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Actions of Surface Groups on the Circle

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Introduction

Let S be a closed oriented surface of genus $g \geq 2$, let $\Gamma := \pi_1(S)$ denote its fundamental group, let $G := \mathrm{Homeo}_+\left(S^1\right)$ be the group of orientation preserving homeomorphisms of the circle. An action of the surface group Γ on the circle $S^1 = \mathbb{R}/\mathbb{Z}$ is a representation

$$\phi : \Gamma \longrightarrow \mathrm{Homeo}_+\left(S^1\right).$$

A classical problem in low dimensional geometry and topology is the study of the properties of circle actions of surface groups and their spaces of parameters

$$\mathrm{Rep}\left(\Gamma, G\right) := \mathrm{Hom}\left(\Gamma, \mathrm{Homeo}_+\left(S^1\right)\right).$$

Some motivations:

- Holonomies $\rho : \Gamma \to \mathrm{PSL}\left(2, \mathbb{R}\right) = \mathrm{Isom}^+\left(\mathbb{H}^2\right)$ of *hyperbolic structures* over S give rise to actions on the boundary $\Gamma \curvearrowright \partial\mathbb{H}^2$.
- *Geometric representations*: a particular interest has recently developed towards representations into transitive Lie subgroups of $\mathrm{Homeo}_+\left(S^1\right)$. By a result of Ghys, such subgroups are exactly the following ones

  $$S^1, \quad \mathrm{PSL}^{(k)}\left(2, \mathbb{R}\right).$$

  Geometric representations are those which are faithful and have discrete image.
- *Space of parameters*. The Teichm̈uller space $\mathrm{Teich}\left(S\right)$ admits an embedding in

  $$\mathrm{Hom}\left(\Gamma, \mathrm{PSL}\left(2, \mathbb{R}\right)\right) / \mathrm{PSL}\left(2, \mathbb{R}\right)$$

  where $\mathrm{PSL}\left(2, \mathbb{R}\right)$ acts on $\mathrm{Hom}\left(\Gamma, \mathrm{PSL}\left(2, \mathbb{R}\right)\right)$ by conjugation.

The aim of this thesis is the proof of some recent *rigidity* results about the *deformation spaces* of surface group actions on the circle, i.e the connected components of $\mathrm{Rep}\left(\Gamma, G\right)$. The precise statement is the following:

THEOREM (K. Mann, S. Matsumoto). *The following holds:*

- *If $\phi : \Gamma \longrightarrow PSL^{(k)}\left(2, \mathbb{R}\right)$ is faithful and has discrete image then its connected component in $Rep\left(\Gamma, G\right)$ consists of a single semi-conjugacy class.*
- *For every $k \mid 2g-2$ there are at least $k^{2g}+1$ connected components containing representations with Euler number $\frac{2g-2}{k}$.*

The thesis is roughly divided into four parts.

In the first one we introduce and describe the *bounded Euler class* $e_b^{\mathbb{Z}}(\phi) \in H_b^2(\Gamma, \mathbb{Z})$, which is the main invariant associated to a group action ϕ on the circle, and the equivalence relation of *semi-conjugacy*. The central results of this section are Ghys Theorem that completely characterize the relation of semi-conjugacy in cohomological terms and Matsumoto's numerical description of the bounded Euler class via rotation numbers and the canonical Euler cocycle.

The second part is devoted to actions of surface groups. We describe some geometric objects, which are *flat circle bundles* over S, naturally associated to a representation. After having defined the *Euler number* $e(\phi) \in \mathbb{Z}$ of a representation $\phi : \Gamma \longrightarrow \mathrm{Homeo}_+\left(S^1\right)$ we prove *Milnor's Formula* which gives an explicit way for computing it as the translation number of a product of lifted commutators.

The Euler number of a representation cannot attain every integral value but lies in a bounded interval. This is the content of the *Milnor-Wood inequality*:

Theorem (Milnor-Wood inequality). *Let S be a closed oriented surface of genus $g \geq 2$. Let $\phi : \Gamma \longrightarrow G$ be a representation. Then*

$$|e(\phi)| \leq |\chi(S)| = 2g - 2.$$

This is a sharp inequality. Moreover it is quite remarkable that equality is reached when ϕ is the holonomy of a hyperbolic structure over S. A classical result by Goldman ensures that maximality is the only obstruction for being geometric:

Theorem (Goldman). *Let $\phi : \Gamma \longrightarrow PSL(2, \mathbb{R})$ be a representation. Then*

$$\phi \textit{ is faithful and has discrete image} \Longleftrightarrow |e(\phi)| = 2g - 2.$$

Further works led to the following:

Theorem (Matsumoto, Burger, Iozzi). *Let $\phi : \Gamma \longrightarrow Homeo_+\left(S^1\right)$ be a maximal representation, i.e. $|e(\phi)| = 2g - 2$. Then ϕ is semi-conjugate to every geometric $PSL(2, \mathbb{R})$ representation.*

In the third part we study the relation between *maximality* and *rigidity*. We end this section with a complete description of the dynamics of a $\mathrm{PSL}^{(k)}(2, \mathbb{R})$ $-$geometric representation due to Matsumoto.

In the last part we address the proof of the main theorem. Also this part is motivated by a work of Goldman in which are completely classified the connected components of the representation variety $\mathrm{Hom}\left(\Gamma, \mathrm{PSL}^{(k)}(2, \mathbb{R})\right)$:

Theorem (Goldman). *The connected components of $Hom\left(\Gamma, PSL^{(k)}(2, \mathbb{R})\right)$ have the following description:*

- *if $k \nmid 2g - 2$ then $e^{-1}(n)$ is a connected component for every $|n| \leq \left\lfloor \frac{2g-2}{k} \right\rfloor$;*

- *if* $k \mid 2g-2$ *then there are* $2k^{2g}$ *connected components on which the Euler number attains the value* $\pm\frac{2g-2}{k}$*. Furthermore they are distinguished by the* $2g-$*tuple of rotation numbers of a standard set of generators of* Γ*.*

First we give coordinates to the representation space $\mathrm{Rep}\,(\Gamma, G)$ using Matsumoto's numerical invariants introduced in the first part. Then we work out a finer analysis of the dynamics of the generators of a geometric $\mathrm{PSL}^{(k)}\,(2,\mathbb{R})\,-$representation finding a common combinatorial structure (*good fixed point sets* and *good representations*). At this stage we develop the technical tools for carrying out the analysis of the deformations of the combinatorial structures. The heart of this part lies in the *Calegari-Walker algorithm.* Finally we are able to prove some stability phenomena of representations that are extremal within a good family. From the stability results we will deduce the main theorem.

CHAPTER 1

Invariants of group actions on the circle

1. Overview

The first chapter covers some background on the theory of general group actions on the circle, group cohomology and bounded cohomology, semi-conjugacy. The principal goal is to introduce and define an invariant of group actions on the circle, the (bounded) Euler class, and classify them up to a suitable natural equivalence relation, the semi-conjugacy relation. There are a couple of central result of this section which we want to highlight: the first is Ghys Theorem which classifies semi-conjugacy in cohomological terms

THEOREM (Ghys). *Let Γ be a group and $\rho_1, \rho_2 : \Gamma \to Homeo_+\left(S^1\right)$ be representations. Then*

$$\rho_1, \rho_2 \text{ are semi-conjugate } \iff e_b\left(\rho_1\right) = e_b\left(\rho_2\right) \in H_b^2\left(\Gamma, \mathbb{Z}\right)$$

The second is Matsumoto's numerical description of the bounded Euler class

THEOREM (Matsumoto). *Let Γ be a group with generators $\left\{\gamma_j\right\}_{j\in I}$, let $\rho : \Gamma \to Homeo_+\left(S^1\right)$ be a representation. Then $\phi^* e_b^{\mathbb{Z}}$ is completely determined by the data*

$$\begin{cases} rot\left(\rho(\gamma_j)\right) & \text{for every } j \in I \\ \rho^*\tau & \text{as an inhomogeneous } 2\text{−cocycle} \end{cases}$$

The $2-$cocycle $\tau : G \times G \longrightarrow \mathbb{R}$ is defined by

$$\tau\left(f, g\right) := \widetilde{rot}\, \widetilde{f}\widetilde{g} - \widetilde{rot}\, \widetilde{f} - \widetilde{rot}\, \widetilde{g}$$

where $\widetilde{f}, \widetilde{g} \in \widetilde{G}$ are arbitrary lifts of f, g.

2. The group $\mathbf{Homeo}_+\left(S^1\right)$

In this section we briefly review some properties of the topological group $\mathrm{Homeo}_+\left(S^1\right)$ that will be extensively used in the next sections. Our main reference is Ghys' article [**Ghy01**].

2.1. Definitions and topology. We denote by $G := \mathrm{Homeo}_+\left(S^1\right)$ the group of orientation preserving homeomorphisms of the circle. The group G is naturally endowed with the compact-open topology which can be described by the metric of uniform convergence. If d denotes the standard metric on S^1, then for every $f, g \in G$ the uniform distance is

$$d_\infty(f, g) := \sup_{t\in S^1} \left\{d(f(t), g(t))\right\}.$$

The universal covering of the topological group $\mathrm{Homeo}_+(S^1)$ can be described as follows

$$\widetilde{G} := \widetilde{\mathrm{Homeo}_+}(S^1) = \left\{ f : \mathbb{Z} \longrightarrow \mathbb{Z} \;\middle|\; \begin{array}{c} f \text{ increasing and continuous} \\ f\tau_1 = \tau_1 f \end{array} \right\}$$

where τ_1 is the unit translation of $\mathbb{R}$. The topology on the group $\widetilde{G}$ is described by uniform convergence. The universal covering projection is given by

$$p : \widetilde{G} \longrightarrow G \qquad , \qquad p(f)[t] := [f(t)].$$

The group $S^1 = \mathbb{R}/\mathbb{Z}$ embeds in G as the subgroup of rotations $[\theta] \in S^1 \longrightarrow \rho_\theta \in G$ where $\rho_\theta[x] := [x+\theta]$. It turns out that $S^1 < \mathrm{Homeo}_+(S^1)$ completely determines the topology of $\mathrm{Homeo}_+(S^1)$:

Theorem 1.1. *The group Homeo$_+(S^1)$ deformation retracts onto S^1. In particular the inclusion $i : S^1 \longrightarrow$ Homeo$_+(S^1)$ is a homotopy equivalence.*

2.2. Algebraic properties of Homeo$_+(S^1)$. The group $\mathrm{Homeo}_+(S^1)$ enjoys some nice algebraic properties which we state without proof. For a more comprehensive treatement we refer to Proposition 5.11 and Theorem 4.3 of Ghys' article [**Ghy01**]. The only property which we will need is uniform perfectness

Definition 1.2. A group Γ is *uniformly perfect* if there is a constant $N > 0$ such that every element $\gamma \in \Gamma$ can be written as a product of less than N commutators.

Theorem 1.3. *The group Homeo$_+(S^1)$ is uniformly perfect of constant $N = 1$.*

Since the unit integral translation τ_1 is a product of 2 commutators we also have:

Corollary 1.4. *The group $\widetilde{Homeo_+}(S^1)$ is perfect.*

2.3. Lie subgroups of Homeo$_+(S^1)$. Lie subgroups $L < \mathrm{Homeo}_+(S^1)$ of dimension $\dim\ L \geq 1$ are well understood. They are completely classified by the following theorem which is proved in [**Ghy01**]:

Theorem 1.5. *Let $L <$ Homeo$_+(S^1)$ be a connected Lie subgroup of dimension* $\dim\ L \geq 1$. *Then L is conjugate to one of the following*

$$S^1, \quad PSL^{(k)}(2,\mathbb{R}), \quad \mathbb{R}, \quad A\mathit{ff}_+(\mathbb{R}), \quad \widetilde{SL}(2,\mathbb{R}).$$

In particular if L acts transitively then it is conjugate to

$$S^1 \quad , \quad PSL^{(k)}(2,\mathbb{R}).$$

The Lie group $\mathrm{PSL}^{(k)}(2,\mathbb{R})$ is the degree k covering of $\mathrm{PSL}(2,\mathbb{R})$. Let us denote by $\pi_k : \mathrm{PSL}^{(k)}(2,\mathbb{R}) \longrightarrow \mathrm{PSL}(2,\mathbb{R})$ the covering projection. We briefly describe the action of $\mathrm{PSL}^{(k)}(2,\mathbb{R})$ on S^1. Let us begin with $\mathrm{PSL}(2,\mathbb{R})$. There are many models to describe the action. If we identify $\mathrm{PSL}(2,\mathbb{R})$ with $\mathrm{Isom}_+(\mathbb{H}^2)$ then $\phi \in \mathrm{PSL}(2,\mathbb{R})$ acts on $\partial\mathbb{H}^2 = S^1$ as the natural extension ϕ to the boundary. Instead if we think of $\mathrm{PSL}(2,\mathbb{R})$ as $\mathrm{PGL}^+(2,\mathbb{R})$ then $\phi \in \mathrm{PSL}(2,\mathbb{R})$ acts on $\mathbb{RP}^1$ as an orientation preserving projective transformation. Another model is given by extensions to the boundary of biholomorphisms of the unit disk Δ or of the upper half plane $H \subseteq \mathbb{C}$.

Consider now the degree k covering of S^1, it is again homeomorphic to S^1. If we think of $S^1 \subseteq \mathbb{C}$ then the covering projection $p_k : S^1 \longrightarrow S^1$ is given by $p_k(z) = z^k$ and the automorphisms group is $\mathbb{Z}/k\mathbb{Z}$ identified with the group of rotations $\rho_{1/k}$. The finite covering $\mathrm{PSL}^{(k)}(2,\mathbb{R})$ of degree k of $\mathrm{PSL}(2,\mathbb{R})$ naturally acts on the finite covering of S^1 of degree k. For every element $\phi \in \mathrm{PSL}^{(k)}(2,\mathbb{R})$ consider the k lifts of ϕp_k along the covering p_k. The set of all lifts has a natural structure of Lie group and projects onto $\mathrm{PSL}(2,\mathbb{R})$ with kernel $\mathbb{Z}/k\mathbb{Z}$. Thus it is a realization of $\mathrm{PSL}^{(k)}(2,\mathbb{R})$. Explicitly $\mathrm{PSL}^{(k)}(2,\mathbb{R})$ acts as piecewise $\mathrm{PSL}(2,\mathbb{R})-$transformations with respect to a fixed subdivision of S^1 into k equal consecutive arcs.

2.4. Minimal exceptional. We describe now the macroscopic behaviour of actions $\phi : \Gamma \longrightarrow \mathrm{Homeo}_+\left(S^1\right)$ and classify them by looking at a natural invariant compact subset associated to the action.

Lemma 1.6. *Let $\phi : \Gamma \longrightarrow Homeo_+\left(S^1\right)$ be a representation without finite orbits. Then there exists a unique minimal invariant compact subset $K_\phi \subseteq S^1$.*

Definition 1.7. Let ϕ be a representation without finite orbits. The compact subset $K_\phi \subseteq S^1$ of Lemma 1.6 is the *minimal exceptional* of ϕ. The representation ϕ is *minimal* if $K_\phi = S^1$.

Remark 1.8. We observe that the closure of every orbit $\overline{\Gamma x} \subseteq S^1$ is an invariant compact subset. Thus, by minimality, the minimal exceptional K_ϕ of ϕ is contained in the intersection $K_\phi \subseteq \bigcap_{x \in S^1} \overline{\Gamma x}$. Conversely, by invariance and compactness the closures of the orbits of points $x \in K_\phi$ are contained in K_ϕ, i.e. $\overline{\Gamma x} \subseteq K_\phi$ for every $x \in K_\phi$. By minimality of K_ϕ among $\Gamma-$invariant compact subsets we must have $K_\phi = \overline{\Gamma x}$ for any $x \in K_\phi$ since $\overline{\Gamma x}$ is $\Gamma-$invariant. In particular minimal actions are those representations such that every orbit is dense $\overline{\Gamma x} = S^1$ for every $x \in S^1$.

We can classify the dynamics of actions by looking at their minimal exceptionals.

Theorem 1.9. *Let $\phi : \Gamma \longrightarrow Homeo_+\left(S^1\right)$ be a representation. We have the following trichotomy:*

(1) ϕ has a finite orbit.
(2) ϕ is minimal or $K_\phi = S^1$.
(3) ϕ has a minimal exceptional which is an invariant Cantor set $K \subsetneq S^1$.

2.5. Rotation and translation numbers. The main invariant for the dynamics of a single homeomorphism $f \in \mathrm{Homeo}_+\left(S^1\right)$ or to the action $\phi_f : \mathbb{Z} \longrightarrow \mathrm{Homeo}_+\left(S^1\right)$ defined by $\phi_f(1) := f$, is the rotation number $\mathrm{rot}(f)$. It measures the average distance f translates points on S^1.

Definition 1.10. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a non decreasing map commuting with integral translations (i.e. such that $f\tau_1 = \tau f_1$). The *translation number* of f is given by

$$\widetilde{\mathrm{rot}}(f) := \lim_{n\to\infty} \frac{f^n(x)}{n} \qquad x \in \mathbb{R} \text{ is chosen arbitrary .}$$

In particular it is well defined for a homeomorphism $f \in \widetilde{\mathrm{Homeo}}_+\left(S^1\right)$.
It is not difficult to see that the limit always exists and is independent of $x \in \mathbb{R}$.
Let $f \in \mathrm{Homeo}_+\left(S^1\right)$ be a homeomorphism. The *rotation number* of f is given by

$$\mathrm{rot}(f) := \left[\widetilde{\mathrm{rot}}(\widetilde{f})\right] \in \mathbb{R}/\mathbb{Z}$$

where $\widetilde{f}$ is any lift of f to $\widetilde{\mathrm{Homeo}}_+\left(S^1\right)$.
It is easy to check that $\mathrm{rot} : \mathrm{Homeo}_+\left(S^1\right) \longrightarrow \mathbb{R}/\mathbb{Z}$ is a continuous function.

3. Group cohomology

In this section we rapidly recall some basic definitions and facts about classical group cohomology. Our framework will be the one of classical homogeneous and inhomogeneous resolution. The topic of interest for us are the relation of group cohomology with cohomology of spaces (as we want to relate them in the case of surfaces and surface groups) and low degree cohomology, in particular degree 2 as the Euler class lives in $H^2(G, \mathbb{Z})$.

3.1. Homogeneous and inhomogeneous resolutions. Let Γ be a group, let R be either $\mathbb{Z}$ or $\mathbb{R}$ thought as a trivial $\Gamma-$module. The set of maps $\mathrm{Map}\left(\Gamma^n, R\right)$ has a natural $\Gamma-$module structure given by $\gamma \cdot f(\bullet) := f\gamma^{-1}(\bullet)$ where $\gamma^{-1} \cdot (\gamma_1, \dots, \gamma_n) := \left(\gamma^{-1}\gamma_1, \dots, \gamma^{-1}\gamma_n\right)$.

Definition 1.11. The *homogeneous complex* $\left(C^\bullet(\Gamma, R), \delta\right)$ is defined as follows

$$C^n(\Gamma, R) := \mathrm{Map}\left(\Gamma^{n+1}, R\right)^\Gamma$$

where the superscript Γ means that we are taking only $\Gamma-$invariants maps.
The boundary δ is described by

$$\delta\phi\left(g_0, \dots, g_{n+1}\right) := \sum_{j=0}^{n+1} (-1)^j \phi\left(g_0, \dots, \widehat{g_j}, \dots, g_{n+1}\right).$$

It is readily seen that δ preserves the invariance of a cochain, thus invariant cochains form a subcomplex of $\left(\mathrm{Map}\left(\Gamma, R\right), \delta\right)$.

Definition 1.12. The *inhomogeneous complex* $\left(\overline{C}^\bullet(\Gamma, R), \overline{\delta}\right)$ is defined as follows

$$\overline{C}^n(\Gamma, R) := \mathrm{Map}\left(\Gamma^n, R\right)$$

with boundary $\overline{\delta}$ given by

$$\begin{aligned}\overline{\delta}\psi\left(g_1, \dots, g_{n+1}\right) := \psi\left(g_2, \dots, g_{n+1}\right) + \sum_{j=1}^{n} (-1)^j \psi\left(g_1, \dots, g_j g_{j+1}, g_{j+2}, \dots, g_{n+1}\right) + \\ + (-1)^{n+1}\psi\left(g_1, \dots, g_n\right).\end{aligned}$$

There is a canonical isomorphism between the inhomogeneous and the homogeneous cochain complexes given by the following cochain morphisms

$$C^\bullet(\Gamma, R) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \overline{C}^\bullet(\Gamma, R)$$

where α, β are defined by

$$\begin{array}{ll} \alpha(\phi) : (g_1, \ldots, g_n) \longrightarrow \phi(1, g_1, g_1 g_2, \ldots, g_1 \cdots g_n) & \phi \in C^n(\Gamma, R) \\ \beta(\psi) : (g_0, \ldots, g_n) \longrightarrow \psi(g_0^{-1} g_1, g_1^{-1} g_2, \ldots, g_{n-1}^{-1} g_n) & \psi \in \overline{C}^n(\Gamma, R). \end{array}$$

Definition 1.13. The *$n-$th cohomology group of Γ with coefficients in R* is thus

$$H^n(\Gamma, R) := H^n(C^\bullet(\Gamma, R)) = H^n\left(\overline{C}^\bullet(\Gamma, R)\right).$$

3.2. Group cohomology and singular cohomology. Let X be a CW-complex with $\pi_1(X) = \Gamma$, let $p : \widetilde{X} \to X$ be the universal covering of X. Choose a set of representatives $F \subseteq \widetilde{X}$ for the action of Γ on $\widetilde{X}$ by deck transformations. For every $x \in \widetilde{X}$ define $g_x \in \Gamma$ as the unique deck transformation such that $x \in g_x F$. Consider the map $r_F : C^\bullet(\Gamma, R) \longrightarrow C^\bullet_{\text{sing}}\left(\widetilde{X}, R\right)$ defined by

$$r_F(\phi)(s) := \phi\left(g_{s(e_0)}, \ldots, g_{s(e_n)}\right).$$

for every $n-$singular simplex $s : \Delta^n \to \widetilde{X}$ with vertices $s(e_0), \ldots, s(e_n)$. It is a routine computation to check that r_F is a morphism of chain complexes of $\Gamma-$modules, i.e. $r_F(\gamma \cdot \phi) = \gamma \cdot r_F(\phi)$ and $r_F \delta = \delta_{\text{sing}} r_F$. In particular r_F induces a morphism on cohomologies which we denote by

$$(r_F)_* : H^\bullet(\Gamma, R) \longrightarrow H^\bullet_{\text{sing}}(X, R).$$

A standard computation ensures that different choices of $F, F' \subseteq \widetilde{X}$ produce chain homotopic morphisms $r_F, r_{F'}$, thus $r := (r_F)_* = (r_{F'})_*$ does not depend on the choice of the particular set of representatives.

Theorem 1.14. *Let Γ be a group, let $X = K(\Gamma, 1)$ be an aspherical CW-complex. Then the map*

$$r : H^\bullet(\Gamma, R) \longrightarrow H^\bullet_{sing}(X, R)$$

is an isomorphism.

Under the canonical isomorphism between the inhomogeneous and homogeneous cochain complexes the map r_F transforms as follows:

$$r_F(\phi) : s \longrightarrow \phi\left(g^{-1}_{s(e_0)} g_{s(e_1)}, \ldots, g^{-1}_{s(e_{n-1})} g_{s(e_n)}\right).$$

3.3. Low degree cohomology and central extensions. We give a brief description of group cohomology in low degrees $n = 1, 2$. Let Γ be any group, let R be either $\mathbb{Z}$ or $\mathbb{R}$.

Degree $n = 1$. Using the inhomogeneous chain complex it is immediate to check that $1-$cocycles $\phi \in \overline{C}^1(\Gamma, R)$ are precisely the homomorphisms $\phi \in \mathrm{Hom}(\Gamma, R)$. Furthermore the boundary of any $0-$cocycle is trivial so we have the following:

LEMMA 1.15. *Let Γ and R be as above. Then $H^1(\Gamma, R) = Hom(\Gamma, R)$.*

Degree $n = 2$. Before stating the main result let us give a definition.

DEFINITION 1.16. A *central extension of* Γ *by* $\mathbb{Z}$ is a short exact sequence of groups

$$0 \longrightarrow \mathbb{Z} \xrightarrow{j} E \xrightarrow{p} \Gamma \longrightarrow 0$$

such that $j(\mathbb{Z})$ is contained in the center of E.
Two such extensions E, E' are said to be isomorphic if there is a homomorphism $\phi : E \longrightarrow E'$ that makes the following diagram commute

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "j"] \arrow[d, equal] & E \arrow[r, "p"] \arrow[d, "\phi"] & \Gamma \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "j'"'] & E' \arrow[r, "p'"'] & \Gamma \arrow[r] & 0.
\end{tikzcd}$$

The main result about degree $n = 2$ cohomology is given by the following correspondence:

THEOREM 1.17. *There is a natural bijection between*

$$H^2(\Gamma, \mathbb{Z}) \longleftrightarrow \left\{ \begin{array}{c} \textit{central extensions} \\ 0 \longrightarrow \mathbb{Z} \xrightarrow{j} E \xrightarrow{p} \Gamma \longrightarrow 0 \end{array} \right\} / \textit{isomorphism}.$$

The correspondence is functorial.

The bijection is explicitly described by the following procedures.
From central extensions to $H^2(\Gamma, \mathbb{Z})$. Consider a central extension

$$0 \longrightarrow \mathbb{Z} \xrightarrow{j} E \xrightarrow{p} \Gamma \longrightarrow 0.$$

Choose $\sigma : \Gamma \to E$ a set-theoretical section $p\sigma = \mathrm{Id}_\Gamma$ and define the inhomogeneous $2-$cochain $c_\sigma : \Gamma \times \Gamma \longrightarrow \mathbb{Z}$ as follows

$$c_\sigma(g, h) := \sigma(g)\sigma(h)\sigma(gh)^{-1} \in j(\mathbb{Z}) \simeq \mathbb{Z}.$$

It is easy to check that c_σ is a cocycle and if σ, σ' are different sections then $c_\sigma = c_{\sigma'} + \delta f$ where $f : \Gamma \longrightarrow \mathbb{Z}$ is defined by $f(g) := \sigma'(g)\sigma(g)^{-1}$.
In particular it is well defined the cohomology class $[c_\sigma] \in H^2(\Gamma, \mathbb{Z})$.
From $H^2(\Gamma, \mathbb{Z})$ *to central extensions.* Consider $e \in H^2(\Gamma, \mathbb{Z})$. Choose an inhomogeneous $2-$cocycle $c : \Gamma \times \Gamma \longrightarrow \mathbb{Z}$ representing e such that $c(1, 1) = 0$ (it always exists) and define the extension

$$0 \longrightarrow \mathbb{Z} \xrightarrow{j} E := \mathbb{Z} \times \Gamma \xrightarrow{p} \Gamma \longrightarrow 0$$

where j, p are the canonical maps and the group structure on E is given by the following formula

$$(n,g)\cdot(m,h) := (n+m+c(g,h), gh)\,.$$

4. Euler class

For group actions on the circle there is essentially a single interesting cohomological invariant, the Euler class. It is defined as the pull-back of a certain cohomology class $e \in H^2\left(\mathrm{Homeo}_+\left(S^1\right), \mathbb{Z}\right)$ called the Euler class of $\mathrm{Homeo}_+\left(S^1\right)$. We construct the Euler class in a purely algebraic setting as the cohomology class associated to a central extension of $\mathrm{Homeo}_+\left(S^1\right)$. In order to motivate the uniqueness assertion we compute the cohomology $H^\bullet\left(\mathrm{Homeo}_+\left(S^1\right), \mathbb{Z}\right)$ and show that it is generated by the euler class $e \in H^2\left(\mathrm{Homeo}_+\left(S^1\right), \mathbb{Z}\right)$.

Notations: throughout the next sections we fix the following shorthands

$$G := \mathrm{Homeo}_+\left(S^1\right) \qquad \text{and} \qquad \widetilde{G} := \widetilde{\mathrm{Homeo}}_+\left(S^1\right).$$

4.1. Euler class of $G = \mathbf{Homeo}_+\left(S^1\right)$ and of a representation. First we define the Euler class of $G := \mathrm{Homeo}_+\left(S^1\right)$. Consider the universal covering central extension

$$0 \longrightarrow \mathbb{Z} = \pi_1\left(\mathrm{Homeo}_+\left(S^1\right)\right) \xrightarrow{j} \widetilde{\mathrm{Homeo}}_+\left(S^1\right) \xrightarrow{p} \mathrm{Homeo}_+\left(S^1\right) \longrightarrow 0.$$

The fundamental group of $\mathrm{Homeo}_+\left(S^1\right)$ is isomorphic to $\mathbb{Z}$ since $\mathrm{Homeo}_+\left(S^1\right)$ deformation retracts to the subgroup of rotations $S^1 < \mathrm{Homeo}_+\left(S^1\right)$.

Definition 1.18. The *Euler class* of $\mathrm{Homeo}_+\left(S^1\right)$ is the cohomology class $e \in H^2\left(\mathrm{Homeo}_+\left(S^1\right), \mathbb{Z}\right)$ corresponding to the universal covering central extension.

Now consider a group Γ and a representation $\phi : \Gamma \longrightarrow \mathrm{Homeo}_+\left(S^1\right)$.

Definition 1.19. The *Euler class of* ϕ is the pull-back of the Euler class of $\mathrm{Homeo}_+\left(S^1\right)$

$$e(\phi) := \phi^* e \in H^2\left(\Gamma, \mathbb{Z}\right).$$

4.2. An obstruction class. The Euler class of a representation measures the obstruction to lift it to the universal covering as precisely stated in the following lemma:

Lemma 1.20. *Let $\rho : \Gamma \longrightarrow G$ be a representation. Consider the following diagram*

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "j"] & \widetilde{G} \arrow[r, "p"] & G \\ & & \Gamma \arrow[u, "\rho"'] \arrow[ul, dashed, "\widetilde{\rho}"] \end{tikzcd}$$

then there exists a lift $\widetilde{\rho}$ of ρ if and only if $\rho^* e = 0$ in $H^2\left(\Gamma, \mathbb{Z}\right)$.
Moreover the lifts are parametrized by $H^1\left(\Gamma, \mathbb{Z}\right)$.

Proof. We prove the two implications separately.
Proof that $\exists\widetilde{\rho} \Rightarrow \rho^*e = 0$. By functoriality $\rho^*e = \widetilde{\rho}^*p^*e$ so it is enough to prove that $p^*e = 0$. Let $\sigma : G \to \widetilde{G}$ be any section and c_σ the associated cocycle representing the Euler class. Then we have

$$\begin{aligned}
&p^*c_\sigma(f,g) = c_\sigma(p(f),p(g)) \\
&= \sigma(p(f))\sigma(p(g))\sigma(p(fg))^{-1} \\
&= \left(f\tau_{u(f)}\right)\left(g\tau_{u(g)}\right)\left(fg\tau_{u(fg)}\right)^{-1} \quad \text{where } \tau_{u(h)} = h^{-1}\sigma(p(h)) \\
&= \tau_{u(f)+u(g)-u(fg)} \\
&= \delta u(f,g).
\end{aligned}$$

Therefore we get $p^*e = [p^*c_\sigma] = [\delta u] = 0$ in $H^2\left(\widetilde{G},\mathbb{Z}\right)$.

Proof that $\rho^*e = 0 \Rightarrow \exists\widetilde{\rho}$. Let $\sigma : G \to \widetilde{G}$ be any section and c_σ the associated cocycle. Since $\rho^*e = 0$ we have $\rho^*c_\sigma = \delta u$ for some $u : \Gamma \to \mathbb{Z}$. Define $\widetilde{\rho}(\gamma) := \sigma(\rho(\gamma))\tau_{-u(\gamma)}$. We claim that $\widetilde{\rho}$ is a homomorphism: in fact for every $\alpha,\beta \in \Gamma$ we have

$$0 = \rho^*c_\sigma(\alpha,\beta) - \delta u(\alpha,\beta)$$

which implies

$$\begin{aligned}
&\tau_0 = \sigma(\rho(\alpha))\sigma(\rho(\beta))\sigma(\rho(\alpha\beta))^{-1}\tau_{u(\alpha)}\tau_{u(\beta)}\tau_{-u(\alpha\beta)} \\
&= \left(\sigma(\rho(\alpha))\tau_{u(\alpha)}\right)\left(\sigma(\rho(\beta))\tau_{u(\beta)}\right)\left(\sigma(\rho(\alpha\beta))\tau_{u(\alpha\beta)}\right)^{-1} \\
&= \widetilde{\rho}(\alpha)\widetilde{\rho}(\beta)\widetilde{\rho}(\alpha\beta)^{-1}.
\end{aligned}$$

Different lifts $\widetilde{\rho}_1, \widetilde{\rho}_2$ are connected by $\phi : \Gamma \to \mathbb{Z}$ given by $\phi(\gamma) := \widetilde{\rho}_1(\gamma)\widetilde{\rho}_2(\gamma)^{-1}$. Using the fact that $\mathbb{Z}$ is in the center of $\widetilde{G}$ it is easy to prove that ϕ is a homomorphism and so defines a class in $H^1(\Gamma,\mathbb{Z})$. Conversely for every homomorphism $\phi \in H^1(\Gamma,\mathbb{Z})$ the function $\widetilde{\rho}_2(\gamma) := \widetilde{\rho}_1(\gamma)\phi(\gamma)$ is a lift of ρ. □

4.3. The Euler class is the unique significant cohomological invariant. Using a theorem due to Mather and Thurston it is possible to explicitly compute the cohomology ring of $\mathrm{Homeo}_+\left(S^1\right)$. In particular the computation shows that $H^\bullet\left(\mathrm{Homeo}_+\left(S^1\right),\mathbb{Z}\right)$ is generated in degree 2 by the Euler class. In order to state the theorem let us introduce the following notations: for a manifold M we denote by $\mathrm{Homeo}_{(+)}(M)$ the group of (orientation preserving) homeomorphisms equipped with the compact-open topology, while we use $\mathrm{Homeo}_{(+)}(M)^\delta$ to describe the same group with the discrete topology.

Theorem 1.21 (Mather-Thurston [**Thu74**]). *Let M be a manifold. Then the natural map of topological groups $Homeo_{(+)}(M)^\delta \longrightarrow Homeo_{(+)}(M)$ induces an isomorphism*

$$H^\bullet\left(BHomeo_{(+)}(M),\mathbb{Z}\right) \xrightarrow{\sim} H^\bullet\left(BHomeo_{(+)}(M)^\delta,\mathbb{Z}\right)$$

on the cohomology rings of the classifying spaces.

Recall now that $\mathrm{Homeo}_+\left(S^1\right)$ deformation retracts onto the subgroup of rotations. Therefore the inclusion $i : S^1 \longrightarrow \mathrm{Homeo}_+\left(S^1\right)$ is a homotopy equivalence (see Theorem 1.1). In particular, by functoriality, the induced map $i_* : BS^1 \longrightarrow B\mathrm{Homeo}_+\left(S^1\right)$ is a homotopy equivalence. Now it is well-known that $BS^1 = \mathbb{CP}^\infty = K(\mathbb{Z}, 2)$. This fact together with Mather-Thurston Theorem enables us to compute the cohomology ring of $\mathrm{Homeo}_+\left(S^1\right)$.

THEOREM 1.22. *The cohomology ring of $Homeo_+\left(S^1\right)$ is a polynomial ring generated by the Euler class $e \in H^2\left(Homeo_+\left(S^1\right), \mathbb{Z}\right)$*

$$H^\bullet\left(Homeo_+\left(S^1\right)^\delta, \mathbb{Z}\right) = H^\bullet\left(Homeo_+\left(S^1\right), \mathbb{Z}\right) = \mathbb{Z}[e].$$

5. Bounded cohomology

In this section we cover the background material on bounded cohomology which we need. As for group cohomology we will work with the homogeneous and inhomogeneous resolutions and coefficients in $R = \mathbb{R}$ or $\mathbb{Z}$. The tools we need are Gersten's sequence and the relations some facts about the comparison map in degree $n = 2$ and (homogeneous) quasi-morphisms.

5.1. Basic definitions. Let Γ be any group. Let R be either $\mathbb{R}$ or $\mathbb{Z}$ with the trivial $\Gamma-$module structure and the trivial norm.

DEFINITION 1.23. The *bounded homogeneous complex* $(C_b^\bullet(\Gamma, R), \delta)$ is defined by

$$C_b^\bullet(\Gamma, R) := \{\phi \in C^\bullet(\Gamma, R) \,|\, \phi \text{ is bounded }\}.$$

The restriction of δ to $C_b^\bullet(\Gamma, R)$ gives the boundary map.
Analogously the *bounded inhomogeneous complex* $\left(\overline{C}_b^\bullet(\Gamma, R), \overline{\delta}\right)$ is given by

$$\overline{C}_b^\bullet(\Gamma, R) := \left\{\phi \in \overline{C}^\bullet(\Gamma, R) \,|\, \phi \text{ is bounded }\right\}.$$

The restriction of $\overline{\delta}$ to $\overline{C}_b^\bullet(\Gamma, R)$ gives the boundary map.
Both the homogeneous and the inhomogeneous complexes are normed complexes when equipped with the l^∞ norm

$$\begin{aligned} |\phi|_\infty &:= \sup_{\gamma_0,\dots,\gamma_n \in \Gamma^{n+1}} \{|\phi(\gamma_0, \dots, \gamma_n)|_R\} \quad \text{if } \phi \in C_b^n(\Gamma, R) \\ |\psi|_\infty &:= \sup_{\gamma_1,\dots,\gamma_n \in \Gamma^{n}} \{|\phi(\gamma_1, \dots, \gamma_n)|_R\} \quad \text{if } \phi \in \overline{C}_b^n(\Gamma, R). \end{aligned}$$

With respect to these norms the boundary maps $\delta, \overline{\delta}$ are bounded. Moreover the morphisms α, β defined respectively on ordinary homogeneous and inhomogeneous complexes restrict to morphisms between the bounded complexes which are isometries when we look at the norms.

DEFINITION 1.24. The *$n-$th bounded cohomology group of* Γ *with coefficients in* R is

$$H_b^n(\Gamma, R) := H^n(C_b^\bullet(\Gamma, R)) = H^n\left(\overline{C}_b^\bullet(\Gamma, R)\right).$$

The infinity norms on the homogeneous and inhomogeneous complexes induce a canonical *seminorm* on the bounded cohomology groups defined as follows:

$$||\alpha||_\infty := \inf_{\phi \in \alpha \in C_b^n(\Gamma, R)} \{|\phi|_\infty\} = \inf_{\psi \in \alpha \in \overline{C}_b^n(\Gamma, R)} \{|\psi|_\infty\}.$$

The inclusions $C_b^\bullet(\Gamma, R) \longrightarrow C^\bullet(\Gamma, R)$ and $\overline{C}_b^\bullet(\Gamma, R) \longrightarrow \overline{C}^\bullet(\Gamma, R)$ are morphisms of complexes and thus induce a map on cohomology:

$$c : H_b^\bullet(\Gamma, R) \longrightarrow H^\bullet(\Gamma, R).$$

We call this map the *comparison map.*

5.2. Gersten's sequence. The short exact sequence of normed trivial $\Gamma-$modules of coefficients

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0$$

gives rise to a long exact sequence in bounded cohomology called *Gersten's sequence*:

$$\dots \xrightarrow{\delta} H_b^i(\Gamma, \mathbb{Z}) \xrightarrow{j} H_b^i(\Gamma, \mathbb{R}) \longrightarrow H^i(\Gamma, \mathbb{R}/\mathbb{Z}) \xrightarrow{\delta} H_b^{i+1}(\Gamma, \mathbb{Z}) \longrightarrow \dots$$

It is easy to see that Gersten's sequence is functorial.

5.3. Low-degree bounded cohomology and quasi-morphisms. We collect and describe some basic facts on low-degree bounded cohomology groups $H_b^n(\Gamma, R)$ as we will make an extensive use of them later. More precisely we focus on $n = 1, 2$ and $R = \mathbb{Z}, \mathbb{R}$.

Degree $n = 1$. Since the only bounded subgroup of $\mathbb{Z}$ is the trivial subgroup, bounded cohomology with $R = \mathbb{Z}, \mathbb{R}$ coefficients always vanishes:

Lemma 1.25. *Let Γ be any group, let R be either $\mathbb{Z}$ or $\mathbb{R}$. Then $H_b^1(\Gamma, R) = 0$.*

Degree $n = 2$. Computing $H_b^2(\Gamma, \mathbb{Z}), H_b^2(\Gamma, \mathbb{R})$ is a hard task, but we can say something about the comparison map

$$c : H_b^2(\Gamma, R) \longrightarrow H^2(\Gamma, R).$$

Definition 1.26. A map $u : \Gamma \longrightarrow R$ is a *quasi-morphism* if δu is bounded or, in other words, there exists $M > 0$ such that

$$|u(\alpha) + u(\beta) - u(\alpha\beta)| \le M \qquad \text{for every } \alpha, \beta \in \Gamma$$

(here we are making use of the inhomogeneous standard resolution of Γ). The best constant $M > 0$ for which the above inequality holds is called the *defect* $M = D(u)$ of the quasi-morphism u.

We denote by $Q(\Gamma, R)$ the $R-$module of quasi-morphisms with values in R.

A quasi-morphism u is *homogeneous* if

$$u(\gamma^n) = nu(\gamma) \qquad \text{for every } \gamma \in \Gamma.$$

We denote by $HQ(\Gamma, R)$ the $R-$module of $R-$valued homogeneous quasi-morphisms.

Easy examples of quasi-morphisms are given by homomorphisms $u \in \mathrm{Hom}(\Gamma, R)$ and bounded functions $u \in \overline{C}_b^1(\Gamma, R)$.

Every quasi-morphism $u \in Q(\Gamma, R)$ defines a class $[\delta u] \in H_b^2(\Gamma, R)$ in the kernel of the comparison map $c([\delta u]) = 0$.

LEMMA 1.27. *Let Γ be any group, let $R = \mathbb{Z}, \mathbb{R}$. Then the map*

$$\frac{Q(\Gamma, R)}{Hom(\Gamma, R) \oplus \overline{C}_b^1(\Gamma, R)} \longrightarrow ker\left(c : H_b^2(\Gamma, R) \longrightarrow H^2(\Gamma, R)\right)$$

sending $[u] \longrightarrow [\delta u]$ is an isomorphism.

In the case of real coefficients $R = \mathbb{R}$ we can refine a quasi-morphism to a homogeneous one via a standard procedure:

LEMMA 1.28. *Let $u \in Q(\Gamma, \mathbb{R})$ be a quasi-morphism. Then the following holds:*

(1) *there exists a unique homogeneous quasi-morphism $\overline{u} \in HQ(\Gamma, \mathbb{R})$ such that $||u - \overline{u}||_\infty < \infty$. Explicitly $\overline{u}$ is defined by*

$$\overline{u}(\gamma) := \lim_{n \to \infty} \frac{u(\gamma^n)}{n}.$$

(2) *The natural map*

$$\frac{HQ(\Gamma, \mathbb{R})}{Hom(\Gamma, \mathbb{R})} \longrightarrow \frac{Q(\Gamma, \mathbb{R})}{Hom(\Gamma, \mathbb{R}) \oplus \overline{C}_b^1(\Gamma, \mathbb{R})}$$

is an isomorphism.

6. Bounded Euler class

This section together to the next one forms the bulk of the first chapter. We define the bounded Euler class of $\text{Homeo}_+\left(S^1\right)$ and of a representation. We will see that for $\text{Homeo}_+\left(S^1\right)$ bounded cohomology injects in classical cohomology, thus the bounded Euler class of $\text{Homeo}_+\left(S^1\right)$ is completely determined by the Euler class of $\text{Homeo}_+\left(S^1\right)$. Then we prove a general criterion which allows us to recover an integral bounded cohomology class from the corresponding real bounded class and some other data. The main result of this section, Matsumoto's criterion, consists of the explicit computation of the correspondence in the case of $e_b^{\mathbb{Z}} \in H_b^2\left(\text{Homeo}_+\left(S^1\right), \mathbb{Z}\right)$. In the last part we describe a geometric realization of the bounded Euler class and its relation with the orientation cocycle.

6.1. Bounded Euler class of $\text{Homeo}_+\left(S^1\right)$ and of a representation. Consider the universal covering central extension

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ j\ } \widetilde{\text{Homeo}}_+\left(S^1\right) \underset{\sigma}{\overset{p}{\rightleftarrows}} \text{Homeo}_+\left(S^1\right) \longrightarrow 0.$$

We show that we can choose (not in a unique way) a section σ such that the cocycle c_σ representing the Euler class is bounded. The cocycle c_σ will depend on the section σ but it will be well-defined up to bounded coboundaries thus defining a bounded class $e_b \in H_b^2(G, \mathbb{Z})$ mapping to $e \in H^2(G, \mathbb{Z})$ under the comparison map.

Let us fix $x \in S^1$ and a lift $\widetilde{x} \in \mathbb{R}$ of x. The section σ is defined as follows: $\sigma(f)$ is the

unique lift of f such that $\sigma(f)(\widetilde{x}) \in [\widetilde{x}, \widetilde{x}+1)$. We call σ a *bounded* section.
It is immediate to check that

$$c_\sigma(f,g) \in \{0,1\}\,.$$

In fact

$$\begin{array}{ll} c_\sigma(f,g) = \sigma(f)\sigma(g)\sigma(fg)^{-1}y - y & \text{for any } y \in \mathbb{R}, \text{ since } c_\sigma(f,g) \text{ is a translation} \\ = \sigma(f)\sigma(g)\widetilde{x} - \sigma(fg)\widetilde{x} & \text{choosing } y = \sigma(fg)\widetilde{x} \\ \Rightarrow c_\sigma \in (-1,2) & \text{since } \sigma(f)\sigma(g)\widetilde{x} \in [\widetilde{x}, \widetilde{x}+2) \text{ and } \sigma(fg)\widetilde{x} \in [\widetilde{x}, \widetilde{x}+1). \end{array}$$

It follows that $c_\sigma \in \{0,1\}$ since c_σ is integral valued.
It is easy to see that different choices of $x, y \in S^1$ and lifts $\widetilde{x}, \widetilde{y} \in \mathbb{R}$ produce sections σ_x, σ_y and cocycles $c_{\sigma_x}, c_{\sigma_y}$ related by

$$c_{\sigma_x} - c_{\sigma_y} = \delta u_{xy}$$

where $u_{xy}(f) = \sigma_x(f)\sigma_y(f)^{-1}$. The function u_{xy} is bounded and satisfies $|u_{xy}| \leq |\widetilde{x} - \widetilde{y}|+1$.

Definition 1.29. The *bounded Euler class* of $\mathrm{Homeo}_+\left(S^1\right)$ is the class $e_b \in H_b^2\left(G, \mathbb{Z}\right)$ represented by c_σ, where σ is any bounded section.

Let $\phi : \Gamma \longrightarrow G$ be any representation.

