bar{t}'|/|\bar{t}| \simeq 1$. It follows that

$$\widetilde{\Theta}(\bar{t}) \overset{\bar{t}\to-\infty}{\sim} - \int_{-\infty}^{\bar{t}} d\bar{t}' \cdot |\bar{t}'|^{1-d_\phi/d_h} \cdot \exp\Big(+ |\bar{t}|^{\kappa}/\kappa - |\bar{t}'|^{\kappa}/\kappa\Big)$$

$$= -e^{+|\bar{t}|^{\kappa}/\kappa} \int_{-\infty}^{\bar{t}} d|\bar{t}'| \cdot |\bar{t}'|^{\kappa-1} \cdot e^{-|\bar{t}'|^{\kappa}/\kappa} = -1.$$

If we assume that the system is at the equilibrium for all times, the contribution on the tail $\bar{t} \to +\infty$ can be computed in the same way and gives $\widetilde{\Theta}(\bar{t}) \propto +1$.

D.5.2 Leading off-equilibrium correction to the asymptotic equilibrium behaviour of the magnetization.

We start from the expression (377) and compute the leaging off-equilibrium correction to the asymptotic equilibrium behaviour under the assumption (316):

$$\Theta(\bar{t}) \sim -|\bar{t}|^{2-\kappa} - e^{|\bar{t}|^{\kappa}/\kappa} \int_{-\infty}^{\bar{t}} d\bar{t}' \cdot e^{-|\bar{t}'|^{\kappa}/\kappa} \cdot \bar{t}' \cdot \Big(\int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot |\bar{t}''|^{2\nu_h} \cdot b \cdot |\bar{t}''|^{a} \cdot \exp(-c|\bar{t}''|^{1+z\nu_h})\Big)$$

$$= -|\bar{t}|^{2-\kappa} + e^{|\bar{t}|^{\kappa}/\kappa} \int_{-\infty}^{\bar{t}} d\bar{t}' \cdot e^{-|\bar{t}'|^{\kappa}/\kappa} \cdot \bar{t}' \cdot \Big(- b \cdot \kappa^{-1} c^{-(a+\kappa)/\kappa} \cdot \Gamma[\frac{a+\kappa}{\kappa}, c|\bar{t}''|^{\kappa}]\Big)\Big|_{\bar{t}'}^{\bar{t}}$$

Using the asympotic expansion (502) and keeping only the leading term we obtain

$$= -|\overline{t}|^{2-\kappa} + e^{|\overline{t}|^{\kappa}/\kappa} \int_{-\infty}^{\overline{t}} d\overline{t}' \cdot e^{-|\overline{t}'|^{\kappa}/\kappa} \cdot \overline{t}' \cdot \Big(- b \cdot \kappa^{-1} c^{-(a+\kappa)/\kappa} \cdot e^{-c|\overline{t}''|^{\kappa}} \cdot (c|\overline{t}''|^{\kappa})^{\frac{a+\kappa}{\kappa}-1}\Big|_{\overline{t}'}^{\overline{t}} \Big)$$

$$= -|\overline{t}|^{2-\kappa} + \Big\{ - b \cdot e^{|\overline{t}|^{\kappa}/\kappa} \cdot \frac{e^{-c|\overline{t}|^{\kappa}}}{c\kappa} \cdot |\overline{t}|^{a} \int_{-\infty}^{\overline{t}} d|\overline{t}'| \cdot e^{-|\overline{t}'|^{\kappa}/\kappa} \cdot |\overline{t}'| \Big\} +$$

$$\Big\{ \frac{b \cdot e^{|\overline{t}|^{\kappa}/\kappa}}{c\kappa} \int_{-\infty}^{\overline{t}} d|\overline{t}'| \cdot e^{-|\overline{t}'|^{\kappa}/\kappa} \cdot e^{-c|\overline{t}'|^{\kappa}} \cdot |\overline{t}'|^{a+1} \Big\};$$

The integrals in the last expression can be solved in terms of Incomplete Gamma functions too:

$$\Theta(\overline{t}) \sim -|\overline{t}|^{2-\kappa} - b \cdot e^{|\overline{t}|^{\kappa}/\kappa} \cdot \frac{e^{-c|\overline{t}|^{\kappa}}}{c\kappa} \cdot |\overline{t}|^{a} \Big((-\kappa)^{2/\kappa-1} \cdot \Gamma[2/\kappa, |\overline{t}'|^{\kappa}/\kappa] \Big|_{-\infty}^{\overline{t}} \Big)$$

$$+ \frac{b \cdot e^{|\overline{t}|^{\kappa}/\kappa}}{c\kappa} \Big(- \kappa^{-1} \cdot (\frac{c\kappa+1}{\kappa})^{-(2+a)/\kappa} \cdot \Gamma[(2+a)/\kappa, \frac{c\kappa+1}{\kappa} |\overline{t}'|^{\kappa}] \Big|_{-\infty}^{\overline{t}} \Big);$$

We keep only the leading asymptotic contribution to the off-equilibrium:

$$\Theta(\overline{t}) \sim -|\overline{t}|^{2-\kappa} - b \cdot e^{|\overline{t}|^{\kappa}/\kappa} \cdot \frac{e^{-c|\overline{t}|^{\kappa}}}{c\kappa} \cdot |\overline{t}|^{a} \Big(- (\kappa)^{2/\kappa-1} \cdot (|\overline{t}|^{2-\kappa}/\kappa) \cdot e^{-|\overline{t}|^{\kappa}/\kappa} \Big)$$

$$+ \frac{b \cdot e^{-|\overline{t}|^{\kappa}/\kappa}}{c\kappa} \Big((1+c\kappa)^{-(2+a)/\kappa} \cdot \kappa^{(2+a)/\kappa-1} \cdot (\frac{(1+c\kappa)}{\kappa} |\overline{t}|^{\kappa})^{(2+a)/\kappa-1} \cdot e^{-|\overline{t}|^{\kappa}/\kappa} \cdot e^{-c|\overline{t}|^{\kappa}} \Big)$$

