Università degli studi di Pisa

Dipartimento di Fisica
Corso di laurea magistrale in Fisica

Tesi di Laurea Magistrale

Holographic computation of the Neutron electric dipole moment

Relatore:
Dott. Francesco Bigazzi

Candidato:
Andrea Manenti

Anno Accademico 2015-2016

This page is intentionally left blank.

Contents

This page is intentionally left blank.

Chapter 1

Introduction

Quantum Field Theory (QFT) is the response to the major theoretical challenge of combining the principles of Quantum Mechanics and Special Relativity. Historically this issue arose from the desire of building a theoretical framework for Fundamental Interactions; one essential ingredient in fact is Lorentz invariance, simply because experiments performed in different reference frames must always agree. The Lorentz group does not contain only rotations and boosts, there is also a subgroup of discrete symmetries: a parity transformation of the spatial coordinates (P) and a reversal of the direction of time (T). There is no reason why we must impose also P and T as symmetries of Nature, in fact experimentally these are observed to be broken by Weak interactions. At present the discrete symmetries of the Weak interactions are not much debated, but for Strong interactions they remain a sort of a mystery.

Every interacting QFT admitting a Lagrangian description possesses a set of parameters g_i, called *couplings*, weighting the strength of the interactions. When the couplings are small, the theory can be studied by means of the *perturbative* approach: all the observables are expressed as a power series in the g_i. If, on the other hand, some g_i approach 1, the perturbative expansion is no longer reliable. The value of the coupling can change as the energies under considerations vary, unless some very strong symmetry prohibits this. This means that the same theory can exhibit either a perturbative (i.e. weakly coupled) or a non perturbative (i.e. strongly coupled) behaviour according to the energy range of the problem. The effective degrees of freedom of the theory in its non perturbative regime are not interpreted as excitations of the fundamental fields in the Lagrangian.

Most of the interesting effects in the landscape of QFT arise in the non perturbative regime. There are many examples in all areas of theoretical physics. In condensed matter physics, for instance, systems near a quantum critical point behave as strongly coupled QFTs. In high energy physics the theory of Quantum Chromodynamics (QCD) has been extensively studied, in both his weakly and strongly coupled regime. This theory describes the Strong interactions between quarks and gluons, which are the fundamental constituents of hadrons: baryons, such as the neutron and the proton, and mesons, such as the pion. The UV description is a non abelian gauge theory on $SU(3)$, with the gluon as a gauge vector and $N_f = 6$ species (referred to as "flavors") of fermions in the fundamental: the quarks (up, down, strange, charm, bottom and top). QCD possesses a coupling g which becomes small at high energies; this behaviour goes under the name of *asymptotic freedom* [1, 2]. At low energies, on the other hand, the theory is strongly coupled.

Among the interesting features that cannot be studied in perturbative QCD we mention confinement and topological effects. The experimental observation that the asymptotic states in QCD are always $SU(3)$ color singlets goes under the name of confinement. This property is checked theoretically by looking for a growing linear dependence on the distance in the quark–anti-quark potential. The topological dependence comes from the so–called θ term: $\frac{\theta}{32\pi^2} \mathrm{Tr}\, F_{\mu\nu} F_{\rho\sigma} \varepsilon^{\mu\nu\rho\sigma}$ (ε is the totally antisymmetric tensor and F is the non abelian field strength), which can be added to the Lagrangian of QCD. This term is a consequence of a class of solutions called *instantons* [3, 4]. These "quasi-particles" interpolate between different topological phases of the theory and they can give rise only to non perturbative effects (the θ term being a total derivative). The presence of a non zero θ–angle has a remarkable effect: the P and T (or CP) symmetry is explicitly broken. This thesis is motivated precisely by the investigation of the θ dependence and CP–breaking effects in QCD.

The most direct non perturbative approach to QCD and Yang–Mills (i.e. QCD without quarks) is the Lattice formulation [5]. It relies on a numerical simulation in the Euclidean spacetime discretized to a finite lattice, which goes under the name of Monte Carlo simulation. The main advantage is the ability to control the systematic and statistical errors of the approximations. Many great results of this approach include the equilibrium properties and the critical behaviour of the confinement/deconfinement transition. The difficulties are mostly technical, arising from the finiteness of the lattice and of the lattice spacing. The Euclidean nature of the model renders very difficult to address both far from equilibrium physics and the topological properties (because the θ term in the Euclidean formulation is imaginary, hence giving rise to what is referred to as a "sign problem").

A successful phenomenological approach is provided by effective theories. Those are based on the idea that the physics at low energy can be effectively described by the light degrees of freedom. Pions are observed experimentally to be very light (with respect to the dynamically generated energy scale of the theory) and they are interpreted as quasi–Goldstone modes of a spontaneously broken approximate symmetry of the theory: the chiral symmetry. The chiral symmetry consists in a flavor dependent transformation on the two chiral (left/right) components of the fermions. This symmetry is explicitly broken by the quark masses and it is also spontaneously broken by quantum effects. Hence the idea is to build the most general Lagrangian which realizes classically this breaking pattern (see [6, 7, 8]). This method, which goes under the name of chiral perturbation theory, provides us an intuitive way to understand the physics at low energies. Moreover it has given predictions on the mass spectrum of mesons.

Another interesting method is based on the idea by 't Hooft [9] to expand the theory in powers of $1/N_c$, N_c being the number of colors (i.e. $N_c = 3$ in nature). A remarkable feature of this expansion is that it can be interpreted as a perturbative expansion of a (yet to be determined) theory of relativistic one dimensional strings. This approach, as opposed to lattice QCD, is not based a priori on numerical simulations. On the other hand it is more difficult to estimate the errors made by extrapolating a result for $N_c = 3$. Remarkably, many non perturbative features can be described in this framework. Among them an explanation for the large mass of the η' [10, 11, 12] (a pseudoscalar flavor singlet meson of the theory), which was previously believed to be a "quasi"-Goldsone mode of a broken flavor symmetry, and the properties of the θ vacuum of QCD [13]. It also provides a interesting description of baryons (i.e. totally antisymmetric combinations of N_c quarks). In fact meson–meson couplings are found to be of order $1/N_c$, while baryon masses scale as N_c; this suggests that baryons can be seen as monopoles in the effective large N_c mesonic Lagrangian (see Skyrme model [14]).

The non perturbative approach that we are going to adopt in this thesis goes under the name of holographic correspondence. It consists in a conjectured equivalence between classes of quantum field theories in d dimensions and quantum gravity (actually string theory) in at least one dimension more [15, 16, 17]. At present there is no rigorous proof of this statement, but in particular classes of theories, such as theories with supersymmetry, it is possible to perform explicit checks and so far no counterexample has been found. The correspondence works as a duality: a particular strongly coupled regime of a quantum field theory (namely large N_c, large $\lambda = g^2 N_c$ in the case of non abelian gauge theories) is mapped into the low energy, classical limit of quantum gravity. As a result, it is sometimes possible to solve non perturbative problems in a QFT by solving easier problems in classical gravity.

Let us now explain these two keywords that play an important role in understanding the holographic correspondence: *duality* and *holography.* A *duality* in Physics is not an uncommon concept, it arises when the same phenomenon can be described equivalently by two different formalisms. This is not only interesting per se but it might be very proficuous if the two descriptions are convenient in different regimes, say one works well at weak coupling while the other at strong coupling. There are many notable examples in simple models (see the introduction of [18]) and also in more complex theories such as dualities in string theory and supersymmetric gauge theories.

The principle of *holography* is also very general, it states that the degrees of freedom of a $d+1$ dimensional system with gravity can be fully accounted for by the degrees of freedom living in the boundary. Gravity is a theory that realizes this principle naturally. This follows from the works of Bekenstein and Hawking regarding black hole entropy and thermodynamics [19, 20, 21].

Originally the correspondence was established between a gravity theory on Anti de Sitter (AdS) space time and a Conformal Field Theory (CFT), hence the name AdS/CFT. However it was soon realized that it could be extended to non–AdS/non–CFT dual pairs. This is particularly relevant if one wants to use the correspondence as a tool to investigate QCD and similar non conformal theories.

All along the past 20 years, holography has emerged as a powerful tool to study both equilibrium and out of equilibrium physics of strongly correlated systems, modeling theories both in the realm of high energy physics [22] and in that of condensed matter [23, 24]. Notable advances in thermo and hydro–dynamics [25], quantum entanglement [26], off–equilibrium dynamics as well as black hole physics on the other side are just a few examples in a wide class of applications. Despite these successes it has to be kept in mind that at present, in the regime in which we are able to perform computations, holography allows us to investigate only model versions of realistic systems. Some reviews on the subject are [27, 28, 29].

The problem we would like to address in this thesis is the computation of the neutron electric dipole moment (NEDM) by means of the holographic correspondence. Electric dipole moments are sensitive observables of CP breaking effects in standard model (SM) and beyond standard model (BSM) theories. The NEDM can be measured by trapping ultra cold neutrons in an electro-magnetic field and looking for a change in the Larmor frequency when the sign of the electric field is flipped. Neutral particles (like the neutron) are chosen because charged particles would be accelerated by the strong electric fields, rendering very difficult the storage.

The first proposal to measure the NEDM is due to Purcell and Ramsey in 1950 [30]. Since then, experiments have been able to provide us only upper bounds on the value of the NEDM. The most recent analyses [31, 32] give $d_n \leq 2.9{\cdot}10^{-26} e{\cdot}\mathrm{cm}$ (90% CL).

The NEDM gets contributions from both the CP breaking phase of the Cabibbo–Kobayashi–Maskawa (CKM) matrix in electroweak theory and the CP breaking θ term in QCD. The CKM phase gives a very small effect, far below the experimental upper bound, while the contribution from QCD can be dominant according to the value of θ. A rough estimate based on dimensional analysis is [6]

$$d_n \sim \theta m_\pi^2 e/M_n^3 \sim 10^{-16}\, \theta\, e \cdot \mathrm{cm}\ , \tag{1.1}$$

where m_π and M_n are, respectively, the pion and the neutron masses. This together with the experimental bound would give an unnaturally small value for θ (i.e. $\theta \lesssim 10^{-10}$). The reason behind the fact that θ is so small (or even zero) is still an open problem, which goes under the name of *Strong CP problem.* A possible resolution of the latter is provided by the Peccei–Quinn mechanism [33], which predicts a new particle: the *axion.*

There are different ways to compute the NEDM: *i)* we can compute the difference in the energy of a neutron with spin up and spin down in presence of an external electric field; *ii)* we can compute the time component of the electromagnetic Noether current and extract the dipole from its expectation value on the baryonic state; *iii)* we can compute the electromagnetic form factors.

Many different non perturbative methods have been used to tackle this problem. The first attempt was done by Baluni in 1979 [34] using the bag model, in which three quarks are confined into a spherical cavity.

In lattice QCD, in order to overcome the sign problem, the θ term is taken to be imaginary; then the final result is obtained by analytic continuation [35]. The systematic and statistical errors however grow when the pion masses m_π approach the physical value, hence usually m_π is taken slightly bigger than the true value. Lattice computations have been done also adopting the external field approach explained above in point *i)* [36].

A notable computation in the framework of chiral effective theories is [37, 38]. The NEDM was found to be proportional to the CP–breaking cubic coupling between nucleons and pions $\overline{g}_{\pi NN}$

$$d_n = \frac{g_{\pi NN}\,\overline{g}_{\pi NN}}{4\pi^2 M_N} \log(M_N/m_\pi) \sim 3.6 \times 10^{-16}\theta\, e\,\mathrm{cm}\,, \tag{1.2}$$

where $g_{\pi NN}$ is the CP conserving coupling. The leading contribution here is a pion loop with three external legs: two neutrons and a photon.

The approach in the Skyrme model (large N_c QCD) is completely different: while in chiral perturbation theory the baryons appear in the model as fundamental fermionic fields, here they arise as solitonic solutions of the model [39]. From the quantization of the soliton, called *Skyrmion*, one can extract the NEDM, as computed by Dixon et al. [40]. Dixon's result is $d_n \propto \theta m_\pi^2 N_c$: the dependence on the pion mass is quadratic, as opposed to the logarithmic behaviour in the chiral effective approach; this is believed to be a consequence of the order in which the two limits, $m_\pi \to 0$ and $N_c \to \infty$, are taken. Moreover the N_c scaling of the cubic couplings πNN is not clear. It was found in [41] that $g_{\pi NN} \sim N_c^{3/2}$ while $\overline{g}_{\pi NN} \sim N_c^{-x}$ with $x \geq \frac{1}{2}$. This, together with the scaling $M_N \sim N_c$, suggests that the Skyrme approach does not yield the pion loop contribution of the chiral effective approach and there is no obvious map between the two computations.

It is important to complement these results with the holographic approach. For this thesis we have focused on the model which is closest to an holographic dual of QCD: the Witten[42]–Sakai–Sugimoto[43, 44] (WSS) model. In the regime where the dual description is provided by classical gravity, the model considered by Witten is a 3+1 dimensional non supersymmetric $SU(N_c)$ Yang–Mills theory coupled with matter fields transforming in the adjoint representation of the gauge group. These fields have masses proportional to an energy scale M_{KK} and they are charged under a global $SO(5)$ symmetry. Despite this microscopic content the low energy properties of the model are the same as expected in large N_c pure Yang–Mills. Sakai and Sugimoto are responsible for the introduction of flavors in the model, i.e. matter fields in the fundamental. In their formulation the quarks are massless, hence the theory enjoys chiral symmetry; it also exhibits the spontaneous chiral symmetry breaking as one would expect. The advantages of this model is that many different effective descriptions, formulated in order to account for different sectors of QCD, are built-in: for instance it describes in an unifying perspective the physics of pions, vector mesons and also baryons. It also shares many features with the large N_c QCD approach, in particular with the Skyrme model. All the parameters in this effective description are analytically expressed in terms of the few bare parameters of the model.

For the purpose of our computation we need to add a few ingredients to the WSS model: the quark masses, the θ parameter and the baryons. With massless quarks all the θ dependence would disappear because of the chiral anomaly. In the WSS model quark masses have been introduced by Aharony and Kutasov [45] and also by Hashimoto et al. [46] with an equivalent method. The θ dependence can be introduced following Witten [47]. Finally baryons arise as solitonic solutions (in this case instantons), as in the Skyrme model [48]. The computations follows essentially the ideas of [40]: we compute the baryonic (instantonic) solution deformed by the presence of the quark mass term, to first order in θ and m_π^2; quantize the solution (following [49]) and then extract the NEDM from the Noether current. We consider the case of two flavors with equal masses. The equations of motion are partial differential equations (PDE) that cannot be solved analytically, hence we adopt numerical methods.

The advantages of this holographic approach in the computation on the NEDM, with respect to the other non perturbative approaches presented above, are several: *i)* the sign problem arising in lattice computations here is absent; *ii)* we have the possibility of taking the limits $m_\pi \to 0$ and $N_c \to \infty$ in both orders; *iii)* everything is expressed in terms of only few parameters, whereas in effective theories each term has an undetermined coefficient which must be fixed by experiments; *iv)* lastly it allows to account not only for the pion contribution, but for that of the whole vector mesonic tower.

The parameters involved in this model are the energy scale M_{KK}, the coupling λ at that scale, the number of colours N_c, the number of quark families ($N_f = 2$ in our case) and the quark mass m_q. Extrapolating those parameters to $N_c = 3$, fitting them with the pion mass, the pion decay constant and with the ρ meson mass we find

$$d_n = 0.79 \cdot 10^{-16}\, \theta\; e \cdot \mathrm{cm}\; . \tag{1.3}$$

If instead instead of using the ρ meson mass we fit the parameters with the nucleon mass the result is slightly different

$$d_n = 0.74 \cdot 10^{-16}\, \theta\; e \cdot \mathrm{cm}\; . \tag{1.4}$$

For large λ and large N_c we find a behaviour

$$d_n \sim \frac{m_q \theta}{\lambda^{3/2} M_{\mathrm{KK}}^2} \sim \frac{N_c m_\pi^2 \theta}{\lambda^2 M_{\mathrm{KK}}}\; , \tag{1.5}$$

respectively if we express d_n in terms of the quark mass m_q or the pion mass m_π. Moreover we find the relation $d_n = -d_p$, d_p being the proton electric dipole moment. Finally we also find that the CP breaking cubic coupling $\overline{g}_{\pi NN}$ in this model is zero (actually subleading in the limits in which we work).

The thesis is organized as follows. In Chapter 2 we introduce the theoretical basis to understand the holographic correspondence, namely string theory, AdS geometry, D–branes and Conformal Field Theories. In the last part of the chapter we introduce the AdS/CFT correspondence along with some simple examples. In Chapter 3 we introduce the theory of Chromodynamics, mainly focusing on the non perturbative aspects and the reasons that brought our interest to the NEDM. We will review the large N_c expansion, the chiral effective approach, the topological dependence and the instantonic solutions (which are not only fundamental to understand the θ term but will also reveal to be very useful for the computations in the baryonic sector of the WSS model). In Chapter 4 we review the Witten–Sakai–Sugimoto model and we study its deformation due to the quark masses and the θ term, mainly focusing on the vacuum solution. In Chapter 5 we introduce baryons in the model, study their quantization and the first correction to the mass spectrum due to θ and the quark masses. Finally in Chapter 6 we study the deformation of the baryonic solution to first order in θ and m_π^2, at leading order in the large N_c limit, quantize it and then compute the NEDM. We also show that $\overline{g}_{\pi NN} = 0$. The last Chapter contains the conclusions. Some technical details are presented in a few appendices.

This page is intentionally left blank.

Chapter 2

The AdS/CFT correspondence

In this Chapter we are going to review the basic elements necessary to understand the holographic correspondence, which will be the framework adopted in the rest of the work. First we will focus on String Theory (see [50, 51, 52]). Next we will take a closer look to the two sides of the correspondence: Anti de Sitter geometry and Conformal symmetry. Finally we will show the AdS/CFT correspondence with some simple examples. Reviews on the subject are [29, 27, 28], see also the book [53].

The holographic correspondence is a duality between a quantum field theory and a gravity theory in a higher number of dimensions. This is an extension of the AdS/CFT correspondence that describes a duality between a *conformal* field theory and a gravity theory in a space which contains an Anti–deSitter (AdS) submanifold. This particular case provides countless quantitative validity checks to the correspondence. The checks can be explicitly performed by computing the same observables in both sides of the correspondence. Often this is technically possible in models constrained by some (super) symmetry. However there is no reason why supersymmetry should be required as a necessary ingredient. This correspondence was first introduced by Juan Maldacena [15] and it did not take long for theoretical physicists to generalize it to a "non-AdS/non–CFT" correspondence. We will need this kind of formulation because QCD is certainly not a conformal theory. Let us refer to it as AdS/CFT anyway hoping not to generate any confusion.

2.1 String theory basics

The most simple object in theoretical physics is the *free particle*, a point–like entity freely moving in space time without interacting with the environment. In

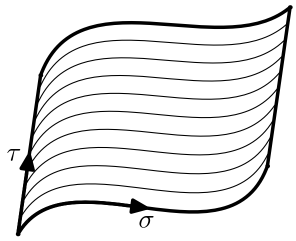


Figure 2.1: String *worldsheet*, the τ and the σ directions are represented.

Special Relativity the action for this object reads

$$S_{\text{point}} = -m \int \mathrm{d}s \sqrt{\eta_{\mu\nu} \frac{\mathrm{d}X^\mu(s)}{\mathrm{d}s} \frac{\mathrm{d}X^\nu(s)}{\mathrm{d}s}} \,. \tag{2.1.1}$$

It is an integral on the world line, where $X^\mu(s)$ is the curve describing its trajectory in space time, parametrized by an affine parameter s, m is the mass of the particle and $\eta_{\mu\nu}$ the Minkowski metric. From a geometrical point of view this is just the length of the world line. This idea can be generalized, if we have an object extended in one dimension: a *string*. We can write its action as the surface area of the world sheet, which is now a two dimensional object. If we call T the tension of the string, this action, which goes under the name of Nambu–Goto action, reads

$$S_{\text{NG}} = -T \int_\Sigma \mathrm{d}^2 s \sqrt{-\det \left(\eta_{\mu\nu} \frac{\partial X^\mu(s)}{\partial s^\alpha} \frac{\partial X^\nu(s)}{\partial s^\beta} \right)} \,. \tag{2.1.2}$$

Now that we have two parameters describing the surface, the area is given by the determinant of the metric induced in the surface. In mathematical terms $X^\mu(s)$ is the *embedding* of Σ in the space time (called target space) and the induced metric is the *pullback* of the embedding. Dealing with a square root is not very pleasant, especially if we want to quantize the theory. It is not hard to prove that the following action, called Polyakov action

$$S_{\text{P}} = -\frac{T}{2} \int_\Sigma \mathrm{d}^2 s \sqrt{-h}\, h^{\alpha\beta} \eta_{\mu\nu} \frac{\partial X^\mu(s)}{\partial s^\alpha} \frac{\partial X^\nu(s)}{\partial s^\beta} \,, \tag{2.1.3}$$

is equivalent to the former, at the price of introducing an auxiliary field: a metric on the world sheet. This replaces the determinant with a trace. The metric $h_{\alpha\beta}$ must have a time like direction (let us call it τ) and a space like direction (let us call it σ) because we want to describe the causal nature of the world sheet. It is interesting to list the invariance properties of this action

- World sheet reparametrization invariance

$$
\begin{aligned}
X^\mu(s) \;&\to\; X'^\mu(s') = X^\mu(s) \;, \\
h_{\alpha\beta}(s) \;&\to\; h'_{\alpha\beta}(s') = h_{\gamma\delta}(s)\frac{\partial s^\gamma}{\partial s'_\alpha}\frac{\partial s^\delta}{\partial s'_\beta} \;.
\end{aligned}
\tag{2.1.4}
$$

- D dimensional Lorentz invariance

$$
\begin{aligned}
X^\mu(s) \;&\to\; X'^\mu(s) = \Lambda^\mu_{\ \nu} X^\nu(s) + a^\mu \;, \\
h_{\alpha\beta}(s) \;&\to\; h'_{\alpha\beta}(s) = h_{\alpha\beta}(s) \;.
\end{aligned}
\tag{2.1.5}
$$

- Weyl invariance (for some arbitrary function Ω)

$$
\begin{aligned}
X^\mu(s) \;&\to\; X'^\mu(s) = X^\mu(s) \;, \\
h_{\alpha\beta}(s) \;&\to\; h'_{\alpha\beta}(s) = \Omega^2(s)\, h_{\alpha\beta}(s) \;.
\end{aligned}
\tag{2.1.6}
$$

The last one, the Weyl invariance, is a special feature of the Polyakov action while the other two subsist also in the Nambu–Goto action. Moreover the Weyl invariance is a special consequence of the fact that the world sheet is bidimensional.

We can easily derive the equations of motion for this action. They read

$$
\begin{cases}
\dfrac{\partial}{\partial s^\alpha}\left(\sqrt{-h}h^{\alpha\beta}\dfrac{\partial X^\mu}{\partial s^\beta}\right) = 0 \\
g_{\alpha\beta} = \dfrac{1}{2}h_{\alpha\beta}h^{\gamma\delta}g_{\gamma\delta}
\end{cases} ,
\tag{2.1.7}
$$

where $g_{\alpha\beta} \equiv \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$ is the induced metric. The first equation is just a D-plet of wave equations in curved space (where D is the dimension of the target space) while the second one is very interesting: it states that $h_{\alpha\beta}$ is proportional to the induced metric. If we can use some invariance property to put the auxiliary field in a simpler form, the equations for X^μ will become easier to solve. A good idea could be to use Weyl invariance, which allows us to put the determinant of h to one, then after diagonalization we can assume $h_{\alpha\beta} = \mathrm{diag}(-1,1) \equiv \eta_{\alpha\beta}$. Now the equation for X^μ is a flat wave equation

$$
\left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X^\mu(s) = \partial_+\partial_- X^\mu(s) = 0 \;,
\tag{2.1.8}
$$

where we have introduced the "light cone" coordinates $s_\pm = \tau \pm \sigma$ and $\partial_\pm = (\partial_\tau \pm \partial_\sigma)/2$. We must not forget the equation for $h_{\alpha\beta}$ (which now is $\eta_{\alpha\beta}$): it consists in a series of constraints called Virasoro constraints (now the D dimensional indices are raised and lowered with $\eta_{\mu\nu}$, while we will keep $\eta_{\alpha\beta}$ explicit).

$$
T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\eta_{\alpha\beta}\left(\eta^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X_\mu\right) = 0 \;.
\tag{2.1.9}
$$

In light cone coordinates, T_{+-} and T_{-+} are identically zero while the other components give the constraints

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu = 0 \ , \quad T_{--} = \partial_- X^\mu \partial_- X_\mu = 0 \ . \tag{2.1.10}$$

Now we have all the elements to pursue a quantization of this theory. For lack of space we will give the main ideas and the results useful to the rest of the discussions. With the above gauge choice for the world sheet metric the $X^\mu(s)$ is an oscillator; it can be expanded in eigenmodes and the coefficients will be quantized via canonical methods. The most general solution to the wave equation is $X^\mu(s) = X^\mu_{(L)}(s_+) + X^\mu_{(R)}(s_-)$, where both functions have an analogous expansion (here $0 \le \sigma \le 2\pi$)

$$\begin{aligned} X^\mu_{(L)}(s_+) &= \frac{\widetilde{x}^\mu_0}{2} + \frac{\alpha'}{2}\widetilde{p}^\mu\, s_+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\widetilde{\alpha}^\mu_n}{n} e^{-ins_+} \ , \\ X^\mu_{(R)}(s_-) &= \frac{x^\mu_0}{2} + \frac{\alpha'}{2} p^\mu\, s_- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha^\mu_n}{n} e^{-ins_-} \ . \end{aligned} \tag{2.1.11}$$

We have defined α' as $T = 1/2\pi\alpha'$: this quantity represents the square of the string length $\alpha' = l_s^2$. The quantities x_0 and p are respectively the center of mass position and momentum of the string, the α_n are the modes that we are going to regard as operators. The canonical quantization requires $[X^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma')] = i\eta^{\mu\nu}\delta(\sigma-\sigma')$, where the momentum Π^μ is the canonical momentum $\delta S_{\mathrm{P}}/\delta\partial_\tau X^\mu$. Imposing this rule gives the following algebra

$$\begin{aligned} [x^\mu_0, p^\nu] &= i\eta^{\mu\nu} \ , \quad [\alpha^\mu_m, \alpha^\nu_n] = m\,\eta^{\mu\nu}\delta_{m,-n} \ , \\ [\widetilde{x}^\mu_0, \widetilde{p}^\nu] &= i\eta^{\mu\nu} \ , \quad [\widetilde{\alpha}^\mu_m, \widetilde{\alpha}^\nu_n] = m\,\eta^{\mu\nu}\delta_{m,-n} \ . \end{aligned} \tag{2.1.12}$$

The $n < 0$ operators can thus be regarded as creation operators while the others as annihilation operators (up to a normalization). Of course we need to get rid of the modes with the wrong sign (due to the signature of η) and this is done by solving the Virasoro constraints (we will not show it here). With this mode expansion we can describe both *closed* strings and *open* strings:

- Closed strings: Imposing periodicity is almost automatic, we just have to set $p^\mu = \widetilde{p}^\mu$. We have both left and right moving modes.
- Open strings: Here we have to impose $\widetilde{\alpha}^\mu_n = \alpha^\mu_n$ for all modes and the expansion is to be regarded only for $0 \le \sigma \le \pi$. If we keep fixed the position of the ends we have Dirichlet boundary conditions ($\delta X^\mu = 0$ at $\sigma = 0, \pi$) while if we keep fixed the normal derivative we have Neumann boundary conditions ($\partial_\sigma X^\mu = 0$ at $\sigma = 0, \pi$).

Now we immediately see that we cannot have any fermionic state at this point. If string theory wants to describe nature we need to introduce fermions and one way to do this is to add to the Polyakov action a fermionic term[1], which is vector in target spacetime and a world sheet Majorana spinor.

$$S_{\mathrm{P}} \to S_{\mathrm{P}} - \frac{T}{2}\int_\Sigma \mathrm{d}^2 s\, \eta^{\alpha\beta}\overline{\Psi}^\mu \gamma_\alpha \partial_\beta \Psi^\nu \eta_{\mu\nu}\ . \tag{2.1.13}$$

The γ_α can be chosen as $\gamma^0 = -i\sigma^2$ and $\gamma^1 = \sigma^1$ so that $\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}\mathbb{1}$ (σ^i being the Pauli matrices). The Majorana 2 dimensional spinor is a duplet $\Psi^\mu = (\psi^\mu_-, \psi^\mu_+)^T$. The equations of motion require the spinor $\psi_\pm$ to depend only on $s_\pm$. According to the different boundary conditions (periodic or antiperiodic) imposed at the ends of the string, we can have the two following expansions: Ramond or Neveu–Schwartz.

$$\begin{aligned}
\text{Ramond (R):}\quad & \psi^\mu_+ = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} \widetilde{d}^\mu_n e^{-ins_+}\ , \qquad \psi^\mu_- = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d^\mu_n e^{-ins_-}\ . \\
\text{Neveu–Schwartz (NS):}\quad & \psi^\mu_+ = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}-\frac{1}{2}} \widetilde{b}^\mu_r e^{-ins_+}\ , \qquad \psi^\mu_- = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}-\frac{1}{2}} b^\mu_r e^{-ins_-}\ .
\end{aligned} \tag{2.1.14}$$

Notice that the r are half integers. The algebra is similar to the one above but now with anticommutators (the same expression holds for the $\widetilde{b}$ and $\widetilde{d}$)

$$\{d^\mu_n, d^\nu_n\} = \eta^{\mu\nu}\delta_{m,-n}\ , \quad \{b^\mu_r, b^\nu_s\} = \eta^{\mu\nu}\delta_{m,-n}\ . \tag{2.1.15}$$

Massless spectrum The most important part of this analysis is the massless spectrum of the theory, made up by the excitations that survive also in the limit of low energies. Let us first compute the mass operator $M^2 = -p^\mu p_\mu$. This is done by studying the Virasoro constraints so we skip the proof and give the final answer[2]:

$$M^2_{\text{bosonic}} = \frac{1}{2\alpha'}\left[\sum_{n=1}^\infty \alpha^i_{-n}\alpha^i_n + \sum_{n=1}^\infty \widetilde{\alpha}^i_{-n}\widetilde{\alpha}^i_n - \frac{D-2}{12}\right]\ , \tag{2.1.16}$$

$$M^2_{\text{super}} = \frac{1}{2\alpha'}\left[\sum_{n=1}^\infty n d^i_{-n}d^i_n + \sum_{r=1/2}^\infty r b^i_{-r}b^i_r + \text{(left movers)} - \frac{D-2}{8}\right]\ . \tag{2.1.17}$$

[1]We introduced the fermions after the gauge choice for $h_{\alpha\beta}$, if we want to be completely general we should introduce them before, but in that case we need to be careful: fermions can couple to a metric tensor only with the vielbein (zweibein in this case) formalism, we preferred to avoid this complication.

[2]The terms $-(D-2)/12$ and $(D-2)/8$ in the two expressions are a result of the central charges in the Virasoro algebra, which is the algebra of the Virasoro constraints after quantization. They are fundamental to ensure a ghost free theory, in fact we will use them to fix the value of D.

The index i goes from 1 to $D-1$: it is a light cone index. The fist line refers to the bosonic theory, the formula is valid in the case of both open and closed strings; in the closed case it has to be supplemented by the level matching condition (which states that the number of $\widetilde{\alpha}$ excitation must be the same of the α ones), while in the open case one has to just set $\alpha_n^i = \widetilde{\alpha}_n^i$. The second line refers to the superstring theory spectrum. The lightest states in the theory are found by applying creation operators to the vacuum (i.e. $\alpha_{-1}^i|0\rangle \sim A^i$). They form an $SO(D-2)$ multiplet, hence they must belong to a massless representation of the Lorentz group (see the insert). In order to avoid gauge anomalies we must impose that these states are massless. This fixes the dimensions to $D = 26$ in the bosonic case and $D = 10$ in the superstring case. The massless states in the open string sector always include a photon A_μ (or, more generally, a Yang–Mills vector), with gauge symmetry $\delta A_\mu = \partial_\mu \Lambda$. While in the closed string sector they always include a graviton $g_{\mu\nu}$, with gauge symmetry $\delta g_{\mu\nu} = \partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu$. There are also other issues that need to be dealt with, namely the presence of tachyonic states (i.e. states with negative square mass). An example in both theories is the vacuum. In bosonic string theory this problem cannot be avoided, so we will focus on superstring theory from now on. A consistent truncation of superstring theory, called GSO projection [54], is able to select only a subset of the possible states. This is sufficient to remove the tachyonic ones. We can build only a few theories both in the closed and open sector, we will focus only on Type IIB, Type IIA and Type I. They are summarized in Table 2.1. The details of the derivation will not be presented in this introduction, it is worth noticing however two things: first the closed sector is made by a tensor product of two open sectors, so for superstring theory we can have RR, RNS, NSR and NSNS. Moreover we see that the vacuum in the R sector is a target spinor (this follows from the observation that the commutation relation for d_n implies that d_0 satisfies a Clifford algebra). As we can see we have two choices in the closed sector: Type IIA and Type IIB, the first one is a non chiral theory and the second one is chiral. They are effectively described at low energies by, respectively, Type IIA and Type IIB supergravity. For open + closed strings instead we have a Type I theory which is described effectively by an $\mathcal{N} = 1$ supersymmetric Yang–Mills plus supergravity (in the Table only the open sector is shown). The fields are listed according to the irreducible representations of $SO(8)$ which is the compact part of the massless little group of the 10 dimensional Poincaré group, see Table 2.2.

> What is the Little group? To study the Poincaré group, and in particular to list its irreducible representations, it is common use to adopt the method of induced representation by Wigner. The first step consists in finding a Casimir operator, an operator commuting with all the algebra. This is the mass operator $-P^2 = M^2$. Secondly we distinguish the cases $P^2 < 0$, $P^2 = 0$ and $P^2 > 0$ (the last one is not physically interesting but is still mathematically well defined). In the third step we define in each case a "standard

Name	Sector	$SO(8)$ Representations	Field content
Type IIA	Closed	$\mathbf{1} \oplus \mathbf{8_v} \oplus \mathbf{28} \oplus \mathbf{56_t} \oplus \mathbf{35} \oplus$ $\oplus \mathbf{8} \oplus \mathbf{8'} \oplus \mathbf{56} \oplus \mathbf{56'}$	ϕ, $C_{(1)}$, $B_{(2)}$, $C_{(3)}$, $g_{\mu\nu}$ λ_L, λ_R, Ψ^μ_L, Ψ^μ_R
Type IIB	Closed	$\mathbf{1}^2 \oplus \mathbf{28}^2 \oplus \mathbf{35} \oplus \mathbf{35}_+ \oplus \mathbf{8'}^2 \oplus \mathbf{56}^2$	$C_{(0)}$, ϕ, $B_{(2)}$, $C_{(2)}$, $g_{\mu\nu}$ $C^+_{(4)}$, $\lambda_{I=1,2}$, $\Psi^\mu_{I=1,2}$
Type I	Open	$\mathbf{8_v} \oplus \mathbf{8} \oplus \mathbf{8'}$	A^a_μ, ψ^a_L, ψ^a_R

Table 2.1: Different medels in String theory: Type IIA and Type IIB are closed superstring theories and Type I is an open supestring theory. We do not mention Heterotic string here. To read this table use the explanations below (Table 2.2).

	Field name	$SO(8)$ IRREP	Name of the IRREP	Sector
ϕ	Dilaton	$\mathbf{1}$	Scalar	NS⊗NS
$C_{(0)}$	Axion	$\mathbf{1}$	Scalar	R⊗R
$C_{(1)}$	RR form	$\mathbf{8_v}$	Vector	R⊗R
$B_{(2)}$	Kalb–Ramond	$\mathbf{28}$	2–form	NS⊗NS
$C_{(2)}$	RR form	$\mathbf{28}$	2–form	R⊗R
$C_{(3)}$	RR form	$\mathbf{56_t}$	3–form	R⊗R
$g_{\mu\nu}$	Graviton	$\mathbf{35}$	Symmetric tensor	NS⊗NS
$C^+_{(4)}$	RR form	$\mathbf{35}_+$	Self–dual 4–form	R⊗R
λ_R	Dilatino	$\mathbf{8}$	MW Right Spinor	NS⊗R
λ_L	Dilatino	$\mathbf{8'}$	MW Left Spinor	R⊗NS
Ψ_R	Gravitino	$\mathbf{56}$	MW Right Spin $\frac{3}{2}$	R⊗NS
Ψ_L	Gravitino	$\mathbf{56'}$	MW Left Spin $\frac{3}{2}$	NS⊗R
A^a_μ	Gauge boson	$\mathbf{8_v}$	Vector	NS
ψ^a_R	Gaugino	$\mathbf{8}$	MW Right Spinor	R
ψ^a_L	Gaugino	$\mathbf{8'}$	MW Left Spinor	R

Table 2.2: Legend of the fields and their irreducible representation (IRREP) according to the Little group $SO(8)$. MW stands for "Majorana–Weyl", p–forms are antisymmetric tensors.

momentum", which is a preferred 4 vector of mass squared, respectively, positive, zero and negative.

Casimir	Standard momentum k^μ	Little group
$P^2 < 0$	$k^\mu = (m, \mathbf{0}_{D-1})$	$SO(D-1)$
$P^2 = 0$	$k^\mu = (E, \mathbf{0}_{D-2}, -E)$	$ISO(D-2)$
$P^2 > 0$	$k^\mu = (0, \nu, \mathbf{0}_{D-1})$	$SO(1, D-2)$

The Little group is defined as the subgroup of the Poincaré group that leaves k^μ invariant $\Lambda^\mu_{\ \nu} k^\nu = k^\mu$. In the case of positive square mass the group is compact, in the other cases it is not. However in the massless case we have an interesting fact ($ISO(D-2)$ is the Euclidean group in $D-2$ dimensions, translations are *not* space time translations), we can look for representations in which the two translations act as the identity, so that the Little group reduces to $SO(D-2)$: this is where the gauge invariance originates from (see [55]). In $D = 4$ we are left with $SO(2)$ which is generated by the helicity operator.