Definition 1.30. The *bounded Euler class of* ϕ is the pull-back of the bounded Euler class of $\mathrm{Homeo}_+\left(S^1\right)$

$$e_b(\phi) := \phi^* e_b \in H_b^2\left(\Gamma, \mathbb{Z}\right).$$

6.2. The Euler class completely determines the bounded Euler class. In order to compare the bounded and the classical Euler classes of $\mathrm{Homeo}_+\left(S^1\right)$ we prove in this section that the comparison map $c : H_b^2\left(\mathrm{Homeo}_+\left(S^1\right), \mathbb{Z}\right) \longrightarrow H^2\left(\mathrm{Homeo}_+\left(S^1\right), \mathbb{Z}\right)$ is injective. Thus there is only a bounded class mapping to the Euler class.

Lemma 1.31. *Let Γ be a group, let R be either $\mathbb{Z}$ or $\mathbb{R}$, let $f \in HQ(\Gamma, R)$ be a homogeneous quasi-morphism with defect $D(f)$. Then*

(1) f is conjugacy invariant: for every $g, x \in \Gamma$

$$f(gxg^{-1}) = f(x).$$

(2) For every $a, b \in \Gamma$

$$|f\left([a,b]\right)| \leq D(f).$$

Proof. We first prove *(1)*. We have:

$$\begin{aligned} &\left|f(gxg^{-1}) - f(x)\right| \\ &= \left|\left(f(gxg^{-1}) - f(gx) - f(g^{-1})\right) + \left(f(gx) + f(g^{-1}) - f(x)\right)\right| \\ &= \left|\left(f(gxg^{-1}) - f(gx) - f(g^{-1})\right) + \left(f(gx) - f(g) - f(x)\right)\right| \\ &\leq \left|f(gxg^{-1}) - f(gx) - f(g^{-1})\right| + \left|f(gx) - f(g) - f(x)\right| \\ &\leq 2D(f). \end{aligned}$$

In the second step we have only added and subtracted the same amount within the norm, while in the third equality we used the fact that $f(g^{-1}) = -f(g)$ since f is homogeneous.

Let us exploit again the homogeneity property together with the previous bound:

$$\begin{aligned} & \left|f(gxg^{-1}) - f(x)\right| \\ &= \frac{1}{n}\left|f(gx^ng^{-1}) - f(x^n)\right| && \text{by homogeneity} \\ &\leq \frac{2D(f)}{n} && \text{by the above inequality.} \end{aligned}$$

Taking the limit for $n \to \infty$ we get the desired property $f(gxg^{-1}) = f(x)$.
The proof of *(2)* follows from a straightforward computation:

$$\begin{aligned} |f([a,b])| &= \left|f\left(aba^{-1}b^{-1}\right)\right| \\ &\leq \left|f(aba^{-1}) + f(b^{-1})\right| + D(f) \\ &\leq \left|f(aba^{-1}) - f(b)\right| + D(f) \\ &= D(f). \end{aligned}$$

In the last step we used the invariance property *(1)*. □

COROLLARY 1.32. *Let Γ be a uniformly perfect group, let R be either $\mathbb{Z}$ or $\mathbb{R}$. Then $HQ(\Gamma, R) = 0$ and the comparison map*

$$c_R : H_b^2(\Gamma, R) \longrightarrow H^2(\Gamma, R)$$

is injective.

PROOF. We recall that non-trivial homogeneous quasi-morphisms are unbounded, so it is enough to show that the hypotesis of uniform perfectness of Γ implies that homogeneous quasi-morphisms are bounded. Let $N \geq 0$ be an integer such that every $\gamma \in \Gamma$ is a product of N commutators (eventually some of them are trivial). Let $f \in HQ(\Gamma, R)$ be a homogeneous quasi-morphism of defect $D = D(f)$. For every $\gamma \in \Gamma$ we have a decomposition $\gamma = \prod_{j=1}^{N} [a_j, b_j]$ for some $a_j, b_j \in \Gamma$, so we get the estimate

$$\begin{aligned} |f(\gamma)| &= \left|f\left(\prod_{j=1}^{N}[a_j,b_j]\right)\right| \\ &\leq \left|f\left(\prod_{j=1}^{N-1}\right)[a_j,b_j]\right| + |f([a_N,b_N])| + D(f) && \text{since } f \text{ is a quasi-morphism} \\ &\leq \sum_{j=1}^{N}|f([a_j,b_j])| + (N-1)D(f) && \text{inductively} \\ &\leq ND(f) + (N-1)D(f) && \text{by Lemma 1.31.} \end{aligned}$$

In conclusion f is bounded.
In order to prove the injectivity of the comparison map in the case $R = \mathbb{R}$ it is enough to

recall that there is an isomorphism (see Lemmas 1.27 and 1.28):

$$\frac{HQ(\Gamma,\mathbb{R})}{\operatorname{Hom}(\Gamma,\mathbb{R})} \xrightarrow{\sim} \ker\left(c : H_b^2(\Gamma,\mathbb{R}) \longrightarrow H^2(\Gamma,\mathbb{R})\right).$$

Now we can prove injectivity of the comparison map for $R = \mathbb{Z}$ with a simple diagram chasing.
Consider the following commutative diagram

$$\begin{tikzcd}
H^1(\Gamma,\mathbb{R}/\mathbb{Z}) \arrow[r] \arrow[d] & H_b^2(\Gamma,\mathbb{Z}) \arrow[r, "j_b"] \arrow[d, "c_{\mathbb{Z}}"'] & H_b^2(\Gamma,\mathbb{R}) \arrow[d, "c_{\mathbb{R}}"'] \\
H^1(\Gamma,\mathbb{R}/\mathbb{Z}) \arrow[r] & H^2(\Gamma,\mathbb{Z}) \arrow[r, "j"] & H^2(\Gamma,\mathbb{R})
\end{tikzcd}$$

where $c_{\mathbb{Z}}, c_{\mathbb{R}}$ are the comparison maps. We have

$$H^1(\Gamma,\mathbb{R}/\mathbb{Z}) = \operatorname{Hom}(\Gamma,\mathbb{R}/\mathbb{Z}) = 0$$

since every homomorphism $\phi : \Gamma \to \mathbb{R}/\mathbb{Z}$ factors throught the abelianization $\frac{\Gamma}{[\Gamma,\Gamma]} = e$ which is trivial as Γ is perfect. In particular the change of coefficients maps j, j_b are both injective. Since the comparison map $c_{\mathbb{R}}$ is also injective we conclude that $c_{\mathbb{Z}}$ is injective too. □

The previous result applies to $\mathrm{Homeo}_+(S^1)$ since it is uniformly perfect with constant $N = 1$ as stated in Theorem 1.3.
We sum up what we have just proved: the bounded Euler class e_b is completely determined by the Euler class. The same holds for representations $\rho : \Gamma \to \mathrm{Homeo}_+(S^1)$ when Γ is uniformly perfect or, more generally, when the comparison map $H_b^2(\Gamma,\mathbb{Z}) \to H^2(\Gamma,\mathbb{Z})$ is injective.

6.3. Canonical Euler cocycle. We describe a canonical Euler cocycle representing the bounded Euler class of $\mathrm{Homeo}_+(S^1)$ and of a representation $\phi : \Gamma \longrightarrow \mathrm{Homeo}_+(S^1)$ for an arbitrary group Γ. The next lemma makes possible to recover an integral bounded class from the knowledge of the corresponding real bounded class and of numerical invariants associated via the real bounded class to a set of generators for the group.

Lemma 1.33. *Let Γ be a group with generators $\{\gamma_j\}_{j\in I}$. Every class $\alpha \in H_b^2(\Gamma,\mathbb{Z})$ is completely determined by $j(\alpha) \in H_b^2(\Gamma,\mathbb{R})$ and $\left\{\xi_j(\alpha) \in H_b^2(\mathbb{Z},\mathbb{Z})\right\}_{j\in I}$ where $j : H_b^2(\Gamma,\mathbb{Z}) \longrightarrow H_b^2(\Gamma,\mathbb{R})$ is the change of coefficients map and $\xi_j^\bullet : H_b^\bullet(\Gamma,\mathbb{Z}) \longrightarrow H_b^\bullet(\mathbb{Z},\mathbb{Z})$ is the map induced by the homomorphism $i_j : \mathbb{Z} \longrightarrow \Gamma$ sending 1 to γ_j.*

Proof. Consider the following segment of Gersten's sequence

$$H_b^1(\Gamma,\mathbb{R}) = 0 \longrightarrow H^1(\Gamma,\mathbb{R}/\mathbb{Z}) = \operatorname{Hom}(\Gamma,\mathbb{R}/\mathbb{Z}) \longrightarrow H_b^2(\Gamma,\mathbb{Z}) \xrightarrow{j} H_b^2(\Gamma,\mathbb{R}).$$

Since $\mathbb{R}/\mathbb{Z}$ is abelian we have:

$$\operatorname{Hom}(\Gamma,\mathbb{R}/\mathbb{Z}) = \operatorname{Hom}\left(\frac{\Gamma}{[\Gamma,\Gamma]},\mathbb{R}/\mathbb{Z}\right).$$

The map $\xi_j^1 : H^1(\Gamma, \mathbb{Z}) = \operatorname{Hom}(\Gamma, \mathbb{Z}) \to H^1(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) = \mathbb{R}/\mathbb{Z}$ induced by the map i_j sends a homomorphism ϕ to $\phi(\gamma_j)$. Since the group $\frac{\Gamma}{[\Gamma,\Gamma]}$ is abelian and is generated by the classes of the γ_j's we have an embedding

$$\bigoplus_{j\in I} \xi_j^1 : H^1(\Gamma, \mathbb{Z}) \longrightarrow \bigoplus_{j\in I} \mathbb{R}/\mathbb{Z}.$$

Now the proof of the lemma reduces to a simple diagram chasing.
By functoriality of the Gersten's sequence we have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[r] & H^1(\Gamma, \mathbb{R}/\mathbb{Z}) \arrow[r, "\delta"] \arrow[d, "\bigoplus_{j\in I}\xi_j^1"'] & H_b^2(\Gamma, \mathbb{Z}) \arrow[r] \arrow[d, "\bigoplus_{j\in I}\xi_j^2"] & H_b^2(\Gamma, \mathbb{R}) \\
\bigoplus_{j\in I} H_b^1(\mathbb{Z}, \mathbb{R}) = 0 \arrow[rr] & & \bigoplus_{j\in I} \mathbb{R}/\mathbb{Z} \arrow[r, "\sim", "\delta"'] & \bigoplus_{j\in I} H_b^2(\mathbb{Z}, \mathbb{Z}) \arrow[r] & \bigoplus_{j\in I} H_b^2(\mathbb{Z}, \mathbb{R}) = 0
\end{tikzcd}$$

where $H_b^1(\mathbb{Z}, \mathbb{R}) = H_b^2(\mathbb{Z}, \mathbb{R}) = 0$ since $\mathbb{Z}$ is amenable. The vertical maps are those described above, the boundary map of the bottom row is an isomorphism by exactness.
Pick $c \in H_b^2(\Gamma, \mathbb{Z})$ such that $j(\alpha) = 0$ and $\xi_j(\alpha) = 0$ for every $j \in I$. We will prove that $\alpha = 0$. By exactness $j(\alpha) = 0$ implies $\alpha = \delta\phi$ for some $\phi \in H^1(\Gamma, \mathbb{R}/\mathbb{Z})$. By commutativity $\delta\left(\bigoplus \xi_j^1\right)\phi = \left(\bigoplus \xi_j^2\right)\delta\phi = 0$. Finally, by injectivity of the leftmost vertical arrow and of the middle bottom arrow we conclude that $\phi = 0$. In conclusion $\alpha = \delta\phi = 0$. □

Let us consider the case where $\Gamma = \operatorname{Homeo}_+(S^1) =: G$ and $\alpha = e_b^{\mathbb{Z}} \in H_b^2(G, \mathbb{Z})$.
We give a more explicit description of the classes $e_b^{\mathbb{R}} = j(e_b^{\mathbb{Z}})$ and $\xi_j^2(e_b^{\mathbb{Z}})$ when we take as a set of generators of G all possible homeomorphisms $\{f_j\}_{j\in I} = \operatorname{Homeo}_+(S^1)$.
First let us discuss $e_b^{\mathbb{R}}$. Let $p : \widetilde{G} \longrightarrow G$ be the universal covering projection (where $\widetilde{G} := \widetilde{\operatorname{Homeo}}_+(S^1)$), and consider the following commutative diagram:

$$\begin{tikzcd}
H_b^2(G, \mathbb{R}) \arrow[r, "c"] \arrow[d, "p^*"'] & H^2(G, \mathbb{R}) \arrow[d, "p^*"'] \\
H_b^2\left(\widetilde{G}, \mathbb{R}\right) \arrow[r, "\widetilde{c}"'] & H^2\left(\widetilde{G}, \mathbb{R}\right)
\end{tikzcd}$$

where $c, \widetilde{c}$ are respectively the comparison maps for $G, \widetilde{G}$.
By construction the pull-back of the integral Euler class $p^*e \in H^2\left(\widetilde{G}, \mathbb{Z}\right)$ is trivial (by Lemma 1.20, the homomorphism p clearly lifts to $\widetilde{G}$ as the identity $\widetilde{p} := \mathrm{Id}_{\widetilde{G}}$). Consequently, by naturality of the change of coefficients, also the pull-back of the real Euler class $p^*e \in H^2\left(\widetilde{G}, \mathbb{R}\right)$ is trivial.
By commutativity of the diagram therefore $\widetilde{c}(e_b^{\mathbb{R}}) = 0$. We recall that (Lemmas 1.27 and

1.28)

$$\ker(\widetilde{c}) = \frac{HQ(\widetilde{G},\mathbb{R})}{\mathrm{Hom}\left(\widetilde{G},\mathbb{R}\right)}.$$

Moreover, since $\widetilde{G}$ is perfect, we have $\mathrm{Hom}\left(\widetilde{G},\mathbb{R}\right) = 0$. In particular there exists a unique homogeneous quasi-morphism $\overline{u} \in HQ\left(\widetilde{G},\mathbb{R}\right)$ such that $e_b^{\mathbb{R}} = p^* e_b^{\mathbb{Z}} = [\delta\overline{u}]$. Let us compute the function $\overline{u} : \widetilde{G} \to \mathbb{R}$ explicitly. Let $\sigma : G \longrightarrow \widetilde{G}$ be the bounded section of p such that $\sigma(f)(0) \in [0,1)$, let $c_\sigma \in \overline{C}_b^2\left(\widetilde{G},\mathbb{Z}\right)$ be the corresponding 2−cocycle representing $e_b^{\mathbb{Z}}$, we have $p^* e_b^{\mathbb{Z}} = [p^* c_\sigma]$. Consider the following straightforward computation where the function $u : \widetilde{G} \to \mathbb{Z}$ is defined by $\tau_{u(h)} = h^{-1}\sigma(p(h)) \in \mathbb{Z}$

$$\begin{aligned}
&p^* c_\sigma(f,g) = c_\sigma(p(f),p(g))\\
&= \sigma(p(f))\sigma(p(g))\sigma(p(fg))^{-1}\\
&= \left(f\tau_{u(f)}\right)\left(g\tau_{u(g)}\right)\left(fg\tau_{u(fg)}\right)^{-1}\\
&= \tau_{u(f)+u(g)-u(fg)}\\
&= \delta u(f,g).
\end{aligned}$$

We have that u is a quasi-morphism (it has bounded coboundary). Finally we find the unique homogeneous quasi-morphism $\overline{u}$ which is at a bounded distance from u. We can construct it as a limit (Lemma 1.28):

$$\overline{u}(f) = \lim_{n\to\infty} \frac{u(f^n)}{n}.$$

Now we have

$$\begin{aligned}
\frac{u(f^n)}{n} &= \frac{f^{-n}\sigma(p(f^n))(0)}{n} && \\
&= \frac{f^{-n}\left\{f^n(0)\right\}}{n} && \text{since } \sigma(p(f^n))(0) = \{f^n(0)\}\\
&= \frac{f^{-n}\left(f^n(0) - \lfloor f^n(0)\rfloor\right)}{n} && \text{simply } f^n(0) = \lfloor f^n(0)\rfloor + \{f^n(0)\}\\
&= \frac{f^{-n}f^n(0) - \lfloor f^n(0)\rfloor}{n} && \text{since } f \text{ commutes with integral translations}\\
&= -\frac{\lfloor f^n(0)\rfloor}{n}.
\end{aligned}$$

Since $\lfloor f^n(0)\rfloor$ is at a bounded distance from $f^n(0)$ we see that taking the limit as n goes to ∞ we obtain

$$\overline{u}(f) = -\lim_{n\to\infty}\frac{f^n(0)}{n} = -\widetilde{\mathrm{rot}}(f).$$

In order to conclude the analysis of $p^* e_b^{\mathbb{R}}$ we need the following general result

Lemma 1.34. *Let $\phi : \Gamma \to \Lambda$ be a surjective homomorphism with amenable kernel. Then the induced maps*

$$\phi^* : H_b^n(\Lambda,\mathbb{R}) \longrightarrow H_b^n(\Gamma,\mathbb{R})$$

are isometric isomorphisms for every $n \in \mathbb{N}$.

In particular Lemma 1.34 applies to the case of the universal covering projection $p : \widetilde{G} \to G$ so $p^* : H_b^2(G, \mathbb{R}) \to H_b^2\left(\widetilde{G}, \mathbb{R}\right)$ is an isometric isomorphism with an inverse which we denote by $\sigma^* := (p^*)^{-1}$. This gives a canonical representative for the real bounded Euler class:

$$e_b^{\mathbb{R}} = -\sigma^* \left[\delta\widetilde{\mathrm{rot}}\right].$$

Now we describe the classes $\xi_j\left(e_b^{\mathbb{Z}}\right)$.

First we observe that the boundary homomorphism $\delta : H^1(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \xrightarrow{\sim} H_b^2(\mathbb{Z}, \mathbb{Z})$ is an isomorphism, since $H_b^1(\mathbb{Z}, \mathbb{R}) = H_b^2(\mathbb{Z}, \mathbb{R}) = 0$. Consequently we have the following identifications:

$$H_b^2(\mathbb{Z}, \mathbb{Z}) \simeq H^1(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z}.$$

Let us make explicit the correspondence.

Equality $H^1(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}, \mathbb{R}/\mathbb{Z})$ simply follows from the fact that

$$\ker\left(\delta : \overline{C}^1(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \to \overline{C}^2(\mathbb{Z}, \mathbb{R}/\mathbb{Z})\right) = \mathrm{Hom}(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \qquad \text{and} \qquad \delta\overline{C}_0(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) = 0.$$

The identification $\mathrm{Hom}(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z}$ is clear: it is the evaluation map $\phi \to \phi(1)$.

Let us now analyze the boundary isomorphism $\delta : H^1(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \to H_b^2(\mathbb{Z}, \mathbb{Z})$: in order to compute the boundary of a class $\alpha \in H^1(\mathbb{Z}, \mathbb{R}/\mathbb{Z})$ we should chase the following diagram

$$\begin{tikzcd}
 & \overline{C}_b^1(\mathbb{Z}, \mathbb{R}) \arrow[r, "\pi"] \arrow[d, "\delta"] & \overline{C}^1(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \arrow[r] & 0 \\
0 \arrow[r] \overline{C}_b^2(\mathbb{Z}, \mathbb{Z}) \arrow[r, "i"] & \overline{C}_b^2(\mathbb{Z}, \mathbb{R}) & &
\end{tikzcd}$$

We will proceed backwards starting from the image $\delta\alpha \in H^2(\mathbb{Z}, \mathbb{Z})$. Let $f \in G$ be a homeomorphism, let $\widetilde{f} \in \widetilde{G}$ be any lift of f, let $\sigma : G \to \widetilde{G}$ be the bounded section such that $\sigma(h)(0) \in [0, 1)$ for every $h \in G$. We have

$$\begin{aligned}
&\phi_f^* c_\sigma(n, m) = c_\sigma(f^n, f^m) \\
&= \sigma(f^n)\sigma(f^m)\sigma(f^{n+m})^{-1} \\
&= \left(\widetilde{f}^n \tau_{u(n)}\right)\left(\widetilde{f}^m \tau_{u(m)}\right)\left(\widetilde{f}^{n+m} \tau_{u(n+m)}\right)^{-1} \quad \text{where } \tau_{u(k)} = \left(\widetilde{f}\right)^{-k} \sigma(f^k) \\
&= \tau_{u(n)+u(m)-u(n+m)} \\
&= \delta u(n, m).
\end{aligned}$$

Thus $i(\phi_f^* c_\sigma) = \delta u$, the function u is an unbounded quasi-morphism. Let us denote by $\overline{u} \in HQ(\mathbb{Z}, \mathbb{R}) = \mathrm{Hom}(\mathbb{Z}, \mathbb{R})$ the unique homomorphism (every homogeneous quasi-morphism on $\mathbb{Z}$ is a homomorphism by definition) at a bounded distance from u . We have

$$\begin{aligned}
&i\phi_f^* c_\sigma = \delta u = \delta\left(u - \lfloor\overline{u}\rfloor\right) + \delta\left(\lfloor\overline{u}\rfloor - \overline{u}\right) + \delta\overline{u} \\
&\Rightarrow i\phi_f^* c_\sigma - \delta\left(u - \lfloor\overline{u}\rfloor\right) = \delta\left(\lfloor\overline{u}\rfloor - \overline{u}\right) + \delta\overline{u} = -\delta\left\{\overline{u}\right\}
\end{aligned}$$

where $\lfloor \overline{u} \rfloor, \{\overline{u}\}$ are respectively the integral and fractional parts of $\overline{u}$. Observe that $\delta \overline{u} = 0$ since $\overline{u}$ is a homomorphism, and that $|\overline{u} - \lfloor \overline{u} \rfloor| = |\{\overline{u}\}| < 1$ so that $\overline{u} - \lfloor \overline{u} \rfloor \in C^1_b(\mathbb{Z}, \mathbb{R})$. Moreover $|u - \lfloor \overline{u} \rfloor| \leq |u - \overline{u}| + |\overline{u} - \lfloor \overline{u} \rfloor| < \infty$ and $u - \lfloor \overline{u} \rfloor$ is integral valued, therefore $u - \lfloor \overline{u} \rfloor \in \overline{C}^1_b(\mathbb{Z}, \mathbb{Z})$, and adding the boundary of $\lfloor \overline{u} \rfloor - u$ does not change the bounded cohomology class of $\phi^*_f c_\sigma$. We are now ready to chase the diagram:

$$\begin{tikzcd} & -\{\overline{u}\} \arrow[r, "\pi"] \arrow[d, "\delta"'] & {[-\overline{u}(1)]} \\ \phi^*_f c_\sigma - \delta(u - \lfloor \overline{u} \rfloor) \arrow[r, "i"] & -\delta\{\overline{u}\} & \end{tikzcd}$$

In conclusion $\left[\phi^*_f c_\sigma\right] = [\overline{u}(1)] \in \mathbb{R}/\mathbb{Z}$. The computation of $\overline{u}(1)$ is completely analogous to the computation of $p^* e^{\mathbb{R}}_b$ we made before and leads to

$$\phi^*_f e^{\mathbb{Z}}_b = [-\overline{u}(1)] = -\mathrm{rot}(f) \in \mathbb{R}/\mathbb{Z}.$$

Let us summarize what we have shown in a proposition:

Proposition 1.35. *Let $G = Homeo_+\left(S^1\right)$. The following holds:*

(1) the real bounded Euler class has a canonical representative induced by a unique homogeneous quasi-morphism on $\widetilde{G} = \widetilde{Homeo_+}\left(S^1\right)$

$$e^{\mathbb{R}}_b = \sigma^*\left[\tau := -\delta\widetilde{rot}\right] \qquad , \qquad \tau(f,g) = \widetilde{rot}\,\widetilde{f}\widetilde{g} - \widetilde{rot}\,\widetilde{f} - \widetilde{rot}\,\widetilde{g}.$$

(2) for every f the integral bounded Euler class of the representation $\phi_f : \mathbb{Z} \longrightarrow Homeo_+\left(S^1\right)$ defined by $\phi_f(n) = f^n$ corresponds to

$$e^{\mathbb{Z}}_b(\phi_f) = -rot(f) \in \mathbb{R}/\mathbb{Z}$$

under the identification $\mathbb{R}/\mathbb{Z} \xrightarrow{\delta} H^2_b(\mathbb{Z}, \mathbb{Z})$.

Definition 1.36. The *canonical Euler cocycle* $\tau : G \times G \longrightarrow$ is the inhomogeneous $2-$cocycle defined by

$$\tau(f,g) = \widetilde{\mathrm{rot}}\,\widetilde{f}\widetilde{g} - \widetilde{\mathrm{rot}}\,\widetilde{f} - \widetilde{\mathrm{rot}}\,\widetilde{g}.$$

6.4. Criteria: numerical invariants for the bounded Euler class. Exploiting the previous results of this section we can easily prove the following criterion:

Theorem 1.37 (Matsumoto, [**Mat86**]). *Let Γ be a group with generators $\{\gamma_j\}_{j \in I}$, let $\rho : \Gamma \to Homeo_+\left(S^1\right)$ be a representation. Then $\phi^* e^{\mathbb{Z}}_b$ is completely determined by the data*

$$\begin{cases} rot(\rho(\gamma_j)) & \text{for every } j \in I \\ \rho^*\tau & \text{as an inhomogeneous } 2-\text{cocycle} \end{cases}$$

Proof. By Lemma 1.33 $\rho^* e^{\mathbb{Z}}_b$ is completely determined from the data $\rho^* e^{\mathbb{R}}_b$ and $\xi_j\left(\rho^* e^{\mathbb{Z}}_b\right)$. By Proposition 1.35 $e^{\mathbb{R}}_b = \left[\tau = -\delta\widetilde{\mathrm{rot}}\right]$ and $\xi_j\left(\rho^* e^{\mathbb{Z}}_b\right) = \phi^*_{\rho(\gamma_j)} e^{\mathbb{Z}}_b = -\mathrm{rot}(\rho(\gamma_j))$. $\square$

6.5. Homogeneous quasi-morphisms on $\widetilde{G} = \widetilde{\mathbf{Homeo}}_+\left(S^1\right)$. The previous results are not so surprising because we have the following characterization of $\widetilde{\mathrm{rot}}$:

PROPOSITION 1.38. *The space of homogeneous quasi-morphisms on $\widetilde{G}$ is generated by $\widetilde{rot}$*

$$HQ\left(\widetilde{G}, \mathbb{R}\right) = \mathbb{R}\widetilde{rot}.$$

PROOF. Let $u : \widetilde{G} \longrightarrow \mathbb{R}$ be a non-trivial homogeneous quasi-morphism. Up to rescaling we can assume $u(\tau_1) = 1$. Define $v := u - \widetilde{\mathrm{rot}}$, by homogeneity $v|_{\mathbb{Z}\subseteq\widetilde{G}} \equiv 0$. We claim that v induces a homogeneous quasi-morphism w on G. If this were the case then using the fact that $HQ(G, \mathbb{R}) = 0$ we would conclude that w is trivial and therefore $0 = v = u - \widetilde{\mathrm{rot}}$. For a fixed $f \in G$ we would like to define $w(f) := v(\widetilde{f})$ for a lift $\widetilde{f} \in \widetilde{G}$, so we need to show that the definition is independent of the lift. This easily follows from the fact that every homogeneous quasi-morphism defined on an amenable group is in fact a homomorphism: since $\mathbb{Z} < \widetilde{G}$ is central, the group $\langle \widetilde{f}, \mathbb{Z}\rangle$ is abelian and, in particular, amenable; consequently $v|_{\langle \widetilde{f},\mathbb{Z}\rangle}$ is a homomorphism and then $v\left(\widetilde{f}\tau_n\right) = v\left(\widetilde{f}\right) + v(\tau_n) = v\left(\widetilde{f}\right)$. Thus w is well-defined and a straightforward computation shows that w is a homogeneous quasi-morphism. $\square$

6.6. Geometric description of the bounded Euler class. Let us fix $[0] \in S^1 = \mathbb{R}/\mathbb{Z}$ be a point. Let $\phi : \Gamma \longrightarrow G$ be any representation. Consider the orbit $\Gamma[0]$. Then the bounded Euler class is equivalent to the complete knowledge of the cyclic ordering of $\Gamma[0]$. Let us explain why. First we introduce a very natural $2-$cocycle defined on G:

DEFINITION 1.39. The (homogeneous) *orientation cocycle* $\overline{\mathrm{Or}}_{[0]} : G \times G \times G \longrightarrow \mathbb{Z}$ is defined as follows

$$\overline{\mathrm{Or}}_0\left(g_0, g_1, g_2\right) = \mathrm{sgn}\left(g_0[0], g_1[0], g_2[0]\right) := \begin{cases} 1 & \text{if positively oriented} \\ -1 & \text{if negatively oriented} \\ 0 & \text{if degenerate.} \end{cases}.$$

Its inhomogeneous version $\mathrm{Or}_0 : G \times G \longrightarrow \mathbb{Z}$ is given by

$$\mathrm{Or}_0(f, g) = \overline{\mathrm{Or}}_0(1, f, fg) = \mathrm{sgn}\left([0], f[0], fg[0]\right).$$

The $G-$invariance of the bounded $2-$cochain $\overline{\mathrm{Or}}_0$ is clear since every element $g \in G$ preserves the orientation on S^1, i.e. $\mathrm{sgn}(gx, gy, gz) = \mathrm{sgn}(x, y, z)$. It is also immediate to see that $\overline{\mathrm{Or}}_t$ is an alternating cochain, i.e. if $\tau \in S_3$ is a transposition $\mathrm{Or}_0\left(g_{\tau(0)}, g_{\tau(1)}, g_{\tau(2)}\right) = -\mathrm{Or}_0(g_0, g_1, g_2)$. A straightforward computation shows that it is a cocycle. We denote by $\mathrm{Or} := \left[\overline{\mathrm{Or}}_t\right] \in H^2_b(G, \mathbb{Z})$ the bounded cohomology class they represent.
Keeping fixed $[0] \in S^1$ there is another natural bounded cochain $\overline{b}_0 : G \times G \longrightarrow \mathbb{Z}$ defined as follows

$$\overline{b}_0(f, g) := \begin{cases} 1 & \text{if } f[0] \neq g[0] \\ 0 & \text{otherwise} \end{cases}.$$

The inhomogeneous version is simply $b_0 : G \longrightarrow \mathbb{Z}$ given by $b_0(f) = \overline{b}_0(1, f)$.
Now consider the representative c_σ of the bounded Euler class corresponding to the bounded

section $\sigma : G \longrightarrow \widetilde{G}$ such that $\sigma(f)(0) \in [0,1)$. The following proposition gives the relation between the bounded Euler class and the orientation cocycle:

PROPOSITION 1.40. *The cocycles Or_0 and c_σ satisfy the following relation*

$$Or_0 = -2c_\sigma + \delta b_0.$$

In particular $2e_b^{\mathbb{Z}} = -Or$ in $H_b^2(G,\mathbb{Z})$.

7. The relation of semi-conjugacy

Semi-conjugacy is a natural equivalence relation on the set of group actions on the circle first introduced by Ghys in [**Ghy01**]. Unfortunately there is an issue with the original definition concerning actions with a global fixed point. Thus many different definitions of semi-conjugacy arose after the publication of Ghys' paper. For a comprehensive treatment of the subject as well as an attempt of making order among the many possible definitions of semi-conjugacy and their relations we refer to [**BFH14**]. In this section we deal with two peculiar avatars of semi-conjugacy and their properties. The main result is Ghys' Theorem that states the bounded Euler class is a complete invariant for the equivalence relation of semi-conjugacy.

7.1. Definitions and properties. Let us identify $S^1 = \mathbb{R}/\mathbb{Z}$ and denote by $\pi : \mathbb{R} \to S^1$ the universal covering projection. We first introduce non-increasing maps of degree 1 which are needed in the definition of semi-conjugacy. The next lemma gives a list of equivalent properties that are required in the definition of these maps:

LEMMA 1.41. *Let $h : S^1 \longrightarrow S^1$ be a map. The following are equivalent:*

(1) there exists a map $\widetilde{h} : \mathbb{R} \to \mathbb{R}$ such that
  - *$\widetilde{h}$ is non-decreasing.*
  - *$\widetilde{h}$ commutes with integral translations τ_n.*
  - *$\widetilde{h}$ lifts h or $\pi\widetilde{h} = h\pi$.*

(2) for every $n \geq 3$ and every cyclic ordered $n-$tuple $x_1 \prec \cdots \prec x_n$ on S^1 the map h preserves the cyclic order

$$h(x_1) \preceq \cdots \preceq h(x_n).$$

(3) for every cyclic ordered $4-$tuple $x_1 \prec x_2 \prec x_3 \prec x_4$ on S^1 the map h preserves the cyclic order

$$h(x_1) \preceq h(x_2) \preceq h(x_3) \preceq h(x_4).$$

DEFINITION 1.42. A map $h : S^1 \longrightarrow S^1$ is said to be

- *non-increasing of degree* 1 or a *semi-conjugacy* if it satisfies the equivalent conditions of Lemma 1.41.
- a *monotone equivalence* if it is a continuous semi-conjugacy of Brower-Hopf degree $\deg(h) = 1$.

Now we are ready to give the following definitions:

DEFINITION 1.43. Representations $\phi, \psi : \Gamma \longrightarrow \mathrm{Homeo}_+\left(S^1\right)$ are said to be

- *semi-conjugate* if there are non-increasing degree 1 maps $h_1, h_2 : S^1 \longrightarrow S^1$ such that for every $\gamma \in \Gamma$ the following holds
$$h_1\phi(\gamma) = \psi(\gamma)h_1 \qquad \text{and} \qquad \phi(\gamma)h_2 = h_2\psi(\gamma).$$
It is clear that semi-conjugacy is an equivalence relation.
- ϕ is *left-monotone equivalent* to ψ if there is a monotone equivalence $h : S^1 \longrightarrow S^1$ such that for every $\gamma \in \Gamma$ the following holds
$$\phi(\gamma)h = h\psi(\gamma).$$
Left-monotone equivalence is not an equivalence relation since it is not symmetric (but it is reflexive and transitive). We call *monotone equivalence* the relation generated by left-monotone equivalence.

The following theorem by Calegari gives a symmetric description of monotone equivalence.

Theorem 1.44 (Calegari, [**Cal07**]). *Let $\phi, \psi : \Gamma \to Homeo_+\left(S^1\right)$ be representations. Then ϕ, ψ are monotone equivalent if and only if there exists a representation ρ such that both ϕ, ψ are left-monotone equivalent to ρ.*

7.2. Relations between semi-conjugacy and monotone equivalence. Semi-conjugacy and monotone equivalence give rise to the same equivalence relation.

Theorem 1.45 (Calegari [**Cal07**]). *Let $\phi, \psi : \Gamma \longrightarrow Homeo_+\left(S^1\right)$ be representations. Then*
$$\phi, \psi \textit{ are semi-conjugate} \iff \phi, \psi \textit{ are monotone equivalent.}$$

For a complete proof of this theorem we refer to the article by M. Bucher, R. Frigerio and T. Hartnick [**BFH14**], we cover here the simplest cases. Sometimes it would be useful to work with semi-conjugacies, instead in other cases, when we want to work with more regularity, we will consider monotone equivalences. Thus, in order to freely move from one setting to the other, it is convenient for us to describe better the relations between them. The easiest relation between semi-conjugacies and monotone equivalences, which follows directly from the property of being non-increasing, is the fact that surjective semi-conjugacies are automatically continuous.
For minimal actions semi-conjugacy is very well behaved:

Proposition 1.46. *Let $\phi, \psi : \Gamma \longrightarrow Homeo_+\left(S^1\right)$ be minimal actions. Then*
$$\phi, \psi \textit{ are semi-conjugate} \iff \phi, \psi \textit{ are topologically conjugate.}$$

Proof. The non-trivial implication is ($\Longrightarrow$). Let $h : S^1 \longrightarrow S^1$ be a left semi-conjugacy between ϕ and ψ, i.e. $\phi h = h\psi$. Fix $t \in S^1$, we have $\phi(\Gamma)h(t) = h\left(\psi(\Gamma)t\right)$. Since the orbit $\phi(\Gamma)h(t)$ is dense and h is non-increasing, we must have h surjective and thus h continuous. We now want to promote h to a homeomorphism, our claim is that we already have one so we prove that h is injective. Proceed by contradiction and suppose that $h(x) = h(y)$ for some $x \neq y$. Since h is non-increasing, it has to be constant on the arc I going from x to y, i.e. $h(z) = h(x) = h(y)$ for every $x < z < y$. Consider now an arbitrary point $t \in S^1$. By minimality of ψ there is an element $\gamma \in \Gamma$ such that $\psi(\gamma)t \in I$. For every $u \in \psi(\gamma)^{-1}I$, i.e. $\psi(\gamma)u = v \in I$, we have $h(x) = h\left(v\right) = h\left(\psi(\gamma)u\right) = \phi(\gamma)h(u)$.

Hence $h|_{\psi(\gamma)^{-1}I} \equiv \phi(\gamma)^{-1}h(x)$ which tells us that h is locally constant. This implies that h is constant by connectedness of S^1, but h is surjective, contradiction. $\square$

REMARK 1.47. Actually we proved more in the proposition since we have seen that every left semi-conjugacy h between a minimal action ϕ and an arbitrary action ψ, i.e. $\phi h = h\psi$, is already a left monotone equivalence. It is also a topological conjugacy if we assume that ψ is minimal too.

In some cases we can reduce an action to a minimal one via a left monotone equivalence:

PROPOSITION 1.48. *Let $\phi : \Gamma \longrightarrow Homeo_+\left(S^1\right)$ be an action. Suppose that every orbit of ϕ is infinite. Then ϕ is right-monotone equivalent to a minimal action ψ, i.e.*

$$\psi h = h\phi \qquad , \qquad h \text{ monotone equivalence.}$$

PROOF. Let K_ϕ be the minimal exceptional of ϕ. If $K_\phi = S^1$ then there is nothing to add as ϕ is already minimal. Suppose $K_\phi \subsetneq S^1$, i.e. K_ϕ is a Cantor set by Lemma 1.6. The complement $S^1 \setminus K_\phi = \bigcup_{j\in\mathbb{N}} I_j$ is a disjoint union of open intervals I_j and Γ acts on them by permutations. If we collapse the closure $\overline{I}_j$ of I_j to a point then the quotient space is again a circle. The action $\Gamma \curvearrowright S^1$ induces through the quotient map an action on the quotient space which has to be minimal by minimality of K_ϕ (otherwise there would be a $\Gamma-$invariant compact set K'_ϕ strictly smaller than K_ϕ). After identifying the quotient space with S^1 we only need to observe that the quotient map is the monotone equivalence we were looking for. $\square$

The last regularity result we need for semi-conjugacies is the following approximation lemma:

LEMMA 1.49. *Let $\phi, \psi : \Gamma \longrightarrow Homeo_+\left(S^1\right)$ be representations such that ϕ is left-monotone equivalent to ψ, i.e. $\phi h = h\psi$ for some monotone equivalence h. Then there exists a continuous family of homeomorphisms $\{h_t\}_{t\in[0,1)} \subseteq G$ such that*

$$h_t\psi(\gamma)h_t^{-1} \longrightarrow \phi(\gamma)$$

uniformly for every $\gamma \in \Gamma$.

7.3. Ghys' Theorem. Finally we state the fundamental result by Ghys that gives a cohomological characterization of the relation of semi-conjugacy:

THEOREM 1.50 (Ghys, [**Ghy01**]). *Let Γ be a group and $\rho_1, \rho_2 : \Gamma \to Homeo_+\left(S^1\right)$ be representations. Then*

$$\rho_1, \rho_2 \text{ are semi-conjugate} \iff e_b(\rho_1) = e_b(\rho_2) \in H_b^2(\Gamma, \mathbb{Z})$$

A proof of this result can be found in [**BFH14**].

As a corollary, using the computations of part *(2)* of Proposition 1.35, we recover the fact that rotation number is a complete invariant for semi-conjugacy for homeomorphisms.

COROLLARY 1.51. *Let $f \in Homeo_+\left(S^1\right)$ be a homeomorphism. If $rot(f) = \theta$ then f is semi-conjugate to ρ_θ.*

CHAPTER 2

Representations of surface groups

1. Overview

In this chapter we focus on representations of surface groups $\Gamma := \pi_1(S)$. Our purpose is to specialize the invariants defined in the first chapter to the surface group case and describe some geometric objects naturally associated to a representation. Thus we first describe the cohomology groups of surface groups and define the Euler number $e(\phi) \in \mathbb{Z}$ of a representation $\phi : \Gamma \longrightarrow \mathrm{Homeo}_+\left(S^1\right)$, which is a numerical invariant equivalent to the Euler class. For every representation ϕ we can construct a circle bundle over the surface which has a flat structure. Hence we will be able to attach new invariants to a representation by looking at cohomological invariants of the associated circle bundle. The main invariant for a circle bundle is again an Euler class. New and old invariants are related, we describe the correspondence and incidentally give a picture of the connections between the algebraic setting and the topological one.

The principal result of the chapter is Milnor's Formula that explicitly computes the Euler number of a representation as the translation number of a product of lifted commutators:

THEOREM (Milnor's Formula). *Let S be a surface of genus $g \geq 2$, let $\phi : \pi_1(S) \to$ $Homeo_+\left(S^1\right)$ be a representation. Then*

$$\tau_{e(\phi)} = \left[\widetilde{\phi}(a_1), \widetilde{\phi}(b_1)\right] \dots \left[\widetilde{\phi}(a_g), \widetilde{\phi}(b_g)\right].$$

where $\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)$ are arbitrary lifts of $\phi(a_j), \phi(b_j)$ to $\widetilde{Homeo_+}\left(S^1\right)$.

1.1. Preliminaries on surface groups. Let us recall that the fundamental group of a closed oriented surface of genus g admits the following standard presentation:

$$\pi_1(S) = \left\langle a_1, b_1, \dots, a_g, b_g \;\middle|\; \prod_{j=1}^{g} [a_j, b_j] = 1 \right\rangle$$

We also recall that if S is a compact oriented surface of finite type with genus g and b boundary components $c_1, \dots, c_b$, then $\pi_1(S)$ admits the following presentation

$$\pi_1(S_{g,b}) = \left\langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_b \;\middle|\; \prod_{j=1}^{g} [a_j, b_j] \cdot c_1 \dots c_b = 1 \right\rangle.$$

2. Surface group cohomology and Euler number

Let S be a closed surface different from the sphere $S \neq S^2$. Since $\widetilde{S}$ is contractible we see that S is a $K(\pi_1(S), 1)$ space. In particular we get the following:

THEOREM 2.1. *Let S be as above, let R be either $\mathbb{R}$ or $\mathbb{Z}$. Then the map*

$$r : H^\bullet(\pi_1(S), \mathbb{Z}) \xrightarrow{\sim} H^\bullet(S, \mathbb{Z})$$

is an isomorphism.

Therefore

$$H^n(\pi_1(S), R) \simeq H^n(S, R) = \begin{cases} R & \text{if } n = 0 \\ R^{2g} & \text{if } n = 1 \\ R & \text{if } n = 2 \\ 0 & \text{if } n \geq 3. \end{cases}$$

DEFINITION 2.2. The *Euler number* of a representation $\phi : \pi_1(S) \longrightarrow \mathrm{Homeo}_+\left(S^1\right)$, which we also denote by $e(\phi) \in \mathbb{Z}$, is given by

$$e(\rho) := \langle r(\phi^* e), [S] \rangle$$

where $r : H^2(\pi_1(S), \mathbb{Z}) \xrightarrow{\sim} H^2(S, \mathbb{Z})$ is the natural identification, $[S] \in H_2(S, \mathbb{Z})$ is the fundamental class of S and $\langle \bullet, \bullet \rangle$ is the duality pairing on $H^2(S, R) \times H_2(S, R)$.

In the following sections, in order to simplify the notations, we avoid writing the natural map r when we identify $H^2(\pi_1(S), \mathbb{Z})$ with $H^2(S, \mathbb{Z})$. Bounded cohomology of surface groups is far more complicated. A description of $H_b^2(\pi_1(S), \mathbb{Z})$ is given in the article [**BS84**]. A result by Ghys (see [**Ghy87**], Theorem B) ensures that every class $\alpha \in H_b^2(\pi_1(S), \mathbb{Z})$ that can be represented by a cocycle $c : \pi_1(S) \times \pi_1(S) \longrightarrow \mathbb{Z}$ with values in $\{0, 1\}$ can be obtained as the bounded Euler class of some representation $\phi : \pi_1(S) \longrightarrow \mathrm{Homeo}_+\left(S^1\right)$.

3. Circle bundles

In this section we are mainly concerned with circle bundles $p : E \longrightarrow S$ over the surface S. For us all circle bundles will be orientable, and their group structure will be $\mathrm{Homeo}_+\left(S^1\right)$. First we describe the class of flat circle bundles and classify them via the holonomy representation which is a homomorphism $\pi_1(S) \longrightarrow \mathrm{Homeo}_+\left(S^1\right)$. Then we construct the Euler class or Euler number of a circle bundle, which is the main cohomological invariant of the bundle, and state a classification theorem.

3.1. Flat circle bundles.

We consider a special class of circle bundles. Before giving the definition we state the following equivalence:

LEMMA 2.3. *Let $p : E \longrightarrow S$ be a circle bundle. The following are equivalent*

(1) there exists a foliation $\mathcal{F}$ of the total space E which is transverse to the fibers.
(2) there exists a trivializing atlas $\mathcal{A}$ with locally constant transition functions.

DEFINITION 2.4. An S^1−bundle $p : E \longrightarrow S$ is *flat* if it satisfies the equivalent properties stated in Lemma 2.3.

Like all bundles, flat bundles have the lifting property for paths and homotopies. Exploiting the flat structure we can construct lifts which lie on a single leaf of the foliation. This gives the following:

Proposition 2.5. *Let $p : E \longrightarrow S$ be a flat circle bundle. Fix $x_0 \in S$ and $e_0 \in E_{x_0} = p^{-1}(x_0)$ as base points. Let $\mathcal{F}_{e_0}$ be the leaf passing through $e_0 \in E$. The following holds*

(1) for every path $\gamma : I \longrightarrow S$ such that $\gamma(0) = x_0$ there exists a unique lift $\widetilde{\gamma} : I \longrightarrow E$ such that $\widetilde{\gamma}(0) = e_0$ and $\widetilde{\gamma}(I) \subseteq \mathcal{F}_{e_0}$.
(2) for every homotopy $H : I^2 \longrightarrow S$ such that $H(0,0) = x_0$ there exists a unique lift $\widetilde{H} : I^2 \longrightarrow E$ such that $\widetilde{H}(0,0) = e_0$ and $\widetilde{H}\left(I^2\right) \subseteq \mathcal{F}_{e_0}$.

In particular for every leaf $\mathcal{F}_e$ the restriction of the projection $p|_{\mathcal{F}_e} : \mathcal{F}_e \longrightarrow S$ is a covering.