Thus,

$$\Theta(\overline{t}) \sim -|\overline{t}|^{2-\kappa} + \frac{b \cdot e^{-c|\overline{t}|^{\kappa}}}{c\kappa} \cdot |\overline{t}|^{a} \cdot |\overline{t}|^{2-\kappa} - \frac{b \cdot e^{-c|\overline{t}|^{\kappa}}}{c\kappa(c\kappa+1)} \cdot |\overline{t}|^{2+a-\kappa}$$

$$= -|\overline{t}|^{2-\kappa} \Big(1 + b \cdot \Big(\frac{-1}{(c\kappa+1)} \Big) \cdot |\overline{t}|^{a} \cdot e^{-c|\overline{t}|^{\kappa}} \Big).$$

D.5.3 About the eq.(356) and the power-law decay ansatz.

Let us consider the scaling constraint-equation (356). The left hand side of the equation involves an integral over the momenta of the transverse two-point function. It is gaussian and can be performed [See Eq.(354)]:

$$\int^{+\infty} \frac{d^d \overline{k}}{(2\pi)^d} \Big[\mathcal{G}_T(\overline{k}, \overline{t}, \mathcal{M}^2) - \mathcal{G}_T(\overline{k}, \overline{t}, 0) \Big]$$

$$= \frac{2}{(4\pi)^{d/2}} \int_{-\infty}^{\overline{t}} d\overline{t}' \cdot \Big\{ \exp \Big(- 2 \int_{\overline{t}'}^{\overline{t}} d\overline{t}'' \cdot \mathcal{M}^2(\overline{t}'') \Big) - 1 \Big\} \cdot |\overline{t} - \overline{t}'|^{-d/2}$$

In the asymptotic limit, under the assumption (316):

$$\frac{2}{(4\pi)^{d/2}} \int_{-\infty}^{\overline{t}} d\overline{t}' \cdot \Big\{ \exp \Big(- 2 \int_{\overline{t}'}^{\overline{t}} d\overline{t}'' \cdot (\mathcal{M}_e^2(\overline{t}'') + \mathcal{M}_o^2(\overline{t}'')) \Big) - 1 \Big\} \cdot |\overline{t} - \overline{t}'|^{-d/2} \overset{\overline{t} \to -\infty}{\sim}$$

$$\frac{2}{(4\pi)^{d/2}} \int_{-\infty}^{\overline{t}} d\overline{t}' \cdot \Big\{ e^{-2 \int_{\overline{t}'}^{\overline{t}} d\overline{t}'' \cdot \mathcal{M}_e^2(\overline{t}'')} \cdot \Big(1 - 2 \int_{\overline{t}'}^{\overline{t}} d\overline{t}'' \cdot \mathcal{M}_o^2(\overline{t}'') \Big) - 1 \Big\} \cdot |\overline{t} - \overline{t}'|^{-d/2}$$

If we insert the asymptotic ansatz for the scaling function $\mathcal{M}^2$:

$$= -\frac{2}{(4\pi)^{d/2}} \int_{-\infty}^{\overline{t}} d|\overline{t}'| \cdot |\overline{t} - \overline{t}'|^{-d/2} \Big[\Big\{ e^{2|\overline{t}|^{\kappa}/\kappa} \cdot e^{-|2\overline{t}'|^{\kappa}/\kappa} - 1 \Big\}$$

$$+2\Big\{e^{2|\overline{t}|^\kappa/\kappa}\cdot e^{-2|\overline{t}'|^\kappa/\kappa}\Big(\int_{\overline{t}'}^{\overline{t}} d|\overline{t}''|\cdot b\cdot|\overline{t}''|^{2\nu+a}\cdot e^{-c|\overline{t}''|^\kappa}\Big)-1\Big\}\Big]$$

$$=-\frac{2}{(4\pi)^{d/2}}\int_{-\infty}^{\overline{t}} d|\overline{t}'|\cdot|\overline{t}-\overline{t}'|^{-d/2}\Big[\Big\{e^{2|\overline{t}|^\kappa/\kappa}\cdot e^{-2|\overline{t}'|^\kappa/\kappa}-1\Big\}$$

$$+2\Big\{e^{2|\overline{t}|^\kappa/\kappa}\cdot e^{-2|\overline{t}'|^\kappa/\kappa}\Big(-b\cdot\kappa^{-1}c^{-(a+\kappa)/\kappa}\cdot\Gamma[(a+\kappa(/\kappa,c|\overline{t}''|^\kappa]\Big|_{\overline{t}'}^{\overline{t}}\Big)-1\Big\}\Big].$$

If we consider the leading order of the asymptotic expansion (502):

$$=-\frac{2}{(4\pi)^{d/2}}\int_{-\infty}^{\overline{t}} d|\overline{t}'|\cdot|\overline{t}-\overline{t}'|^{-d/2}\Big[\Big\{e^{2|\overline{t}|^\kappa/\kappa}\cdot e^{-2|\overline{t}'|^\kappa/\kappa}-1\Big\}$$

$$-2\Big\{e^{2|\overline{t}|^\kappa/\kappa}\cdot e^{-2|\overline{t}'|^\kappa/\kappa}\Big(\frac{b\cdot e^{-c|\overline{t}|^\kappa}}{c\kappa}\cdot|\overline{t}|^a-\frac{b\cdot e^{-c|\overline{t'}|^\kappa}}{c\kappa}\cdot|\overline{t'}|^a\Big)-1\Big\}\Big];$$