Perturbation theory The quantization described so far is able to account only for free strings: what happens when we want to make them interact? The idea is very simple but the actual realization is far too complicated to be portrayed here, we will again give only some hints. Suppose we have a certain initial state in String theory, take only closed strings for definiteness; this initial state will be in general a compact 1–manifold (i.e. an union of S^1) and after time evolution this will go to a final state. The world sheet described by this evolution is a 2–manifold whose boundary is made by the initial and final states. So the true dynamical object is the world sheet: this is somewhat analogous to what would happen in Quantum Gravity, where the initial and final states are fixed and we want to study the evolution of space time, which in this case is a 4–manifold. With this idea in mind let us work in analogy: in quantum gravity one writes down the action, which is the Einstein–Hilbert action (here we define it in 2 dimensions for h)

$$S_{\mathrm{EH}} = \frac{\kappa}{4\pi} \int_\Sigma \mathrm{d}^2 s \sqrt{h}\, R(h) \ , \tag{2.1.18}$$

where R is the Ricci scalar. The two dimensional case is very peculiar: the integral of the Ricci scalar is a topological invariant, the Gauss–Bonnet theorem states that $S_{\mathrm{EH}} = \kappa \chi_E$ where $\chi_E = 2 - 2g$ is the Euler characteristic of the surface, while g is the genus, equal to the number of "handles" of the surface. If we want to quantize the theory via functional methods at this point we would like to define a path integral, however since this is a topological invariant the integral becomes a sum on disconnected regions,

$$Z = \int [\mathcal{D}X^\mu][\mathcal{D}h_{\alpha\beta}]\, e^{-S_{\mathrm{P}} - S_{\mathrm{EH}}} = \sum_{g=0}^{\infty} \int_{\Sigma_g} [\mathcal{D}X^\mu][\mathcal{D}h_{\alpha\beta}]\, e^{-S_{\mathrm{P}} - \kappa(2-2g)} \ , \tag{2.1.19}$$

Where the Σ_g selects only the surfaces whose genus is g. The exponential can be further simplified defining $g_s = e^{-\kappa}$,

$$Z = \sum_{g=0}^{\infty} g_s^{2-2g} \int_{\Sigma_g} [\mathcal{D}X^\mu][\mathcal{D}h_{\alpha\beta}]\, e^{-S_{\mathrm{P}}} \,. \tag{2.1.20}$$

Then we observe that the perturbative expansion of the theory in the parameter g_s is actually a genus expansion, see for instance Figure 2.2. This is already a little evidence for the AdS/CFT correspondence, in fact we shall see in Chapter 3 that the large N limit of gauge theories admits a genus expansion in the same way, where χ_E is the Euler characteristic of the Feynman graph.

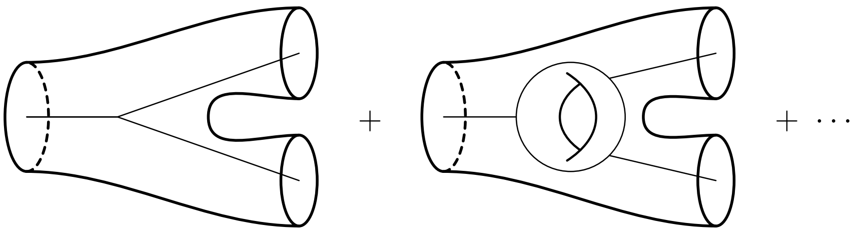


Figure 2.2: Worldsheet amplitudes. Process $1 \to 1+1$ at "tree" level and at "one loop".

2.2 D-branes

In this section we want to introduce an object that will be fundamental in our study: the D–branes. These are actually contained String theory but we prefer to treat them separately in more detail. When we discussed the quantization of the bosonic string, we said that we can in principle impose two different boundary conditions for open strings (open strings are for $0 \leq \sigma \leq \pi$)

- Dirichlet boundary conditions (D b.c.): Fixed ends, $\delta X^\mu(\tau, \bar{\sigma}) = 0$ for $\bar{\sigma} = 0$ or π.

- Neumann boundary conditions (N b.c.): Fixed derivative, $\partial_\sigma X^\mu(\tau, \bar{\sigma}) = 0$ for $\bar{\sigma} = 0$ or π.

If we impose N b.c. on, say, directions x^0 to x^p and D b.c. on the others we have defined a $p+1$ dimensional hypersurface where the end of the string can move. If we want a dynamical string we must impose N b.c. on the x^0 coordinate so this hypersurface will always have Lorentzian signature. We call this hypersurface a Dp–brane. In principle these are just mathematical entities, but there is more:

if we require momentum conservation at the ends of the string we are forced to make some momentum flow into these branes; as a consequence they become dynamical objects and, since they have their own mass, they can warp space time as General Relativity imposes. We can write an action for D–branes with the same philosophy used for strings, but before doing that let us see in more detail the excited open string states in presence of D boundary conditions. Consider a bosonic string whose last $D-p-1$ coordinates have D b.c. on both ends that fix their positions at a space separation Δx^m. Then

$$M^2 = \frac{1}{\alpha'}\left[\sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n - \frac{D-2}{24}\right] + \frac{\sum_m (\Delta x^m)^2}{(2\pi\alpha')^2} \ . \tag{2.2.1}$$

The $(\Delta x^m)^2$ is just the distance between the D branes and we can regard the extra term as the elastic energy of the string stretching between the two branes. When the two hypersurfaces coincide we have a massless gauge vector as the lightest state, but if the branes are separated the gauge vector becomes massive (here there is something similar to an Higgs phenomenon). The picture becomes much richer if we consider a non abelian generalization, so we will discuss this in the more general case. To introduce a gauge group in this theory we can add *global* degrees of freedom to the string, the so–called Chan–Paton factors. The idea is to endow the ends of the string of an additional label i, so that the lightest open string states would represent a matrix V_{ij}. We can perform a global rotation of these factors in the world sheet by means of unitary matrices

$$V_{ij} \to (U^{-1})_{ik} V_{kl} U_{lj} \ . \tag{2.2.2}$$

Remarkably the global symmetry in the world sheet becomes a local symmetry in the target space, so we are dealing with a gauge theory. Rather than considering the Chan–Paton factors as mere labels we can assign a different D–brane to each of them. In this picture a field V_{ij} would be a string stretching between the ith and jth branes. If we have, say, N branes in different positions we have a $U(1)^N$ gauge theory, but whenever k of them come to coincide the symmetry group is enhanced to a $U(k) \times U(1)^{N-k}$. This is intuitive from the world sheet point of view because we can rotate the k Chan–Paton indices amongst themselves when they live in the same D–brane. In the target space perspective the picture is the following: we see from the mass formula (2.2.1) that the vectors V_{ij} where i, j belong to the first k branes, are massless, thus being the $U(k)$ gauge fields, while V_{ia} and $V_{ab,\, a\neq b}$, where a, b belong to the other separated D–branes, acquire a mass, like a W boson; finally vectors V_{aa} are the massless $U(1)$ fields. Here we must be careful at the counting of the degrees of freedom. The vectors V_{ij} actually divide into two different representations of the Poincaré group $SO(1, D-1)$ when looked from the brane point of view: the components lying on the k branes A^μ_{ij} realize a $SO(1,p)$ symmetry on the world volume, while the other $D-p-1$ are just scalars Φ^m_{ij} belonging to an internal symmetry $SO(D-p-1)$. When solving the constraints in performing the quantization we are left with $p-1$ components of A^μ_{ij}: it has thus the correct number of degrees of freedom for a massless $SO(1,p)$ vector. If we follow the same reasoning for the massive bosons we find again $D-p-1$ massive scalars Φ^m_{ij} and a massive vector W^μ_{ij} with $p-1$ polarizations: this is not the correct number! In this case however the scalar along the direction $\bar{m}$ of separation of the branes acquires a v.e.v. (it is in fact the embedding of the brane in the target space time); this induces an Higgs phenomenon and the field $\Phi^{\bar{m}}$ must be included in the same multiplet as W^μ: now the number of degrees of freedom is correct.

2.2.1 Dirac–Born–Infeld action

We have just learned that D–branes naturally contain gauge fields living on them. Actually there is another very profound aspect hiding underneath, although we will not be able to develop this concept thoroughly it is worth to mention it. String theory exhibits a duality called the T–duality. It arises when some coordinates are compactified (which means that one periodically identifies $x \equiv x + 2\pi R$). T–duality states that the theories on circles of radius R and α'/R are equivalent. We can imagine a compactification as a theory in which D–branes are inserted in $x = 0$, $x = 2\pi R, \ldots$; in fact these two descriptions are T–dual. The positions of the D–branes are interpreted in the T–dual theory as the background value of some gauge field A_m along the compactified directions (perpendicular to the brane), more precisely to the value of the Wilson line of this gauge field. For instance, in $D = 26$ with X^{25} compactified and D24 branes in positions $\theta_1, \ldots \theta_N$:

$$\mathcal{W} = \exp\left(iq \oint A_{25} X^{25}_{\text{T–dual}} \right) , \quad A_{25} = \operatorname{diag}\left(\theta_i / 2\pi R\right) , \tag{2.2.3}$$

$$i\text{th D–brane in position} \quad X^{25} \longleftrightarrow 2\pi\alpha' A_{25,\, ii} . \tag{2.2.4}$$

The action for a D–brane is built in order to preserve this T–duality; it is not a surprise thus that in some way gauge fields must appear in the action. The complete derivation uses also the background furnished by the NS⊗NS, namely the metric, the Kalb–Ramond form and the dilaton, here we finally present the action, called Dirac–Born–Infeld action

$$S_p = -T_p \int \mathrm{d}^{p+1}\xi \, e^{-\phi} \sqrt{-\det\left(\mathcal{P}[g] + \mathcal{P}[B] + 2\pi\alpha' F\right)} , \tag{2.2.5}$$

where

$$T_p = \frac{1}{(2\pi)^p (\alpha')^{\frac{p+1}{2}}} , \qquad \mathcal{P}[g]_{ab} = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} g_{\mu\nu} , \qquad \text{Similarly for } \mathcal{P}[B]_{ab} , \tag{2.2.6}$$

The idea is similar to the Nambu–Goto action studied in the previous section, the $X^\mu(\xi)$ represent the *embedding* of the brane[3] in the target space and the action is just the surface area, however here the situation is much richer: as noted above the gauge fields must appear (both to ensure T–duality and to account for the gauge theory living in the world volume), moreover this action is the result of tree level open string amplitudes, the dilaton ϕ is a mark of this aspect (in fact the v.e.v. of the dilaton corresponds to the string coupling g_s appeared in (2.1.20)), and also since the brane floats in space time it feels the background generated by the Kalb–Ramond field and the metric, which is nothing but a result of the closed string low energy behaviour. The present action is for the abelian case, the non abelian generalization is much more complicated because the embedding functions are non abelian fields themselves. We will use in the following a simplified version of this non abelian action in which the embedding is fixed. The only subtlety arising in this case is that the trace (that must be taken in order to ensure gauge invariance) has to be defined properly. In the following expression we are dropping terms involving $[\Phi^m, \Phi^n]$ where Φ^m are the embedding functions

$$S_p = -T_p \int \mathrm{d}^{p+1}\xi \, e^{-\phi} \mathrm{STr} \sqrt{-\det\left(\mathcal{P}[g] + \mathcal{P}[B] + 2\pi\alpha' F\right)} \, . \tag{2.2.7}$$

The “STr” stands for symmetrized trace, the gauge indices must be symmetrized before taking the trace when expanding the action[4].

2.2.2 Chern–Simons terms

The DBI action must be supplemented by some corrective terms, called Chern–Simons (CS) terms. They account for the anomalies arising from the presence of defects in space time due to the branes. These anomalies render ill defined the RR forms $C_{(p)}$ to which any D–brane configuration couples, so we need some extra terms to cancel them. Before proceeding let us see the formalism of p–form electrodynamics: the coupling with $C_{(p)}$ forms and D–branes can be easily described in these terms.

> A p–form is an antisymmetric covariant tensor $C_{\mu_1 \ldots \mu_p}$. We denote by $\wedge$ the wedge product (antisymmetrized tensor product).
>
> $$C_{(p)} \equiv \frac{1}{p!} C_{\mu_1 \ldots \mu_p} \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p} \, . \tag{2.2.8}$$
>
> Two operations are naturally defined on p–forms:

[3]The scalar fields Φ^m mentioned in the last paragraph are precisely these embedding functions, one can in fact choose the first $p+1$ X^a to be just ξ^a.

[4]$\mathrm{STr}\left(A_{a_1} \cdots A_{a_n}\right) = \sum_{\text{perm. } \pi} \mathrm{Tr}\left(A_{a_{\pi(1)}} \cdots A_{a_{\pi(n)}}\right)$

- Exterior derivative d

  The d operator is the antisymmetrized derivative, it turns a p–form into a $(p+1)$–form $\mathrm{d}:\Omega_p \mapsto \Omega_{p+1}$ (notice that it is a regular derivative, not a covariant derivative)

$$\begin{aligned}\mathrm{d}C_{(p)} &= \frac{1}{p!}\partial_\nu C_{\mu_1\ldots\mu_p}\mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_p} \\ &= \frac{1}{(p+1)!}\partial_{[\mu_1}C_{\mu_2\ldots\mu_{p+1}]}\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_{p+1}}\ .\end{aligned} \tag{2.2.9}$$

  The exterior derivative is nilpotent $\mathrm{d}^2=0$. Forms such that $\mathrm{d}C=0$ are called *closed* and forms D that are a derivative $D=\mathrm{d}C$ are called *exact*. The nilpotency says that exact $\Rightarrow$ closed, while in general the converse is not true; under certain conditions the Poicaré's lemma states also closed $\Rightarrow$ exact.

- Hodge dual *

  The Hodge dual is a map $^\star:\Omega_p\mapsto\Omega_{d-p}$ in a d–dimensional manifold equipped with a metric $g_{\mu\nu}$ defined as

$$^\star C_{(p)} = \frac{1}{(d-p)!}\left(\frac{\sqrt{g}}{p!}\varepsilon_{\nu_1\ldots\nu_p\mu_{p+1}\ldots\mu_d}g^{\nu_1\rho_1}\cdots g^{\nu_p\rho_p}C_{\rho_1\ldots\rho_p}\right)\mathrm{d}x^{\mu_{p+1}}\wedge\ldots\wedge\mathrm{d}x^{\mu_d}\ . \tag{2.2.10}$$

  The Hodge dual defines an inner product on p forms in this way (calling ω the d volume form $\omega=\sqrt{g}\,\mathrm{d}x^1\wedge\ldots\wedge\mathrm{d}x^d$)

$$\langle C_{(p)},D_{(p)}\rangle\,\omega \equiv C_{(p)}\wedge{}^\star D_{(p)}\ . \tag{2.2.11}$$

  It is also involutive, more precisely in a space with signature $s=\pm1$ it gives $^{\star\,\star}C_{(p)}=(-1)^{p(d-p)}sC_{(p)}$.

In complete analogy with electromagnetism let us define a field strength $F_{(p+1)}$ satisfying the Bianchi identity (i.e. homogeneous Maxwell equations) $\mathrm{d}F_{(p+1)}=0$. Poincaré's lemma ensures us that locally we can define a potential $C_{(p)}$ such that $\mathrm{d}C_{(p)}=F_{(p+1)}$, but we have an ambiguity in the definition: in fact $C_{(p)}$ and $C_{(p)}+\mathrm{d}\Lambda_{(p-1)}$ are completely equivalent (because of $\mathrm{d}^2=0$), this being what is usually called gauge invariance

$$\mathrm{d}C_{(p)}=F_{(p+1)}\ ,\qquad \delta_\Lambda C_{(p)}=\mathrm{d}\Lambda_{(p-1)}\ . \tag{2.2.12}$$

The interaction between gauge fields and matter is often mediated by a current coupling $J_{(p)}$, which in this case is a p–form, whose conservation law is generalized to $\mathrm{d}\,^\star J_{(p)}=0$. The action for this theory is

$$S=\frac{(-1)^p}{2}\int F_{(p+1)}\wedge{}^\star F_{(p+1)}+\int C_{(p)}\wedge{}^\star J_{(p+1)} \tag{2.2.13}$$

and this allows us to write the equivalent of the inhomogeneous Maxwell equations

$$\mathrm{d}\,^\star F_{(p+1)}={}^\star J_{(p)} \tag{2.2.14}$$

When the source is zero $F_{(p+1)}$ and ${}^\star F_{(p+1)} \equiv G_{(d-p-1)} = \mathrm{d}C_{(d-p-2)}$ are completely equivalent, the Maxwell equation and the Bianchi identity are interchanged under this duality.

Standard electrodynamics is the theory of one forms, a system of a point particle can be described by a current $J = e\delta^{d-1}(x^i - X^i(t))\mathrm{d}t$ where $X(t)$ is the trajectory (world line) of the particle, the same idea can be generalized to higher dimensions, the current is a δ function with support on the world volume

$$\mathrm{d}C_{(p+1)} = F_{(p+2)} \ , \qquad J_{(p+1)} = \mu_p \, \delta^{(d-p-1)}(x - X(\xi)) \, \mathrm{d}\xi^1 \wedge \ldots \wedge \mathrm{d}\xi^{p+1} \quad (2.2.15)$$

The action is rewritten as

$$S_p = -\frac{(-1)^p}{4\kappa_0^2} \int F_{(p+2)} \wedge {}^\star F_{(p+2)} - \mu_p \int_{Dp} C_{(p+1)} \ , \qquad (2.2.16)$$

where $\mu_p = (2\pi)^{-p} l_s^{-p-1}$ is the charge of the Dp–brane and $2\kappa_0^2$ is the Newton constant coming from the supergravity action. The analogy with electrodynamics stops here because now we have to deal with anomalies; a complete discussion of this topic is beyond the scope of this work, it can be found in Chapter 9 of [52] and in [56]. Anomalies spoil the gauge invariance $\delta_\Lambda C_{(p)} = \mathrm{d}\Lambda_{(p-1)}$, forcing us to add the term

$$\mu_p \int_{Dp} C_{(p-2n+1)} \frac{(2\pi\alpha')^n}{n!} \mathrm{Tr}\, F^n \ . \qquad (2.2.17)$$

Where $F^n = F \wedge \ldots \wedge F$ n times. The most general term would have to include also the Kalb–Ramond field, this is achieved by substituting $2\pi\alpha' F \to B + 2\pi\alpha' F$, but we will ignore it. A consistent string theory contains all RR forms of the same parity (for instance, Type IIA contains only forms for p odd while Type IIB only for p even), so we ought to put all of them

$$S_{\mathrm{CS}} = \mu_p \int_{Dp} \sum_{n=0} C_{(p-2n+1)} \wedge \frac{(2\pi\alpha')^n}{n!} \mathrm{Tr}\, F^n = \mu_p \int_{Dp} \sum_{q=0} C_{(q+1)} \wedge \mathrm{Tr}\, e^{2\pi\alpha' F} \ . \qquad (2.2.18)$$

In the last equality the expansion of the exponential is by means of the wedge product, the integral over the Dp world volume picks up only the terms that are $p+1$ forms.

This expression is actually very interesting also from a mathematical point of view. The exponential $\mathrm{Tr}\, e^{iF/2\pi} \equiv \mathrm{ch}(F)$ (this normalization is usually adopted in mathematics) is known as the *Chern character* of the gauge bundle[5]. It generates the *Chern classes*, which are topological invariants; in fact for abelian

[5] A gauge bundle is the structure of a base manifold $\mathcal{M}$ (space time) over which a gauge group G is built (a fiber). Locally it is just $\mathcal{M} \times G$ but globally the bundle is not trivial. The gauge 1–form A is a connection on this space and F its curvature.

groups $\mathrm{Tr}\, F/2\pi$ (the first class) represents, for instance, the charge of a magnetic monopole; more interestingly the class $\mathrm{Tr}\, F \wedge F/8\pi^2$ for non abelian groups is the instanton number (we will discuss it in the next Chapter), etc.... All these classes are closed forms (the first one only in the abelian case): in fact they are locally the derivative of a form ω_{2k-1}, called Chern–Simons form

$$\mathrm{d}\omega_{2k-1} = \mathrm{Tr}\, F^k \ . \tag{2.2.19}$$

As globally a closed form can be non exact, they are in fact representatives of the *cohomology group*[6] of the gauge bundle. From a physical point of view, when $\mathrm{Tr}\, F^k$ is not exact it means that there is no potential A that can be written globally in the whole space: we are forced then to cover the space with different "patches" and impose that the transition from a patch to another is just a gauge transformation for A. For completeness we should say that $\mathrm{Tr}\, e^{F/2\pi}$ generates all the Chern classes but they appear in non trivial linear combinations. The correct definition for Chern classes $c_p(F)$ is

$$\mathrm{Tr}\left(t\mathbb{1} + \frac{iF}{2\pi}\right) = \sum_{j=0}^{N} c_{N-j}(F)t^j \ , \tag{2.2.20}$$

where the group G is $U(N)$ (or any other N dimensional compact group) and F can be regarded as an $N \times N$ matrix, forgetting that it is a 2 form. The coefficients of the expansion are the Chern classes and can be easily computed. The Chern character has an expansion

$$\mathrm{ch}(F) = N + c_1(F) + \frac{1}{2}(c_1(F)^2 - 2c_2(F)) + \ldots + \frac{1}{n!}c_N(F) \ . \tag{2.2.21}$$

The first class is c_1, the class $\mathrm{Tr}\, F^2$ is actually the combination $c_1^2 - 2c_2$, the Nth class is just $\det F$.

There are also anomaly terms related to the curvature. They topopogical invariants built from powers of the Riemann tensor $R^{\lambda}_{\mu\nu\rho\sigma}$. Since we will not need them in the following we refer the reader to [52].

2.2.3 p–brane solutions

We now would like to present an aspect that apparently has not much in common with D–branes, but it will reveal to be strictly related to them, resulting fundamental in the derivation of the AdS/CFT correspondence: p–brane solutions to supergravity. It is better to take a few steps back and talk about charged black holes, also known as Reissner–Nordström black holes.

[6] Which is the set of closed form modulo exact forms, equipped with the sum as a group structure.

The Reissner–Nordström black hole is a spherically symmetric solution to the Einstein–Maxwell system of equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \ , \quad T_{\mu\nu} = \frac{1}{4\pi} \left(g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) \ . \tag{2.2.22}$$

The parameters of the solution are the total mass M of the black hole and the electrical charge Q. The charge is found obviously integrating the electric field over a sphere at infinity and dividing by its area. In more general terms we can express this condition in the language of differential forms

$$Q = \lim_{R\to\infty} \frac{1}{4\pi R} \int_{S^2(R)} {}^\star F \ . \tag{2.2.23}$$

Also, keep in mind that $F = \mathrm{d}A$ and the electromagnetic coupling is given by $\sim Q \int \mathrm{d}tA$. The solution is given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) \mathrm{d}t^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega_2^2 \ , \tag{2.2.24}$$

where Ω_2 is the line element of S^2. The limit $Q \to 0$ is the Schwarzschild metric. The electric field is simply the point charge:

$$F = \frac{Q}{r^2} \mathrm{d}t \wedge \mathrm{d}r \ . \tag{2.2.25}$$

If the sign of $M - Q$ is negative we do not have null surfaces (i.e. surfaces with a normal vector n^μ of null length $n_\mu n^\mu = 0$), which means that there is no horizon. We exclude this case because it is believed that no naked singularities exist in nature ($r = 0$ is a singularity of this solution, a "naked" singularity is a singularity which is not surrounded by an event horizon). If $M - Q$ is non negative then we have two null surfaces (one if $M = Q$)

$$r = r_\pm \ , \quad r_\pm = M \pm \sqrt{M^2 - Q^2} \ . \tag{2.2.26}$$

The situation concerning $M = Q$ is very interesting: in this case a bound called Bogomol'nyi –Prasad –Sommerfield (BPS) bound is saturated and this means that the symmetry of the solution is enhanced. We will not prove here this statement, the argument consists in finding a Killing spinor[a] of the metric. Some evidence of this additional symmetry can be seen in the near horizon limit, choosing $R = r - Q$ and taking $R \to 0$ we have

$$\mathrm{d}s^2 = -\frac{R^2}{Q^2} \mathrm{d}t^2 + \frac{Q^2}{R^2} \left(\mathrm{d}R^2 + R^2 \mathrm{d}\Omega_2^2 \right) \tag{2.2.27}$$

This is, as we will see in the next section, an $AdS_2 \times S^2$ space (AdS being the (t, R) part). This space has a much larger symmetry than regular Minkowski space.

[a] Killing spinors are the spinorial generalization of the concept of Killing vectors. A Killing vector is the generator of a symmetry of the metric under a certain group of diffeomorphisms.

As we have showed in Section 2.1, the massless excitations in string theory always contain a graviton. However, the theory describing the degrees of freedom at low energies is not pure Einstein gravity because the metric interacts with the p forms and with a spin 3/2 field (the gravitino). The specific form of these couplings is fixed by a local supersymmetry, this is why the theory is called *Supergravity*. Type IIA and Type IIB superstring theory reduce, at low energies, to Type IIA and Type IIB supergravity respectively. A possible way to build these theories from the world sheet action is to consider the Weyl invariance of the Polyakov action. First of all one has to rewrite the Polyakov action with the backreaction of the fields $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ (respectively the metric, the Kalb–Ramond and the Dilaton arising from the massless NSNS sector). It takes the form

$$\begin{aligned} S_{\mathrm{P}} = -\frac{T}{2}\int \mathrm{d}^2 s\sqrt{-h}\,\big(&h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu g_{\mu\nu}(X)+ \\ &+iB_{\mu\nu}(X)\varepsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu+\alpha'\phi(X)R^{(2)}\big)\ , \end{aligned} \tag{2.2.28}$$

where $\varepsilon^{\alpha\beta}$ is the antisymmetric tensor and $R^{(2)}$ is the Ricci scalar in the two dimensional worldsheet. Now the theory is not automatically invariant under Weyl rescalings, but we can write the beta functions of $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ, regarding them as couplings of the action S_{P}. The equations

$$\begin{aligned} \beta(G) &= \mu\frac{\partial g(\mu)}{\partial\mu}=0\ , \\ \beta(B) &= \mu\frac{\partial B(\mu)}{\partial\mu}=0\ , \\ \beta(\phi) &= \mu\frac{\partial \phi(\mu)}{\partial\mu}=0\ , \end{aligned} \tag{2.2.29}$$

encode the Weyl invariance of the Polyakov action. It turns out that they can be obtained as a minimization of a Lagrangian, which is precisely the Lagrangian of Supergravity. The bosonic parts of the actions of Type IIA and Type IIB supergravity are, respectively,

$$\begin{aligned} S_{\mathrm{IIA}} = \frac{1}{2\kappa_{10}^2}\Bigg[&\int \mathrm{d}^{10}X\,\sqrt{-g}\left(e^{-2\phi}\left(R+4\partial_M\phi\partial^M\phi-\frac{1}{2}|H_{(3)}|^2\right)\right. \\ &\left.-\frac{1}{2}|F_{(2)}|^2-\frac{1}{2}|\widetilde{F}_{(4)}|^2\right)-\frac{1}{2}\int B_{(2)}\wedge F_{(4)}\wedge F_{(4)}\Bigg]\ , \end{aligned} \tag{2.2.30}$$

$$\begin{aligned} S_{\mathrm{IIB}} = \frac{1}{2\kappa_{10}^2}\Bigg[&\int \mathrm{d}^{10}X\,\sqrt{-g}\left(e^{-2\phi}\left(R+4\partial_M\phi\partial^M\phi-\frac{1}{2}|H_{(3)}|^2\right)\right. \\ &\left.-\frac{1}{2}|F_{(1)}|^2-\frac{1}{2}|\widetilde{F}_{(3)}|^2-\frac{1}{2}|\widetilde{F}_{(5)}|^2\right) \\ &-\frac{1}{2}\int C_{(4)}\wedge H_{(3)}\wedge F_{(3)}\Bigg]\ , \end{aligned} \tag{2.2.31}$$

where we have defined for brevity

$$\int \mathrm{d}^{10}X\,\sqrt{-g}\,|F|^2 \equiv \int F \wedge {}^\star F\ . \tag{2.2.32}$$

The constant κ_{10} is the Newton constant, related to the string tension as

$$2\kappa_{10} = (2\pi)^7 \alpha'^4\ . \tag{2.2.33}$$

For Type IIA we have the following definitions

$$F_{(p)} = \mathrm{d}C_{(p-1)}\ , \quad \widetilde{F}_{(4)} = \mathrm{d}C_{(3)} - C_{(1)} \wedge F_{(3)}\ . \tag{2.2.34}$$

While for Type IIB we have

$$\begin{aligned} &F_{(p)} = \mathrm{d}C_{(p-1)}\ , \quad H_{(3)} = \mathrm{d}B_{(2)}\ , \quad \widetilde{F}_{(3)} = F_{(3)} - C_{(0)}H_{(3)}\ , \\ &\widetilde{F}_{(5)} = F_{(5)} - \frac{1}{2}C_{(2)} \wedge H_{(3)} + \frac{1}{2}B_{(2)} \wedge F_{(3)}\ , \end{aligned} \tag{2.2.35}$$

and $\widetilde{F}_{(5)}$ must satisfy a self–duality condition ${}^\star\widetilde{F}_{(5)} = \widetilde{F}_{(5)}$.

It is interesting to mention that the theory Type IIA can be obtained as a dimensional reduction of eleven dimensional supergravity over a circle of radius $g_s l_s$ (g_s being the vacuum value of the dilaton ϕ). The bosonic part of eleven dimensional supergravity is

$$S_{11} = \frac{1}{2\kappa_{11}}\left[\int \mathrm{d}^{11}x\,\sqrt{-g}\left(R - \frac{1}{2}|F_{(4)}|^2\right) - \frac{1}{6}\int C_{(3)} \wedge F_{(4)} \wedge F_{(4)}\right]\ , \tag{2.2.36}$$

where $F_{(4)} = \mathrm{d}C_{(3)}$ and κ_{11} is the eleven dimensional Newton constant.

The solution showed in the insert is the forerunner of many other solutions is Supergravity[7]. In fact the following generalization has been studied (see refs. 94, 95 of [52]): a solution rotationally symmetric on a $9-p$ dimensional spacelike subspace of $\mathbb{R}^{1,9}$, with a RR $C_{(p+1)}$ form acting as an electric source and a dilaton appearing in the Einstein–Hilbert coupling in the standard way $S_{\mathrm{EH}} \sim \int \sqrt{g}\, e^{-2\phi} R$. The gravitino and the Kalb–Ramond are set to zero in these solutions. The solution, called 10 dimensional black p–brane solution, reads

$$\mathrm{d}s^2 = Z_p^{-1/2}(r)\left(-K(r)\mathrm{d}t^2 + \sum_{i=1}^{p} \mathrm{d}x_i^2\right) + Z_p^{1/2}(r)\left(\frac{\mathrm{d}r^2}{K(r)} + r^2 \mathrm{d}\Omega_{8-p}^2\right)\ , \tag{2.2.37}$$

where Ω_{8-p}^2 is the line element of S^{8-p} and

$$\begin{aligned} Z_p(r) &= 1 + \alpha_p \left(\frac{r_p}{r}\right)^{7-p}\ , \\ K(r) &= 1 - \left(\frac{r_H}{r}\right)^{7-p}\ , \\ e^{2\phi} &= g_s^2 Z_p(r)^{\frac{3-p}{2}}\ , \\ C_{(p+1)} &= g_s^{-1}\left(Z_p(r)^{-1} - 1\right)\mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^p\ . \end{aligned} \tag{2.2.38}$$

[7] Type IIA if p is even or Type IIB if p is odd.

The parameter r_H is the horizon radius and the other parameters are given by

$$r_p^{7-p} = (4\pi)^{\frac{5-p}{2}}\Gamma\left(\frac{7-p}{2}\right) g_s N(\alpha')^{\frac{7-p}{2}} ,$$
$$\alpha_p = \sqrt{1+\frac{1}{4}\left(\frac{r_H}{r_p}\right)^{2(7-p)}} - \frac{1}{2}\left(\frac{r_H}{r_p}\right)^{7-p} . \tag{2.2.39}$$

Here N as a free parameter representing the units of charge of the black brane. The charge is given, in analogy as before, by the following condition

$$\frac{1}{(2\pi)^{7-p}(\alpha')^{\frac{7-p}{2}}}\int_{S^{8-p}} {}^{\star}\mathrm{d}C_{(p+1)} = \alpha_p N , \tag{2.2.40}$$

where S^{8-p} is a sphere surrounding the black p–brane (here the normalization of the RR form is the standard one, in the following we will use a different normalization where the unpleasant α' and π factors disappear). The BPS condition analogous to $M = Q$ here is realized by imposing $\alpha_p = 1$. The physical interpretation of this class of solutions is very interesting: it is the warped geometry in presence of a p dimensional extended object in the origin of space charged under N units of a RR $p+1$ form, or simply called p–brane. We argued before that D–branes do warp space time and they are charged under RR forms via the standard coupling $\int_{Dp} C_{(p+1)}$ on the $p+1$ dimensional world volume. So this solution is a good candidate to study, in the low energy limit, the behaviour of space time in presence of D–branes. The charge N is interpreted as the number of D–branes. In the same spirit of the Reissner–Nordström solution we would like to study the near horizon limit of this metric. Sadly for generic p the horizon located at $r = 0$ shrinks to a point and it is a singularity, however for $p = 3$ a special feature appears: the horizon is a sphere with finite volume (there is a cancellation between the divergent warp factor Z_p and the sphere metric $\propto r^2$). Not by chance the metric in this limit is again an AdS metric times a compact manifold, namely $AdS_5 \times S^5$. This feature is not only interesting as it is: it furnishes a bridge between string theory/D–branes and classical gravity and it will be crucial in the formulation of the AdS/CFT correspondence, as we shall see shortly.

2.3 *AdS* geometry

We obviously cannot talk about AdS/CFT without knowing the geometry of Anti De Sitter (AdS) space. First of all a little background on Maximally symmetric spaces in general is presented.

Infinitesimal symmetries in space time can be described via the technology of Killing vectors. Suppose the metric is symmetric under the transformation

$$x^\mu \to x'^\mu = x^\mu - \epsilon\,\xi^\mu\ , \quad \epsilon \ll 1\ . \tag{2.3.1}$$

This means that

$$g'_{\mu\nu}(x') - g_{\mu\nu}(x') = \epsilon\, g_{\mu\nu,\lambda}\,\xi^\lambda + \epsilon\, g_{\mu\sigma}\,\xi^\sigma{}_{,\nu} + \epsilon\, g_{\rho\nu}\,\xi^\rho{}_{,\mu} + \mathcal{O}(\epsilon^2) \equiv \pounds_\xi g_{\mu\nu} = 0\ . \tag{2.3.2}$$

Notice that the two arguments are the same in both terms: we need to compare the new and old metric in the same point. The symbol $\pounds_\xi$ goes under the name of *Lie derivative*. If on the space is defined a connection ∇ (such that $\nabla_\mu \xi^\rho = \partial_\mu \xi^\rho + \Gamma^\rho_{\mu\nu}\xi^\nu$) we can rewrite the last expression as

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = \pounds_\xi g_{\mu\nu} = 0\ . \tag{2.3.3}$$

This is called the *Killing equation* and its solution are the *Killing vectors*. To each Killing vector is associated a symmetry of space time. This equation gives a very strong statement about multiple derivatives of ξ; in fact it is known that the commutator of two derivatives is just the Riemann curvature

$$[\nabla_\mu, \nabla_\nu]\xi_\rho = R^\lambda_{\ \rho\mu\nu}\xi_\lambda\ . \tag{2.3.4}$$

The Killing equation combined with this and the ciclicity property of the Riemann tensor $R^\lambda_{[\mu\nu\rho]} = 0$ allows us to write

$$\nabla_\rho \nabla_\nu \xi_\mu = R^\lambda_{\ \rho\mu\nu}\xi_\lambda\ . \tag{2.3.5}$$

Since the second derivatives of ξ are just linear combinations of ξ, via a Taylor expansion we can reconstruct $\xi^\mu(x)$ in each point in space just by knowing $\xi^\mu(x_0)$ and $\xi^\mu{}_{,\nu}(x_0)$ for some fixed point x_0. In terms of some universal functions $A(x)$ and $B(x)$ we can write:

$$\xi_\mu(x) = A^\lambda_\mu(x)\xi_\lambda(x_0) + B^{\rho\lambda}_\mu(x)\nabla_\lambda \xi_\rho(x_0)\ , \tag{2.3.6}$$

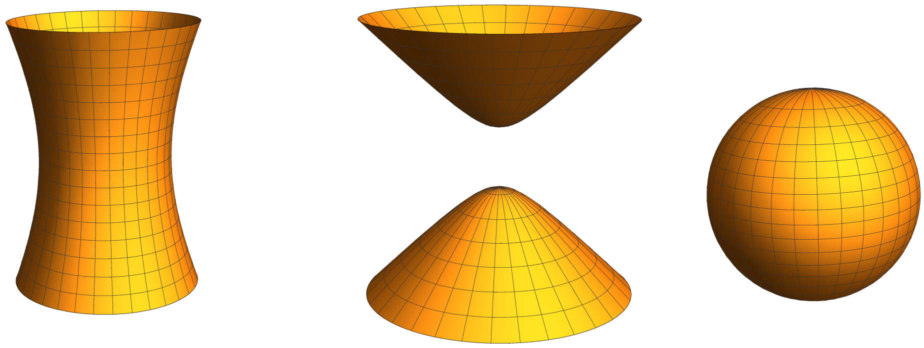
so the maximal number of linearly independent Killing vectors (with constant coefficients!) is the total number of independent vectors and derivatives of vectors. The Killing equation reduces the number to $n + n(n-1)/2 = n(n+1)/2$ where n is the dimension of space. A Maximally symmetric spaces is defined as *a space whose number of Killing vectors is maximal*. When the number of Killing vectors is maximal we have many interesting properties, in fact it can be shown that any Maximally symmetric space is *isotropic* and *homogeneous*, which means that the space enjoys translational and rotational invariance, hence the general coordinate scalars should not depend on the point. The scalar curvature $R(x) = R$ is always a constant in these class of spaces and the Riemann tensor can be cast in the following form

$$R_{\lambda\rho\sigma\nu} = \frac{R}{n(n-1)}(g_{\nu\rho}g_{\lambda\sigma} - g_{\sigma\rho}g_{\lambda\nu})\ . \tag{2.3.7}$$

A fundamental theorem of unicity about Maximally symmetric spaces will be enounced.

Theorem A Maximally symmetric space is uniquely determined by 3 characteristics: the dimension n, the scalar curvature R and the signature of the metric.

Since Maximally symmetric spaces are unique, we can build them in a stan-



(a) AdS or dS space (b) Hyperbolic space (c) Sphere

Figure 2.3: Three different kinds of hyperspheres. The first has undefined signature, the last two have positive definite signature with curvature respectively negative and positive.

dard way. Suppose we need a space in $p+q$ dimensions with signature $(p+, q-)$; then we can go in a space of $p+q+1$ dimensions and embed an hypersphere: the metric restricted to the hypersphere will be that of the space we are looking for. To be more concrete let us distinguish between positive and negative curvature.