Fix $x_0 \in S$ a base point and identify E_{x_0} with S^1 by trivializing the bundle near x_0.
For every $[\gamma] \in \pi_1(S, x_0)$ we can define the following map

$$\eta_{[\gamma]} : E_{x_0} \longrightarrow E_{x_0} \qquad , \qquad \eta_{[\gamma]}(e) := \widetilde{\gamma}_e^{-1}(1)$$

where $\widetilde{\gamma}_e$ is the unique lift to $\mathcal{F}_{e_0} \subseteq E$ with base point $e \in E_{x_0}$ of the representative $\gamma \in [\gamma]$. Uniqueness of the lifts of paths and homotopies guarantees that $\eta_{[\gamma]}$ is well-defined on classes of paths with respect to homotopies with fixed endpoints. Moreover it is not difficult to prove the following

Proposition 2.6. *Let $x_0 \in S$ and $E_{x_0} \simeq S^1$ as above. Then*

(1) for every $\alpha, \beta \in \pi_1(S, x_0)$ we have $\eta_\alpha \eta_\beta = \eta_{\alpha\beta}$.
(2) for every $\alpha \in \pi_1(S, x_0)$ we have $\eta_\alpha \in Homeo_+\left(E_{x_0} = S^1\right)$.

In particular there is a well defined homomorphism

$$\eta : \pi_1(S, x_0) \longrightarrow Homeo_+\left(S^1\right).$$

Definition 2.7. The homomorphism $\eta : \pi_1(S, x_0) \longrightarrow \mathrm{Homeo}_+\left(S^1\right)$ described above is the *holonomy representation* of the flat circle bundle $p : E \longrightarrow S$.

Now we describe a general procedure to construct flat circle bundles $p : E_\phi \longrightarrow S$ with a prescribed holonomy $\phi : \pi_1(S) \longrightarrow \mathrm{Homeo}_+\left(S^1\right)$.

Let S be an oriented surface. Let $\pi : \widetilde{S} \longrightarrow S$ be the universal covering. The fundamental group $\pi_1(S)$ acts on $\widetilde{S}$ as deck transformations and on S^1 as prescribed by ϕ. Hence $\pi_1(S)$ acts diagonally on $\widetilde{S} \times S^1$ and the action is free and properly discontinuous since it is free and properly discontinuous on the first factor. The total space of the bundle is given by

$$E_\phi := \frac{\widetilde{S} \times S^1}{\pi_1(S)} \qquad \text{where } \gamma \cdot (\widetilde{x}, t) := (\gamma\widetilde{x}, \phi(\gamma)t).$$

The space E_ϕ has a natural projection $p : E_\phi \longrightarrow S$ such that $p\,[\widetilde{x}, t] := \pi(\widetilde{x})$.
The fibers are circles:

$$p^{-1}(x) = \frac{\pi^{-1}(x) \times S^1}{\pi_1(S)} \simeq \{\widetilde{x}\} \times S^1$$

choosing $\widetilde{x} \in \pi^{-1}(x)$. Furthermore E_ϕ admits a natural foliation transverse to the fibers:

$$\mathcal{F} := \left\{ \frac{\widetilde{S} \times \{t\}}{\operatorname{Stab}_{\pi_1(S)}(t)} \right\}_{t \in S^1}.$$

Thus E_ϕ is a flat bundle by Lemma 2.3.
Finally, it is not difficult to compute the holonomy. Fix $\gamma \in \pi_1(S, x_0)$. Let $e = [\widetilde{x}, t] \in E_{\phi, x_0}$ be any point in the fiber, the leaf passing through e is $\mathcal{F}_e = \left[\widetilde{S} \times \{t\}\right]$. The unique lift of γ^{-1} to E lying on $\mathcal{F}_e$ is precisely $\left[\widetilde{\gamma}_{\widetilde{x}}^{-1}, t\right]$ where $\widetilde{\gamma}_{\widetilde{x}}$ is the unique lift of γ to $\widetilde{S}$ starting from $\widetilde{x}$. The final point $\widetilde{\gamma}_{\widetilde{x}^{-1}}(1) = \gamma^{-1}\widetilde{x}$, thus the final point of the lift of γ to $\mathcal{F}_e$ is $[\gamma\widetilde{x}, t]$.
To understand the holonomy we have to trivialize E_{ϕ, x_0}, as before this is done by choosing $\widetilde{x}_0 \in \pi^{-1}(x_0)$. Under the identification $E_{\phi, x_0} \simeq \{\widetilde{x}_0\} \times S^1$ we have

$$t \in S^1 \longrightarrow [\widetilde{x}_0, t] \in E_{\phi, x_0} \longrightarrow \left[\gamma^{-1}\widetilde{x}_0, t\right] = [\widetilde{x}_0, \phi(\gamma)t] \in E_{\phi, x_0} \longrightarrow \phi(\gamma)t.$$

Thus $\eta_\gamma = \phi(\gamma)$.

Theorem 2.8. *Let $p : E \longrightarrow S$ be a flat circle bundle, let $\phi : \pi_1(S) \longrightarrow \mathit{Homeo}_+\left(S^1\right)$ be the holonomy representation. Then we have an isomorphism*

$$E \simeq E_\phi$$

as flat circle bundles (the isomorphism preserves fibers and leaves).

If we think of $\widetilde{S}$ as the set of homotopy classes of paths based at a fixed basepoint $x_0 \in S$ (homotopies have fixed endpoints) and trivialize the bundle E near x_0 (thus we identify $E_{x_0} \simeq S^1$ in the trivialization), then the isomorphism $E_\phi \simeq E$ is induced by the map

$$F : \widetilde{S} \times S^1 \longrightarrow E \qquad , \qquad F([\gamma], t) := \widetilde{\gamma}_t(1)$$

where $\widetilde{\gamma}_t$ is the unique lift of γ on the leaf $\mathcal{F}_t$ starting from $\widetilde{\gamma}_t(0) = t \in E_{x_0}$.

3.2. An obstruction class for circle bundles. The main invariant associated to an oriented circle bundle $p : E \longrightarrow S$ is the Euler class or Euler number $e(E) \in H^2(S, \mathbb{Z}) \simeq \mathbb{Z}$ which measures the obstruction of finding a section of the bundle. We describe the procedure to contruct the Euler class in the framework of cellular cohomology.
Let $p : E \longrightarrow S$ be an oriented circle bundle. Suppose that S is endowed with a CW-complex structure. First we observe that it is always possible to contruct a section over the $1-$skeleton $\sigma : S^{(1)} \longrightarrow E$. Let e_α^2 be a $2-$cell with attaching map ψ_α. The pull-back bundle $p_\alpha : \psi_\alpha^* E \longrightarrow D_\alpha^2$ trivializes as the base is contractible. We assign to the $2-$cell e_α^2 the following number

$$c_\sigma\left(e_\alpha^2\right) := \deg\left(\sigma|_{\partial D_\alpha^2} \xrightarrow{\sigma_\alpha} \psi_\alpha^* E = D_\alpha^2 \times S^1 \xrightarrow{\pi_2} S^1 \right)$$

where σ_α is the section of the pull-back bundle induced on the boundary ∂D_α^2 by the fixed section σ. It is an easy computation to prove that $c_\sigma : C_2^{CW}(S, \mathbb{Z}) \longrightarrow \mathbb{Z}$ is a well-defined $2-$cocycle independent of the chosen orientation preserving trivializations $\psi_\alpha^* E = D_\alpha^2 \times S^1$.

Moreover different choices of $\sigma, \sigma' : S^{(1)} \longrightarrow E$ produce cocycles $c_\sigma, c_{\sigma'}$ that differ by a coboundary.

DEFINITION 2.9. The *Euler class* of a circle bundle $p : E \longrightarrow S$ is the cohomology class represented by the $2-$cocycle c_σ just described:

$$e(E) := [c_\sigma] \in H^2(S, \mathbb{Z}).$$

The *Euler number* of E, also denoted by $e(E)$, is the integer corresponding to the Euler class unde the identification $H^2(S, \mathbb{Z}) \simeq \mathbb{Z}$.

As we stated above the Euler class classifies topologically oriented circle bundles.

THEOREM 2.10. *There is a natural bijection between*

$$H^2(S, \mathbb{Z}) \simeq \mathbb{Z} \longleftrightarrow \left\{ \begin{array}{c} \textit{Oriented } S^1-\textit{bundles} \\ p : E \longrightarrow S \end{array} \right\} / \textit{isomorphism.}$$

The correspondence is given by the Euler number.

3.3. Relative Euler class. Every compact surface with non-empty boundary deformation retracts to a one dimensional CW-complex, i.e. a finite graph. Since every circle bundle over a one dimensional CW-complex trivializes (it always admit a section) we have the following

LEMMA 2.11. *Every circle bundle $p : E \longrightarrow S$ over a compact surface with non-empty boundary $\partial S \neq \emptyset$ is trivial $E \simeq S \times S^1$.*

Suppose that S is endowed with a CW-complex structure such that ∂S is a subcomplex. It is possible to define a relative version for the bounded Euler class of a circle bundle $p : E \longrightarrow S$ over a surface with boundary $(S, \partial S)$. In order to avoid trivialities we should at least fix some boundary data, so let us fix a trivialization of E over the boundary given by a section $\sigma : \partial S \longrightarrow E$. As before we first extend σ to a section over the $1-$skeleton $\sigma : S^{(1)} \longrightarrow E$, then we define for every $2-$cell e_α the number $c_\sigma\left(e^2_\alpha\right)$ exactly like the absolute case. It turns out that c_σ is a relative $2-$cocycle in $C^2(S, \partial S; \mathbb{Z})$ and up to coboundaries it is independent of the extension of σ. Thus

DEFINITION 2.12. The *relative Euler class* of a circle bundle $p : E \longrightarrow S$ is the cohomology class represented by the $2-$cocycle c_σ just described:

$$e(E; \sigma) := [c_\sigma] \in H^2(S, \partial S; \mathbb{Z}).$$

The *relative Euler number* of E, also denoted by $e(E; \sigma)$, is the integer corresponding to the Euler class under the identification $H^2(S, \partial S; \mathbb{Z}) \simeq \mathbb{Z}$.

There is a nice additivity property of the Euler number which we now describe. Let S be a closed oriented surface, suppose that we decompose S as the gluing of a family of compact subsurfaces

$$S = \bigcup_{j=1}^{n} S_j$$

along boundary components. Suppose we have chosen coherently trivializations of E along the boundaries $\bigcup_{j=1}^{n} \partial S_j$ through sections σ_j. Then we have

$$e(E) = \sum_{j=1}^{n} e\left(E|_{S_j};\sigma_j\right).$$

4. Milnor's Theorem

This section is the core of the second chapter. We prove Milnor's Formula first in the topological setting and then in the algebraic one.

4.1. Milnor's Theorem: flat circle bundles. For representations of surface groups Milnor gave an explicit and useful way to compute the Euler number. In this section we state and prove this theorem.

THEOREM 2.13 (Milnor's Formula, [**Mil58**]). *Let S be a surface of genus $g \geq 2$, let $\rho : \pi_1(S) \to Homeo_+\left(S^1\right)$ be a representation. Then*

$$\tau_{e(\rho)} = [\widetilde{\rho}(a_1), \widetilde{\rho}(b_1)] \dots [\widetilde{\rho}(a_g), \widetilde{\rho}(b_g)]\,.$$

where $\widetilde{\rho}(a_j), \widetilde{\rho}(b_j)$ are arbitrary lifts of $\rho(a_j), \rho(b_j)$ to $\widetilde{Homeo_+}\left(S^1\right)$.

PROOF. Let $\pi : E_\rho \to S$ be the flat bundle associated to ρ. The total space is defined as the quotient $E_\rho := \frac{\widetilde{S}\times S^1}{\pi_1(S)}$ with respect to the diagonal action of $\pi_1(S)$, while the map $\pi : E_\rho \to S$ is induced by the universal covering projection $p : \widetilde{S} \to S$. By construction we have a natural identification $p^*E_\rho = \widetilde{S} \times S^1$ given by the quotient map $q : \widetilde{S} \times S^1 \to E_\rho$.
Let us describe S as a quotient of a fundamental $4g-$agon $P \subseteq \widetilde{S}$ with respect to the standard identifications on the edges $a_j, b_j, a_j^{-1}, b_j^{-1}$ which represent the standard generators of $\pi_1(S)$. We endow S_g with a CW-complex structure with a single $2-$cell P attached to the $2g$ $1-$cells $\{a_j, b_j\}_{j=1}^{g}$ with attaching map described by the word $[a_1, b_1] \dots [a_g, b_g]$ as in Figure 1.
A fundamental class $[S] \in H_2(S, \mathbb{Z})$ of S is representad by the $2-$cycle P in cellular homology.
We lift the only $0-$cell x_0 to the point $e_0 = [\widetilde{x_0}, [0]] \in E_{\rho,x_0}$ where $\widetilde{x_0}$ is the left vertex of $\widetilde{a_1} \subseteq P$.
Now we describe trivializations of E_ρ over the $1-$singular simplices corresponding to the $1-$cells. For simplicity let us describe the procedure for a_1. A trivialization $a_1^*E_\rho$ over the $1-$singular simplex $a_1 : [0, 1] \to S$, corresponding to the attaching map of the $1-$cell a_1, is given by

$$(t, x) \to [\widetilde{a_1}(t), x]$$

where $\widetilde{a_1}(t)$ is the lift of the attaching map $a_1 = p\widetilde{a_1}$ to the universal cover $\widetilde{S}$ with starting point $\widetilde{x_0}$. With respect to this trivialization the points in the fibers $p^*E_{\rho,0}, p^*E_{\rho,1}$ corresponding to $e_0 \in E_{\rho,x_0}$ are respectively $(0, [0])\,, (1, \rho(a_1)[0])$, so we need to construct a

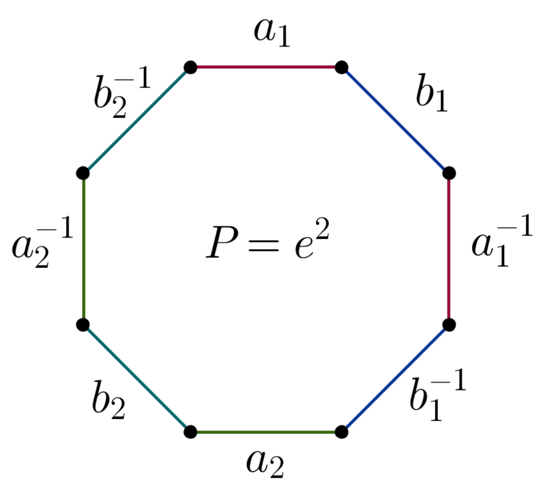

FIGURE 1.

section over $[0,1]$ that connects these points. We choose the path given by

$$A_1 : t \to (t, [t\widetilde{\rho}(a_1)0]) .$$

where $\widetilde{\rho}(a_1)$ is any lift of $\rho(a_1)$.
Consider now b_1. As a trivialization of the bundle over the attaching map $b_1 : [0,1] \to S$ we choose the map $[0,1] \times S^1 \to E_\rho$ described by

$$(t,x) \to \left[a_1\widetilde{b_1}(t), x\right]$$

where, as before, $\widetilde{b_1}$ is the lift of b_1 to the universal cover with starting point $\widetilde{x_0}$, while $a_1 \circ \widetilde{b_1}$ is the lift starting from $\widetilde{a_1}(1) = a_1\widetilde{x_0}$. We choose this lift in order to match the one defined for a_1. With respect to this trivialization e_0 corresponds over $0, 1 \in [0,1]$ respectively to $(0, \phi(a_1)[0]), (1, \phi(b_1)\phi(a_1)[0])$. Finally we construct the section

$$B_1 : t \to (t, [(1-t)\widetilde{\rho}(a_1)0 + t\widetilde{\rho}(a_1)\widetilde{\rho}(b_1)0]) .$$

Define analogously the sections $A_j, B_j, \alpha_j, \beta_j$ over the other $1-$cells $a_j, b_j, a_j^{-1}, b_j^{-1}$ so that they satisfy the matching property

$$\begin{aligned}
&A_j(t) := (t, [(1-t)\widetilde{\rho}(a_1)\dots\widetilde{\rho}(b_{j-1})0 + t\widetilde{\rho}(a_1)\dots\widetilde{\rho}(b_{j-1})\widetilde{\rho}(a_j)0])\\
&B_j(t) := (t, [(1-t)\widetilde{\rho}(a_1)\dots\widetilde{\rho}(a_j)0 + t\widetilde{\rho}(a_1)\dots\widetilde{\rho}(a_j)\widetilde{\rho}(b_j)0])\\
&\alpha_j(t) := \left(t, \left[(1-t)\widetilde{\rho}(a_1)\dots\widetilde{\rho}(b_j)0 + t\widetilde{\rho}(a_1)\dots\widetilde{\rho}(b_j)\widetilde{\rho}(a_j^{-1})0\right]\right)\\
&\beta_j(t) := \left(t, \left[(1-t)\widetilde{\rho}(a_1)\dots\widetilde{\rho}(b_j^{-1})0 + t\widetilde{\rho}(a_1)\dots\widetilde{\rho}(a_j^{-1})\widetilde{\rho}(b_j^{-1})0\right]\right).
\end{aligned}$$

Consider now the attaching map of the 2−cell $p : P \to S$. A trivialization of the bundle p^*E_ρ is induced by the quotient map

$$q : P \times S^1 \to E_\rho.$$

With respect to this trivialization the sections constructed above patch together and give rise to the map $s : \partial P \to \widetilde{P} \times S^1 \to S^1$. The boundary is naturally parametrized by $[0, 4g]$ patching together the lifts $\widetilde{a_1}, a_1\widetilde{b_1}, \ldots, \prod_{j=1}^{g} [a_j, b_j]\, b_g\widetilde{b_g^{-1}}$, and the section s is piecewise linear with respect to this parametrization.
Finally we can easily compute the degree of the section s over the boundary:

$$\deg(s) = \left(\prod_{j=1}^{g} [\widetilde{\rho}(a_j), \widetilde{\rho}(b_j)] \right)(0) = \tau_{e(\rho)}(0) = e(\rho).$$

□

4.2. Milnor's Theorem: relative case. The same construction may be refined to include boundary data. Let $S = S_{g,b}$ be a compact surface with genus g and $b \geq 1$ boundary components $\partial S = c_1 \cup \cdots \cup c_b$. Denote by $c_1, \ldots, c_b \in \pi_1(S)$ a collection of loops representing the boundary curves in a standard presentation of $\pi_1(S)$. Let $\sigma : G \longrightarrow \widetilde{G}$ be an arbitrary section.

Definition 2.14. The *relative Euler number* of a representation $\phi : \Gamma = \pi_1(S) \longrightarrow \mathrm{Homeo}_+\left(S^1\right)$ with respect to the section $\sigma : G \longrightarrow \widetilde{G}$ is the integer $e\left(\phi, \sigma\right)$ given by the following formula

$$\tau_{e(\rho,\sigma)} = \left[\widetilde{\phi}(a_1), \widetilde{\phi}(b_1)\right] \ldots \left[\widetilde{\phi}(a_g), \widetilde{\phi}(b_g)\right] \sigma(\rho(c_1)) \ldots \sigma(\rho(c_b))$$

where $\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)$ are arbitrary lifts of $\phi(a_j), \phi(b_j)$.
We state without proof for representations of the addition formula

Theorem 2.15 (Addition Formula). *Let $S = \bigcup_{j=1}^{n} S_j$ be as above a decomposition of S into compact subsurfaces with boundary. The graph of the gluing gives identifications $\pi_1(S_j) \subseteq \pi_1(S_g)$. Let $\rho : \pi_1(S) \longrightarrow Homeo_+\left(S^1\right)$ be a representation such that every boundary curve is mapped to a homeomorphism with a fixed point. If $\sigma : G \longrightarrow \widetilde{G}$ is a special section (i.e. $0 \leq \widetilde{rot}\, \sigma(f) < 1$ for every $f \in G$), then*

$$e\left(\rho\right) = \sum_{j=1}^{n} e\left(\rho|_{\pi_1(S_j)}, \sigma\right).$$

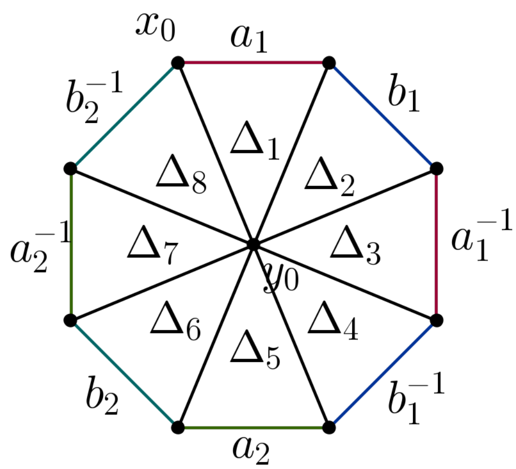

FIGURE 2.

4.3. Milnor's Theorem: cohomological approach. We describe how to compute the Euler number of a representation within a cohomological framework.
Consider a central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow E \longrightarrow \Gamma \longrightarrow 1.$$

Let $\sigma : \Gamma \longrightarrow E$ be any section, denote by c_σ the inhomogeneous $2-$cocycle associated to the central extension.

THEOREM 2.16 (Milnor). *Let $[c_\sigma] \in H^2(\pi_1(S), \mathbb{Z})$ be as above. Then under the identifications*

$$H^2(\pi_1(S), \mathbb{Z}) \simeq H^2(S, \mathbb{Z}) \simeq \mathbb{Z}$$

we have

$$[c_\sigma] = \prod_{j=1}^{g} [\sigma(a_j), \sigma(b_j)] \in \mathbb{Z}.$$

PROOF. Let us describe S as a quotient of a $4g-$gon $P \subseteq \widetilde{S} = \mathbb{R}^2$ with respect to standard identifications on the edges. We endow S with a $\Delta-$complex structure as in Figure 2 with triangles $\Delta_1, \ldots, \Delta_{4g}$.
As set of representatives for the action of $\pi_1(S)$ on $\widetilde{S}$ we choose

$$F := \mathrm{int}(P) \cup \left(\bigcup_{j=1}^{g} \mathrm{int}(a_j) \cup \mathrm{int}(b_j) \right) \cup \{x_0\}$$

where x_0 is the first end of a_1. Actually we will only need x_0 and the interior point $y_0 \in P$ (inner vertex of the Δ_j's). A fundamental class for S in $H_2^{\text{sing}}(S,\mathbb{Z})$ is given by

$$[S] = \left[\sum_{j=1}^{4g} s_j\right] \in H_2(S,\mathbb{Z})$$

where s_j is the attaching map of the simplex Δ_j.
We want to compute

$$\langle [c_\sigma], [S]\rangle = \sum_{j=1}^{4g} r_F(c_\sigma)(s_j).$$

By definition of r_F (inhomogeneous case) we have

$$r_F(c_\sigma)(s_1) = c_\sigma\left(g_{s_1(e_0)}^{-1} g_{s_1(e_1)}, g_{s_1(e_1)}^{-1} g_{s_1(e_2)}\right).$$

Looking at Figure 2 we see that for triangles $\Delta_{4j-3}, \Delta_{4j-2}, \Delta_{4j-1}, \Delta_{4j}$ with perimetral edges $a_j, b_j, a_j^{-1}, b_j^{-1}$ we have

	$g_{s(e_0)}$	$g_{s(e_1)}$	$g_{s(e_2)}$
Δ_{4j-3}	1	w_j	$w_j a_j$
Δ_{4j-2}	1	$w_j a_j$	$w_j a_j b_j$
Δ_{4j-1}	1	$w_j a_j b_j$	$w_j a_j b_j a_j^{-1}$
Δ_{4j}	1	$w_j a_j b_j a_j^{-1}$	w_{j+1}

where $w_j \in \pi_1(S)$ is the word $w_j := \prod_{i \le j-1} [a_i, b_i]$ with the convention $w_1 := 1$ (in the table we dropped the index from $s(e_0), s(e_1), s(e_2)$ to simplify the notations).
Thus

$$\begin{aligned}
&r_F(c_\sigma)(\Delta_{4j-3} + \Delta_{4j-2} + \Delta_{4j-1} + \Delta_{4j})\\
&= c_\sigma(w_j, a_j) + c_\sigma(w_j a_j, b_j) + c_\sigma\left(w_j a_j b_j, a_j^{-1}\right) + c_\sigma\left(w_j a_j b_j a_j^{-1}, b_j^{-1}\right)\\
&= \left(\sigma(w_j)\sigma(a_j)\sigma(w_j a_j)^{-1}\right) \cdot \left(\sigma(w_j a_j)\sigma(b_j)\sigma(w_j a_j b_j)^{-1}\right) \cdot \left(\sigma(w_j a_j b_j)\sigma(a_j^{-1})\sigma(w_j a_j b_j a_j^{-1})^{-1}\right)\\
&\qquad \cdot \left(\sigma(w_j a_j b_j a_j^{-1})\sigma(b_j^{-1})\sigma(w_{j+1})^{-1}\right)\\
&= \sigma(w_j)\sigma(a_j)\sigma(b_j)\sigma(a_j^{-1})\sigma(b_j^{-1})\sigma(w_{j+1})^{-1}.
\end{aligned}$$

Thus evaluating $\langle r_F(c_\sigma), [S]\rangle$ we get something like a telescopic sum

$$\begin{aligned}
\langle r_F(c_\sigma), [S]\rangle &= \sum_{j=1}^{g} r_F(c_\sigma)(\Delta_{4j-3} + \Delta_{4j-2} + \Delta_{4j-1} + \Delta_{4j})\\
&= \prod_{j=1}^{g} \sigma(w_j)\sigma(a_j)\sigma(b_j)\sigma(a_j^{-1})\sigma(b_j^{-1})\sigma(w_{j+1})^{-1}\\
&= \prod_{j=1}^{g} \sigma(a_j)\sigma(b_j)\sigma(a_j^{-1})\sigma(b_j^{-1}).
\end{aligned}$$

$\square$

5. Correspondence between Euler classes of extensions and bundles

In this section we describe the correspondence between circle bundles, central extensions and Euler numbers. Every circle bundle

$$S^1 \xrightarrow{\ i\ } E \xrightarrow{\ p\ } S$$

gives rise to a fibration sequence (after fixing base points $x_0 \in S$ and $e_0 \in E_{x_0}$)

$$0 = \pi_2(S, x_0) \longrightarrow \pi_1(S^1, e_0) = \mathbb{Z} \longrightarrow \pi_1(E, e_0) \longrightarrow \pi_1(S, x_0) \longrightarrow 0.$$

where $\pi_2(S, x_0) = 0$ since the universal covering $\widetilde{S}$ is contractible.
This exact sequence of groups is also central.

LEMMA 2.17. *Let* $S^1 \xrightarrow{\ i\ } E \xrightarrow{\ p\ } S$ *be as above. Then the extension of* $\pi_1(S)$

$$0 \longrightarrow \pi_1(S^1, e_0) \longrightarrow \pi_1(E, e_0) \longrightarrow \pi_1(S, x_0) \longrightarrow 0$$

is central.

PROOF. Let $\sigma : \pi_1(S) \longrightarrow \pi_1(E)$ be any section.
We need to show that the action of $\pi_1(S)$ on $\pi_1(S^1)$ is trivial. First let us recall the definition of the action of $\pi_1(S)$ on $i\left(\pi_1(S^1)\right)$: for every $\gamma \in \pi_1(S)$ and $\alpha \in \pi_1(S^1)$ set

$$i\,(\gamma \cdot \alpha) := \sigma(\gamma) i(\alpha) \sigma(\gamma)^{-1}.$$

Fix $\gamma \in \pi_1(S)$ and represent it as a loop $\gamma : \left(S^1, t_0\right) \longrightarrow (S, x_0)$. Represent a section $\sigma(\gamma)$ as a lifted loop $\sigma(\gamma) : \left(S^1, t_0\right) \longrightarrow (E, e_0)$; also represent α as a loop $\alpha : \left(S^1, t_0\right) \longrightarrow (E_{x_0}, e_0)$, so that α projects to the identity of $\pi_1(S, x_0)$.
We observe that the loops $\sigma(\gamma), \alpha$ correspond to sections $\overline{\sigma(\gamma)}, \overline{\alpha}$ of the pull-back

$$\begin{tikzcd}
(\gamma^* E, \overline{e_0}) \arrow[r, "g"] \arrow[d, "\pi"'] & (E, e_0) \arrow[d, "p"] \\
\left(S^1, t_0\right) \arrow[r, "\gamma"'] & (S, x_0)
\end{tikzcd}$$

moreover, if we call g the bundle map covering γ, we see that

$$\overline{\sigma(\gamma)}, \overline{\alpha} \in \pi_1\left(\gamma^* E, \overline{e_0}\right) \xrightarrow{\ g_*\ } \sigma(\gamma), \alpha \in \pi_1\left(E, e_0\right).$$

The total space of $\gamma^* E$ bundle is topologically a torus, in particular it has abelian fundamental group $\pi_1\left(\gamma^* E, \overline{e_0}\right) = \mathbb{Z}^2$, hence we conclude

$$\sigma(\gamma) \alpha \sigma(\gamma)^{-1} = g_*\left(\overline{\sigma(\gamma)}\,\overline{\alpha}\,\overline{\sigma(\gamma)}^{-1}\right) = g_*\left(\overline{\alpha}\right) = \alpha.$$

$\square$

Lemma 2.18. *The Euler number of the central extension $\pi_1(E, e_0)$ coincides with the Euler number of the circle bundle E.*

Proof. To see this fact let us endow S with a standard CW-structure with a single $0-$cell $x_0 \in S$, $2g$ $1-$cells $a_1, b_1, \ldots, a_g, b_g$ and a single $2-$cell e with boundary attaching map given by $[a_1, b_1] \ldots [a_g, b_g]$. Let $\sigma : \pi_1(S) \longrightarrow \pi_1(E)$ be any section, denote by c_σ the associated $2-$cocycle.
Let us compute the Euler number of E. By definition we first have to choose a lift of x_0 to E, so let us fix $e_0 \in E_{x_0}$ as a base point. Then we should fix sections over the $1-$cells, but this is exactly the same as choosing sections of the generators $a_j, b_j \in \pi_1(S, x_0)$ to $\pi_1(E, e_0)$ with respect to the projection $p_* : \pi_1(E, e_0) \longrightarrow \pi_1(S, x_0)$. Thus we can choose $\sigma(a_j), \sigma(b_j)$. Finally the Euler number is given by the degree of the composition of the attaching map of e $\partial D^2 \longrightarrow S^{(1)}$ with the section over the $1-$skeleton, which we also denote by $\sigma : S^{(1)} \longrightarrow E$, and the projection to the fiber. This is precisely the number

$$\prod_{j=1}^{g} [\sigma(a_j), \sigma(b_j)] \in \pi_1(S, e_0).$$

Now the result follows from the computation given by Milnor's Theorem 2.16. $\square$

Thus we have the following commutative diagram of equivalences

$$\begin{tikzcd}
H^2(\Gamma, \mathbb{Z}) \arrow[r, leftrightarrow] \arrow[d, "r"] & \text{Central extensions} \\
H^2(S, \mathbb{Z}) \arrow[r, leftrightarrow] & \text{Oriented } S^1\text{−bundles.} \arrow[u]
\end{tikzcd}$$

6. Some constructions of surface groups actions on S^1

In this final section we present some natural ways for producing surface group actions on S^1 from topology and geometry. We also give some explicit examples.

6.1. Hyperbolic structures. The most important example for us comes from hyperbolic geometry. Let S be a closed oriented surface of genus $g \geq 2$. Every hyperbolic metric on S gives rise to a $\left(\mathbb{H}^2, \mathrm{Isom}_+\left(\mathbb{H}^2\right)\right)$ complete structure over S (in the sense of Thurston's $(G, X)-$structures). In particular we get a developing map $\delta : \widetilde{S} \longrightarrow \mathbb{H}^2$ and the corresponding holonomy representation $\phi : \pi_1(S) \longrightarrow \mathrm{Isom}_+\left(\mathbb{H}^2\right) = \mathrm{PSL}\left(2, \mathbb{R}\right) < \mathrm{Diff}_+\left(\partial\mathbb{H}^2\right)$.

6.2. Foliated circle bundles. Another way to produce representations of surface groups into $\mathrm{Homeo}_+\left(S^1\right)$ from topology is provided by circle bundle as we explained before. In general it is not so easy to produce a transverse foliation on a circle bundle and moreover there is no reason for the foliation to be unique (up to a suitable notion of equivalence of flat foliated circle bundles). However the following result by Wood [**Woo71**] which generalizes a theorem by Milnor [**Mil58**] gives a necessary and sufficient condition for a circle bundle to admit a flat structure:

THEOREM 2.19 (Wood, [**Woo71**]). *Let S be a closed oriented surface of genus $g \geq 2$. Let $p : E \longrightarrow S$ be an oriented circle bundle with group structure Homeo$_+$ $\left(S^1\right)$. The following are equivalent:*

(1) *The Euler number satisfies $|e(E)| \leq -\chi(S) = 2g - 2$.*
(2) *The circle bundle admits a flat structure.*

We will prove later the implication "flat $\Longrightarrow |e(E)| \leq \chi(S)$". For a detailed proof of the converse statement we refer to Wood's article [**Woo71**], later we present a proof based upon a theorem by Eisenbud-Hirsch-Neumann (see Theorem 2.22). Constructing a circle bundle with a given Euler number is rather easy:

EXAMPLE 2.20. Let S be a closed oriented surface. Let $\Delta \subseteq S$ be an embedded disk, denote by $\Sigma := S \setminus \mathrm{int}\,(\Delta)$ the complement of Δ. Consider the trivial bundles $\Sigma \times S^1$ and $\Delta \times S^1$. We want to glue them along the boundary tori in a non-trivial way with a map $\phi : \partial\Sigma \times S^1 \longrightarrow \partial\Delta \times S^1$ which is an isomorphism of bundles over $\partial\Sigma = \partial\Delta$. Since we glued the bundles along an isomorphism of bundles the resulting object $E = \Sigma \times S^1 \cup_\phi \Delta \times S^1$ has again a natural circle bundle structure over the gluing of the bases $S = \Sigma \cup \Delta$.
The map ϕ has necessarily the following form $\phi\,(x, \theta) = (x, f(x, \theta))$. Fix $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$. The canonical section $\sigma : \Sigma \longrightarrow \Sigma \times S^1$ defined by $\sigma(x) = (x, \theta)$ gives a section of the bundle $\Delta \times S^1|_{\partial\Delta}$ via the identification ϕ. This section is defined by $\phi^{-1}\sigma$. Thus the Euler number of E is computed by

$$\begin{aligned}
e\,(E) &= e\,(E|_S, \sigma) + e\,(E|_\Delta, \sigma) = e\,(E|_\Delta, \sigma) \\
&= \deg\left(\partial\Delta \xrightarrow{\ \sigma\ } \Delta \times S^1 \xrightarrow{\ \phi^{-1}\ } \partial\Sigma \times S^1 \longrightarrow S^1\right) \\
&= \deg\left(f\,(\bullet, \theta) : \partial\Delta \longrightarrow S^1\right).
\end{aligned}$$

It is clear that we can realize every integral value $n \in \mathbb{Z}$ with this procedure. It is enough to consider the family of maps $f_n\,(x, \theta) = [nx + \theta] \in S^1 = \mathbb{R}/\mathbb{Z}$.

REMARK 2.21. Every circle bundle $p : E \longrightarrow S$ can be constructed with the procedure of Example 2.20: in fact $E = E_1 \cup E_2 = E|_\Delta \cup_\phi E|_{\Sigma = S \setminus \mathrm{int}(\Delta)}$ for an embedded disk $\Delta \subseteq S$ where E_1, E_2 are glued along a bundle isomorphism of $\phi : \partial E_1 \longrightarrow \partial E_2$ (we see ∂E_i as a bundle over $\partial\Delta$). The bundles E_1, E_2 are trivial as Δ is contractilble and Σ deforms to a one dimensional complex. Thus the argument used in the previous example can be adapted to give a proof of the classification Theorem 2.10.

6.3. A Theorem by Eisenbud-Hirsch-Neumann. The next theorem by Eisenbud-Hirsch-Neumann [**EHN81**] provides another abstract way to produce surface group representations into Homeo$_+$ $\left(S^1\right)$:

THEOREM 2.22 (Eisenbud-Hirsch-Neumann [**EHN81**]). *Let $f \in \widetilde{G}$ be a homeomorphism. The following are equivalent:*

(1) *f can be written as a product of N commutators.*

(2) the following holds

$$\underline{m}(f) := \inf_{x\in\mathbb{R}} \{f(x) - x\} < 2N - 1 \qquad \textit{and} \qquad \overline{m}(f) := \sup_{x\in\mathbb{R}} \{f(x) - x\} > 1 - 2N.$$

REMARK 2.23. Observe that Theorem 2.22 is stronger than uniform perfectness of $\mathrm{Homeo}_+\left(S^1\right)$ (Theorem 1.32). In fact for every $f \in \mathrm{Homeo}_+\left(S^1\right)$ there exists a lift $\widetilde{f} \in \widetilde{\mathrm{Homeo}_+}\left(S^1\right)$ such that $\widetilde{f}(0) \in [0,1)$. In particular $\underline{m}\left(\widetilde{f}\right) \leq \widetilde{f}(0) - 0 < 1$ and $\overline{m}\left(\widetilde{f}\right) \geq \widetilde{f}(0) - 0 > -1$. Hence, by Theorem 2.22, we get $\widetilde{f} = \left[\widetilde{a}, \widetilde{b}\right]$ for some $\widetilde{a}, \widetilde{b} \in \widetilde{\mathrm{Homeo}_+}\left(S^1\right)$. In conclusion $f = [a,b]$ where a, b are respectively the projections of $\widetilde{a}, \widetilde{b}$ to $\mathrm{Homeo}_+\left(S^1\right)$.

An easy computation shows the following:

LEMMA 2.24. *Let $f \in \widetilde{G}$ be a homeomorphism. Then there exists a point $t \in \mathbb{R}$ such that $f(t) = t + \widetilde{rot}(f)$. In particular*

$$\underline{m}(f) \leq \widetilde{rot}(f) \quad \textit{and} \quad \overline{m}(f) \geq \widetilde{rot}(f).$$

We use the theorem to make some examples:

EXAMPLE 2.25. For every integer $e \in \mathbb{Z}$ such that $|e| \leq 2g - 2$ there exists a representation $\phi : \pi_1(S) \longrightarrow \mathrm{Homeo}_+\left(S^1\right)$ with Euler number $e(\phi) = e$. In fact the integral translation τ_e satisfies $\overline{m}\left(\tau_e\right) = \underline{m}\left(\tau_e\right) = e$, and hence, by Theorem 2.22, it can be written as a product of g commutators $\tau_e = \left[\widetilde{a}_1, \widetilde{b}_1\right] \ldots \left[\widetilde{a}_g, \widetilde{b}_g\right]$ where $\widetilde{a}_j, \widetilde{b}_j \in \widetilde{\mathrm{Homeo}_+}\left(S^1\right)$.

REMARK 2.26. The previous example together with Theorem 2.10 gives a proof of Theorem 2.19. Suppose we are given an oriented circle bundle $p : E \longrightarrow S$ of Euoler number $|e := e(E)| \leq 2g - 2$. By Example 2.25 we can find a representation $\phi : \pi_1(S) \longrightarrow \mathrm{Homeo}_+\left(S^1\right)$ with Euler number $e(\phi) = e$. By Milnor's Formula 2.13 also the associated flat bundle $q : E_\phi \longrightarrow S$ has Euler number equal to $e\left(E_\phi\right) = e$. Thus by Theorem 2.10 we can find an isomorphism of oriented circle bundles $E_\phi \simeq E$. This transport the foliation $\mathcal{F}$ transverse to the fibers of E_ϕ to a foliation with the same property for E.

EXAMPLE 2.27. Consider $2g - 2$ homeomorphisms $a_1, b_1, \ldots, a_{g-1}, b_{g-1} \in \widetilde{\mathrm{Homeo}_+}\left(S^1\right)$. Form the product $f := [a_1, b_1] \ldots [a_{g-1}, b_{g-1}] \in \widetilde{\mathrm{Homeo}_+}\left(S^1\right)$ and compute $\widetilde{\mathrm{rot}}(f) = e - 1 + \alpha$ where $e - 1 := \lfloor \widetilde{\mathrm{rot}}(f) \rfloor \in \mathbb{Z}$ and $\alpha := \left\{\widetilde{\mathrm{rot}}(f)\right\} \in [0,1)$ are respectively the integral and fractional part of $\widetilde{\mathrm{rot}}(f)$. Suppose that $\alpha > 0$ and consider the homeomorphism $\tau_e f^{-1}$. We have $\widetilde{\mathrm{rot}}\left(\tau_e f^{-1}\right) = \widetilde{\mathrm{rot}}\left(\tau_e\right) - \widetilde{\mathrm{rot}}(f) = 1 - \alpha \in (0,1)$, in particular $\overline{m}(\tau_e f^{-1}) \geq 1 - \alpha > -1$ and $\underline{m}(\tau_e f^{-1}) \leq 1 - \alpha < 1$. Therefore by Theorem 2.22 we can write $\tau_e f^{-1} = [a_g, b_g]$. This gives a representation $\phi : \pi_1(S) \longrightarrow \mathrm{Homeo}_+\left(S^1\right)$ where the rotation numbers $\mathrm{rot}(a_j)$ and $\mathrm{rot}(b_j)$ are prescribed for every $j \leq g - 1$. Actually we observe that the same result can be obtained using only the uniform perfectness property of $\mathrm{Homeo}_+\left(S^1\right)$.

EXAMPLE 2.28. The same procedure used in Example 2.27 can be used to produce a representation $\phi : \pi_1(S) \longrightarrow \mathrm{Homeo}_+\left(S^1\right)$ where the quantities $\widetilde{\mathrm{rot}}\left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)\right]$ attain presecribed values for every $j \leq g - 1$. It is enough to observe that for every $\alpha \in [-1,1]$

there is a pair $a, b \in \widetilde{\mathrm{Homeo}_+}\left(S^1\right)$ that realizes $\widetilde{\mathrm{rot}}\,[a,b] = \alpha$. This follows again from Theorem 2.22 and Lemma 2.24.

CHAPTER 3

Milnor-Wood and maximality

1. Overview

This chapter is entirely devoted to the relations between maximality and rigidity.
A fundamental theorem by Milnor and Wood gives bounds on the possible values of the Euler number of a representation

THEOREM (Milnor-Wood inequality). *Let S be a closed oriented surface of genus $g \geq 2$. Let $\rho : \pi_1(S) \to Homeo_+ \left(S^1\right)$ be a representation. Then*

$$|e\left(\rho\right)| \leq 2g - 2.$$

The inequality is sharp and the upper bound is realized by the holonomies of hyperbolic structures over S. A representation $\phi : \Gamma := \pi_1(S) \longrightarrow \mathrm{Homeo}_+ \left(S^1\right)$ is called maximal if its Euler number attains the maximal value allowed by the Milnor-Wood inequality. A classical theorem by Goldman states that maximal representations $\phi : \Gamma \longrightarrow \mathrm{PSL}\left(2, \mathbb{R}\right)$ are faithful and have discrete image. This is a far reaching result and has given impulse to the field of Higher Teichmüller Theory. Goldman's Theorem has many proofs involving very different tecniques, we will describe an elementary one due to Matsumoto.
Maximality forces some rigidity phenomena on the dynamics of the action, in particular on some lifts of commutators and on rotation numbers of generators. It turns out that the knowledge of maximality of a representation is sufficient to determine its bounded Euler class, thus we will prove the following rigidity theorem:

THEOREM (Matsumoto). *All maximal $PSL\left(2, \mathbb{R}\right)$ $-$representations are semi-conjugate.*

We also describe an important trick exploiting the fact that maximality is preserved when passing to finite index subgroups: using a theorem by Scott we will be able to reduce the proof of an assertion on every element $\gamma \in \Gamma$ to the particular case where $\gamma = a_1$ is a standard generator.

2. Milnor-Wood inequality

Using the results on the translation numbers of commutators given in part *(2)* of Lemma 1.31 we can give a simple proof of the following fundamental inequality:

THEOREM 3.1 (Milnor-Wood inequality). *Let S be a closed oriented surface of genus $g \geq 2$. Let $\rho : \pi_1(S) \to Homeo_+ \left(S^1\right)$ be a representation. Then*

$$|e\left(\rho\right)| \leq 2g - 2.$$

Proof. We have the following chain of inequalities

$$
\begin{aligned}
|e(\rho)| &= \left|\widetilde{\mathrm{rot}}\left(\prod_{j=1}^{g}[\widetilde{\rho}(a_j),\widetilde{\rho}(b_j)]\right)\right| \\
&= \left|\widetilde{\mathrm{rot}}\left(\prod_{j=1}^{g-1}[\widetilde{\rho}(a_j),\widetilde{\rho}(b_j)]\right)+\widetilde{\mathrm{rot}}\,[\widetilde{\rho}(a_g),\widetilde{\rho}(b_g)]\right| && \text{in fact } \prod_{j=1}^{g-1}[\widetilde{\rho}(a_j),\widetilde{\rho}(b_j)],[\widetilde{\rho}(a_g),\widetilde{\rho}(b_g)] \text{ commute} \\
&\leq \sum_{j=1}^{g}\left|\widetilde{\mathrm{rot}}\,[\widetilde{\rho}(a_j),\widetilde{\rho}(b_j)]\right|+g-2 && \widetilde{\mathrm{rot}} \text{ quasi-morphism, defect } D\left(\widetilde{\mathrm{rot}}\right)=1 \\
&\leq 2g-2 && \text{by Lemma 1.31 } \left|\widetilde{\mathrm{rot}}\,[\widetilde{\rho}(a_j),\widetilde{\rho}(b_j)]\right|\leq 1.
\end{aligned}
$$

We observe that $\chi(S)=2-2g$. □

Motivated by this result we give the following definition:

Definition 3.2. A representation $\rho:\pi_1(S)\to \mathrm{Homeo}_+\left(S^1\right)$ is *maximal* if $e(\rho)=|\chi(S)|$.