The term due to the integral over the momenta makes the relation above not writable in terms of special functions.
However, a different ansatz like (382) leads to inconsistences. If we perform the leading off-equilibrium corrections in the magnetization under the assumption (382):

$$\Theta(\overline{t})=-|\overline{t}|^{2-\kappa}+e^{|\overline{t}|^\kappa/\kappa}\int_{-\infty}^{\overline{t}} d\overline{t}'\cdot\overline{t}'\cdot e^{-|\overline{t}'|^\kappa/\kappa}\cdot\Big(-\int_{\overline{t}'}^{\overline{t}} d\overline{t}''\cdot|\overline{t}''|^{2\nu_h}\cdot b|\overline{t}|^{-\alpha}\Big)$$

$$=-|\overline{t}|^{2-\kappa}+e^{|\overline{t}|^\kappa/\kappa}\int_{-\infty}^{\overline{t}} d|\overline{t}'|\cdot|\overline{t}'|\cdot e^{-|\overline{t}'|^\kappa/\kappa}\Big(\frac{b|\overline{t}''|^{\kappa-\alpha}}{(\kappa-\alpha)}\Big|_{\overline{t}'}^{\overline{t}}\Big)$$

$$=-|\overline{t}|^{2-\kappa}+\Big\{e^{|\overline{t}|^\kappa/\kappa}\cdot\frac{b|\overline{t}|^{\kappa-\alpha}}{(\kappa-\alpha)}\int_{-\infty}^{\overline{t}} d|\overline{t}'|\cdot|\overline{t}'|\cdot e^{-|\overline{t}'|^\kappa/\kappa}\Big\}-\Big\{\frac{b\cdot e^{|\overline{t}|^\kappa/\kappa}}{(\kappa-\alpha)}\cdot\int_{-\infty}^{\overline{t}} d|\overline{t}'|\cdot|\overline{t}'|\cdot e^{-|\overline{t}'|^\kappa/\kappa}\cdot|\overline{t}'|^{\kappa-\alpha}\Big\}$$

$$=-|\overline{t}|^{2-\kappa}+e^{|\overline{t}|^\kappa/\kappa}\cdot\frac{b|\overline{t}|^{\kappa-\alpha}}{(\kappa-\alpha)}\cdot\Big(-\kappa^{2/\kappa-1}\cdot\Gamma[2/\kappa,|\overline{t}'|^\kappa/\kappa]\Big|_{-\infty}^{\overline{t}}\Big)-$$

$$\frac{b\cdot e^{|\overline{t}|^\kappa/\kappa}}{(\kappa-\alpha)}\cdot\Big(-\kappa^{(2-\alpha)/\kappa}\cdot\Gamma[(2-\alpha+\kappa)/\kappa,|\overline{t}'|^\kappa/\kappa]\Big|_{-\infty}^{\overline{t}}\Big)$$

We keep only the leading part of the asymptotic expansion (502):

$$\Theta(\overline{t})\sim-|\overline{t}|^{2-\kappa}+e^{|\overline{t}|^\kappa/\kappa}\cdot\frac{b|\overline{t}|^{\kappa-\alpha}}{(\kappa-\alpha)}\cdot\Big(-\kappa^{2/\kappa-1}\cdot(|\overline{t}|^\kappa/\kappa)^{2/\kappa-1}\cdot e^{-|\overline{t}|^\kappa/\kappa}\Big)$$

$$-\frac{b\cdot e^{|\overline{t}|^\kappa/\kappa}}{(\kappa-\alpha)}\cdot\Big(-\kappa^{(2-\alpha)/\kappa}\cdot(|\overline{t}|^\kappa/\kappa)^{(2-\alpha+\kappa)/\kappa-1}\cdot e^{-|\overline{t}|^\kappa/\kappa}\Big)$$

$$=-|\overline{t}|^{2-\kappa}-\frac{b|\overline{t}|^{\kappa-\alpha}}{(\kappa-\alpha)}\cdot|\overline{t}|^{2-\kappa}+\frac{b|\overline{t}|^{2-\alpha}}{(\kappa-\alpha)}=-|\overline{t}|^{2-\kappa}.$$

The off-equilibrium terms cancel themselves and therefore the relation (379) cannot be satisfied asymptotically.

E Appendix: Cross-over behaviour in the momenta below the critical temperature.

We consider the transverse two-point correlation function $G_T(k,t,t_s)$ below the critical temperature. Its general expression is given by the relation (345):

$$G_T(k,t,t_s) = 2\int_{t_0}^{t} dt' \cdot \exp\Big(-2\int_{t'}^{t} dt'' \cdot k^2 + m^2(t'',t_s)\Big).$$

Since we consider the asymptotic behaviour of the system, we can roughly approximate the mass term as:

$$m^2(t,t_s) \approx \frac{h(t,t_s)}{\Sigma(t,t_s)} \approx \frac{|t|}{t_s\sigma}.$$

We already know that this approximation breaks down when $t \simeq 0$ where the magnetization cannot be considered a constant. However, this relation permits to compute explicitly the correlation functions. In particular we study the transverse two-point correlation function:

$$G_T(k,t,t_s) \sim 2\int_{t_0}^{t} dt' \cdot \exp\Big(2\int_{t'}^{t} |dt''| \cdot \Big(k^2 + \frac{|t''|}{t_s\sigma}\Big)\Big) = 2\int_{t_0}^{t} dt' \cdot e^{2k^2(|t|-|t'|)} \cdot \exp\Big(\frac{1}{t_s\sigma}(|t|^2 - |t'|^2)\Big)$$

$$= 2e^{2k^2|t| + (|t|^2/t_s\sigma)} \cdot \int_{t_0}^{t} dt' \cdot \exp\Big(-2k^2|t'| - \frac{|t'|^2}{t_s\sigma}\Big)$$

$$= 2e^{2k^2|t| + (|t|^2/t_s\sigma)} \cdot \Big\{ \frac{\sqrt{\pi}}{2}\sqrt{\sigma t_s}\exp\Big(\sqrt{\sigma t_s}k^4\Big) \cdot \mathrm{Erfc}\Big(\frac{\sigma t_s k^2 + |t'|}{\sqrt{\sigma t_s}}\Big|_{t_0}^{t}\Big)\Big\}.$$