- Negative curvature with signature $(p+, q-)$, Anti de Sitter $AdS_{p,q}$

$$\begin{aligned} \text{Ambient metric:} \qquad & \mathrm{d}s^2 = \sum_{i=1}^{p} \mathrm{d}X_i^2 - \sum_{j=1}^{q+1} \mathrm{d}T_j^2 \,, \\ \text{Hypersphere:} \qquad & -L^2 = \sum_{i=1}^{p} X_i^2 - \sum_{j=1}^{q+1} T_j^2 \,. \end{aligned} \tag{2.3.8}$$

- Positive curvature with signature $(p+, q-)$, de Sitter $dS_{p,q}$

$$\begin{aligned} \text{Ambient metric:} \qquad & \mathrm{d}s^2 = \sum_{i=1}^{p+1} \mathrm{d}X_i^2 - \sum_{j=1}^{q} \mathrm{d}T_j^2 \,, \\ \text{Hypersphere:} \qquad & -L^2 = \sum_{i=1}^{p+1} X_i^2 - \sum_{j=1}^{q} T_j^2 \,. \end{aligned} \tag{2.3.9}$$

The special cases $q = 0$ with positive and negative curvature are, respectively, the p–Sphere S^p and the Hyperbolic space $\mathbb{H}^p$. In Figure 2.3 there is a depiction of these spaces in $p+q = 2$. Strictly speaking Anti de Sitter has Lorentian signature, so in our notation is $AdS_{p+1} \equiv AdS_{p,1}$: it originates from the metric induced over

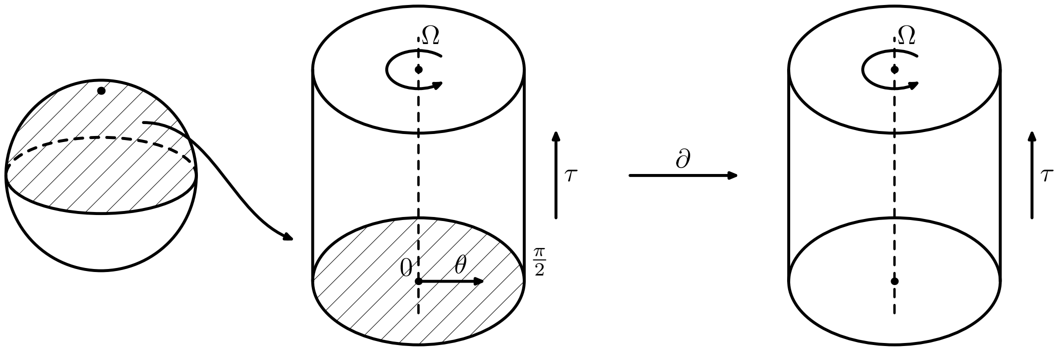


Figure 2.4: Half of $\mathbb{R} \times S^p$ is conformally equivalent to AdS_{p+1}. The figure shows $p = 2$. Next to it also the boundary is shown.

an hyperboloid of negative radius with two times. The isometry group for AdS_{p+1} is $SO(2, p)$; the construction showed here renders this symmetry evident because this is the symmetry of the hyperboloid. The following coordinate chart covers the whole surface only once (if $\rho > 0$)

$$\begin{aligned} &T_1 = L \cosh\rho \cos\tau \ , \quad T_2 = L\cosh\rho\sin\tau \ , \\ &X_i = L\sinh\rho\,\Omega_i \ , \qquad \Big(i = 1, \ldots p \ , \ \sum_i \Omega_i^2 = 1\Big) \ . \end{aligned} \tag{2.3.10}$$

Actually one subtlety must be fixed: the coordinate τ is periodic so this coordinate chart admits closed time like curves. In order to avoid this paradoxical consequence it is preferable to cut the hyperboloid at $\tau = 2\pi$ and analytically continue the metric for $\tau \in]-\infty, \infty[$. After the change of variables $\tan\theta = \sinh\rho$, $\theta \in [0, \frac{\pi}{2}]$, we get the following induced metric on the surface

$$\mathrm{d}s^2 = \frac{L^2}{\cos^2\theta}\left(-\mathrm{d}\tau^2 + \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\Omega_{p-1}^2\right) \tag{2.3.11}$$

If we are interested in studying the causal structure of the space we are allowed to make a conformal rescaling (Weyl transformation), which allows us to get rid of the factor $1/\cos^2\theta$ in front, leaving us with $\mathbb{R} \times S^p$. Actually this is not quite true, since $\theta \in [0, \frac{\pi}{2}]$ and we are covering half of the S^p, which is a B^p (p dimensional ball). In Figure 2.4 the case AdS_3 is shown. The space $\mathbb{R} \times S^p$ is conformally equivalent to $\mathbb{R}^{1,p}$ (this can be proved in a similar way). This is crucial in the AdS/CFT correspondence because we just discovered that AdS_{p+1} has a conformal boundary which is $\mathbb{R} \times S^{p-1}$ (simply from $\partial B^p = S^{p-1}$), which is conformally equivalent to $\mathbb{R}^{1,p-1}$. The causal structure of the boundary is of the Lorentzian type, so it is suitable to describe causal dynamics. When we refer to the "boundary" of AdS we mean conformal boundary in the sense exposed here.

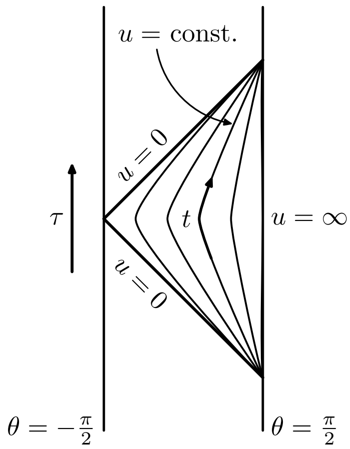


Figure 2.5: The strip AdS_2 and the Poincaré patch (the triangular region). The Poincaré patch covers only half of the strip if is thought to be periodically continued for $\tau > 2\pi$, $\tau < 0$. The lines at $u = \text{const.}$ are drawn, those are also the lines along which t grows.

We now show a change of variables to express the metric of AdS in a more familiar form. This new chart of coordinates is called the "Poincaré patch" and it does not cover the whole hyperboloid but only half of it ($u \geq 0$, $\vec{x} \in \mathbb{R}^{p-1}$):

$$\begin{aligned} T_1 &= \frac{1}{2u}\left(1 + u^2(L^2 + \vec{x}^2 - t^2)\right) \;, \quad T_2 = Lut \;, \\ X^i &= Lux^i \;, \quad (i = 1, \dots p-1) \;, \\ X^p &= \frac{1}{2u}\left(1 - u^2(L^2 - \vec{x}^2 + t^2)\right) \;. \end{aligned} \tag{2.3.12}$$

The metric reads

$$\begin{aligned} \mathrm{d}s^2 &= L^2\left(\frac{\mathrm{d}u^2}{u^2} + u^2(-\mathrm{d}t^2 + \mathrm{d}\vec{x}^2)\right) \\ &= \frac{\mathrm{d}z^2 - \mathrm{d}t^2 + \mathrm{d}\vec{x}^2}{z^2} \;. \end{aligned} \tag{2.3.13}$$

The second line is also a familiar form of the metric, where the substitution $Lu = 1/z$ has been made. In Figure 2.5 is shown the part of the hyperboloid covered by the Poincaré patch in the case $p = 1$, where AdS is just a strip $\mathbb{R} \times I$: we can see that indeed only half of it is covered.

It turns out that $(A)dS$ is the Maximally symmetric solution of Einstein equations with (negative)positive cosmological constant. The Einstein–Hilbert action in n dimensions with a cosmological constant term is

$$S_{\text{EH}} = -\frac{1}{2\kappa_0^2}\int \mathrm{d}^n x \sqrt{g}(R - 2\Lambda) \;. \tag{2.3.14}$$

The equation in the vacuum is just

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0 \ . \qquad (2.3.15)$$

The AdS metric (2.3.13) is a solution of the above equation where the parameter L (which in the second line appears implicitly in the units of the adimensional coordinate z) is related to the cosmological constant:

$$R_{\mu\nu} = -\frac{n-1}{L^2} g_{\mu\nu} = \frac{2\Lambda}{n-2} g_{\mu\nu} \ , \quad \Lambda = -\frac{(n-2)(n-1)}{2L^2} \ . \qquad (2.3.16)$$

As mentioned before the isometry group of AdS_{n+1} is the orthogonal group $SO(2, n)$. We prefer to postpone the discussion of this group to the next section because, as we shall see, it will be related to conformal symmetry.

2.4 Conformal group

In this section we will give a brief introduction to the conformal group, mostly in relation to the previously cited connection with Anti de Sitter spaces. Conformal symmetry is a symmetry that preserves only angles, so it is an extension of isometries: also scale transformations $\vec{x} \to \lambda^2 \vec{x}$ and inversions $\vec{x} \to \vec{x}/|\vec{x}|$ are allowed. Inversions for instance map the inside of a sphere to the outside, the origin is thus sent to the point at infinity. For this reason conformal transformations are always defined on the conformal compactification of the space. Under a conformal transformation the metric transforms as a Weyl rescaling[8]

$$g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu} \ . \qquad (2.4.1)$$

We will be interested in the conformal group for dimension $n > 2$. The case $n = 2$ is very peculiar because it turns out that the conformality condition (that we will express in a moment) is equivalent to the Cauchy–Riemann equations for holomorphic functions. It is known that the space of solutions to these equations is infinite dimensional, so only for $n = 2$ the conformal group is infinite dimensional and its algebra (or, to be more precise, the central extension of the algebra) is called the Virasoro algebra. From now on we focus on the $n > 2$ case. A general infinitesimal conformal transformation is $x^\mu \to x'^\mu = x^\mu + \epsilon\, \xi^\mu(x)$, where $\epsilon \ll 1$. For this discussion we restrict to flat space time, $g_{\mu\nu} = \eta_{\mu\nu}$. Hence the metric transforms as:

$$\eta_{\mu\nu} \to \eta_{\mu\nu} + \epsilon\, \partial_\mu \xi_\nu + \epsilon\, \partial_\nu \xi_\mu \ . \qquad (2.4.2)$$

[8]A conformal transformation differs from a Weyl rescaling because it is an actual transformation on the coordinates $\vec{x} \to \vec{x}'(x)$ while a Weyl rescaling just transforms the metric, and a priori there is no coordinate transformation related to it.

Name	$\xi^\mu(x)$	$\sigma(x)$	Operator
Translations	a^μ	0	P_μ
Lorentz transformations	$\omega^\mu_{\ \nu} x^\mu$	0	$J_{\mu\nu}$
Dilations	λx^μ	λ	D
Special conformal transformations	$b^\mu x^2 - 2(b \cdot x)x^\mu$	$-2(b \cdot x)$	K_μ

Table 2.3: List of the generators of the conformal group.

Now we also know that, defining $\Omega = 1 + \epsilon\, \sigma(x)$, one has

$$\eta_{\mu\nu} \to \eta_{\mu\nu}(1 + 2\epsilon\, \sigma(x)) \ . \tag{2.4.3}$$

Combining the two equations at order ϵ and contracting the indices we find the conformality condition, called *conformal Killing equation*

$$\big(\eta_{\mu\nu}\Box + (n-2)\partial_\mu\partial_\nu\big)\partial \cdot \xi = 0 \ . \tag{2.4.4}$$

The most general solution to this equation is

$$\xi^\mu(x) = a^\mu + \omega^\mu_{\ \nu} x^\nu + \lambda x^\mu + b^\mu x^2 - 2(b \cdot x)x^\mu \ . \tag{2.4.5}$$

It contains, obviously, the Poincaré transformations (translations a^μ and boosts / rotations $\omega_{\mu\nu}$), but we also have in addition dilations λ and a new generator b^μ that is related to *special conformal transformations*. Inversions are not connected to the identity so we do not see them, but it can be shown that a composition of an inversion and a translation gives indeed a special conformal transformation. In Table 2.3 there is a list of the generators and the conformal factor σ associated to them.

Using the notation expressed in the Table above we list the commutation relations of the conformal algebra, showing that P_μ and $J_{\mu\nu}$ indeed form a Poincaré subalgebra.

$$\begin{aligned}
[J_{\mu\nu}, J_{\rho\sigma}] &= i\left(\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma}\right) \ , \\
[J_{\mu\nu}, P_\rho] &= i\left(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\right) \ , \qquad [P_\mu, P_\nu] = 0 \ , \\
[J_{\mu\nu}, K_\rho] &= i\left(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu\right) \ , \qquad [K_\mu, K_\nu] = 0 \ , \\
[K_\mu, P_\nu] &= -2i\left(\eta_{\mu\nu}D - J_{\mu\nu}\right) \ , \qquad [D, J_{\mu\nu}] = 0 \ , \\
[D, P_\mu] &= iP_\mu \ , \qquad [D, K_\mu] = -iK_\mu \ .
\end{aligned} \tag{2.4.6}$$

This algebra seems quite complicated, but it turns out to be isomorphic to an $SO(2,n)$ algebra, precisely the isometry group of AdS_{n+1}! To prove this claim we first need to see the commutation relations of $SO(2,n)$. Let us consider the metric with two time like directions $\zeta = \mathrm{diag}(-1, 1, \ldots, 1, -1)$. A set of generators J_{ab}

of $SO(2,n)$ satisfies[9]

$$e^{i\epsilon^{ab}(J^{ab})^T}\zeta e^{i\epsilon^{ab}J_{ab}} = \zeta \quad \Longrightarrow \quad \zeta(J_{ab})^T\zeta = -J_{ab} \ . \tag{2.4.7}$$

It is easy to see that, if $J^{(E)}_{ab}$ are generators of $SO(n+2)$, which are simply a base of antisymmetric matrices, then the generators J_{ab} are just

$$J_{ab} = \zeta J^{(E)}_{ab} \ , \qquad (J^{(E)}_{ab})_{ij} = -i\left(\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi}\right) \ . \tag{2.4.8}$$

Of course among these generators only $\binom{n+2}{2}$ are independent because $J^{(E)}_{ab} = -J^{(E)}_{ba}$. One can use this definition to write down the commutation relations, the result being

$$[J_{ab}, J_{cd}] = i\left(\zeta_{ac}J_{bd} + \zeta_{bd}J_{ac} - \zeta_{ad}J_{bc} - \zeta_{bc}J_{ad}\right) \ , \tag{2.4.9}$$

As one might have expected in analogy with the Lorentz group. The isomorphism is easily constructed defining the generators J_{ab} in the following way:

$$J_{ab} = \begin{cases} J_{\mu\nu} & a,b=\mu\ ,\ \nu\ <n \\ \frac{1}{2}(K_\mu - P_\mu) & a=\mu\ ,\ b=n \\ \frac{1}{2}(K_\mu + P_\mu) & a=\mu\ ,\ b=n+1 \\ D & a=n\ ,\ b=n+1 \end{cases} \tag{2.4.10}$$

We will not prove it explicitly but J_{ab} defined in this way satisfies (2.4.9). This isomorphism is one of the key aspects that lead to the AdS/CFT correspondence.

2.4.1 Conformal field theories

A quantum field theory enjoying conformal invariance is called *Conformal Field Theory* (CFT). Some references on CFT are the book [57], Chapter 2 of [50] and the reviews [58, 59]. If the conformal symmetry is realized at the quantum level we can organize the operators diagonalizing the dilation operator D. Let us consider a local operator $\mathcal{O}_\Delta(x)$ such that:

$$[D, \mathcal{O}_\Delta(0)] = i\Delta\,\mathcal{O}(0) \ , \tag{2.4.11}$$

where Δ is called *conformal dimension*. From the last two commmutators of (2.4.6) follows that P_μ is a *raising* operator for Δ and K_μ is a *lowering* operator for Δ. If we restrict ourselves to unitary theories, the conformal dimension is bounded from below. In particular it must be always ≥ 0 (even though the optimal bound is usually greater than zero and it depends on the spin). Acting

[9] Be careful: ab are not indices, they label different generators, which are matrices.

with K_μ recursively we will always get to zero at a certain point. We then call *conformal primary operators* those operators that satisfy

$$[K_\mu, \mathcal{O}(0)] = 0 \; . \tag{2.4.12}$$

The infinite tower obtained by acting with $P_\mu = -i\partial_\mu$ composes the *descendant operators*. In particular, the operator at any point $x \neq 0$ can be obtained as an infinite (Taylor) series of descendants. Operators can also have non zero spin. In this case they will carry an index a (collectively denoting a set of indices $\mu_1 \dots \mu_s$ for instance) and they will transform according to a representation of the rotation group $(\mathcal{S}_{\mu\nu})^a_{\ b}$. If one perform any of the conformal transformations listed in Table 2.3 with Killing vector ξ^μ, the result on a conformal primary operator is the following:

$$[Q_\xi, \mathcal{O}^a_\Delta(x)] = \left(\xi \cdot \partial + \frac{\Delta}{d}(\partial \cdot \xi) - \frac{1}{2}(\partial^\mu \xi^\nu)(\mathcal{S}_{\mu\nu})^a_{\ b}\right) \mathcal{O}^b_\Delta(x) \; , \tag{2.4.13}$$

where Q_ξ is the generator related to the Killing vector ξ^μ and d the number of dimensions.

A very interesting feature of CFTs is that there exist an exact map between states and operators, which is called *state–operator map*. We will not give the proof but just a hint on how this map is realized. First of all it is clear that any local operator $\mathcal{O}(x)$ can be mapped to a state just by considering the functional integral on a ball $B \subset \mathbb{R}^d$ with an $\mathcal{O}$ insertion at the origin.

$$\Psi[\varphi_0] = \int_{\varphi = \varphi_0|_{\partial B}} [\mathcal{D}\varphi]\, \mathcal{O}(0) e^{iS[\varphi]} \; , \tag{2.4.14}$$

where the subscript of the integral sets the boundary condition for the field at the boundary of the ball. The converse is still true and it can be seen easily in the *radial quantization*. In this approach D acts as an Hamiltonian for the field theory and the time evolution is interpreted as a scaling of the coordinates. Different time slices are actually different radii r. Any state defined in a sphere ∂B can be evolved back in "time" by the unitary operator $e^{-iD\log r}$ until it "shrinks" to a point. Hence we have $\Psi[\varphi_0] \leftrightarrow \mathcal{O}(0)$. An important consequence of this map is that operators, in the same way as states, satisfy a completeness relation. This means that any operator can be expanded as a series of local operators, with appropriate coefficients. This is particularly useful for the product of two local operators, in this case it goes under the name of *Operator Product Expansion* (OPE). It can be expressed as:

$$\mathcal{O}_a(x)\mathcal{O}_b(y) = \sum_c C_{abc}(|x-y|, \partial)\mathcal{O}(x) \; , \tag{2.4.15}$$

where $\mathcal{O}_a$ are conformal primaries and the equality is regarded as an operator equality, hence valid in any correlation function with insertions of local operators in points sufficiently far from x and y. The functions C_{abc} contain all the conformal descendants (coming from the "∂" dependence) and the functional form in $|x-y|$ is fixed by conformal symmetry up to a numerical factor f_{abc}. The numbers f_{abc} are called *OPE coefficients*.

It turns out that the conformal symmetry is powerful enough to fix uniquely (up to a normalization) all the two point functions. For example, if $\mathcal{O}(x)$ is a scalar operator of dimension Δ we have

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{1}{|x-y|^{2\Delta}} \,. \tag{2.4.16}$$

Two point functions of operators with different dimensions are zero. When operators carry a spin the result is slightly more involved but still uniquely determined. By means of an iterative application of the OPE decomposition we can always turn any n point function to a 2 point function. Hence the whole set of observables of the theory is reduced to an (infinite) set of real numbers

$$\{\Delta_a, f_{abc}\} \tag{2.4.17}$$

called *CFT data*. This is why conformal symmetry is a very strong constraint on a quantum theory.

A particularly interesting conformal theory is $\mathcal{N}=4$ Super Yang–Mills in $d=4$ on the gauge group $SU(N)$. $\mathcal{N}$ stands for the number of fermionic charges of the supersymmetry algebra while N is the number of colors. The fermionic supercharges are rotated one into another by an internal $SU(4)_R$ global symmetry called R symmetry. As the name Yang–Mills suggests the theory contains a gauge vector A_μ belonging to the adjoint of $SU(N)$. His superpartner is a spin 1/2 massless spinor λ^a called *gaugino* in the adjoint of $SU(N)$ and in the fundamental of $SU(4)_R$. The theory also contains six scalars ϕ^i in the adjoint representation of $SU(N)$ and in the vector representation of $SU(4)_R$ (i.e. fundamental[10] of $SO(6)_R$). The coupling g_{YM} does not run when the energy scale is changed because the theory is conformal (i.e. the β function is identically zero). The bosonic part of the action reads

$$\begin{aligned} S_{\mathrm{SYM}} = \int \mathrm{d}^4x \,\mathrm{Tr}\,\Bigg(&-\frac{1}{2g_{\mathrm{YM}}^2} F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{32\pi^2}\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} + \\ &-\sum_i D_\mu\phi^i D^\mu\phi^i + \frac{g_{\mathrm{YM}}^2}{2}\sum_{ij}\left[\phi^i,\phi^j\right]^2\Bigg) \,, \end{aligned} \tag{2.4.18}$$

[10] $SU(4)$ is the universal covering of $SO(6)$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ is the field strength and $D_\mu = \partial_\mu + i[A_\mu, \cdot\,]$ is the covariant derivative. This theory can be obtained as a dimensional reduction over a six dimensional torus (i.e. a product of six S^1) of $\mathcal{N} = 1$ Super Yang–Mills in ten dimensions. The action reads

$$S_{10\mathrm{D}} = \int \mathrm{d}^{10}x \left(-\frac{1}{2} F_{mn} F^{mn} + \frac{i}{2} \overline{\Psi} \Gamma^m D_m \Psi \right) , \tag{2.4.19}$$

where Ψ is a Majorana–Weyl spinor and Γ^m the ten dimensional Dirac matrix. The definitions are similar as above: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$ and $D_\mu = \partial_\mu + ig[A_\mu, \cdot\,]$. Here g is related to g_{YM} and the radii of compactification.

2.5 The correspondence

At last, after a long way we put together all the tools needed to understand the AdS/CFT correspondence. It beautifully combines string theory, classical gravity, the geometry of D–branes and also the physics of strongly interacting quantum field theories. Not only this is an huge theoretical discovery, as it consists in a bridge between two apparently different areas in physics (classical gravity and strongly interacting QFTs), but it is also very useful in practical applications, such as in modelling low energy QCD or hydrodynamics of quantum systems, where the perturbative approach does not work. The original formulation of the correspondence [15] considered the following system: N D3–branes in flat 10 dimensional space time. Since D3–branes couple naturally with $C_{(4)}$ forms we are in Type IIB Superstring theory. As we saw previously D3–branes are not only hypersurfaces on which the strings end, they can also warp space time and excite a 3–brane gravitational background. We will refer to the string theoretical description as “String picture” and to the effective gravitational description as “Gravity picture”. From our study on D–branes we have no doubt that these two descriptions are in some sense equivalent, the statement of the AdS/CFT correspondence consists in making precise this equivalence and in giving an explicit prescription by which one picture is mapped into the other. This is all we need to establish a duality.

The correspondence in its strongest form is a precise statement about two quantum theories, namely:

> $\mathcal{N} = 4$ Super Yang–Mills (SYM) theory with gauge group $SU(N)$ and coupling constant g_{YM}
>
> *is dynamically equivalent to*
>
> Type IIB superstring theory with string length $l_s = \sqrt{\alpha'}$ and coupling constant g_s on $AdS_5 \times S^5$ with radius of curvature L and N units of $F_{(5)}$ flux on S^5.

The map between parameters on the two theories is given by

$$g_{\mathrm{YM}}^2 = 2\pi g_s \ , \qquad 2 g_{\mathrm{YM}}^2 N = L^4/\alpha'^2 \ . \tag{2.5.1}$$

This statement is now a conjecture because we do not have a complete knowledge of the UV completion of both theories. However an enormous amount of quantitative validity checks is being collected in the literature, with no counterexample so far. By "dynamically equivalent" we mean that the algebras of the observables $\mathscr{A}$ and the Hilbert spaces $\mathscr{H}$ of the two theories are isomorphic.

$$\mathscr{A}_{\mathrm{SYM}} \cong \mathscr{A}_{\mathrm{string}} \ , \qquad \mathscr{H}_{\mathrm{SYM}} \cong \mathscr{H}_{\mathrm{string}} \ . \tag{2.5.2}$$

Now the name AdS/CFT becomes clear: the gravity part is a product of an AdS space and a compact submanifold, while the gauge theory part is $\mathcal{N} = 4$ SYM which is a conformal theory (the theory enjoys conformal invariance which remains unbroken also at the quantum level: the β function is zero to all orders in perturbation theory). More interesting is the low energy limit, in which we can both understand the correspondence and have a concrete workshop for computations. The parameters of the theory are the string length $\sqrt{\alpha'}$, the AdS radius L and the coupling constant g_s. The first two govern the interaction of the strings with the background geometry: when $\sqrt{\alpha'}/L$ is small strings are almost point like and they do not see the curvature. The last parameter concerns the genus perturbative expansion in string theory: when it is small only tree diagrams (i.e. the first one of Figure 2.2) are important. Let us see these two limits in both pictures.

String picture The limit $g_s \ll 1$, as said, selects only the tree diagrams and this allows us to treat the string excitations as a small perturbation. Hence the D–brane dynamics is able to describe the theory effectively. The low energy limit $\sqrt{\alpha'} \to 0$ (i.e. $E \ll 1/\sqrt{\alpha'}$) gives very large masses to massive stringy states, so the spectrum of the theory is just the massless spectrum (summarized in Table 2.1). We have both closed and open strings, however the closed strings necessarily decouple from the theory because closed strings are described by gravity; the low energy expansion is by means of the coupling κ_0 (the Newton constant) with negative dimensions, hence the coefficients are vanishing in the low energy limit. We have massless open strings, hence we can describe this theory effectively by the Dirac–Born–Infeld action[11]

$$S_{\mathrm{D3}} = -\frac{1}{(2\pi)^2 \alpha'^2 g_s} \int \mathrm{d}^4\xi \, e^{-\phi} \mathrm{STr} \sqrt{-\det\left(\mathcal{P}[g] + \mathcal{P}[B] + 2\pi\alpha' F\right)} \ . \tag{2.5.3}$$

[11] The definition of the tension T_3 has been changed to include $e^{-\phi_0} = 1/g_s$ the background value of the dilaton i.e. the string coupling.

Expanding the square root to second order we get

$$-\frac{1}{2\pi g_s}\int \mathrm{d}^4\xi\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}\eta^{\mu\nu}\partial_\mu\Phi^m\partial_\nu\Phi^m+\cdots\right), \tag{2.5.4}$$

where the dilaton is put to zero (the asymptotic value g_s has been factorized) and the fields Φ^m are the embedding of the D–brane, living in the orthogonal directions. This is part of the bosonic action of $\mathcal{N}=4$ super Yang–Mills with coupling

$$g_{\mathrm{YM}}^2 = 2\pi g_s\ . \tag{2.5.5}$$

Since we have N D3–branes the gauge group considered here is $SU(N)$ (for $N\to\infty$ the difference between $U(N)$ and $SU(N)$ is negligible).

Gravity picture To understand the same limit in the gravity picture we must look at the 3–brane solution (2.2.37) (in the case $\alpha_p=1$, $K(r)=1$ and $p=3$). The limit $L\to\infty$ (large curvature radius) ensures us that classical supergravity is a good approximation as higher derivative corrections are suppressed by factors $\sim 1/L$. This gravity solution can be divided into two regions: the flat asymptotic region ($r\to\infty$) and the "throat" region ($r\to 0$). The observer at infinity sees particles with energies redshifted by the factor $\sqrt{g_{tt}} = Z_3^{-1/4}(r)$ so in the flat region and in the throat we have different behaviours.

$$E_{\mathrm{observer}\to\infty} = Z_3^{-1/4}(r)E_{\mathrm{particle}}\ . \tag{2.5.6}$$

We have two kinds of low energetic excitations: low energetic particles in the flat region and any kind of excitations in the throat (because they get infinitely redshifted). These two sectors decouple. Since we are dealing with gravity we only have closed strings in this picture. In the throat region the metric can be approximated by the near horizon limit:

$$\mathrm{d}s^2=\frac{r^2}{L^2}\left(-\mathrm{d}t^2+\mathrm{d}\vec{x}^2\right)+\frac{L^2}{r^2}\mathrm{d}r^2+L^2\mathrm{d}\Omega_5^2\ . \tag{2.5.7}$$

This is just the metric (2.3.13) with the substitution $r=L^2u$ times an S^5 factor with the same radius of AdS. By looking at (2.2.39) the radius L is given by

$$L^4=r_3^4=4\pi g_s N\alpha'^2\ . \tag{2.5.8}$$

Combining both pictures The correspondence is almost settled, we have in both pictures two different sectors in the low energy limit, mutually decoupled. As summarized in Figure 2.6, we have open and closed strings in the string picture and closed strings in the throat and in the flat region in the gravity picture. The claim is that the closed string in the first side describe the same physics as the closed strings in the flat region on the other side, as a consequence:

> Open strings in the String picture are dual to closed string in the throat in the Gravity picture.

We can then describe the same physics either with the DBI action that tends to SYM theory on $SU(N)$ with coupling g_{YM} or with the supergravity action on a background $AdS_5 \times S^5$ with radius L. This is the precise statement expressed by the AdS/CFT correspondence. Summarizing, the limits which have been taken are $L/\sqrt{\alpha'} \to \infty$ (that turns superstring theory in supergravity) and $g_s \to 0$ (that turns quantum gravity in classical gravity). The dictionary is

$$g_{\mathrm{YM}}^2 = 2\pi g_s \ , \qquad \lambda = g_{\mathrm{YM}}^2 N = \frac{L^4}{2\alpha'^2} \ , \tag{2.5.9}$$

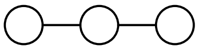
where N is both the number of D3–branes and the units of $F_{(5)}$ flux on S^5 (recall that $F_{(5)} = \mathrm{d}C_{(4)}$ is self dual)

$$N = \int_{S^5} {}^\star F_{(5)} = \int_{S^5} F_{(5)} \ . \tag{2.5.10}$$

The limit $g_s \to 0$ (classical limit) translates to $N \to \infty$ with λ fixed, while the limit $L/\sqrt{\alpha'} \to \infty$ (gravity limit) translates to $\lambda \to \infty$. We have thus just established a duality between *classical gravity* and a *large N strongly interacting quantum field theory*. The large N limit was believed to be a good expansion prescription much before AdS/CFT correspondence by the work of 't Hooft on QCD [9] (see also [60]), the next Chapter is dedicated to this subject.

At the moment it looks like the AdS part of the metric has the only relevant role in this correspondence, while the S^5 factor is left aside. It turns out however that it is important to describe the *global symmetries* of the dual theory. In fact $\mathcal{N} = 4$ SYM has a global $SU(4)$ symmetry called R symmetry, arising from the possibility of making unitary transformations in the space of supercharges. As a general principle in AdS/CFT, global symmetries of the quantum field theory are reproduced by gauge symmetries in supergravity. Here in particular we have the local isometry group of the sphere $SO(6)$ (in the same way as the local Lorentz group in Minkowski). The Lie Algebras of $SO(6)$ and $SU(4)$ are isomorphic because they have the same Dynkin diagrams[12]

$$SO(6) \cong SU(4) \ : \tag{2.5.11}$$



Thus the S^5 factor of the metric is necessary in order to reproduce the R symmetry of the dual theory.

[12] We will not discuss the theory of Lie Algebras, a good introduction can be found in [61].

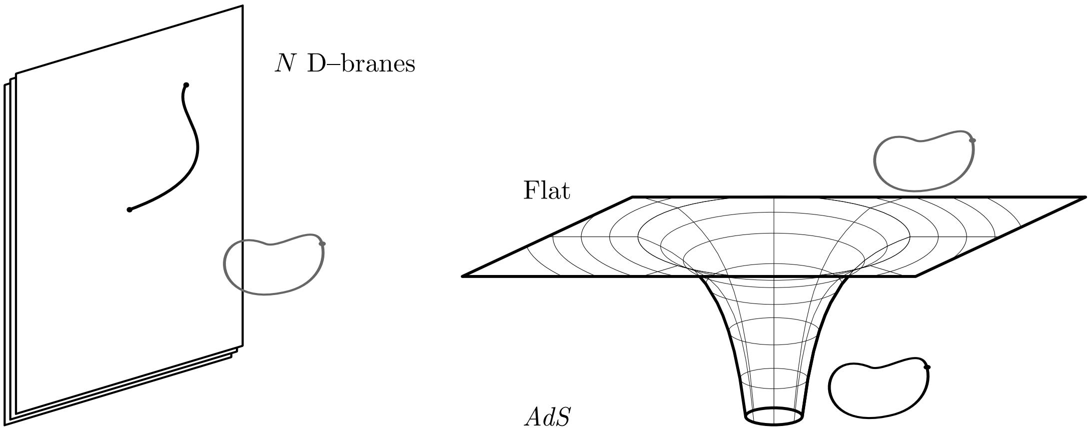


Figure 2.6: Intuitive depiction of the duality, on the left we have the String picture (N D–branes in flat space with open and closed strings) and on the right the Gravity description (warped space with only closed strings).

2.5.1 Concrete realization of the correspondence

We have just stated the equivalence of two theories but we need to give a concrete prescription to perform some actual computations. The idea of holography was not emphasized in the preceding discussions, but it enters crucially in the present one. The space $AdS_5 \times S^5$ can be *dimensionally reduced* to AdS_5. Dimensional reduction, or Kaluza–Klein reduction, is the act of turning compact components of the space time into internal degrees of freedom of the theory (in the following paragraph this is briefly explained).

Though we have to perform a dimensional reduction on S^5, we prefer to analyse the simpler case of S^1 since the main ideas are identical. Suppose our space is $\mathcal{M} \times S^1$ and the coordinates are $x^\mu \in \mathcal{M}$ and $\theta \in S^1$. Any field $\varphi(x^\mu, \theta)$ can be expanded in a Fourier series

$$\varphi(x^\mu, \theta) = \sum_{n \in \mathbb{Z}} \phi^{(n)}(x^\mu)\, e^{in\theta/R} \, , \tag{2.5.12}$$

where R is the radius of the S^1. The Laplacian in $\mathcal{M} \times S^1$ (denoted by $\Box_{x,\theta}$) acts on the field in the following way

$$\Box_{x,\theta}\varphi(x^\mu, \theta) = \sum_{n \in \mathbb{Z}} \left(\Box_x - \frac{n^2}{R^2} \right) \phi^{(n)}(x^\mu) \, . \tag{2.5.13}$$

So each $\phi^{(n)}$ for $n \neq 0$ acquires effectively a mass proportional to $1/R$ and the label n is interpreted as an internal degree of freedom of the theory: the idea of the compact direction θ disappears. This is the paradigm of dimensional reduction, the general case of S^p is similar but instead of a Fourier expansion on complex exponentials we should perform an expansion on the spherical harmonics $Y_l(\theta_1, \dots \theta_p)$.

After performing a dimensional reduction on S^5, we are left with AdS_5. We have seen in the discussions in Section 2.3 that the *boundary* of AdS_5 is conformally equivalent to 4 dimensional Minkowski spacetime, hence the field theory side of the duality can be found there. The theory on AdS_5 in the limits considered ($g_s \ll 1$ and $E \ll 1/\sqrt{\alpha'}$) is a classical theory, so we are interested in Cauchy problems: the causal structure of the boundary forces us to impose an initial condition everywhere on the boundary and not just on a single surface[13]. The initial data, or the *source*, is thus a function living on the boundary $\mathcal{J}(x^\mu)$ where $\mu = 0, 1, 2, 3$ is a Minkowski space time index. The picture is different in the field theory side of the correspondence where the system is strongly quantum. If we define the partition function $Z[J]$ we introduce again an external *source* $J(x^\mu)$ coupled to an operator $\mathcal{O}(x)$. The correspondence states that $\mathcal{J}(x^\mu)$ and $J(x^\mu)$ describe the same field, but AdS geometry more precisely asserts that they are related by a conformal transformation (because this is the transformation that maps AdS_5 boundary into Minkowski space time). We can thus formulate the correspondence in a concrete way expressing the exact equivalence not only of the sources but also of all the correlation functions, namely [16]:

$$Z_{\mathrm{CFT}}[J(x^\mu)] \simeq e^{iS_{\mathrm{grav}}[\phi(x^\mu,u)]}\Big|_{\phi \text{ on } \partial AdS_5 = \mathcal{J}(x^\mu)} , \tag{2.5.14}$$

when the equivalence holds in the large N, large λ regime. Here $Z_{\mathrm{CFT}}[J]$ is the partition function, which can be expressed in the path integral formalism or in the canonical formalism

$$\begin{aligned} Z_{\mathrm{CFT}}[J(x^\mu)] &= \frac{1}{\int [\mathcal{D}\varphi]\, e^{iS_{\mathrm{CFT}}}} \int [\mathcal{D}\varphi(x^\mu)]\, e^{iS_{\mathrm{CFT}} + i\int J(x^\mu)\mathcal{O}(x^\mu)} \\ &= \frac{\langle 0|T e^{i\int J(x^\mu)\mathcal{O}(x^\mu)}|0\rangle}{\langle 0|0\rangle} , \end{aligned} \tag{2.5.15}$$

while the *r.h.s.* means: “solve the equations of motion for ϕ with a boundary data $\mathcal{J}(x^\mu)$ on ∂AdS_5 and then compute the exponential of the classical action on the on–shell value of the field”. The correlation functions of the operator $\mathcal{O}(x)$ are defined as

$$\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle = (-i)^n \frac{\delta^n}{\delta J(x_1)\cdots\delta J(x_n)} Z_{\mathrm{CFT}}[J] \tag{2.5.16}$$

The gravity counterpart involves functional derivatives with respect to the boundary conditions,

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = (-i)^n \frac{\delta^n}{\delta \mathcal{J}(x_1)\cdots\delta \mathcal{J}(x_n)} e^{iS_{\mathrm{grav}}[\phi[\mathcal{J}]]} , \tag{2.5.17}$$

[13] A surface in which we can impose an initial condition and then determine the future completely and uniquely is called *Cauchy surface*. If this surface exists the space is said to be *globally hyperbolic*. *AdS* space is *not* globally hyperbolic.

where we regard the solution $\phi(x^\mu, u)$ as a functional of the boundary value $\phi[\mathcal{J}]$. The precise mapping $J \leftrightarrow \mathcal{J}$ and $\mathcal{O} \leftrightarrow \phi$ depends on the problem. A simple example where everything can be worked out exactly is the scalar particle in *AdS*. Understanding this exercise is sufficient to have a good comprehension of the main ideas.