We state a relative version of the Milnor-Wood inequality:

Theorem 3.3 (Relative Milnor-Wood inequality). *Let S be a compact oriented surface of genus $g\geq 0$ with $b\geq 1$ boundary components $c_1,\dots,c_b$. Let $\rho:\pi_1(S)\to Homeo_+\left(S^1\right)$ be a representation such that $\rho(c_i)$ has a fixed point for every $i\leq b$. If σ is a special section (i.e. $0\leq \widetilde{rot}\,\sigma(f)<1$ for every $f\in Homeo_+\left(S^1\right)$) then*

$$|e(\rho,\sigma)|\leq 2g+b-2.$$

Proof. The computation is completely analogous to the previous one

$$
\begin{aligned}
|e(\rho),\sigma| &= \left|\widetilde{\mathrm{rot}}\left(\prod_{j=1}^{g}[\widetilde{\rho}(a_j),\widetilde{\rho}(b_j)]\sigma(\rho(c_1))\dots\sigma(\rho(c_b))\right)\right| \\
&= \left|\widetilde{\mathrm{rot}}\left(\prod_{j=1}^{g}[\widetilde{\rho}(a_j),\widetilde{\rho}(b_j)]\cdot\prod_{i=1}^{b-1}\sigma(\rho(c_i))\right)+\widetilde{\mathrm{rot}}\,\sigma(\rho(c_b))\right| \\
&\leq \sum_{j=1}^{g}\left|\widetilde{\mathrm{rot}}\,[\widetilde{\rho}(a_j),\widetilde{\rho}(b_j)]\right|+\sum_{i=1}^{b}\widetilde{\mathrm{rot}}\,\sigma(\rho(c_i))+g+b-2 \\
&\leq 2g+b-2.
\end{aligned}
$$

In the last step we used that $\sigma(\rho(c_i))$ has a fixed point for every $i\leq b$ as σ is a special section. We observe that $\chi(S)=2-2g-b$. □

2.1. Holonomies of hyperbolic structures are maximal. The prominent example of a maximal action comes from hyperbolic geometry. Let S be a hyperbolic surface. We can identify S with the surface $S=\mathbb{H}^2/\Gamma$ through the isometry induced by a developing map $\delta:\widetilde{S}\longrightarrow\mathbb{H}^2$. The subgroup $\Gamma<\mathrm{PSL}(2,\mathbb{R})$ is discrete and acts freely on $\mathbb{H}^2$. We can identify Γ with $\phi(\pi_1(S))$ where $\phi : \pi_1(S) \longrightarrow \mathrm{PSL}(2,\mathbb{R}) = \mathrm{Isom}_+\left(\mathbb{H}^2\right)$ is the holonomy representation corresponding to the developing map δ.

Proposition 3.4. *Let S be a closed oriented surface of genus $g \geq 2$. Let $\phi : \pi_1(S) \longrightarrow PSL(2,\mathbb{R})$ be the holonomy of a hyperbolic structure. Then*

$$e(\phi) = \chi(S) = 2 - 2g.$$

Proof. We compute the Euler number of the associated flat bundle $p : E_\phi \longrightarrow S$. Let us identify $\widetilde{S}$ with $\mathbb{H}^2$ and S^1 with $\partial\mathbb{H}^2$. Thus

$$E_\phi = \mathbb{H}^2 \times \partial\mathbb{H}^2/\pi_1(S)$$

where $\gamma \in \pi_1(S)$ acts on $\mathbb{H}^2$ as the isometry $\phi(\gamma)$ and on $\partial\mathbb{H}^2$ as the extension of $\phi(\gamma)$ to the boundary. Consider the unit tangent bundle $T_1\mathbb{H}^2$. We have a natural action of $\pi_1(S)$ on $T_1\mathbb{H}^2$ induced by taking the differential of the action $\pi_1(S) \curvearrowright \mathbb{H}^2$, i.e. the element γ acts as $\gamma \cdot v := d\phi(\gamma)_{\pi(v)}v$ where $\pi : T_1\mathbb{H}^2 \longrightarrow \mathbb{H}^2$ is the tangent bundle projection. Observe that we have a natural identification $dp : T_1\mathbb{H}^2/\pi_1(S) \longrightarrow T_1S$ induced by the universal covering projection $p : \mathbb{H}^2 \longrightarrow S$. If we think of $\partial\mathbb{H}^2$ as the set of classes of arc-lenght parametrized geodesic rays up to asymptotic equivalence, then we have a natural map

$$F : T_1\mathbb{H}^2 \longrightarrow \mathbb{H}^2 \times \partial\mathbb{H}^2$$

defined by $F(v) := \left(\pi(v), \left[\exp_{\pi(v)}(tv)\right]\right)$. It is easy to check that F is indeed a well-defined isomorphism of bundles. Moreover F is $\pi_1(S)-$ equivariant since an easy computation shows that $\pi\left(d\phi(\gamma)_{\pi(v)}v\right) = \phi(\gamma)\pi(v)$ and using the fact that $\phi(\gamma)$ is an isometry $\exp_{\phi(\gamma)\pi(v)}\left(d\phi(\gamma)_{\pi(v)}(tv)\right) = \phi(\gamma)\exp_{\pi(v)}(tv)$ (as we described it, the element γ acts on $\partial\mathbb{H}^2$ by $\gamma \cdot [a(t)] := [\phi(\gamma)a(t)]$). By equivariance F induces an isomorphism of circle bundles

$$T_1S = T_1\mathbb{H}^2/\pi_1(S) \xrightarrow{\sim} E_\phi.$$

Now the result follows from the standard computation of $e(T_1S)$ carried out via Poincaré-Hopf Theorem. $\square$

A classical theorem by Goldman ensures that maximality is the only obstruction for being the holonomy of some hyperbolic structure on S.

Theorem (Goldman,[**GUoC80**]). *Let $\phi : \Gamma \longrightarrow PSL(2,\mathbb{R})$ be a representation. Then*

$$\phi \text{ is faithful with discrete image } \iff |e(\phi)| = 2g - 2.$$

We will prove this theorem later.

3. Maximality and coverings

We study now some operations that preserve maximality.
The first one regards finite coverings:

Lemma 3.5. *Let $\rho : \pi_1(S) \to Homeo_+\left(S^1\right)$ be a maximal representation with $e(\rho) = |\chi(S)|$. Let $p : S' \to S$ be a finite covering. Then the composition $\rho' = \rho p_* : \pi_1(S') \to Homeo_+\left(S^1\right)$ is maximal, i.e. $e(\rho') = |\chi(S')|$, where $p_* : \pi_1(S') \to \pi_1(S)$ is the inclusion of the subgroup $\pi_1(S')$ corresponding to the covering.*

Proof. The proof is a straightforward formal computation. We have

$$\begin{aligned} e(\rho') &= \langle (p_*)^* \rho^* e, [S'] \rangle \\ &= \langle \rho^* e, p_*[S'] \rangle \\ &= \langle \rho^* e, d[S] \rangle = -d\chi(S) \\ &= -\chi(S') \end{aligned}$$

so ρ' is maximal. $\square$

More generally maximality passes to subgroups with some care about boundary conditions. Let $F < \pi_1(S)$ be a finitely generated subgroup. Let $p : X \to S$ be the covering corresponding to the conjugacy class of F in $\pi_1(S)$. By a Scott's Theorem [] we can find a finite covering $\overline{S} \to S$ and a compact π_1-injective surface $\Sigma \subseteq \overline{S}$ such that $\pi_1(\Sigma) = F \subseteq \pi_1(\overline{S})$. The natural embedding $j : \Sigma \to X$ lifts along the covering $p : X \to \overline{S}$ since $j_*\pi_1(\Sigma) = p_*\pi_1(X) = F \subseteq \pi_1(\overline{S})$

$$\begin{tikzcd} & X \arrow[d, "p"] \\ \Sigma \arrow[r, "j"'] \arrow[ur, dashed, "f"] & \overline{S}. \end{tikzcd}$$

The map $f : \Sigma \to X$ is an embedding which induces an isomorphism $f_* : \pi_1(\Sigma) \to \pi_1(X)$, in particular $f(\Sigma) \subseteq X$ is a homotopy equivalence and X deformation retracts to $f(\Sigma)$. Consider a maximal representation $\rho : \pi_1(S) \to \mathrm{Homeo}_+\left(S^1\right)$ and assume that $\rho(c)$ has a fixed point for every $c \in \pi_1(S)$. Later we will see that this assumption is redundant as every maximal representation enjoys this property. Since maximality passes to finite coverings, also the induced representation $\rho := \rho|_{\pi_1(\overline{S})} : \pi_1(\overline{S}) \to \mathrm{Homeo}_+\left(S^1\right)$ is maximal. Cutting S along the boundary of $\Sigma \subseteq S$ gives a decomposition

$$S = \Sigma \cup \bigcup_{j=1}^{n} S_j \qquad , \qquad \chi(S) = \chi(\Sigma) + \sum_{j=1}^{n} \chi(S_j).$$

Let σ be a special section. By the Addition Formula 2.15 we get

$$e(\rho) = e\left(\rho|_{\pi_1(\Sigma)}, \sigma\right) + \sum_{j=1}^{n} e\left(\rho|_{\pi_1(S_j)}, \sigma\right).$$

Since ρ is maximal also $\rho|_{\pi_1(S_j)}$ and $\rho|_{\pi_1(\Sigma)}$ are maximal or, in other words,

$$\left| e\left(\rho|_{\pi_1(S_j)}, \sigma\right) \right| = |\chi(S_j)| \qquad \left| e\left(\rho|_{\pi_1(\Sigma)}, \sigma\right) \right| = |\chi(\Sigma)| = |\chi(X)|.$$

Lemma 3.6. *Let $\rho : \pi_1(S) \longrightarrow \mathit{Homeo}_+\left(S^1\right)$ be a maximal representation such that $\rho(c)$ has a fixed point for every $c \in \pi_1(S)$. Let $F < \pi_1(S)$ be a finitely generated subgroup corresponding to a covering $p : X \longrightarrow S$. Then $\rho|_F$ is maximal or*

$$|e(\rho, \sigma)| = |\chi(X)|$$

where $\sigma : G \longrightarrow \widetilde{G}$ is a special section.

4. Dynamics of maximal representations

Proposition 3.7. *Let S be a surface of genus g. The following hold:*

(1) if ρ is maximal, i.e. $e(\rho) = 2g-2$, then

$$\widetilde{rot}\,[\widetilde{\rho}(a_j), \widetilde{\rho}(b_j)] = 1.$$

(2) suppose that $k \mid 2g-2$, let ρ be a representation which $k-$covers a maximal representation, then $e(\rho) = \frac{2g-2}{k}$ and

$$\widetilde{rot}\,[\widetilde{\rho}(a_j), \widetilde{\rho}(b_j)] = \frac{1}{k}.$$

Proof. We first prove *(1)*. Let ρ be a maximal representation. We have

$$\begin{aligned}
2g-2 &= \widetilde{\mathrm{rot}}\left(\prod_{j=1}^{g} [\widetilde{\rho}(a_j), \widetilde{\rho}(b_j)]\right) \\
&= \widetilde{\mathrm{rot}}\left(\prod_{j=1}^{g-1} [\widetilde{\rho}(a_j), \widetilde{\rho}(b_j)]\right) + \widetilde{\mathrm{rot}}\,[\widetilde{\rho}(a_g), \widetilde{\rho}(b_g)] && \text{as } \prod_{j=1}^{g-1} [a_j, b_j], [a_g, b_g] \text{ commute in } \pi_1(S) \\
&\leq \sum_{j=1}^{g} \widetilde{\mathrm{rot}}\,[\widetilde{\rho}(a_j), \widetilde{\rho}(b_j)] + g - 2 && \widetilde{\mathrm{rot}} \text{ is a quasi-morphism with defect } D\left(\widetilde{\mathrm{rot}}\right) = 1 \\
&\Rightarrow \sum_{j=1}^{g} \widetilde{\mathrm{rot}}\,[\widetilde{\rho}(a_j), \widetilde{\rho}(b_j)] \geq g.
\end{aligned}$$

Since $\left|\widetilde{\mathrm{rot}}\,[\widetilde{\rho}(a_j), \widetilde{\rho}(b_j)]\right| \leq 1$ for every $j \leq g$ by Lemma 1.31, we conclude that $\widetilde{\mathrm{rot}}\,[\widetilde{\rho}(a_j), \widetilde{\rho}(b_j)] = 1$. Now we prove *(2)*. Let ρ be a representation which $k-$covers a maximal representation. The equalities follow formally from the previous case. Denote by $G^{(k)} := \mathrm{Homeo}_+^{(k)}(S^1)$ the degree k cover of $G := \mathrm{Homeo}_+\left(S^1\right)$ which we identify with the group of homeomorphisms of S^1 which commute with the rotation r_k of angle $\frac{1}{k}$

$$\begin{tikzcd}
\pi_1(S) \arrow[dr, "\psi"] \arrow[drr, bend left, "\rho"] \arrow[ddr, bend right, "\phi"'] & & \\
& G^{(k)} \arrow[r, "j"] \arrow[d, "p_k"'] & G \\
& G &
\end{tikzcd}$$

By assumption ρ factors through $G^{(k)}$, i.e. $\rho = j\psi$, and $\phi = p_k\psi$ is a maximal representation where p_k is the covering and $j : G^{(k)} \to G$ is the natural inclusion.
We first prove that $e(\rho) = \frac{2g-2}{k}$. Since

$$\mathrm{eu}\,(\rho) = \rho^* e = \psi^* j^* e \qquad , \qquad e\,(\phi) = \phi^* e = \psi^* p_k^* e$$

it is enough to show that $kj^*e = p_k^*e$. Observe that the lifts of $p_k(f) \in G$ to the universal cover $\widetilde{G}$ are precisely of the form $c_k(\widetilde{f})$ where c_k is the conjugation induced by the multiplication by $k \in \mathbb{N}$, and $\widetilde{f}$ is some lift of f to $\widetilde{G}$, explicitly

$$c_k\left(\widetilde{f}\right)(x) = k\widetilde{f}\left(\frac{x}{k}\right).$$

We prove this assertion. Denote by $\pi_k : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ the covering projection of degree k such that $\pi_k[x] = [kx]$. By definition, the induced map $p_k(f)$ evaluated at $[x] \in \mathbb{R}/\mathbb{Z}$ is defined as $\pi_k f[y]$ where $\pi_k[y] = x$. Choosing $y = \frac{x}{k}$ we recover $p_k(f)[x] = \left[kf\left[\frac{x}{k}\right]\right] \in \mathbb{R}/\mathbb{Z}$. It is now straightforward to prove that the lifts of $p_k(f)$ to the universal cover $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ are precisely those of the form $c_k\left(\widetilde{f}\right)$ where $\widetilde{f}$ is a lift of f.

Fix a section $\sigma : G \to \widetilde{G}$, we get

$$\sigma(p_k(f)) = c_k(\sigma(f))\tau_{n(f)} \qquad , \qquad n : G^{(k)} \to \mathbb{Z}.$$

The section σ defines a cocycle $e = [c_\sigma]$. Finally we get

$$\begin{aligned}
&p_k^* c_\sigma(f,g) \\
&= \sigma(p_k(f))\sigma(p_k(gf))^{-1}\sigma(p_k(g)) \\
&= c_k\left(\sigma(f)\sigma(gf)^{-1}\sigma(g)\right)\tau_{n(f)+n(g)-n(gf)} \\
&= k\sigma(f)\sigma(gf)^{-1}\sigma(g)\tau_{\delta n(f,g)} \\
&= kj^* c_\sigma(f,g) + \delta n(f,g) \\
&\Rightarrow kj^*e = p_k^*e.
\end{aligned}$$

This concludes the computation of $e(\rho)$.

We remark that if we had chosen the section $\sigma : G \longrightarrow \widetilde{G}$ such that $\sigma(f)(0) \in [0,1)$ then the same argument above would have proved that $kj^*e_b^{\mathbb{Z}} = p_k^*e_b^{\mathbb{Z}}$. This follows from the fact that the function $n : G^{(k)} \to \mathbb{Z}$ is bounded with the special choice of σ. In particular we get the more general result

$$ke_b^{\mathbb{Z}}(\rho) = e_b^{\mathbb{Z}}(\phi).$$

We prove that $\widetilde{\mathrm{rot}}\,[\widetilde{\rho}(a_j), \widetilde{\rho}(b_j)] = \frac{1}{k}$. We have that $\widetilde{\phi}(a_j) = c_k(\widetilde{\rho}(a_j))$ and $\widetilde{\phi}(b_j) = c_k(\widetilde{\rho}(b_j))$ for some $\widetilde{\rho}(a_j), \widetilde{\rho}(b_j)$ so we get

$$\begin{aligned}
1 &= \widetilde{\mathrm{rot}}\left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)\right] && \text{by maximality of } \phi \\
&= \widetilde{\mathrm{rot}}\left[c_k(\widetilde{\rho}(a_j)), c_k(\widetilde{\rho}(b_j))\right] && \text{by the above argument} \\
&= \widetilde{\mathrm{rot}}\ c_k([\widetilde{\rho}(a_j), \widetilde{\rho}(b_j)]) && \text{since } c_k \text{ is a homomorphism} \\
&= k\ \widetilde{\mathrm{rot}}\,[\widetilde{\rho}(a_j), \widetilde{\rho}(b_j)] && \text{by a straightforward computation.}
\end{aligned}$$

□

Corollary 3.8. *Let S be a surface of genus g, and $\rho : \pi_1(S) \to Homeo_+\left(S^1\right)$ be a representation. Then the following hold*

(1) if ρ is maximal, then $rot(c) = 0$ for every $c \in \pi_1(S)$.
(2) if ρ $k-$covers a maximal representation, then $rot(c) = \frac{m_c}{k}$ for every $c \in \pi_1(S)$.

Proof. We prove *(1)*. Let ρ be maximal. By Proposition 3.7 we have $\widetilde{\mathrm{rot}}\left[\widetilde{a}_j, \widetilde{b}_j\right] = 1$. Suppose by contradiction that $\mathrm{rot}(a_j) \neq 0$ and choose a lift $\widetilde{a}_j$ such that $0 < \widetilde{\mathrm{rot}}\, \widetilde{a}_j < 1$. Choose arbitrary the lift $\widetilde{b}_j$. By the conjugacy invariance and homogeneity properties of $\widetilde{\mathrm{rot}}$ we have $\widetilde{\mathrm{rot}}\, \widetilde{b}_j \widetilde{a}_j^{-1} \widetilde{b}_j^{-1} = \widetilde{\mathrm{rot}}\, \widetilde{a}_j^{-1} = -\widetilde{\mathrm{rot}}\, \widetilde{a}_j < 0$. In particular $\widetilde{b}_j \widetilde{a}_j^{-1} \widetilde{b}_j^{-1} x < x$ for every $x \in \mathbb{R}$. Hence we get

$$\left[\widetilde{a}_j, \widetilde{b}_j\right] x = \widetilde{a}_j \widetilde{b}_j \widetilde{a}_j^{-1} \widetilde{b}_j^{-1} x < \widetilde{a}_j x.$$

The last inequality implies $\widetilde{\mathrm{rot}}\left[\widetilde{a}_j, \widetilde{b}_j\right] < \widetilde{\mathrm{rot}}\, \widetilde{a}_j < 1$, a contradiction.
So if $c \in \{a_1, b_1, \ldots, a_g, b_g\}$ is a standard generator then we are done. In general we observe that for every loop $c \in \pi_1(S)$ there is a finite cover $p : S' \to S$ along which c lifts to a loop which is represented by a non separating simple curve c' by Scott's Theorem 3.9 (see below). The covering $p : S' \to S_g$ corresponds to a finite index subgroup $p_* : \pi_1(S') \to \pi_1(S)$, say $d := [\pi_1(S) : p_*\pi_1(S')]$, and $p_*[c'] = [c]$. Call $\rho' = \rho p_*$ the induced representation of $\pi_1(S')$. By Lemma 3.5 ρ' is maximal.
Since every non separating simple closed curve can be completed to a standard generating set for $\pi_1(S')$ we can use the same argument of the previous point and conclude by the above computations that $\mathrm{rot}(c) = 0$.
The proof of *(2)* is now straightforward. Assume that ρ $k-$covers a maximal representation. Then by *(1)* every homeomorphism $p_k\rho(c)$ has vanishing rotation number and therefore has a fixed point. Any homeomorphism that $k-$covers a homeomorphism with fixed points has a periodic orbit of period k. □

Scott's Theorem can be stated in the following form

Theorem 3.9 (Scott, [**Sco78**]). *Let S be a surface. Let $F < \pi_1(S)$ be a finitely generated subgroup. Let $\gamma \in \pi_1(S) \setminus F$ be an element outside F. Then there exists a finite covering $p : S' \longrightarrow S$ such that*

(i) $F < p_*\pi_1(S')$ *and* $\gamma \notin p_*\pi_1(S')$;
(ii) the lift of F to $\pi_1(S')$ is the fundamental group of an incompressible surface $X \subseteq S'$, i.e. $(p_*)^{-1}(F)\pi_1(S') = \pi_1(X)$.

Remark 3.10. We observe that the scheme of the proof of Corollary 3.8 can be generalized to this fact: *if maximality implies a certain property for every $\rho(c)$ where $c \in \{a_1, b_1, \ldots, a_g, b_g\}$ is a standard generator then maximality implies the same property for every loop $c \in \pi_1(S)$*. This follows from Scott's Theorem 3.9 and from the fact that maximality is preserved when passing to finite coverings. Every loop $c \in \pi_1(S)$ lifts to a loop which is represented by some simple closed non separating curve in some finite degree covering $p : S' \to S$ and the induced representation ρp_* is again maximal.

Remark 3.11. Corollary 3.8 is already enough to prove that all maximal $\mathrm{PSL}(2, \mathbb{R})$ representations are semi-conjugate. We will prove this fact later in Theorem 4.18.

4.1. Maximal representations avoid tame elements. As in the remark another interesting property is being *tame*:

Definition 3.12. A homeomorphism $f \in \widetilde{\mathrm{Homeo}}_+\left(S^1\right)$ is *tame* if there is an integer $n \in \mathbb{Z}$ such that

$$n \leq f(x) - x \leq n+1$$

Remark 3.13. Every non-tame element $f \in \widetilde{\mathrm{Homeo}}_+\left(S^1\right)$ has at least an *unstable* fixed point and a *stable* fixed point. The elements of $\mathrm{PSL}(2,\mathbb{R})$ with this property are precisely the hyperbolic ones.

The following proposition witnesses that maximal representations into $\mathrm{Homeo}_+\left(S^1\right)$ are similar to hyperbolic representations:

Proposition 3.14. *Let ρ be a maximal representation. Then for every non-trivial $c \in \pi_1(S) \setminus \{1\}$ the homeomorphism $\rho(c)$ is not tame.*

Proof. As before (see Remark 3.10) it is enough to prove the corollary for all generators in $\{a_1, b_1, \ldots, a_g, b_g\}$; for notation sake let us call $a := \widetilde{\rho}(a_1), b := \widetilde{\rho}(b_1)$ (these are the preferred lifts of $\rho(a_1), \rho(b_1)$) and $c = ba^{-1}b^{-1}$.
Suppose by contradiction that a is tame. By our choice of the lifts the following hold

$$0 \leq a(x) - x \leq 1 \qquad -1 \leq a^{-1}(x) - x \leq 0;$$

then conjugating by b we get from the second inequality $-1 \leq c(x) - x \leq 0$. Consider $ac = [a,b]$. We have the inequalities

$$\begin{array}{ll} ac(x) - x \leq a(x) - x \leq 1 & \text{since } a \text{ is increasing and } c(x) \leq x \\ ac(x) - x \geq a(x-1) - x \geq -1 & \text{since } a \text{ is increasing and } c(x) \geq x-1. \end{array}$$

The inequalities $-1 \leq ac(x) - x \leq 1$ are strict: let us consider $ac(x) - x \leq 1$, looking at the intermediate steps above we observe that, in order to achieve equality, we should have $ba^{-1}b^{-1}(x) = c(x) = x$ and $a(x) = x+1$ or, equivalently, $ab^{-1}(x) = b^{-1}(x), a(x) = x+1$. This cannot occur since otherwise we can find $y = b^{-1}(x) + k \in \mathbb{R}$ (with $k \in \mathbb{Z}$) such that $x < y < x+1$ and $a(y) = y$ which gives $x + 1 = a(x) < a(y) = y < x+1$. Analogously we exclude the equality case $ac(x) - x = -1$.
In conclusion $-1 < [a,b](x) - x < 1$ which implies $\left|\widetilde{\mathrm{rot}}\,[a,b]\right| < 1$, and this contradicts maximality by Proposition 3.7. □

4.2. Goldman's Theorem. The previous discussion allows to recover Goldman's Theorem (the proof is due to Matsumoto in [**Mat87**]):

Theorem 3.15 (Goldman). *Let $\phi : \Gamma \longrightarrow PSL(2,\mathbb{R})$ be a representation. If $e(\phi) = 2g-2$ then ϕ is faithful and has discrete image.*

Before giving the proof let us state the following fact about subgroups of $\mathrm{PSL}(2,\mathbb{R})$ containing only hyperbolic isometries:

Proposition 3.16. *Let $\Lambda < PSL(2,\mathbb{R})$ be a subgroup. Suppose that Λ does not fix any point on $\partial\mathbb{H}^2$ and contains only hyperbolic isometries. Then Λ is discrete.*

For a proof of the proposition see [**Rat13**]. Now we prove the theorem:

Proof. Proposition 3.14 implies that ϕ is injective since $\mathrm{Id} \in \mathrm{PSL}(2,\mathbb{R})$ is tame. It also implies by Remark 3.13 that for every non-trivial $\gamma \in \Gamma \setminus \{1\}$ the homeomorphism $\phi(\gamma) \in \mathrm{PSL}(2,\mathbb{R})$ is a hyperbolic motion. We observe that since $e(\phi) = 2g-2 \neq 0$ the subgroup $\phi(\Gamma) < \mathrm{PSL}(2,\mathbb{R})$ does not fix any point on $\partial\mathbb{H}^2$. Thus by Proposition 3.16 the subgroup $\phi(\Gamma) < \mathrm{PSL}(2,\mathbb{R})$ is discrete. □

5. Bounded Euler class of a maximal representation

We compute the canonical Euler cocycle $\rho^*\tau$ described in Proposition 1.35 representing the real bounded Euler class for a maximal $\mathrm{PSL}(2,\mathbb{R})$ representation ρ of $\Gamma := \pi_1(S)$. In particular we show that it is possible to define $\rho^*\tau$ in terms of the topology of S.
Let us recall the definition of $\tau : G \times G \longrightarrow \mathbb{R}$:

$$f, g \in \mathrm{Homeo}_+\left(S^1\right) \qquad \tau(f,g) = \widetilde{\mathrm{rot}}\ \widetilde{f}\widetilde{g} - \widetilde{\mathrm{rot}}\ \widetilde{f} - \widetilde{\mathrm{rot}}\ \widetilde{g}$$

where $\widetilde{f}, \widetilde{g} \in \widetilde{\mathrm{Homeo}_+}\left(S^1\right)$ are arbitrary lifts of f, g. We have $\sigma^*\left[\tau = -\delta\widetilde{\mathrm{rot}}\right] = e_b^{\mathbb{R}} \in H_b^2(G,\mathbb{R})$ where $\sigma^* = (p^*)^{-1}$ and $p : \widetilde{G} \longrightarrow G$ is the universal covering projection.
We remark that τ is a well defined continuous conjugacy invariant function on $G \times G$. Moreover it is symmetric since for every $f, g \in \widetilde{G}$ we have

$$\widetilde{\mathrm{rot}}\ fg = \widetilde{\mathrm{rot}}\ g(fg)g^{-1} = \widetilde{\mathrm{rot}}\ gf.$$

5.1. Subgroups of $\pi_1(S)$. Let S be a surface of genus $g \geq 2$ and let $\gamma \in \Gamma := \pi_1(S) \setminus \{e\}$ be a non-trivial loop. Let us fix a complete hyperbolic metric on S which corresponds to a (conjugacy class of) discrete subgroup $\Gamma \subseteq \mathrm{PSL}(2,\mathbb{R})$ which acts freely on $\mathbb{H}^2$. Under this identification the loop $\gamma \in \Gamma$ corresponds to a hyperbolic isometry $\gamma \in \mathrm{PSL}(2,\mathbb{R}) = \mathrm{Isom}^+\left(\mathbb{H}^2\right)$ which has an axis that ends on $\partial\mathbb{H}^2 = S^1$ with two fixed points.

Definition 3.17. Let $a, b \in \pi_1(S)$ be two loops. Consider the axes of the corresponding hyperbolic isometries $a, b \in \mathrm{PSL}(2,\mathbb{R})$. We say that

- a, b have *intersecting axes* if the axes of the hyperbolic motions intersect.
- a, b have *disjoint axes* if the axes of the hyperbolic motions are disjoint.

Remark 3.18. We observe that this definition can be equivalently stated in terms of the possible cyclic orderings of the four fixed points of a, b on the boundary $\partial\mathbb{H}^2$. Since semi-conjugacies (weakly) preserve the cyclic ordering on the boundary we see that the notion is well defined up to semi-conjugacy.

This definition does not depend on the particular geometry we put on S. As we will see having disjoint or intersecting axes may be characterized in purely topological terms looking at the covering $p : X \longrightarrow S$ corresponding to the conjugacy class of the subgroup $\langle a, b\rangle < \pi_1(S)$.
A classical theorem about hyperbolic surfaces ensures that there are only a few possibilities for the topological type of X.

Theorem 3.19. *Let S be surface with finitely generated fundamental group $\pi_1(S)$. Then S is a surface of finite type.*

One can prove this theorem by means of Morse Theory.

COROLLARY 3.20. *Let $\Gamma = \langle \alpha, \beta \rangle < PSL(2, \mathbb{R})$ be a non-trivial torsion free discrete subgroup. Let $X = \mathbb{H}^2/\Gamma$ be the quotient surface. Then X is either a cylinder, a punctured torus or a pair of pants.*

PROOF. Since $\pi_1(X) = \Gamma$ is finitely generated, by Theorem 3.19 we have $X = S_{g,p}$ where g denotes the genus and p denotes the number of punctures. Consider $H_1(X, \mathbb{Z}) = \frac{\Gamma}{[\Gamma,\Gamma]}$. Necessarily $p > 1$ otherwise we would have a closed oriented surface S_g with $2g = \text{rank } H_1(S_g, \mathbb{Z}) = \text{rank } H_1(X, \mathbb{Z}) \leq 2$. This is possible only if $g = 1$, hence X is a torus, but no torus supports a hyperbolic structure by Gauss-Bonnet Theorem. Thus $\Gamma = \pi_1(S_{g,p})$ is a free group and X is not closed. If Γ is infinite cyclic, i.e. $\Gamma = \mathbb{Z}\gamma$, generated by a hyperbolic or parabolic isometry γ then X is either a hyperbolic cylinder or a hyperbolic cusp (in both cases $X = S_{0,2}$). If Γ is a free group of rank $\Gamma = 2$ then we have $\chi(X) = -1$, hence $X = S_{0,3}$ is a pair of pants or $X = S_{1,1}$ is a punctured torus. □

Now we give a description of the possible cases in terms of the generators of Γ.

PROPOSITION 3.21. *Let S be a hyperbolic surface. Let $\alpha, \beta \in \pi_1(S) \setminus \{e\}$, let $F = \langle \alpha, \beta \rangle$ be the subgroup they generate, and let $p : X \to S_g$ the covering corresponding to the conjugacy class of $F < \pi_1(S)$, so that $\Gamma = \pi_1(X)$. We have the following cases*

(1) if α, β have the same axes then $\Gamma = \mathbb{Z}$ and X is a cylinder.
(2) if α, β have disjoint axes then $\Gamma = F_2$ and X is a pair of pants.
(3) if α, β have intersecting axes then $\Gamma = F_2$ and X is a punctured torus.

PROOF. By Corollary 3.20 the space X can be a cylinder, a punctured torus or a pair of pants. Suppose that X is a cylinder. Then $\Gamma = \pi_1(X) = \mathbb{Z}$ is an abelian discrete subgroup of hyperbolic isometries of $\mathrm{PSL}(2, \mathbb{R})$. It is well known that two commuting hyperbolic isometries in a discrete subgroup have the same axes.
Suppose that X is a punctured torus. Since $\langle \alpha, \beta \rangle = \Gamma = \pi_1(X)$ we have that $[\alpha], [\beta]$ generate $H_1(X, \mathbb{Z}) = \mathbb{Z}^2$. Since the cup product is non trivial on the first group of cohomology with compact support $H^1_c(X, \mathbb{Z})$ we see that the intersection product $[\alpha] \cdot [\beta] \neq 0$ is non trivial. Hence every pair of curves in the free homotopy classes of α, β intersect. In particular the geodesic representatives of α and β intersect, but this means exactly that the axes of α and β intersect.
Suppose that X is a pair of pants. Assume by contradiction that the closed geodesics representing the free homotopy classes of α, β intersect. Lift them to transversely intersecting geodesic l_α, l_β of $\mathbb{H}^2$. For the next constructions we refer to Figure 1. Call $p = l_\alpha \cap l_\beta$ the intersection point. Let $A_1, A_2 \in l_\alpha$ be points which are symmetric with respect to P and such that $\alpha(A_1) = A_2$. Choose $B_1, B_2 \in l_\beta$ with the analogous property $\beta(B_1) = B_2$. Consider the unique lines $r_1, r_2 \subseteq \mathbb{H}^2$ which are orthogonal to l_α in A_1, A_2 respectively. Define the lines $t_1, t_2 \subseteq \mathbb{H}^2$ in an analogous way for l_β.
We claim that the lines r_1 and t_1 do not intersect. Assume the contrary, let $Q_1 = r_1 \cap t_1$ be the intersection point. By the symmetry of the configuration with respect to the lines l_α, l_β we have

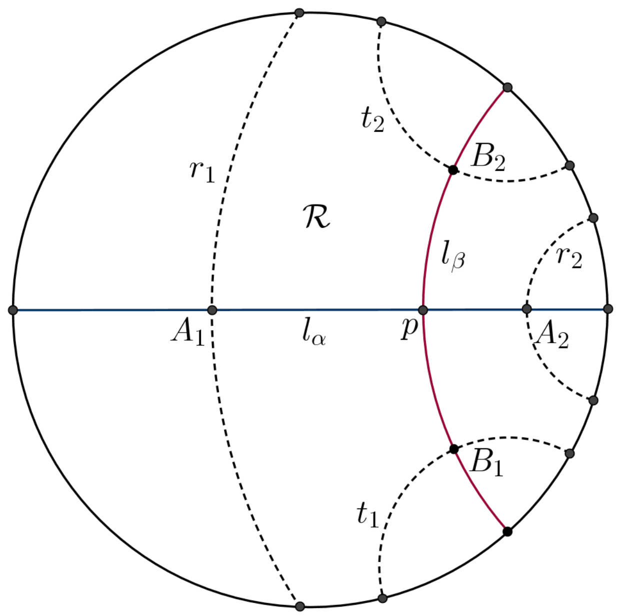

FIGURE 1.

- the lines t_1, r_2 intersect in the point $\beta(Q_1) = t_1 \cap r_2$.
- the lines t_2, r_1 intersect in the point $\alpha(Q_1) = t_1 \cap r_2$.
- the lines t_2, r_2 intersect in a point $Q_2 = t_1 \cap r_2$.

Again by symmetry we have $Q_2 = \alpha\beta(Q_1) = \beta\alpha(Q_1)$. Hence $[\alpha, \beta]\, Q_1 = Q_1$, and the isometry $[\alpha, \beta] \in \Gamma$ is elliptic. Since $\Gamma = \pi_1(X)$ does not contain elliptics we should have $[\alpha, \beta] = e$, but this contraddicts the fact that Γ is free nonabelian. Therefore the four lines r_1, r_2, t_1, t_2 are pairwise disjoint. Denote by R the region delimited by those lines containing the point $P = l_\alpha \cap l_\beta$. It is not difficult to prove that R is a fundamental domain for Γ and R/Γ is a punctured torus. The last assertion produces the contradiction since we assumed that X is a pair of pants. $\square$

REMARK 3.22. We can distinguish the pair of pants case from the punctured torus case by non-trivialities of cup products in compactly supported cohomology. Unfortunately triviality of the algebraic intersection is not enough to prove that the geodesic representatives of α, β do not intersect in the case of a pair of pants.

COROLLARY 3.23. *Let S be a surface. Let $\alpha, \beta \in \pi_1(S) \setminus \{e\}$ be non-trivial loops. Then the property of having intersecting or disjoint axes for α, β does not depend on the particular hyperbolic structure used.*

Since we have characterized the property of having intersecting or disjoint axes for α, β in terms of the cyclic ordering of the four points on $\partial\mathbb{H}^2$ corresponding to the fixed points of the hyperbolic motions, Corollary 3.23 gives the first evidence that the holonomies of hyperbolic structures are all semi-conjugate to each other.

Using the characterization given by Proposition 3.21 we will be able to describe the dynamics of maximal actions in purely topological terms. Let us fix a hyperbolic structure on S corresponding to the holonomy representation $\rho : \pi_1(S) \longrightarrow \mathrm{PSL}\,(2,\mathbb{R})$. We study in detail the three possible cases of Proposition 3.21. In what follows we have to deal with surfaces with punctures $S_{g,p}$. Since $S_{g,p}$ deformation retracts onto a compact surface $S_{g,b}$ with $b = p$ boundary components we will call a loop $\gamma \in \pi_1(S_{g,p})$ *boundary curve* if its free homotopy class is represented by a boundary curve in $S_{g,b}$.

5.2. Hyperbolic cylinder. Suppose that $\alpha, \beta \in \pi_1(S)$ generate an infinite cyclic group $F = \langle \alpha, \beta \rangle = \mathbb{Z}$ that corresponds to a covering $p : X \longrightarrow S$ where X is a cylinder. Then $\rho|_F^* e_b^{\mathbb{R}} = 0$ because F is amenable. In particular $\rho|_F^* \tau = 0$. Thus we have

LEMMA 3.24. *If $\alpha, \beta \in \pi_1(S) \setminus \{e\}$ have the same axes or equivalently generate an infinite cyclic subgroup $\langle \alpha, \beta \rangle = \mathbb{Z}$ that corresponds to a cover $p : X \longrightarrow S$ where X is a cylinder, then*

$$\rho^* \tau(\alpha, \beta) = 0.$$

REMARK 3.25. Lemma 3.24 holds for every maximal representation ρ whether it has image contained in $\mathrm{PSL}\,(2,\mathbb{R})$ or not.

5.3. Hyperbolic punctured torus. Suppose that $\alpha, \beta \in \pi_1(S)$ generate a free group $F = \langle \alpha, \beta \rangle = F_2$ that corresponds to a cover $X \to S$ where X is a punctured torus. Then α, β have intersecting axes and the fixed points of the hyperbolic motions $\rho(\alpha), \rho(\beta)$ divide $\partial\mathbb{H}^2$ into four arcs. Consider the arc $I \subseteq \mathbb{H}^2$ delimited by the attractive fixed points of α, β. If we lift I to $I' \subseteq \mathbb{R}$ and $\rho(\alpha), \rho(\beta)$ to $\widetilde{\rho}(\alpha), \widetilde{\rho}(\beta)$ in such a way that both lifts have fixed points (and in particular they fix the extrema of I'), then we have $\widetilde{\rho}(\alpha)I' \subseteq I'$ and $\widetilde{\rho}(\beta)I' \subseteq I'$ (the extrema of I' are attractive). In particular $\widetilde{\rho}(\alpha)\widetilde{\rho}(\beta)I' \subseteq I'$ which implies that $\widetilde{\rho}(\alpha)\widetilde{\rho}(\beta)$ has a fixed point in I'. Hence we have

LEMMA 3.26. *If $\alpha, \beta \in \pi_1(S) \setminus \{e\}$ have intersecting axes or equivalently generate a free subgroup $\langle \alpha, \beta \rangle = F_2$ that corresponds to a cover $p : X \longrightarrow S$ where X is a punctured torus, then*

$$\rho^* \tau(\alpha, \beta) = 0.$$

Suppose now that the lifts of the loops $\alpha, \beta \in \pi_1(S)$ to X are such that the free homotopy class of $[\alpha, \beta] \in \pi_1(X)$ is represented by the boundary curve of X. Since ρ is maximal and $\rho(c)$ has a fixed point for every $c \in \pi_1(S)$ also $\rho|_F$ is maximal, i.e. $e\,(\rho|_F, \sigma) = -\chi(X)$,

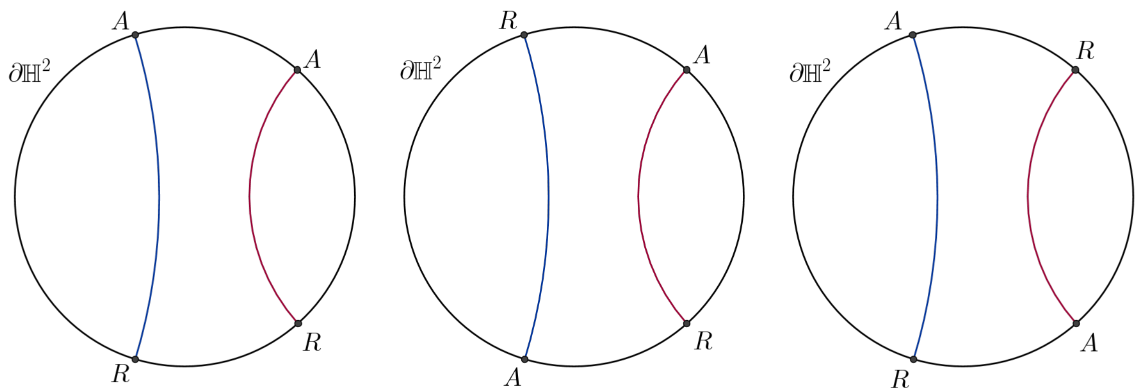

FIGURE 2.

where $\sigma : G \longrightarrow \widetilde{G}$ is a special section such that $0 \leq \widetilde{\text{rot}}\ \sigma(f) < 1$. Hence

$$\begin{aligned}
-1 &= e\,(\rho,\sigma) \\
&= \widetilde{\text{rot}}\,[\sigma(\rho(\alpha)),\sigma(\rho(\beta))]\,\sigma\,([\rho(\beta),\rho(\alpha)]) \\
&= \widetilde{\text{rot}}\,[\sigma(\rho(\alpha)),\sigma(\rho(\beta))] + \widetilde{\text{rot}}\ \sigma\,([\rho(\beta),\rho(\alpha)]) && \text{since } [\sigma(\rho(\alpha)),\sigma(\rho(\beta))]\,\sigma\,([\rho(\beta),\rho(\alpha)]) = \tau_{-1} \\
&= \widetilde{\text{rot}}\,[\sigma(\rho(\alpha)),\sigma(\rho(\beta))] && \text{by assumption } \widetilde{\text{rot}}\ \sigma\,([\rho(\beta),\rho(\alpha)]) = 0.
\end{aligned}$$

5.4. Hyperbolic pair of pants. Suppose that $\alpha,\beta \in \pi_1(S)$ generate a free group $F = \langle\alpha,\beta\rangle = F_2$ that corresponds to a cover $X \to S$ where X is a pair of pants. Let us consider for a moment the case where the geodesics in the free homotopy classes of $a, b \in \pi_1(X)$ represent boundary curves with the orientation matching the orientation of X. Let $\gamma \in \pi_1(X)$ be a loop representing the third boundary curve. From the relative Milnor's Formula **??** for the Euler number we get

$$e\,(\rho,\sigma) = \widetilde{\text{rot}}\ \widetilde{\rho}(\alpha)\widetilde{\rho}(\beta)\widetilde{\rho}(\gamma)$$

where $\widetilde{\rho}(\alpha),\widetilde{\rho}(\beta),\widetilde{\rho}(\gamma)$ are lifts of $\rho(\alpha),\rho(\beta),\rho(\gamma)$ such that $\widetilde{\text{rot}}\ \widetilde{\rho}(\alpha) = \widetilde{\text{rot}}\ \widetilde{\rho}(\beta) = \widetilde{\text{rot}}\ \widetilde{\rho}(\gamma) = 0$. In particular, by Lemma 3.6 and maximality of ρ, we have

$$\begin{aligned}
-1 &= e\,(\rho,\sigma) \\
&= \widetilde{\text{rot}}\ \widetilde{\rho}(\alpha)\widetilde{\rho}(\beta)\widetilde{\rho}(\gamma) \\
&= \widetilde{\text{rot}}\ \widetilde{\rho}(\alpha)\widetilde{\rho}(\beta) + \widetilde{\text{rot}}\ \widetilde{\rho}(\gamma) && \text{since } \widetilde{\rho}(\alpha)\widetilde{\rho}(\beta)\widetilde{\rho}(\gamma) = \tau_{-1} \\
&= \widetilde{\text{rot}}\ \widetilde{\rho}(\alpha)\widetilde{\rho}(\beta) && \text{by assumption } \widetilde{\text{rot}}\ \widetilde{\rho}(\gamma) = 0 \\
&= \tau\,(\rho(\alpha),\rho(\beta))\,.
\end{aligned}$$

Not every choice of generators for $\pi_1(X)$ produces a pair of geodesics representing boundary components. Consider the axes l_α, l_β of $\rho(\alpha),\rho(\beta) \in \mathrm{PSL}\,(2,\mathbb{R})$, they are disjoint by Proposition 3.21. Their extrema, the fixed points of $\rho(\alpha),\rho(\beta)$, divide the boundary $\partial\mathbb{H}^2$ into four arcs (see Figure 2, where every A denotes an attractive fixed point, while every R indicates a repelling one).

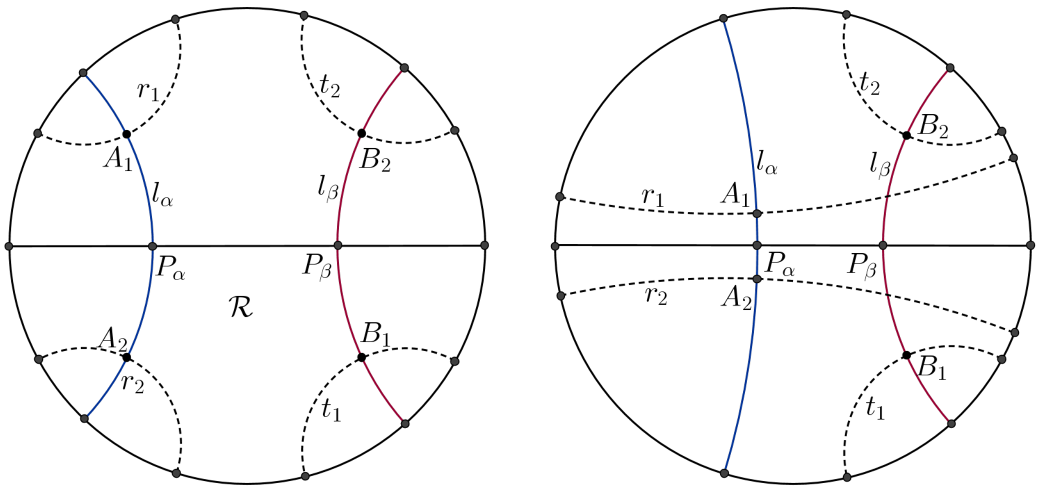

FIGURE 3.

If the attractive fixed points are adiacent then we can adopt the same argument used for hyperbolic cylinders to show that

$$\rho^*\tau(\alpha,\beta) = \widetilde{\mathrm{rot}}\ \widetilde{\rho}(\alpha)\widetilde{\rho}(\beta) = 0.$$

Otherwise we can assume, up to replacing α, β with α^{-1}, β^{-1} which changes the sign to the canonical Euler cocycle $\tau(\alpha,\beta) = -\tau(\alpha^{-1},\beta^{-1})$, that we have the configuration described by Figure 2 in the center. For notation sake $\alpha := \rho(\alpha)$ and $\beta := \rho(\beta)$. Let $s \subseteq \mathbb{H}^2$ be the unique line orthogonal to both l_α, l_β. Denote the intersection points of s with l_α, l_β respectively by $P_\alpha := s \cap l_\alpha$ and $P_\beta := s \cap l_\beta$. As in Proposition 3.21 choose $A_1, A_2 \in l_\alpha$ and $B_1, B_2 \in l_\beta$ in a symmetric position with respect to P_α, P_β and such that $\alpha(A_1) = A_2$ and $\beta(B_1) = B_2$. Let r_1, r_2 and t_1, t_2 be the lines orthogonal to l_α, l_β respectively in A_1, A_2 and B_1, B_2. Discreteness of F ensures that the four lines r_1, r_2, t_1, t_2 are disjoint just like in Proposition 3.21 (argue by contradiction and find an elliptic element in F using symmetries). Thus we have two possible configurations as in Figure 3.