If we set $s = (k^2 t_s\sigma + |t|)/\sqrt{t_s\sigma}$, the result is:[72]

$$G_T(k,t,t_s) \sim e^{s^2} \cdot \sqrt{\pi\sigma t_s} \cdot \Big\{\mathrm{Erfc}(s) - \mathrm{Erfc}(s_0)\Big\}. \tag{514}$$

If we take the leading part of the asymptotic expansion (513) for s large[73] , we obtain:

$$G_T(s,t_s) \sim e^{s^2} \cdot \sqrt{\pi\sigma t_s} \cdot \frac{e^{-s^2}}{\sqrt{\pi}s} = \frac{\sqrt{\sigma t_s}}{s} = \frac{1}{k^2 + |h(t,t_s)|/\sigma} \sim \frac{1}{k^2 + m^2(t,t_s)}.$$

We recover the equilibrium value of the transverse two-point correlation function.
Note that the asymptotic limit which leads to the last expression is $s \to \infty$ that does not necessary imply $|\bar{t}| \to \infty$. Since we consider large-momenta, the system appear at instant thermal equilibrium at all the times [see ref.[21] and [28]].
Below the critical temperature the fluctuations over the reference ground state are essentially given by the Goldstone-waves $\varphi(k,t)$. We perform a *spin-wave approximation* for large momenta: we distinguish the short and the long-wavelength fluctuations:

$$\varphi(k,t) = \varphi(k < k^\diamond, t) + \varphi(k > k^\diamond, t) = \theta(k,t) + \delta\theta(k,t). \tag{515}$$

In the Fourier space we have differentiated the modes in terms of the momenta: $\theta(k,t)$ has only modes $k < k^\diamond$ and $\delta\theta(k,t)$ is a short-wave $k > k^\diamond$ fluctuation.
The short-distance fluctuations are roughly at instantaneus thermal equillibrium at all times. Thus, they are quite well-described by the non-interacting spin-wave approximation. The long-wavelength fluctuations are affected by the non-linearities present in the system and

[72] The constant $s_0 = k^2 t_s\sigma + t_0)/\sqrt{t_s\sigma}$

[73] We have neglected the value of the Erfc at s_0 because since $|t_0|$ is very large $|t_0| \gg |t|$, the value of the Erfc tends to zero.

therefore cannot be treated in the same way. In order to investigate the off-equilibrium behaviour of the system and the following hysteresis phenomena, we are interested only to the long-wavelength modes θ.
The value of momentum $k^\diamond$ defines a cross-over behaviour and separetes the local-fluctuation regime from the spin-waves as we have qualitatively seen in Sec.1,1.2.8. Its value can be computed self-consistently: if we assume that the modes $\delta\theta(k,t)$ are at the instantaneous equilibrium for all times, the previous approximation for the mass term remains valid even when $t \to 0$. Thus, we can identify $k^\diamond$ as the value in which $s \sim O(1)$ at $t = 0$ [see ref.[21]]. It follows that

$$k^\diamond = (1/t_s\sigma)^{1/4} \tag{516}$$

When we perform the KZ scaling limit, we enlarge the interval of the small momenta: $k \cdot (t_s)^{1/2d}$ remains fixed when $t_s \to \infty$. In particular, the rescaled value of $k^\diamond$ is:

$$k^\diamond \cdot (t_s)^{1/2d} \sim (1/t_s\sigma)^{1/4} \cdot (t_s)^{1/2d} \propto (t_s)^{-(d-2)/4d} \to 0 \tag{517}$$

because $2 < d < 4$. Since we consider very-low frequency protocols $t_s \to \infty$, the cross-over value of the momenta $k^\diamond$ tends to zero and this is equivalent to consider the momenta as subleading terms in the Kibble-Zurek scaling limit.

F Algebraic computations of Sec.7

F.1 Asymptotic equilibrium behaviours:

F.1.1 Magnetization.

We check the consistence of the relation (448) in the asymptotic equilibrium limit i.e. assuming that the scaling function of the mass term is given by (455):

$$\cos(\vartheta(\bar{t})) = \frac{1}{\sigma}\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot \exp\Big(-\int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot \mathcal{M}^2(\bar{t}'')\Big) \underset{\mathcal{M}^2\simeq\mathcal{M}_e^2}{\overset{\bar{t}\to-\infty}{\sim}}$$

$$\frac{1}{\sigma}\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot \exp\Big(\frac{1}{\sigma}\int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot \bar{t}''\Big) = \frac{1}{\sigma}\int_{-\infty}^{\bar{t}} d|\bar{t}'| \cdot |\bar{t}'| \cdot \exp\Big(\frac{1}{\sigma}\int_{\bar{t}'}^{\bar{t}} d|\bar{t}''| \cdot |\bar{t}''|\Big)$$

$$= \frac{1}{\sigma}\int_{-\infty}^{\bar{t}} d|\bar{t}'| \cdot |\bar{t}'| \cdot \exp\Big\{\frac{1}{2\sigma}\Big(|\bar{t}|^2 - |\bar{t}'|^2\Big)\Big\} = \frac{1}{\sigma} e^{|\bar{t}|^2/2\sigma}\int_{-\infty}^{\bar{t}} d|\bar{t}'| \cdot |\bar{t}'| \cdot e^{-|\bar{t}'|^2/2\sigma} = -1.$$