2.5.2 Correlation function of scalar operators

Consider a scalar field $\phi(x^\mu, z)$ in the metric (2.3.13) (with variable z) in AdS_{d+1}. The action is

$$S_\phi = -\frac{1}{2\kappa}\int \mathrm{d}^{d+1}x\sqrt{|g|}\left(g^{mn}\partial_m\phi\partial_n\phi + m^2\phi^2\right) \ . \tag{2.5.18}$$

We are interested in a boundary problem with boundary condition at $z = 0$. The conjugated momentum Π is given by the z component:

$$\Pi = \frac{\delta S}{\delta(\partial_z\phi)} = \frac{1}{\kappa}\sqrt{|g|}g^{zz}\partial_z\phi \ , \tag{2.5.19}$$

where the fact that the metric is diagonal has been used. Let us now introduce an UV cutoff $z > \epsilon$ with $\epsilon \ll 1$ (in the end the result will not depend on the cutoff). By doing a simple integration by parts we can rewrite the action in the following form

$$S_\phi = \frac{1}{2\kappa}\int \mathrm{d}^dx\,\phi\sqrt{|g|}g^{zz}\partial_z\phi\Big|_{z=\epsilon} + \int \mathrm{d}^{d+1}x\Big(\text{Euler–Lagrange eq. for } \phi\Big) \ . \tag{2.5.20}$$

If the field ϕ satisfies its equation of motion the action becomes simply the integral $\frac{1}{2}\int \phi\,\Pi$ on the boundary $z = \epsilon$. The equation of motion is just the Klein–Gordon equation in warped space, which we can write in Fourier space as

$$z^{d+1}\partial_z\left(\frac{1}{z^{d-1}}\partial_z\widetilde{\phi}\right) + z^2k^\mu k_\mu\widetilde{\phi} - m^2\widetilde{\phi} = 0 \ , \tag{2.5.21}$$

where

$$\phi(x^\mu, z) = \int \frac{\mathrm{d}^dk}{(2\pi)^d}\, e^{ik^\mu x_\mu}\widetilde{\phi}(k^\mu, z) \ . \tag{2.5.22}$$

Near $z = 0$ we can neglect the kinetic k^2 term and we have a complete set of solutions depending on the coefficients $A(k)$ and $B(k)$

$$\widetilde{\phi}(k^\mu, z) \underset{z\ll 1}{\simeq} A(k)z^{d-\Delta} + B(k)z^\Delta \ , \tag{2.5.23}$$

where we have defined

$$\Delta \equiv \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2L^2} \equiv \frac{d}{2} + \nu \ . \tag{2.5.24}$$

Here the length scale L has been reintroduced explicitly. Notice that in AdS the curvature is able to render stable also configurations with negative square masses, as long as they do not violate the Breitenlohner–Freedman bound [62] $m^2L^2 > -d^2/4$. The quantity Δ is precisely the conformal dimension of the operator $\mathcal{O}(x)$. The reason for this identification will be clear in a moment. It is now easy to compute the conjugated momentum

$$\widetilde{\Pi}(k^\mu, z) \underset{z\ll 1}{\simeq} \frac{1}{\kappa}\frac{1}{z^{d-1}}\big((d-\Delta)A(k)z^{d-1-\Delta} + \Delta B(k)z^{\Delta-1}\big) \ . \tag{2.5.25}$$

Since this field is on shell we can ignore the second term of (2.5.20). The action, after eliminating the terms vanishing in the limit $z \to 0$ and turning in Fourier transform becomes

$$S_\phi^{\text{on-shell}} = \frac{1}{2\kappa}\int \frac{\mathrm{d}^dk}{(2\pi)^d}\big((d-\Delta)A(-k)A(k)\epsilon^{-2\nu} + dA(-k)B(k)\big) \ . \tag{2.5.26}$$

Now we have the on shell action depending on the boundary conditions $A(k)$ and $B(k)$, which are the equivalent of what we called $\mathcal{J}$ previously. We are almost ready to take functional derivatives, but there is still one issue to solve: the divergent factor $\epsilon^{-2\nu}$ must be somehow cured. This task is accomplished by *holographic renormalization* [63]. As in usual renormalization theory we make use of an arbitrariness of the theory, but this case is somewhat different: instead of adding local counterterms we add covariant boundary terms to the action, whose presence do not modify the equations of motion. In this particular case we add a boundary mass term

$$S_\phi \ \longrightarrow \ S_\phi^{\text{ren}} \equiv S_\phi + \int \frac{\mathrm{d}^dk}{(2\pi)^d}\sqrt{|\gamma|}\, C\widetilde{\phi}(k)\widetilde{\phi}(-k) \ , \tag{2.5.27}$$

where γ is g restricted to ∂AdS. The choice

$$C = \frac{1}{2\kappa}\frac{\Delta - d}{L} \tag{2.5.28}$$

does the job (where again L is introduced via dimensional analysis). The renormalized on shell action now is given by

$$S_\phi^{\text{on-shell, ren}} = \frac{1}{2\kappa}\int \frac{\mathrm{d}^dk}{(2\pi)^d}\,(2\Delta - d)A(-k)B(k) \ . \tag{2.5.29}$$

Now we can safely take the limit $\epsilon \to 0$. The boundary value of ϕ simply reduces to the singular part because the other is vanishing for $z \to 0$, hence we have the identification

$$\mathcal{J}(k, z) = A(k)z^{d-\Delta} \equiv J(k)z^{d-\Delta} \tag{2.5.30}$$

The second equality is the AdS/CFT prescription: J is precisely[14] the source appearing in $Z_{\mathrm{CFT}}[J]$. The derivative with respect to J on Z_{CFT} is the derivative with respect to A in $S_\phi^{\text{on-shell, ren}}$. The operator of which we are computing the expectation value is the one canonically conjugated to the current J, namely in this case a scalar $\mathcal{O}(x)$ that appears in a term like $\int \mathcal{O}\, J$. Now the interpretation of Δ can be understood: a global rescaling $x^\mu \to \lambda x^\mu$ of the quantum field theory can be seen as a global rescaling in AdS $(x^\mu, z) \to (\lambda x^\mu, \lambda z)$. Since $J(x)$ is multiplied by $z^{d-\Delta}$ its scaling dimension is $d-\Delta$. The coupling

$$\int \mathrm{d}^d x\, \mathcal{O}(x) J(x) \tag{2.5.31}$$

must remain invariant under this rescaling, so we conclude that $\mathcal{O}(x)$ has conformal dimension Δ. Coming back to our task: the one point function reads

$$\langle \widetilde{\mathcal{O}}(k) \rangle_J = \frac{\delta S_\phi^{\text{on-shell, ren}}[A]}{\delta A}\bigg|_{A=J} . \tag{2.5.32}$$

If, as in this case, the one point function for $J=0$ is zero we automatically have the two point function

$$\langle \widetilde{\mathcal{O}}(k) \widetilde{\mathcal{O}}(k) \rangle_{J=0} = \lim_{J\to 0} \frac{\langle \widetilde{\mathcal{O}}(k) \rangle_J}{J(k)} , \tag{2.5.33}$$

where the limit is intended in a functional sense. This comes from the simple statement of linear response theory: under a linear perturbation the system reacts with the Green's function[15] $\widetilde{G}_E(k) \equiv \langle \widetilde{\mathcal{O}}(k) \widetilde{\mathcal{O}}(k) \rangle_{J=0}$

$$\langle \mathcal{O}(x) \rangle_J = \langle \mathcal{O}(x) \rangle_{J=0} + \int \mathrm{d}^d y\, G_E(x-y) J(y) \tag{2.5.34}$$

Or in Fourier transform

$$\langle \widetilde{\varphi}(k) \rangle_J = \langle \widetilde{\varphi}(k) \rangle_{J=0} + \widetilde{G}_E(k) J(-k) \tag{2.5.35}$$

The remaining part of the discussion now is only technical, we need to find the functional dependence of $B[A]$ in order to compute the derivative of the on shell action. This is done by solving exactly the equations and imposing regularity (in particular at the origin $z \to \infty$). Luckily the solutions are known exactly, they are:

$$\widetilde{\phi}(k^\mu, z) = z^{d/2} \left(\widehat{A}(k) K_\nu(kz) + \widehat{B}(k) I_\nu(kz) \right) , \tag{2.5.36}$$

[14] Let us be loose with the notation $\widetilde{J}$ with respect to J for Fourier transform, it will be clear from the context.

[15] To be precise this works in the Euclidean formalism, but we have tacitly assumed to define the path integral as the Wick rotated Euclidean path integral.

where K_ν and I_ν are the modified Bessel functions, ν being defined in (2.5.24). Regularity in the bulk imposes $\widehat{B}(k) = 0$ and $\widehat{A}(k)$ arbitrary. The asymptotic expansion of the Bessel K_ν gives

$$K_\nu(kz) = \frac{1}{2}\left[\left(\frac{kz}{2}\right)^{-\nu}\Gamma(\nu) - \left(\frac{kz}{2}\right)^{\nu}\frac{\Gamma(1-\nu)}{\nu}\right] + \ldots \tag{2.5.37}$$

Comparing with the asymptotic expansion (2.5.23) we have the identification

$$-\frac{1}{2}\left(\frac{k}{2}\right)^{\nu}\frac{\Gamma(1-\nu)}{\nu}\widehat{A}(k) = B(k) = -\left(\frac{k}{2}\right)^{2\nu}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}A(k)\ . \tag{2.5.38}$$

This gives the correct functional relation between $B(k)$ and $A(k)$ (it is actually a proportionality relation). The derivative of the on shell action is now very easy to compute, yielding to the final result

$$\langle\widetilde{\mathcal{O}}(k)\widetilde{\mathcal{O}}(k)\rangle_{J=0} = -\frac{1}{\kappa}L^{d-1}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}2\nu\left(\frac{k^2}{4}\right)^{\nu}\ . \tag{2.5.39}$$

This is, once Fourier transformed back, precisely the expression of the 2 point function of a conformal primary scalar operator of dimension Δ in a CFT. For higher order correlation functions this formalism is a bit cumbersome: there exist generalizations with an approach similar to Feynman diagrams, where the propagators are *bulk–to–bulk* propagators or *bulk–to–boundary* propagators, but we will not need this formalism in the following so it will not be discussed.

2.5.3 Wilson loops

The analysis above covers only local operators: what about non local operators? There is indeed a particular class of non local operators in gauge theories: the *Wilson loops*, which enjoy a special treatment in AdS/CFT. A complete description of Wilson lines and loops is postponed to the next Chapter; here we will give a prescription on how to compute them [64]. Consider a closed curve C in the d dimensional field theory space time, the Wilson loop operator is defined as

$$W_C = \mathrm{Tr}\,\mathcal{P}\exp\left(-i\oint_C A^a_\mu T^a \mathrm{d}x^\mu\right)\ , \tag{2.5.40}$$

where $\mathcal{P}$ is the path ordering symbol. In string theory we can define a similar quantity: the curve C is a curve on the boundary ∂AdS_{d+1} to which a string is attached. The world sheet of this string is a surface Σ whose boundary is $\partial\Sigma = C$, but now this surface can stretch in a $d+1$ dimensional space. The action is simply the Nambu–Goto action hence the surface Σ prefers to minimize its area. Since

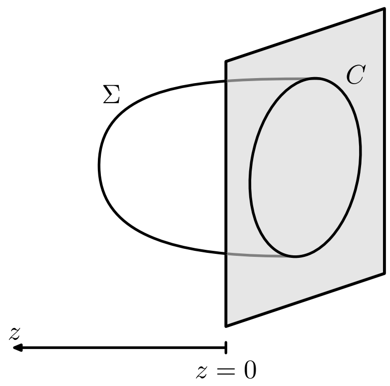


Figure 2.7: Representation of the Wilson loop C and its surface Σ. The plane pictured represents the boundary $z = 0$ of AdS_3 and the bulk is in the horizontal direction.

the metric of AdS diverges on the boundary the surface is pushed deep in the bulk (see Figure 2.7). The AdS/CFT prescription states the equivalence:

$$W_C \simeq e^{iS_{\text{NG}}[\Sigma_0]} , \tag{2.5.41}$$

where again the equivalence holds in the low energy gravity regime. Here $S_{\text{NG}}[\Sigma_0]$ is the Nambu–Goto action computed on the minimal surface Σ_0 such that $\partial\Sigma_0 = C$. A difficult problem in gauge theory is now turned in a minimization problem, easily accessible via numerical (and sometimes analytical) methods. We will show the example in pure AdS_{d+1} even though interesting effects arise when one considers black holes in AdS or defects such as an "hard wall".

Consider a simple curve C along a spatial and a time like direction: a square with sides $x^0 \in [0, T]$ and $x^1 \in [-s/2, s/2]$ (we work in Euclidean time here). The Nambu–Goto action, written using $z = z(x^1)$ as ansatz for the embedding of Σ, reads:

$$S_{\text{NG}} = \frac{L^2}{2\pi\alpha'} \int_0^T \mathrm{d}x^0 \int_{-s/2}^{s/2} \mathrm{d}x \, \frac{\sqrt{1 + z'(x)^2}}{z^2} . \tag{2.5.42}$$

The integral in x^0 is trivial and the one above becomes a one dimensional problem. It can be integrated using the energy as a constant of motion (define $S = \int \mathrm{d}x\mathcal{L}$):

$$\mathcal{E} = \frac{\partial\mathcal{L}}{\partial z} z' - \mathcal{L} = \text{const} \implies z^2\sqrt{1 + z'^2} = \text{const} . \tag{2.5.43}$$

Impose now the boundary conditions $z = 0$ at $x = \pm s/2$. As a consequence of the symmetry $x \to -x$ we can assume that the stationary point where $z' = 0$ is at $x = 0$. Setting $z(x = 0) \equiv z_*$, we get the equations

$$\begin{aligned} z_* &= z^2\sqrt{1 + z'^2} , \\ x &= \int_{z_*}^{z(x)} \frac{\mathrm{d}x}{z'} . \end{aligned} \tag{2.5.44}$$

The first one can be substituted in the second one, which in turn can be evaluated in $x = s/2$. The integral implicitly gives z_* as a function of s

$$z_* = \frac{s}{2\sqrt{2}\pi^{3/2}} \left[\Gamma\left(\frac{1}{4}\right)\right]^2 . \tag{2.5.45}$$

Now that we have z' as a function of z we can recast the on shell action as

$$S = \frac{TL^2 z_*^2}{2\pi\alpha'} \int \frac{\mathrm{d}x}{\mathrm{d}z} \frac{\mathrm{d}z}{z^4} = \frac{TL^2 z_*^2}{2\pi\alpha'} 2 \cdot \int_\epsilon^{z_*} \frac{\mathrm{d}z}{z^2\sqrt{z_*^2 - z^4}} . \tag{2.5.46}$$

Moreover a factor "2" appeared because of the sign ambiguity when inverting the first equation of (2.5.44). ϵ is a UV regulator that can be sent to zero after a renormalization procedure similar to the one performed before. The result is:

$$S = \frac{(\text{something})}{\epsilon} - \frac{TL^2 4\pi^2}{\alpha' \left[\Gamma\left(\frac{1}{4}\right)\right]^4} \frac{1}{s} . \tag{2.5.47}$$

The physical interpretation of this result is very interesting: it can be stated that

$$S = TV(s) , \tag{2.5.48}$$

where $V(s)$ is the potential between two non–dynamical very massive quarks interacting with a Yang–Mills gluon background. We will not have the time to prove the correspondence Wilson loop $\leftrightarrow$ Quark potential, see for instance [5]. The pole $1/\epsilon$ can be subtracted: in fact it represents the contribution to the energy due to the infinite masses of the quarks at positions $\pm s/2$. The other term states that the potential between the quarks is long ranged as it goes like $1/s = 1/\text{distance}$. This is actually the Coulomb–like behaviour we expect in a CFT. A theory is said to be *confining* if the potential grows linearly with s (hence the Wilson loop is proportional to the area of the enclosed region). The explanation is simple: if the potential grows with s there must be a maximum separation for any initial energy of the quarks, hence any state is bound. We do not obtain this result in *AdS*, because it is dual to a CFT, but if we study other set ups in which there is a maximal value for the coordinate z: $z \le z_{\text{max}}$, so that $g_{00}(z_{\text{max}}) \neq 0$, such a behaviour can arise and the dual theory will be confining.

2.6 Thermal theory and conformality breaking

In this section we will study how to introduce a finite temperature in the theory. Then we will also explain how a similar approach can also be used to break the conformal invariance. The conformality breaking will be essential for the rest of the work because we are ultimately interested in QCD, hence a "non–AdS/non–CFT" correspondence is necessary.

First of all let us recall how a finite temperature is introduced in quantum field theory. This is better explained in the functional integral formalism, where the partition function in d dimensions is expressed as

$$Z = \mathrm{Tr}\, e^{-\beta H} = \int_{\varphi(0)=(-)^F\varphi(\beta)} [\mathcal{D}\varphi]\, e^{-\int_0^\beta \int \mathrm{d}^d x\, \mathcal{L}_E[\varphi]} \;, \qquad \begin{matrix} F=1 \text{ fer.} \\ F=0 \text{ bos.} \end{matrix} \;. \tag{2.6.1}$$

Here φ stands for the fields in the theory, β is the inverse of the temperature (with $k_B = 1$) and $\mathcal{L}_E$ is the Euclidean Largangian. The boundary condition for the integral at Euclidean times $t_E = 0$ and β is *periodic* (+) for bosons and *antiperiodic* (−) for fermions. The (anti)periodicity comes from the fact that we are computing a Trace in the space of quantum configurations. Fermions acquire a minus sign because they are Grassmann (anticommuting) variables in the path integral formulation. There is a geometrical interpretation of this: Z is the partition function of the Euclidean theory with the Euclidean time compactified on a circle of circumference β where (anti)periodic boundary conditions on the bosons (fermions) are imposed.

Since we are interested in an holographic realization of this phenomenon we could try to perform a Wick rotation (i.e. $t \to -it_E$) on the gravity side of the correspondence. Black holes are a good candidate because they have a thermodynamical description, as it was discovered by Bekenstein and Hawking [20, 21]. Let us first consider a generic black hole with the metric

$$\mathrm{d}s^2 = g(r)\left(-f(r)\mathrm{d}t^2 + \mathrm{d}x^i\mathrm{d}x^i\right) + \frac{1}{h(r)}\mathrm{d}r^2 \;, \tag{2.6.2}$$

where $i = 1, \dots d$. In order to have a black hole there must be an horizon r_0 where the warp functions f and h behave as

$$f(r) \simeq f'(r_0)(r - r_0) \;, \quad h(r) \simeq h'(r_0)(r - r_0) \;. \tag{2.6.3}$$

After a Wick rotation the surface $r = r_0$ is no longer an horizon because the metric becomes positive definite. Near $r = r_0$ we can rewrite the Euclidean metric as

$$\mathrm{d}s^2 = g(r_0)\left(f'(r_0)(r - r_0)\mathrm{d}t_E^2 + \mathrm{d}x^i\mathrm{d}x^i\right) + \frac{1}{h'(r_0)(r - r_0)}\mathrm{d}r^2 \;. \tag{2.6.4}$$

After the change of variables

$$\rho = \frac{2}{\sqrt{h'(r_0)}}\sqrt{r - r_0} \;, \quad \theta = \frac{1}{2}t_E\sqrt{h'(r_0)f'(r_0)g(r_0)} \;, \tag{2.6.5}$$

the metric takes the form

$$\mathrm{d}s^2 = g(r_0)\mathrm{d}x^i\mathrm{d}x^i + \rho^2\mathrm{d}\theta^2 + \mathrm{d}\rho^2 \;. \tag{2.6.6}$$

The space (ρ, θ) is an $\mathbb{R}^2$ plane provided θ is periodically identified $\theta \sim \theta + 2\pi$. Hence the space is naturally compactified on a circle. If the period is not 2π there is a conical singularity arising from the angle defect at the origin. Coming back to the Euclidean time we have

$$t_E \sim t_E + \frac{4\pi}{\sqrt{h'(r_0) f'(r_0) g(r_0)}} \equiv t_E + \beta_{bh} \ . \tag{2.6.7}$$

The temperature of the black hole β_{bh} is interpreted holographically as the temperature of the field theory $\beta = \beta_{bh}$. The finite temperature correlation functions are computed with this Euclidean black hole metric in the same way as in Section 2.5.2.

A further step is required if one wants to describe a conformal symmetry breaking. Let us perform another Wick rotation on one of the x^i coordinates (say x^1). Now the metric is again of Minkowskian signature $(d-1, 1)$, but with no event horizon. The extra coordinate t_E plays the role of an S^1 submanifold whose radius defines an energy scale for the theory. This energy scale can be interpreted as the renormalization group scale (the one analogous to Λ_{QCD}), which always arises in quantum field theories when the β function is non zero. This trick is also able to break supersymmetry by giving different masses to bosons and fermions, even at the tree level. This follows from a KK reduction on $S^1(t_E)$. Since bosons and fermions have different boundary conditions their fourier expansion is, respectively,

$$\begin{aligned} \varphi(x^\mu, \theta) &= \sum_{n \in \mathbb{Z}} \phi^{(n)}(x^\mu)\, e^{in\theta/R} \ , \\ \psi(x^\mu, \theta) &= \sum_{r \in \mathbb{Z}+\frac{1}{2}} \chi^{(r)}(x^\mu)\, e^{ir\theta/R} \ . \end{aligned} \tag{2.6.8}$$

As explained in the insert in Section 2.5.1, the bosons acquire masses

$$m_\phi^2 = \frac{4\pi^2 n^2}{\beta^2} \ , \tag{2.6.9}$$

analogously fermions acquire masses

$$m_\chi^2 = \frac{4\pi^2 r^2}{\beta^2} \ . \tag{2.6.10}$$

While there is a massless boson in the spectrum, the lightest fermion $(r = 1/2)$ has a non zero mass. This clearly breaks supersymmetry because the supersymmetric multiplets should be formed by particles of the same mass. These ideas open a wider range of possibilities because now we can address problems on non conformal and non supersymmetric theories by means of holography. One of the goals in the current research is to be able to build an exact dual to QCD, but at the present status we are quite far. However, the Witten–Sakai–Sugimoto model that we are going to present in Chapter 4 is a good candidate for the low energy physics. The gravitational background is build using the same simple ideas explained above, as we will review next.

2.6.1 2+1 Yang–Mills: D3–branes on a circle

We anticipate that the Witten's metric is build by compactifying D4–branes on a circle. Just as a warm up we think it could be useful to make an example in one dimension less: D3–branes on a circle. Before the compactification we know that the metric is $AdS_5 \times S^5$ and the dual conformal field theory is $\mathcal{N} = 4$ SYM in 3+1 dimensions. The first step consists in finding a black hole solution in AdS_5, this will describe the thermodynamics of $\mathcal{N} = 4$ SYM. The metric is

$$\begin{aligned} \mathrm{d}s^2 &= \frac{L^2}{z^2}\left(\mathrm{d}x^i\mathrm{d}x^i - f(z)\mathrm{d}t^2 + \frac{\mathrm{d}z^2}{f(z)}\right) , \\ f(z) &= 1 - \left(\frac{z_h}{z}\right)^4 , \end{aligned} \tag{2.6.11}$$

where $i = 1, 2, 3$ and $z = z_h$ is the event horizon. After a Wick rotation $t \to -it_E$ the metric becomes Euclidean. The thermodynamics is obtained by the holographic correspondence

$$F(\beta) \simeq -S_E^{\text{on–shell, ren}} , \tag{2.6.12}$$

where F is the Helmholtz free energy and, again, the equivalence holds at strong coupling. Our next step now involves a second Wick rotation, on the coordinate, say, $x^3 \to -ix^0$. The metric becomes

$$\mathrm{d}s^2 = \frac{L^2}{z^2}\left(\mathrm{d}x^\mu\mathrm{d}x_\mu + f(z)\mathrm{d}t_E^2 + \frac{\mathrm{d}x^2}{f(x)}\right) , \tag{2.6.13}$$

where now $\mu = 0, 1, 2$ is a Lorentz index. The dual theory $\mathcal{N} = 4$ SYM, with action

$$S_{\text{SYM}} = \frac{1}{2g_{\text{YM}}^2}\int \mathrm{d}^3x\,\mathrm{d}t_E \mathrm{Tr}\,\left[F_{\mu\nu}F^{\mu\nu} + \cdots\right] , \tag{2.6.14}$$

can now be compactified on the circle $S^1(t_E)$. As we have seen before the circle has radius

$$R_{t_E} = \frac{2}{f'(z_h)} = \frac{z_h}{2} . \tag{2.6.15}$$

We have the following relation between the 3 dimensional coupling and the four dimensional one:

$$g_3^2 = \frac{g_{\text{YM}}^2}{2\pi R_{t_E}} . \tag{2.6.16}$$

Moreover, now the dimensional reduction produces KK fields with masses multiples of $1/R_{t_E}$ in the same way as explained above. The radius can also be seen as an UV cutoff because if we want to send $R_{t_E} \to 0$ by keeping fixed $g_3^2 N$ (N being the number of colors) then we must have $g_{\rm YM}^2 N \to 0$, which is the opposite limit of the supergravity description. This model hence is useful only for the low energy, non perturbative, regime.

2.6.2 3+1 Yang–Mills: D4–branes on a circle

In the main body of this thesis we will work on a dual of 3+1 dimensional Yang–Mills. We are thus interested in building an holographic dual of this theory. At the moment it is not known how to do this, but we can try to build a dual to a theory that resembles Yang–Mills at least in the low energy limit. In analogy of what we did before, we will look for solutions representing D4–branes wrapped on S^1 with antiperiodic boundary conditions for the fermions.

It will be useful for this purpose to go in an higher number of dimensions: eleven instead of ten. This will allow us to use the technology of AdS spaces studied before because in this case, as we will see, we can find an $AdS_7 \times S^4$ solution. Eleven dimensional supergravity is the low energy limit of a theory called M–theory. From M–theory we can obtain all known string theories by means of dualities, in this sense it is usually referred to as the “mother” of all string theories. The bosonic part of eleven dimensional supergravity is given by

$$S_{11} = \frac{1}{2\kappa_{11}} \left[\int \mathrm{d}^{11}x \sqrt{-g} \left(R - \frac{1}{2}|F_{(4)}|^2 \right) - \frac{1}{6} \int C_{(3)} \wedge F_{(4)} \wedge F_{(4)} \right] \ , \qquad (2.6.17)$$

where $F_{(4)} = \mathrm{d}C_{(3)}$ and $\kappa_{11} = (2\pi)^8 l_P^9$ is the Newton constant in 11 dimensions.

It is possible to find an extremal solution of this theory, similar to the black p–brane solution in ten dimensional supergravity. Here, on the other hand, we are considering $M5$–branes (extended objects in M–theory equivalent to D–branes). It is, not surprisingly, a metric asymptotically $AdS_7 \times S^4$ (see paragraph 3.1 of [15]). Since we look for a D4–brane solution we need to go to Type IIA supergravity. This is remarkably easy to do: it suffices to perform a supersymmetry conserving dimensional reduction on a small S^1[65]:

$$11 \text{ dim SUGRA} \xrightarrow{S^1(R_y)} \text{Type IIA} \ , \qquad (2.6.18)$$

where the radius R_y is sent to zero. Under this dimensional reduction $M5$–branes are mapped to D4–branes. Finally we need to perform a supersymmetry breaking dimensional reduction on $S^1(R_\tau)$, where the radius R_τ instead will remain finite, furnishing an energy scale $M_{\rm KK} = 1/R_\tau$. This is achieved by the double Wick rotation as showed in the previous Section.

Before deriving explicitly the desired solution let us outline the procedure described above.

M–theory (N M5–branes)

$D = 6$ gauge theory with supersymmetry (0,2)	$D = 11$ supergravity with geometry $AdS_7 \times S^4$

$\downarrow$ Compactify on $S^1(R_y)$ with periodic b.c.

Type IIA superstring (N D4–branes)

$D = 5$, $SU(N)$ super Yang–Mills with coupling $g_5^2 = R_y$	$D = 10$ Type IIA supergravity

$\downarrow$ Compactify on $S^1(R_\tau)$ with anti periodic b.c.

N D4–branes wrapped around S^1

$D = 4$, $SU(N)$ Yang–Mills with coupling $g^2_{\mathrm{YM}} = R_y/R_\tau$	$D = 10$ Euclidean black hole solution $T \propto 1/R_\tau$

The radii R_y and R_τ are chosen so that $\lambda = g^2_{\mathrm{YM}} N$ remains fixed for $N \to \infty$, one of them is arbitrary and the other fixed:

$$g^2_{\mathrm{YM}} = \frac{\lambda}{N}\,, \qquad R_y = \frac{\lambda R_\tau}{N}\,. \tag{2.6.19}$$

The AdS/CFT correspondence relates the various parameters in the following way:

- Coupling $g^2_{\mathrm{YM}} = R_y/R_\tau$. This is interpreted as the YM coupling at the energy scale M_{KK}.
- AdS radius $L = 2\,l_P(\pi N)^{1/3}$, where l_P is the eleven dimensional Planck length.
- Type IIA radius $L' = (\text{const.}) \cdot \lambda$. This is the radius of the coordinate coming from AdS_7 before the compactification.
- Dilaton $\phi = [R(\rho)]^{3/2}$ where ρ is the (former) AdS coordinate and $R(\rho)$ is the radius of the circle parametrized by τ. The string coupling runs, so the dilaton has a non constant background value.

We previously said, imprecisely, that the radius of the circle parametrized by τ was R_τ. Actually the radius grows with ρ so R_τ is the length scale at which it shrinks to zero size.

The construction is rather straightforward. Although it is more logical to compactify on R_y first and then on R_τ, we will proceed in the opposite order because it is easier. The $AdS_7 \times S^4$ solution of eleven dimensional supergravity is

$$\mathrm{d}s^2 = \frac{\rho^2}{b^2}\left(1-\frac{b^6}{\rho^6}\right)\mathrm{d}\tau^2 + \frac{\mathrm{d}\rho^2}{\frac{\rho^2}{b^2}\left(1-\frac{b^6}{\rho^6}\right)} + \frac{\rho^2}{b^2}\sum_{i=1}^{5}\mathrm{d}x_i^2 + \frac{b^2}{4}\mathrm{d}\Omega_4^2\ , \tag{2.6.20}$$

where $b = 2\,l_P(\pi N)^{1/3}$ and $l_P = 1$ is the Planck length. The coordinate τ must be periodically identified to get rid of the conical singularity in the origin of the (τ,ρ) plane. We have

$$\tau \sim \tau + \delta\tau\ , \qquad \delta\tau = \frac{4\pi b}{6} \equiv 2\pi R_\tau\ . \tag{2.6.21}$$

The coordinate x_5 is now compactified in a circle. The radius R_y has to be (according to KK theory)

$$R_y = \frac{\lambda}{N}R_\tau\ . \tag{2.6.22}$$

We have the usual ansatz for the dimensional reduction on S^1:

$$\mathrm{d}s_{11}^2 = {}^{(11)}g_{mn}\mathrm{d}x^m\mathrm{d}x^n = e^{-\frac{2}{3}\phi}\ {}^{(10)}g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + e^{\frac{4}{3}\phi}(\mathrm{d}x_5 + C_\mu\mathrm{d}x^\mu)^2\ . \tag{2.6.23}$$

We have two extra fields appearing, a scalar ϕ (dilaton) and a 1–form C. The form can be set to zero while the dilaton has a precise expression

$$e^{\frac{2}{3}\phi} = \frac{\rho}{b}\ . \tag{2.6.24}$$

The coordinate x_5 can be integrated away and after this we have to multiply the metric by R_y. We can now perform the change of variables $\rho/b = \sqrt{U/U_{\mathrm{KK}}}$ (now U_{KK} is an explicit length scale). The metric now reads:

$$\mathrm{d}s^2 = \left(\frac{U}{R}\right)^{3/2}\left(\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + f(U)\mathrm{d}\tau^2\right) + \left(\frac{R}{U}\right)^{3/2}\left(\frac{\mathrm{d}U^2}{f(U)} + U^2\mathrm{d}{\Omega_4}^2\right)\ , \tag{2.6.25}$$

$$e^\phi = g_s\left(\frac{U}{R}\right)^{3/4}\ , \qquad F_{(4)} = dC_{(3)} = \frac{2\pi N_c}{\mathrm{Vol}(S^4)}\omega_{S^4}\ , \tag{2.6.26}$$

where $f(U)$ is given by

$$f(U) = 1 - \frac{U_{\mathrm{KK}}^3}{U^3} \tag{2.6.27}$$

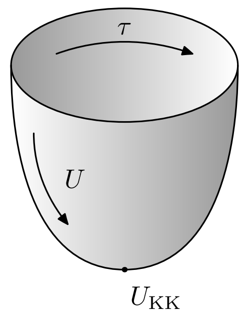


Figure 2.8: Representation of the "cigar" $\sim \mathbb{R}^2$ realized by the coordinates U and τ, the bottom of the cigar sets the scale $U_{\rm KK}$.

and R is related simply to R_y and b (we will not write this explicitly). The plane (U, τ) is homeomorphic to $\mathbb{R}^2$. Even if at first sight one may think it is a $\mathbb{R} \times S^1$ (a cylinder), the radius of the circle shrinks to zero size at $U = U_{\rm KK}$, so it forms what is usually called a *cigar* (see Figure 2.8), regularity at the tip being realized by the condition (2.6.29). The charge of the D4 brane, which is given by the integral of $F_{(4)}$ in S^4, is normalized in units of 2π (see appendix A of [43])

$$\frac{1}{2\pi}\int_{S^4} F_{(4)} = N_c \; . \tag{2.6.28}$$

The coordinate τ is the same as before, whose period is now

$$\tau \sim \tau + \delta\tau \; , \qquad \delta\tau = \frac{4\pi}{3}\frac{R^{3/2}}{U_{\rm KK}^{1/2}} = \frac{2\pi}{M_{\rm KK}} \; . \tag{2.6.29}$$

The parameter $U_{\rm KK}$ is arbitary while R and g_s are given by the AdS/CFT dictionary, l_s being the string length

$$R^3 = \frac{1}{2}\frac{\lambda l_s^2}{M_{\rm KK}} \; , \quad U_{\rm KK} = \frac{2}{9}\lambda M_{\rm KK} l_s^2 \; , \quad g_s = \frac{1}{2\pi}\frac{g_{\rm YM}^2}{M_{\rm KK} l_s} \; . \tag{2.6.30}$$

These relations are found by expanding to second order and then compactifying on $S^1(\tau)$ the DBI action of the D4 branes; which is given by

$$\frac{1}{(2\pi)^4 l_s^5\, g_s}\int_{D4} \mathrm{d}^4\xi \mathrm{d}\tau\, e^{-\phi}\mathrm{STr}\sqrt{|\det(\mathcal{P}[g] + 2\pi l_s^2 F)|} \; , \tag{2.6.31}$$

$\mathcal{P}[g]$ being the pullback of the metric in the worldvolume and STr denoting the trace symmetrized in the gauge group indices. The energy $M_{\rm KK}$ plays the role of b defined before[16], the coupling $g_{\rm YM}$ is the Yang–Mills running coupling computed

[16] $M_{\rm KK} = 1 \Leftrightarrow b = 3$

at energy $M_{\rm KK}$. Also, following [44], we observe that l_s does not appear in the supergravity action if it is written in terms of $M_{\rm KK}$ and $g_{\rm YM}$. Therefore, without loss of generality, we can set

$$\frac{2}{9}M_{\rm KK}^2 l_s^2 = (g_{\rm YM}^2 N_c)^{-1} \equiv \lambda^{-1} \; . \tag{2.6.32}$$

To simplify the formulas it is convenient to set $M_{\rm KK} = 1$ and recover it in the final results by dimensional analysis. We thus finally obtain:

$$R^3 = \frac{9}{4} \; , \quad U_{\rm KK} = M_{\rm KK} = 1 \; , \quad g_s = \frac{1}{2\pi}\frac{g_{\rm YM}^2}{M_{\rm KK} l_s} \; . \tag{2.6.33}$$

Chapter 3

Non perturbative QCD

In this Chapter we are going to the introduce the theory of Quantum Chromodynamics (QCD), mostly regarding both the interesting features that are preserved in the effective description and the motivations that led us to perform a computation of the neutron electric dipole moment (NEDM). We will in particular study the instantonic solution of Yang–Mills and the topological properties (θ dependence). Then the main non perturbative tools available in QCD, namely the large N_c expansion and the chiral perturbation theory, are briefly reviewed. Some books about QCD are [60, 66, 6], see also the reviews [67, 14, 68, 8] about large N, chiral perturbation theory and the Skyrme model. Some aspects that we will not consider here include the lattice approach [5] and the perturbative approach [69].

3.1 Introduction

Quantum Chromodynamics (QCD) is the theory of strong interactions. It describes the interactions between quarks and gluons which make up the hadrons: the baryons (like the nucleons, proton and neutron) and the mesons (like the pion). It is a non abelian gauge theory: the force is mediated by a massless vector boson which is a connection over the non abelian group $SU(3)$. The quarks are Dirac fermions belonging to the fundamental representation of $SU(3)$; they come in $N_f = 6$ different species called *flavors*. The gauge fields are called gluons and the theory describing exclusively gluon dynamics is called Yang–Mills (YM) theory. The Lagrangian of QCD has the following form:

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{2g^2}\operatorname{Tr} F_{\mu\nu}F^{\mu\nu} + \sum_{f=1}^{N_f} \overline{q}_f(i\not{D} - m_f)q_f \ , \tag{3.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \ , \quad D_\mu = \partial_\mu + iA_\mu \ , \tag{3.2}$$

and A_μ is a matrix valued field belonging to the adjoint representation of $SU(3)$: $A_\mu = A_\mu^a T^a$, T^a being the generators of the group. It is possible to define the parallel transport of the gauge connection A_μ, usually called Wilson line (loop if it is on a closed path). First of all an infinitesimal parallel transport from x to $x + \mathrm{d}x$ is defined as

$$h_{x+\mathrm{d}x \leftarrow x}(A) = \mathbb{1} - iA_\mu(x)\mathrm{d}x^\mu \ . \tag{3.3}$$

Let γ be a curve in space time with $\gamma(0) = x_I$ and $\gamma(1) = x_F$. The Wilson loop is the *ordered* product

$$\mathcal{W}_\gamma(A) = \lim_{n\to\infty} h_{x_F \leftarrow x_{n-1}}(A) \cdots h_{x_3 \leftarrow x_2}(A) h_{x_2 \leftarrow x_I}(A) \ , \tag{3.4}$$

where the points x_j lay in sequence on the curve γ and, as n goes to infinity, they all come very close to each other. This ordering is called *path ordering* because it is based on the order in which the points appear on γ. This long product is often denoted as

$$\mathcal{W}_\gamma(A) = \mathcal{P} \exp\left(-i \int_\gamma A_\mu \, \mathrm{d}x^\mu\right) \in SU(N_c) \ . \tag{3.5}$$

The gauge transformation property of $\mathcal{W}_\gamma$ depends only on the endpoints (because in the product all gauge functions Ω between two consecutive points cancel).

$$\mathcal{W}_\gamma(A) \underset{\Omega}{\longrightarrow} \Omega(x_F)\mathcal{W}_\gamma(A)\Omega^{-1}(x_I) \ . \tag{3.6}$$

For a closed curve $x_I \equiv x_F$ so the Wilson loop is gauge covariant. The trace

$$W_\gamma(A) = \mathrm{Tr}\, \mathcal{W}_\gamma(A) \tag{3.7}$$

is a gauge invariant quantity and for a generic curve γ it contains all the information of the pure gluon theory.

The key feature of the theory is asymptotic freedom [1, 2]: it is weakly coupled (and thus amenable for a perturbative treatment) at high energies (i.e. in the UV), while it becomes strongly coupled at low energies (i.e. in the IR). This implies that non–perturbative tools are necessary to describe the low energy features (bound states spectra, static properties and so on).