Suppose we have the configuration on the left. Then the region R bounded by the four lines r_1, r_2, t_1, t_2 is the fundamental domain for the action $F \curvearrowright \mathbb{H}^2$. The quotient R/F is a pair of pants and the geodesics representing α, β are simple closed curves representing boundary components that match the orientation of X. These geodesics are freely homotopic to a, b. Thus we are in the first special case we depicted and

$$\widetilde{\mathrm{rot}}\ \widetilde{\rho}(a)\widetilde{\rho}(b) = e\left(\rho|_F, \sigma\right) = -1.$$

Suppose now we have the configuration on the right. Let us take a closer look at the dynamics of $\widetilde{\rho}(a)$ and $\widetilde{\rho}(b)$. Consider the points $x_1, x_2, y_1, y_2, z_1, z_2, w_1, w_2$ as in Figure 4

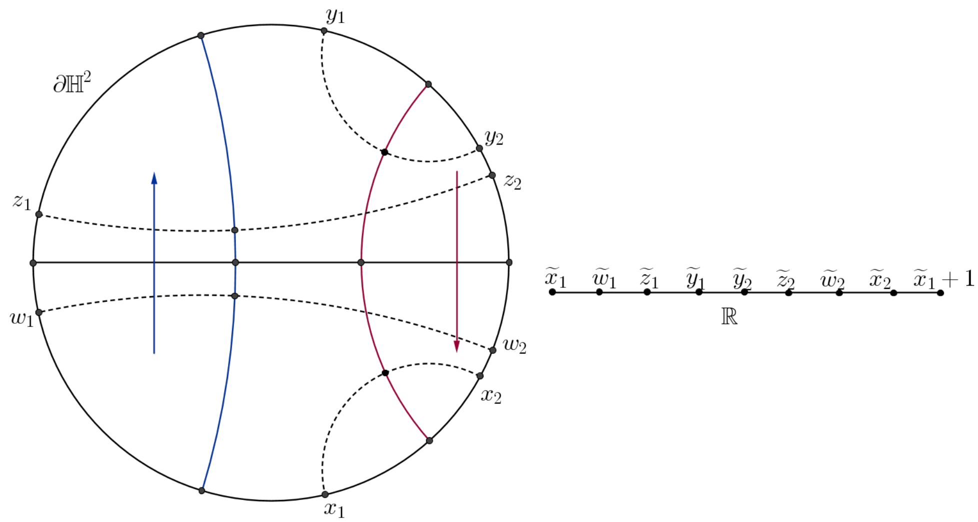

FIGURE 4.

on the right and lift them to $\mathbb{R}$ in such a way that

$$\widetilde{x}_1 < \widetilde{w}_1 < \widetilde{z}_1 < \widetilde{y}_1 < \widetilde{y}_2 < \widetilde{z}_2 < \widetilde{w}_2 < \widetilde{x}_2 < \widetilde{x}_1 + 1.$$

Then we have

$$\widetilde{x}_1 \xrightarrow{\widetilde{\rho}(a)} \widetilde{y}_1 \xrightarrow{\widetilde{\rho}(b)} \widetilde{\rho}(b)\widetilde{y}_1 \geq \widetilde{w}_1 \geq \widetilde{x}_1.$$

$$\widetilde{x}_2 \xrightarrow{\widetilde{\rho}(a)} \widetilde{y}_2 \xrightarrow{\widetilde{\rho}(b)} \widetilde{\rho}(b)\widetilde{y}_2 \leq \widetilde{w}_2 \leq \widetilde{x}_2.$$

Thus on one hand $\widetilde{\rho}(a)\widetilde{\rho}(b)x_1 \geq x_1$ tells us $\widetilde{\mathrm{rot}}\ \widetilde{\rho}(a)\widetilde{\rho}(b) \geq 0$. On the other hand $\widetilde{\rho}(a)\widetilde{\rho}(b)x_2 \leq x_2$ implies that $\widetilde{\mathrm{rot}}\ \widetilde{\rho}(a)\widetilde{\rho}(b) \leq 0$. Therefore

$$\rho^*\tau(a,b) = \widetilde{\mathrm{rot}}\ \widetilde{\rho}(a)\widetilde{\rho}(b) = 0.$$

LEMMA 3.27. *If $\alpha, \beta \in \pi_1(S)\backslash\{e\}$ have disjoint axes or equivalently generate a free subgroup $\langle\alpha, \beta\rangle = F_2$ that corresponds to a cover $p : X \longrightarrow S$ where X is a punctured torus, then we have the following three cases:*

(1) *If α, β are freely homotopic to boundary curves and their orientation matches the orientation of the boundary (α, β positively oriented), then $\rho^*\tau(\alpha, \beta) = 1$.*
(2) *If α, β are freely homotopic to boundary curves and their orientation matches the opposite orientation of the boundary (α, β negatively oriented), then $\rho^*\tau(\alpha, \beta) = -1$.*
(3) *Otherwise $\rho^*\tau(\alpha, \beta) = 0$.*

5.5. Canonical maximal integral Euler cocycle. Consider $\alpha, \beta \in \pi_1(S)$ and the subgroup $F = \langle \alpha, \beta \rangle$ they generate. Let $p : X \to S$ be the covering corresponding to F. We define

$$\theta(\alpha, \beta) = \begin{cases} +1 & \text{if } X \text{ is a pair of pants and } \alpha, \beta \text{ are positively oriented} \\ -1 & \text{if } X \text{ is a pair of pants and } \alpha, \beta \text{ are negatively oriented} \\ 0 & \text{otherwise} \end{cases}$$

Lemmas 3.24, 3.26 and 3.27 together allow us to prove the following.

PROPOSITION 3.28. *Let $\rho : \pi_1(S) \to PSL\,(2, \mathbb{R})$ be a maximal representation. Then*

$$\rho^* \tau = \theta.$$

PROOF. If ρ is maximal then by Lemma 3.8 $\rho(\gamma)$ has a fixed point for every element $\gamma \in \pi_1(S)$. Choose special lifts $\widetilde{\rho}(\gamma)$ of $\rho(\gamma)$ that have fixed points. With such choices the canonical Euler cocycle is computed by

$$\rho^* \tau\,(\alpha, \beta) = \widetilde{\mathrm{rot}}\ \widetilde{\rho}(\alpha)\widetilde{\rho}(\beta) - \widetilde{\mathrm{rot}}\ \widetilde{\rho}(\alpha) - \widetilde{\mathrm{rot}}\ \widetilde{\rho}(\beta) = \widetilde{\mathrm{rot}}\ \widetilde{\rho}(\alpha)\widetilde{\rho}(\beta) \in \{0, \pm 1\}$$

By Proposition 3.21 the subgroup $F = \langle \alpha, \beta \rangle \subseteq \pi_1(S)$ is trivial $\{e\}$, cyclic $\mathbb{Z}$ or free F_2. In this last case let us denote by $X \to S$ the covering corresponding to F. By the computations of the previous sections we deduce the following

- *case $F = \{e\}$.* It is clear that $\widetilde{\mathrm{rot}}\ \widetilde{\rho}(\alpha)\widetilde{\rho}(\beta) = 0$.
- *case $F = \mathbb{Z}$.* By Lemma 3.24 we have $\rho^*\tau(\alpha, \beta) = \widetilde{\mathrm{rot}}\ \widetilde{\rho}(\alpha)\widetilde{\rho}(\beta) = 0$.
- *case $F = F_2$ and X is a punctured torus.* By Lemma 3.26 we have $\rho^*\tau(\alpha, \beta) = \widetilde{\mathrm{rot}}\ \widetilde{\rho}(\alpha)\widetilde{\rho}(\beta) = 0$.
- *case $F = F_2$ and X is a pair of pants.* By Lemma 3.27 if α, β do not lift to loops freely homotopic to boundary curves which match the orientation of X or the opposite one then we have $\rho^*\,(\alpha, \beta) = \widetilde{\mathrm{rot}}\ \widetilde{\rho}(\alpha)\widetilde{\rho}(\beta) = 0$. In the remaining cases $\rho^*\tau\,(\alpha, \beta) = \widetilde{\mathrm{rot}}\ \widetilde{\rho}(\alpha)\widetilde{\rho}(\beta) = \pm 1$ where the sign depends on the orientation.

□

CHAPTER 4

Space of representations

1. Overview

In this chapter we introduce the space of representations $\mathrm{Rep}(\Gamma)$ and the character space $\overline{\mathrm{Rep}}(G)$. The topology of this space is not well understood, it is not even known if its connected components are finitely many or infinitely many or if the space is locally connected. We use Matsumoto's description of the bounded Euler class in terms of the rotation numbers of generators and the canonical Euler cocycle to give coordinates for the character space or at least a finite dimensional approximation of it. We also introduce other useful natural functionals on the space of representations like the Euler number and translation numbers of lifted commutators. In the last part we come back to rigidity, we define the class of locally stable representations, give criteria for local stability and prove that the class is non-empty.

Henceforth $\Gamma := \pi_1(S)$ where S is a closed oriented surface of genus $g \geq 2$.

2. The Lie group case $\mathbf{PSL}^{(k)}(2, \mathbb{R})$

In this section we present known results about the Lie group case, i.e. representations into $\mathrm{PSL}^{(k)}(2, \mathbb{R})$. This section should provide some motivations and insight also in the topological case of our interest. Let us mention that the vast subject of Higher Teichmüller Theory, whose description goes far beyond our purposes, prompted from the results we are going to describe and is greatly indebted to the work of Goldman (see [**GUoC80**], [**Gol88**], [**Gol84**],[**Gol09**]).

2.1. Representation variety. Let us consider the set of representations

$$\mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{R})).$$

We can give this set, the *representation variety*, many different structures.
First we can identify it with the algebraic subset

$$\mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{R})) = \left\{ (A_1, B_1, \ldots, A_g, B_g) \in \mathrm{PSL}(2, \mathbb{R})^{2g} \,\middle|\, \prod_{j=1}^{g} [A_j, B_j] = I \right\} \subseteq G^{2g}.$$

In particular it has the structure of a manifold near every point that is regular for the commutator function. In general there will be many singular points, for example $(I, I, \ldots, I, I) \in G^{2g}$. We can also put on $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{R}))$ the structure of a quasi-projective algebraic

variety over $\mathbb{R}$ (the space is precisely the set of the real points of this variety).
Thinking of geometric structures it is natural to identify representations that are conjugate. Thus it is also interesting to consider the *character variety*

$$\operatorname{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$$

where $\mathrm{PSL}(2,\mathbb{R})$ acts on the representation variety by conjugation. Also this quotient can be identified with the set of real points of a quasi-projective real algebraic variety.

2.2. Trace coordinates. It is natural to ask for a parametrization of the character variety. There are some easy conjugacy invariant functionals defined on the representation variety: they are trace functions $\mathrm{tr}_\gamma(\phi) := \mathrm{tr}(\phi(\gamma))$.
If we consider representations $\mathbb{Z} \longrightarrow \mathrm{SL}(2,\mathbb{R})$ then trace functions are already sufficient to determine the conjugacy class of non-trivial elements. It is a highly non-trivial result that trace functions generates the ring of polynomial invariant functions on the character variety (see [**Gol09**]).

2.3. Connected components. An important functional defined on the representation variety is given by the Euler number. It is continuous with respect to the strong topology and thus it is constant on the connected components. It turns out that the Euler number almost distinguishes the connected components of the representation variety. In fact we have the following theorem by Goldman :

Theorem 4.1 (Goldman [**Gol88**]). *The connected components of Hom* $\left(\Gamma, PSL^{(k)}(2,\mathbb{R})\right)$ *have the following description:*

- *if* $k \nmid 2g-2$ *then* $e^{-1}(n)$ *is a connected component for every* $|n| \leq \left\lfloor \frac{2g-2}{k} \right\rfloor$*;*
- *if* $k \mid 2g-2$ *then there are* $2k^{2g}$ *connected components on which the Euler number attains the value* $\pm\frac{2g-2}{k}$*. Furthermore they are distinguished by the* $2g-$*tuple of rotation numbers of a standard set of generators of* Γ*.*

3. The topological case $\mathrm{Homeo}_+\left(S^1\right)$

Now we work in the topological setting.

3.1. Definition and topology. Let us begin with a definition:

Definition 4.2. The *space of representations* of Γ into $G := \mathrm{Homeo}_+\left(S^1\right)$ is the set

$$\operatorname{Rep}(\Gamma) := \operatorname{Hom}\left(\Gamma, \mathrm{Homeo}_+\left(S^1\right)\right).$$

We give to $\operatorname{Rep}(\Gamma)$ a topology as follows: for any choice of a representation $\phi \in \operatorname{Rep}(\Gamma)$, a positive $\epsilon > 0$ and a finite set of elements $F = \{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$ define

$$U(\phi, \epsilon, F) := \left\{ \psi \in \operatorname{Rep}(\Gamma) \,\middle|\, \begin{array}{c} d_\infty(\psi(\gamma), \phi(\gamma)) < \epsilon \\ \text{for every } \gamma \in F \end{array} \right\}.$$

The collection

$$\{U(\phi, \epsilon, F)\}_{\epsilon > 0, F \subseteq \Gamma \text{ finite}}$$

forms a fundamental system of neighborhoods of ϕ for a topology on the representation space $\mathrm{Rep}(\Gamma)$. Given representations $\{\phi_n\}_{n\in\mathbb{N}}$ and ϕ we have the following characterization of the convergence with respect to the topology of $\mathrm{Rep}(\Gamma)$:

$$\phi_n \xrightarrow{\mathrm{Rep}} \phi \Longleftrightarrow \phi_n(\gamma) \xrightarrow{d_\infty} \phi(\gamma) \text{ in } G \text{ for every } \gamma\in\Gamma.$$

If we fix a set of generators $\{\gamma_i\}_{i\in I}$ for Γ it is easy to prove that

$$\phi_n(\gamma) \xrightarrow{d_\infty} \phi(\gamma) \text{ in } G \text{ for every } \gamma\in\Gamma \Longleftrightarrow \phi_n(\gamma_i) \xrightarrow{d_\infty} \phi(\gamma_i) \text{ in } G \text{ for every } i\in I$$

using the continuity of d_∞ and of the product operation on G.
The definition of $\mathrm{Rep}(\Gamma)$ given above has the advantage of being intrinsic. For computational purposes it will be more convenient to introduce the following description which relies on the choice of a set of generators for Γ. We have the following embedding result.

Lemma 4.3. *Let $\{\gamma_1,\dots,\gamma_n\}\subseteq\Gamma$ be a finite set of generators. Then the map*

$$J : Rep(\Gamma) \longrightarrow G^n \qquad , \qquad J(\phi) := (\phi(\gamma_1),\dots,\phi(\gamma_n))$$

is an embedding. If we choose standard generators $a_1,b_1,\dots,a_g,b_g$ for Γ then the image of J coincides with the subspace

$$\left\{ (A_1,B_1,\dots,A_g,B_g)\in G^{2g} \;\middle|\; \prod_{j=1}^{g}[A_j,B_j] = Id \right\}.$$

3.2. Character space. The group G acts on $\mathrm{Rep}(\Gamma)$ by conjugation. If we were interested in topological conjugacy classes we should have studied the quotient $\mathrm{Rep}(\Gamma)/G$. Instead we will consider the quotient $\overline{\mathrm{Rep}}(G)$ of $\mathrm{Rep}(\Gamma)$ with respect to the relation of semi-conjugacy which is a space less fine with respect to $\mathrm{Rep}(\Gamma)/G$, but is much more well behaved.

Definition 4.4. The *character space* of semi-conjugacy classes is defined by

$$\overline{\mathrm{Rep}}(G) := \mathrm{Rep}(\Gamma) \Big/ \begin{array}{c} \phi\sim\psi \\ \Longleftrightarrow \phi \text{ semi-conjugate to } \psi \end{array} .$$

4. Coordinates

Let us discuss some interesting natural functionals defined on $\mathrm{Rep}(\Gamma)$.

4.1. Rotation numbers and bounded cocycles. The most important functionals for our purposes are rotation numbers and bounded cocycles. They have the role of natural coordinates on the character space.

Definition 4.5. For every $\gamma\in\Gamma$ define the function

$$\mathrm{rot}_\gamma : \mathrm{Rep}(\Gamma) \longrightarrow \mathbb{R}/\mathbb{Z} \qquad , \qquad \mathrm{rot}_\gamma(\phi) := \mathrm{rot}(\phi(\gamma)).$$

Fix an auxiliary section $\sigma : G \longrightarrow \widetilde{G}$. For every $\alpha,\beta\in\Gamma$ define the function

$$\tau_{\alpha,\beta} : \mathrm{Rep}(\Gamma) \longrightarrow \mathbb{R} \qquad , \qquad \tau_{\alpha,\beta}(\phi) := \widetilde{\mathrm{rot}}\,\sigma(\phi(\alpha))\sigma(\phi(\beta)) - \widetilde{\mathrm{rot}}\,\sigma(\phi(\alpha)) - \widetilde{\mathrm{rot}}\,\sigma(\phi(\beta)).$$

We have already observed that $\tau_{\alpha,\beta}$ is independent of the section σ.
The first important property of these functionals is continuity:

LEMMA 4.6. *The functionals rot_γ and $\tau_{\alpha,\beta}$ are continuous for every $\alpha, \beta, \gamma \in \Gamma$.*

PROOF. It is an easy check that rot_γ is continuous since it is the composition of continuous maps: $\mathrm{rot}_\gamma = \mathrm{rot} \circ v_\gamma$ where $v_\gamma : \mathrm{Rep}\,(\Gamma) \longrightarrow G$ is the evaluation map $v_\gamma(\phi) := \phi(\gamma)$, and $\mathrm{rot} : G \longrightarrow \mathbb{R}/\mathbb{Z}$ is the rotation number.
Let us prove that $\tau_{\alpha,\beta}$ is continuous. Fix $\phi \in \mathrm{Rep}\,(\Gamma)$. Let v_α, v_β be the evaluation maps of α, β. Let $U_\alpha, U_\beta \subseteq G$ be simply connected neighborhoods of $v_\alpha(\phi), v_\beta(\phi)$ such that we can define a continuous section $\sigma : U_\alpha \cup U_\beta \longrightarrow \widetilde{G}$. By continuity of evaluations we can find a neighborhood W of ϕ such that $v_\alpha(W) \subseteq U_\alpha$ and $v_\beta(W) \subseteq U_\beta$. Then $\tau_{\alpha,\beta} = \widetilde{\mathrm{rot}}\ \sigma(v_\alpha(\bullet))\sigma(v_\beta(\bullet)) - \widetilde{\mathrm{rot}}\ \sigma(v_\alpha(\bullet)) - \widetilde{\mathrm{rot}}\ \sigma(v_\beta(\bullet))$ is clearly well-defined and continuous on W. $\square$

Theorems 1.50 and 1.37 guarantee that the functionals rot_γ and $\tau_{\alpha,\beta}$ pass to the quotient space being constant on the equivalence classes. Moreover it says that the map

$$S : \overline{\mathrm{Rep}}\,(G) \longrightarrow (\mathbb{R}/\mathbb{Z})^{2g} \times \mathbb{R}^{\Gamma\times\Gamma}$$

defined by sending a class X_ϕ to its invariants

$$\left(\mathrm{rot}_{a_1}(\phi), \mathrm{rot}_{b_1}(\phi), \ldots \mathrm{rot}_{a_g}(\phi), \mathrm{rot}_{b_g}(\phi)\right) \in (\mathbb{R}/\mathbb{Z})^{2g} \quad , \quad \{\tau_{\alpha,\beta}(\phi)\}_{\alpha,\beta\in\Gamma} \in \mathbb{R}^{\Gamma\times\Gamma}$$

is continuous and injective (we put on $\mathbb{R}^{\Gamma\times\Gamma}$ the weak topology which makes continuous the finite dimensional projections).

4.2. Euler number. Another important functional on $\mathrm{Rep}\,(\Gamma)$ is given by the Euler number map which we now define.

DEFINITION 4.7. The Euler number function

$$e : \mathrm{Rep}\,(\Gamma) \longrightarrow \mathbb{Z}$$

is the map assigning to ϕ its Euler number $e(\phi) := \langle \phi^* e, [S] \rangle$.

Again it is easy to prove continuity using Milnor's formula:

LEMMA 4.8. *The functional e is continuous.*

PROOF. Let $\sigma : G \longrightarrow \widetilde{G}$ be an auxiliary section. For every $a, b \in \Gamma$ the element $[\sigma(a), \sigma(b)]$ does not depend on the particular choice of σ. As before we can choose locally on ϕ continuous determinations of $\sigma(v_a(\bullet)), \sigma(v_b(\bullet))$. Thus it is clear that the function $[\sigma(v_a(\bullet)), \sigma(v_b(\bullet))]$ is continuous.
By Milnor's Theorem 2.13 we have

$$e(\phi) = \widetilde{\mathrm{rot}} \left(\prod_{j=1}^{g} [\sigma(\phi(a_j), \sigma(b_j)] \right).$$

From this formula we see that e is the composition of continuous functions. $\square$

Since $e : \mathrm{Rep}\,(\Gamma) \longrightarrow \mathbb{R}$ is continuous but attains values only in a discrete set, it has to be constant on the connected components of the representation space.

4.3. Translation numbers of lifted commutators. The following functionals will play a prominent role in the next sections. Recall that there is a well-defined continuous lifted commutator function

$$L : G \times G \longrightarrow \widetilde{G}$$

sending $L\,(f,g) := \left[\widetilde{f},\widetilde{g}\right]$ where $\widetilde{f},\widetilde{g}$ are arbitrary lifts of f,g.

If S is a closed surface of genus $g \geq 2$ and $\phi : \Gamma := \pi_1(S) \longrightarrow G$ is any representation, then we recall that by Milnor's Formula 2.13 the Euler number is computed by

$$e(\phi) = \widetilde{\mathrm{rot}}\left(\prod_{j=1}^{g}\left[\widetilde{\phi}(a_j),\widetilde{\phi}(b_j)\right]\right)$$

where $\{a_1, b_1, \ldots, a_g, b_g\}$ is a standard set of generators for $\pi_1(S)$. For every $j \leq g$ define $c_j(\phi) := \left[\widetilde{\phi}(a_j),\widetilde{\phi}(b_j)\right]$. We have

$$\begin{aligned}
&e(\phi) = \widetilde{\mathrm{rot}}\,(c_1(\phi)\ldots c_g(\phi))\\
&= \widetilde{\mathrm{rot}}\ c_g(\phi)\,(c_1(\phi)\ldots c_g(\phi))\,c_g^{-1}(\phi) = \widetilde{\mathrm{rot}}\,(c_g(\phi)c_1(\phi)\ldots c_{g-1}(\phi))\\
&= \ldots\\
&= \widetilde{\mathrm{rot}}\ c_j\,(c_{j+1}(\phi)\ldots c_j(\phi))\,c_j^{-1} = \widetilde{\mathrm{rot}}\,(c_j(\phi)c_{j+1}(\phi)\ldots c_{j-1}(\phi))\\
&= \widetilde{\mathrm{rot}}\ c_j(\phi) + \widetilde{\mathrm{rot}}\,(c_{j+1}(\phi)\ldots c_{j-1}(\phi))\,.
\end{aligned}$$

We give a name to the summands:

Definition 4.9. The $j-$commutator traslation number function is defined by

$$r_j : \mathrm{Rep}\,(\Gamma) \longrightarrow \mathbb{R} \qquad , \qquad r_j(\phi) := \widetilde{\mathrm{rot}}\ c_j(\phi).$$

Definition 4.10. The $j-$complement translation number function is defined by

$$R_j : \mathrm{Rep}\,(\Gamma) \longrightarrow \mathbb{R} \qquad , \qquad R_j(\phi) := \widetilde{\mathrm{rot}}\,(c_{j+1}(\phi)\ldots c_{j-1}(\phi))\,.$$

With these notations $e = R_j + r_j$ for every j.
These functionals are continuous as they are compositions of continuous functions.

5. Topology of semi-conjugacy classes

Let $\phi \in \mathrm{Rep}\,(\Gamma)$ be a representation. We denote by

$$X_\phi := \left\{\psi \in \mathrm{Rep}\,(\Gamma)\ \middle|\ \begin{array}{c}\psi \text{ semi-conjugate to } \phi\\ \psi h_1 = h_1\phi \quad , \quad h_2\psi = \phi h_2\end{array}\right\} \subseteq \mathrm{Rep}\,(\Gamma)$$

the semi-conjugacy class of ϕ.

We describe some general properties of these subspaces.

Lemma 4.11. *The subspace X_ϕ is path-connected.*

Proof. Let $\psi \in X_\phi$ be a representation. By Calegari's Theorem 1.44 and Theorem 1.45 there exists $\rho \in \mathrm{Rep}\,(\Gamma)$ which is left-monotone-equivalent to both ϕ, ψ through non-increasing continuous maps of Brouwer-Hopf degree 1 h_0, h_1 respectively. Consider a path of homeomorphisms $f_t : [0,1) \to \mathrm{Homeo}_+\left(S^1\right)$ such that $f_0 = \mathrm{Id}$ and $f_t \xrightarrow{t\to 1} h_0$ uniformly (see Lemma 1.49) and set $\rho_t := f_t \rho f_t^{-1}$. By construction $\rho_0 = \rho$ and $\rho_t \in X_\phi$ for every $t \in [0,1)$.

The path of representations $\rho_t : [0,1) \longrightarrow X_\phi$ extends in $t = 1$ and connects ρ to ψ. We need to prove that for every $\gamma \in \pi_1(S_g)$ the homeomorphisms $\phi_t(\gamma)$ uniformly converge to $\psi(\gamma)$:

$$\begin{aligned}
&||\psi(\gamma) - \rho_t(\gamma)||_\infty = \left|\left|\psi(\gamma) - f_t^{-1}\phi(\gamma)f_t^{-1}\right|\right|_\infty \\
&= \left|\left|(\psi(\gamma)f_t - f_t\rho(\gamma))f_t^{-1}\right|\right|_\infty = ||\psi(\gamma)f_t - f_t\rho(\gamma)||_\infty \quad \text{since } f_t^{-1} \text{ is surjective} \\
&||\psi(\gamma)f_t - f_t\rho(\gamma)||_\infty \xrightarrow{t\to 1} ||\psi(\gamma)h_0 - h_0\rho(\gamma)||_\infty = 0 \quad \text{since } ||f_t - h_0||_\infty \to 0.
\end{aligned}$$

Analogously we can connect ρ to ϕ in X_ϕ. □

Lemma 4.12. *The subspace X_ϕ is closed.*

Using the technology we developed so far we can give a short proof in coordinates:

Proof. We can identify X_ϕ with the following space:

$$X_\phi = \left(\bigcap_{\alpha,\beta\in\Gamma} \{\tau_{\alpha,\beta} = \tau_{\alpha,\beta}(\phi)\}\right) \cap \left(\bigcap_{\gamma\in\Gamma} \{\mathrm{rot}_\gamma = \mathrm{rot}_\gamma(\phi)\}\right).$$

This follows from Theorem 1.37. The sets

$$\{\tau_{\alpha,\beta} = \tau_{\alpha,\beta}(\phi)\} \qquad , \qquad \{\mathrm{rot}_\gamma = \mathrm{rot}_\gamma(\phi)\}$$

are the fibers of continuous maps (see Lemma 4.6) hence they are closed. □

6. Local stability

In this section we introduce locally stable representations. We give some criteria for local stability and provide an examples of a locally stable representation.

Definition 4.13. A representation $\phi \in \mathrm{Rep}\,(\Gamma)$ is *locally stable* if there exists a neighborhood U of ϕ such that every $\psi \in U$ is semi-conjugate to ϕ.

6.1. Properties of local stability.

Local stability is generically preserved under semi-conjugacy as stated in the next proposition:

Proposition 4.14. *Let $\phi \in Rep\,(\Gamma)$ be a locally stable representation. Then there is an open subset $U \subseteq Rep\,(\Gamma)$ such that U is a dense subset of X_ϕ and U is made of locally stable representations.*

Proof. Let U be a neighborhood of ϕ made entirely of representations which are semi-conjugate to ϕ. We prove the result by small steps.

First we show that every $\psi \in \mathrm{Rep}\,(\Gamma)$ topologically conjugate to ϕ is locally stable: suppose

that $\psi = h\phi h^{-1}$, then hUh^{-1} is a neighborhood of ψ such that every $\rho \in hUh^{-1}$ is semi-conjugate to ρ.
Now we prove that every representation $\psi \in \mathrm{Rep}(\Gamma)$ such that ϕ is left-monotone equivalent to ψ is locally stable. Suppose that $h\psi = \phi h$ for some monotone equivalence h. Then by Lemma 1.49 there are representations conjugate to ψ which are arbitrarily close to ϕ, in particular we can find one of these representations ρ in the neighborhood U. Hence ψ is topologically conjugate to a locally stable representation $\rho \in U$.
We are now ready to treat the general case. Let ψ be semi-conjugate to ϕ. By Calegari's Theorem 1.44 there exists a third representation ρ such that ψ, ϕ are both left-monotone equivalent to ρ i.e. $\psi h_1 = h_1\rho$ and $\phi h_2 = h_2\rho$ for some monotone equivalences h_1, h_2. Using the previous discussion we see that ρ is locally stable since ϕ is left-monotone equivalent to ρ. By Lemma 1.49 we can find arbitrarily near to ψ representations which are conjugate to ρ (so they are locally stable). Define

$$X_\phi^s := \{\psi \in X_\phi \,|\psi \text{ locally stable }\}.$$

The previous argument shows that X_ϕ^s is dense in X_ϕ, on the other hand being locally stable is an open property so X_ϕ^s is an open subset. □

If we had $X_\phi = X_\phi^s$ then we could prove that X_ϕ is a connected component which is also path-connected: in fact X_ϕ, which is a path-connected set, would be simultaneously open and closed. The above condition is surely satisfied if ϕ is minimal as stated in the following:

PROPOSITION 4.15. *Let $\phi \in Rep(\Gamma)$ be a locally stable minimal representation. Then $X_\phi \subseteq Rep(\Gamma)$ is a connected component.*

PROOF. Since having a finite orbit is preserved by semi-conjugacy every representation $\psi \in X_\phi$ does not have any finite orbit. Thus every $\psi \in X_\phi$ is right-monotone equivalent to a minimal action $\overline{\psi}$, or $h\overline{\psi} = \psi h$. Looking at the proof of Proposition 4.14 we see that if $\overline{\psi}$ is locally stable then ψ is locally stable. Local stability for $\overline{\psi}$ follows from the fact that $\overline{\psi}$ is semi-conjugate to ψ and semi-conjugacy equals topological conjugacy for minimal actions (Lemma 1.46). □

6.2. Criteria for local stability. By Theorem 1.37 locally stable representations $\phi \in \mathrm{Rep}(\Gamma)$ are precisely those representations such that the functionals rot_γ and $\tau_{\alpha,\beta}$ are constant in a neighborhood U of ϕ.

LEMMA 4.16. *Let $C \subseteq Rep(\Gamma)$ be a connected component. If for every $\gamma \in \Gamma$ the function*

$$rot_\gamma : C \longrightarrow \mathbb{R}/\mathbb{Z}$$

is constant, then C is a single semi-conjugacy class. In particular C is path-connected.

PROOF. By Lemma 4.11 semi-conjugate representations lie in the same connected component of $\mathrm{Rep}(\Gamma)$, so we are left to prove that every pair ϕ, ψ of representations in C are semi-conjugate. By Theorem 1.37 it is enough to show that $\mathrm{rot}_\gamma(\phi) = \mathrm{rot}_\gamma(\psi)$ as γ varies in a set of generators of Γ and $\tau_{\alpha,\beta}(\phi) = \tau_{\alpha,\beta}(\psi)$ for every $\alpha, \beta \in \Gamma$ where τ is the canonical real $2-$cocycle representing $e_b^{\mathbb{R}}$.

By assumption rot_γ is constant on C for every $\gamma \in \Gamma$ so the first condition clearly holds.
Proof that $\phi^*\tau = \psi^*\tau$: we recall that

$$\tau(f,g) = \widetilde{\mathrm{rot}}\ \widetilde{f}\widetilde{g} - \widetilde{\mathrm{rot}}\ \widetilde{f} - \widetilde{\mathrm{rot}}\ \widetilde{g}$$

Fix $\alpha, \beta \in \Gamma$ and consider the composition of the function $d_{\alpha,\beta} : C \times C \to \mathbb{R}$ given by

$$d_{\alpha,\beta}(\phi,\psi) = \phi^*\tau(\alpha,\beta) - \psi^*\tau(\alpha,\beta)$$

with the projection $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$. We have

$$\begin{aligned}
&\pi d_{\alpha,\beta}(\phi,\psi)\\
&= \pi\left(\widetilde{\mathrm{rot}}\ \widetilde{\phi}(\alpha)\widetilde{\phi}(\beta) - \widetilde{\mathrm{rot}}\ \widetilde{\phi}(\alpha) - \widetilde{\mathrm{rot}}\ \widetilde{\phi}(\beta) - \left(\widetilde{\mathrm{rot}}\ \widetilde{\psi}(\alpha)\widetilde{\psi}(\beta) - \widetilde{\mathrm{rot}}\ \widetilde{\psi}(\alpha) - \widetilde{\mathrm{rot}}\ \widetilde{\psi}(\beta)\right)\right)\\
&= \mathrm{rot}\ \phi(\alpha)\phi(\beta) - \mathrm{rot}\ \phi(\alpha) - \mathrm{rot}\ \phi(\beta) - (\mathrm{rot}\ \psi(\alpha)\psi(\beta) - \mathrm{rot}\ \psi(\alpha) - \mathrm{rot}\ \psi(\beta))\\
&= [\mathrm{rot}_{\alpha\beta}(\phi) - \mathrm{rot}_{\alpha\beta}(\psi)] + [\mathrm{rot}_\alpha(\phi) - \mathrm{rot}_\alpha(\psi)] + [\mathrm{rot}_\beta(\phi) - \mathrm{rot}_\beta(\psi)]\\
&= 0
\end{aligned}$$

in particular $d_{\alpha,\beta} : C \times C \to \mathbb{R}$ has image in $\mathbb{Z}$. By the connectedness of $C \times C$ we conclude that d is constant and that the value it assumes is precisely $d_{\alpha,\beta}(\phi,\phi) = 0$.
Since $\alpha, \beta \in \Gamma$ are arbitrary, we conclude that $\phi^*\tau = \psi^*\tau$ for every $\phi, \psi \in C$. □

The viceversa is clear: if a connected component consists of a single semi-conjugacy class then the functionals rot_γ are constant on it since they are semi-conjugacy invariant.

6.3. Local stability for maximal representations. The first example of locally stable representations is given by maximal representations:

THEOREM 4.17. *Let $\phi \in Rep(\Gamma)$ be a maximal representation. Then ϕ is locally stable. In particular the connected components of the subset of maximal representations $e^{-1}(2g-2) \subseteq Rep(\Gamma)$ are semi-conjugacy classes.*

PROOF. Since the function e is continuous, the fiber $e^{-1}(2g-2)$ is open and closed at the same time so it is a union of connected components of $\mathrm{Rep}(\Gamma)$. By Corollary 3.8 $\mathrm{rot}_\gamma \equiv 0$ on $e^{-1}(2g-2)$ for every $\gamma \in \Gamma$ hence we can apply Lemma 4.16. □

Using a finer argument it is possible to show that there is more connectedness:

THEOREM 4.18 (Matsumoto [**Mat87**], Burger [**Bur11**], Iozzi [**Ioz02**]). *The subset of maximal representations $e^{-1}(2g-2)$ is path-connected and consists of a single semi-conjugacy class.*

CHAPTER 5

Stability phenomena for geometric representations

1. Overview

The main goal of this chapter is the proof of some stability phenomena for representations that look like $\mathrm{PSL}^{(k)}(2,\mathbb{R})$ −maximal representations. First we introduce the class of geometric representations and relate it to the class of $\mathrm{PSL}^{(k)}(2,\mathbb{R})$ −maximal ones. Then we give a finer analysis of the dynamics of a $\mathrm{PSL}^{(k)}(2,\mathbb{R})$ −geometric representation and locate some common dynamical properties shared by all geometric representations. This description suggests to consider the set of all representations that exhibit these properties (representations with good fixed point sets and good representations) and try to study its deformation space . The technical tool to carry out the analysis is the Calegari-Walker Algorithm which produces an upper bound for the rotation number of a product of homeomorphisms of the circle given as input the combinatorial data of the dynamics of periodic orbits of each homeomorphism. It provides the following fundamental results:

THEOREM. *For every $a, b \in \widetilde{Homeo}_+\left(S^1\right)$ we have the following*

(1) If $\widetilde{rot}(a) \notin \mathbb{Q}$ or $rot(b) \notin \mathbb{Q}$ then $\widetilde{rot}\,[a,b] = 0$.
(2) If $\widetilde{rot}(a) = \frac{p}{q}$ or $rot(b) = \frac{p}{q}$ then $\left|\widetilde{rot}\,[a,b]\right| \leq \frac{1}{q}$. Furthermore, if $a, b \in \widetilde{Homeo}_+\left(S^1\right)$ realize the maximal translation number for $[a,b]$, then they have periodic orbits with prescribed combinatorial structure.

THEOREM. *Let $c_1, \ldots, c_n \in \widetilde{Homeo}_+\left(S^1\right)$ be homeomorphisms such that*

(i) $\widetilde{rot}\ c_j = \frac{1}{k}$, and every c_j has an orbit $X_j = \left\{x_j^i\right\}_{i\in\mathbb{Z}}$ periodic modulo $\mathbb{Z}$ such that $c_j(x_j^i) = x_j^{i+1}$ and $x_j^{i+k} = x_j^i + 1$;
(ii) the set $X = \bigcup_{j=1}^n X_j$ can be ordered as

$$\cdots \leq x_1^i \leq x_2^i \leq \cdots \leq x_n^i \leq x_1^{i+1} \leq \ldots$$

Then the following holds

(1) $\widetilde{rot}\,(c_1 \ldots c_n) \leq \frac{2n-1}{k}$.
(2) If $\gcd(k, 2n-1) = 1$ and equality holds in (ii) *between two consecutive points in X then $\widetilde{rot}\,(c_1 \ldots c_n) < \frac{2n-1}{k}$. In particular if we have equality X has a prescribed combinatorial structure.*

We will apply these results to the class of good representations.

2. Geometric representations

In this section we describe geometric representations and their dynamics on the circle.

DEFINITION 5.1. A representation $\phi \in \mathrm{Rep}(\Gamma)$ is *geometric* if it is faithful and has discrete image $\phi(\Gamma)$ contained in a transitive Lie subgroup of G.

By Theorem 1.5, up to conjugacy, the only transitive Lie subgroups of $G := \mathrm{Homeo}_+\left(S^1\right)$ are:

$$S^1, \qquad \mathrm{PSL}^{(k)}(2,\mathbb{R}).$$

Let us briefly describe representations into these subgroups.

2.1. Representations in S^1. No representation $\Gamma \longrightarrow S^1$ can be faithful since S^1 is abelian. The study of these representations is not very interesting as

$$\mathrm{Hom}\left(\Gamma, S^1\right) = \mathrm{Hom}\left(\frac{\Gamma}{[\Gamma,\Gamma]}, S^1\right) = \mathrm{Hom}\left(\mathbb{Z}^{2g}, S^1\right) = \left(S^1\right)^{2g}.$$

Representations $\Gamma \longrightarrow S^1$ are essentially classified up to semi-conjugacy by the rotation numbers of the generators $a_1, b_1, \ldots, a_g, b_g \in \Gamma$. This follows from Lemma 1.33 and the fact that the real bounded Euler class of $\phi : \mathbb{Z}^{2g} \longrightarrow G$ always vanishes as $H_b^2\left(\mathbb{Z}^{2g}, \mathbb{R}\right) = 0$ by amenability of $\mathbb{Z}^{2g}$.

2.2. Representations in $\mathbf{PSL}^{(k)}(2,\mathbb{R})$. Far more interesting is the case of $\mathrm{PSL}(2,\mathbb{R})$. Every hyperbolic structure on S gives rise to a holonomy representation $\phi : \Gamma \longrightarrow \mathrm{PSL}(2,\mathbb{R}) = \mathrm{Isom}_+\left(\mathbb{H}^2\right)$ which is faithful and has discrete image. The lifts of these representations to the finite sheeted covering $\pi_k : \mathrm{PSL}^{(k)}(2,\mathbb{R}) \longrightarrow \mathrm{PSL}(2,\mathbb{R})$ when $k \mid 2g-2$ give examples of geometric representations into $\mathrm{PSL}^{(k)}(2,\mathbb{R})$.

LEMMA 5.2. *Let $\phi : \Gamma \longrightarrow PSL^{(k)}(2,\mathbb{R})$ be a representation. Let $\pi_k : PSL^{(k)}(2,\mathbb{R}) \longrightarrow PSL(2,\mathbb{R})$ be the degree k covering of $PSL(2,\mathbb{R})$. Then*

$$\pi_k\phi \text{ is geometric } \Longleftrightarrow \phi \text{ is geometric.}$$

PROOF. Let us prove the double implication for injectivity. Assume that ϕ is injective and suppose that $\pi_k(\phi(\gamma)) = 1$ for some $\gamma \in \Gamma$. Then $\phi(\gamma) \in \ker \pi_k = \mathbb{Z}/k\mathbb{Z}$ and therefore $\phi(\gamma^k) = \phi(\gamma)^k = 1$. By injectivity of ϕ we get $\gamma^k = 1$ in Γ. We conclude that $\gamma = 1$ by observing that Γ has no torsion elements. The converse statement is clear: if a composition of two maps is injective then the inner one is injective.

Now we prove the double implication for dicreteness. Assume that $\phi(\Gamma)$ is discrete. In order to prove that $\pi_k\phi(\Gamma)$ is discrete we only need to show that $1 \in \pi_k\phi(\Gamma)$ is an isolated point. Proceed by contradiction and let $\{\pi_k\phi(\gamma_n)\}_{n\in\mathbb{N}}$ be a sequence (containing infinite distinct terms) converging to $\pi_k\phi(\gamma_n) \longrightarrow 1$. Since $\ker \pi_k = \mathbb{Z}/k\mathbb{Z}$ is finite, up to passing to a subsequence we may assume that $\phi(\gamma_n) \longrightarrow \delta \in \ker \pi_k$ (preserving the fact that $\{\pi_k\phi(\gamma_n)\}_{n\in\mathbb{N}}$ contains infinite distinct terms). By discreteness of $\phi(\Gamma)$ the limit point δ lies in $\phi(\Gamma)$ (the image is closed). Again by discreteness this implies that $\phi(\gamma_n)$ definitely stabilize. This contradicts the fact that the sequence $\{\pi_k\phi(\gamma_n)\}_{n\in\mathbb{N}}$ contains infinitely many distinct elements.

The converse implication is again easier. The map $\pi_k : \mathrm{PSL}^{(k)}(2,\mathbb{R}) \longrightarrow \mathrm{PSL}(2,\mathbb{R})$ is a covering, so if $S \subseteq \mathrm{PSL}(2,\mathbb{R})$ is a discrete subset then $\pi_k^{-1}S$ is again discrete. Thus $\pi_k\phi(\Gamma)$, being a subset of the discrete set $\pi_k^{-1}(\pi_k\phi(\Gamma))$, has the discrete topology. In particular $1 \in \phi(\Gamma)$ is an isolated point which implies that $\phi(\Gamma)$ is discrete (it is closed). $\square$

We recall that by Goldman's Theorem a representation $\phi : \Gamma \longrightarrow \mathrm{PSL}(2,\mathbb{R})$ is geometric if and only if it is maximal $|e(\phi)| = 2g-2$. Hence by Lemma 5.2 we obtain

Corollary 5.3. *Let $\phi : \Gamma \longrightarrow PSL^{(k)}(2,\mathbb{R})$ be a representation. Suppose that $k \mid 2g-2$, then*

$$\phi \textit{ is geometric } \iff |e(\phi)| = \frac{2g-2}{k}.$$

The following is a quite formal consequence of Proposition 3.28.