The limit $\bar{t}\to+\infty$ cannot be performed because of memory effects. However, if we consider the system at the equilibrium for all times, the relation (455) permits to compute also the asymptotic limit after the transition:

$$\cos(\vartheta(\bar{t})) = \frac{1}{\sigma}\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot \exp\Big(-\int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot \mathcal{M}^2(\bar{t}'')\Big) \underset{\mathcal{M}^2=\mathcal{M}_e^2}{\overset{\bar{t}\to+\infty}{\sim}}$$

$$\frac{1}{\sigma}\Big\{\int_{-\infty}^{0} d\bar{t}' \cdot \bar{t}' \cdot \exp\Big(-\frac{1}{\sigma}\int_{\bar{t}'}^{0} d\bar{t}'' \cdot |\bar{t}''|\Big) + \int_{0}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot \exp\Big(-\frac{1}{\sigma}\int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot \bar{t}''\Big)\Big\}$$

$$= \frac{1}{\sigma}\Big\{\int_{-\infty}^{0} d|\bar{t}'| \cdot |\bar{t}'| \cdot \exp\Big(-\frac{1}{2\sigma}|\bar{t}'|^2\Big) + \int_{0}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot \exp\Big(-\frac{1}{2\sigma}(\bar{t}^2 - \bar{t}'^2)\Big)\Big\}$$

$$= \frac{1}{\sigma}\Big\{\int_{-\infty}^{0} d|\bar{t}'| \cdot |\bar{t}'| \cdot e^{-|\bar{t}'|^2/2\sigma} + e^{-\bar{t}^2/2\sigma}\int_{0}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot e^{+\bar{t}'^2/2\sigma}\Big\}$$

The first term represent the value of the magnetization before the transition:

$$\frac{1}{\sigma}\int_{-\infty}^{0} d|\bar{t}'| \cdot |\bar{t}'| \cdot e^{-|\bar{t}'|^2/2\sigma} = -1.$$

The second term represent the value of the magnetization after the transition:

$$\frac{1}{\sigma} e^{-\bar{t}^2/2\sigma}\int_{0}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot e^{+\bar{t}'^2/2\sigma} = \frac{1}{\sigma} e^{-\bar{t}^2/2\sigma}\Big(-1 + e^{+\bar{t}^2/2\sigma}\Big) \overset{\bar{t}\to+\infty}{\sim} +1.$$

which is in agreement with our expectations. The computation does not lead to the correct result because the magnetization present a non-analicity in zero. The correct interpretation is to divide the two contributes or equivently assume that the value in zero is zero for both the sides.

F.1.2 Transverse susceptibility.

We consider the transverse two-point correlation function (345) at zero momenta and we investigate the asymptotic equilibrium behaviour using (455):

$$\mathcal{G}_T(\bar{t}) = 2\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \exp\Big(-2\int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot \mathcal{M}^2(\bar{t}'')\Big) \underset{\mathcal{M}^2\simeq\mathcal{M}_e^2}{\overset{\bar{t}\to-\infty}{\sim}} 2\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \exp\Big(-2\int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot \frac{|\bar{t}''|}{\sigma}\Big)$$

$$= 2\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \exp\Big(\frac{1}{\sigma}(|\bar{t}|^2 - |\bar{t}'|^2\Big) = 2e^{|\bar{t}|^2/\sigma}\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot e^{-|\bar{t}'|^2/\sigma} =$$

$$2e^{|\bar{t}|^2/\sigma} \cdot \Big\{\frac{\sqrt{\pi}}{2} \cdot \sqrt{\sigma} \cdot \mathrm{Erfc}\Big(|\bar{t}'|/\sqrt{\sigma}\Big)\Big|_{-\infty}^{\bar{t}}\Big\}.$$

The function Erfc is called *complementary Error function* and its general features are reported into the appendix C. Using the relations (510) and taking the leading term of the asymptotic expansion (513), we find:

$$\mathcal{G}_T(\bar{t}) \sim 2e^{|\bar{t}|^2/\sigma} \cdot \Big\{\frac{\sqrt{\pi}}{2} \cdot \sqrt{\sigma} \cdot \frac{e^{-|\bar{t}|^2/\sigma}}{(\sqrt{\pi} \cdot |\bar{t}|/\sqrt{\sigma})}\Big\} = \sigma/|\bar{t}|.$$

F.2 Phase dynamics:

F.2.1 Equation for the phase dynamics.

The equation of motion for the planar vector follows from (460):

$$\dot{\phi}(x,t) = \sigma i\dot{\theta}(x,t)e^{i\theta(x,t)}(1+\varrho(x,t)) + \sigma e^{i\theta(x,t)}\dot{\varrho}(x,t) =$$

$$\sigma\nabla^2\Big(e^{i\theta(x,t)}(1+\varrho(x,t))\Big) - \sigma \cdot m^2(t,t_s)(1+\varrho(x;t))e^{i\theta(x,t)} + h(t,t_s) + \varsigma(x,t)$$

where the magnetic field is composed only by a real component and acts on the dynamics of the real part of ϕ which is $\sigma(1+\varrho)\cos(\theta)$. The noise ς is a complex vector $\varsigma(x,t) = \varsigma_1(x,t) + i\varsigma_2(x,t)$ with cumulants given by (206) and (207). Thus,

$$= \sigma\nabla\Big(i\nabla\theta(x,t)e^{i\theta(x,t)}(1+\varrho(x,t)) + e^{i\theta(x,t)} \cdot \nabla\varrho(x,t)\Big)$$

$$-\sigma \cdot m^2(t,t_s)(1+\varrho(x;t))e^{i\theta(x,t)} + h(t,t_s) + \varsigma(x,t)$$

$$= \sigma\Big(i\nabla^2\theta(x,t)e^{i\theta(x,t)}(1+\varrho(x,t)) + (\nabla\theta(x,t))^2e^{i\theta(x,t)}(1+\varrho(x;t)) + 2i\nabla\theta(x,t)\cdot\nabla\varrho(x,t)e^{i\theta(x,t)}\Big)$$

$$-\sigma \cdot m^2(t,t_s)(1+\varrho(x;t))e^{i\theta(x,t)} + h(t,t_s) + \varsigma(x,t)$$