3.2 Instantons

Let us consider classical solutions of the pure gauge theory, i.e. Yang–Mills (YM) theory

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2g^2} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} \ . \tag{3.2.1}$$

The equations of motion are simply

$$D_\mu F^{\mu\nu} \equiv \partial_\mu F^{\mu\nu} + i[A_\mu, F^{\mu\nu}] = 0 \ , \tag{3.2.2}$$

where now $D_\mu = \partial_\mu + i[A_\mu, \cdot]$ is the covariant derivative in the adjoint representation (whereas the one appearing in (3.1) is in the fundamental representation). When the theory is not coupled to any external source a possible solution is the trivial $F_{\mu\nu} = 0$ vacuum. This is indeed the vacuum configuration around which the perturbative expansion is performed. One could be interested in finding non trivial solutions to these equations. These solutions are called *instantons* and they first appeared in [3]. The equations (3.2.2) are non linear, hence before aimlessly looking for a solution we should make some observations to simplify the problem. To obtain a minimum of the action it is clear that $S_{\rm YM} = \int \mathrm{d}^4x\, \mathcal{L}_{\rm YM}$ should be finite, hence $F_{\mu\nu}$ at $|\vec{x}| \to \infty$ must approach zero. The theory however does not distinguish between a certain configuration A_μ and its gauge transformed one A_μ^Ω defined as

$$A_\mu^\Omega \equiv \Omega A_\mu \Omega^{-1} - i\Omega\partial_\mu\Omega \ , \tag{3.2.3}$$

where Ω is any $SU(N)$ valued function of the coordinates, $SU(N)$ being the gauge group. Let us switch to Euclidean signature (by means of a Wick rotation on the time coordinate). So asymptotically the gauge vector should approach

$$A_\mu \underset{|x|\to\infty}{\longrightarrow} 0^\Omega = -i\Omega\partial_\mu\Omega \tag{3.2.4}$$

The function Ω now depends only on $\hat{x} = \vec{x}/|x|$ which is the direction along which the limit is taken; topologically it means that Ω is a map

$$\Omega(\hat{x}) : \ S^3 \ \longrightarrow \ SU(N) \ . \tag{3.2.5}$$

Let us assume from now on that $N = 2$, which is the simplest case. All the other $SU(N > 2)$ groups have many $SU(2)$ subgroups, so we can always find solutions by embedding the $SU(2)$ one in one of those subgroups. It is known that topologically $SU(2) \cong S^3$, the maps $S^3 \to S^3$ being arranged in homotopy classes, which are classes of maps that can be continuously deformed one into another. The set of homotopy classes endowed with a group structure (the composition of maps) is called *homotopy group*. In general the homotopy group for the maps $S^n \to \mathcal{M}$ is $\pi_n(\mathcal{M})$ (where $\mathcal{M}$ is a manifold). In this case we have

$$\pi_3(S^3) = \mathbb{Z} \ . \tag{3.2.6}$$

Thus the instantons are naturally classified by integer numbers. The integer number associated to an instantonic configuration is called *instanton number* or *topological charge*. It could be useful to find a representation of the homotopy group, i.e. a functional $\mathcal{Q}[\Omega]$ of Ω (the asymptotic limit of A_μ) that provides the instanton number of the solution and satisfies a property

$$\mathcal{Q}[\Omega\,\Omega'] = \mathcal{Q}[\Omega] + \mathcal{Q}[\Omega'] \tag{3.2.7}$$

Let us define the Maurer–Cartan integral,

$$\mathcal{I}[\Omega] = \int \mathrm{d}^3\theta\,\varepsilon^{abc}\mathrm{Tr}\left[\Omega^{-1}\frac{\partial\Omega}{\partial\theta^a}\Omega^{-1}\frac{\partial\Omega}{\partial\theta^b}\Omega^{-1}\frac{\partial\Omega}{\partial\theta^c}\right] , \tag{3.2.8}$$

where the variables θ^1, θ^2 and θ^3 are the angles on the sphere S^3, we have thus $\Omega \equiv \Omega(\vec{\theta})$. Using the fact $A \to -i\Omega^{-1}\partial_\mu\Omega$ as $|x| \to \infty$ we get

$$\mathcal{I}[\Omega] = -i \lim_{|x|\to\infty} \int \mathrm{d}^3\theta\,\varepsilon^{abc}\frac{\partial\hat{x}^\mu}{\partial\theta^a}\frac{\partial\hat{x}^\nu}{\partial\theta^b}\frac{\partial\hat{x}^\rho}{\partial\theta^c}\mathrm{Tr}\left[A_\mu A_\nu A_\rho\right] . \tag{3.2.9}$$

As we can see the integration measure is just the measure on the sphere. Using Stokes theorem we can recast this expression in a full space integral; it is convenient to add $\int \mathrm{Tr}\left[A \wedge F\right]$ which is zero because of $F_{\mu\nu} \to 0$. After using the identity

$$\mathrm{Tr}\left[F \wedge F\right] \equiv \mathrm{d}\,\mathrm{Tr}\left[A \wedge F - \frac{i}{3}A \wedge A \wedge A\right] \tag{3.2.10}$$

we obtain

$$\mathcal{I}[\Omega] = 3\int \mathrm{Tr}\left[F \wedge F\right] = \frac{3}{4}\int \mathrm{d}^4x\,\varepsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\,F_{\mu\nu}F_{\rho\sigma} . \tag{3.2.11}$$

It can be checked that $\mathcal{I}[\Omega]$ furnishes a representation of $\pi_3(S^3)$: in fact it is invariant under continuous deformations of Ω and if two functions Ω and Ω' are composed the result is the sum of the two integrals

$$\mathcal{I}[\Omega \circ \Omega'] = \mathcal{I}[\Omega] + \mathcal{I}[\Omega'] . \tag{3.2.12}$$

In order to check the normalization take a standard map of instanton number 1; the most simple one–to–one mapping from S^3 to $SU(2)$ is:

$$\Omega^1 = i\vec{\sigma}\cdot\vec{\theta} + \mathbb{1}(1 - |\vec{\theta}|^2) , \qquad \mathcal{I}[\Omega^1] = 24\pi^2 . \tag{3.2.13}$$

In general a solution with instanton number ν will have a boundary behaviour

$$(\Omega^1)^\nu \text{ if } \nu > 0 , \qquad (\Omega^{1\dagger})^{-\nu} \text{ if } \nu < 0 . \tag{3.2.14}$$

The normalized definition of $\mathcal{Q}[\Omega]$ is

$$\mathcal{Q}[\Omega] = \frac{1}{32\pi^2}\int \mathrm{d}^4x\,\varepsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\,F_{\mu\nu}F_{\rho\sigma} . \tag{3.2.15}$$

The dependence on A_μ is fictitious because it is a total derivative and only the boundary limit enters.

The other remark concerns a standard method used to find topological solutions in these kind of systems. Since we want to minimize the action we might try and find a global minimum by means of an inequality. To do this define

$$^{\star}F_{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \ . \tag{3.2.16}$$

The dual * applied two times is the identity in Euclidean space. Now

$$\begin{aligned} S_{\rm YM} &= \frac{1}{2g^2}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} = \\ &= \frac{1}{4g^2}\int \mathrm{d}^4x \left[\mathrm{Tr}\,(F_{\mu\nu} \mp {}^{\star}F_{\mu\nu})^2 \pm 2\mathrm{Tr}\, F_{\mu\nu}\, {}^{\star}F^{\mu\nu}\right] \geq \\ &\geq \pm\frac{1}{4g^2}\int \mathrm{d}^4x\, \mathrm{Tr}\, F_{\mu\nu}F_{\rho\sigma}\,\varepsilon^{\mu\nu\rho\sigma} = \\ &= \frac{8\pi^2}{g^2}|\nu||;, \end{aligned} \tag{3.2.17}$$

where ν is the instanton number and the $\pm$ sign is chosen to obtain a positive lower bound. We see that the bound is saturated if $F_{\mu\nu} = {}^{\star}F_{\mu\nu}$ for $\nu > 0$ and $F_{\mu\nu} = -{}^{\star}F_{\mu\nu}$ for $\nu < 0$. This condition is much easier to satisfy because now it is a first order equation for A_μ. It is easy to see that this implies the Maxwell equations; the Bianchi identity in fact can be written as

$$D_\mu {}^{\star}F^{\mu\nu} = 0 \ . \tag{3.2.18}$$

This is always satisfied and implies the Maxwell equation if F is (anti)self–dual.

Let us see now the explicit solution for $SU(2)$ in 4 Euclidean dimensions. The most reasonable way to proceed is to find a basis of (anti)self–dual tensors and then reduce the problem to an ordinary differential equation (ODE). This basis is given by the 't Hooft symbols

$$\begin{aligned} \eta^a_{\mu\nu} &= \varepsilon_{a\mu\nu4} + \delta_{a\mu}\delta_{\nu4} - \delta_{a\nu}\delta_{\mu4} \ , \\ \overline{\eta}^a_{\mu\nu} &= \varepsilon_{a\mu\nu4} - \delta_{a\mu}\delta_{\nu4} + \delta_{a\nu}\delta_{\mu4} \ , \end{aligned} \tag{3.2.19}$$

where $a = 1, 2, 3$ and $\mu = 1, 2, 3, 4$. Notice that

$$\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\eta^a_{\rho\sigma} = \eta^a_{\mu\nu} \ , \qquad \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\overline{\eta}^a_{\rho\sigma} = -\overline{\eta}^a_{\mu\nu} \ , \tag{3.2.20}$$

with $\varepsilon_{1234} = +1$. Hence $\eta^a_{\mu\nu}$ is self dual while $\overline{\eta}^a_{\mu\nu}$ is anti–self dual. Some useful properties are (the same hold for $\overline{\eta}$)

$$\begin{aligned} \varepsilon_{abc}\eta^b_{\mu\rho}\eta^c_{\nu\sigma} &= \delta_{\mu\nu}\eta^a_{\rho\sigma} - \delta_{\mu\sigma}\eta^a_{\rho\nu} + \delta_{\rho\sigma}\eta^a_{\mu\nu} - \delta_{\rho\nu}\eta^a_{\mu\sigma} \ , \\ \eta^a_{\mu\nu}\eta^a_{\mu\nu} &= 12 \ . \end{aligned} \tag{3.2.21}$$

A whole list of them can be found in the Appendix of [70]. Using these simple definitions we can propose an ansatz for A^a_μ:

$$A^a_\mu = -\eta^a_{\mu\nu}\partial_\nu \log\phi(x) \ . \tag{3.2.22}$$

This works if we are interested in instantons (self dual tensors): for anti–instantons (anti self–dual) just substitute η with $\overline{\eta}$. The field strength reads

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \varepsilon_{abc}A^b_\mu A^c_\nu = \\ &= \left[-\eta^a_{\nu\rho}\left(\frac{\partial_\mu\partial_\rho\phi}{\phi}\right) + \eta^a_{\mu\rho}\left(\frac{\partial_\nu\partial_\rho\phi}{\phi}\right)\right] - \eta^a_{\mu\nu}(\partial_\rho\log\phi)^2 \ . \end{aligned} \tag{3.2.23}$$

This must be proportional to $\eta^a_{\mu\nu}$, hence

$$\frac{\partial_\mu\partial_\rho\phi}{\phi} \propto \delta_{\mu\rho} \ . \tag{3.2.24}$$

The result is a quadratic polynomial

$$\phi(x) = c + \frac{(x-X)^2}{\rho^2} \ . \tag{3.2.25}$$

Since A^a_μ depends only on the derivative of the logarithm of ϕ, the solution depends only on $c\rho^2$ and X^μ. We might as well set $c = 1$ without loss of generality, the parameter ρ in this case is a free parameter and it will turn out to be very important in the physics of instantons. The vector X^μ is simply the position of the center of mass of the instanton. Explicitly the solution reads

$$A^a_\mu = -\frac{2(x-X)_\nu\eta^a_{\mu\nu}}{\xi^2+\rho^2} \ , \qquad F^a_{\mu\nu} = -\eta^a_{\mu\nu}\frac{4\rho^2}{(\xi^2+\rho^2)^2} \ , \tag{3.2.26}$$

where we have defined $(x-X)^2 = \xi^2$. Note that, although A^a_μ may seem not to have a sufficiently fast decay at infinity (being $\sim x^{-1}$), the function F^2 is integrable and the fact that A^a_μ is not well behaved at infinity is merely a gauge artifact. It is not hard to compute the instanton number

$$\mathcal{Q}[A] = \frac{1}{64\pi^2}\int \mathrm{d}^4x\, F^a_{\mu\nu}\,{}^\star F^a_{\mu\nu} = 1 \ . \tag{3.2.27}$$

If one prefers regularity at infinity it is possible to perform a gauge transformation on A^a_μ, since we have $A_\mu \to -i\Omega\partial_\mu\Omega^{-1}$ we could gauge Ω away transforming with Ω^{-1}, the result

$$A^a_\mu = -\frac{2\rho^2(x-X)_\nu}{\xi^2(\xi^2+\rho^2)^2}\overline{\eta}^a_{\mu\nu} \ , \tag{3.2.28}$$

is rather counterintuitive because now the solution is expressed in terms of $\overline{\eta}$ tensors and diverges at $x^\mu = X^\mu$. The reason for this is that we are somehow "forcing" the solution to be regular at infinity and the divergence must reappear somewhere. This is known to be the *singular gauge*, whereas the former was the *regular gauge*. Another nice way to present the solution (in regular gauge) is in terms of the function Ω directly,

$$A_\mu = -if(\xi)\,\Omega\partial_\mu\Omega^{-1} \,, \tag{3.2.29}$$

where (σ^a are the Pauli matrices) and

$$f(\xi) = \frac{\xi^2}{\xi^2+\rho^2} \,, \qquad \Omega(x) = \frac{(x-X)_4\mathbb{1} + i(x-X)_a\sigma^a}{\xi} \,. \tag{3.2.30}$$

We will use this expression in most of the work. In this way the asymptotic limit of A_μ is rather clear, and it is also evident that the configuration has instantonic number 1 without having to make the integral (because Ω is a 1 to 1 mapping from S^3 to $SU(2)$).

Instantons are interpreted as pseudoparticle solutions to the Yang–Mills equations: as every particle they have their own degrees of freedom. As we have seen in fact $\vec{X}$ and ρ are free parameters of the solution. We can also make a group transformation, a so called *large gauge transformation*, which is indeed a true degree of freedom because it is not connected to the identity, as opposed to a small gauge transformation (one that approaches $\mathbb{1}$ at infinity) that is just another way to write A_μ. Take an element $g \in SU(2)$ represented as a 2 by 2 matrix, we have

$$A_\mu = -\frac{(x-X)_\nu \eta^a_{\mu\nu}}{\xi^2+\rho^2}\,(g\sigma^a g^{-1}) \,. \tag{3.2.31}$$

This is the most general solution with instanton number 1. It has 8 parameters (4 X^μ, 1 ρ and 3 g), spanning the space of all possible configurations called *moduli space*. Since they correspond to global symmetries they do not modify the action, hence the energy of the instanton remains the same as we vary one of these parameters. We could say that these are *zero modes* of the theory (excitations with zero potential energy). It would be correct to interpret them as Goldstone modes as well (X^μ breaks translational invariance, ρ scale invariance and g global $SU(2)$ invariance). To be a little bit more general let us define the moduli space of a instanton of number ν in the group $SU(N)$, which we will call $\mathcal{M}_{N,\nu}$. The Atiyah Singer theorem, that we will not prove, states that (here the proof of the theorem with the original reference can be found [71])

$$\dim \mathcal{M}_{N,\nu} = 4\nu N \,. \tag{3.2.32}$$

In our case this is indeed 8. In order to achieve a dynamical description of the instanton we must study the moduli space and add some structure to it. This is because the motion of the instanton is equivalent to the motion of a free particle in the moduli space. If we want to address this problem we must introduce a metric on $\mathcal{M}_{N,\nu}$ and render it a manifold. Anticipating the result we have

$$\mathcal{M}_{2,1} \cong \mathbb{R}^4 \times \mathbb{R}^4/\mathbb{Z}_2 \ , \tag{3.2.33}$$

where the first $\mathbb{R}^4$ is spanned by X^μ and the quotient $\mathbb{R}^4/\mathbb{Z}_2$ is the 4–plet $(\vec{\theta}, \rho)$ with the $\mathbb{Z}_2$ acting as $\vec{\theta} \to -\vec{\theta}$ (which does not have any effect on the instanton[1]). The metric is

$$\mathrm{d}s^2 = \frac{g^2}{2} S_{\mathrm{YM}}\, \delta_{\mu\nu} \mathrm{d}X^\mu \mathrm{d}X^\nu + g^2 S_{\mathrm{YM}} \left(\mathrm{d}\rho^2 + \rho^2 \delta_{ij} \mathrm{d}\theta^i \mathrm{d}\theta^j\right) \ , \tag{3.2.34}$$

where S_{YM} is the on shell Yang–Mills action. Let us prove this result. The main idea is that, once we have a solution with fixed moduli, any excitation of a modulus (say $\rho \to \rho + \delta\rho$) is a linear perturbation over the solution

$$A_\mu \to A_\mu + \delta A_\mu \ , \tag{3.2.35}$$

where δA_μ solves the linearized $F = {}^\star F$ equation and it is called *zero mode*. These modes must be genuine excitations of the solution, we do not want them to be gauge transformations, i.e. $\delta A_\mu = D_\mu \Lambda$; in order to solve this issue it is convenient to fix the gauge

$$D_\mu^{(0)}(\delta A_\mu) = 0 \ , \tag{3.2.36}$$

where the superscript (0) reminds us to compute the covariant derivative in the unperturbed solution. Let us call X^α the collective coordinates $X^\alpha = (X^\mu, \vec{\theta}, \rho)$. The zero mode associated the the modulus X^α is

$$\delta_\alpha A_\mu = \frac{\partial A_\mu}{\partial X^\alpha} + D_\mu^{(0)} \Lambda_\alpha \ , \tag{3.2.37}$$

where Λ_α has the purpose to enforce the gauge fixing condition. The set of all $\delta_\alpha A_\mu$ constitutes a basis of the tangent space of $\mathcal{M}_{N,\nu}$; a suitable choice for a metric is given by the following scalar product (which is the standard L^2 product in space x^μ, the Euclidean scalar product in indices μ and the Killing form on the group indices)

$$g_{\alpha\beta} \equiv \int \mathrm{d}^4x \, \mathrm{Tr}\, [\delta_\alpha A_\mu \, \delta_\beta A^\mu] \ . \tag{3.2.38}$$

The gauge fixing condition renders the metric diagonal (we will not show it explicitly). To see if the overall factors are the ones we claimed in (3.2.34) we must do the explicit computation (following [72]).

[1] The elements belonging to the $\mathbb{Z}_2$ quotiented out make up the center of $SU(2)$, $Z(SU(2)) = \pm\mathbb{1}$.

- Translational modes:
$$\delta_\nu A_\mu = \frac{\partial A_\mu}{\partial X^\nu} + D^{(0)}_\mu \Lambda_\nu \ . \tag{3.2.39}$$
It is easy to notice that
$$\frac{\partial A_\mu}{\partial X^\nu} = -\frac{\partial A_\mu}{\partial x^\nu} \ . \tag{3.2.40}$$
Choosing $\Lambda_\nu = A_\nu$ we obtain for the gauge fixing condition
$$D^{(0)}_\mu \left(-\partial_\nu A_\mu + D^{(0)}_\mu A_\nu\right) = D^{(0)}_\mu F_{\mu\nu} = 0 \ , \tag{3.2.41}$$
due to the Yang–Mills equations. A simple computation shows
$$\begin{aligned} g_{\mu\nu} &= \int \mathrm{d}^4x \, \mathrm{Tr}\left[F_{\rho\mu} F_{\rho\nu}\right] \\ &= \int \mathrm{d}^4x \, \mathrm{Tr}\left[F_{\rho\sigma} F_{\rho\sigma}\right] \frac{1}{4}\delta_{\mu\nu} = \frac{g^2}{2} S_{\mathrm{YM}} \ . \end{aligned} \tag{3.2.42}$$

- Scale modes:
$$\delta_\rho A_\mu = \frac{\partial A_\mu}{\partial \rho} \tag{3.2.43}$$
Luckily these modes do not require any Λ_ρ because they solve automatically the gauge fixing condition. The metric element has to be computed explicitly
$$\int \mathrm{d}^4x \, \mathrm{Tr} \, \frac{\partial A_\mu}{\partial \rho} \frac{\partial A^\mu}{\partial \rho} = 4\pi^2 = g^2 S_{\mathrm{YM}} \ . \tag{3.2.44}$$

- Lastly, the gauge group orientation:
$$\delta_a A_\mu = D_\mu \Lambda_a \ . \tag{3.2.45}$$
The transformation $D_\mu \Lambda$ is a linearized gauge transformation of A_μ (that does *not* approach zero at infinity), the component a is just $\Lambda_a = \mathrm{Tr}\,(\sigma^a \Lambda)$. The gauge fixing condition amounts to the equation
$$D_\mu D_\mu \Lambda = 0 \ . \tag{3.2.46}$$
Curiously it is easier to find a solution to this equation in singular gauge. Calling $\overline{\Lambda}$ the function in singular gauge we have
$$\overline{\Lambda}_a = \frac{\xi^2}{\xi^2 + \rho^2} \sigma^a = f(\xi) \sigma^a \ . \tag{3.2.47}$$
It is easy to go back: $\Lambda = \Omega \overline{\Lambda} \Omega^{-1}$ (with the definitions in (3.2.30)) but for our purposes one gauge is as good as another since we have to compute a

trace in the end. The solution is not hard to find in singular gauge because it is possible to make an ansatz $\overline{\Lambda}_a = u(\xi)$ for each a to obtain an equation

$$\overline{D}_\mu \overline{D}_\mu \overline{\Lambda} = \frac{3}{\xi} u'(\xi) + u''(\xi) - \frac{8}{\xi^2}(1 - f(\xi))^2 u(\xi) = 0 \; . \tag{3.2.48}$$

The equation is solved by $u(\xi) = f(\xi)$. The metric element is calculated as before

$$\int \mathrm{d}^4 x \, \mathrm{Tr}\, D_\mu \Lambda_i D_\mu \Lambda_j = 4\pi^2 \delta_{ij} \rho^2 = g^2 S_{\mathrm{YM}} \rho^2 \; . \tag{3.2.49}$$

These result prove the claim (3.2.34). The X^μ part of the metric is clearly that of an $\mathbb{R}^4$, while the $(\vec{\theta}, \rho)$ part of the metric is proportional to

$$\mathrm{d}\rho^2 + \rho^2 \delta_{ij} \mathrm{d}\theta_i \mathrm{d}\theta_j \; , \tag{3.2.50}$$

which is again $\mathbb{R}^4$ in spherical coordinates (with radius ρ). The $\mathbb{Z}_2$ factor has to be quotiented away because $\pm\mathbb{1}$ do not correspond to different configurations for the instanton. The singularity arising at $\rho \to 0$ of the resulting orbifold is interpreted as the singularity of the solution when $\rho \to 0$ (*small instanton limit*).

3.3 Topological term

The most important consequence of these instantonic solutions is that they oblige us to introduce a new term in the Yang–Mills Lagrangian, the so called *topological term*. Its name is due to the fact that it will be a total derivative, hence depending only on the boundary condition on the fields. We will first analyse a somewhat different system that shows many features similar to the ones encountered with instantons and then use a purely field theoretical argument to show that indeed it is necessary to add a topological term.

Let us start with this elementary example: quantum mechanics on S^1 or, equivalently, a one dimensional crystal. The Hamiltonian has a periodic potential and can be written as

$$\mathcal{H} = \frac{p^2}{2} + V(q) \; , \qquad V(q + 2\pi) = V(q) \tag{3.3.1}$$

A consistency condition on the wave function must be imposed, after a full turn on S^1 it has to go back to itself, modulo a phase

$$\Psi(q + 2\pi) = e^{i\theta} \Psi(q) \tag{3.3.2}$$

This is usually called the *Bloch theorem*, which is formally proven by defining the operator U that makes a full turn on S^1. U is a symmetry of the Hamiltonian so

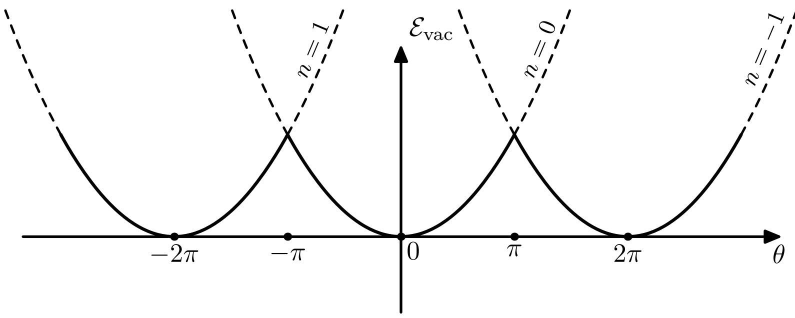


Figure 3.1: Plot of the θ–vacuum for $V(q) = 0$. As we can see we have discontinuities in the derivative of $\mathcal{E}_{\rm vac}$ in odd multiples of π.

the energy eigenstates can be chosen to be diagonal under its action; moreover U has to be unitary (for Wigner's theorem) hence the eigevalues must be $e^{i\theta}$. Also, eigenstates with different eigenvalues under the action of U (let us call them Ψ_θ and $\Psi_{\theta'}$) are orthogonal, being different eigenstates of an unitary operator. There is actually more, there exist no operator interpolating between them

$$\langle \Psi_{\theta'} | \mathcal{O} | \Psi_\theta \rangle = 0 \quad \forall\, \mathcal{O} \in \text{Observables} \;, \tag{3.3.3}$$

so we have a *superselection rule*: once the world has chosen a θ it remains the same forever. It may be interesting to see the solution for the vacuum energy $\mathcal{E}_{\rm vac}$ in the trivial case $V(q) = 0$. We have

$$\Psi_n^\theta = \frac{1}{\sqrt{2\pi}} e^{i(n+\theta/2\pi)q} \;, \qquad \mathcal{E}_n^\theta = \frac{(n+\theta/2\pi)^2}{2} \;. \tag{3.3.4}$$

The vacuum selects a different $\mathcal{E}_n$ for different values of θ so that the energy remains minimal, see Figure 3.1. The discontinuities at odd multiples of π are a special feature of these systems. It is interesting, for the purpose of building an analogy to QCD, to perform a path integral quantization of this system. The path integral that computes the amplitude of the transition $q_i \to q_f$ in a time t is defined by

$$K(q_i, q_f; t) = \int_{q(0)=q_i}^{q(t)=q_f} \mathcal{D}q \, e^{iS[q(t)]} \;. \tag{3.3.5}$$

However our theory lives on S^1, which is not simply connected

$$\pi_1(S^1) = \mathbb{Z} \;. \tag{3.3.6}$$

The path integral is decomposed into a sum, each term belonging to a specific homotopy class, which will be multiplied by a certain weight

$$\begin{aligned} K(q_i, q_f; t) &= \sum_{\alpha\in\pi_1(S^1)} w(\alpha) \int_{q(0)=q_i}^{q(t)=q_f} \mathcal{D}q\, e^{iS[q(t)]}\Big|_\alpha = \\ &= \sum_{\alpha\in\pi_1(S^1)} w(\alpha) K_\alpha(q_i, q_f; t) \ , \end{aligned} \tag{3.3.7}$$

where each integral now is restricted to the class α. For the S^1 the interpretation is very simple because the homotopy classes correspond to the number of windings around the circle, hence each integral is constrained to wind a fixed number of times around it. The weight factors are not arbitrary, they can be computed by means of the theorem by Laidlaw and DeWitt [73] (see also [74]). The theorem states that $w(\alpha)$ are 1 dimensional representations of the homotopy group (i.e. its characters). If α is the class corresponding to n windings we have

$$w(\alpha) = e^{in\theta} \ . \tag{3.3.8}$$

The θ is exactly the same parameter we have seen before because the phase acquired under the action of U can be seen as the amplitude of propagating around the circle once. We have finally

$$K(q_i, q_f; t) = \sum_{n\in\mathbb{Z}} e^{in\theta} K_n(q_i, q_f; t) \ . \tag{3.3.9}$$

We would like to get rid of this sum and go back to an ordinary path integral. This can be done in the following way: suppose $q(t)$ is a path winding around the circle n times, then

$$\frac{1}{2\pi}\int_0^t \mathrm{d}t'\, \dot q(t') = \frac{q(t)-q(0)}{2\pi} = n \ . \tag{3.3.10}$$

We can include the term $in\theta$ in the action using this simple trick

$$S \equiv \int \mathrm{d}t \left[\frac{1}{2}\dot q^2(t) - V(q(t))\right] \longrightarrow S' \equiv S + \frac{1}{2\pi}\int \mathrm{d}t\, \dot q(t) \ . \tag{3.3.11}$$

The path integral by means of S' is defined without the weight factors

$$K'(q_i, q_f; t) = \int_{q(0)=q_i}^{q(t)=q_f} \mathcal{D}q\, e^{iS'[q(t)]} \tag{3.3.12}$$

and no constraint is imposed on the number of windings of q. Notice that the term we added to the action is a total derivative (a topological term). We should expect a similar result in QCD.

Finally let us analyse one last feature: in this model we can see some interesting solutions which, not by chance, are called *instantons*. They arise from the periodicity of $V(q)$ alone, but in order to have an explicit solution we will assume a specific form of the potential. It should be clear that the qualitative results hold also in general, precisely because these instantons play the role of interpolating between different vacua (of course if $V(q)$ has a minimum q_0 then all $q_0 + 2n\pi$ will be minima as well). Suppose that the potential $V(q)$ is such that between two minima we can approximate it with a "double well" potential

$$V(q) = \frac{\omega^2}{2a^2}(q^2 - a^2)^2 \ . \tag{3.3.13}$$

Notice that in the Euclidean formulation the sign of the potential is reversed, this representing a particle evolving along an imaginary time in the classically forbidden region separating the two vacua. The energy is defined as

$$\mathcal{E} = \frac{1}{2}\dot{q}^2 - \frac{\omega^2}{2a^2}(q^2 - a^2)^2 \ . \tag{3.3.14}$$

A solution with zero energy (vacuum) is given by

$$\dot{q} = \pm\frac{\omega}{a}(q^2 - a^2) \quad \Longrightarrow \quad q(t) = a\tanh\left(\omega(t - t_0)\right) \ . \tag{3.3.15}$$

This solution interpolates between the two vacua $q = -a$ at $t \to -\infty$ and $q = a$ at $t \to \infty$. Here the name *instanton* acquires a simple meaning: this solution may represent an excitation that appears from the vacuum and remains for a brief time, or "instant", after which it disappears in the vacuum again (but in a different one!). When we include topologically non trivial configurations in the path integral we are considering precisely these instantons because, as they go from $-a$ to a, they are winding around the circle S^1.

Now we are ready to list all the similarities with QCD encountered in this simple exercise:

- The configuration space is of course much more complicated than S^1 (just consider that it is infinite dimensional). But suppose that we move in the "direction" identified by $\mathcal{Q}$ (3.2.15), we get that all the values $\mathcal{Q} = 0, \pm 1, \pm 2, \ldots$ are related by a large gauge transformation, so they are all equivalent. We have the topology of S^1!

- The wave functional $\Psi[A]$ satisfies the Bloch theorem as $\Psi(q)$: consider the operator U that changes the instanton number $\mathcal{Q}$ to $\mathcal{Q}+1$. We again have

$$U\Psi[A] = e^{i\theta}\Psi[A] \ . \tag{3.3.16}$$

- There are instantonic solutions that interpolate between two vacua with different $\mathcal{Q}$ as $t = x_4$ passes from $-\infty$ to ∞: the instantons analyzed in the previous Section.

Working in analogy with the previous example it is reasonable to write down the path integral in this way:

$$K(A_i, A_f; x_4) = \sum_{n\in\mathbb{Z}} e^{in\theta} K_n(A_i, A_f; x_4) \ , \tag{3.3.17}$$

where K_n is given by the following path integral restricted only to configurations with instanton number n (forget about the gauge fixing and the ghosts for this simple discussion). Then

$$K_n(A_i, A_f; x_4) = \int \mathcal{D}A \, e^{-S_{\mathrm{YM}}}\Big|_{\mathcal{Q}=n} \ . \tag{3.3.18}$$

But we already know that the instanton number n can be written as a total derivative in this way:

$$n = \frac{1}{32\pi^2}\int \mathrm{d}^4x \, \varepsilon^{\mu\nu\rho\sigma} \mathrm{Tr}\, F_{\mu\nu}F_{\rho\sigma} = \frac{1}{32\pi^2}\int_{S^3} \mathrm{Tr}\, \left[A\wedge F - \frac{i}{3} A\wedge A\wedge A\right] \ . \tag{3.3.19}$$

The action must be modified by

$$S_{\mathrm{YM}} \ \rightarrow \ S'_{\mathrm{YM}} \equiv S_{\mathrm{YM}} + \frac{i\theta}{32\pi^2}\int \mathrm{d}^4x \, \varepsilon^{\mu\nu\rho\sigma} \mathrm{Tr}\, F_{\mu\nu}F_{\rho\sigma} \ . \tag{3.3.20}$$

The new term is called the *topological term* or the *θ term*. So we have just seen that the θ parameter appears naturally also in QCD. it is superselected (3.3.3) and it defines the vacuum energy. Being a free parameter of the theory it is interesting from the physical point of view to be able to measure it. The first step is to find some observables that depend directly on θ and hope that there exist experiments capable of measuring them. One of these observables is the Neutron electric dipole moment, which is the main subject of this thesis. This is essentially the motivation of the present work. We will not discuss this further because it is addressed in detail in the next Chapters.

As promised we also give a field theoretical proof of the presence of the θ term in the Lagrangian. This proof is based on the *cluster decomposition principle*, which is a fundamental requirement that any consistent field theory should satisfy.

The cluster decomposition principle states: if multi particle processes $\alpha_1 \to \beta_1$, $\alpha_2 \to \beta_2, \dots$ are studied in N very distant laboratories then the S–matrix element between them factorizes. Defining the S–matrix element

$$S_{\alpha_1\alpha_2\ldots\to\beta_1\beta_2\ldots} \equiv \langle\Psi_{\alpha_1\alpha_2\ldots}|e^{-it\mathcal{H}}|\Psi_{\beta_1\beta_2\ldots}\rangle \ , \tag{3.3.21}$$

we have

$$S_{\alpha_1\alpha_2\ldots\to\beta_1\beta_2\ldots} = S_{\alpha_1\to\beta_1}\cdot S_{\alpha_2\to\beta_2}\cdots \ . \tag{3.3.22}$$

A theory that does not satisfy this requirement is inconsistent because that would mean that an outcome of an experiment made here would depend on an infinity of other experiments performed arbitrarily far away.

Let us again write the path integral in terms of an unknown weight function of the instanton number

$$K(q_i, q_f; t) = \sum_{n\in\mathbb{Z}} w(n) K_n(q_i, q_f; t) \ . \tag{3.3.23}$$

Suppose we have an operator localized in a volume Ω, of which we are interested in the expectation value

$$\langle\mathcal{O}\rangle = \frac{\sum_n w(n)\int_n \mathcal{D}A\, e^{-S_{\mathrm{YM}}}\mathcal{O}[A]}{\sum_n w(n)\int_n \mathcal{D}A\, e^{-S_{\mathrm{YM}}}} \ . \tag{3.3.24}$$

Since the operator is localized in Ω we can divide the integral in two regions:

$$S_{\mathrm{YM}} = \int_\Omega \mathcal{L}_{\mathrm{YM}} + \int_{\overline{\Omega}} \mathcal{L}_{\mathrm{YM}} \equiv S^{\Omega}_{\mathrm{YM}} + S^{\overline{\Omega}}_{\mathrm{YM}} \ , \tag{3.3.25}$$

and, using the fact that the instanton number is additive, we can assume that in Ω there are n_1 units of instanton number and in $\overline{\Omega}$ there are n_2 units, such that $n = n_1 + n_2$. We obtain

$$\langle\mathcal{O}\rangle = \frac{\sum_{n_1,n_2} w(n_1+n_2)\int_{n_1}\mathcal{D}A\, e^{-S^{\Omega}_{\mathrm{YM}}}\mathcal{O}[A]\int_{n_2}\mathcal{D}A\, e^{-S^{\overline{\Omega}}_{\mathrm{YM}}}}{\sum_{n_1,n_2} w(n_1+n_2)\int_{n_1}\mathcal{D}A\, e^{-S^{\Omega}_{\mathrm{YM}}}\int_{n_2}\mathcal{D}A\, e^{-S^{\overline{\Omega}}_{\mathrm{YM}}}} \ . \tag{3.3.26}$$

Since the operator is localized in Ω, as a consequence of the cluster decomposition principle, the expectation value should not depend on what happens on $\overline{\Omega}$. Hence the numerator and the denominator should factorize so that $\int \mathcal{D}A e^{-S^{\overline{\Omega}}_{\mathrm{YM}}}$ cancels. This is possible only if

$$w(n_1+n_2) = w(n_1)w(n_2) \ . \tag{3.3.27}$$

The only solution to this equation is (modulo a completely irrelevant constant multiplicative factor)

$$w(n) = e^{in\theta} \ . \tag{3.3.28}$$

With the same trick explained before we can introduce this phase in the action and the same result as before is obtained.

The topological term defined above

$$\frac{i\theta}{32\pi^2}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\, F_{\mu\nu}F_{\rho\sigma}\ , \tag{3.3.29}$$

has a peculiarity which will be fundamental in the following analysis. It breaks two of the discrete symmetries: parity P and time reversal T. This is simple to see since the Levi–Civita tensor transforms under a Lorentz transformation $\Lambda^{\nu}{}_{\mu}$ in this way

$$\varepsilon_{\mu\nu\rho\sigma} \to \det\Lambda\, \varepsilon_{\mu\nu\rho\sigma}\ . \tag{3.3.30}$$

While it is invariant under $SO(1,3)$, it changes sign under P and T. The charge conjugation is obviously a symmetry because it is quadratic in F. A priori we could say that the theory is CP conserving for $\theta = 0$ or π, however there are physical observations that exclude the $\theta = \pi$ case (see for instance [37]). The present observations suggest either $\theta = 0$ or θ very small, of order $\sim 10^{-10}$. It is rather puzzling for a fundamental constant to be this small, this is usually referred to as the *Strong CP problem.*

If we want to find observables capable of giving us information about θ we must find CP breaking quantities. The Neutron electric dipole moment is one of them (see Chapter 6) and currently it gives the most stringent bound on θ. Being able to compute it with high precision is important in order to render even more rigorous these experimental bounds.

3.4 Chiral symmetry

So far we have only considered the Yang–Mills term of $\mathcal{L}_{\mathrm{QCD}}$; now it is time to introduce the quarks. Quarks come into 6 different species, called *flavors*: up, down, charm, strange, top and bottom. In principle all of them are important but if we concentrate on the low energy regime the heavier quarks will be almost non dynamical. Then it is possible to exclude them from the degrees of freedom of the theory. Usually suitable low energy limits include $N_f = 2$ (up and down) or $N_f = 3$ (up, down and strange) flavours. Under the Lorentz group, the quarks are Dirac fermions, so we can decompose them into the Weyl basis

$$q_L = \frac{1-\gamma_5}{2}q\ , \qquad q_R = \frac{1+\gamma_5}{2}q\ . \tag{3.4.1}$$

If we define $Q = (q_1, \dots q_{N_f})$ the N_f–plet of quarks, the Lagrangian term is easily written

$$\mathcal{L}_q = \overline{Q}\left(i\not{D} - M\right)Q\ , \tag{3.4.2}$$

where flavor indices are contracted and $M = \mathrm{diag}\,(m_f)$ is the quark mass matrix. The term $\mathcal{L}_q$ clearly enjoys a $[U(1)]^{N_f}$ symmetry, where each flavor has a conserved number associated. If M is proportional to the identity matrix this symmetry is enhanced to a $U(N_f)$ symmetry

$$U(N_f)_V = U(1)_B \otimes SU(N_f)_V \ , \tag{3.4.3}$$

acting as $Q \to UQ$ with $U \in U(N_f)$. The abelian subgroup is simply the baryon number symmetry. There is also a further generalization of the symmetry when $M = 0$. In this case left and right chiral components decouple and we can transform them independently. The group is

$$U(N_f)_L \otimes U(N_f)_R \ , \tag{3.4.4}$$

acting on Q as

$$Q_L \to AQ_L \ , \quad Q_R \to BQ_R \ , \qquad A, B \in U(N_f) \ . \tag{3.4.5}$$

The vectorial group $U(N_f)_V$ is clearly a subgroup, given by $A = B$. The complement (which is *not* a group) is usually denoted $U(N_f)_A$, corresponding to the transformations such that $A = B^\dagger$. The abelian subgroup $U(1)_A$, composed by the transformation of $U(N_f)_A$ generated by the identity, acts on Q simply as $Q \to e^{i\gamma_5 \alpha} Q$, $\alpha \in [0, 2\pi]$. This is a purely classical treatment and it is far from being applicable to the quantum theory. In particular we have a breaking of $U(N_f)_L \otimes U(N_f)_R$ due to two different quantum effects.