Proposition 5.4. *Let $\rho : \pi_1(S) \to PSL^{(k)}(2,\mathbb{R})$ be a maximal representation. Then*

$$\rho^*\tau = \frac{1}{k}\theta.$$

Suppose that $k \mid 2g-2$. By Proposition 5.4 the canonical Euler cocycle of a maximal $\mathrm{PSL}^{(k)}(2,\mathbb{R})$ representation ϕ is

$$\phi^*\tau = \frac{1}{k}\theta$$

where θ is the canonical Euler cocycle of maximal $\mathrm{PSL}(2,\mathbb{R})$ representations. Thus, by Theorem 1.37, the semi-conjugacy class of a maximal $\mathrm{PSL}^{(k)}(2,\mathbb{R})$ representation ϕ is completely detected by the $2g-$tuple

$$\left(\mathrm{rot}_{a_1}(\phi), \mathrm{rot}_{b_1}(\phi), \ldots, \mathrm{rot}_{a_g}(\phi), \mathrm{rot}_{b_g}(\phi)\right).$$

Under the assumption $k \mid 2g-2$, every $2g-$tuple $\left(\mathrm{rot}_{a_1}(\phi), \mathrm{rot}_{b_1}(\phi), \ldots, \mathrm{rot}_{a_g}(\phi), \mathrm{rot}_{b_g}(\phi)\right) \in (\mathbb{R}/\mathbb{Z})^{2g}$ with $\mathrm{rot}_{a_j}(\phi), \mathrm{rot}_{b_j}(\phi) \in \left\{0, \frac{1}{k}, \ldots, \frac{k-1}{k}\right\}$ is realized by some maximal representation as follows from the next lemma:

Lemma 5.5. *Let $\phi \in Hom(\Gamma, PSL(2,\mathbb{R}))$ be a maximal representation and suppose $k \mid 2g-2$. Let $\overline{\phi}(a_j), \overline{\phi}(b_j)$ be arbitrary lifts of $\phi(a_j), \phi(b_j)$ to $PSL^{(k)}(2,\mathbb{R})$. Then the assignation $\psi(a_j) := \overline{\phi}(a_j)$ extends to a maximal representation $\psi : \Gamma \longrightarrow PSL^{(k)}(2,\mathbb{R})$.*

Proof. Fix lifts $\sigma(\phi(a_j)), \sigma(\phi(b_j))$ of $\phi(a_j), \phi(b_j)$ to $\widetilde{G}$ and denote by $c_k : \widetilde{G} \longrightarrow \widetilde{G}$ the homomorphism given by conjugation $c_k(f) := \mu_{\frac{1}{k}} f \mu_k$ where $\mu_\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ is the multiplication by α defined by $\mu_\alpha(x) := \alpha x$. We can choose as lifts of $\overline{\phi}(a_j), \overline{\phi}(b_j)$ the homeomorphisms $\sigma\left(\overline{\phi}(a_j)\right) := c_k\sigma(\phi(a_j))$ and $\sigma\left(\overline{\phi}(b_j)\right) := c_k\sigma(\phi(a_j))$ (as we have already seen in Lemma 3.7). Thus

$$\prod_{j=1}^{g}\left[\sigma\left(\overline{\phi}(a_j)\right), \sigma\left(\overline{\phi}(b_j)\right)\right] = c_k\left(\prod_{j=1}^{g}\left[\sigma\left(\phi(a_j)\right), \sigma\left(\phi(b_j)\right)\right]\right) = c_k\left(\tau_{e(\phi)}\right) = \tau_{\frac{e(\phi)}{k}}.$$

Since $k \mid e(\phi) = 2g-2$ the last translation is an integral translation. In particular

$$\prod_{j=1}^{g}\left[\overline{\phi}(a_j), \overline{\phi}(b_j)\right] = p\left(\prod_{j=1}^{g}\left[\sigma\left(\overline{\phi}(a_j)\right), \sigma\left(\overline{\phi}(b_j)\right)\right]\right) = p\left(\tau_{\frac{2g-2}{k}}\right) = \mathrm{Id}$$

where $p : \widetilde{G} \longrightarrow G$ denotes the universal covering projection. The last equality tells us that ψ gives rise to a representation into $\mathrm{PSL}^{(k)}(2,\mathbb{R})$. Maximality of ψ is clear from the computations above. $\square$

Throughout the rest of the chapter we will work under the assumption $k \mid 2g-2$.
The last property of geometric representations on which we want to put a stress is the following:

PROPOSITION 5.6. *Let $\phi : \Gamma \longrightarrow PSL^{(k)}(2,\mathbb{R})$ be a geometric representation. Then ϕ is minimal.*

2.3. Geometric representations: dynamics of generators. By Lemma 5.2 we know that $\mathrm{PSL}^{(k)}(2,\mathbb{R})-$geometric representations are precisely the lifts to $\mathrm{PSL}^{(k)}(2,\mathbb{R})$ of geometric $\mathrm{PSL}(2,\mathbb{R})-$representations, i.e. holonomies of hyperbolic structures over S. Moreover every choice of rotation numbers for generators in the set $\left\{0, \frac{1}{k}, \dots, \frac{k-1}{k}\right\}$ is realized by some geometric representation into $\mathrm{PSL}^{(k)}(2,\mathbb{R})$. Let $\psi : \Gamma \longrightarrow \mathrm{PSL}^{(k)}(2,\mathbb{R})$ be a geometric representation that lift $\phi : \Gamma \longrightarrow \mathrm{PSL}(2,\mathbb{R})$. Fix a standard set of generators $\{a_1, b_1, \dots, a_g, b_g\}$ for $\pi_1(S)$.
The element $\psi(a_j)$ is a lift of a hyperbolic motion $\phi(a_j)$. Suppose that $\mathrm{rot}(\psi(a_j)) = 0$. Since every hyperbolic isometry has a pair of fixed points on $S^1 = \partial\mathbb{H}^2$ the homeomorphism $\psi(a_j)$ has $2k$ fixed points which are precisely the lifts of the fixed points of $\phi(a_j)$ to the degree k cover of S^1. Since the fixed points of $\phi(a_j)$ are a source and a sink, their $2k$ lifts are naturally divided into k sinks and k sources for $\psi(a_j)$ and they are disposed in an alternating way. See the picture in the center in Figure 1, where every A_j and R_j denotes an attractive or repulsive fixed point for $\psi(a_1)$. The picture on the right shows the dynamics for the lift of $\phi(a_1)$ satisfying $\mathrm{rot}(\psi(a_1)) = \frac{1}{3}$).
Since we are looking for properties that make a representation similar to a maximal $\mathrm{PSL}^{(k)}(2,\mathbb{R})-$maximal one we gather the previous descriptions in a precise definition:

DEFINITION 5.7. Let $\phi \in \mathrm{Rep}(\Gamma)$ be a representation. We say that ϕ has a *$j-$good fixed set* if $\phi(a_j)$ has a fixed point set $X_1(\phi) \subseteq \mathrm{Fix}(\phi(a_j))$ such that

(1) The fixed point set $X_1(\phi)$ has $2k$ points. Denote by $\left\{x_1^i(\phi)\right\}_{i\in\mathbb{Z}} \subseteq \mathbb{R}$ the lift of $X_1(\phi)$ to $\mathbb{R}$.
(2) If we consider $X_2(\phi) := \phi(b_j)X_1(\phi) \subseteq \mathrm{Fix}\left(\phi(b_j a_j^{-1} b_j^{-1})\right)$ and call $\left\{x_2^i(\phi)\right\}_{i\in\mathbb{Z}} \subseteq \mathbb{R}$ the lift of $X_2(\phi)$ to $\mathbb{R}$ then the set $X_1(\phi) \cup X_2(\phi)$ can be ordered as follows:
$$\dots < x_1^0(\phi) < x_1^1(\phi) < x_2^0(\phi) < x_2^1(\phi) < x_1^2(\phi) < x_1^3(\phi) < \dots$$
(3) With respect to the previous ordering the lift $\widetilde{\phi}(b_j)$ acts as follows:
$$\widetilde{\phi}(b_j)\left(x_1^{2i}\right) = x_2^{2i+2n-1} \qquad , \qquad \widetilde{\phi}(b_j)\left(x_1^{2i+1}\right) = x_2^{2i+2n}.$$

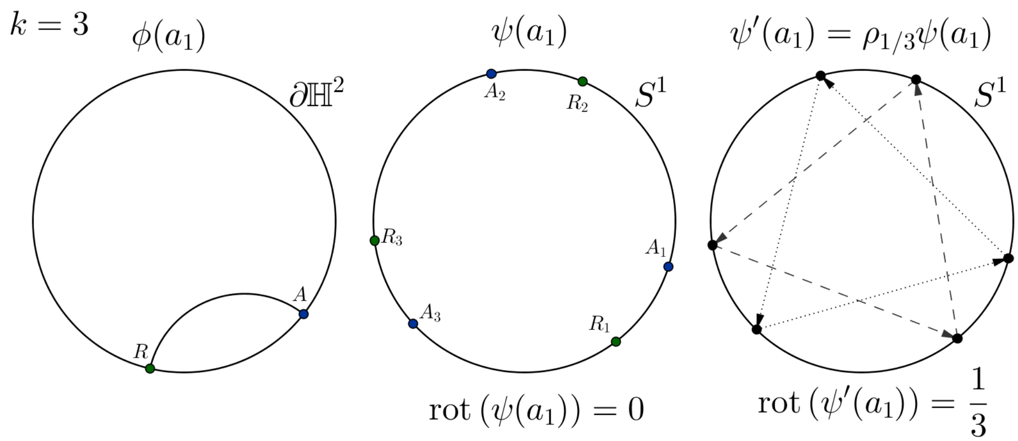

FIGURE 1.

Denote by N_j the closure in $\mathrm{Rep}\,(\Gamma)$ of the set of representations with $j-$good fixed sets

$$N_j := \overline{\{\phi \in \mathrm{Rep}\,(\Gamma)\,|\phi \text{ has } j-\text{good fixed set }\}}.$$

Requirements *(2)* and *(3)* will be clear from the discussion about maximal rotation numbers of commutators given in the following sections, in particular from the proof of Lemma 5.17.

Now we represent S as a quotient of a fundamental $4g-$regular hyperbolic polygon (see Figure 2, where for simplicity geodesics are pictured as straight lines and segments, this is actually the real picture of the configuration in Klein model of $\mathbb{H}^2$).
Consider the commutator $[\phi(a_j), \phi(b_j)]$, it is a hyperbolic motion with two fixed point on $\partial\mathbb{H}^2$. Its axis corresponds to the geodesic line passing through the extrema of the broken segment $a_j b_j a_j^{-1} b_j^{-1}$ (this geodesic bounds a hyperbolic pentagon together with the segments $a_j, b_j, a_j^{-1}, b_j^{-1}$, see Figure 2). The fixed points $A_1, R_1, \dots, A_g, R_g$ of the commutators $[\phi(a_1), \phi(b_1)], \dots, [\phi(a_g), \phi(b_g)]$ can be arranged on the circle in two alternating sets of g attractive fixed points $A_1 < A_2 < \dots < A_g$ and g repelling fixed points $R_1 < R_2 < \dots < R_g$ (see Figure 2).

Every fixed point A_j lifts to a periodic orbit $\mathcal{A}_j := \left\{A_j^i\right\}_{i=1}^k$ for $[\psi(a_j), \psi(b_j)]$ when we lift $\phi(a_j)$ to $\psi(a_j) \in \mathrm{PSL}^{(k)}\,(2,\mathbb{R})$. We note that $\mathrm{rot}\,[\psi(a_j), \psi(b_j)] = \frac{1}{k}$ since $\widetilde{\mathrm{rot}}\left[\widetilde{\psi}(a_j), \widetilde{\psi}(b_j)\right] = \frac{1}{k}$ by Proposition 3.7. Thus if we choose the lifts $\psi(a_j)$ of $\phi(a_j)$ in a way such that $\mathrm{rot}\,(\psi(a_j)) = 0$ for every $j \le g$, then $[\psi(a_j), \psi(b_j)]$ will act on $\mathcal{A}_j$ as $[\psi(a_j), \psi(b_j)]\,A_j^i = A_j^{i+1}$ (where indices are thought modulo k). Moreover it is immediate to check (see Figure 2) that the sets $\mathcal{A}_1, \dots, \mathcal{A}_g$ can be ordered as follows:

$$\dots < A_g^{j-1} < A_1^j < A_2^j < \dots < A_g^j < A_1^{j+1} < \dots$$

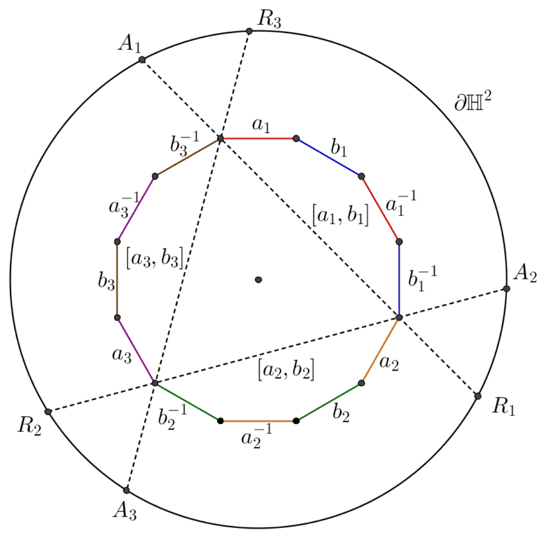

FIGURE 2.

Again we summarize these properties in a definition for representations that look like $\mathrm{PSL}^{(k)}(2,\mathbb{R})$ −maximal ones:

DEFINITION 5.8. Let $\phi \in \mathrm{Rep}(\Gamma)$ be a representation. We say that ϕ is *good* if

(1) $\widetilde{\mathrm{rot}}_{[a_j,b_j]}(\phi) = \widetilde{\mathrm{rot}}\left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)\right] = \frac{1}{k}$ for every $j \leq g-1$.

(2) There are periodic orbits (modulo $\mathbb{Z}$) $X_j(\phi)$ of $c_j(\phi) := \left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)\right]$ that can be ordered as

$$\cdots < x_{g-1}^{j-1}(\phi) < x_1^j(\phi) < x_2^j(\phi) < \cdots < x_{g-1}^j(\phi) < x_1^{j+1}(\phi) < \ldots$$

Denote by N_0 the closure in $\mathrm{Rep}(\Gamma)$ of the set of good representations

$$N_0 := \overline{\{\phi \in \mathrm{Rep}(\Gamma)\,|\phi \text{ is good }\}}.$$

Define $N := N_0 \cap \bigcap_{j=1}^{g} N_j$.

3. The Calegari-Walker algorithm

In this section we present the Calegari-Walker Algorithm which is a simple technical tool that works as follows:

- **Input**: a positive word $w \in F_2^+$ and the combinatorial data of periodic orbits X, Y of homeomorphisms $f, g \in \mathrm{Homeo}_+(S^1)$.
- **Output**: the best possible upper bound for $\widetilde{\mathrm{rot}}\, w(f,g)$.

Moreover sometimes it gives strong combinatorial constraints on the periodic orbit structures on homeomorphisms realizing the maximal value. The Algorithm generalizes in a straightforward way to inputs that are not periodic orbits but finite sets on the given homeomorphism acts. Incidentally we state as abstract and general applications of the algorithm two nice results even if we won't need after: the Rationality Theorem and the $ab-$Theorem.

3.1. Positive words. Let F_2 be the free group on generators $\{a, b\}$.
We introduce the following notation: given a word $w \in F_2$, a group Λ and two elements $f, g \in \Lambda$ we denote by $w(f, g) \in \Lambda$ the element $w(f, g) := \rho(w)$ obtained evaluating at w the homomorphism $\rho : F_2 \to \Lambda$ defined by the assignation $\rho(a) := f, \rho(b) := g$.

Definition 5.9. A word $w \in F_2$ is *positive* if it does not contain a^{-1}, b^{-1}. We denote the semigroup of positive words by F_2^+.

Words $w \in F_2$ that are positive enjoy the following simple property: consider maps $A, B, a, b : \mathbb{R} \to \mathbb{R}$ that are non-decreasing and commute with integral translations τ_n. If $A \geq a$ and $B \geq b$ then $w(A, B) \geq w(a, b)$ as maps $\mathbb{R} \longrightarrow \mathbb{R}$, and $\widetilde{\mathrm{rot}}\, w(A, B) \geq \widetilde{\mathrm{rot}}\, w(a, b)$. Exploiting this easy property Calegari and Walker in [**CW11**] constructed a discrete dynamical system which computes the maximal value of $\widetilde{\mathrm{rot}}\, w(\widetilde{\rho}(a), \widetilde{\rho}(b))$ that can be attained by a representation $\rho : F_2 \longrightarrow \mathrm{Homeo}_+ \left(S^1\right)$ with given constraints $\mathrm{rot}\, \rho(a) = r, \mathrm{rot}\, \rho(b) = s$ with $r, s \in \mathbb{R}$ (we will treat only the case where $r, s \in \mathbb{Q}$). Here we denote by $\widetilde{\rho}(a), \widetilde{\rho}(b)$ the unique lifts of $\rho(a), \rho(b)$ such that $\widetilde{\mathrm{rot}}\, \widetilde{\rho}(a) \in [0, 1)$ and $\widetilde{\mathrm{rot}}\, \widetilde{\rho}(b) \in [0, 1)$.

3.2. Associated dynamical system. Consider a representation $\rho : F_2 \to \mathrm{Homeo}_+ \left(S^1\right)$, suppose that $\rho(a), \rho(b)$ have periodic orbits

$$X = \{[x_0], \dots, [x_{q-1}]\} \qquad Y = \left\{[y_0], \dots, [y_{q'-1}]\right\}$$

(cyclically ordered), on which they act as follows

$$a[x_i] = [x_{i+p}] \qquad b[y_j] = [y_{j+p'}]$$

(indices are thought resp. modulo q, q').
We construct an *associated dynamical sistem* on $\mathbb{R}$ as follows. First we lift the orbits X, Y respectively to

$$\begin{array}{ll} \overline{X}: & \dots < x_{q-1} - 1 = x_{-1} < x_0 < x_1 < \dots < x_{q-1} < x_q = x_0 + 1 < \dots \\ \overline{Y}: & \dots < y_{q'-1} - 1 = y_{-1} < y_0 < y_1 < \dots < y_{q'-1} < y_{q'} = y_0 + 1 < \dots \end{array}$$

Then we lift the homeomorphisms $\rho(a), \rho(b)$ to $\widetilde{a}, \widetilde{b} \in \widetilde{\mathrm{Homeo}_+} \left(S^1\right)$ as above, i.e. $\widetilde{\mathrm{rot}}\, \widetilde{a}, \widetilde{\mathrm{rot}}\, \widetilde{b} \in [0, 1)$. With these coices the action of $\widetilde{a}, \widetilde{b}$ on $\overline{X}, \overline{Y}$ is given by

$$\widetilde{a} x_i = x_{i+p} \qquad , \qquad \widetilde{b} y_j = y_{j+p'} \qquad \text{where } 0 \leq p < q \text{ and } 0 \leq p' < q'.$$

We define non-decreasing, $\mathbb{Z}-$equivariant (discontinuous) maps A, B as follows:

$$A\left(x \in (x_{i-1}, x_i]\right) := \widetilde{a}(x_i) = x_{i+p} \qquad B\left(y \in (y_{j-1}, y_j]\right) := \widetilde{b}(y_j) = y_{j+p'}.$$

By construction A, B have the following properties:

(1) $A \geq \widetilde{a}, B \geq \widetilde{b}$, in particular $\widetilde{\mathrm{rot}}\ w\,(A,B) \geq \widetilde{\mathrm{rot}}\ w\left(\widetilde{a},\widetilde{b}\right)$ for every positive word w;
(2) A, B act on $\overline{X}, \overline{Y}$ exactly like $\widetilde{a}, \widetilde{b}$; in particular $\widetilde{\mathrm{rot}}(A) = \widetilde{\mathrm{rot}}(\widetilde{a}), \widetilde{\mathrm{rot}}(B) = \widetilde{\mathrm{rot}}(\widetilde{b})$;
(3) $A(\mathbb{R}) \subseteq \overline{X}$ and $B(\mathbb{R}) \subseteq \overline{Y}$.

Finally we denote by $Z \subseteq \mathbb{R}$ the union of the orbits $\overline{Z} = \overline{X} \cup \overline{Y}$ and define an action of the semigroup of positive words F_2^+ on Z setting

$$w \cdot z := w(A,B)z.$$

We call this dynamical system with orbit space $\overline{Z} \subseteq \mathbb{R}$ the *associated dynamical system* to the representation ρ. In a similar way we also define a dynamical system with orbit space $Z = X \cup Y \subseteq S^1$ simply by passing the action of F_2^+ to the quotient $S^1 = \mathbb{R}/\mathbb{Z}$.

3.3. Computation of $\widetilde{\mathbf{rot}}$ $w(A,B)$. Let us fix a representation ρ and a positive word $w \in F_2^+$ as above, in particular $\rho(a), \rho(b)$ have periodic orbits.
Consider the associated dynamical systems. Since $A \geq \widetilde{a}$ and $B \geq \widetilde{b}$ we have

$$\widetilde{\mathrm{rot}}\ w\,(A,B) \geq \widetilde{\mathrm{rot}}\ w\left(\widetilde{a},\widetilde{b}\right).$$

Observe that since the orbit space $Z = X \cup Y$ is finite the induced dynamical system on S^1 has a periodic orbit: in fact for every $[z] \in Z$ we have $w^m[z] = w^n[z]$ for some $m > n$ and therefore $w^k[t] = [t]$ where $[t] := w^{m-n}[z]$ and $k = m - n$. The orbit of the lifted point $t \in \overline{Z}$ is periodic modulo $\mathbb{Z}$ i.e. $w^k \cdot t = t + u$ with $u \in \mathbb{Z}$. We call k the *period* and u the *lenght* of the orbit. By the definition of $\widetilde{\mathrm{rot}}$ we get

$$\widetilde{\mathrm{rot}}\ w\,(A,B) = \frac{u}{k}.$$

Observe that $\widetilde{\mathrm{rot}}\ w(A,B)$ depends only on the dynamical system of $\overline{Z} = \overline{X} \cup \overline{Y}$, therefore we may define

$$\widetilde{\mathrm{rot}}\,(w, X \cup Y) := \widetilde{\mathrm{rot}}\ w(A,B).$$

3.4. Realization by homeomorphisms. Viceversa suppose we are given two actions of $\mathbb{Z}$ on some finite subsets $X, Y \subseteq S^1$ which cyclically permute the points of X, Y. Lift $\mathbb{Z} \curvearrowright X$ and $\mathbb{Z} \curvearrowright Y$ to actions on the lifts of the orbits $\overline{X}, \overline{Y} \subseteq \mathbb{R}$ in a way such that $\overline{x} \leq 1 \cdot \overline{x} < \overline{x} + 1$ for any $x \in \overline{X}$ and $\overline{y} \leq 1 \cdot \overline{y} < \overline{y} + 1$ for any $y \in \overline{Y}$. Observe that such actions have well-defined rotation numbers

$$\widetilde{\mathrm{rot}}(w := 1 \in \mathbb{Z}, X) = \lim_{n\to\infty} \frac{n \cdot \overline{x}}{n} \qquad \widetilde{\mathrm{rot}}(w := 1 \in \mathbb{Z}, Y) = \lim_{n\to\infty} \frac{n \cdot \overline{y}}{n}$$

(where $\overline{x} \in \overline{X}, \overline{y} \in \overline{Y}$ are arbitrary points and $n \cdot \overline{x}, n \cdot \overline{y}$ denote the actions of $\mathbb{Z}$ on $\overline{X}, \overline{Y}$). These rotation numbers are given, as above, by $\frac{u}{k}$ where u, k are respectively the length and the period of periodic orbits. The $\mathbb{Z}-$actions together give rise to an action of F_2^+ on the union of the lifted orbits $\overline{Z} = \overline{X} \cup \overline{Y} \subseteq \mathbb{R}$ just like the one described in Section 2.1. For every word $w \in F_2^+$ we can compute

$$\widetilde{\mathrm{rot}}\,(w, Z = X \cup Y) = \lim_{n\to\infty} \frac{w^n \cdot \overline{z}}{n}.$$

We show that for every word $w \in F_2^+$ the translation number $\widetilde{\mathrm{rot}}\,(w, X \cup Y)$ is realizable as

$$\widetilde{\mathrm{rot}}\ w\,(\widetilde{\rho}(a), \widetilde{\rho}(b)) = \widetilde{\mathrm{rot}}\,(w, X \cup Y)$$

for some representation $\widetilde{\rho} : F_2 \to \widetilde{\mathrm{Homeo}_+}\left(S^1\right)$ such that $\widetilde{\rho}(a), \widetilde{\rho}(b)$ admit as "periodic" (modulo $\mathbb{Z}$) orbits the sets $\overline{X}, \overline{Y}$ (respectively) on which they act as prescribed by the dynamical systems $\mathbb{Z} \curvearrowright \overline{X}$ and $\mathbb{Z} \curvearrowright \overline{Y}$. In particular

$$\widetilde{\mathrm{rot}}\ \widetilde{\rho}(a) = \widetilde{\mathrm{rot}}(w := 1 \in \mathbb{Z}, X) \qquad \widetilde{\mathrm{rot}}\ \widetilde{\rho}(b) = \widetilde{\mathrm{rot}}(w := 1 \in \mathbb{Z}, Y).$$

Proposition 5.10. *Let $w \in F_2^+$ be a positive word. Let $X, Y \subseteq S^1$ be finite subsets and denote by $\overline{X}, \overline{Y} \subseteq \mathbb{R}$ their lifts to $\mathbb{R}$. Let $\mathbb{Z} \curvearrowright X$ and $\mathbb{Z} \curvearrowright Y$ be actions. Consider the associated dynamical system $F_2 \curvearrowright \overline{X} \cup \overline{Y}$ described by maps A, B corresponding to the single actions of $\mathbb{Z}$ on $\overline{X}, \overline{Y}$ respectively. Then there are homeomorphisms $f, g \in \widetilde{Homeo_+}\left(S^1\right)$ such that*

$$\widetilde{rot}\,(w(A, B)) = \widetilde{rot}\,(w(f, g))\,.$$

Proof. Fix $\epsilon > 0$. We can find $f \in \widetilde{\mathrm{Homeo}_+}\left(S^1\right)$ such that

(1) $f \leq A$ and $f \equiv A$ on the orbit X, $f(x_i) = A(x_i) = \widetilde{a}(x_i)$.
(2) $f \geq A - \epsilon$ on every interval $(x_{i-1} + \epsilon, x_i]$.

For example we can choose f piecewise linear with respect to a suitable refinement of the subdivision $\overline{X}$. Analogously find $g \in \widetilde{\mathrm{Homeo}_+}\left(S^1\right)$ with the corresponding properties with respect to B. By construction $\widetilde{\mathrm{rot}}(f) = \widetilde{\mathrm{rot}}(A)$ and $\widetilde{\mathrm{rot}}(g) = \widetilde{\mathrm{rot}}(B)$. We claim that $\widetilde{\mathrm{rot}}\ w(f, g) = \widetilde{\mathrm{rot}}\ w(A, B)$ if $\epsilon > 0$ is sufficiently small.
Choose $\epsilon < \frac{1}{2} \min \left\{ d(z_1, z_2) \mid z_1 < z_2 \in \overline{Z} \right\}$ and set

$$I_z := (z - \epsilon, z] \qquad , \qquad U := \bigcup_{z \in \overline{Z}} I_z.$$

We define $A \cdot I_z := I_{Az}$ and $B \cdot I_z := I_{Bz}$. Observe that by our choice of ϵ we have $f(I_z) \subseteq A \cdot I_z$ and $g(I_z) \subseteq B \cdot I_z$. In fact, if $t \in U$ then there exist $x_i \in \overline{X}, y_j \in \overline{Y}$ s.t.

$$x_{i-1} + \epsilon < z - \epsilon < t \leq z \leq x_i \quad , \quad y_{j-1} + \epsilon < z - \epsilon < t \leq z \leq y_j.$$

By the properties of f, g we get

$$Ax_i - \epsilon = Az - \epsilon < f(t) \leq Ax_i = Az \quad , \quad By_j - \epsilon = Bz - \epsilon < g(t) \leq By_j = Bz.$$

In conclusion $w(f, g)I_z \subseteq w(A, B) \cdot I_z$ for every positive word w; for a fixed $t \in U$ this implies that for any $n \in \mathbb{N}$ we have

$$|w^n(f, g)t - w^n(A, B)t| < \epsilon$$

which gives $\widetilde{\mathrm{rot}}\ w(f, g) = \widetilde{\mathrm{rot}}\ w(A, B)$. ☐

Remark 5.11. Consider an action by cyclic permutations of $\mathbb{Z}$ on a finite subset $X \subset S^1$. We prove that the set $R(X) \subseteq \mathrm{Homeo}_+\left(S^1\right)$ consisting of homeomorphisms that acts on $X \subseteq S^1$ in the way prescribed by $1 \in \mathbb{Z}$ is connected. Consider two homeomorphisms $f_1, f_2 \in R(X)$ and lift them in a way such that both $\widetilde{f}_1, \widetilde{f}_2 \in \widetilde{\mathrm{Homeo}_+}\left(S^1\right)$ act the same way on a lift $\overline{X} \subseteq \mathbb{R}$ of the finite subset X. Define $f_t := tf_1 + (1-t)f_2$, observe that for every $t \in [0,1]$ we have $f_t \in \widetilde{\mathrm{Homeo}}_+\left(S^1\right)$ and f_t acts on $\overline{X}$ exactly like $\widetilde{f}_1$ and $\widetilde{f}_2$. Denote by $p : \widetilde{\mathrm{Homeo}}_+\left(S^1\right) \to \mathrm{Homeo}_+\left(S^1\right)$ the universal covering projection. The path $\{pf_t\}_{t\in[0,1]}$ joins f_1 to f_2 in $R(X)$. Carefully looking at the construction in the proof of Proposition 5.10 we conclude that every representation $\rho : F_2 \to \widetilde{\mathrm{Homeo}}_+\left(S^1\right)$ as above can be continuously deformed by $\{\rho_t\}_{t\in[0,1]}$ to a representaion that realizes $\widetilde{\mathrm{rot}}\,(w, X \cup Y)$ though representations ρ_t that exhibit the same finite orbits for $\rho_t(a), \rho_t(b)$ as $\rho(a), \rho(b)$.

3.5. Maximal rotation number given constraints $\widetilde{\mathbf{rot}}(\widetilde{a}) = \frac{p_1}{q_1}, \widetilde{\mathbf{rot}}(\widetilde{b}) = \frac{p_2}{q_2}$. Fix $w \in F_2^+$ a positive word. We want to compute

$$R\left(w, \frac{p_1}{q_1}, \frac{p_2}{q_2}\right) = \sup\left\{\widetilde{\mathrm{rot}}\ w\left(\widetilde{a}, \widetilde{b}\right) \,\middle|\, \begin{array}{c} \widetilde{a}, \widetilde{b} \in \widetilde{\mathrm{Homeo}}_+\left(S^1\right) \\ \widetilde{\mathrm{rot}}(\widetilde{a}) = \frac{p_1}{q_1}, \widetilde{\mathrm{rot}}(\widetilde{b}) = \frac{p_2}{q_2} \end{array}\right\}.$$

Every homeomorphism $f \in \mathrm{Homeo}_+\left(S^1\right)$ with rational translation number $\widetilde{\mathrm{rot}}\ f = \frac{p}{q}$ (with $\gcd(p,q) = 1$) has a periodic orbit

$$X = \{[x_0] \ldots, [x_{q-1}]\} \subseteq S^1$$

(cyclically ordered) on which it acts by $f[x_i] = [x_{i+p}]$. In particular, by our assumptions on $\widetilde{a}, \widetilde{b} \in \widetilde{\mathrm{Homeo}}_+\left(S^1\right)$, we always find periodic orbits $X, Y \subseteq S^1$ with the following prescribed actions

$\widetilde{\mathrm{rot}}(\widetilde{a}) = \frac{p_1}{q_1}$	$\widetilde{\mathrm{rot}}(\widetilde{b}) = \frac{p_2}{q_2}$
$X = \{[x_0] \ldots, [x_{q_1-1}]\}$ $\widetilde{a}[x_i] = [x_{i+p_1}]$	$Y = \{[y_0] \ldots, [y_{q_2-1}]\}$ $\widetilde{b}[y_j] = [y_{j+p_2}]$

which induce an action of F_2^+ on the lifted orbits $\overline{Z} = \overline{X} \cup \overline{Y}$ as above (see Section 2.1). Observe that the resulting dynamical system only depends on the *cyclic orders* of X, Y and $Z = X \cup Y$. Since the possible cyclic orders are finite we have only a finite number of possible dynamical systems arising for a couple $\widetilde{a}, \widetilde{b}$ with the prescribed translation numbers. Furthermore, by the realization procedure discussed in Proposition 5.10, every such configuration occurs as the associated dynamical system for some $\widetilde{a}, \widetilde{b}$.
In conclusion we have

PROPOSITION 5.12. *Let $w \in F_2^+$ be a positive word. Then*

$$R\left(w, \frac{p_1}{q_1}, \frac{p_2}{q_2}\right) = \max\left\{\widetilde{rot}\left(w, \overline{X} \cup \overline{Y}\right) \,\middle|\, \begin{array}{c} X, Y \subset S^1 \\ \widetilde{rot}(1 \in \mathbb{Z}, \overline{X}) = \frac{p_1}{q_1}, \widetilde{rot}(1 \in \mathbb{Z}, \overline{Y}) = \frac{p_2}{q_2} \end{array}\right\}.$$

3.6. Rationality theorem. The following rationality theorem easily follows from the previous discussion.

THEOREM 5.13 (Rationality theorem, [**CW11**]). *Let $w \in F_2$ be a positive word, $r = \frac{p_1}{q_1}, s = \frac{p_2}{q_2} \in \mathbb{Q}$. Then*

(1) $R(w,r,s)=\frac{u}{v}$ *for some* $\frac{u}{v}\in\mathbb{Q}$.
(2) if $w\neq a^n, b^m$ *then* $v\leq\min\{q_1,q_2\}$.

Proof. Let us prove *(1)*. By Proposition 5.12 $R(w,r,s)$ is the maximum of a finite set of rational numbers. Now we prove *(2)*. By construction $\widetilde{\mathrm{rot}}\left(w,\overline{X}\cup\overline{Y}\right)$ is computed by a periodic orbit of $X\cup Y$, and the denominator of $\widetilde{\mathrm{rot}}\,(w,X\cup Y)$ is the period of the orbit. Since $w\neq a^n,b^m$ then w is conjugate in F_2 to both $w_1=ac_1, w_2=bc_2$ with $c_1,c_2\in F_2$ positive. The period of a periodic orbit of w is equal to the period of a periodic orbit of w_1 or w_2. By construction $w_1X\subseteq X$ and $w_2Y\subseteq Y$, in particular the period of a periodic orbit of w_1,w_2 is smaller than or equal to $|X|=q_1,|Y|=q_2$ respectively.

□

3.7. $ab-$**Theorem.** A remarkable case is given by the positive word $w=ab$. A computation carried out using the Calegari-Walker algorithm (Proposition 5.12) allows to explicitly describe the value $R\,(ab,r,s)$:

Theorem 5.14 ($ab-$theorem, [**CW11**]). *The following holds:*

$$R(ab,r,s)=\sup_{\frac{p_1}{q}\leq r\ ,\ \frac{p_2}{q}\leq s}\left\{\frac{p_1+p_2+1}{q}\right\}.$$

The supremum is taken over all rational $\frac{p_j}{q}$ *such that* $\gcd(p_j,q)=1$ *for* $j=1,2$.

A proof of the $ab-$Theorem can be found in [**CW11**].

3.8. Fixed point sets input. It is not difficult to check that the Calegari-Walker algorithm works equally well if we replace the input of finite orbits of f and g with finite sets X and Y over which the homeomorphism f and g acts. In other words: Suppose that $\widetilde{f}$ and $\widetilde{g}$ act on the periodic sets $\overline{X}$ and $\overline{Y}$ respectively. For example $\overline{X}$ could be the lift of the union of a finite number of periodic orbits of f. Then the action of the positive word $w\in F_2^+$ on $\overline{X}\cup\overline{Y}$ is well defined and has an associated rotation number $\widetilde{\mathrm{rot}}\ \left(w,\overline{X}\cup\overline{Y}\right)$. The discussion on the Calegari-Walker Algorithm leads to the estimate:

$$\widetilde{\mathrm{rot}}\ w\left(\widetilde{f},\widetilde{g}\right)\leq\widetilde{\mathrm{rot}}\ \left(w,\overline{X}\cup\overline{Y}\right).$$

In particular we will be interested in the case where $\overline{X}$ and $\overline{Y}$ are the lifts of fixed point sets of $f\in G$ and $g\in G$.

3.9. Algorithm for higher rank free groups. There are straightforward generalizations of the Calegari-Walker algorithm to positive words $w\in F_n^+$ in the larger alphabet freely generated by $\{a_1,\ldots,a_n\}$. If we have homeomorphisms $f_1,\ldots,f_n\in G$ with lifted periodic orbits $\overline{X}_1,\ldots,\overline{X}_n$ then, with the same constructions, we get the sharp bound

$$\widetilde{\mathrm{rot}}\ w\left(\widetilde{f}_1,\ldots,\widetilde{f}_n\right)\leq\widetilde{\mathrm{rot}}\ \left(w,\overline{X}_1\cup\cdots\cup\overline{X}_n\right).$$

In particular Proposition 5.12 generalizes to:

PROPOSITION 5.15. *Let $w \in F_n^+$ be a positive word. Then*

$$R\left(w, \frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}\right) = \max\left\{ \widetilde{rot}\left(w, \overline{X}_1 \cup \cdots \cup \overline{X}_n\right) \;\middle|\; \begin{array}{c} X_1, \ldots, X_n \subseteq S^1 \\ \widetilde{rot}(1 \in \mathbb{Z}, \overline{X}_j) = \frac{p_j}{q_j} \end{array} \right\}.$$

4. Rotation number of a commutator

In this section we deal with a particular case of independent interest: the rotation number of a commutator. A commutator $[a, b]$ is clearly a non-positive words, but we will think of it as the product of the words a and $ba^{-1}b^{-1}$. This example allows us to apply different tecniques in the spirit of Calegari-Walker Algorithm and also exploit the relations between monotone equivalence and semi-conjugacy. It also exhibits a "maximality implies rigidity" phenomenon.

In general knowing that an element $f \in G := \mathrm{Homeo}_+\left(S^1\right)$ is a commutator, i.e. $f = [a, b]$, does not give any information since by Theorem 1.32 we know that G is uniformly perfect of constant $N = 1$. However if we have constraints on $\mathrm{rot}(a), \mathrm{rot}(b)$ then we can estimate $\widetilde{\mathrm{rot}}\left[\widetilde{a}, \widetilde{b}\right]$: this is the content of Lemma 5.16 and Lemma 5.17.

4.1. Vanishing of $\widetilde{\mathrm{rot}}\left[\widetilde{a}, \widetilde{b}\right]$.

Let $a, b \in \mathrm{Homeo}_+\left(S^1\right)$ be homeomorphisms. If either a or b does not have rational rotation number then the rotation number of their commutator always vanishes:

LEMMA 5.16. *Let $a, b \in Homeo_+\left(S^1\right)$ be homeomorphisms. If $rot(a) \notin \mathbb{Q}/\mathbb{Z}$ or $rot(b) \notin \mathbb{Q}/\mathbb{Z}$, then*

$$\widetilde{rot}\left[\widetilde{a}, \widetilde{b}\right] = 0$$

where $\widetilde{a}, \widetilde{b} \in \widetilde{Homeo}_+\left(S^1\right)$ are arbitrary lifts of a, b.

Before going on with the proof of the lemma let us remark that the function

$$G \times G \longrightarrow \mathbb{R} \qquad , \qquad (a, b) \longrightarrow \widetilde{\mathrm{rot}}\left[\widetilde{a}, \widetilde{b}\right]$$

is well-defined (independent of the lifts) and continuous.

PROOF. We divide the proof in three cases.
Case $a = \rho_\theta$. We can choose as lift of a the translation $\widetilde{a} = \tau_\theta$. We have $\left[\tau_\theta, \widetilde{b}\right] = \tau_\theta \widetilde{b} \tau_{-\theta} \widetilde{b}^{-1}$. Since $\widetilde{\mathrm{rot}}\left(\widetilde{b}\tau_{-\theta}\widetilde{b}^{-1}\right) = -\theta$ there exists $x \in \mathbb{R}$ such that $\widetilde{b}\tau_{-\theta}\widetilde{b}^{-1}x = x - \theta$ (by elementary properties of $\widetilde{\mathrm{rot}}$). Thus we get $\left[\tau_\theta, \widetilde{b}\right] x = \tau_\theta(x - \theta) = x$.
Case a conjugate to ρ_θ. We can reduce to the case where $a = \rho_\theta$. Since $\widetilde{\mathrm{rot}}$ is a conjugacy invariant we have $\widetilde{\mathrm{rot}}\left[\widetilde{a}, \widetilde{b}\right] = \widetilde{\mathrm{rot}}\; g\left[\widetilde{a}, \widetilde{b}\right] g^{-1} = \widetilde{\mathrm{rot}}\;\left[g\widetilde{a}g^{-1}, g\widetilde{b}g^{-1}\right]$. Choosing $g \in \widetilde{G}$ such that $g\widetilde{a}g^{-1} = \tau_\theta$ (since a is conjugate to a rotation, $\widetilde{a}$ is conjugate to a translation) we get the desired reduction.
Case a left-monotone-equivalent to ρ_θ. Again we reduce to the case where a is conjugate to

ρ_θ. Let $h : S^1 \to S^1$ be a monotone equivalence such that $ah = h\rho_\theta$. Choose $\{h_n\}_{n\in\mathbb{N}} \subseteq G$ a sequence of homeomorphisms that uniformly approximate h (see Lemma 1.49). Define $a_n = h_n\rho_\theta h_n^{-1}$. We claim that $a_n \to a$ uniformly:

$$\begin{aligned}
&||a - a_n||_\infty = \left|\left|a - h_n\rho_\theta h_n^{-1}\right|\right|_\infty \\
&= \left|\left|(ah_n - h_n\rho_\theta)h_n^{-1}\right|\right|_\infty = ||ah_n - h_n\rho_\theta||_\infty \quad \text{since } h_n^{-1} \text{ is surjective} \\
&||ah_n - h_n\rho_\theta||_\infty \overset{n\to\infty}{\longrightarrow} ||ah - h\rho_\theta||_\infty = 0 \qquad \text{since } ||h_n - h||_\infty \to 0.
\end{aligned}$$

Finally we get

$$\begin{aligned}
&0 = \widetilde{\text{rot}}\left[\widetilde{a}_n, \widetilde{b}\right] && \text{since } a_n \text{ is conjugate to } \rho_\theta \\
&\widetilde{\text{rot}}\left[\widetilde{a}_n, \widetilde{b}\right] \overset{n\to\infty}{\longrightarrow} \text{rot}\left[\widetilde{a}, \widetilde{b}\right] = 0 && \text{by continuity of } \widetilde{\text{rot}}\,[\bullet, b], \text{ together with } a_n \to a.
\end{aligned}$$

Thus $\widetilde{\text{rot}}\left[\widetilde{a}, \widetilde{b}\right] = 0$. □

4.2. Upper bound for $\widetilde{\textbf{rot}}\left[\widetilde{a}, \widetilde{b}\right]$. We develop now the case where $a \in G$ or $b \in G$ has rational rotation number, and thus a periodic orbit:

Lemma 5.17. *Let $a, b \in Homeo_+\left(S^1\right)$ be homeomorphisms. Let $\widetilde{a}, \widetilde{b} \in \widetilde{Homeo_+}\left(S^1\right)$ be arbitrary lifts of a, b. If $\widetilde{rot}\,\widetilde{a} = \frac{p}{q} \in \mathbb{Q}$ or $\widetilde{rot}\,\widetilde{b} = \frac{p}{q} \in \mathbb{Q}$ with* $\gcd(p, q) = 1$ *then*

$$\left|\widetilde{rot}\left[\widetilde{a}, \widetilde{b}\right]\right| \le \frac{1}{q}.$$

Proof. As we will work only on $\mathbb{R}$, we redefine for notation sake $a := \widetilde{a}$ and $b := \widetilde{b}$. Up to exchanging a and b we may assume $\widetilde{\text{rot}}\, a \in \mathbb{Q}$ with a periodic (modulo $\mathbb{Z}$) orbit $\overline{X} \subseteq \mathbb{R}$ with prescribed dynamics. In the proof we will establish the following facts: first we will describe two possible patterns for the action of $b \in \widetilde{\text{Homeo}_+}\left(S^1\right)$ on the orbit $\overline{X}$ which are of independent interest for our purpose. Then, using this description we will prove the inequality $|\text{rot}\,[a, b]| \le \frac{1}{q}$.

Assume $\widetilde{\text{rot}}\ a = \frac{p}{q}$. Thus the map $a : \mathbb{R} \to \mathbb{R}$ has an orbit $\overline{X} = \{x_i\} \subseteq \mathbb{R}$ of lenght p and period q

$$\dots < x_{q-1} - 1 = x_{-1} < x_0 < x_1 < \dots < x_{q-1} < x_q = x_0 + 1 < \dots$$

on which acts by sending $a(x_j) = x_{j+p}$ (with the rule $x_{j+uq} = x_j + u$, $a(x_j) = x_{j+p}$).

Let us study the action of b on the orbit $\overline{X}$: we will show that one of the two following configurations holds:

(1) b *preserves the distances for* a between every pair $\left(a^{-1}x_i, x_i\right) = (x_{i-p}, x_i)$ in the following sense: if $b(x_i) \in (x_j, x_{j+1}]$ then
$$b(a^{-1}x_i) = b(x_{i-p}) \in a(x_j, x_{j+1}] = (x_{j-p}, x_{j-p+1}]$$
Furthermore if $b(x_u) \in (x_j, x_{j+1}]$ then $bx_i \in (x_{j+i-u}, x_{j+i-u+1}]$ for every $i \in \mathbb{Z}$.

(2) there exists a pair (x_{j-p}, x_j) for which b *increases the distance* with respect to a: if $b(x_i) \in (x_j, x_{j+1}]$ then $b(x_{i-p}) \in (x_k, x_{k+1}]$ for some $k \in \mathbb{Z}$ with $k < j - p$.

Before proving that the above dichotomy holds let us establish the estimate $\left|\widetilde{\mathrm{rot}}\,[a,b]\right| \leq \frac{1}{q}$ of the lemma in the two specific cases *(1)* and *(2)*.

- Case *(1)*. We have $\widetilde{\mathrm{rot}}[a,b] \leq \frac{1}{q}$: fix $x_i \in \overline{X}$ and suppose that $b(x_i) \in (x_j, x_{j+1}]$. By the second property of *(1)* we have $b(x_u) \in (x_{j+u-i}, x_{j+u-i+1}]$ for every $u \in \mathbb{Z}$. In order to estimate $\widetilde{\mathrm{rot}}\,[a,b]$ we prove the following inequality that holds for arbitrary $x_i \in \overline{X}$:

$$\begin{aligned}
&[a,b]\,b(x_i) = aba^{-1}b^{-1}(bx_i) \\
&= aba^{-1}x_i \\
&= abx_{i-p} \\
&\leq ax_{j-p+1} = x_{j+1} && \text{since } b(x_{i-p}) \in (x_{j-p}, x_{j-p+1}] \\
&\leq b(x_{i+1}) && \text{since } b(x_{i+1}) \in (x_{j+1}, x_{j+2}].
\end{aligned}$$

In particular

$$\begin{aligned}
&[a,b]^q\,b(x_i) \leq b(x_{i+q}) && \text{by the above inequality} \\
&= b(x_i) + 1 && \text{since } x_{i+q} = x_i + 1.
\end{aligned}$$

That implies $\mathrm{rot}\,[a,b] \leq \frac{1}{q}$.
Observe now that since b preserves the distances between the pairs (x_{i-p}, x_i), the same argument used to prove that $[a,b]\,b(x_i) \leq b(x_{i+1})$ shows that $\left[a^{-1},b\right]b(x_i) \leq b(x_{i+1})$:

$$\begin{aligned}
&\left[a^{-1},b\right]b(x_i) = a^{-1}bab^{-1}(bx_i) \\
&= a^{-1}bax_i \\
&= abx_{i+p} \\
&< ax_{j-p+1} = x_{j+1} && \text{since } b(x_{i+p}) \in (x_{j+p}, x_{j+p+1}] \\
&\leq b(x_{i+1}) && \text{since } b(x_{i+1}) \in (x_{j+1}, x_{j+2}].
\end{aligned}$$

In particular $\widetilde{\mathrm{rot}}\left[a^{-1},b\right] \leq \frac{1}{q}$.
Finally, putting together the previous inequalities and using the simple algebraic identity $[b,a] = a\left[a^{-1},b\right]a^{-1}$ we are able to prove that $\widetilde{\mathrm{rot}}\,[a,b] \geq -\frac{1}{q}$:

$$\begin{aligned}
&-\widetilde{\mathrm{rot}}\,[a,b] \\
&= \widetilde{\mathrm{rot}}\,[b,a] && \text{since } [b,a] = [a,b]^{-1} \\
&= \widetilde{\mathrm{rot}}\;a\left[a^{-1},b\right]a^{-1} && \text{since } [b,a] = a\left[a^{-1},b\right]a^{-1} \\
&= \widetilde{\mathrm{rot}}\left[a^{-1},b\right] && \text{since } \widetilde{\mathrm{rot}} \text{ is conjugacy invariant} \\
&\leq \tfrac{1}{q} && \text{by the inequalities proved above.}
\end{aligned}$$

- Case *(2)*. We have $\widetilde{\mathrm{rot}}[a,b] \leq 0$. Let (x_i, x_{i-p}) be an increasing couple, $b(x_i) \in (x_j, x_{j+1}]$ and $b(x_{i-p}) \in (x_k, x_{k+1}]$ with $k \leq j-p-1$. Then

$$\begin{aligned} & [a,b]\, b(x_i) = aba^{-1}b^{-1}b(x_i) \\ & = aba^{-1}x_i \\ & = abx_{i-p} \\ & \leq ax_{k+1} = x_{k+p+1} && \text{since } b(x_{i-p}) \in (x_k, x_{k+1}] \\ & \leq x_j && \text{since } k \leq j-p-1 \\ & \leq b(x_i) && \text{since } b(x_i) \in (x_j, x_{j+1}]. \end{aligned}$$

  That implies $\widetilde{\mathrm{rot}}\,[a,b] \leq 0$.
  Analogously to the case *(1)* we observe that since b doesn't preserve the distances for a, it doesn't preserve the distances for a^{-1} too, in particular we get from the dichotomy an increasing couple for a^{-1} which gives the inequality

$$\widetilde{\mathrm{rot}}\,\left[a^{-1}, b\right] \leq 0.$$

  Finally, using the same trick as before, we have

$$-\widetilde{\mathrm{rot}}\,[a,b] = \widetilde{\mathrm{rot}}\,[b,a] = \widetilde{\mathrm{rot}}\; a\left[a^{-1}, b\right]a^{-1} = \widetilde{\mathrm{rot}}\,\left[a^{-1}, b\right] \leq 0.$$

  In conclusion $\widetilde{\mathrm{rot}}\,[a,b] = 0$.