It follows that

$$i\dot{\theta}(x,t)(1+\varrho(x,t)) + \dot{\varrho}(x,t) =$$

$$i\nabla^2\theta(x,t)(1+\varrho(x,t)) + (\nabla\theta(x,t))^2(1+\varrho(x,t)) + 2i\nabla\theta(x,t)\cdot\nabla\varrho(x,t)$$

$$-m^2(t,t_s)(1+\varrho(x;t)) + \frac{h(t,t_s)}{\sigma}e^{-i\theta(x,t)} + \frac{\varsigma(x,t)}{\sigma}e^{-i\theta(x,t)};$$

We can obtain an equation for the phase dynamics by keeping the immaginary part of the previous equation:

$$\dot{\theta}(x,t)(1+\varrho(x,t)) = \nabla^2\theta(x,t)(1+\varrho(x,t)) + 2\nabla\theta(x,t)\cdot\nabla\varrho(x,t)$$

$$-\frac{h(t,t_s)}{\sigma}\sin\Big(\theta(x,t)\Big) - \frac{\varsigma_1(x,t)}{\sigma}\sin(\theta(x,t)) + \frac{\varsigma_2(x,t)}{\sigma}\cos(\theta(x,t))$$

and from the real part

$$\dot{\varrho}(x,t) = (\nabla\theta(x,t))^2(1+\varrho(x,t)) - m^2(t,t_s)(1+\varrho)$$

$$+\frac{h(t,t_s)}{\sigma}\cos\Big(\theta(x,t)\Big) + \frac{\varsigma_1(x,t)}{\sigma}\cos(\theta(x,t)) + \frac{\varsigma_2(x,t)}{\sigma}\sin(\theta(x,t)).$$

F.2.2 Solution for the phase dynamics.

Let us find a solution to the equation (464). Firstly, we solve the homogeneous one:

$$\int_{\theta_0}^{\theta(t)} \frac{d\theta}{\sin\theta} = \log(\tan(\theta/2))\Big|_{\theta_0}^{\theta(t)} = -\int_{t_0}^{t} dt' \cdot \frac{h(t',t_s)}{\sigma} = -\int_{t_0}^{t} dt' \cdot \frac{(t'/t_s)}{\sigma};$$

It follows that the homogenous solution is:

$$\theta(t) = 2\arctan\Big(\tan(\theta_0/2)\cdot e^{-(t^2-t_0^2)/2t_s\sigma}\Big).$$

Now we try to find a particular solution in the following form:

$$\tan(\theta(t)/2) = \tan(\theta_0/2)\cdot e^{-A(t)}\cdot w(t);$$

where $A(t) = -(t^2 - t_0^2)/2t_s\sigma$. Let us consider again the equation (464) in terms of $\tan(\theta(t)/2)$

$$\frac{d}{dt}\tan(\theta(t)/2) = \frac{1}{\cos^2(\theta(t)/2)}\cdot\Big(\frac{\dot{\theta(t)}}{2}\Big) \overset{(464)}{=} \frac{1}{2\cos^2(\theta(t)/2)}\cdot\Big[-\sigma^{-1}(t/t_s)\sin(\theta(t)) + \varsigma_\theta(t)\Big];$$

We set $q = \theta(t)/2$ and we substitute the guess above in the equation:

$$\frac{d}{dt}\tan(q) = \tan(\theta_0/2)\cdot e^{-A(t)}\cdot\Big[-\sigma^{-1}(t/t_s)w(t) + \dot{w}(t)\Big]$$

$$\overset{(464)}{=} \frac{1}{2\cos^2(q)}\cdot\Big[-\sigma^{-1}(t/t_s)2\sin(q)\cos(q) + \varsigma_{2q}(t)\Big] = \Big[-\sigma^{-1}(t/t_s)\tan(q) + \frac{\varsigma_{2q}(t)}{2\cos^2(q)}\Big];$$

Thus,

$$\tan(\theta_0/2)\cdot e^{-A(t)}\cdot\Big[-\sigma^{-1}(t/t_s)w(t)+\dot{w}(t)\Big] = \Big[-\sigma^{-1}(t/t_s)\tan(\theta_0/2)\cdot e^{-A(t)}\cdot w(t) + \frac{\varsigma_{2q}(t)}{2\cos^2(q)}\Big];$$

$$\dot{w}(t) = \frac{e^{+A(t)}}{\tan(\theta_0/2)}\Big(\frac{\varsigma_\theta(t)}{2\cos^2(\theta(t)/2)}\Big) = e^{+A(t)}\cdot\varsigma'_\theta(t).$$

$$w(t) = \frac{1}{\tan(\theta_0/2)}\int_{t_0}^{t} dt'\cdot e^{+A_\alpha(t')}\cdot\varsigma'_{\theta,\alpha}(t') + w_0;$$

where we have redefined again the noise distribution according to,

$$P(\varsigma'_\theta) \propto \int \mathcal{D}\varsigma'\cdot\exp\Big(-\frac{\sigma^2}{2\Omega}\int dt\cdot\int d^dx\cdot|\varsigma'_\theta(x,t)|^2\Big).$$

which has the same cumulants of the distribution (466). It follows that the solution at zero-momenta for the phase $\theta(t)$ is:

$$\theta(t) = 2\arctan\Big\{\tan(\theta_0/2)\cdot\exp\Big(-\int_{t_0}^{t} dt'\cdot h(t',t_s)/\sigma\Big) + \int_{t_0}^{t} dt'\cdot\exp\Big(-\int_{t'}^{t} dt''\cdot h(t'',t_s)/\sigma\Big)\cdot\varsigma'_\theta(t')\Big\}$$

F.2.3 Variance of the phase distribution at early times.