Let us briefly review how can a symmetry of the classical action be realized in the quantum theory. There are essentially three different situations.

Symmetry realized à la Wigner–Weyl A symmetry G with generators Q_a is realized in this way if there is an unitary operator U in the Hilbert space that induces the transformation on the physical states,

$$U(g)|\Psi\rangle = |g\Psi\rangle \ , \qquad g \in G \ . \tag{3.4.6}$$

A necessary condition is that the vacuum must be left invariant under the action of Q_a: $Q_a|0\rangle = 0 \ \forall\, Q_a$. The conservation of the Noether current, which is a classical requirement, has a quantum analogous: the Ward identities. If we compute an expectation value of a monomial of fields times the divergence of the current, such as

$$\mathcal{A} = \langle \varphi(x_1) \cdots \varphi(x_n)\, \partial_\mu J^\mu_a(y) \rangle \ , \tag{3.4.7}$$

the result is not zero but is a sum of "*contact terms*", given by

$$\mathcal{A} = \sum_{j=1}^{n} \delta(y - x_j) \langle \varphi(x_1) \cdots (Q_a \varphi(x_j)) \cdots \varphi(x_n) \rangle \ . \tag{3.4.8}$$

Spontaneous symmetry breaking A spontaneous breaking arises when the fundamental state of the theory does not have the same symmetries of the action. In this case the system is forced to "choose" a direction in the broken subgroup. As a consequence there are, say, k generators such that $Q_a|0\rangle \neq 0$ for $a = 1, \ldots k$. The Ward identities for the Noether current conservation remain intact also at the quantum level. A remarkable consequence of this situation goes under the name of *Nambu–Goldstone theorem*. It states that when a theory with symmetry G exhibits a spontaneous breaking $G \to H \subsetneq G$, the spectrum contains massless scalar particles with the same quantum numbers of $Q_a|0\rangle$, Q_a, $a = 1, \ldots, k$ being the generators of G/H.

Anomalous symmetry breaking The quantization procedure can break a symmetry in another way: a violation of the Ward identities for the Noether currents J^μ_a. In the functional formalism this is a direct consequence of a non invariance of the measure $[\mathcal{D}\varphi]$. In the canonical formalism instead it can be seen as a divergence in triangular diagrams related to the current J^μ_a, whose counterterms break the symmetry of the renormalized action. Either way we have a smaller symmetry in the physical observables. This behaviour is fundamentally different from the spontaneous breaking case because we do not have any Goldstone mode appearing.

To be rigorous we do not have any of these symmetries in the real world because $M \neq 0$. However we can think in terms of a perturbative approach in M. If the symmetry is exact at $M = 0$ the real theory should be the one at $M = 0$ plus corrections at $\mathcal{O}(M)$ for small masses. In this case small has a precise meaning: it means $M \ll \Lambda_{\mathrm{QCD}}$, where the latter is the energy scale of QCD arising from the the renormalization group flow. For instance, if the chiral symmetric theory has massless Goldstone modes, then the real theory would show some "quasi"–Goldstone pseudoscalar particles with a very small mass. This approach is called *Chiral perturbation theory*. Let us now see what are the surviving symmetries of the quantum theory:

- Spontaneous symmetry breaking. The group $SU(N_f)_L \otimes SU(N_f)_R$ is spontaneously broken to the diagonal subgroup

$$SU(N_f)_L \otimes SU(N_f)_R \to SU(N_f)_V \; . \tag{3.4.9}$$

  The Goldstone modes associated to this breaking are called *pions*. They are rigorously massless pseudoscalar particles only when the mass matrix M is zero. For quark masses much smaller than Λ_{QCD} we can expect them to have a small mass. In fact experimentally $m_{\pi^0} = 135$ MeV, while $\Lambda_{\mathrm{QCD}} = 217$ MeV (in the MS subtraction scheme).

- Anomalous breaking. We have a similar breaking pattern

$$U(1)_L \otimes U(1)_R \to U(1)_B \; . \tag{3.4.10}$$

  The axial abelian group $U(1)_A$ breaking cannot be described in terms of a spontaneous breaking. If this was the case the η' meson would be the "quasi"–Goldstone mode associated to it, but we observe that $m_{\eta'}$ is too big to be considered as such. This was noted by Weinberg [10]: assuming the η' was a pseudo–Goldstone boson, he found an upper limit on its mass, which was experimentally exceeded. The anomaly associated to this transformation had already been computed in electromagnetism [75, 76] (to explain the otherwise prohibited decay $\pi^0 \to \gamma\gamma$). Even though it is a different physical phenomenon the computation is identical since both are $U(1)$ groups. The result is very interesting so we will discuss it now.

Under a transformation $U(1)_A$ such that

$$Q \to Q' = e^{i\gamma^5\alpha} Q \ , \qquad \alpha \in [0, 2\pi] \ , \tag{3.4.11}$$

we have a non invariance of the functional measure (this is one way to see it). This feature is expressed in the following way:

$$\int [\mathcal{D}Q][\mathcal{D}\overline{Q}][\mathcal{D}A] \, e^{iS_{\mathrm{QCD}}} \longrightarrow \int [\mathcal{D}Q'][\mathcal{D}\overline{Q}'][\mathcal{D}A] \, e^{iS'_{\mathrm{QCD}}} \ . \tag{3.4.12}$$

The action is invariant by definition. We can exponentiate the variation of the measure $[\mathcal{D}Q][\mathcal{D}\overline{Q}]$ to get the anomaly $\mathscr{A}$

$$\int [\mathcal{D}Q][\mathcal{D}\overline{Q}][\mathcal{D}A] \, e^{iS_{\mathrm{QCD}} + i\int \mathrm{d}^4x \, \alpha \mathscr{A}} \ . \tag{3.4.13}$$

It goes under the name of Adler–Bell–Jakiw anomaly and its expression is

$$\mathscr{A} = \frac{N_f}{16\pi^2} \varepsilon^{\mu\nu\rho\sigma} \mathrm{Tr}\, F_{\mu\nu} F_{\rho\sigma} = 2N_f \mathcal{Q} \ . \tag{3.4.14}$$

We have already encountered this expression! It is the topological term: as a result a $U(1)_A$ transformation with parameter α is *equivalent* to the following shift in the θ parameter

$$\theta \to \theta' = \theta + 2N_f\alpha \ . \tag{3.4.15}$$

It follows that θ is not really a parameter of the theory: first of all if *at least one* of the quarks is massless, say the u, then a $U(1)_A$ rotation on the flavour u would be a symmetry of the theory. Choosing $\alpha = \theta/2$ the parameter θ completely disappears. If on the other hand all quarks are massive, then the true physical combination that can be observed in the laboratory is

$$\theta_{\mathrm{phys}} = \theta + \arg\det M \ , \tag{3.4.16}$$

because $\arg\det M$ is the sum of the phases $e^{i\varphi}$ of the mass matrix elements (and it is well defined only for $\det M \neq 0$).

3.5 Large *N* expansion

As mentioned in the introduction, the perturbative approach, which is the expansion in powers of $g^2_{\rm YM}$, works only at high energies (i.e. energies much greater than $\Lambda_{\rm QCD}$, whose definition can be found below). When flowing to smaller energies, the coupling $g_{\rm YM}$ grows because it is not a Renormalization Group (RG) invariant. Hence the perturbative expansion is doomed to break down eventually. One possible way to study the low energy regime is to find another expansion parameter or, equivalently, to sum the diagrams in a different way. Strictly speaking $g_{\rm YM}$ is not a free parameter in QCD because of dimensional transmutation: due to the RG, in fact, $g_{\rm YM}$ is reabsorbed into defining the scale masses, so we must find a not obvious free parameter. QCD is a non abelian theory with gauge group $SU(N_c = 3)$ where N_c is the number of colours. Let us consider N_c as an arbitrary parameter. Of course if $N_c \neq 3$ we are dealing with a different theory, but perhaps the behaviour at large N_c is qualitatively and quantitatively similar to $N_c = 3$. This idea was born in the work by 't Hooft [9]. We will try to motivate this expansion (that at first sight might appear slightly nonsensical) and briefly show the main results.

First of all we must bear in mind that different N_c correspond to different theories. If we want to compare them what should we keep fixed? The physics is mostly dictated by the energy scale $\Lambda_{\rm QCD}$. It is the RG invariant

$$\mu \exp\left(\frac{1}{2\beta_0 g^2_{\rm YM}(\mu)}\right) = \Lambda_{\rm QCD} \ , \tag{3.5.1}$$

where β_0 is the first coefficient of the β function

$$\beta_0 = -\frac{1}{16\pi^2}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right) \ . \tag{3.5.2}$$

The number of flavours N_f is fixed so at large N_c we have $\beta_0 \propto N_c$. The correct large N_c limit ('t Hooft limit) is thus

$$N_c \to \infty \ , \quad g^2_{\rm YM} \to 0 \ ; \qquad \lambda \equiv g^2_{\rm YM} N = \text{fixed} \ , \tag{3.5.3}$$

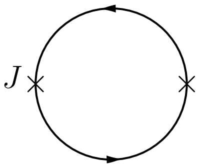
where λ is called the 't Hooft coupling and it remains fixed. To understand how this limit affects the observables we need to study the N_c scaling of the vertices and the propagators. The Largangian, ignoring the gauge fixing, is the following

$$\mathcal{L}_{\rm YM} = \frac{1}{g^2_{\rm YM}}\left[\frac{1}{2}A^a_\mu(\Box g^{\mu\nu} - \partial^\mu\partial^\nu)A^a_\nu + f_{abc}\partial^\mu A^\nu_a A^b_\mu A^c_\nu - \frac{1}{4}f_{abc}f_{ade}A^b_\mu A^c_\nu A^\mu_d A^\nu_e\right] \ . \tag{3.5.4}$$

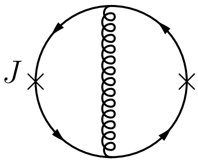
All vertices scale as $1/g^2_{\mathrm{YM}}$ and the gluon propagator is $\propto \delta^{ab} g^2_{\mathrm{YM}}$. With this normalization of the gauge field, no explicit g_{YM} appears in the quark propagators. Let us compute an expectation value of two insertions of an operator J:

$$\langle JJ\rangle(k) = \int \mathrm{d}^4x\, e^{ik\cdot x} \langle 0|J(x)J(0)|0\rangle \ , \tag{3.5.5}$$

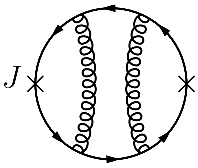
where J is a fermionic bilinear, for instance ($J = \overline{Q}\Gamma Q$). We have many choices as intermediate states, let us just see some of them.



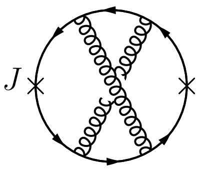
$$J \sim \mathrm{Tr}\,[\mathbb{1}] = \mathcal{O}(N_c) \ ,$$



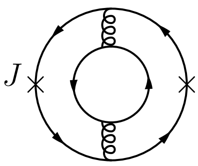
$$J \sim g^2_{\mathrm{YM}} \mathrm{Tr}\,[T_a T_b]\delta_{ab} = \frac{\lambda}{N_c} C_F N_c = \mathcal{O}(N_c) \ ,$$



$$J \sim g^4_{\mathrm{YM}} \mathrm{Tr}\,[T_a T_b T_b T_a] = C_F^2 \frac{\lambda^2}{N_c^2} N_c = \mathcal{O}(N_c) \ , \tag{3.5.6}$$



$$J \sim g^4_{\mathrm{YM}} \mathrm{Tr}\,[\underbrace{T_a T_b T_a}_{-\frac{1}{2N_c}T_b} T_b] \propto C_F \frac{\lambda^2}{N_c^2} = \mathcal{O}(1/N_c) \ ,$$



$$J \sim g^4_{\mathrm{YM}} \big(\mathrm{Tr}\,[T_a T_b]\big)^2 \propto \frac{\lambda^2}{N_c^2}(N_c^2 - 1) T_F^2 = \mathcal{O}(1) \ .$$

> Here follows a brief summary of some useful $SU(N)$ identities. Given an irreducible representation ρ with generators: $T^a_{(\rho)}$, $a = 1, \ldots, N^2 - 1$, we can define the Dynkin index as
>
> $$\mathrm{Tr}\, T^a_{(\rho)} T^b_{(\rho)} = T(\rho)\, \delta_{ab} \ . \tag{3.5.7}$$
>
> The quadratic Casimir operator is defined as:
>
> $$C^{(2)} = \sum_{a=1}^{N^2-1} T^a_{(\rho)} T^a_{(\rho)} \equiv \mathbb{1}_{d_\rho} C(\rho) \ . \tag{3.5.8}$$
>
> It will be always proportional to the identity (where $\dim \rho = d_\rho$) in an irreducible

representation for the Schur's Lemma. The following relation always holds:

$$T(\rho) = \frac{1}{N^2-1} d_\rho C(\rho) \; . \tag{3.5.9}$$

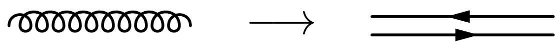
In the fundamental representation we have

$$T(\mathbf{fund}) = T_F = \frac{1}{2} \; , \qquad C(\mathbf{fund}) = C_F = \frac{N^2-1}{2N} \; , \tag{3.5.10}$$

while for the adjoint

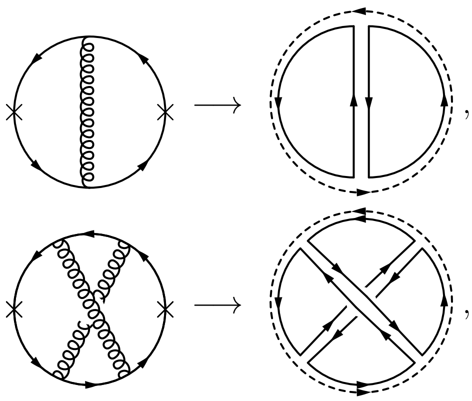
$$T(\mathbf{adj}) = T_G = N \; , \qquad C(\mathbf{adj}) = C_G = N \; . \tag{3.5.11}$$

We could keep going, but it is better to learn a more systematic way to get the correct factors. It is easy to put the $g_{\rm YM}$ to the right power but the N_c coming from the traces are not so immediate, switching to this new notation is of great help. The gauge field A^a_μ can be seen as the product of a fundamental and an antifundamental[2] $(A_\mu)^i_j$, so we can draw the gluon propagator as two colour lines going in opposite directions.



(3.5.12)

With this simple idea it is clear how to count the factors N_c coming from the traces: they are the number of closed color line loops. See for instance



(3.5.13)

where the external dashed line is a loop of flavour to remind us that the single line comes from the quarks. While the first diagram has two colour loops, the second only has one. This explains the difference in the N_c scaling. Calling P the number of propagators, $V = V_3 + V_4$ the total number of gluon vertices and I the number of internal colour loops we find

$$\text{Diagram} \sim g_{\rm YM}^{2P-2V} N_c^I = \lambda^{P-V} N_c^{I-P+V} \; . \tag{3.5.14}$$

The power of N_c has a very nice geometrical interpretation (in terms of the

[2] Actually the product $\mathbf{fund} \otimes \overline{\mathbf{fund}} = \mathbf{1} \oplus \mathbf{adj}$, where the $\mathbf{1}$ is the trace degree of freedom, for large N_c this difference is negligible.

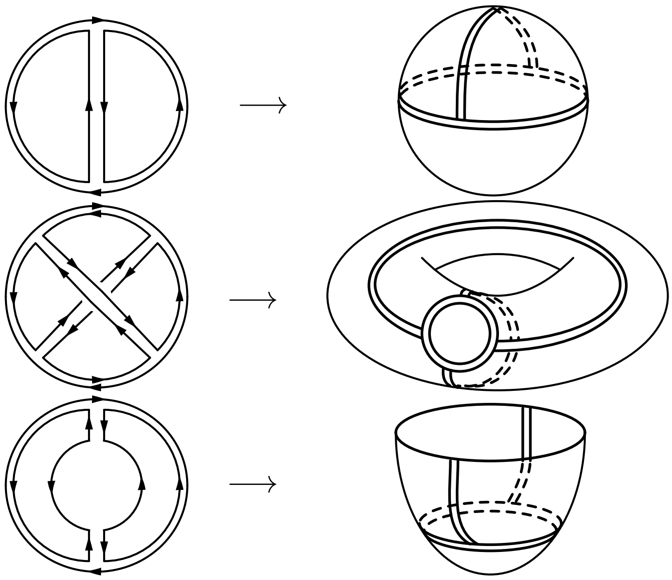


Figure 3.2: Graphical representation of the Feynman graphs interpreted as a surface with genus g. The case of genus 0 (Sphere), genus 1 (Torus) and open surface (Disk) are presented. The first is leading order $\mathcal{O}(N_c^2)$, the second $\mathcal{O}(1)$ and the third $\mathcal{O}(N_c)$.

theory of graphs). Consider each loop as a face, together with the whole region external to the diagram, and consider each colour line as the boundary of a face. For the moment let us restrict to diagrams without fermion loops. If we add the point at infinity the graph closes in a solid figure (see Figure 3.2) and the quantity

$$I - P + V = \text{Faces} - \text{Edges} + \text{Vertices} = \chi_E = 2 - 2g \ , \tag{3.5.15}$$

is the Euler characteristic of the surface. It is equal to $2 - 2g$ where g is the genus (number of "handles") of the surface. We have already encountered it in the loop expansion of the string amplitudes. There is indeed a strong relation between the large N_c expansion in QCD and the worldsheet genus expansion in string theory amplitudes. In the former the genus is related to the Feynman diagram topology and the expansion parameter is $1/N_c$, while in the latter it is related to the worldsheet topology and the expansion parameter is g_s. This suggests to investigate the possibility of a string theory dual of QCD (at least at low energies).

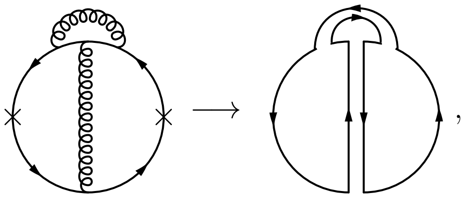
If we include the case of fermion loops the result is not too different: since the fermion propagator has a single line it has to be seen as boundary of the resulting surface, which means that the surface is missing a face. The Euler characteristic is lowered by one so the final result is

$$I - P + V = 2 - 2g - \big(\text{number of fermion loops}\big) \ . \tag{3.5.16}$$

When $g = 0$ the surface is homeomorphic to the sphere S^2 which is equal to $\mathbb{R}^2 \cup \{\infty\}$. For this reason such diagrams are called "*planar*": because we can draw them on a sheet of paper without intersections. Each crossing of lines renders the diagram non–planar and raises the genus. We have learned that:

- Each "crossing" suppresses the N_c scaling by a power two,
- each fermion line suppresses the N_c scaling by a power one.

The correlator $\langle JJ \rangle$ studied above and the pure gluon diagrams differ by an overall power of N_c but they both have a dependence N_c^{2-2g-f} (where f are the fermion loops). There is just another minor difference that we should point out: if in a quark bilinear we draw a gluon propagator *outside* the edge defined by the quark lines we get a subleading contribution. Consider for instance the planar diagram



(3.5.17)

this does not have a leading N_c scaling as the other planar diagrams.

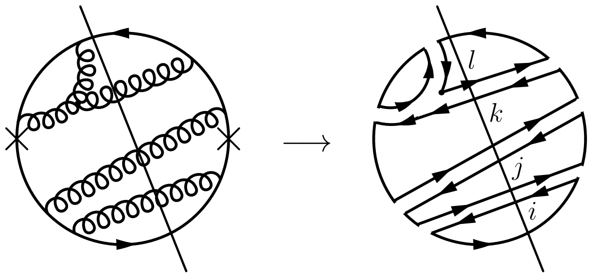
To summarize, the leading diagrams are planar diagrams with the minimum number of fermion lines and, if they correspond to a quark bilinear correlator, with only quarks in the edge. In this sense the large N_c expansion is just another way to rearrange the Feynman diagrams. Hence it is justified to adopt this technique also for N_c not too big. This of course is not the only motivation of the large N_c limit: we can have a physical confirmation that all of this makes sense. Consider for this purpose only the leading contribution to a quark bilinear correlator $\langle JJ \rangle$. We can always think this correlator in terms of a sum on the intermediate states

$$\langle JJ \rangle = \sum_n \langle 0|J|n\rangle\langle n|J|0\rangle \ . \tag{3.5.18}$$

These states can, a priori, be single particle states or multi particle states. It is possible to understand their matter composition by looking at the diagrams that contribute to the leading behaviour of $\langle JJ \rangle$. In particular we can cut the diagram in two and see what are the particles in both halves. It turns out that only single particle states appear, which can be rephrased as

$$\langle JJ \rangle(k) = \sum_n \frac{a_n^2}{k^2 - m_n^2} \ , \tag{3.5.19}$$

i.e. the correlator has only poles. But there is more: we could considers also three point correlators $\langle JJJ \rangle$, four points $\langle JJJJ \rangle$ etc... and the striking result is that all the contributions are poles in each channel. In the three point function $\langle JJJ\rangle(p,q,r)$ we could have two simultaneous poles in two channels p and q for instance, or three simultaneous poles: a three legged vertex. The same goes for the four point function. It turns out that all these results are consistent with unitarity and crossing symmetry. What we are dealing with is a local Largangian where the degrees of freedom are $\overline{q}q$ states, which are called *mesons*. This is precisely what we see in the chiral effective theory description of QCD, which is based on completely different assumptions (we will analyse this topic in the next section). For brevity and for the fact that the idea is much similar in the general case, we will prove this set of statements only for the two point correlator. The idea is very simple: let us cut the diagram in half in an arbitrary way



. (3.5.20)

Of course we have drawn a planar diagram with internal gluons. These are the only diagrams allowed in the $N_c \to \infty$ theory. As it can be easily seen the intermediate states that run across the cut are

$$\overline{q}_l A^l_k A^k_j A^j_i q^i \ . \tag{3.5.21}$$

We must now remember that QCD is a *confining* theory. This means that the only possible asymptotic states are colourless ($SU(N_c)$ singlets). The only possible way to build a singlet in this case is to consider the whole product, hence we have a single particle. On the other hand, states like this

$$\overline{q}_l A^l_k q^k A^i_j A^j_i \ , \tag{3.5.22}$$

namely a meson and a glueball, will never appear in planar diagrams. As it is well known, single particle states correspond to poles in the two point function. The argument goes in a similar way for higher correlators. A more thorough discussion of this can be found in [77].

3.5.1 θ dependence in the large N approach

Let us write the Euclidean action of Yang–Mills as

$$S_{\text{YM}} = \frac{N_c}{\lambda}\int \mathrm{d}^4x\, \mathrm{Tr}\left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - i\frac{\lambda}{32\pi^2}\frac{\theta}{N_c}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right) \ . \tag{3.5.23}$$

As we can see the θ dependence comes through the combination θ/N_c. The large N_c limit thus has to be taken keeping θ/N_c fixed. The vacuum energy density is proportional to N_c^2 (the number of degrees of freedom), thus it can be expressed as

$$\varepsilon(\theta) = N_c^2\, h(\theta/N_c)\ , \tag{3.5.24}$$

where h is a regular function that has to satisfy the property of invariance under shifts $\theta \to \theta + 2\pi$. It follows that h is either a constant or a multi-branched function [47, 13] given by

$$h(\theta/N_c) = \min_{k\in\mathbb{Z}} \tilde{h}((\theta + 2\pi k)/N_c)\ , \tag{3.5.25}$$

for some regular function $\tilde{h}$. The vacuum always chooses the branch in which the energy is minimized. As a result we have a periodic energy density with possible singularities when there is a jump from one k to another.

We define the topological susceptibility of Yang–Mills as

$$\chi_g = \left.\frac{\mathrm{d}^2\varepsilon(\theta)}{\mathrm{d}\theta^2}\right|_{\theta=0}\ . \tag{3.5.26}$$

In the Euclidean path integral formulation $\varepsilon(\theta)$ can be defined as

$$\exp(-\varepsilon(\theta)) = \int [\mathcal{D}A]\ \exp\left(-\frac{N_c}{2\lambda}\int \mathrm{d}^4x\, \mathrm{Tr}\, F^2 - i\theta\int \mathrm{d}^4x\, \mathcal{Q}\right)\ , \tag{3.5.27}$$

where $\mathcal{Q}$ is the topological charge density (3.2.15). By taking two derivatives w.r.t. θ one can show that χ_g is related to the two point function of $\mathcal{Q}$ by

$$\chi_g = \frac{1}{\mathcal{V}_4}\int \mathrm{d}^4x \langle \mathcal{Q}(x)\mathcal{Q}(0)\rangle\ , \tag{3.5.28}$$

where $\mathcal{V}_4$ is the 4 dimensional volume, serving as an IR cutoff. It is easy to show that $\varepsilon(\theta)$ must have a minimum in $\theta = 0$. In fact for any integral this inequality holds

$$\left|\int f\right| \leq \int |f|\ . \tag{3.5.29}$$

Hence, since the θ term is just a phase, one finds

$$\varepsilon(\theta) \geq \varepsilon(0)\ . \tag{3.5.30}$$

For small θ mod 2π we find this expression for the energy density:

$$\varepsilon(\theta) \simeq \frac{1}{2}\chi_g \min_{k\in\mathbb{Z}}(\theta + 2k\pi)^2 + \mathcal{O}(1/N_c)\ . \tag{3.5.31}$$

One of the most important results of the large N_c approach is the so–called Witten–Veneziano relation [11, 12]. We limit ourselves to enunciate the formula without the proof:

$$m_{\eta'}^2 = \frac{2N_f}{f_\pi^2}\chi_g \ , \tag{3.5.32}$$

where f_π is the pion decay constant, i.e. the matrix element of the axial current between the pion and the vacuum. A part form being an interesting result per se, it is also an ending to the famous $U(1)_A$ problem. In fact, before the discovery of the axial anomaly, it was believed that the η' meson should be a Goldstone boson of the spontaneously broken $U(1)_A$ symmetry. This idea wasn't experimentally supported because of its large mass: $m_{\eta'} = 958$ MeV. The above formula (whose proof indeed uses the anomaly argument) gives a non zero mass to the η' also in the chiral limit $m_q \to 0$, so the problem is solved.

At $N_c = \infty$ the $U(1)_A$ anomaly disappears and η' becomes a Goldstone boson. This fact is consistent with the result because we have $f_\pi \sim \mathcal{O}(\sqrt{N_c})$, hence $m_{\eta'}^2 \sim \mathcal{O}(1/N_c)$.

3.5.2 Large *N* Baryons

In the paper [77] Witten shows also how to fit baryons in this picture. We will just give a few remarks about the main results. Baryons in the large N_c limit are bound states of N_c quarks (the totally antisymmetric combination). The main difference from mesons is thus the fact that the number of constituents is not fixed but grows with N_c. This may be a disaster if one considers the N_c scaling of the diagrams, made by N_c parallel lines on the left and other N_c parallel lines on the right (with the same orientation if one is interested in baryon–baryon interaction and with opposite orientation for baryon–antibaryon). It turns out that the first contributions *grow* with N_c, and the situation becomes even worse at higher orders. This is a result of the combinatoric factors arising from the high number of components, in fact there are $\frac{1}{2}N_c(N_c-1)$ pairs of quarks in the baryon. The correct approach is completely different, it is based on the simple observation that the mass of a baryon is the sum of the mass of its constituents plus their average kinetic energies plus a potential (which is of order $1/N_c$) summed over the pairs, so we have:

$$\begin{aligned} M_B &= N_c m_q + N_c K + \frac{N_c(N_c-1)}{2}\frac{V_0}{N_c} = \\ &= N_c(m_q + K + V_0) \ . \end{aligned} \tag{3.5.33}$$

From this, two observations follow: first of all the baryon becomes very massive at large N_c, hence a non relativistic description becomes reliable; secondly as N_c

factorizes it appears as an overall factor in the Hamiltonian $\mathcal{H} = N_c H$, so we can study directly H, by means of, for instance, an Hartree–Fock method.

AdS/CFT has provided the first explicit realization of the relation between the large N_c limit of a gauge theory and a string model. The road for a string description of QCD is still long, but the first steps can actually be obtained by deformations of conformal models. The Witten–Sakai–Sugimoto model we will focus on in the next chapters is a prominent example in this class. Many of the expected large N_c features of QCD, including the Witten–Veneziano relation, naturally emerge in this framework.

3.6 Chiral effective Lagrangian

We will present a brief review of the chiral effective Lagrangian. It will not be used in this work but it will be useful to know the basics since the Sakai–Sugimoto model (that we will introduce in the next chapter) reduces to this one in the low energy limit. We will also introduce the chiral effective description of Baryons, which goes under the name of Skyrme model. A review on the chiral Lagrangian is [8], see also [13] for the large N approach and [14] for the Skyrme model.

The most natural way to deal with a theory in its low energy regime is to consider only the effective degrees of freedom. Those are the ones which dominate in the energy range of our interest. For QCD this is the case of *pions*. Ideally we are performing a partial integration of the system, but this is practically impossible so we must necessarily build the most general local Lagrangian satisfying all the internal symmetries of the fundamental theory ($\mathcal{L}_{\text{QCD}}$) as well as CPT and Lorentz invariance. Only a few terms will be important if we adopt a perturbative approach on the number of derivatives

$$\mathcal{L}_{\text{eff}} \sim \mathcal{O}(\partial^2) + \mathcal{O}(\partial^4) + \cdots \ . \tag{3.6.1}$$

This expansion in justified because the pion coupling is dominant only at low energies (*pion pole dominance hypothesis*). Typically effective Lagrangians are not renormalizable, so we have to consistently include all terms up to the correct number of derivatives and renormalize each of them up to the correct number of loops. Each loop ℓ raises the order in the derivatives by a factor $p^{\ell(d-2)}$ (d being the number of dimensions). For example, let us call $\mathcal{L}_{\text{eff}}^{(m)}$ the chiral effective Lagrangian up to $\mathcal{O}(\partial^m)$. If we want to do a computation at $\mathcal{O}(\partial^4)$, then we would have to use tree diagrams from $\mathcal{L}_{\text{eff}}^{(4)}$ and tree and one loop diagrams from $\mathcal{L}_{\text{eff}}^{(2)}$, hence this last term must contain counterterms up to one loop and the first needs no counterterm.

The symmetry that plays a key role in the building of this effective theory is the chiral symmetry explained before. The fact that the symmetries are realized non linearly in these contexts is a general feature. Without going through the proof of the generality of this result we will just give the final answer. The pion matrix is defined as

$$U = \exp\left(\frac{2i}{f_\pi}\sum_{a=1}^{N_f^2-1}\pi_a t_a\right) \ , \tag{3.6.2}$$

where f_π is the pion decay constant, π_a the pion fields and t_a the generators of $SU(N_f)$. The transformation law under a chiral transformation $(L,R) \in SU(N_f)_L \otimes SU(N_f)_R$ is

$$U \to LUR^\dagger \ . \tag{3.6.3}$$

If $N_f = 3$ the first two pions are a mixing of $\pi^\pm$, the π_3 and π_8, a mixing η and π^0, and $\pi_{4,5,6,7}$ a mixing of $K^\pm$ and K^0, $\overline{K}^0$. If the singlet η' becomes important in some physical process it can be easily included by enlarging the group from $SU(N_f)$ to $U(N_f)$ (so that one of the t_a is $\mathbb{1}/\sqrt{2N_f}$), the $U(1)$ factor corresponding to the η'. This works only in the approximation in which $f_\pi \simeq f_S$ (the decay constant of the singlet), which is a result strictly valid only in the limit $N_c \to \infty$, otherwise η' is just another pseudoscalar field in the theory. It is called chiral singlet but it transforms under $U(1)_A$ in this way

$$\eta' \to \eta' + \alpha\sqrt{2N_f}f_\pi \ , \tag{3.6.4}$$

where α is the parameter of the transformation. To the lowest order in the derivatives we have the *non linear sigma model*

$$\mathcal{L}^{(2)}_{\text{eff}} = \frac{f_\pi^2}{4}\mathrm{Tr}\left[\partial_\mu U^\dagger \partial^\mu U\right] + \frac{1}{2}\partial_\mu\eta'\partial^\mu\eta' \ . \tag{3.6.5}$$

This theory enjoys full chiral invariance, so it is not completely realistic because we know that quarks have masses. Is there a way to recover a formal chiral invariance even in presence of a mass term? The answer is yes, we just have to assume that also the mass matrix is formally changing in this way

$$\overline{Q}_{R,L} \to \overline{Q}_{R,L}V^\dagger_{R,L} \ , \qquad Q_{R,L} \to V_{R,L}Q_{R,L} \ , \qquad M \to V_R M V_L^\dagger \ . \tag{3.6.6}$$

The mass term, assuming a general complex mass matrix, is written as

$$\overline{Q}_R M Q_L + \overline{Q}_L M^\dagger Q_R \ . \tag{3.6.7}$$

The second term has to be $M^\dagger$ for C invariance. It is easy to check that this term is invariant. If we want to emulate a mass term in the chiral effective description we just have to introduce a term that enjoys this formal invariance when also the mass matrix is allowed to change. The answer is

$$\mathcal{L}_{\text{mass}} = \frac{f_\pi^2}{4}2B\,\mathrm{Tr}\left[M(U+U^\dagger)\right] \ , \tag{3.6.8}$$

where B is a new parameter fixed by the experiments. This is one of the major drawbacks of an effective approach to field theories in general: each time we add terms we are also forced to introduce new parameters that cannot be fixed by symmetry. At a certain point the theory loses its predictive power. As we will see next, the Sakai–Sugimoto holographic model for QCD indeed reduces in a particular limit to the Largangian $\mathcal{L}_{\text{eff}} + \mathcal{L}_{\text{mass}}$, but the coefficients are fixed by AdS/CFT and are not arbitrary.

3.6.1 Skyrme model

What we have discussed above is all we need to describe (pseudo)scalar mesons at the lowest level. The Skyrme approach uses the same Largangian to describe baryons, but now the degrees of freedom we are interested in do not appear as classical fields, they arise as *solitonic solutions* of the model. A soliton is a finite energy solution of a certain model with some topological properties. They are localized in space (or in time, in that case are called *instantons*). The Lagrangian $\mathcal{L}_{\text{eff}}$ is, from a mathematical point of view, the theory of *mappings*:

$$U : \mathbb{R}^3 \to SU(N_f) \ . \tag{3.6.9}$$

The mapping itself is the canonical variable that evolves in time. The finiteness of the action imposes that U must have a constant limit when $|\vec{x}| \to \infty$. Hence U is well defined in the point at infinity. Considering $\mathbb{R}^3 \cup \{\infty\} = S^3$ in place of $\mathbb{R}^3$ we have

$$U : S^3 \to SU(N_f) \ / \sim \, = \pi_3(SU(N_f)) \cong \mathbb{Z} \ . \tag{3.6.10}$$

Again, similarly to the case of instantons, we may classify all classical solutions according to an integer number. The current associated to the conservation of this topological number is

$$B^\mu = \frac{i}{24\pi^2} \varepsilon^{\mu\nu\rho\sigma} \mathrm{Tr}\left[L_\mu L_\rho L_\sigma\right] \ , \quad \text{Where} \quad L_\mu = U^\dagger \partial_\mu U \ , \tag{3.6.11}$$

where $\partial_\mu B^\mu = 0$ and $\int \mathrm{d}^3x B^0(x) = n \in \mathbb{Z}$. The story however is not complete: if we look for solution with $n > 0$ we are not able to find stable, finite energy, configurations. This is essentially due to a scale symmetry of the model (see Derrick's theorem [78]). If we introduce a new higher derivative term this scale invariance is broken and the solution is stabilized. The only term allowed by symmetry is

$$\mathcal{L}^{(4)}_{\text{eff}} = \frac{1}{32e^2} \mathrm{Tr}\left([U^\dagger \partial_\mu U, U^\dagger \partial_\nu U]^2\right) \ , \tag{3.6.12}$$

where e is a new adimensional parameter. The solution (called *Skyrmion*) is known in the case of $SU(2)$ [39], but only numerically. Solutions in $SU(3)$ are found as an embedding of the former in the isospin subgroup. The ansatz used to solve the equation is called the *hedgehog ansatz* because it assumes a combined spin and isospin symmetry. This is a spherically symmetric ansatz given by

$$U(x) = \exp\left(i\frac{\vec{\tau}\cdot\vec{r}}{r}F(r)\right) , \tag{3.6.13}$$

where $\vec{\tau}$ are the isospin Pauli matrices and $F(r)$ is a radial function only known numerically. The symmetry properties are very easy: under spin and isospin, respectively, U transforms as

$$U(\vec{x}) \to U(R\cdot\vec{x}) , \qquad U(\vec{x}) \to AU(\vec{x})A^{\dagger} . \tag{3.6.14}$$

Calling $\vec{J}$ the current associated to spin and $\vec{I}$ the one associated to isospin we have (call $\hat{r} = \vec{x}/r$)

$$\begin{aligned} [I^a, U(\vec{x})] &= -\varepsilon^{abc}\hat{r}_b\tau_c \sin F(r) , \\ [J^a, U(\vec{x})] &= +\varepsilon^{abc}\hat{r}_b\tau_c \sin F(r) , \end{aligned} \tag{3.6.15}$$

Clearly $\vec{I} + \vec{J}$ is a symmetry of the solution.

The Skyrmion can be quantized via a time dependent isorotation $A(t)$. The process is so similar to the quantization of the baryon in the Sakai–Sugimoto model that we prefer to skip it and analyse more deeply the subject in the model of our interest. We would just like to point the attention to a simple feature: the quantized Skyrmion is

$$U'(\vec{x},t) = A(t)U(\vec{x})A^{\dagger}(t) . \tag{3.6.16}$$

A *left* transformation on A is equivalent to a global isospin rotation

$$A(t) \to LA(t) \implies U'(\vec{x},t) \to LU'(\vec{x},t)L^{\dagger} , \tag{3.6.17}$$

whereas a *right* transformation on A is equivalent to a global rotation

$$A(t) \to A(t)R \implies U'(\vec{x},t) \to A(t)RU(\vec{x})R^{\dagger}A(t) = U'(R\cdot\vec{x},t) . \tag{3.6.18}$$

Here we have used the hedgehog property that states that an isorotation on $U(\vec{x})$ is equivalent to a spatial rotation. The important thing to keep in mind is that spin and isospin rotations of the Baryon are interpreted as right/left transformation of the quantization function $A(t)$.