Proof of the *dichotomy.* We should exclude that b only decreases the distances (and strictly decreases the distance for some pair), in other words this means that for every $x_i \in \overline{X}$ we have $x_j < b(x_i) \leq x_{j+1}$ and $x_k < b(x_{i-p}) \leq x_{k+1}$ with $k \geq j-p$. We proceed by contradiction: choose $x_i, x_{i-p} \in \overline{X}$ a strictly decreasing couple, for which $b(x_i) \in (x_{j_0}, x_{j_0+1}]$ and $b(x_{i-p}) \in (x_{j_1}, x_{j_1+1}]$ with $j_1 > j_0 - p$. Then

$$\begin{aligned} & b(x_{i-2p}) \in (x_{j_2}, x_{j_2+1}] \text{ with } j_2 \geq j_1 - p \geq j_0 - 2p && \text{since } b \text{ doesn't decrease distances} \\ & b(x_{i-3p}) \in (x_{j_3}, x_{j_3+1}] \text{ with } j_3 \geq j_2 - p && \text{analogously} \\ & \Rightarrow \dots \\ & \Rightarrow b(x_{i-qp}) \in (x_{j_q}, x_{j_q+1}] \text{ with } j_q \geq j_{q-1} - p \end{aligned}$$

but $x_{i-pq} = x_i - p$ so we get

$$b(x_{i-pq}) = b(x_i) - p \in (x_{j_0} - p, x_{j_0+1} - p] = (x_{j_0-pq}, x_{j_0+1-pq}].$$

In particular $j_q = j_0 - pq$, so

$$j_0 - pq = j_q \geq j_{q-1} - p \geq \cdots \geq j_1 - p(q-1) > j_0 - p + p(q-1) = j_0 + pq,$$

a contradiction.
Proof of the second part of point *(1)* of the dichotomy. In order to simplify the notation we shift the indices so that $u = 0$. Suppose that $bx_0 \in (x_j, x_{j+1}]$. Fix $i \in \mathbb{Z}$ since $\gcd(p,q) = 1$ then we can find $u, v \in \mathbb{Z}$ s.t. $i = up + vq$, using the fact that b^{-1} preserves distances we

get $b^{-1}x_{up} \in (x_{j+up}, x_{j+1+up}]$ and finally

$$\begin{aligned}
&b^{-1}x_i = b^{-1}x_{up+vq} = b^{-1}(x_{up}+v) = b^{-1}x_{up}+v \\
&\Rightarrow b^{-1}x_{up}+v \in (x_{j+up}+v, x_{j+1+up}+v] && \text{since } b^{-1}x_{up} \in (x_{j+up}, x_{j+1+up}] \\
&\Rightarrow b^{-1}(x_i = x_{up}+v) \in (x_{j+up+vq}, x_{j+1+up+vq}] && \text{since } x_k + e = x_{k+eq} \\
&\Rightarrow b^{-1}x_i \in (x_{j+i}, x_{j+1+i}].
\end{aligned}$$

We conclude that $b^{-1}x_i \in (x_{j+i}, x_{j+i+1}]$ for every $i \in \mathbb{Z}$. □

The upper bound given in Lemma 5.17 is sharp. In fact we see that for every $0 \le p, p' < q$ we can construct, using Lemma 5.2, a geometric $\mathrm{PSL}^{(q)}(2,\mathbb{R})$ −representation $\phi : \pi_1(S) \longrightarrow \mathrm{PSL}^{(q)}(2,\mathbb{R})$ with $\mathrm{rot}(\phi(a_1)) = \frac{p}{q}$ and $\mathrm{rot}(\phi(b_1)) = \frac{p'}{q}$. Since ϕ is geometric it realizes $\widetilde{\mathrm{rot}}\left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)\right] = \frac{1}{q}$. Hence:

THEOREM 5.18. *For every $a, b \in \widetilde{Homeo_+}\left(S^1\right)$ we have the following*

(1) if $\widetilde{rot}(a) \notin \mathbb{Q}$ or $rot(b) \notin \mathbb{Q}$ then $\widetilde{rot}[a,b] = 0$.

(2) if $\widetilde{rot}(a) = \frac{p}{q}$ or $rot(b) = \frac{p}{q}$ then $\left|\widetilde{rot}[a,b]\right| \le \frac{1}{q}$.

(3) $R\left([a,b], \frac{p}{q}, \frac{p'}{q}\right) = \frac{1}{q}$. Furthermore, if $a, b \in \widetilde{Homeo_+}\left(S^1\right)$ realize the maximal translation number for $[a,b]$ then they have periodic orbits with prescribed combinatorial structure.

5. Applications to geometric representations

Finally, in this last part, we give as input to the Calegari-Walker Algorithm the combinatorial data of representations with j−good fixed point sets and of good representations. From the analysis of the extremal cases we will be able to prove some stability results of orbit configurations.

5.1. Good fixed point sets. Having a j−good fixed set gives some constraints on the lifted translation numbers:

LEMMA 5.19. *Let $\phi \in N_j$ be a representation with a j−good fixed set. Then*

$$\widetilde{rot}_{[a_j,b_j]}(\phi) \le \frac{1}{k}.$$

In particular every representation $\phi \in N_j$ satisfies the same bound by continuity.

PROOF. The proof is an application of Calegari-Walker Algorithm 5.12. Let ϕ be a representation with a j−good fixed set, define $u := \widetilde{\phi}(a_j)$ and $v := \widetilde{\phi}(b_j)\widetilde{\phi}(a_j)^{-1}\widetilde{\phi}(b_j)^{-1}$ where $\widetilde{\mathrm{rot}}\ u = \widetilde{\mathrm{rot}}\ v = 0$. The fixed point sets of u, v are $X_1(\phi) = \left\{x_1^i(\phi)\right\}_{i\in\mathbb{Z}}$ and $X_2(\phi) = \left\{x_2^i(\phi)\right\}_{i\in\mathbb{Z}}$ and can be ordered as follows

$$\dots < x_1^0(\phi) < x_1^1(\phi) < x_2^0(\phi) < x_2^1(\phi) < x_1^2(\phi) < x_1^3(\phi) < \dots$$

Moreover, by property *(3)* the action of $\widetilde{\phi}(b_j)$ on $X_1(\phi)$ is given by

$$\widetilde{\phi}(b_j)\left(x_1^{2j}\right) = x_2^{2i+2n-1} \qquad , \qquad \widetilde{\phi}(b_j)\left(x_1^{2i+1}\right) = x_2^{2i+2n}.$$

The dynamics of the associated dynamical system is described by

$$\begin{cases} u \cdot x_1^i(\phi) = x_1^i(\phi) \\ u \cdot x_2^{2i}(\phi) = x_1^{2i+2}(\phi) \\ u \cdot x_2^{2i+1}(\phi) = x_1^{2i+2}(\phi) \end{cases} \qquad \text{and} \qquad \begin{cases} v \cdot x_2^i(\phi) = x_2^i(\phi) \\ v \cdot x_1^{2i}(\phi) = x_2^{2i}(\phi) \\ v \cdot x_1^{2i+1}(\phi) = x_2^{2i}(\phi). \end{cases}$$

Consider the positive word $w := uv = \left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)\right]$. Observe that

$$w \cdot x_1^i(\phi) = uv \cdot x_1^i(\phi) = u \cdot x_2^i(\phi) = x_1^{i+2}(\phi).$$

The orbit of $x_1^0(\phi)$ under the action of w is given by

$$x_1^0(\phi) \xrightarrow{w} x_1^2(\phi) \xrightarrow{w} \dots \xrightarrow{w} x_1^{2k-2}(\phi) \xrightarrow{w} x_1^{2k}(\phi) = x_1^0(\phi) + 1.$$

Hence $w^k \cdot x_1^0(\phi) = x_1^0(\phi) + 1$ and $\widetilde{\mathrm{rot}}\ w \leq \frac{1}{k}$. □

From Lemma 5.17 we know that, under the constraints on the rotation numbers $\mathrm{rot}(a)$ and $\mathrm{rot}(b)$, when maximality is attained we have a particular rigid configuration on the periodic sets of a, b. The next lemma is completely analogous:

Lemma 5.20. *Let $\phi \in Rep(\Gamma)$ be a representation satisfying property* (2) *with weak inequalities and property* (3)*. In other words $\phi(a_j)$ has a fixed point set with $2k$ non necessarily distinct points that can be lifted to $X_1(\phi) = \left\{x_1^i(\phi)\right\}_{i\in\mathbb{Z}} \subseteq Fix\left(\widetilde{\phi}(a_j)\right)$ for a suitable lift $\widetilde{\phi}(a_j)$ of $\phi(a_j)$, and if we call $X_2(\phi) := \widetilde{\phi}(b_j)X_1(\phi) = \left\{x_2^i(\phi)\right\}_{i\in\mathbb{Z}}$ then the set $X := X_1(\phi) \cup x_2(\phi)$ can be ordered as follows:*

$$\dots \leq x_1^0(\phi) \leq x_1^1(\phi) \leq x_2^0(\phi) \leq x_2^1(\phi) \leq x_1^2(\phi) \leq x_1^3(\phi) \leq \dots$$

Moreover the action of $\widetilde{\phi}(b_j)$ on $X_1(\phi)$ is given by

$$\widetilde{\phi}(b_j)x_1^{2i}(\phi) = x_2^{2i+2n-1}(\phi), \qquad \widetilde{\phi}(b_j)x_1^{2i+1}(\phi) = x_2^{2i+2n}(\phi).$$

Suppose that equality holds at some point. Then

$$\widetilde{rot}\left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)\right] = 0.$$

Proof. If $x_1^u(\phi) = x_2^t(\phi)$ then

$$\widetilde{\phi}(a_j)\widetilde{\phi}(b_j)\widetilde{\phi}(a_j)^{-1}\widetilde{\phi}(b_j)^{-1}x_2^t = \widetilde{\phi}(a_j)x_2^t = \widetilde{\phi}(a_j)x_1^u = x_1^u = x_2^t.$$

Thus $\widetilde{\mathrm{rot}}\left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)\right] = 0$, a contradiction. In particular $x_1^u(\phi) \neq x_2^t(\phi)$ for every u, t.
If $x_1^{2i}(\phi) = x_1^{2i+1}(\phi)$ then $x_2^{2i+2n-1} = \widetilde{\phi}(b_j)x_1^i(\phi) = \widetilde{\phi}(b_j)x_1^{i+1}(\phi) = x_2^{2i+2n}$. From

$$x_2^{2i+2n-1} \leq x_1^{2i+2n} \leq x_1^{2i+2n+1} \leq x_2^{2i+2n} = x_2^{2i+2n-1}$$

we get $x_1^{2i+2n} = x_2^{2i+2n-1}$ that contradicts the previous case.
Lastly the case $x_2^{2i} = x_2^{2i+1}$ reduces to the previous one once we apply $\widetilde{\phi}(b_j)$. □

Corollary 5.21. *Let $\phi \in N_j$ be a representation such that*

$$\widetilde{rot}\left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)\right] = \frac{1}{k}.$$

Then ϕ has a $j-$good fixed point set.

Proof. Let $\{\phi_n\}_{n\in\mathbb{N}} \subseteq N_j$ be a sequence of good representations converging to ϕ. Denote by $X_1(\phi_n) = \left\{x_1^i(\phi_n)\right\}_{i\in\mathbb{Z}}$ their $j-$good fixed point sets which satisfy properties *(2)* and *(3)*. Up to passing to a subsequence we may assume that $x_1^i(\phi_n) \longrightarrow x_1^i$ for every $i \in \mathbb{Z}$. The points $x_1^i(\phi) := x_1^i$ in the set $X_1(\phi) = \left\{x_1^i\right\}_{i\in\mathbb{Z}}$ are fixed by $\phi(a_j)$ since $\phi_n(a_j)$ uniformly converges to $\phi(a_j)$. Moreover, since $\phi_n(b_j) \longrightarrow \phi(b_j)$ uniformly, we have $x_2^u(\phi_n) = \phi_n(b_j)x_1^v(\phi_n) \longrightarrow \phi(b_j)x_1^v(\phi)$. Define $X_2(\phi) := \left\{x_2^i(\phi)\right\}_{i\in\mathbb{Z}}$. The set $X := X_1(\phi) \cup X_2(\phi)$ can be ordered as follows:

$$\dots \leq x_1^{j-1}(\phi) \leq x_1^j(\phi) \leq x_2^j(\phi) \leq \dots \leq x_{g-1}^j(\phi) \leq x_1^{j+1}(\phi) \leq \dots$$

as every $X_j(\phi_n)$ satisfies the strong form of the same inequality.
Finally, using again the uniform convergence of $\widetilde{\phi}_n(b_j)$ to $\widetilde{\phi}(b_j)$, we see that

$$\widetilde{\phi}(b_j)x_1^{2i}(\phi) = x_2^{2i+2n-1}(\phi), \qquad \widetilde{\phi}(b_j)x_1^{2i+1}(\phi) = x_2^{2i+2n}(\phi).$$

Now $\widetilde{\mathrm{rot}}\left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)\right] = \frac{1}{k}$, therefore, by Lemma 5.20 the inequalities are strict and thus the representation ϕ has a $j-$good fixed point set. $\square$

The next proposition (whose proof, being rather technical, is delayed to the Appendix) guarantees that having a $j-$good fixed is stable under perturbations in a neighborhood of a representation with maximal translation number of the lifted commutator.

Proposition 5.22. *Let $\phi \in N_j$ be a representation such that*

$$\widetilde{rot}_{[a_j,b_j]}(\phi) = \frac{1}{k}.$$

Then there exists an open neighborhood U of ϕ in $Rep(\Gamma)$ such that $U \subseteq N_j$.

5.2. Good representations. As for the $j-$good fixed set, being good gives quantitative constraints on the representation. In particular we have a bound on the translation number of the product of lifted commutators and on the Euler number:

Lemma 5.23. *Let $\phi \in Rep(\Gamma)$ be a good representation. Then*

$$\widetilde{rot}\left(\prod_{j=1}^{g-1}\left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)\right]\right) \leq \frac{2g-3}{k}.$$

The same estimate holds for every $\phi \in N_0$ by continuity.

Proof. The proof is again an application of Calegari-Walker Algorithm in the case of higher rank. Let ϕ be a good representation, define $c_j(\phi) := \left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)\right]$. Every $c_j(\phi)$

has a periodic orbit $X_j(\phi) = \left\{x_j^i(\phi)\right\}_{i\in\mathbb{Z}}$ (modulo $\mathbb{Z}$) of lenght 1 and period k. By property *(2)* the set $X := \bigcup_{j=1}^{g-1} X_j(\phi)$ can be ordered as follows

$$\dots x_{g-1}^{j-1}(\phi) < x_1^j(\phi) < x_2^j(\phi) < \dots < x_{g-1}^j(\phi) < x_1^{j+1}(\phi) < \dots$$

The dynamics of the associated dynamical system is described by

$$c_j \cdot x_{j'}^i(\phi) = \begin{cases} x_{j'}^{i+1}(\phi) & \text{if } j \geq j' \\ x_{j'}^{i+2}(\phi) & \text{if } j < j'. \end{cases}$$

Consider the positive word $w := c_1 \dots c_{g-1}$. We have

$$\begin{aligned} & w \cdot x_1^0(\phi) = c_1 \dots c_{g-1} \cdot x_1^0(\phi) \\ & = c_1 \dots c_{g-2} \cdot x_{g-1}^1(\phi) \\ & = c_1 \dots c_{g-3} \cdot x_{g-2}^3(\phi) \\ & = \dots \\ & = c_1 \cdot x_2^{2g-5}(\phi) = x_1^{2g-3}(\phi). \end{aligned}$$

Hence the orbit of $x_1^0(\phi)$ under the action of w is given by

$$x_1^0(\phi) \xrightarrow{w} x_1^{2g-3}(\phi) \xrightarrow{w} \dots \xrightarrow{w} x_1^{(k-1)(2g-3)}(\phi) \xrightarrow{w} x_1^{k(2g-3)}(\phi) = x_1^0 + 2g - 3.$$

Therefore $w \cdot x_1^0(\phi) = x_1^0(\phi) + 2g - 3$ which implies $\widetilde{\mathrm{rot}}\ w \leq \frac{2g-3}{k}$. □

The next result indicates that also in this case, as in Lemma 5.17, maximality implies rigidity on the dynamics of the lifted commutators:

LEMMA 5.24. *Suppose that* $\gcd(k, 2g-3) = 1$. *Let* $\phi \in Rep(\Gamma)$ *be a representation satisfying property* (1) *and property* (2) *with weak inequalities, i.e. there are periodic orbits (modulo $\mathbb{Z}$) $X_j(\phi)$ of $c_j(\phi) := \left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)\right]$ that can be ordered as*

$$\dots \leq x_{g-1}^{j-1}(\phi) \leq x_1^j(\phi) \leq x_2^j(\phi) \leq \dots \leq x_{g-1}^j(\phi) \leq x_1^{j+1}(\phi) \leq \dots$$

If $x_1^0(\phi) = x_2^0(\phi)$ then

$$\widetilde{rot}\left(\prod_{j=1}^{g-1}\left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)\right]\right) < \frac{2g-3}{k}.$$

PROOF. For notation sake let us drop ϕ from the notation $x_i^j(\phi)$, thus we set $x_i^j := x_i^j(\phi)$. The equality $x_1^0(\phi) = x_2^0(\phi)$ changes the dynamics of the associad dynamical system in Calegari-Walker Algorithm. Suppose that $x_i^j = x_{i+1}^j$ then

$$c_i \cdot x_{i+1}^j = x_i^{j+1}.$$

Thus $w \cdot x_1^j = x_1^{j+2g-3-u(j)}$ for some $u(j) > 0$ if there is some occurrence of $x_i^t = x_{i+1}^t$ in the sequence

$$x_1^j \longrightarrow c_{g-1} \cdot x_1^j \longrightarrow \dots \longrightarrow c_1 \dots c_{g-2} \cdot x_1^j \longrightarrow c_1 \dots c_{g-1} \cdot x_1^j.$$

We prove now that $w^k \cdot x_1^j = x_1^{j+n(j)}$ for some $n(j) < k(2g-3)$. Assume by contradiction that $w^k \cdot x_1^j = x_1^{j+k(2g-3)}$. Then, by the computation above, we should have an orbit like the following one

$$x_1^j \xrightarrow{w} x_1^{j+(2g-3)} \xrightarrow{w} \dots \xrightarrow{w} x_1^{j+(k-1)(2g-3)} \xrightarrow{w} x_1^{j+k(2g-3)} = w^k \cdot x_1^j.$$

In other words $u(j+d(2g-3)) = 0$ for every $d \leq k$. Since $\gcd(k, 2g-3) = 1$, for some $0 \leq d < k$ we have $k \mid j + d(2g-3)$. Then $x_1^{j+d(2g-3)} = x_1^{ks}$ for some $s \geq 0$ which implies that $u(j+d(2g-3)) \geq 1$ because $x_1^{ks} = x_1^0 + s = x_2^0 + s = x_2^{ks}$ by assumption.
Define recursively the sequence $\alpha_0 = 0$ and $\alpha_j = \alpha_{j-1} + n\left(\alpha_{j-1}\right)$. Necessarily we find $j \geq i$ such that $\alpha_j \equiv \alpha_i \pmod{k}$, i.e. $\alpha_j = \alpha_i + kv$ for some $v \geq 0$. By construction $w^{k(j-i)} x_1^{\alpha_i} = x_1^{\alpha_j} = x_1^{\alpha_i + kv} = x_1^{\alpha_i} + v$. In particular by Calegari-Walker Algorithm $\widetilde{\mathrm{rot}}\, w \leq \frac{v}{k(j-i)}$, thus, in order to get the desired strict inequality it is sufficient to prove that $\dfrac{v}{k(j-i)} < \dfrac{2g-3}{k}$ or equivalently $v < (j-i)(2g-3)$. The last inequality holds by a straightforward computation:

$$v = \frac{\alpha_j - \alpha_i}{k} = \frac{1}{k} \sum_{t=i+1}^{j} n(\alpha_t) < \frac{1}{k} \sum_{t=i+1}^{j} k(2g-3) = (j-i)(2g-3).$$

where we used $n(t) < k(2g-3)$ for every $t \in \mathbb{Z}$. $\square$

Remark 5.25. It is clear that the same result holds if we replace $x_1^0 = x_2^0$ with an equality at some point in the chain of weak inequalities.

Since every representation in N satisfies the weak form of properties *(1)* and *(2)* we get:

Corollary 5.26. *Suppose that* $\gcd(k, 2g-3) = 1$. *Let* $\phi \in N_0$ *be a representation such that*

$$\widetilde{rot}\left(\prod_{j=1}^{g-1} \left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j) \right] \right) = \frac{2g-3}{k}.$$

Then ϕ *is good.*

Proof. Let $\{\phi_n\}_{n\in\mathbb{N}} \subseteq N_0$ be a sequence of good representations converging to ϕ. Denote by $X_j(\phi_n) = \left\{ x_j^i(\phi_n) \right\}_{i\in\mathbb{Z}}$ their periodic sets (modulo $\mathbb{Z}$) which satisfy property *(2)*. Up to passing to a subsequence we may assume that $x_j^i(\phi_n) \longrightarrow x_j^i$ for every $j \leq g-1$ and $i \in \mathbb{Z}$. The set $X_j(\phi) = \left\{ x_j^i(\phi) := x_j^i \right\}_{i\in\mathbb{Z}}$ is a periodic orbit of $c_j(\phi)$ that satisfies weak property *(2)* as every $X_j(\phi_n)$ satisfies the strong form of the same property, i.e.

$$\dots \leq x_{g-1}^{j-1}(\phi) \leq x_1^j(\phi) \leq x_2^j(\phi) \leq \dots \leq x_{g-1}^j(\phi) \leq x_1^{j+1}(\phi) \leq \dots$$

Moreover $\widetilde{\mathrm{rot}}\ c_j(\phi) = \lim_{n\to\infty} \widetilde{\mathrm{rot}}\ c_j(\phi_n) = \frac{1}{k}$ by continuity of $\widetilde{\mathrm{rot}}$. Suppose that in the chain of weak inequalities above we have an equality at some point. The representation ϕ satisfies the hypothesis of Lemma 5.24 therefore $\widetilde{\mathrm{rot}}\left(\prod_{j=1}^{g-1}\left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)\right]\right) < \dfrac{2g-3}{k}$, a contradiction. $\square$

The following are high dimensional versions of cases *(1)* and *(2)* of Lemma 5.17, they will allow us to prove a stability result for representations in N with maximal Euler number.

PROPOSITION 5.27. *Suppose that* $\gcd(k, 2g-3) = 1$. *Let* $\phi \in N$ *be a good representation. Denote by* $c_j := \left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)\right]$ *for* $j \le g-1$ *the lifted commutators and with* $X_j(\phi) = \left\{x_j^i := x_j^i(\phi)\right\}_{i\in\mathbb{Z}}$ *the periodic set of* c_j. *These sets can be ordered as*

$$\cdots < x_{g-1}^{j-1} < x_1^j < x_2^j < \cdots < x_{g-1}^j < x_1^{j+1} < \ldots$$

Suppose that

$$\frac{2g-3}{k} = \widetilde{rot}\left(\prod_{j=1}^{g-1}\left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)\right]\right) = \widetilde{rot}\left(\prod_{j=1}^{g-1} c_j\right).$$

Then we have

$$c_{g-1}(x_1^i) > x_{g-2}^{i+1}.$$

PROOF. Proceed by contradiction and assume that $c_{g-1}(x_1^i) \le x_{g-2}^{i+1}$ for some $i \in \mathbb{Z}$. For simplicity $i = 0$ and $w := c_1 \ldots c_{g-1}$. Thinking of w as a homeomorphism we have

$$\begin{aligned}
&c_1 \ldots c_{g-1}(x_1^0) \le c_1 \ldots c_{g-2}(x_{g-2}^1)\\
&= c_1 \ldots c_{g-3}(x_{g-2}^2) < c_1 \ldots c_{g-3}(x_{g-3}^3)\\
&= c_1 \ldots c_{g-4}(x_{g-3}^4) < c_1 \ldots c_{g-4}(x_{g-4}^5)\\
&\le \ldots\\
&\le c_1(x_2^{2g-6}) < c_1(x_1^{2g-5}) = x_1^{2g-4}.
\end{aligned}$$

Thus $w(x_1^0) \le x_1^{2g-4}$. On the other hand if we think of w as a positive word we have

$$\begin{aligned}
&w \cdot x_1^i = (c_1 \ldots c_{g-1}) \cdot x_1^i\\
&= (c_1 \ldots c_{g-2}) \cdot x_{g-1}^{i+1}\\
&= (c_1 \ldots c_{g-3}) \cdot x_{g-2}^{i+3}\\
&= \ldots\\
&= c_1 \cdot x_2^{i+2g-5} = x_1^{i+2g-3}
\end{aligned}$$

for every $i \in \mathbb{Z}$. Hence $w \cdot x_1^i = x_1^{i+2g-3}$.

Now consider w^k. As homeomorphism it is controlled from above by the associated dynamical system, i.e. $w^k(x_j^i) \le w^k \cdot x_j^i$. Since $\gcd(k, 2g-3) = 1$ we can find $u, v \ge 1$ such that $v(2g-3)-uk=1$, therefore

$$\begin{aligned}&w^v(x_1^0)=w^{v-1}(w(x_1^0))\\&\le w^{v-1}(x_1^{2g-4})\\&\le w^{v-1}\cdot x_1^{2g-4}\\&=x_1^{2g-4+(v-1)(2g-3)}=x_1^{uk}=x_1^0+u.\end{aligned}$$

The previous computation implies that $\widetilde{\mathrm{rot}}\ w\le\frac{u}{v}=\frac{2g-3}{k}-\frac{1}{vk}$, but we already know that $\widetilde{\mathrm{rot}}\ w=\frac{2g-3}{k}$, a contradiction. $\square$

Remark 5.28. We stated the result of Proposition 5.27 only for c_{g-1} and x_1^i. Since $\widetilde{\mathrm{rot}}$ is conjugacy invariant we have

$$\begin{aligned}&\widetilde{\mathrm{rot}}\ c_1\dots c_{g-1}\\&=\widetilde{\mathrm{rot}}\ c_{g-1}\,(c_1\dots c_{g-1})\,c_{g-1}^{-1}=\widetilde{\mathrm{rot}}\ c_{g-1}\dots c_1\\&=\widetilde{\mathrm{rot}}\ c_{g-2}\,(c_{g-1}\dots c_{g-2})\,c_{g-2}^{-1}=\widetilde{\mathrm{rot}}\ c_{g-2}c_{g-1}\dots c_{g-3}\\&=\dots\\&=\widetilde{\mathrm{rot}}\ c_{g-1}\dots c_1.\end{aligned}$$

Thus it is clear that we can obtain the same result for every choice of c_j and x_{j+1}^i where indices are thought modulo $g-1$. Therefore the general case is

$$\begin{cases}c_1(x_2^i)>x_{g-1}^{i+1}\\c_j(x_{j+1}^i)>x_{j-1}^{i+2}\\c_{g-1}(x_1^i)>x_{g-2}^{i+1}.\end{cases}$$

Remark 5.29. Suppose that $\gcd(k,2g-3)=1$. Let ϕ be a representation as in Proposition 5.27. We observe that

$$\begin{aligned}&c_1^k(x_2^0)>c_1^{k-1}(x_2^1) && \text{as } x_2^1<x_n^1<c_1(x_2^0)\\&>c_1^{k-2}(x_2^2) && \text{analogously}\\&>\dots\\&>c_1(x_2^{k-1})>x_2^k=x_2^0+1.\end{aligned}$$

In particular $\widetilde{\mathrm{rot}}\ c_j\ge\frac{1}{k}$. The only property involved in the argument is $c_1(x_2^j)>x_n^j$ for every $j\in\mathbb{Z}$. We observe that this property (as those of Remark 5.28) is open with respect to the topology of the representations space. Thus it is satisfied for every ψ sufficiently close to the good representation ϕ. In conclusion there is an open neighborhood V of ϕ such that $\widetilde{\mathrm{rot}}\ c_j(\psi)\ge\frac{1}{k}$ for every $\psi\in V$.

If a representation $\phi\in N$ is good with respect to some choice of fixed set $X_j(\phi)$ then it is not clear (and in general not true) that it is good for any choices of $X_j(\phi)$. This stronger property holds for good representations with maximal Euler number:

Corollary 5.30. *Let $\phi \in N_0$ be a good representation such that*

$$\widetilde{rot}\left(\prod_{j=1}^{g-1}\left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)\right]\right) = \frac{2g-3}{k}.$$

Suppose that $Y_j(\phi) = \left\{y_j^i := y_j^i(\phi)\right\}$ *is a periodic orbit of* $c_j(\phi)$*. Then the set* $Y := \bigcup_{j=1}^{g-1} Y_j(\phi)$ *can be ordered in order to satisfy property* (2)

$$\cdots < y_{g-1}^{j-1}(\phi) < y_1^j(\phi) < y_2^j(\phi) < \cdots < y_{g-1}^j(\phi) < y_1^{j+1}(\phi) < \ldots$$

Proof. It is enough to prove the case where $Y_j(\phi)$ is arbitrary and $Y_u(\phi) = X_u(\phi)$ for every $u \neq j$. Moreover, since the order of Y is invariant up to conjugacy it is enough to prove the result for a fixed j. Thus fix $1 < j < g$.
First let us prove that for every $y \in Y_j(\phi)$ we have $y \in (x_{j-1}^i, x_{j+1}^i)$ for some $i \in \mathbb{Z}$. Proceed by contradiction and suppose that for every $i \in \mathbb{Z}$ we have $y \notin (x_{j-1}^i, x_{j+1}^i)$. Thus $y \in [x_{j+1}^i, x_{j-1}^{i+1}]$ for some $i \in \mathbb{Z}$. Hence we get

$$\begin{aligned}
&c_j^k(y) \geq c_j^k(x_{j+1}^i)\\
&> c_j^{k-1}(x_{j-1}^{i+2}) > c_j^{k-1}(x_{j+1}^{i+1})\\
&> c_j^{k-2}(x_{j-1}^{i+3}) > c_j^{k-2}(x_{j+1}^{i+2})\\
&> \ldots\\
&> c_j(x_{j-1}^{i+k}) > x_{j+1}^{i+k+1}\\
&= x_{j-1}^{i+1} + 1 \geq y + 1
\end{aligned}$$

where every line is a consecutive application of Remark 5.28 and $x_{j-1}^u > x_{j+1}^{u-1}$.
On the other hand we have $y \leq x_{j-1}^{i+1}$, therefore

$$c_j^k(y) \leq c_j^k(x_{j-1}^{i+1}) < c_j^k \cdot x_{j-1}^{i+1} = x_j^{i+1+k} < x_j^{i+1} + 1 < x_{j+1}^{i+1} + 1 \leq x_{j+1}^{i+k} + 1 = x_{j+1}^i + 2 \leq y + 2.$$

Since y is a periodic point for c_j modulo $\mathbb{Z}$ of period k we have $c_j^k(y) - y \in \mathbb{Z}$, but we have just shown that $y + 1 < c_j^k(y) < y + 2$, a contradiction.
Now we prove that the set $Y_j(\phi)$ can be ordered in such a way that $y_j^i \in (x_{j-1}^i, x_{j+1}^i)$ for every $i \in \mathbb{Z}$. We will show that if $y \in Y(\phi)$ satisfies $y \in (x_{j-1}^i, x_{j+1}^i)$ then $c_j(y) \in (x_{j-1}^{i+1}, x_{j+1}^{i+1})$. We already know that $c_j(y) \in (x_{j-1}^u, x_{j+1}^u)$ by the previous argument for some $u \in \mathbb{Z}$. We observe that there are only two of possible values for u: from

$$c_j(y) < c_j(x_{j+1}^i) \leq c_j \cdot x_{j+1}^i = x_j^{i+2}$$

we get $u \leq i + 2$. On the other side Remark 5.28 gives

$$c_j(y) > c_j(x_{j-1}^i) > c_j(x_{j+1}^{i-1}) > x_{j-1}^{i+1}.$$

Thus $u \geq i + 1$. We show how to exclude the case $u = i + 2$, the method is completely analogous to the one used in the first part of the proof. Let us divide two cases: if $y \leq x_j^i$

then $c_j(y) < x_j^{i+1} < x_{j-1}^{i+2}$ which implies in turn $u = i+1$ so there is nothing to add. Now suppose that $u = i+2$ and $x_j^i < y$ and consider $c_j^k(y)$. Using repeatedly Remark 5.28 we find

$$\begin{aligned} & c_j^k(y) \geq c_j^{k-1}(x_{j-1}^{i+2}) \\ & > c_j^{k-1}(x_{j+1}^{i+1}) > c_j^{k-2}(x_{j-1}^{i+3}) \\ & > \dots \\ & > c_j(x_{j+1}^{i+k-2}) > x_{j-1}^{i+k} \\ & = x_{j-1}^i + 1 > y + 1. \end{aligned}$$

On the other hand

$$c_j^k(y) \leq c_j^k(x_{j+1}^i) < c_j^k \cdot x_{j+1}^i = x_j^{i+1+k} \leq x_j^{i+2k} = x_j^i + 2 < y + 2.$$

Hence $1 < c_j^k(y) - y < 2$, but $c_j(y) - y \in \mathbb{Z}$, a contradiction. $\square$

The stability results for $\widetilde{\mathrm{rot}}_{[a_j,b_j]}(\phi)$ and for $\widetilde{\mathrm{rot}}\left(\prod_{j=1}^{g-1}\left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)\right]\right)$ together allow to prove the following fundamental stability property for good representations that are close to a maximal one:

Proposition 5.31. *Let $\phi \in N$ be a representation such that*

$$\widetilde{rot}\left(\prod_{j=1}^{g-1}\left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)\right]\right) = \frac{2g-3}{k}.$$

Then there exists an open neighborhood U of ϕ in $Rep(\Gamma)$ such that $U \subseteq N$.

Proof. First we observe that by Corollary 5.26 the representation ϕ is good.
Since $\phi \in N_0$ we have $\widetilde{\mathrm{rot}}\left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)\right] = \dfrac{1}{k}$ for every $j \leq g$. By Proposition 5.22, we can find open sets U_j in $\mathrm{Rep}(\Gamma)$ such that $U_j \subseteq N_j$ for every $j \leq g$. Define $U := \displaystyle\bigcap_{j=1}^{g} U_j$.
Thus it is enough to find an open subset $U_0 \subseteq \mathrm{Rep}(\Gamma)$ such that $U_0 \subseteq N_0$. Suppose by contradiction that ϕ is a boundary point of N_0. Then we can find a sequence $\{\phi_n\}_{n\in\mathbb{N}} \subseteq \mathrm{Rep}(\Gamma) \setminus N_0$ approaching ϕ i.e. $\phi_n \longrightarrow \phi$. Up to passing to a subsequence we may assume that $\phi_n \in U \setminus N_0$ for every $n \in \mathbb{N}$, in particular $\widetilde{\mathrm{rot}}\left[\widetilde{\phi}_n(a_j), \widetilde{\phi}_n(b_j)\right] \leq \frac{1}{k}$ for every $j \leq g$. Since ϕ is good and attains the maximal value for the translation number of the product of lifted commutators allowed for good representations we have by Remark 5.29 that $wrot\left[\widetilde{\psi}(a_j), \widetilde{\psi}(b_j)\right] \geq \frac{1}{k}$ for every ψ in a neighborhood V of ϕ and every $j \leq g-1$. Up to passing to a further subsequence we may assume that $\phi_n \in V \cap U$ for every $n \in \mathbb{N}$, hence $\widetilde{\mathrm{rot}}\left[\widetilde{\phi}_n(a_j), \widetilde{\phi}_n(b_j)\right] = \frac{1}{k}$ for every $n \in \mathbb{N}$.
Every $c_j(\phi_n)$ has a $k-$periodic orbit $Y_j(\phi_n)$. By compactness of $k-$tuples of points on S^1, up to passing to a subsequence, we may assume that $Y_j(\phi_n)$ converges to some set Y_j. It

is clear that Y_j is a periodic orbit of ϕ. By Corollary 5.30 the set $Y(\phi) := \bigcup_{j=1}^{g-1} Y_j$ can be ordered as follows

$$\cdots < y_{g-1}^{j-1}(\phi) < y_1^j(\phi) < y_2^j(\phi) < \cdots < y_{g-1}^j(\phi) < y_1^{j+1}(\phi) < \ldots$$

with the same properties as in the definition of a good representation. Finally if $Y_j(\phi_n)$ are sufficiently close to $Y_j(\phi)$, then the set $Y(\phi_n) := \bigcup_{j=1}^{g-1} Y_j(\phi_n)$ can be ordered exactly like $Y(\phi)$. In particular ϕ_n is a good representation for n sufficiently large, but this contradicts the assumption $\phi_n \notin N_0$. $\square$

REMARK 5.32. If $\phi \in N$ is a representation as in Proposition 5.31 we can choose the open neighborhood U such that the equality

$$\widetilde{\mathrm{rot}}\left(\prod_{j=1}^{g-1}\left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)\right]\right) = \frac{2g-3}{k}$$

holds in the whole neighborhood U. In fact it suffices to shrink U to $U \cap e^{-1}\left(\frac{2g-2}{k}\right)$. Let us prove this assertion. First we notice that $e(\phi) = \frac{2g-2}{k}$. Then we observe that if $\psi \in N$ and $e(\psi) = \left(\frac{2g-2}{k}\right)$ then we have

$$\widetilde{\mathrm{rot}}\left(\prod_{j=1}^{g-1}\left[\widetilde{\psi}(a_j), \widetilde{\psi}(b_j)\right]\right) = \frac{2g-3}{k} \qquad \text{and} \qquad \widetilde{\mathrm{rot}}\left[\widetilde{\psi}(a_j), \widetilde{\psi}(b_j)\right] = \frac{1}{k}.$$

This follows from

$$\widetilde{\mathrm{rot}}\left[\widetilde{\psi}(a_j), \widetilde{\psi}(b_j)\right] + \widetilde{\mathrm{rot}}\left(\prod_{j=1}^{g-1}\left[\widetilde{\psi}(a_j), \widetilde{\psi}(b_j)\right]\right) = e(\psi)$$

and the fact that, on N, the first summand is smaller than $\frac{2g-3}{k}$ while the second is lesser than $\frac{1}{k}$. In particular Proposition 5.31 tells us that $N \cap e^{-1}\left(\frac{2g-2}{k}\right)$ is open in $\mathrm{Rep}(\Gamma)$.

CHAPTER 6

A Theorem by Mann, Matsumoto

In this final chapter we prove the main rigidity result for $\mathrm{PSL}^{(k)}(2,\mathbb{R})-$geometric representations due to K. Mann and S. Matsumoto. The precise statement is the following

THEOREM (Mann, Matsumoto). *Let S be a closed oriented surface of genus $g \geq 2$. Let $\phi : \Gamma := \pi_1(S) \longrightarrow Homeo_+\left(S^1\right)$ be a geometric representation. Then the connected component of ϕ in $Rep\,(\Gamma)$ is a single semi-conjugacy class.*

First we describe some easy facts about the constancy of rotation numbers and of the standard cocycles near a geometric representation. Then we describe a procedure, an Euclidean Algorithm also due to K. Mann, to reduce the proof of local stability for geometric representations to the case where $\mathrm{rot}\,(\phi(a_j)) = 0$ for every $j \leq g$. Finally in the last part we exploit the work done in the previous chapters to give a proof of the main theorem.

1. Local stability and geometric representations

In this section we give some evidences that geometric representations are good candidates for being locally stable. Unfortunately the results we are going to prove do not hold uniformly for every element $\gamma \in \Gamma$.

1.1. Rotation numbers are locally constant near a geometric representation.

The next lemma is an easy consequence of Lemmas 5.16 and 5.17:

LEMMA 6.1. *Let $a \in G$ be an element for which there exists $b \in G$ such that $\widetilde{rot}\left[\widetilde{a}, \widetilde{b}\right] \neq 0$ where $\widetilde{a}, \widetilde{b} \in \widetilde{G}$ are arbitrary lifts of a, b. Then there exists an open neighborhood $U \subseteq G$ of $a \in G$ such that $rot|_U$ is constant.*

PROOF. Proceed by contradiction. Suppose that for every neighborhood U of a there exists an element $a_U \in U$ such that $\mathrm{rot}(a_U) \neq \mathrm{rot}(a)$. Let us fix a sequence of neighborhoods $\{U_n\}_{n\in\mathbb{N}}$ of a such that $\bigcap_{n\in\mathbb{N}} U_n = \{a\}$. Let $a_1 \in U_1$ be an element such that $\mathrm{rot}(a_1) \neq \mathrm{rot}(a)$. Consider U_2, then $V_2 = U_2 \cap \mathrm{rot}^{-1}\left(S^1 \setminus \{\mathrm{rot}(a_1)\}\right)$ is again an open neighborhood of a thus we can find $a_2 \in V_2$ such that $\mathrm{rot}(a_2) \notin \{\mathrm{rot}(a), \mathrm{rot}(a_1)\}$. Analogously we find $a_3 \in V_3 = U_3 \cap \mathrm{rot}^{-1}\left(S^1 \setminus \{\mathrm{rot}(a_1), \mathrm{rot}(a_2)\}\right)$ such that $\mathrm{rot}(a_3) \notin \{\mathrm{rot}(a), \mathrm{rot}(a_1), \mathrm{rot}(a_2)\}$ and so on. We end up with a sequence $\{a_n\}_{n\in\mathbb{N}}$ that satisfies the following properties: firstly we have the convergence $a_n \longrightarrow a$, secondly for every $n \neq m$ we have $\mathrm{rot}(a_n) \neq \mathrm{rot}(a_m)$ and also $\mathrm{rot}(a_n) \neq \mathrm{rot}(a)$. Since $\mathrm{rot}(a_n) \longrightarrow \mathrm{rot}(a)$ the sequence $\{\mathrm{rot}(a_n)\}_{n\in\mathbb{N}}$ contains either a subsequence of irrational numbers or a subsequence of rational numbers whose denominator diverges to ∞. In both cases Lemmas 5.16 and 5.17 imply $\widetilde{\mathrm{rot}}\left[\widetilde{a}_n, \widetilde{b}\right] \longrightarrow 0$, but by continuity $\widetilde{\mathrm{rot}}\left[\widetilde{a}_n, \widetilde{b}\right] \longrightarrow \widetilde{\mathrm{rot}}\left[\widetilde{a}, \widetilde{b}\right] \neq 0$, contradiction. $\square$

Using Lemma 6.1 and Scott's trick we are able to prove the following:

THEOREM 6.2. *Let $\phi : \Gamma \longrightarrow PSL^{(k)}(2, \mathbb{R})$ be a geometric representation. For every $\gamma \in \Gamma$ there is an open neighborhood $U_\gamma \subseteq Rep(\Gamma)$ of ϕ such that $rot_\gamma|_{U_\gamma}$ is constant.*

PROOF. Let us prove the theorem for a generator $\gamma := a_1$. Let $j : \mathbb{Z} \longrightarrow \Gamma$ be the homomorphism defined by $j(1) := a_1$. Denote by $J : \mathrm{Hom}(\Gamma, G) \longrightarrow \mathrm{Hom}(\mathbb{Z}, G) = G$ the map induced by composition $J(\psi) := \psi(a_1)$. Since ϕ is geometric we have $\widetilde{\mathrm{rot}}\left[\widetilde{\phi}(a_1), \widetilde{\phi}(b_1)\right] = \frac{1}{k}$ thus, by Lemma 6.1, there is an open neighborhood $V \subseteq G$ of $j(1) = \phi(a_1)$ such that $\mathrm{rot}|_V$ is constant. Define $U := J^{-1}(V)$. For every $\psi \in U$ we have $J(\psi) = \phi(a_1) \in V$ therefore $\mathrm{rot}_{a_1}(\psi) = \mathrm{rot}_{a_1}(\phi)$. The general result follows from Scott's trick. Let $c \in \Gamma \setminus \{1\}$ be any non-element loop, by Scott's Theorem 3.9 there exists a finite covering $p : S' \longrightarrow S$ such that c lifts to a non separating simple closed curve c'. In particular we can include c' in a standard set of generators of $\pi_1(S')$ as $c' = a'_1$. Denote by $p_* : \pi_1(S') \longrightarrow \Gamma$ the inclusion homomorphism and by $P : \mathrm{Hom}(\Gamma, G) \longrightarrow \mathrm{Hom}(\pi_1(S'), G)$ the map induced by composition $P(\psi) := \psi p_*$. Since ϕ is geometric also $\rho := \phi p_* : \pi_1(S') \longrightarrow \mathrm{PSL}^{(k)}(2, \mathbb{R})$ is geometric. In particular $\widetilde{\mathrm{rot}}\,[\widetilde{\rho}(a'_1), \widetilde{\rho}(b'_1)] = \frac{1}{k}$. By the previous case we can find an open neighborhood $V \subseteq \mathrm{Hom}(\pi_1(S'), G)$ of ρ such that $\mathrm{rot}_{a'_1}|_V$ is constant. Define $U := P^{-1}(V)$. For every $\psi \in U$ we have $P(\psi)(a'_1) = \phi(c) \in V$ therefore $\mathrm{rot}_c(\psi) = \mathrm{rot}_{a'_1}(P(\psi)) = \mathrm{rot}_{a'_1}(P(\phi)) = \mathrm{rot}_c(\phi)$. $\square$

Theorem 6.2 has the following consequence:

THEOREM 6.3. *Let $\phi : \Gamma \longrightarrow PSL^{(k)}(2, \mathbb{R})$ be a geometric representation. For every $\alpha, \beta \in \Gamma$ there is an open neighborhood $V_{\alpha,\beta} \subseteq Rep(\Gamma)$ of ϕ such that $\tau_{\alpha,\beta}|_{V_{\alpha,\beta}}$ is constant.*

PROOF. Since ϕ is a geometric $\mathrm{PSL}^{(k)}(2, \mathbb{R})$ −representation, by Corollary 3.8 for every $\gamma \in \Gamma$ we have $\mathrm{rot}_\gamma(\phi) = \frac{p_\gamma}{k}$ for some $p_k \in \mathbb{N}$. The function $\tau_{\alpha,\beta}$ restricted to $U_\alpha \cap U_\beta$ (where U_α, U_β are given by Theorem 6.2) is continuous and has image contained in $\dfrac{1}{k}\mathbb{Z} := \left\{\dfrac{m}{k} : m \in \mathbb{Z}\right\}$ which is a discrete set. Thus it is locally constant. $\square$

As we have already pointed out Theorems 6.2 and 6.3 are not enough to prove local stability since the neighborhoods U_γ and $V_{\alpha,\beta}$ depend on the elements $\alpha, \beta, \gamma \in \Gamma$.