Using the solution (470) and the cumulants of the noise distribution (471), (472) we obtain:

$$\gamma^2(t,t_s) = \int_{t_0}^{t} dt'\cdot\exp\Big(\int_{t'}^{t} dt''\cdot h(t'',t_s)/\sigma\Big)\cdot\int_{t_0}^{\tau} d\tau\cdot\exp\Big(\int_{\tau}^{t} dt''\cdot h(t'',t_s)/\sigma\Big)\cdot < \varsigma'_\theta(t')\cdot\varsigma'_\theta(\tau) >_{\varsigma'_\theta}$$

$$= \frac{2}{\sigma^2}\int_{t_0}^{t} dt' \cdot \exp\Big(2\int_{t'}^{t} dt'' \cdot h(t'', t_s)/\sigma\Big).$$

We perform the integral

$$\gamma^2(t, t_s) = \frac{2}{\sigma^2} e^{t^2/t_s\sigma} \int_{t_0}^{t} dt' \cdot e^{-t'^2/t_s\sigma} = \frac{2}{\sigma^2} e^{t^2/t_s\sigma} \cdot \Big\{\frac{\sqrt{\pi}}{2} \cdot \sqrt{\sigma t_s} \cdot \mathrm{Erfc}\Big(|t'|/\sqrt{\sigma t_s}\Big)\Big|_{t_0}^{t}\Big\}.$$

If we consider the KZ scaling limit: $t_s \to \infty$ keeping $\bar{t}$ fixed, we obtain:

$$\gamma^2(t, t_s) = t_Q \cdot \frac{1}{\sigma^2} e^{\bar{t}^2/\sigma} \cdot \sqrt{\pi\sigma} \cdot \mathrm{Erfc}\Big(|\bar{t}|/\sqrt{\sigma}\Big) = t_Q \cdot \gamma'^2(\bar{t}).$$

The approximation (468) is consistent with the asymptotic limit $\bar{t} \to -\infty$. Thus, the Erfc can be expanded through (513). The variance of the phase distribution in the limit $\bar{t} \to -\infty$ is:

$$\gamma'^2(\bar{t}) \overset{\bar{t}\to-\infty}{\sim} \frac{1}{\sigma|\bar{t}|}.$$

F.3 Leading off-equilibrium corrections to the asymptotic equilibrium behaviour:

F.3.1 Magnetization.

We compute the leading asymptotic deviations of the longitudinal magnetization using the ansatz (478) in (448):

$$\cos\Big(\vartheta(\bar{t})\Big) \sim \frac{1}{\sigma}\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot \exp\Big(-\int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot \frac{|\bar{t}''|}{\sigma}(1 + b|\bar{t}''|^a \cdot e^{-c|\bar{t}''|^2})\Big) \simeq$$

$$\frac{1}{\sigma}\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot e^{(|\bar{t}|^2 - |\bar{t}'|^2)/2\sigma}\Big(1 + \int_{\bar{t}'}^{\bar{t}} d|\bar{t}''| \cdot \frac{b}{\sigma}|\bar{t}''|^{a+1} \cdot e^{-c|\bar{t}''|^2}\Big)$$

$$= -1 + \frac{1}{\sigma} e^{|\bar{t}|^2/2\sigma}\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot e^{-|\bar{t}|^2/2\sigma} \cdot \Big(-\frac{b}{2\sigma} c^{-a/2-1} \cdot \Gamma[(2+a)/2, c|\bar{t}''|]\Big|_{\bar{t}'}^{\bar{t}}\Big);$$

Keeping the leading term of the asymptotic expansion (502):

$$\cos\Big(\vartheta(\bar{t})\Big) \sim -1 + \frac{1}{\sigma} e^{|\bar{t}|^2/2\sigma}\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \bar{t}' \cdot e^{-|\bar{t}|^2/2\sigma} \cdot \Big(-\frac{b}{2\sigma} c^{-a/2-1} \cdot (c|\bar{t}''|^2)^{(2+a)/2-1} \cdot e^{-c|\bar{t}''|^2}\Big|_{\bar{t}'}^{\bar{t}}\Big)$$

$$= -1 + \Big\{-\frac{b}{2c\sigma^2} e^{|\bar{t}|^2/2\sigma} \cdot |\bar{t}|^a \cdot e^{-c|\bar{t}|^2}\int_{-\infty}^{\bar{t}} d|\bar{t}'| \cdot |\bar{t}'| \cdot e^{-|\bar{t}'|^2/2\sigma}\Big\}$$

$$+\Big\{\frac{b}{2c\sigma^2} e^{|\bar{t}|^2/2\sigma}\int_{-\infty}^{\bar{t}} d|\bar{t}'| \cdot |\bar{t}'|^{1+a} \cdot e^{-|\bar{t}'|^2/2\sigma} \cdot e^{-c|\bar{t}'|^2}\Big\}$$

$$= -1 - \frac{b}{2c\sigma^2} e^{|\bar{t}|^2/2\sigma} \cdot |\bar{t}|^a \cdot e^{-c|\bar{t}|^2}\Big(-e^{|\bar{t}'|^2/2\sigma} \cdot \sigma\Big|_{-\infty}^{\bar{t}}\Big)$$

$$+\frac{b}{2c\sigma^2} e^{|\bar{t}|^2/2\sigma}\Big(-2^{a/2} \cdot \Big(\frac{1+2c\sigma}{\sigma}\Big)^{a/2-1} \cdot \Gamma[(2+a)/2, (1+2c\sigma)|\bar{t}'|^2/2\sigma]\Big|_{-\infty}^{\bar{t}}\Big);$$

Using the relation (502):

$$\cos\Big(\vartheta(\bar{t})\Big) \sim -1 + \frac{b}{2c\sigma}|\bar{t}|^a \cdot e^{-c|\bar{t}|^2}$$

$$+\frac{b}{2c\sigma^2} e^{|\bar{t}|^2/2\sigma}\Big(-2^{a/2} \cdot \Big(\frac{1+2c\sigma}{\sigma}\Big)^{a/2-1} \cdot \Big(\frac{1+2c\sigma}{2\sigma}|\bar{t}'|^2\Big)^{(2+a)/2-1} \cdot e^{-|\bar{t}|^2/2\sigma} \cdot e^{-c|\bar{t}|^2}\Big);$$

Thus,

$$\cos\Big(\vartheta(\bar{t})\Big) \sim -1 + \frac{b}{2c\sigma}|\bar{t}|^a \cdot e^{-c|\bar{t}|^2} - \frac{b}{2c\sigma^2} \cdot \Big(\frac{\sigma}{1+2c\sigma}\Big) \cdot |\bar{t}|^a \cdot e^{-c|\bar{t}|^2} = -1\Big(1 + \Big(-\frac{b}{1+2c\sigma}\Big)|\bar{t}|^a \cdot e^{-c|\bar{t}|^2}\Big).$$

F.3.2 Transverse susceptibility.