3.6.2 θ dependence in the chiral effective Lagrangian

Another ingredient necessary in our computation is the θ term. It is instructive to see how to introduce it in the chiral effective Lagrangian because it will present many analogies with the holographic description. To study the θ dependence we must find an effective term able to describe $U(1)$ anomaly explained in Section

3.4. Since this anomaly breaks $U(N_f)$ to $SU(N_f)$, we might look for a function of $\det U$. Moreover, large N_c counting rules impose that this term should be purely quadratic in the η'. The term to be added reads

$$\frac{f_\pi^2 a}{4N_c}(i\log\det U)^2 \ , \tag{3.6.19}$$

where a is a dimensionless constant of order 1 in the large N_c expansion (N_c being the number of colors). This scaling can be understood by looking at the anomaly term (3.4.14) in the Yang–Mills Lagrangian:

$$\mathcal{L}_{\mathrm{YM}} = \frac{1}{g_{\mathrm{YM}}^2}\mathrm{Tr}\left[-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{g_{\mathrm{YM}}^2 N_f}{16\pi^2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right] . \tag{3.6.20}$$

It is multiplied by a g_{YM}^2, which, in the large N_c expansion, goes as $1/N_c$. For this reason as $N_c \to \infty$ the anomaly disappears (and, accordingly, the η' becomes massless). The full chiral effective Lagrangian (Skyrme term excluded) now reads:

$$\mathcal{L}_{\mathrm{eff}} = \frac{f_\pi^2}{4}\left(\mathrm{Tr}\,\partial_\mu U^\dagger \partial^\mu U + 2B\mathrm{Tr}\,(\mathcal{M}U + \mathcal{M}^\dagger U^\dagger) - \frac{a}{N_c}(-i\log\det U)^2\right) . \tag{3.6.21}$$

A consequence of the anomaly is that the θ dependence can be moved from the topological term to the mass matrix. This feature is nicely reproduced here: let us suppose that $\mathcal{M}$ has a phase

$$\mathcal{M} = e^{i\theta/N_f}M \ , \tag{3.6.22}$$

with M real, positive and diagonal. Then we can perform the transformation

$$U \to e^{-i\theta/N_f}U \ . \tag{3.6.23}$$

Only the last two terms in the Lagrangian are sensible to this. They become

$$2B\mathrm{Tr}\,(MU + M^\dagger U^\dagger) - \frac{a}{N_c}(-i\log\det U - \theta)^2 \ . \tag{3.6.24}$$

The θ angle is now explicitly displayed in the Lagrangian.

We can use this Lagrangian to find the vacuum configuration of the theory. As an ansatz we could take a diagonal matrix for the vacuum value of U:

$$U = \mathrm{diag}\left(e^{i\varphi_j}\right) , \qquad j = 1, \ldots, N_f \ . \tag{3.6.25}$$

Let us call m_j the diagonal entries of M. The potential energy to be minimized is

$$V(\varphi_j) = \frac{f_\pi^2}{2}\left(-2B\sum_{j=1}^{N_f} m_j \cos\varphi_j + \frac{a}{2N_c}\left(\sum_{k=1}^{N_f}\varphi_k - \theta\right)^2\right) . \tag{3.6.26}$$

The variation of this potential will give us the following equations

$$2Bm_j \sin\varphi_j = \frac{a}{N_c}\left(\theta - \sum_{k=1}^{N_f} \varphi_k\right) \ , \tag{3.6.27}$$

where j is not summed over. As we said previously, the θ dependence completely disappears if one of the quarks is massless. This can be seen from the above equation: let us suppose $m_1 = 0$, then we can solve the equation by putting $\varphi_1 = \theta$ and the others to zero. The parameters appearing above have a very simple physical interpretation. For instance, in the $N_f = 2$ case, we have:

$$m_\pi^2 = 2B\frac{m_1 + m_2}{2} \ , \quad m_{\eta'}^2 = \frac{N_f a}{N_c} = \frac{2N_f \chi_g}{f_\pi^2} \ , \tag{3.6.28}$$

where χ_g is the topological susceptibility of the gluon theory, obtained differentiating the Lagrangian two times by θ. The discussion of the solution is postponed to Section 4.8. In that Section we will study the solution of the vacuum in the holographic model, but we will see that the two descriptions are indeed very similar.

This page is intentionally left blank.

Chapter 4

Witten Sakai Sugimoto model

In this Chapter we are going to introduce the Witten–Sakai–Sugimoto model [42, 43, 44]. We start by studying some properties of the Witten's background obtained by N_c D4 branes wrapped on S^1 (already introduced in Section 2.6.2), then we review the addition of massless flavors as a stack of N_f D8–$\overline{\text{D8}}$ branes. We finally introduce the mass deformation of the model [45, 46], as well as the θ term, and study the vacuum of the theory.

4.1 Witten background

We are interested in finding a dual to Yang–Mills. As argued in Section 2.6.2 the closest known approximation to Yang-Mills, at least in the low energy limit, is given the solution of a consistent truncation of Type IIA Supergravity (where only the metric, the dilaton and the $F_{(4)}$ form are switched on)

$$S_{\text{IIA}} = \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}X \sqrt{-g} \left(e^{-2\phi} \left(R + 4\partial_M \phi \partial^M \phi \right) - \frac{1}{2} |F_{(4)}|^2 \right) , \tag{4.1.1}$$

corresponding to N_c D4–branes wrapped on S^1. We rewrite the metric:

$$\mathrm{d}s^2 = \left(\frac{U}{R}\right)^{3/2} \left(\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + f(U)\mathrm{d}\tau^2\right) + \left(\frac{R}{U}\right)^{3/2} \left(\frac{\mathrm{d}U^2}{f(U)} + U^2 \mathrm{d}{\Omega_4}^2\right) , \tag{4.1.2}$$

$$e^\phi = g_s \left(\frac{U}{R}\right)^{3/4} , \qquad F_{(4)} = dC_{(3)} = \frac{2\pi N_c}{\text{Vol}(S^4)} \omega_{S^4} , \tag{4.1.3}$$

where $f(U)$ is given by

$$f(U) = 1 - \frac{U_{\text{KK}}^3}{U^3} . \tag{4.1.4}$$

With the AdS/CFT dictionary

$$R^3 = \frac{9}{4} \,, \quad U_{\rm KK} = M_{\rm KK} = 1 \,, \quad g_s = \frac{1}{2\pi} \frac{g_{\rm YM}^2}{M_{\rm KK} l_s} \,, \quad \frac{2}{9} M_{\rm KK}^2 l_s^2 = \lambda^{-1} \,. \qquad (4.1.5)$$

This metric, in particular the "cigar" described in Figure 2.8, not only provides an explicit energy scale, but it is also able to describe qualitatively most of the properties of low energy Yang–Mills. In particular we have

- Mass gap: the mass spectrum of the theory is discrete hence there are no arbitrarily light glueballs (i.e. colourless bound states of gluons).
- Confinement: the potential of an external heavy quark–antiquark couple grows linearly with the distance. There is also a deconfined solution that we won't discuss. There is a phase transition between these two called Hawking–Page phase transition [42, 79].
- As we will see when D8 branes are introduced, it also describes the spontaneous chiral symmetry breaking.

We should point out that this model is not precisely the dual of YM essentially for two reasons. The first one is that the compactification along τ creates a whole towers of massive ($\sim M_{\rm KK}$) fields in the adjoint representation of $SU(N_c)$, which we are unable to decouple. The reason for this is that the parameter weighting the decoupling is $T_s/M_{\rm KK}^2 \sim \lambda$ (T_s being the string tension from the rectangular Wilson loop); sadly however, in order to have a reliable supergravity description, we must take $\lambda \gg 1$, which is the opposite of the decoupling limit. Secondly the S^4 factor has an holographic interpretation as a global $SO(5)$ symmetry, of which there is no trace in YM theory. Implicitly we will take a multipole expansion in the $SO(5)$ spherical harmonics retaining only the $\ell = 0$ term, so that these extra degrees of freedom do not appear in the holographic description.

Let us now show explicitly some of the properties claimed above, namely *confinement* and *mass gap*.

4.1.1 Confinement in the Witten model

The potential between an external heavy quark–antiquark couple can give us information abound the confinement of the theory: if asymptotically it grows linearly we say that the theory is confining. In order to compute the potential one has to take the expectation value of the Wilson loop along a rectangle with one (Euclidean) time direction.

$$\langle W_C \rangle = e^{-TV(l)} \,, \qquad (4.1.6)$$

T and l being, respectively, the time and space sides of the rectangle C. As we said in Chapter 2 this expectation value can be computed holographically as the on shell Nambu–Goto action of a string worldsheet whose boundary is C. The key element in this computation is that the U coordinate has a minimum value U_{KK}. Hence, since the metric diverges at the boundary $U \to \infty$ where C stands, the worldsheet will try to minimize its area by laying on U_{KK} as much as possible. This results in a "bathtub" shaped worldsheet (see Figure 4.1). The Wilson loop in only the integral in the bottom of the bathtub, the two walls being the (infinite) contributions related to the heavy quark masses.

Let us embed the wolrdsheet in the space (x_E^0, x^1), x_E^0 being the Euclidean time, with $U \equiv U(x^1)$. The Nambu–Goto action reads

$$S_{\mathrm{NG}} = \frac{1}{2\pi\alpha'} \int_0^T \mathrm{d}x_E^0 \int_{-l/2}^{l/2} \mathrm{d}x^1 \sqrt{g_{00}\left(g_{11} + U'^2(x^1) g_{UU}\right)} \, . \tag{4.1.7}$$

In the limit $l \to \infty$ the worldsheet is laying on $U = U_{\mathrm{KK}}$, a part from the divergent contribution of the walls that we ignore. The integral becomes simply

$$S_{\mathrm{NG}} \underset{l \gg 1}{\sim} \frac{T\, l}{2\pi\alpha'} \left(\frac{U_{\mathrm{KK}}}{R}\right)^{3/2} + (\text{divergent part}) \, . \tag{4.1.8}$$

We have obtained that $V(l)$ grows linearly, with a string tension

$$T_s = \frac{1}{2\pi\alpha'} \left(\frac{U_{\mathrm{KK}}}{R}\right)^{3/2} = \frac{2\lambda}{27\pi} M_{\mathrm{KK}}^2 \, . \tag{4.1.9}$$

This indicates that the theory is confining, as announced.

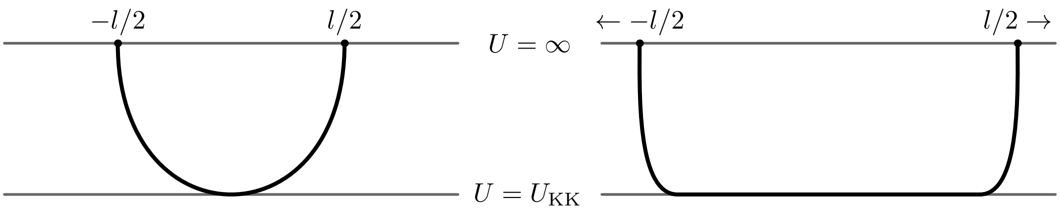


Figure 4.1: Section $x_E^0 =$ const. of the worldsheet. The vertical direction is U and the horizontal direction is x^1. When l is not too large the surface does not lie completely of $U = U_{\mathrm{KK}}$; as l grows it forms the "bathtub" shape.

4.1.2 Mass gap in the Witten model

Yang–Mills is expected to exhibit a mass gap, i.e. a lower bound on the mass spectrum of the theory above zero. The bound states (which are colourless due to confinement) are called *glueballs*. In this simple argument let us consider a scalar parity even glueball, which will be related to the operator $\mathrm{Tr}\, F^2$. The holographic correspondence tells us that this operator is dual to a scalar field in the gravity side. Let us then consider a scalar field ϕ in the Witten's background. We look for solutions to the Klein–Gordon equation

$$\Box\,\phi - m^2\phi = 0\ , \tag{4.1.10}$$

with $k^2 = -M_g^2 < 0$. M_g is the mass of the glueball while m is related to the dimension of the operator $\mathrm{Tr}\, F^2$ by means of the holographic correspondence (2.5.24).

Let us suppose that ϕ does not depend on the S^4 angles, we choose the ansatz

$$\phi(x, U) \equiv e^{ik\cdot x}\varphi(U)\ . \tag{4.1.11}$$

Expanding the Klein–Gordon equation one finds an equation for φ

$$\left(-\frac{3}{2U^{3/2}}k^2 - m^2\right)\varphi + \frac{1}{3U^{5/2}}\left(7U^3 - 1\right)\varphi' + \frac{2}{3U^{3/2}}(U^3 - 1)\varphi'' = 0\ , \tag{4.1.12}$$

where the primes are derivatives w.r.t. U and we have used the dictionary in (2.6.33). Now we must impose some suitable boundary conditions:

i) Normalizability of φ at $U \to \infty$.

ii) At the tip of the cigar we must have $\varphi'(U_{\mathrm{KK}}) = 0$ otherwise there would be a cusp in the plane (U, τ).

Now we have reduced our problem to a one dimensional Schrödinger problem with a potential that grows both at $U \to \infty$ and $U \to U_{\mathrm{KK}}$. It is known that the eigenvalues M_g^2 are discrete and with a finite gap. We can compute some of them numerically with the shooting method:

$$M_g^2 = \{4.33,\ 10.11,\ 17.78, \ldots\}\ , \tag{4.1.13}$$

all expressed in units of M_{KK}^2. The theory exhibits a mass gap as announced.

4.2 Introducing flavors

The Witten's model outlined above is a supergravity description of the system obtained by N_c D4 branes in Type IIA string theory wrapped around a compactified coordinate, here called τ. With this model we can describe a 4 dimensional Yang Mills theory. The massless gauge fields are the 4 dimensional gauge fields on the D4–brane. The compactification we have done on S^1 actually gives masses

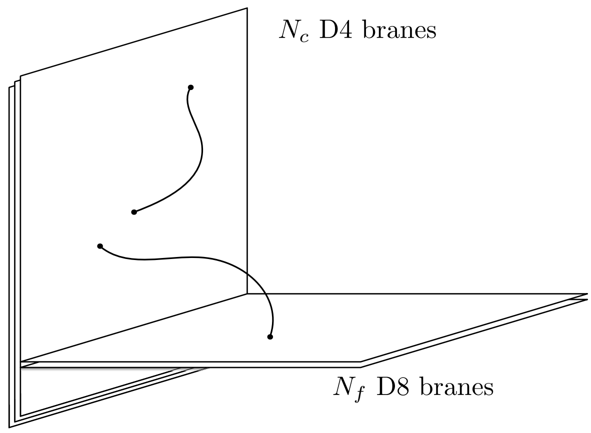


Figure 4.2: Intuitive depiction of the D4/D8 system, the $\overline{\text{D8}}$ can be added in the same way. The strings on the D4 correspond to the gauge field and the string from the D8 ($\overline{\text{D8}}$) to the D4 are (anti)quarks.

to all scalar and fermionic fields which transform in the adjoint representation of the group. To describe QCD properly we need to introduce fermionic fields in the fundamental representation. In order to introduce fields in the fundamental, Sakai and Sugimoto in [43] proposed to add N_f D8 and N_f $\overline{\text{D8}}$ branes (the bar on $\overline{\text{D8}}$ means that they have an opposite charge with respect to the D8's). These branes should be added at the level of "probes", which means that they do not backreact on the D4 branes, allowing us to use the same background metric. This is realized concretely by imposing $N_f \ll N_c$. More concretely, the parameter weighting the backreaction of the D8–branes in the Witten's background is

$$\epsilon_f \sim N_f M_{\text{KK}} T_{D8} \kappa_{10} \sim \lambda^2 \frac{N_f}{N_c} \ll 1 \,, \tag{4.2.1}$$

where T_{D8} is the D8–brane tension and κ_{10} the 10 dimensional Newton constant. In this work $\mathcal{O}(\epsilon_f)$ corrections are neglected. In [80] the first corrections in ϵ_f are considered. Strings from the D8/$\overline{\text{D8}}$ branes to the D4 will have "Chan-Paton" indices belonging to $U(N_f)_{\text{D8}} \times U(N_f)_{\overline{\text{D8}}}$ (as well as a color index), the two factors are interpreted as the *left* and the *right* factors of the chiral group $U(N_f)_L \times U(N_f)_R$. So strings from a D8 ($\overline{\text{D8}}$) brane to a D4 brane are interpreted as left (right) quarks because the lowest states in superstring theory are fermions (2.1.17) and the components on the D4 are the only ones not acquiring mass from the S^1 compactification. As we will see from the embedding chosen for the D8/$\overline{\text{D8}}$, there is no room for the D8 to be spatially separated from the D4 so these string excitation are necessarily massless. This model is an effective description of quantum Chromodynamics with massless quarks at low energies.

Schematically, we embed the D8/$\overline{\text{D8}}$ branes in the Witten's background as in Table 4.1 ($\circ$ = point-like, $-$ = extended).

0,1,2,3	$=\mathbb{R}^{1,3}$	spacetime in which the QFT lives
4	$=\tau$	compactified coordinate $\sim M_{\text{KK}}^{-1}$
5	$=U$	former AdS_7 radius
6,7,8,9	$=S^4$	S^4 sphere

	0	1	2	3	4	5	6	7	8	9
D4	–	–	–	–	–	○	○	○	○	○
D8/$\overline{\text{D8}}$	–	–	–	–	○	–	–	–	–	–

Table 4.1: D8/$\overline{\text{D8}}$ branes embedding.

The massless string excitations contain:

- $A_\mu^{(\text{D4})}$ gauge field on the D4, $\mathbb{R}^{1,3}$ components
- a_4 trace part[1] of the gauge field on the D4, τ component
- ϕ_i trace part[2] of the D4 embeddings: scalars with index on the transverse coordinates
- q_L, q_R quarks in the fundamental representation with opposite chiralities depending whether they come from the D8 or the $\overline{\text{D8}}$. See [81].

There are also strings from D8 to $\overline{\text{D8}}$: in flat spacetime they would contain a tachyon in their spectrum. When D8–branes are placed on Witten background these states acquire a positive mass and thus are decoupled. The gauge groups are $U(N_f)_L \times U(N_f)_R$ and $U(N_c)$ and the spacetime symmetry groups are $SO(1,3)$ (of $\mathbb{R}^{1,3}$) and $SO(5)$ (of S^4). The fields considered above belong to irreducible representations of these groups, they are summarized in Table 4.2.

	$U(N_c)$	$SO(1,3)$	$SO(5)$	$U(N_f)_L \times U(N_f)_R$
$A_\mu^{(\text{D4})}$	**adj**	$\mathbf{4}$	$\mathbf{1}$	$(\mathbf{1},\mathbf{1})$
a_4	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$(\mathbf{1},\mathbf{1})$
ϕ_i	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{5}$	$(\mathbf{1},\mathbf{1})$
q_L	**fund**	$\mathbf{2}_+^{(\text{Weyl})}$	$\mathbf{1}$	$(\mathbf{fund},\mathbf{1})$
q_R	**fund**	$\mathbf{2}_-^{(\text{Weyl})}$	$\mathbf{1}$	$(\mathbf{1},\mathbf{fund})$

Table 4.2: Irreducible representations of the massless fields originated by the low energy string excitations of the theory.

[1] Only the trace (i.e. abelian) part of these gauge fields remains massless because of a gauge symmetry $A_4 \to A_4 + \alpha \mathbb{1}_{N_c}$, where we call $a_4 = \operatorname{Tr} A_4$ and $\mathbb{1}_{N_c}$ the $N_c \times N_c$ unit matrix. The non abelian components on the other hand acquire a mass.

[2] Same as footnote 1 where $\phi_i = \operatorname{Tr} \Phi$.

As we will see, the embedding of the D8/$\overline{\text{D8}}$ requires the two stacks to join at low values of U, in order to minimize the energy. This feature has a very beautiful interpretation in terms of the boundary theory, in fact when the stacks are separated the flavor symmetry group is $U(N_f)_L \times U(N_f)_R$, but when the two stacks join the group is broken to $U(N_f)$. This is precisely the spontaneous chiral symmetry breaking in QCD, which is a very important aspect of its low energy physics. We will indeed see that Goldstone modes arise as in the chiral effective theory.

4.3 Probe D8 brane

In the supergravity description the D4 branes are substituted by the Witten's curved background. The pair of stacks D8/$\overline{\text{D8}}$ joins at values of $U \sim U_{\text{KK}}$ becoming a single stack of N_f D8 branes. We want to see how precisely they join and whether this is a stable solution or not. In order to study the embedding of the probe D8 branes we use the ansatz $U = U(\tau)$. For now we keep writing the metric with the explicit factors R and U_{KK} and we do not impose $M_{\text{KK}} = 1$ yet. The induced metric in this ansatz reads ($U' = \partial_\tau U$):

$$\mathrm{d}s^2_{\text{D8}} = \left(\frac{U}{R}\right)^{3/2} \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \left(\left(\frac{U}{R}\right)^{3/2} f(U) + \left(\frac{R}{U}\right)^{3/2} \frac{U'^2}{f(U)}\right) \mathrm{d}\tau^2 + R^{3/2}U^{1/2}\mathrm{d}\Omega_4^2 \,. \tag{4.3.1}$$

Actually the metric should be along U and not τ according to Table 4.1. We write it in this way because now we can solve for $\tau(U)$ using the method of quadratures. The equation of motion for $U(\tau)$ is given by the action

$$S_{\text{D8}} \propto \int \mathrm{d}^4x \mathrm{d}\tau \omega_{S^4}\, e^{-\phi}\sqrt{\det g} \propto \int \mathrm{d}\tau\, U^4 \sqrt{f(U) + \left(\frac{R}{U}\right)^3 \frac{U'^2}{f(U)}} \,. \tag{4.3.2}$$

This is a one dimensional action so it is sufficient to impose $E = \text{const}$, where E is the Hamiltonian

$$\frac{\partial \mathcal{L}}{\partial U'} U' - \mathcal{L} = \frac{-U^4 f(U)}{\sqrt{f(U) + \left(\frac{R}{U}\right)^3 \frac{U'^2}{f(U)}}} = \text{const.} \,. \tag{4.3.3}$$

We can write in an integral form the solution to the Cauchy problem $U(\tau = 0) \equiv U_0$ and $U'(\tau = 0) = 0$. There is a $\pm$ ambiguity which is interpreted as the solution for the D8 stack or the $\overline{\text{D8}}$ stack. The two solutions join at $U = U_0$:

$$\tau(U_1) = \pm \int_{U_0}^{U_1} \mathrm{d}U \, \frac{U_0^4 f(U_0)^{1/2}}{\left(\frac{U}{R}\right)^{3/2} f(U)\sqrt{U^8 f(U) - U_0^8 f(U_0)}} \,. \tag{4.3.4}$$

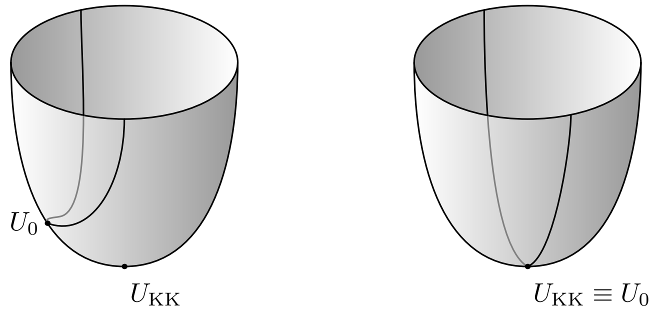


Figure 4.3: Graphical representation of the embedding of the D8/$\overline{\rm D8}$ branes in the Witten background, in the first picture it is shown a general embedding and in the second the antipodal one, with $U_0 = U_{\rm KK}$

The simplest embedding is the so called *antipodal embedding*, which is given by $\tau(U) = \text{const}$. We will employ this embedding throughout the whole work so it is better to study how it is realized. It can be easily checked that $\tau'(U)$ is identically zero for $U \neq U_{\rm KK}$ if $U_0 = U_{\rm KK}$, so the $\tau = \text{const}$ solution is for $U_0 = U_{\rm KK}$. We should also be able to check that the constant is $\pm\delta\tau/4$, hence the name "antipodal" (it corresponds to $\pm\pi/2$ in terms of a regular angle). The integral expression for $\tau(U)$ is ill defined for $U_0 = U_{\rm KK}$. We make it well defined via the change of variables

$$\frac{U}{U_{\rm KK}} = x \ , \quad f_0 = 1 - \left(\frac{U_{\rm KK}}{U_0}\right)^3 \ , \quad y = \frac{f(U)}{f_0} \ , \quad x(y) = \frac{1}{(1-f_0 y)^{1/3}} \ . \tag{4.3.5}$$

Substituting this in the integral we find

$$\tau = \frac{R^{3/2}}{3U_{\rm KK}^{1/2}} \int_1^{1/f_0} {\rm d}y \frac{x^{5/2}(y)}{y\sqrt{x^8(y)\cdot y - 1}} \ . \tag{4.3.6}$$

In the limit $U_0 \to U_{\rm KK}$ the constant $f_0 \to 0$ and so $x(y) \to 1$. The integral is then easy to compute[3] and, if we compare with (2.6.29), we indeed have $\tau = \delta\tau/4$.

The only thing to check is the stability of the solution. The computation is carried on in [43]. The idea is simple: we just have to consider a more general embedding as a perturbation over this solution, write down the action to the leading order on the perturbation and check the positivity of the energy. Let us change the parametrization of the (U, τ) plane for convenience: first we move to

[3]
$$\int_1^\infty \frac{1}{y\sqrt{y-1}} = \pi$$

the origin the singularity $U = U_{\rm KK}$ then we rescale τ to a regular angle φ by setting

$$U^3 = U^3_{\rm KK} + U_{\rm KK} r^2 \ , \quad \varphi = \frac{2\pi}{\delta\tau}\tau \ , \tag{4.3.7}$$

finally we parametrize the (r, φ) plane in cartesian coordinates (y, z)

$$y = r\cos\varphi \ , \quad z = r\sin\varphi \ . \tag{4.3.8}$$

The metric reads

$$\mathrm{d}s^2_{(y,z)} = \frac{4}{9}\left(\frac{R}{U}\right)^{3/2}\left[\left(1 - q(r)z^2\right)\mathrm{d}z^2 + \left(1 - q(r)y^2\right)\mathrm{d}y^2 - 2zy\, q(r)\mathrm{d}x\mathrm{d}y\right] \ , \tag{4.3.9}$$

with U given as a function of z and y by (4.3.7) and $q(r)$ defined by $q(r) = \frac{1}{r^2}\left(1 - \frac{U_{\rm KK}}{U}\right)$. The embedding studied above corresponds to $y = 0$. In this case the metric becomes simpler and the function U reduces to $U(z) = (U^3_{\rm KK} + U_{\rm KK} z^2)^{1/3}$. We consider as a perturbation the more general embedding $y = y(x^\mu, z)$, with $\mathcal{O}(y^2) \sim 0$. Computing the DBI action to leading order in $y(x^\mu, z)$ we find the following (modulo positive constants)

$$S \propto -\int \mathrm{d}^4x\mathrm{d}z \left[U^2(z) + \frac{2}{9}\frac{R^3}{U(z)}\partial_\mu y\partial_\nu y\eta^{\mu\nu} + y^2 + \frac{U^3(z)}{2U_{\rm KK}}(\partial_z y)^2\right] \ . \tag{4.3.10}$$

Denoting with $\mathcal{L}$ the integrand of (4.3.10) we find the energy

$$\mathcal{E} = \int \mathrm{d}^3x\mathrm{d}z\, \frac{4}{9}\frac{R^3}{U(z)}(\partial_0 y)^2 - \int \mathrm{d}^3x\mathrm{d}z\, \mathcal{L} \geq 0 \ . \tag{4.3.11}$$

We can now forget about $y(x^\mu, z)$ and keep the original embedding, the metric becomes:

$$\mathrm{d}s^2_{(z,y=0)} = \frac{4}{9}\left(\frac{R}{U}\right)^{3/2}\frac{U_{\rm KK}}{U}\mathrm{d}z^2 \ . \tag{4.3.12}$$

4.4 Supergravity action

The first thing we are going to do is to write the complete supergravity action of the model, this will be done in presence of a finite θ term. For now it is easier to keep the parameters $R, U_{\rm KK}, l_s$, etc., the AdS/CFT dictionary will be applied afterwards.

Let us start from the *bulk + D8 brane* action. This is the sum of the kinetic terms of the p–forms and the Dirac–Born–Infeld plus Chern–Simons terms of the

D8 brane. The Einstein–Hilbert term will be neglected since we are working at fixed background.

$$
\begin{aligned}
S_{\text{bulk+D8}} = -\frac{1}{4\pi}\sum_{p\ \text{odd}}(2\pi l_s)^{2(p-4)}\int F_{(p+1)}\wedge {}^\star F_{(p+1)}+ \\
+\int_{D8}\sum_{k=1}^{4} C_{(9-2k)}\wedge\frac{1}{k!(2\pi)^k}\operatorname{Tr}\mathcal{F}^k+ \\
-\frac{1}{(2\pi)^8 l_s^9}\int_{D8}\mathrm{d}^9\xi\, e^{-\phi}\operatorname{STr}\sqrt{|\det(\mathcal{P}[g]+2\pi l_s^2\mathcal{F})|}
\end{aligned}
\tag{4.4.1}
$$

The $F_{(p+1)}$ are the RR field strengths of the bulk $C_{(p)}$ forms while the $\mathcal{F}$ are the $U(N_f)$ field strengths of the gauge fields living on the D8 branes $\mathcal{F}=\mathrm{d}\mathcal{A}+i\mathcal{A}\wedge\mathcal{A}$. Powers of differential forms are done by means of the wedge product $\wedge$. The symbol $\mathcal{P}[g]$ denotes the *pullback* on the D8 worldvolume and the symbol "STr" denotes the symmetrized trace on the gauge group indices.

DBI Term Let us explicitly write the DBI term in the action. The D8 brane is embedded in the way we have seen before, with $y=0$, so

$$
-T_{D8}\int_{D8}\mathrm{d}^4x\mathrm{d}z\mathrm{d}^4\psi\,\frac{1}{g_s}\left(\frac{R}{U(z)}\right)^{3/4}\operatorname{STr}\sqrt{|\det(g_{D8}+2\pi l_s^2\mathcal{F})|}\,, \tag{4.4.2}
$$

with the definition $T_{D8}=1/(2\pi)^8 l_s^9$. The pullbacked metric has the form

$$
\mathrm{d}s^2_{D8}=\left(\frac{U}{R}\right)^{3/2}\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu+\frac{4}{9}\left(\frac{R}{U}\right)^{3/2}\frac{U_{\text{KK}}}{U}\mathrm{d}z^2+R^{3/2}U^{1/2}\mathrm{d}{\Omega_4}^2\,, \tag{4.4.3}
$$

with $U\equiv U(z)$. We look for solution such that the S^4 components of the $\mathcal{F}$ field are zero and the other components do not depend on the angular coordinates ψ_i. In this way the determinant factorizes and we can integrate over the sphere:

$$
-T_{D8}\mathrm{Vol}(S^4)\frac{R^{15/4}}{g_s}\int\mathrm{d}^4x\mathrm{d}z\,U^{1/4}(z)\operatorname{STr}\sqrt{|\det(g_{(5)}+2\pi l_s^2\mathcal{F})|}\,. \tag{4.4.4}
$$

Expanding the square root to first order gives a five dimensional Yang–Mills action:

$$
-\frac{\widetilde{T}_{D8}}{4}\int\mathrm{d}^4x\mathrm{d}z\left(\frac{R^3}{U(z)}\operatorname{Tr}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}+\frac{9U^3(z)}{2U_{\text{KK}}}\operatorname{Tr}\mathcal{F}_{\mu z}{\mathcal{F}^{\mu}}_{z}\right)\,, \tag{4.4.5}
$$

where we have defined

$$
\widetilde{T}_{D8}=T_{D8}16\pi^2R^{3/2}U_{\text{KK}}^{1/2}\frac{(2\pi l_s^2)^2}{9g_s}=\frac{N_cM_{\text{KK}}l_s^{-2}}{54\pi^3}\,. \tag{4.4.6}
$$

Now we adopt the AdS/CFT dictionary with (2.6.33). The function $U(z)$ becomes simply $(1+z^2)^{1/3}$. The above term becomes

$$-\frac{N_c\lambda}{216\pi^3}\int \mathrm{d}^4x\mathrm{d}z\left(\frac{1}{2}\frac{1}{(1+z^2)^{1/3}}\,\mathrm{Tr}\,\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}+(1+z^2)\mathrm{Tr}\,\mathcal{F}_{\mu z}\mathcal{F}^{\mu}{}_{z}\right)\,. \tag{4.4.7}$$

In many papers [48, 49, 82, 83] the following notation has become popular, we shall adopt it here as well:

$$-\kappa\int \mathrm{d}^4x\mathrm{d}z\left(\frac{1}{2}h(z)\,\mathrm{Tr}\,\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}+k(z)\mathrm{Tr}\,\mathcal{F}_{\mu z}\mathcal{F}^{\mu}{}_{z}\right)\,, \tag{4.4.8}$$

with obvious definitions.

$C_{(1)}$ and $C_{(7)}$ terms Let us study the $C_{(1)}$ and $C_{(7)}$ terms. First of all we need to discuss the gauge transformations for $C_{(1)}$ and $\mathcal{A}$. As discussed in Section 2.2, the presence of D8 branes spoils the gauge invariance of the RR forms, as a consequence an anomaly arises. In order to cancel this anomaly we introduce the CS couplings in the action, but we also have to redefine the gauge transformation of the RR forms. The $C_{(1)}$ transformation reads

$$\delta_\Lambda C_{(1)}=\mathrm{Tr}\,(\Lambda)\delta(y)\mathrm{d}y\,,\qquad \delta_\Lambda\mathcal{A}=i[\Lambda,\mathcal{A}]-\mathrm{d}\Lambda\,. \tag{4.4.9}$$

This renders ill defined the RR form $C_{(1)}$ because its kinetic term is not gauge invariant anymore. A way out of this issue is to work with the dual form $C_{(7)}$ which is related to $C_{(1)}$ when no anomalies are present by the following condition (here we write the duality for generic p for completeness, this normalization follows Appendix A of [43])

$$^{\star}\mathrm{d}C_{(p+1)}=(2\pi l_s)^{2(3-p)}\mathrm{d}C_{(7-p)}\,. \tag{4.4.10}$$

The action now reads

$$S_{C_7}=-\frac{1}{4\pi}(2\pi l_s)^6\int \mathrm{d}C_{(7)}\wedge{}^{\star}\mathrm{d}C_{(7)}+\frac{1}{2\pi}\int C_{(7)}\wedge\mathrm{Tr}\,\mathcal{F}\wedge\omega_y\,, \tag{4.4.11}$$

where we have defined the 1–form $\omega_y=\delta(y)\mathrm{d}y$, in order to extend the D8 integral to the whole spacetime. We would expect a term like this also

$$\int_{D8} C_{(1)}\wedge\mathrm{Tr}\,\mathcal{F}^4\,, \tag{4.4.12}$$

however if we assume that $\mathcal{F}$ lives only on the first 5 coordinates of the D brane, this term is necessarily zero.

What used to be the Maxwell equation for $F_{(2)} = \mathrm{d}C_{(1)}$ is now the Bianchi identity for $F_{(8)} = \mathrm{d}C_{(7)}$, which is automatically satisfied by construction. Conversely the presence of the D8 Chern–Simons coupling acts as a source for $C_{(7)}$, resulting in a violation of the Bianchi identity for $F_{(2)}$. The equation for $C_{(7)}$ is

$$\mathrm{d}\,{}^\star\mathrm{d}C_{(7)} = \frac{1}{(2\pi l_s)^6}\mathrm{Tr}\,\mathcal{F}\wedge\delta(y)\mathrm{d}y\ . \tag{4.4.13}$$

It is convenient to split every $U(N_f)$ matrix in its abelian and non abelian parts

$$\mathcal{A} = \widehat{A}\frac{\mathbb{1}}{\sqrt{2N_f}} + A^a T^a\ , \tag{4.4.14}$$

where T^a are $SU(N_f)$ generators satisfying $\mathrm{Tr}\,T^aT^b = \frac{1}{2}\delta^{ab}$. Notice that the anomaly transformation mentioned above is driven only by the "hatted" (i.e. abelian) component of this decomposition. The variation of the action S_{C_7} with respect to the abelian fields is given by[4]

$$\begin{aligned}\frac{\delta S_{C_7}}{\delta\widehat{A}_z} &= \delta(y)\frac{1}{2\pi}\mathrm{Vol}(S^4)\sqrt{\frac{N_f}{2}}[\mathrm{d}C_{(7)}]_{0123,S^4}\ ,\\ \frac{\delta S_{C_7}}{\delta\widehat{A}_\mu} &= \delta(y)\frac{1}{2\pi}\mathrm{Vol}(S^4)\sqrt{\frac{N_f}{2}}[\mathrm{d}C_{(7)}]_{..\widehat{\mu}..z,S^4}\ ,\end{aligned} \tag{4.4.15}$$

which must be added to the variation of the DBI action to obtain the equations of motion for the gauge fields. Now the form $C_{(1)}$ does not exist anymore, but we can define a two form as

$$\widetilde{F}_{(2)} = (2\pi l_s)^6\,{}^\star\mathrm{d}C_{(7)}\ . \tag{4.4.16}$$

Clearly only the components "$y0,\dots y3, yz$" of the $\widetilde{F}_{(2)}$ form couple to the gauge field equations. We can solve the Bianchi identity/Maxwell equation (4.4.13) with an ansatz analogous to the one used by Hořava and Witten in [84], without touching the equations (4.4.15). This is the only way to avoid δ like singularities

$$[\widetilde{F}_{(2)}]_{AB} = \Theta(y)\sqrt{\frac{N_f}{2}}\widehat{F}_{AB} + \cdots\ , \qquad A, B \neq y\ , \tag{4.4.17}$$

where $\Theta(y)$ is the step function, $\Theta(y) = |y|/2y$, while the dots are terms vanishing at $y = 0$. These extra terms are necessary to impose $\mathrm{d}\,{}^\star\widetilde{F}_{(2)} = 0$, they vanish at $y = 0$ because they are regular there and because the function is assumed to be odd under $y \to -y$. This solution does not prevent us to add zero modes (i.e.

[4] $..\widehat{\mu}..$ means indices from 0 to 3 except μ

solutions to $d\widetilde{F}_{(2)} = d\,{}^\star\widetilde{F}_{(2)} = 0$) to the above expression. Such zero modes are given by the following expression [47]

$$[\widetilde{F}_{(2)}]_{zy} = \frac{C}{U^6} \ , \tag{4.4.18}$$

where C is an arbitrary constant. A more thorough exposition of the details and the logic of this Hořava–Witten ansatz can be found in Appendix C.

At the moment the theory does not have any source of θ dependence, which is crucial for our work. In the next section we will see how it arises in the massless theory and we will check that the physics does not depend on it. The complete discussion can be then found after the introduction of a mass term for the quarks in Section 4.7.