2. Euclidean algorithm

An Euclidean Algorithm Theorem permits us to reduce the proof of the Main Theorem to the case where of geometric representations $\phi : \Gamma \longrightarrow \mathrm{PSL}^{(k)}(2, \mathbb{R})$ with the further property $\mathrm{rot}(\phi(a_j)) = 0$ for every $j \leq g$. In order to prove the Euclidean Algorithm Theorem we first introduce crossed pairs and use them to do some explicit computations of rotation numbers.

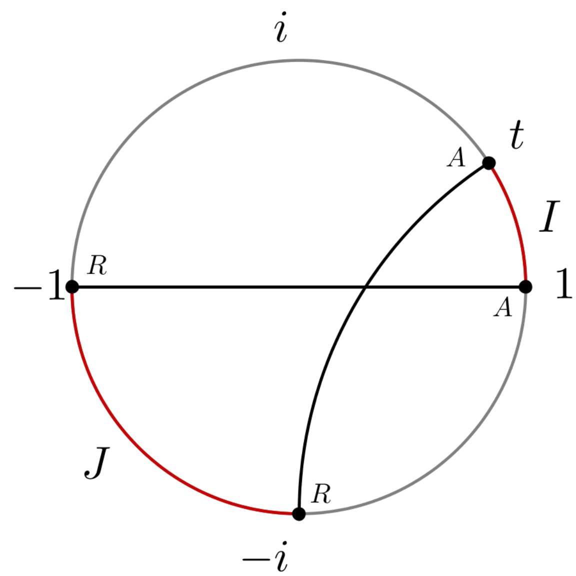

FIGURE 1.

2.1. Crossed pairs. Let us introduce the following terminology:

DEFINITION 6.4. A pair $(a,b) \in \mathrm{PSL}^{(k)}(2,\mathbb{R}) \times \mathrm{PSL}^{(k)}(2,\mathbb{R})$ is a *crossed pair* if a,b are the lifts of hyperbolic motions in $\mathrm{PSL}(2,\mathbb{R})$ with negatively intersecting axes.

There are simple operations on pairs that preserve the property of being crossed.

LEMMA 6.5. *Let $(a,b) \in PSL^{(k)}(2,\mathbb{R}) \times PSL^{(k)}(2,\mathbb{R})$ be a crossed pair. Both (ab,b) and (a,ba) are crossed pairs.*

PROOF. Let $\overline{a}, \overline{b}$ be the projections of a,b to $\mathrm{PSL}(2,\mathbb{R})$. We think of $\overline{a}, \overline{b}$ as hyperbolic isometries of the Poincaré disk model of $\mathbb{H}^2 \subseteq \mathbb{R}^2 = \mathbb{C}$. Up to conjugacy (which does not change the property of being a crossed pair) we can assume that $\overline{a}$ has a repelling fixed point in -1 and an attractive fixed point in $+1$, furthermore we can also assume that g has as repelling fixed point $-i$ and as attractive fixed point some $t \in S^1$ in the upper half circle delimited by the axes of $\overline{a}$. Consider the short arcs $I := [1,t]$ and $J := [-1,-i]$ (see Figure 1). By construction $\overline{a}I, \overline{b}I \subseteq I$ and $\overline{a}^{-1}J, \overline{b}^{-1}J \subseteq J$. In particular $\overline{a}\overline{b}I \subseteq I$ and $\left(\overline{a}\overline{b}\right)^{-1} J \subseteq J$. The last property implies that $\overline{a}\overline{b}$ has an attractive fixed point in I and a repelling fixed point in J, therefore it is a hyperbolic motion whose axes negatively intersects the axes of $\overline{b}$. Thus (ab,b) is a crossed pair. The same argument shows that (a,ba) is a crossed pair. $\square$

The advantage of considering crossed pairs comes from the fact that they allow simple explicit computations of rotation numbers:

LEMMA 6.6. *Let $(a,b) \in PSL^{(k)}(2,\mathbb{R}) \times PSL^{(k)}(2,\mathbb{R})$ be a crossed pair. Then*

(1) If $\widetilde{a}, \widetilde{b}$ are the lifts of a, b to $\widetilde{G}$, then

$$\widetilde{rot}\left[\widetilde{a}, \widetilde{b}\right] \geq 0.$$

(2) If $rot(a) = rot(b) = 0$ then for every positive word $w \in F_2$

$$rot(w(a,b)) = 0.$$

(3) Let $w := x^{\alpha_1} y^{\beta_1} \dots x^{\alpha_t} y^{\beta_t}$ be a positive word. If $rot(a) = \frac{p}{k}$ and $rot(b) = \frac{q}{k}$ then

$$rot(w(a,b)) = \frac{p(\alpha_1 + \dots + \alpha_t) + q(\beta_1 + \dots + \beta_t)}{k} = l_x(w) rot(a) + l_y(w) rot(b).$$

where $l_x(w)$ and $l_y(w)$ are respectively the numbers of x's and y's in w.

PROOF. Let us prove *(1)*. Let $\sigma : G \longrightarrow \widetilde{G}$ be an auxiliary section, we have

$$\widetilde{\mathrm{rot}}\left[\sigma(a), \sigma(b)\right] = \frac{1}{k} \widetilde{\mathrm{rot}}\left[\sigma(\overline{a}), \sigma\left(\overline{b}\right)\right].$$

Thus it is sufficient to show *(1)* when $k = 1$ or $a, b \in \mathrm{PSL}(2,\mathbb{R})$. Since $[\sigma(a), \sigma(b)]$ is independent of the section we can assume that $\sigma(a), \sigma(b)$ have both fixed points, say $x_r, x_a \in [0,1]$ and $y_r, y_a \in [0,1]$ where the first is the repelling one and the second is the attractive one. Moreover, since (a,b) is a crossed pair, the four fixed ponts are ordered as $x_r < y_r < x_a < y_a$. Observe that on the interval (y_a, x_a) the homeomorphism $\sigma(b)$ moves the points to the right (the same is true for $\sigma(a)$), hence there exists a point $t \in (y_a, x_a)$ such that $\sigma(b)^{-1} t = x_a$. Finally

$$\sigma(a)\sigma(b)\sigma(a)^{-1}\sigma(b)^{-1} t = \sigma(a)\sigma(b)\sigma(a)^{-1} x_a = \sigma(a)\sigma(b) x_a = \sigma(a) t \geq t$$

which implies $\widetilde{\mathrm{rot}}\left[\sigma(a), \sigma(b)\right] \geq 0$.
Now we prove *(2)*. Let $I \subseteq S^1$ be an interval such that $\overline{a}I, \overline{b}I \subseteq I$ as in Lemma 6.5. Since $\mathrm{rot}(a) = 0$ and $\mathrm{rot}(b) = 0$ we can lift I to an interval $I' \subseteq S^1$ on the $k-$fold cover of S^1 such that $aI', bI' \subseteq I'$. Thus for any positive word $w \in F_2$ we have $w(a,b)I' \subseteq I'$ which implies that $w(a,b)$ has a fixed point in I' and $\mathrm{rot}(w(a,b)) = 0$.
Point *(3)* can be reduced to point *(2)* by a trick. Let $r := \rho_{\frac{1}{k}}$ denote the rotation of $\frac{1}{k}$. Both a, b commute with r. Since a, b are $k-$lifts of homeomorphisms (hyperbolic motions) that have fixed points we have $\mathrm{rot}(a) = \frac{p}{k}$ and $\mathrm{rot}(b) = \frac{q}{k}$ for some $0 \leq p, q \leq k-1$. Define $c := ar^{-p}$ and $d := br^{-q}$. The projections of c, d on $\mathrm{PSL}(2,\mathbb{R})$ coincide with those of a, b so (c,d) is again a crossed pair. Since r commutes with a and b we have

$$\mathrm{rot}(c) = \mathrm{rot}(a) + \mathrm{rot}(r^{-p}) = 0, \qquad \mathrm{rot}(d) = \mathrm{rot}(b) + \mathrm{rot}(r^{-q}) = 0.$$

Hence by point *(2)* we get $\mathrm{rot}(w(c,d)) = 0$, but

$$w(c,d) = w(a,b) r^{-p(\alpha_1 + \dots + \alpha_t) - q(\beta_1 + \dots + \beta_t)}$$

therefore

$$
\begin{aligned}
0 = \mathrm{rot}\,(w(c,d)) &= \mathrm{rot}\,(w(a,b)) + \mathrm{rot}\left(r^{-p(\alpha_1+\dots+\alpha_t)-q(\beta_1+\dots+\beta_t)}\right) \\
&= \mathrm{rot}\,(w(a,b)) - \frac{p(\alpha_1+\dots+\alpha_t)+q(\beta_1+\dots+\beta_t)}{k}.
\end{aligned}
$$

□

2.2. The Euclidean Algorithm. Now we are ready to prove the following theorem:

THEOREM 6.7 (Euclidean Algorithm, Mann [**Man14**]). *Let $(a,b) \in PSL^{(k)}(2,\mathbb{R})$ be a crossed pair. Then there exist positive words $u, v \in F_2$ that satisfy the following conditions:*

(1) $[a,b] = [u(a,b), v(a,b)]$.
(2) $rot(u(a,b)) = 0$.
(3) $(u(a,b), v(a,b))$ is a crossed pair.

PROOF. Consider the rotation numbers $\mathrm{rot}(a) = \frac{p}{k}$ and $\mathrm{rot}(b) = \frac{q}{k}$.
By the standard Euclidean algorithm for $\gcd(p,q)$ we can recursively find m_j, r_j such that

$$
\begin{aligned}
& m_0 := p\ ,\ r_0 := q \\
& m_0 = m_1 r_0 + r_1 \\
& m_1 = m_2 r_1 + r_2 \\
& r_1 = m_3 r_2 + r_3 \\
& \dots \\
& r_{n-1} = m_{n+1} r_n
\end{aligned}
$$

where $r_n = \gcd(p,q)$. Define the elements

$$
\begin{array}{ll}
a_0 := a & b_0 := b \\
a_1 := a_0 b_0^{\overline{m_1}} & b_1 := b_0 a_1^{\overline{m_2}} \\
a_2 := a_1 b_1^{\overline{m_3}} & b_2 := b_1 a_2^{\overline{m_4}} \\
\dots &
\end{array}
$$

where $\overline{m_j}$ is a positive integer such that $\overline{m_j} \equiv -m_j \pmod{k}$.
It is immediate to check that every a_j, b_j is a positive word of a, b.
By point *(3)* of Lemma 6.6 we have

$$
\begin{array}{ll}
\mathrm{rot}(a_0) = \frac{p}{k} & \mathrm{rot}(b_0) = \frac{q}{k} \\
\mathrm{rot}(a_1) = \frac{p+\overline{m_1}q}{k} \equiv \frac{r_1}{k} \pmod{\mathbb{Z}} & \mathrm{rot}(b_1) = \frac{q+\overline{m_2}r_1}{k} \equiv \frac{r_2}{k} \pmod{\mathbb{Z}} \\
\mathrm{rot}(a_2) = \frac{r_1+\overline{m_3}r_2}{k} \equiv \frac{r_3}{k} \pmod{\mathbb{Z}} & \mathrm{rot}(b_2) = \frac{r_2+\overline{m_4}r_3}{k} \equiv \frac{r_4}{k} \pmod{\mathbb{Z}} \\
\dots &
\end{array}
$$

the last term appearing in the sequence has rotation number $\mathrm{rot} \equiv 0 \pmod{\mathbb{Z}}$.
An easy induction using Lemma 6.5 proves that every pair in the following sequence is a crossed pair

$$
(a_0,b_0)\,,\ (a_1,b_0)\,,\ (a_1,b_1)\,,\ (a_2,b_1)\ (a_2,b_2)\,,\dots,(a_j,b_{j-1})\,,\ (a_j,b_j)\,,\dots
$$

Finally using the identity $[a,b]=[ab^n,b]=[a,ba^m]$ we have

$$\begin{aligned}
[a_0,b_0] &= \left[a_0 b_0^{\overline{m_1}}, b_0\right] = [a_1,b_0]\\
[a_1,b_0] &= \left[a_1, b_0 a_1^{\overline{m_2}}\right] = [a_1,b_1]\\
&\dots
\end{aligned}$$

This proves that given (a,b) we can find positive words $u,v\in F_2$ such that properties *(1)* and *(3)* hold for u,v while property *(2)* holds for u or for v. Suppose that $\mathrm{rot}(v(a,b))=0$ let us prove that without losing properties *(1)* and *(3)* we can impose property *(2)* for the word u. We claim that the pair $\left(uvu^{k-1}, vu^{k-1}\right)$ satisfies all the conditions: property *(1)* follows from the chain of identities $[a,b]=[u,v]=\left[u,vu^{k-1}\right]=\left[u\left(vu^{k-1}\right),vu^{k-1}\right]$. Property *(3)* follows from Lemma 6.5 since the new pair is obtained from the old with the following sequence of operations $(u,v)\longrightarrow\left(u,vu^{k-1}\right)\longrightarrow\left(u\left(vu^{k-1}\right),vu^{k-1}\right)$. Finally property *(2)* follows from point *(3)* of Lemma 6.6 since $\mathrm{rot}\left(uvu^{k-1}\right)=k\mathrm{rot}(u)+\mathrm{rot}(v)\equiv 0 \pmod{\mathbb{Z}}$. ☐

2.3. Reduction to the case where $\mathbf{rot}_{a_j}(\phi)\equiv 0$. We use the Euclidean algorithm to reduce the proof of the main theorem to a particular case:

LEMMA 6.8. *If every maximal $PSL^{(k)}(2,\mathbb{R})$ representation ϕ with $rot_{a_j}(\phi)=0$ for every j is locally stable then all maximal $PSL^{(k)}(2,\mathbb{R})$ representations are locally stable.*

PROOF. Assume that every $\mathrm{PSL}^{(k)}(2,\mathbb{R})$ maximal representation with $\mathrm{rot}_{a_j}(\phi)=0$ for every generator $a_j\in\Gamma$ is locally stable. Let ψ be another maximal $\mathrm{PSL}^{(k)}(2,\mathbb{R})$ representation. By Theorem 6.7 we can find $u_j,v_j\in F_2$ relative to the crossed pairs $(\psi(a_j),\psi(b_j))$ with the properties

(1) $[\rho(a_j),\rho(b_j)]=[u_j(\rho(a_j),\rho(b_j)),v_j(\rho(a_j),\rho(b_j))]$ for every ρ.
(2) $\mathrm{rot}(u_j(\psi(a_j),\psi(b_j)))=0$.
(3) $(u_j(\psi(a_j),\psi(b_j)),v_j(\psi(a_j),\psi(b_j)))$ is a crossed pair.

Define the map

$$F:\mathrm{Rep}\,(\Gamma)\longrightarrow\mathrm{Rep}\,(\Gamma)\qquad\text{where } F(\rho) \text{ extends}\qquad\begin{cases}F(\rho)(a_j):=u_j\left(\rho(a_j),\rho(b_j)\right)\\ F(\rho)(b_j):=v_j\left(\rho(a_j),\rho(b_j)\right).\end{cases}$$

Property *(1)* ensures that $F(\rho)$ is well defined

$$\prod_{j=1}^{g}\left[F(\rho)(a_j),F(\rho)(b_j)\right]=\prod_{j=1}^{g}\left[u_j(\phi(a_j),\phi(b_j)),v_j(\phi(a_j),\phi(b_j))\right]=\prod_{j=1}^{g}\left[\phi(a_j),\phi(b_j)\right]=1.$$

Moreover the function F is continuous.
We prove that $F(\psi)$ is a locally stable maximal $\mathrm{PSL}^{(k)}(2,\mathbb{R})$ representation. By Milnor's Formula 2.13 the Euler number is computed by

$$e\left(F(\psi)\right)=\widetilde{\mathrm{rot}}\left(\prod_{j=1}^{g}\left[\widetilde{F(\psi)}(a_j),\widetilde{F(\psi)}(b_j)\right]\right).$$

Thus, since ψ is maximal, in order to prove that $F(\psi)$ is maximal it would be enough to show that $\left[\widetilde{F(\psi)}(a_j), \widetilde{F(\psi)}(b_j)\right] = \left[\widetilde{\psi(a_j)}, \widetilde{\psi}(b_j)\right]$. By construction we have $[F(\psi)(a_j), F(\psi)(b_j)] = [\psi(a_j), \psi(b_j)]$, therefore

$$\left[\widetilde{F(\psi)}(a_j), \widetilde{F(\psi)}(b_j)\right] = \left[\widetilde{\psi}(a_j), \widetilde{\psi}(b_j)\right] \tau_n$$

for some $n \in \mathbb{Z}$. Computing translation numbers from both sides of the equality we get

$$\widetilde{\mathrm{rot}} \left[\widetilde{F(\psi)}(a_j), \widetilde{F(\psi)}(b_j)\right] = \widetilde{\mathrm{rot}} \left[\widetilde{\psi}(a_j), \widetilde{\psi}(b_j)\right] + n = \frac{1}{k} + n.$$

Since $(F(\psi)(a_j), F(\psi)(b_j))$ is a crossed pair (property *(3)*) we have $\widetilde{\mathrm{rot}} \left[\widetilde{F(\psi)}(a_j), \widetilde{F(\psi)}(b_j)\right] \geq 0$, on the other hand we always have the general estimate $\left|\widetilde{\mathrm{rot}} \left[\widetilde{F(\psi)}(a_j), \widetilde{F(\psi)}(b_j)\right]\right| \leq 1$. Hence the only possibility for $n \in \mathbb{Z}$ is $n = 0$. This proves maximality of $F(\psi)$. Finally, by construction $\mathrm{rot}_{a_j} F(\psi) = 0$ for every j (property *(2)*) thus $F(\psi)$ is locally stable by our assumption.
Now we prove that ψ itself is locally stable. Again by construction we have

$$\mathrm{rot}_{[a_j,b_j]} \circ F = \mathrm{rot}_{[u_j(a_j,b_j),v_j(a_j,b_j)]} = \mathrm{rot}_{[a_j,b_j]}$$

(property *(1)*). By continuity of F the connected component of ψ is mapped to the connected component of $F(\psi)$ which is the semi-conjugacy class $X_{F(\psi)}$ since $F(\psi)$ is locally stable. Thus the function $\mathrm{rot}_{[a_j,b_j]} = \mathrm{rot}_{[a_j,b_j]} \circ F$ is constant on the connected component of ψ, its constant value is $\mathrm{rot}_{[a_j,b_j]} \equiv \mathrm{rot}_{[a_j,b_j]}(\psi) = \frac{1}{k}$.
We are now ready to prove constancy of rot_{a_j}. Suppose by contradiction that there exists a representation ρ in the connected component of ψ such that $\mathrm{rot}_{a_j}(\rho) \neq \mathrm{rot}_{a_j}(\psi)$. Then the image under rot_{a_j} of the connected component containing ψ is some interval of the real line. In particular we find a representation ρ' in the connected component such that $\mathrm{rot}_{a_j}(\rho') \notin \mathbb{Q}$. By Lemma 5.16 irrational rotation number $\mathrm{rot}_{a_j}(\rho')$ implies the vanishing of $\mathrm{rot}_{[a_j,b_j]}(\rho')$, and this contradicts the fact that $\mathrm{rot}_{[a_j,b_j]} \equiv \frac{1}{k}$. $\square$

REMARK 6.9. The same reduction holds (with the same proof) if we replace local stability with the following property: rot_γ is constant on the connected component of ϕ for every $\gamma \in \Gamma$. In other words: if rot_γ is constant for every $\gamma \in \Gamma$ on the connected component of every geometric representation $\phi : \Gamma \longrightarrow \mathrm{PSL}^{(k)}(2, \mathbb{R})$ with $\mathrm{rot}_{a_j}(\phi) = 0$ for every $j \leq g$, then the same property holds for every geometric $\mathrm{PSL}^{(k)}(2, \mathbb{R})$ −representation.

3. Main Theorem

The connected component of a geometric $\mathrm{PSL}^{(k)}(2, \mathbb{R})$ −representation lies in the intersection of good representations and representations with good fixed sets as stated in the next proposition:

PROPOSITION 6.10. *Let $\phi \in N$ be a $PSL^{(k)}(2, \mathbb{R})$ −geometric representation. Let $N(\phi)$ the connected component of ϕ in N. Then $N(\phi)$ is a connected component of $Rep(\Gamma)$.*

Proof. By Lemma 5.23 and continuity of the Euler number for every $\psi \in N(\phi) \subseteq N$ we have
$$\widetilde{\mathrm{rot}}\left(\prod_{j=1}^{g-1}\left[\widetilde{\psi}(a_j),\widetilde{\psi}(b_j)\right]\right) \leq \frac{2g-3}{k} \qquad \text{and} \qquad e\left(\psi\right) = \frac{2g-2}{k}.$$
Thus for every $\psi \in N(\phi)$ we get
$$\widetilde{\mathrm{rot}}_{[a_g,b_g]}(\psi) = e\left(\psi\right) - \widetilde{\mathrm{rot}}\left(\prod_{j=1}^{g-1}\left[\widetilde{\psi}(a_j),\widetilde{\psi}(b_j)\right]\right) \geq e\left(\psi\right) - \frac{2g-3}{k} = \frac{1}{k}.$$
On the other hand, since $\psi \in N_g$, we know that $\widetilde{\mathrm{rot}}_{[a_g,b_g]}(\psi) \leq \dfrac{1}{k}$. Therefore
$$\widetilde{\mathrm{rot}}_{[a_g,b_g]}(\psi) = \frac{1}{k} \qquad \text{and} \qquad \widetilde{\mathrm{rot}}\left(\prod_{j=1}^{g-1}\left[\widetilde{\psi}(a_j),\widetilde{\psi}(b_j)\right]\right) = \frac{2g-3}{k}.$$
By Proposition 5.31 every $\psi \in N(\phi)$ has an open neighborhood U_ψ in $\mathrm{Rep}\left(\Gamma\right)$ entirely contained in $U_\psi \subseteq N$. In particular there is an open subset U of $\mathrm{Rep}\left(\Gamma\right)$ such that $N(\phi) \subseteq U \subseteq N$. The result now follows from some general topology. The subspace $N := N_0 \cap \bigcap_{j=1}^{g} N_g$ is closed as it is the intersection of closed subsets. Also the set $N(\phi)$ is closed: being a connected component of N, it is closed in N and thus, since N is closed, it is closed in $\mathrm{Rep}\left(\Gamma\right)$. Let us denote by $C(\phi)$ the connected component of ϕ in $\mathrm{Rep}\left(\Gamma\right)$. We have
$$C(\phi) = \left(N \cap C(\phi)\right) \cup \left(C(\phi) \setminus N(\phi)\right).$$
The set $C(\phi) \setminus N$ is open in $C(\phi)$ since N is closed. By Proposition 5.31 also $C(\phi) \cap N$ is open in $C(\phi)$. In fact for every $\psi \in N \cap C(\phi)$ we have $e\left(\psi\right) = \frac{2g-2}{k}$ thus the following eqaulity holds
$$\widetilde{\mathrm{rot}}\left(\prod_{j=1}^{g-1}\left[\widetilde{\psi}(a_j),\widetilde{\psi}(b_j)\right]\right) = \frac{2g-3}{k}.$$
By Proposition 5.31 there is an open neighborhood U of ψ such that $U \subseteq N$ thus $U \cap C(\phi) \subseteq N \cap C(\phi)$. By connectedness of $C(\phi)$ we must have $C(\phi) \cap N = C(\phi)$ or $C(\phi) \setminus N = C(\phi)$, since $\phi \in C(\phi) \cap N$ it is the first relation the one that holds. This proves that $C(\phi) \subseteq N$ and therefore $C(\phi) = N(\phi)$. $\square$

Corollary 6.11. *Let $\phi \in N$ be a $PSL^{(k)}\left(2,\mathbb{R}\right)-$geometric representation. Let $N(\phi)$ be the connected component of ϕ in $Rep\left(\Gamma\right)$. Then rot_{a_g} is constant on $N(\phi)$.*

Proof. Argue by contradiction and assume that rot_{a_g} is not constant on $N(\phi)$. By continuity of rot_{a_g} and connectedness of $N(\phi)$ the image $\mathrm{rot}_{a_g}\left(N(\phi)\right)$ is the whole $\mathbb{R}/\mathbb{Z}$ or is an interval $I \subseteq \mathbb{R}/\mathbb{Z}$. In either case we can find a representation $\psi \in N(\phi)$ such that

$\mathrm{rot}_{a_g}(\psi) \notin \mathbb{Q}/\mathbb{Z}$. By Lemma 5.16 then $\widetilde{\mathrm{rot}}_{[a_g,b_g]}(\psi) = 0$, but $\psi \in N_g$ by Proposition 6.10 thus $\widetilde{\mathrm{rot}}_{[a_g,b_g]}(\psi) = \frac{1}{k}$, a contradiction. $\square$

COROLLARY 6.12. *Let $\phi \in N$ be a $PSL^{(k)}(2,\mathbb{R})-$geometric representation. Let $N(\phi)$ be the connected component of ϕ in $Rep(\Gamma)$. Then rot_c is constant on $N(\phi)$ for every $c \in \pi_1(S)$.*

PROOF. The result follows from Corollary 6.11 using a trick by Scott. For every loop $c \in \pi_1(S)$ there exists a finite covering $p : S' \longrightarrow S$ such that c lifts to a loop represented by a simple closed curve on S'. In particular we can choose a standard set of generators $a_1, b_1, \ldots, a_{g'}, b_{g'}$ of $\pi_1(S')$ such that $p_*(a_{g'}) = c$. The inclusion map $p_* : \pi_1(S') \longrightarrow \pi_1(S)$ induces by pre-composition a continuous function

$$P : \mathrm{Hom}\,(\pi_1(S), G) \longrightarrow \mathrm{Hom}\left(\pi_1(S'), G\right) \qquad , \qquad P(\psi) := \psi p_*.$$

For the representation $P(\phi) : \pi_1(S') \longrightarrow \mathrm{PSL}^{(k)}(2,\mathbb{R})$ we have

$$e\left(\phi p_*\right) = \langle (p_*)^* \phi^* e, [S'] \rangle = \langle \phi^* e, p_*[S'] \rangle = \langle \phi^* e, \deg(p)[S] \rangle = \deg(p) e\left(\phi\right).$$

Since the Euler characteristic obeys the same rule $\chi(S') = \deg(p)\chi(S)$ we see that

$$\frac{e\left(\phi p_*\right)}{\chi(S')} = \frac{e(\phi)}{\chi(S)}.$$

Finally we observe that the representation $P(\phi)$ is again a geometric and $P\left(N(\phi)\right) \subseteq N(P(\phi))$ by continuity of P. By Corollary 6.11 we have that $\mathrm{rot}_{a_{g'}}$ is constant on $N(P(\phi))$ thus $\mathrm{rot}_c|_{N(\phi)} = \mathrm{rot}_{a_{g'}}|_{N(P(\phi))} \circ P|_{N(\phi)}$ is constant on $N(\phi)$. $\square$

Finally we get the following results that generalize Theorem 4.17

THEOREM 6.13 (Mann [**Man14**], Matsumoto [**Mat14**]). *Let $\phi \in Rep(\Gamma)$ be a geometric representation. Then the connected component of ϕ is a single semi-conjugacy class.*

PROOF. We can assume that $\phi \in \mathrm{Hom}\left(\Gamma, \mathrm{PSL}^{(k)}(2,\mathbb{R})\right)$. By Lemma 6.8 it is enough to reduce to the case where $\mathrm{rot}_{a_j}(\phi) = 0$ for every $j \leq g$. The theorem now follows from Corollary 6.12 and Lemma 4.16. $\square$

With more work it is possible to prove that the open set U given by Proposition 5.31 can be chosen to lie in the connected component of $N(\phi)$. This implies that the connected component $N(\phi)$ is also open in $\mathrm{Rep}(\Gamma)$. Therefore

THEOREM 6.14. *Let $\phi \in Rep(\Gamma)$ be a geometric representation. Then ϕ is locally stable.*

Using Lemma 5.5 we get the following immediate consequence:

THEOREM 6.15. *For every positive divisor $k \mid 2g-2$ there are at least k^{2g} connected components of $Rep(\Gamma)$ with Euler number $e = \frac{2g-2}{k}$.*

Assuming more regularity an analogous result for $\mathrm{Hom}\left(\Gamma, \mathrm{Diff}_+\left(S^1\right)\right)$ has been proved by J. Bowden in [**Bow13**] (Theorem C of the article). We cite the result:

THEOREM 6.16 (Bowden [**Bow13**]). *Let* $\phi \in Hom\left(\Gamma, Diff_{+}\left(S^1\right)\right)$ *be a representation. If* ϕ *lies in the same* C^0-*path component of an Anosov representation then it is itself an Anosov representation. In particular it is conjugate to a geometric* $PSL^{(k)}\left(2,\mathbb{R}\right)-$*representation for some* $k \geq 1$.

APPENDIX A

Proof of the stability property

The appendix is devoted to the proof of Proposition 5.22:

PROPOSITION. *Let $\phi \in N_j$ be a representation such that*

$$\widetilde{rot}_{[a_j,b_j]}(\phi) = \frac{1}{k}.$$

Then there exists an open neighborhood U of ϕ in $Rep(\Gamma)$ such that $U \subseteq N_j$.

We already know by Corollary 5.21 that ϕ has a $j-$good fixed set. We want to prove that nearby representations enjoy the same property. This can be done by studying exploiting further the consequences of $\widetilde{\mathrm{rot}}\left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)\right]$ on the dynamics of $\widetilde{\phi}(a_j)$ and $\widetilde{\phi}(b_j)$. The next lemmas will give a description of Figures 1 and 2 that shows the dynamics of $\widetilde{\phi}(a_j)$.

LEMMA A.1. *Let $\phi \in N_j$ be a representation with a $j-$good fixed set. Suppose that*

$$\widetilde{rot}\left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)\right] = \frac{1}{k}.$$

Then the following holds:

(1) *the homeomorphism $g := \widetilde{\phi}(b_j)\widetilde{\phi}(a_j)^{-1}\widetilde{\phi}(b_j)^{-1}$ restricted to the interval $\left[x_1^{2i}, x_1^{2i+1}\right]$ satisfies*

$$g(x) > x \text{ for every } x \in \left[x_1^{2i}, x_1^{2i+1}\right].$$

(2) *the homeomorphism $f := \widetilde{\phi}(a_j)$ restricted to the interval $\left[x_2^{2i}, x_2^{2i+1}\right]$ satisfies*

$$f(x) > x \text{ for every } x \in \left[x_2^{2i}, x_2^{2i+1}\right].$$

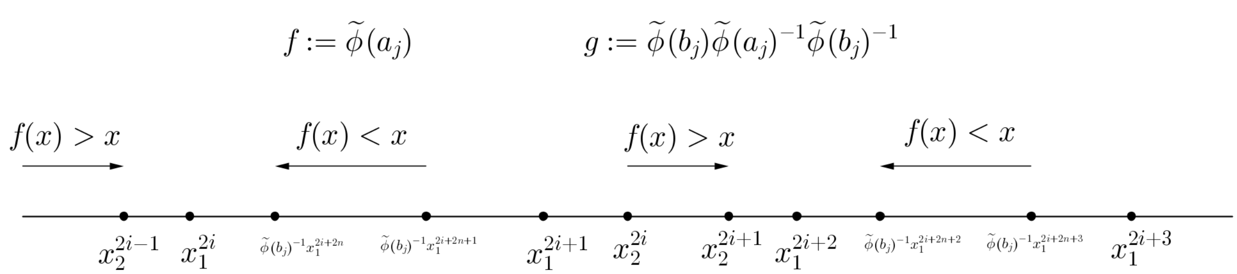

FIGURE 1.

Proof. Let us prove *(2)*, the proof of *(1)* is completely analogous. There are two cases: either f has a fixed point $f(x) = x$ for some $x \in \left[x_2^{2i}, x_2^{2i+1}\right]$ or f satisfies one of the following strict inequalities $f(y) < y$ for every $y \in \left[x_2^{2i}, x_2^{2i+1}\right]$ or $f(y) > y$ for every $y \in \left[x_2^{2i}, x_2^{2i+1}\right]$ (the graph of $f|_{\left[x_2^{2i}, x_2^{2i+1}\right]}$ can cross the bisector line or stays always above or always under that line). Let us consider the case where $f(y) < y$. We have $fg\left(x_2^{2i+1}\right) = f\left(x_2^{2i+1}\right) < x_2^{2i+1}$, therefore $\widetilde{\text{rot}}\ fg \leq 0$ contradicting the fact that $\widetilde{\text{rot}}\ fg = \frac{1}{k}$. Consider now the fixed point case $f(y) = y$ for some $y \in \left[x_2^{2i}, x_2^{2i+1}\right]$. If y is one of the extrema of the interval then $fg(y) = f(y) = y$, thus $\widetilde{\text{rot}}\ fg = 0$, again a contradiction. Suppose then $x_2^{2i} < y < x_2^{2i+1}$. Consider the sets $X_1(\phi)' := X_1(\phi) \cup (\{y\} + \mathbb{Z})$ and $X_2(\phi) := \widetilde{\phi}(b_j)\left(X_1(\phi)\right)$. The set $X := X_1(\ phi)' \cup X_2(\phi)$ can be ordered as follows:

$$\begin{aligned}\cdots < x_1^{2i+1} < x_2^{2i} < y < x_2^{2i+1} < x_1^{2i+2} < \cdots < x_1^{2i+1+2k} < x_2^{2i+2k} < \\ < y + 1 < x_2^{2i+1+2k} < x_1^{2i+2+2k} < \dots\end{aligned}$$

The associated dynamical system for f, g with respect to these fixed point sets exhibits the following behaviour (we use the positive word $w := fg$):

$$y \longrightarrow fg \cdot y = x_1^{2i+2} \longrightarrow (fg)^2 \cdot y = x_1^{2i+4} \longrightarrow \dots$$

$$\dots \longrightarrow (fg)^k \cdot y = x_1^{2i+2k} \longrightarrow (fg)^{k+1} \cdot y = y + 1.$$

Thus from the Calegari-Walker algorithm we get $\widetilde{\text{rot}}\ fg \leq \frac{1}{k+1}$, a contradiction since by assumption we have $\widetilde{\text{rot}}\ fg = \frac{1}{k}$. □

Remark A.2. From Lemma A.1 the homeomorphisms $f := \widetilde{\phi}(a_j)$ and $g := \widetilde{\phi}(b_j)\widetilde{\phi}(a_j)^{-1}\widetilde{\phi}(b_j)^{-1}$ satisfy the following properties for every $t \in \mathbb{Z}$:

$$f(x) > x \text{ for every } x \in \left[x_2^{2t}, x_2^{2t+1}\right] \quad , \quad g(x) > x \text{ for every } x \in \left[x_1^{2t}, x_1^{2t+1}\right].$$

The second one, i.e. $\widetilde{\phi}(b_j)\widetilde{\phi}(a_j)^{-1}\widetilde{\phi}(b_j)^{-1}(x) > x$ for every $x \in \left[x_1^{2t}, x_1^{2t+1}\right]$, is equivalent to $\widetilde{\phi}(a_j)^{-1}\widetilde{\phi}(b_j)^{-1}(x) > \widetilde{\phi}(b_j)^{-1}(x)$ for every $x \in \left[x_1^{2t}, x_1^{2t+1}\right]$, i.e. $f^{-1}(x) > x$ (or equivalently $f(x) < x$) for every $x \in \widetilde{\phi}(b_j)^{-1}\left[x_1^{2t}, x_1^{2t+1}\right]$.

Properties *(1)* and *(2)* of Lemma A.1 are open with respect to the topology of $\text{Rep}(\Gamma)$, i.e. they hold in every sufficiently small neighborhood of ϕ.

Lemma A.3. *Let $\phi \in N_j$ be a representation with a $j-$good fixed sets $X_1(\phi) = \left\{x_1^t\right\}_{t\in\mathbb{Z}}$ and $X_2(\phi) = \left\{x_2^t\right\}_{t\in\mathbb{Z}}$. Define*

$$y^{2i} := \max\left\{\mathit{Fix}\left(\widetilde{\phi}(a_j)\right) \cap \left[x_2^{2i-1}, \widetilde{\phi}(b_j)^{-1}x_1^{2i+2n}\right]\right\}, \ \ y^{2i+1} := \min\left\{\mathit{Fix}\left(\widetilde{\phi}(a_j)\right) \cap \left[\widetilde{\phi}(b_j)^{-1}x_1^{2i+2n+1}, x_2^{2i}\right]\right\}$$

and

$$z^{2i} := \min\left\{\mathit{Fix}\left(\widetilde{\phi}(a_j)\right) \cap \left[x_2^{2i-1}, \widetilde{\phi}(b_j)^{-1}x_1^{2i+2n}\right]\right\}, \ \ z^{2i+1} := \max\left\{\mathit{Fix}\left(\widetilde{\phi}(a_j)\right) \cap \left[\widetilde{\phi}(b_j)^{-1}x_1^{2i+2n+1}, x_2^{2i}\right]\right\}.$$

Then the following holds:

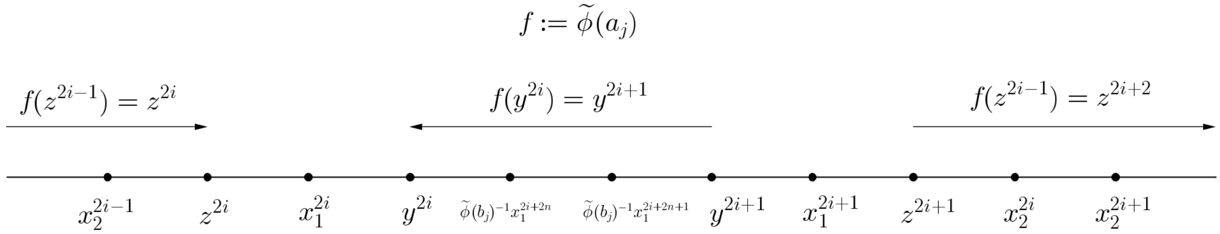

FIGURE 2.

(1) We have $\widetilde{\phi}(b_j)y^{2i} < z^{2i+2n}$.
(2) We have $\widetilde{\phi}(b_j)y^{2i+1} > z^{2i+2n+1}$.

PROOF. We prove only point *(1)* as point *(2)* is completely analogous.
First we prove that y^{2i}, z^{2i} are well defined. Fix $t = 2i + 2n$. We have

$$\widetilde{\phi}(b_j)x_1^{2t} = x_2^{2t+2n-1} < x_1^{2t+2n} < x_1^{2t+2n+1} < x_2^{2t+2n} = \widetilde{\phi}(b_j)x_1^{2t+1}.$$

Therefore $\widetilde{\phi}(b_j)^{-1}\left[x_1^{2t+2n}, x_1^{2t+2n+1}\right] \subset \widetilde{\phi}(b_j)^{-1}\left(x_2^{2t+2n-1}, x_2^{2t+2n}\right) = \left(x_1^{2t}, x_1^{2t+1}\right)$ for every $t \in \mathbb{Z}$. In particular the interval $\left[x_2^{2i}, \widetilde{\phi}(b_j)^{-1}x_1^{2i+2n}\right]$ contains a fixed point of f, i.e. $x_2^{2i-1} < x_1^{2i} < \widetilde{\phi}(b_j)^{-1}x_1^{2i+2n}$.
Proceed now by contradiction and assume $\widetilde{\phi}(b_j)y^{2i} \geq z^{2i}$. If we have equality then $\widetilde{\phi}(b_j)y^{2i}$ is simultaneously a fixed point of $f := \widetilde{\phi}(a_j)$ since it is equal to z^{2i+2n}, and a fixed point of $g := \widetilde{\phi}(b_j)\widetilde{\phi}(a_j)^{-1}\widetilde{\phi}(b_j)^{-1}$ as it is the image under $\widetilde{\phi}(b_j)$ of the fixed point y^{2i} of $\widetilde{\phi}(a_j)^{-1}$. In particular $\widetilde{\mathrm{rot}}\, fg = 0$, but we know by assumption that $\widetilde{\mathrm{rot}}\, fg = \frac{1}{k}$, a contradiction.
Suppose now that $\widetilde{\phi}(b_j)y^{2i} > z^{2i+2n}$. We have

$$x_2^{2i+2n} < z^{2i+2n} < \widetilde{\phi}(b_j)y^{2i} < x_1^{2i+2n}$$

Define the set $X_2(\phi)' := X_2(\phi) \cup \left(\left\{\widetilde{\phi}(b_j)y^{2i}\right\} + \mathbb{Z}\right)$ and also $X_1(\phi)' := X_1(\phi) \cup \left(\left\{z^{2i+2n}\right\} + \mathbb{Z}\right)$. The set $X := X_1(\phi)' \cup X_2(\phi)'$ can be ordered as follows

$$\begin{aligned} \cdots < x_2^{2i+2n} < z^{2i+2n} < \widetilde{\phi}(b_j)y^{2i} < x_1^{2i+2n} < \cdots < x_2^{2i+2n+2k} < \\ < z^{2i+2n} + 1 < \widetilde{\phi}(b_j)y^{2i} + 1 < x_1^{2i+2n+2k} < \ldots \end{aligned}$$

A computation (using the Calegari-Walker algorithm) similar to the one carried out in Lemma A.1 shows that $\widetilde{\mathrm{rot}}\left[\widetilde{\phi}(a_j), \widetilde{\phi}(b_j)\right] \leq \frac{1}{k+1}$, again a contradiction. $\square$

Again properties *(1)* and *(2)* of Lemma A.3 are open with respect to the topology of the representation space, i.e. they still hold (for fixed y^t and z^t) in every sufficiently small neighborhood of ϕ in $\mathrm{Rep}\,(\Gamma)$. We are now ready to prove Proposition 5.22:

Proof. Using Lemmas A.1 and A.3 we can find a sufficiently small $\epsilon > 0$ and a neighborhood U of ϕ in $\operatorname{Rep}(\Gamma)$ such that the folloing four inequalities hold for every $\psi \in U$ (and for suitable lifts $\widetilde{\psi}(a_j)$ and $\widetilde{\psi}(b_j)$):

(1) $\widetilde{\psi}(a_j)x < x$ for every $x \in \left[y^{2i} + \epsilon, y^{2i-1} - \epsilon\right]$.
(2) $\widetilde{\psi}(a_j)x > x$ for every $x \in \left[z^{2i-1} + \epsilon, z^{2i} - \epsilon\right]$.
(3) $\widetilde{\psi}(b_j)\left(y^{2i+1} - \epsilon\right) > z^{2i+2n+1} + \epsilon$.
(4) $\widetilde{\psi}(b_j)\left(y^{2i} + \epsilon\right) > z^{2i+2n} - \epsilon$.

Using *(1)* and *(2)* we see that $\widetilde{\psi}(a_j)$ has a fixed point $x_1^{2i-1}(\psi)$ in $\left(y^{2i-1} - \epsilon, z^{2i-1} + \epsilon\right)$ and also a fixed point $x_1^{2i-1}(\psi)$ in $\left(z^{2i} - \epsilon, y^{2i} + \epsilon\right)$. We choose the fixed points $\left\{x_1^i(\psi)\right\}_{i\in\mathbb{Z}}$ in the following way: first we choose $2k$ fixed points $x_1^{2i-1}(\psi), x_1^{2i}(\psi)$ in consecutive intervals for $i = 1, \ldots, k$. Then we choose the others exploiting the fact that $\widetilde{\psi}(a_j)$ commutes with integral translations, i.e. $X_1(\psi) := \bigcup_{i=1}^k \left(\left\{x_1^{2i-1}(\psi)\right\} + \mathbb{Z}\right) \cup \left(\left\{x_1^{2i}(\psi)\right\} + \mathbb{Z}\right)$.
Define $X_2(\psi) := \widetilde{\psi}(b_j)X_1(\psi)$. We order $X_2(\psi)$ by requiring that property *(3)* of the definition of $j-$good fixed sets hold with the same n as the one of ϕ, i.e. $x_2^{2i+2n-1}(\psi) := \widetilde{\psi}(b_j)x_1^{2i}(\psi)$ and $x_2^{2i+2n}(\psi) := \widetilde{\psi}(b_j)x_1^{2i+1}(\psi)$. In order to conclude that $X_1(\phi)$ is a $j-$good fixed set for ψ we need to prove that the set $X := X_1(\psi) \cup X_2(\psi)$ can be ordered as

$$\cdots < x_1^0(\psi) < x_1^1(\psi) < x_2^0(\psi) < x_2^1(\psi) < x_1^2(\psi) < x_1^3(\psi) < \ldots$$

This follows from properties *(3)* and *(4)*. Let us explain why:

$$\begin{aligned} & x_2^{2i+2n}(\psi) = \widetilde{\psi}(b_j)x_1^{2i+1}(\psi) \\ & > \widetilde{\psi}(b_j)\left(y^{2i+1} - \epsilon\right) \\ & > z^{2i+2n+1} + \epsilon && \text{by property } \textit{(3)} \\ & > x_1^{2i+2n+1}(\psi). \end{aligned}$$

Analogously using property *(4)* we get:

$$\begin{aligned} & x_2^{2i+2n-1}(\psi) = \widetilde{\psi}(b_j)x_1^{2i}(\psi) \\ & < \widetilde{\psi}(b_j)\left(y^{2i} + \epsilon\right) \\ & < z^{2i+2n} - \epsilon && \text{by property } \textit{(4)} \\ & < x_1^{2i+2n}(\psi). \end{aligned}$$

In both the chains of inequalities we used $x_1^{2t-1}(\psi) \in \left(y^{2t-1} - \epsilon, z^{2t-1} + \epsilon\right)$ and $x_1^{2t}(\psi) \in \left(z^{2t} + \epsilon, y^{2t} + \epsilon\right)$ for every $t \in \mathbb{Z}$. Thus the points of X are ordered as follows

$$\cdots < x_1^{2t+1}(\psi) < x_2^{2t}(\psi) < x_2^{2t+1}(\psi) < x_1^{2t+2}(\psi) < \ldots$$

□

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