We investigate the first corrections to equilibrium behaviour also in the transverse susceptibility using the ansatz (478) in the definition (345) at zero momenta:

$$\mathcal{G}_T(\bar{t}) \sim 2\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot \exp\Big(-2\int_{\bar{t}'}^{\bar{t}} d\bar{t}'' \cdot \frac{|\bar{t}''|}{\sigma}(1+b|\bar{t}|^a \cdot e^{-c|\bar{t}|^2}\Big) \simeq$$

$$2\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot e^{(|\bar{t}|^2-|\bar{t}'|^2)/\sigma} \cdot \Big(1+2\int_{\bar{t}'}^{\bar{t}} d|\bar{t}''| \cdot \frac{b}{\sigma} \cdot |\bar{t}|^{1+a} \cdot e^{-c|\bar{t}|^2}\Big)$$

$$= \frac{\sigma}{|\bar{t}|} + 4e^{|\bar{t}|^2/\sigma}\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot e^{-|\bar{t}'|^2/\sigma} \cdot \Big(-\frac{b}{2\sigma}c^{a/2-1} \cdot \Gamma[(2+a)/2, c|\bar{t}''|]\Big|_{\bar{t}'}^{\bar{t}}\Big);$$

Taking the leading part of the asymptotic expansion (502), we obtain:

$$\mathcal{G}_T(\bar{t}) \sim \frac{\sigma}{|\bar{t}|} + 4e^{|\bar{t}|^2/\sigma}\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot e^{-|\bar{t}'|^2/\sigma} \cdot \Big(-\frac{b}{2\sigma}c^{a/2-1} \cdot (c|\bar{t}''|^2)^{(2+a)/2-1} \cdot e^{-c|\bar{t}''|^2}\Big|_{\bar{t}'}^{\bar{t}}\Big)$$

$$= \frac{\sigma}{|\bar{t}|} + \Big\{-\frac{2b \cdot e^{|\bar{t}|^2/\sigma}}{\sigma c} \cdot |\bar{t}|^a \cdot e^{-c|\bar{t}|^2}\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot e^{-|\bar{t}'|^2/2\sigma}\Big\} + \Big\{\frac{2b \cdot e^{|\bar{t}|^2}/\sigma}{\sigma c}\int_{-\infty}^{\bar{t}} d\bar{t}' \cdot e^{-|\bar{t}'|^2/\sigma} \cdot |\bar{t}'|^a \cdot e^{-c|\bar{t}'|^2}\Big\}$$

$$= \frac{\sigma}{|\bar{t}|} - \frac{2b \cdot e^{|\bar{t}|^2/\sigma}}{\sigma c} \cdot |\bar{t}|^a \cdot e^{-c|\bar{t}|^2}\Big(\frac{\sqrt{\pi}}{2} \cdot \sqrt{\sigma} \cdot \mathrm{Erfc}\Big(\frac{|\bar{t}'|}{\sigma}\Big)\Big|_{-\infty}^{\bar{t}}\Big)$$

$$-\frac{2b \cdot e^{|\bar{t}|^2}/\sigma}{\sigma c} \cdot \Big(-\frac{1}{2}\Big(\frac{1+c\sigma}{\sigma}\Big)^{-(1+a)/2} \cdot \Gamma[(1+a)/2, (1+c\sigma)|\bar{t}'|^2/\sigma]\Big|_{-\infty}^{\bar{t}}\Big);$$

Using (502) in the last expression:

$$\mathcal{G}_T(\bar{t}) \sim \frac{\sigma}{|\bar{t}|} - \frac{2b \cdot e^{|\bar{t}|^2/\sigma}}{\sigma c} \cdot |\bar{t}|^a \cdot e^{-c|\bar{t}|^2}\Big(\frac{\sqrt{\pi}}{2} \cdot \sqrt{\sigma} \cdot \frac{e^{-|\bar{t}|^2/\sigma}}{\sqrt{\pi}|\bar{t}|/\sigma}\Big) +$$

$$\frac{b \cdot e^{|\bar{t}|^2}/\sigma}{\sigma c} \cdot \Big(\Big(\frac{1+c\sigma}{\sigma}\Big)^{-(1+a)/2} \cdot \Big(\frac{1+c\sigma}{\sigma}|\bar{t}|^2\Big)^{(1+a)/2-1} \cdot e^{-|\bar{t}|^2/\sigma} \cdot e^{-c|\bar{t}|^2}\Big);$$

Thus,

$$\mathcal{G}_T(\bar{t}) \sim \frac{\sigma}{|\bar{t}|} - \frac{b}{c} \cdot \frac{1}{|\bar{t}|} \cdot |\bar{t}|^a \cdot e^{-c|\bar{t}|^2} + \frac{b}{\sigma c}\Big(\frac{\sigma}{1+c\sigma}\Big)|\bar{t}|^{a-1} \cdot e^{-c|\bar{t}|^2} = \frac{\sigma}{|\bar{t}|}\Big(1+\Big(-\frac{b}{1+c\sigma}\Big)|\bar{t}|^a \cdot e^{-c|\bar{t}|^2}\Big);$$

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