$C_{(3)}$ and $C_{(5)}$ terms Now let us consider the forms $C_{(3)}$ and $C_{(5)}$. Without the D4 branes in the background those would be related by the hodge duality in the ordinary way (4.4.10), however the term $\int C_{(3)} \wedge \mathcal{F}^3$ is ill defined since the field strength $F_{(4)} = dC_{(3)}$ should have a non trivial flux on S^4. An easy way out is to write $\mathcal{F}^3$ as a total derivative (which is always possible locally by the Poincaré's lemma since these forms are all closed), and then to integrate by parts and consider the new term as the correct one, namely:

$$\frac{1}{3!(2\pi)^3}\int_{D8} C_{(3)} \wedge \mathcal{F}^3 = \frac{1}{3!(2\pi)^3}\int_{D8} F_{(4)} \wedge \omega_5(\mathcal{A}) \ . \tag{4.4.19}$$

Here we are assuming that the $\mathcal{A}$ field vanishes at $z = \pm\infty$, we decide from now on to work in this gauge. Let us recall that we still have a residual gauge in which the parameter Λ of the transformation approaches a constant at infinity $\partial_M \Lambda \to 0$. These are exactly the holographic equivalent of the global $U(N_f)$ transformations in the gauge theory. The *Chern-Simons form* $\omega_5(\mathcal{A})$ is given by

$$\omega_5(\mathcal{A}) = \mathrm{Tr}\left(\mathcal{A}\wedge\mathcal{F}^2 - \frac{i}{2}\mathcal{A}^3\wedge\mathcal{F} - \frac{1}{10}\mathcal{A}^5\right) \ , \quad d\omega_5(\mathcal{A}) = \mathrm{Tr}\,\mathcal{F}^3 \ . \tag{4.4.20}$$

If we integrate on the S^4 the answer is simple

$$= \frac{N_c}{3!(2\pi)^2}\int_{(5)} \omega_5(\mathcal{A}) \ . \tag{4.4.21}$$

The $C_{(5)}$ term can be simplified using the same trick and the Hodge duality ${}^\star dC_{(5)} = F_{(4)}$:

$$\begin{aligned} \int_{D8} C_{(5)} \wedge \mathcal{F} \wedge \mathcal{F} &= \int_{D8} dC_{(5)} \wedge \omega_3(\mathcal{A}) = \\ &\propto \int_{D8} {}^\star F_{(4)} \wedge \omega_3(\mathcal{A}) \ . \end{aligned} \tag{4.4.22}$$

As above, $\mathcal{A}$ is assumed to have components only on the 5 dimensional subspace spanned by x^μ, z, thus this last term is forced to vanish since neither ${}^\star F_{(4)}$ nor ω_3 have components along the S^4 directions.

To summarize, the whole action, takes contributions from the CS term associated to $C_{(3)}$ and $C_{(7)}$ and the DBI action of the D8 brane, which is considered at the leading order, neglecting higher derivative corrections. The action, after assuming a trivial dependence on the S^4 coordinates, reads:

$$\begin{aligned} S_{\text{bulk+D8}} = -\kappa \int \mathrm{d}^4x\mathrm{d}z \left(\frac{1}{2}h(z)\,\mathrm{Tr}\,\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + k(z)\mathrm{Tr}\,\mathcal{F}_{\mu z}\mathcal{F}^{\mu}{}_{z}\right) + \\ + S_{C_7} + \frac{N_c}{24\pi^2}\int \omega_5(\mathcal{A})\ . \end{aligned} \tag{4.4.23}$$

4.5 Holographic θ term

To understand how to introduce a θ term in the holographic description it is necessary to go back to the “string picture” of the Witten’s background. The curved spacetime is given by the presence of D4 branes. They have their own DBI action yielding the gluon dynamics, as well as the CS couplings. There we can find the following term

$$S_{\text{D4}} \supset \frac{1}{8\pi^2}\int_{D4} C_{(1)} \wedge F^2\ , \tag{4.5.1}$$

where now $F_{\alpha\beta}$ ($\alpha, \beta = 0, 1, 2, 3, \tau$) is the field strength whose dimensional reduction on $S^1(\tau)$ yelds the gluon field. We can assume $F_{\mu\nu}$ independent on τ and integrate $C_{(1)}$ in $\mathrm{d}\tau$. This clearly leads to the holographic map

$$\oint_{S^1(\tau)} C_{(1)} = \theta + 2k\pi\ , \tag{4.5.2}$$

where $k \in \mathbb{Z}$ expresses the periodicity of θ and, in the unflavored case, accounts for the expected multibranched structure of the vacuum energy density in the large N_c limit.

When flavors are not considered one can regard $C_{(1)}$ as a probe in Witten’s gravitational background and study the properties of the θ vacuum, such as energy density and topological susceptibility [47]. The probe approximation is valid for small θ mod 2π. The energy density is given by the on–shell value of the action with the opposite sign and the topological susceptibility is obtained by taking the second derivative with respect to θ. If the D8 branes are not present the Hodge duality

$${}^\star\mathrm{d}C_{(1)} = (2\pi l_s)^6\mathrm{d}C_{(7)} \tag{4.5.3}$$

holds. The solution to $C_{(1)}$ compatible with the integral constraint (4.5.2) is the zero mode (4.4.18)

$$\mathrm{d}C_{(1)} = \frac{1}{\pi U}(\theta + 2k\pi)\,\mathrm{d}z \wedge \mathrm{d}y \,. \tag{4.5.4}$$

The relevant part of the action here is the kinetic term in S_{C_7}

$$S_{C_7} \supset -\frac{1}{4\pi(2\pi l_s)^6}\int \mathrm{d}C_{(1)} \wedge {}^{\star}\mathrm{d}C_{(1)} \,, \tag{4.5.5}$$

where the Hodge duality has been used. Computing the on–shell value of this term with changed sign gives

$$\varepsilon_k(\theta) = \frac{\lambda^3}{8(3\pi)^6}(\theta + 2\pi k)^2 \,. \tag{4.5.6}$$

This is the kth branch of the vacuum energy density. To obtain the correct value on has to take the minimum $\varepsilon(\theta) = \min_k \varepsilon_k(\theta)$. The topological susceptibility is

$$\chi_g = \frac{\lambda^3}{4(3\pi)^6} \,. \tag{4.5.7}$$

As predicted in Section 3.5.1 the function $\varepsilon(\theta)$ is smooth everywhere except in the points $\theta = \pi$ where it "jumps" from one branch to another, rendering the first derivative discontinuous.

In presence of the D8 branes we have the gauge freedom explained in the previous section that allows us to put the y component of $C_{(1)}$ to zero, hence removing from the theory the θ angle. This is consistent with the fact that the theory with massless flavours is θ independent, due to the $U(1)_A$ anomaly. We can say that *the holographic equivalent of the $U(1)_A$ anomaly is the anomalous CS coupling to the D8 branes.* From now on we will take $k = 0$ for simplicity, thus focusing on $|\theta| \ll 1$ values.

In order to be quantitative let us perform the following gauge transformation of the field $\widehat{A}_z$

$$\delta_\Lambda C_{(1)} = \sqrt{\frac{N_f}{2}}\Lambda\,\delta(y)\mathrm{d}y \,, \qquad \delta_\Lambda \widehat{A}_z = -\partial_z \Lambda \,, \tag{4.5.8}$$

so that

$$\left(\lim_{z\to\infty} - \lim_{z\to-\infty}\right)\Lambda = \sqrt{\frac{2}{N_f}}\theta \,. \tag{4.5.9}$$

The integral of $C_{(1)}$ can be computed via the Stokes' theorem

$$\delta_\Lambda \oint_{S^1(\tau)} C_{(1)} = -\delta_\Lambda \int \mathrm{d}z\mathrm{d}y\,\partial_z C_y = -\theta \tag{4.5.10}$$

After this gauge transformation the integral of $C_{(1)}$ is zero, but we have a non zero Wilson line for $\widehat{A}_z$:

$$\frac{1}{\sqrt{2N_f}} \int_{-\infty}^{\infty} \mathrm{d}z\, \widehat{A}_z = -\frac{\theta}{N_f} \ . \tag{4.5.11}$$

In the massless theory theory everything is gauge invariant, so we just showed that any θ dependence can be removed for free. As we will see next, the mass term for the fermions will contain the above Wilson line in the Lagrangian, making this θ dependence physical. In that case however we must find a more clever way to define θ since, as we observed earlier, $C_{(1)}$ is not a well defined field in the theory, so we must be able to use only $C_{(7)}$ (or, equivalently, $\widetilde{F}_{(2)}$).

What we just did can be understood in terms of an Higgs mechanism, more precisely a *Stuckelberg mechanism*, in which a massless vector field $\widehat{A}_M$ "eats" a scalar (from the D8 point of view) field C_y. Acquiring a new degree of freedom $\widehat{A}_M$ becomes massive, hence explaining the mass of the η' arising from the $U(1)_A$ anomaly. The mass has been computed in [43]. It suffices to consider the zero mode of $\widetilde{F}_{(2)}$ (4.4.18) and interpret it as a background value of the η' meson

$$\widetilde{F}_{(2)} = \frac{1}{\pi U^6} \left(\theta + \frac{\sqrt{N_f}}{f_\pi} \eta' \right) \ . \tag{4.5.12}$$

Computing the kinetic term in S_{C_7} (recall (4.4.16)) on this solution gives a mass to the field η' equal to

$$m^2_{\eta'} = \frac{2N_f}{f_\pi^2} \chi_g = \frac{N_f \lambda^2}{27\pi^2 N_c} \ , \tag{4.5.13}$$

where χ_g is the topological susceptibility of the *flavourless* theory. This is precisely the Witten–Veneziano formula [11, 12]. As we can see the squared mass of the η' is proportional to ϵ_f (4.2.1), which is the parameter that weights the backreaction of the probe D8 branes in the Witten's metric. We will assume in the following that this parameter is small, which amounts to have

$$m_{\eta'} \ll M_{\mathrm{KK}} \ . \tag{4.5.14}$$

This is then another limit in which we are forced to work.

4.6 Meson physics

In this section we will give a brief review of the Sakai–Sugimoto model's description of mesons. For the sake of this discussion only the DBI term is important. Even though we are not interested in mesons in this work, some tools defined here reveal to be useful in the following. We will also prove that the $\lambda \to \infty$ limit of the Sakai–Sugimoto model coincides with the Skyrme model. Consider the action (4.4.8) with $N_f = 1$ for the moment.

$$S = -\frac{\kappa}{2} \int \mathrm{d}^4x \mathrm{d}z \left(\frac{1}{2} h(z)\, \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} + k(z) \mathcal{F}_{\mu z} \mathcal{F}^{\mu}{}_{z} \right) . \tag{4.6.1}$$

Let us make a fourier mode expansion of the fields $\mathcal{A}_\mu$ and $\mathcal{A}_z$

$$\begin{aligned} \mathcal{F}_{\mu\nu}(x^\mu, z) &= \sum_n \left(\partial_\mu B^{(n)}_\nu(x^\mu) - \partial_\nu B^{(n)}_\mu(x^\mu) \right) \psi_n(z) \\ &\equiv \sum_n \mathcal{F}^{(n)}_{\mu\nu}(x^\mu) \psi_n(z) , \\ \mathcal{F}_{\mu z}(x^\mu, z) &= \sum_n \left(\partial_\mu \varphi^{(n)}(x^\mu) \phi_n(z) - B^{(n)}_\mu(x^\mu) \psi'_n(z) \right) . \end{aligned} \tag{4.6.2}$$

The functions ψ_n and ϕ_n will be defined in a moment and ψ' means $\partial_z \psi$. For now we set the $\varphi^{(n)}$ to zero. The action then becomes:

$$S = -\frac{\kappa}{2} \int \mathrm{d}^4x \mathrm{d}z \left(\frac{1}{2} h(z) \sum_{m,n} \mathcal{F}^{(n)}_{\mu\nu} \mathcal{F}^{\mu\nu\,(m)} \psi_n \psi_m + k(z) \sum_{m,n} B^{(n)}_\mu B^{\mu\,(m)} \psi'_n \psi'_m \right) . \tag{4.6.3}$$

We require ψ to be a complete orthonormal set with a normalization condition; moreover we impose a condition over the ψ' to obtain a mass term

$$\kappa \int \mathrm{d}z\, h(z) \psi_n(z) \psi_m(z) = \delta_{mn} , \qquad \kappa \int \mathrm{d}z\, k(z) \psi'_n(z) \psi'_m(z) = \lambda_n \delta_{mn} . \tag{4.6.4}$$

Integrating by parts (the ψ_n approach zero for $z \to \pm\infty$ because of the normalization) we get the following eigenvalue equation

$$- h(z)^{-1} \partial_z \left(k(z) \partial_z \psi_n \right) = \lambda_n \psi_n . \tag{4.6.5}$$

When the λ_n are ordered such that $\lambda_1 < \lambda_2 < \cdots$ it can be shown that ψ_n has positive (negative) parity for n odd (even) under the transformation $z \to -z$. The transformation $x^\mu, z \to -x^\mu, -z$ is interpreted as the holographic equivalent of the parity transformation in the boundary theory.

If we use these relations we find a Proca action for the fields $B^{(n)}_\mu$, with masses $m_n^2 = \lambda_n$. These fields are interpreted as the vector mesons of the theory. Now it is easy to include scalar fields $\varphi^{(n)}$ as well. As before define the following scalar product

$$\kappa \int \mathrm{d}z\, k(z) \phi_n \phi_m = \delta_{mn} . \tag{4.6.6}$$

We can take ϕ_n to be just $\phi_n = \psi'_n/\sqrt{\lambda_n}$. However there is a zero mode which is orthogonal to all the ψ'

$$\phi_0 = \frac{1}{\sqrt{\kappa\pi}}\frac{1}{k(z)} \ . \tag{4.6.7}$$

In fact the ψ_0 mode whose derivative would be ϕ_0 is $\propto \arctan(z)$: this is not normalizable by means of the integral (4.6.4), however we can easily see that ϕ_0 has the correct normalization with respect to (4.6.6). The μz field strength is rewritten as

$$\mathcal{F}_{\mu z} = \partial_\mu \varphi^{(0)} \frac{1}{\sqrt{\kappa\pi}}\frac{1}{k(z)} + \sum_{n\geq 1}\left(m_n^{-1}\partial_\mu\varphi^{(n)} - B_\mu^{(n)}\right) \ . \tag{4.6.8}$$

The gauge transformation $B_\mu^{(n)} \mapsto B_\mu^{(n)} + m_n^{-1}\partial_\mu\varphi^{(n)}$ can be used to eliminate all the $\varphi^{(n)}$ with $n \geq 1$ from the theory; the $\varphi^{(0)}$ mode survives instead. All in all we get the following Lagrangian

$$S = -\kappa \int \mathrm{d}^4x \left[\sum_{n\geq 1}\left(\frac{1}{4}\mathcal{F}_{\mu\nu}^{(n)}\mathcal{F}^{\mu\nu\,(n)} + \frac{1}{2}m_n^2 B_\mu^{(n)}B^{\mu\,(n)}\right) + \frac{1}{2}\partial_\mu\varphi^{(0)}\partial^\mu\varphi^{(0)}\right] \ . \tag{4.6.9}$$

The massless field $\varphi^{(0)}$ is associated to the mode $\psi_0 \propto \arctan z$ which is an odd function: it is thus a pseudoscalar field and we interpret it as the pion field, which is the Goldstone boson of the spontaneous chiral symmetry breaking.

A similar analysis can be performed to include also the massive scalar mesons: they arise as fluctuations of the embedding of the D8 branes in the Witten's background.

It is possible also to generalize the pion effective action to $N_f > 1$ flavors. As discussed in Section 4.4 the gauge fields $\mathcal{A}_\alpha$ approach zero at $z \to \pm\infty$, but we still have a residual gauge symmetry for gauge functions that approach constants as $z \to \pm\infty$. This residual gauge symmetry is interpreted as the global symmetry of the boundary theory $G^{\mathrm{glob}} = U(N_f)_L \times U(N_f)_R$ (see Section 4.9)

$$\begin{aligned}
&\mathcal{A}_\alpha(x^\mu, z) \mapsto g(x^\mu, z)\mathcal{A}_\alpha(x^\mu, z)g^{-1}(x^\mu, z) - ig(x^\mu, z)\partial_\alpha g^{-1}(x^\mu, z) \ , \\
&\lim_{z\to\pm\infty} g(x^\mu, z) = g_\pm \ , \qquad \lim_{z\to\pm\infty} \partial_\alpha g(x^\mu, z) = 0 \ , \\
&(g_+, g_-) \in G^{\mathrm{glob}} \ .
\end{aligned} \tag{4.6.10}$$

We know that the Wilson line from a point x_A to a point x_B transforms with the gauge function evaluated at the two points. If in particular we consider the path

$$\mathcal{U}(x^\mu) = \mathcal{P}\exp\left(-i\int_{-\infty}^{\infty} \mathrm{d}z\, \mathcal{A}_z(x^\mu, z)\right) \ , \tag{4.6.11}$$

then the transformation law is $\mathcal{U} \mapsto g_+ \mathcal{U} g_-^{-1}$. This is precisely the transformation law for the pion matrix. We can thus define the pion field as

$$\mathcal{U}(x^\mu) \equiv \exp\left(\frac{2i}{f_\pi}\pi^a(x^\mu)T^a\right) , \tag{4.6.12}$$

where T^a are $U(N_f)$ generators normalized to $\mathrm{Tr}\,(T^a T^b) = \frac{1}{2}\delta_{ab}$ and f_π is the pion decay constant[5]. Let us move to a particular gauge, called the $\mathcal{A}_z = 0$ gauge. This is done using a gauge function g defined as

$$g(x^\mu, z) = \mathcal{P}\exp\left(i\int_0^z \mathrm{d}z'\,\mathcal{A}_z(x^\mu, z')\right) . \tag{4.6.13}$$

Under this gauge transformation also $\mathcal{A}_\mu$ changes, but now the requirement $\mathcal{A}_\mu \to 0$ as $z \to \pm\infty$ is not satisfied anymore (this is not a problem since we are dropping the CS terms). We obtain in fact

$$\begin{aligned} &\mathcal{A}_z \mapsto g\mathcal{A}_z g^{-1} - ig\partial_z g^{-1} = 0 , \\ &\mathcal{A}_\mu \mapsto g\mathcal{A}_\mu g^{-1} - ig\partial_\mu g^{-1} \underset{z\to\pm\infty}{\longrightarrow} -i\xi_\pm \partial_\mu \xi_\pm^{-1} , \end{aligned} \tag{4.6.14}$$

where we have defined $\xi_\pm$ as the limit for $z \to \pm\infty$ of g. As a result the expansion in terms of the ψ_n is not valid anymore (because all those functions approach zero): we have to include the non–normalizable zero mode $\psi_0 = \frac{2}{\pi}\arctan z$. This has the limit $\psi_0 \to \pm 1$ as $z \to \pm\infty$. For simplicity we will drop the vector mesons in $\mathcal{A}_\mu$. The following expansion matches the limit properly (we have defined $\psi_\pm(z) = -\frac{i}{2}(1 \pm \psi_0(z))$)

$$\mathcal{A}_\mu(x^\mu, z) = \xi_+\partial_\mu\xi_+^{-1}\,\psi_+(z) + \xi_-\partial_\mu\xi_-^{-1}\,\psi_-(z) . \tag{4.6.15}$$

There is a further residual gauge given by all the functions $h(x^\mu)$ that are independent of z: it is possible to impose $\xi_- = 1$, but in this case ξ_+ becomes exactly the inverse of the pion matrix: $\mathcal{U}^{-1}$

$$\mathcal{A}_\mu(x^\mu, z) = \mathcal{U}^{-1}\partial_\mu\mathcal{U}\,\psi_+ . \tag{4.6.16}$$

We can finally substitute these fields in the DBI action. The field strengths read

$$\begin{aligned} \mathcal{F}_{\mu\nu} &= -i[\mathcal{U}^{-1}\partial_\mu\mathcal{U}, \mathcal{U}^{-1}\partial_\nu\mathcal{U}]\psi_+\psi_- , \\ \mathcal{F}_{\mu z} &= \mathcal{U}^{-1}\partial_\mu\mathcal{U}\psi_+' . \end{aligned} \tag{4.6.17}$$

Using the normalization conditions given at the beginning of this section we find

$$S = -\kappa\int \mathrm{d}^4x\,\mathrm{Tr}\,\left(a\,(\mathcal{U}^{-1}\partial_\mu\mathcal{U})^2 + b\,([\mathcal{U}^{-1}\partial_\mu\mathcal{U}, \mathcal{U}^{-1}\partial_\nu\mathcal{U}])^2\right) , \tag{4.6.18}$$

[5] This implies that the decay constant for the singlet f_S equals f_π: remember that this is true only in the $N_c \to \infty$ limit.

where a and b are constants given by

$$a = \int \mathrm{d}z\, k(z)(\psi_+')^2 = \frac{1}{\pi}\,, \quad b = \int \mathrm{d}z \frac{1}{2} h(z)(\psi_+\psi_-)^2 = -\frac{1}{2\pi^4}\cdot 15.25\ldots\,. \tag{4.6.19}$$

The constant $15.25\ldots$ is the integral $\int \mathrm{d}z \frac{1}{1+z^2}\left(\frac{\pi^2}{4} - \arctan^2 z\right)^2$. We see that we have obtained the Skyrme model (see [14] for a review) with parameters

$$f_\pi = 2\sqrt{\frac{\kappa}{\pi}}\,, \quad e = \frac{1}{32\kappa b} \tag{4.6.20}$$

The Sakai–Sugimoto model without the vector mesons reduces to the Skyrme model in the limit $\lambda \to \infty$. In fact we stopped at the first order in the expansion of the square root in the DBI action, and $\kappa \propto \lambda$. The higher derivative terms are suppressed by further powers of $l_s^2 \propto \lambda^{-1}$, they are thus subleading.

4.7 Mass term for the fermions

As explained above, the Witten–Sakai–Sugimoto model offers an effective description of QCD with massless flavors. In order to introduce a mass term we will follow the proposal of [46, 45]. The key observation in [45] is that the $U(N_f)$ holonomy matrix serves as an order parameter for the chiral symmetry breaking. A possible way to construct such an operator is to insert a fundamental string (actually a worldsheet instanton) stretching between the branes.

In [46] the same setup is described by means of the introduction of a stack of N' D6 branes at the probe level ($N' \ll N_c$). These branes extend on the directions x^μ, τ and two directions ψ_1, ψ_2 of S^4. This setting explicitly breaks chiral symmetry to its diagonal subgroup; moreover, if the D6 branes are not coincident also the $U(N_f)_V$ group is broken to the subgroup $[U(1)]^{N_f}$. On the field theory side the sets of joined D branes D4–D8–D6–$\overline{\text{D8}}$ form a square, so it is possible to define a worldsheet whose boundary lies on this square. Since quarks from D8–D4 are of the opposite chiralities than the ones from $\overline{\text{D8}}$–D4, the worldsheet action involves a $q_L\, q_R$ vertex: the quark mass term. In the gravity point of view the D4–branes are replaced by the curved geometry, the $\overline{\text{D8}}$–D8 are joined together and the D6 are still probe branes.

Correspondingly this modifies the action with an extra term which is the worldsheet amplitude given by the Nambu–Goto action. The tuning of the number of D6 branes N' and their position changes the amplitude accordingly and thus allows us to choose the quark masses.

The worldsheet instanton amplitude is given by

$$N' \frac{1}{g_s}\left(\frac{U_{\mathrm{KK}}}{R}\right)^{9/4} \frac{1}{(2\pi)^3 l_s^4} \int \mathrm{d}^4x\, e^{-S_{\mathrm{NG}}} \mathrm{Tr}\left[\mathcal{P}\exp\left(-i\int_{-\infty}^{\infty} \mathcal{A}_z \mathrm{d}z\right) + \text{c.c.}\right]. \tag{4.7.1}$$

The exponential $e^{-S_{\mathrm{NG}}}$ is proportional to the quark mass matrix, if the masses are degenerate. Otherwise we would need to add scalar fields (the embedding fields of the D6 branes) and make them condensate in the desired positions in order to obtain a flavor dependent quark mass matrix. The final result can be recast in the form

$$S_{\mathrm{mass}} = c \int \mathrm{d}^4 x \,\mathrm{Tr}\, \mathcal{P} \left[M \exp\left(-i \int_{-\infty}^{\infty} \mathcal{A}_z \mathrm{d}z \right) + \mathrm{c.c.} \right] , \tag{4.7.2}$$

where c is a constant whose value has been calculated in [46]

$$c = \frac{1}{3^{9/2}\pi^3} g_{\mathrm{YM}}^3 N_c^{3/2} M_{\mathrm{KK}}^3 \mathcal{N}^{-1} . \tag{4.7.3}$$

The dimensionless number $\mathcal{N}$ is a normalization factor, we can take it to be 1.

Now that the theory contains massive flavors we expect that the θ dependence emerges again. As argued in Section 4.5, the θ term can be introduced as an integral of $C_{(1)}$. But now we only have $\widetilde{F}_{(2)}$. We could attempt a formal solution to the Bianchi identity as

$$\widetilde{F}_{(2)} = \mathrm{d}C_{(1)} + \sqrt{\frac{N_f}{2}} \widehat{A} \wedge \delta(y) \mathrm{d}y . \tag{4.7.4}$$

This is clearly incompatible with the equation of motion, but we can take the integral of this expression on the cigar and regard its asymptotic value for $|\vec{x}| \to \infty$ as a boundary condition on $\widetilde{F}_{(2)}$. As we argued in Section 4.4 we are free to add any "zero mode" on top of the Hořava–Witten solution, so this boundary condition has the purpose of selecting one particular vacuum.

$$\lim_{|\vec{x}| \to \infty} \int \mathrm{d}z \mathrm{d}y \, \widetilde{F}_{zy} = \theta + \lim_{|\vec{x}| \to \infty} \sqrt{\frac{N_f}{2}} \int \mathrm{d}z \, \widehat{A}_z . \tag{4.7.5}$$

This is what defines for us the θ parameter. We can make a gauge transformation on $\widehat{A}_z$ as the one described in Section 4.5 together with a particular redefinition of the mass matrix

$$M' = M e^{i\theta/N_f} . \tag{4.7.6}$$

In this way the "topological" θ disappears, but the mass matrix becomes complex. This exactly reproduces the property of QCD where the physical θ parameter is not the coefficient of $F \wedge F$ but the combination

$$\theta_{\mathrm{phys}} = \theta + \arg\det M , \tag{4.7.7}$$

where $\arg\det M$ is just a compact way of denoting the sum of the complex phases of the mass matrix eigenvalues.

4.8 θ dependence of the vacuum energy

Now let us see how this mass deformation modifies the vacuum solution. In the chiral effective theory [68], in the degenerate case (all quark masses are equal), we would expect a dependence of the kind

$$F(\theta) - F(0) = N_f m_q \Sigma \left[1 - \cos(\theta/N_f)\right] \ , \quad \Sigma = -\frac{\langle\overline{\psi}\psi\rangle}{N_f} \ , \tag{4.8.1}$$

where $F(\theta)$ is the free energy at zero temperature (which corresponds to the vacuum energy) in function of θ, Σ is proportional to the chiral condensate (summed over the flavors) and m_q is the common mass of the quarks $\mathrm{Tr}\, M = N_f m_q$. The vacuum energy per unit volume in holography is computed by the prescription $F = -\mathcal{L}^{\text{on-shell}}_{\text{sugra}}$ ($S = \int \mathrm{d}^4x\, \mathcal{L}$). We see that the term S_{mass} can reproduce this behaviour, we just need to check the coefficient and to make sure that the other terms in the action give negligible contributions.

In [46, 45] is given an estimate of the chiral condensate using the Gell-Mann–Oakes–Renner relation[6] , the result is

$$-\sum_{f=1}^{N_f} \langle\overline{\psi}_f \psi_f\rangle = 2cN_f \tag{4.8.2}$$

Where c is defined in (4.7.3). The mass deformation S_{mass} is given by (4.7.2). In the particular case of degenerate flavors $M_{ij} = m_q\delta_{ij}$ we can assume that the minimum of the energy is given only by the abelian part, i.e. $\mathcal{U} = e^{-i\varphi}\mathbb{1}$, where $\mathcal{U}$ is the holonomy or pion matrix: $\mathcal{P}\exp\left(-i\int \mathcal{A}_z\right)$. We thus have

$$S_{\text{mass}} = 2cN_f m_q \int \mathrm{d}^4x\, \cos\left(\frac{1}{\sqrt{2N_f}}\int_{-\infty}^{\infty} \widehat{A}_z \mathrm{d}z\right) \ . \tag{4.8.3}$$

Since in this Section we will look for vacuum solutions, the field $\mathcal{A}$ is taken to be pure gauge, hence $\delta S_{\text{DBI}}/\delta\mathcal{A}$ is zero, being proportional to $\mathcal{F}$. The only terms appearing are S_{mass} and S_{C_7}. The z component of the equations (4.4.15) yields

$$\begin{aligned} \frac{\delta S_{\text{mass}}}{\delta \widehat{A}_z} &= -\frac{4\pi}{3}\sqrt{\frac{N_f}{2}}[\mathrm{d}C_{(7)}]_{0123,S^4} = \\ &= -\frac{4\pi}{3}\sqrt{\frac{N_f}{2}}\frac{1}{(2\pi l_s)^6}\frac{3}{2}k^2(z)[\widetilde{F}_{(2)}]_{yz} \ . \end{aligned} \tag{4.8.4}$$

[6]For N_f degenerate light quarks: $f_\pi^2 m_\pi^2 = 2\Sigma m_q = 4cm_q$

Using the zero mode found before (4.4.18) the equation becomes simply[7]

$$\frac{\delta S_{\text{mass}}}{\delta \widehat{A}_z} = \sqrt{\frac{N_f}{2}} \frac{2\pi}{(2\pi l_s)^6} C \ , \tag{4.8.5}$$

where we have used $U^3\big|_{y=0} = k(z)$. Expressing the *l.h.s.* as well we find

$$-2cN_f m_q \frac{1}{\sqrt{2N_f}} \sin\left(\frac{1}{\sqrt{2N_f}} \int \mathrm{d}z \widehat{A}_z\right) = \sqrt{\frac{N_f}{2}} \frac{2\pi}{(2\pi l_s)^6} C \ . \tag{4.8.6}$$

Let us define the argument of the sine as φ

$$\varphi \equiv \frac{1}{\sqrt{2N_f}} \int \mathrm{d}z \widehat{A}_z \ . \tag{4.8.7}$$

The (4.7.5) combined with this equation gives

$$-2cN_f m_q \frac{1}{\sqrt{2N_f}} \sin\varphi = \sqrt{\frac{N_f}{2}} \frac{2\pi}{(2\pi l_s)^6} C = \sqrt{\frac{N_f}{2}} \frac{2}{(2\pi l_s)^6} (\theta + N_f \varphi) \ , \tag{4.8.8}$$

where we have also used the integral $\int \mathrm{d}z\mathrm{d}y\,(1+y^2+z^2)^{-2} = \pi$. The coefficients appearing are the masses of the pion and the η' (with mass of the η' we mean only the Witten–Veneziano contribution). In fact they can be found to be

$$m_\pi^2 = \frac{\pi c m_q}{\kappa} \ , \qquad m_{\eta'}^2 = \frac{\pi N_f}{2\kappa} \frac{2}{(2\pi l_s)^6} = \frac{\pi N_f}{2\kappa} \chi_g \ , \tag{4.8.9}$$

where χ_g is the topological susceptibility of the pure gluon theory. We get finally

$$-N_f m_\pi^2 \sin\varphi = m_{\eta'}^2 (\theta + N_f \varphi) \ . \tag{4.8.10}$$

This result is consistent with that of [67, 13][8]. There are two possible limits in which this equation can be solved exactly, in general one would have to solve it graphically (see Figure 4.4). Let us see then the two extreme cases

i) $m_{\eta'}^2 \ll m_\pi^2$: we have a multi-branched solution

$$\varphi = -\frac{\chi_g}{2cm_q}(\theta - 2\pi k) + 2k\pi + \mathcal{O}(m_{\eta'}^4 m_\pi^{-4}) \ , \qquad k \in \mathbb{Z} \ . \tag{4.8.11}$$

This is the limiting case which arises if we take the large N_c limit first. In a sense, this limit is analytically connected with the limit in which the quark

[7] Notice the sign difference due to $\widetilde{F}_{zy}$ instead of $\widetilde{F}_{yz}$

[8] With the difference $\varphi = -\phi_i$([13])

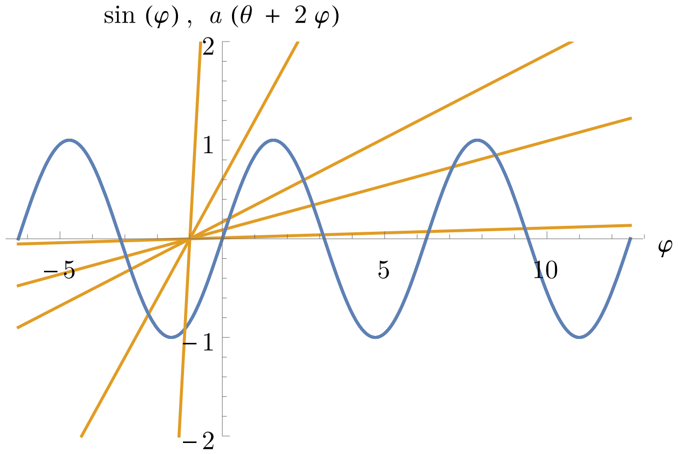


Figure 4.4: Graphical solution to the equation (4.8.10), for $N_f = 2$. As one can easily see for $m_{\eta'} \ll m_\pi$ the orange line is almost horizontal and the solution is multi branched. If the opposite limit is considered, $m_{\eta'} \gg m_\pi$ then the orange line is almost vertical and the solution is roughly $-\theta/N_f$.

mass is so large that the flavors can be integrated out. Correspondingly, the vacuum energy density around $\theta = 0$ goes, to leading order, like

$$F(\theta) - F(0) \sim \frac{\chi_g}{2}\theta^2 , \tag{4.8.12}$$

which is the same as the unflavored theory.

ii) $m_{\eta'}^2 \gg m_\pi^2$: in this case the solution is unique

$$\varphi = -\frac{\theta}{N_f} + \mathcal{O}(m_\pi^2 m_{\eta'}^{-2}) . \tag{4.8.13}$$

This limit is actually closer to the physically acceptable case because $m_\pi \simeq 135\,\text{MeV}$ while $m_{\eta'} \simeq 958\,\text{MeV}$. In this physically acceptable case we have for the Free Energy $F(\theta)$ the following result

$$\mathcal{L}_{\text{mass}}^{\text{on-shell}} = -F(\theta) = 2cN_f m_q \cos\left(\frac{\theta}{N_f}\right) + \mathcal{O}(m_\pi^4) . \tag{4.8.14}$$

The topological susceptibility is thus

$$\chi = \frac{\partial^2 F(\theta)}{\partial\theta^2}\bigg|_{\theta=0} = \frac{2cm_q}{N_f} , \tag{4.8.15}$$

as expected from chiral perturbation theory.

In any case, expanding around the vacuum solution, we can obtain the following θ dependent mass spectrum

$$m^2_{\eta'}(\varphi) = m^2_{\eta'} + m^2_\pi(\varphi) \ , \quad m^2_\pi(\theta) \equiv m^2_\pi \cos\varphi \ , \tag{4.8.16}$$

where by $m_{\eta'}$ we mean the Witten–Veneziano term. Considering the physically acceptable case we see that the meson masses decrease quadratically with θ for θ around zero. This behaviour reflects the general trend already observed in [85] for other mass scales in the unflavored theory.

In the case of non degenerate flavors the phase θ/N_f redistributes unequally in the entries of M. Only the diagonal elements can be taken to be non zero, $\mathcal{U} = \mathrm{diag}\,(e^{-i\varphi_i})$. The equations now give

$$-2cm_i \sin\varphi_i = \chi_g \left(\theta + \textstyle\sum_j \varphi_j\right) \ , \quad \forall i = 1 \cdots N_f \tag{4.8.17}$$

The general case is somewhat cumbersome, we will give the result for $\theta \ll 1$, in the limit *ii)*. Since the *r.h.s.* does not depend on i, it is the same for all equations, we can call it λ. The Largangian to first order in m_q and θ is given by the sum $\sum_i \lambda \cot\varphi_i$, then it suffices to Taylor expand for $\varphi_i \ll 1$. The result is

$$\mathcal{L}^{\mathrm{on-shell}}_{\mathrm{mass}} = -F(\theta) = 2c \left[\sum_i m_i - \frac{1}{2}\left(\sum_i \frac{1}{m_i}\right)^{-1} \theta^2\right] \tag{4.8.18}$$

The topological susceptibility of the full theory is thus

$$\chi = \frac{\partial^2 F(\theta)}{\partial\theta^2}\Big|_{\theta=0} = 2c \left(\sum_i \frac{1}{m_i}\right)^{-1} \tag{4.8.19}$$

4.9 Observations on the flavor symmetry

In this model, as in many other holographic models, the following principle manifests explicitly: *local* gauge symmetries of the bulk theory map into *global* symmetries of the boundary theory. In the Sakai–Sugimoto model the symmetry $U(N_f)_L \times U(N_f)_R$ embodies this idea:

- Bulk theory: gauge invariance with gauge group

$$G^{\mathrm{loc}} = U(N_f)_L \times U(N_f)_R \ , \tag{4.9.1}$$

where $g(x^\mu, z) \in G^{\mathrm{loc}}$ is a group valued function of x^μ and z (the S^4 dependence is suppressed).

- Bulk theory after the gauge fixing: to render the theory easier we make a partial gauge fixing, requiring the fields $\mathcal{A}_\alpha \to 0$ for $z \to \pm\infty$ ($\alpha = \mu, z$). This means that they can be fourier expanded by means of the functions ψ_n and ϕ_n defined in Section 4.6. The gauge group is reduced to

$$G_0^{\rm loc} = \left\{ g(x^\mu, z) \in G^{\rm loc} \text{ s.t. } \lim_{z\to\pm\infty} \partial_\alpha g(x^\mu, z) = 0 \right\} . \tag{4.9.2}$$

Which means that the gauge transformation can approach only a constant when $z \to \pm\infty$.

- Boundary theory: global symmetry

$$G^{\rm glob} = U(N_f)_L \times U(N_f)_R \,, \tag{4.9.3}$$

where $g \in G^{\rm glob}$ is a constant group element. The relation between $G^{\rm glob}$ and $G_0^{\rm loc}$ is the following: consider the subgroup of $G_0^{\rm loc}$ defined as

$$H_0^{\rm loc} = \left\{ g(x^\mu, z) \in G_0^{\rm loc} \text{ s.t. } \lim_{z\to\pm\infty} g(x^\mu, z) = \mathbb{1} \right\} , \tag{4.9.4}$$

which is a normal subgroup. Let us define thus

$$G^{\rm glob} = G_0^{\rm loc}/H_0^{\rm loc} \,. \tag{4.9.5}$$

This means that two gauge transformations $g(x^\mu, z)$ and $g'(x^\mu, z)$ are identified if they both have the same limits for $z \to \pm\infty$.

Usually in the literature gauge transformations approaching $\mathbb{1}$ at infinity (i.e. $H_0^{\rm loc}$) are called “small gauge transformations”, while general transformations approaching a constant are called “large gauge transformations” and they correspond to global symmetries of the action. In the following we will calculate some quantities (such as Noether currents associated to chiral transformations). The requirement for all these quantities to be well defined is that they must be invariant under $H_0^{\rm loc}$, which ensures that, when a global transformation is performed, they are compatible with the quotient in (4.9.5).

Whenever a new term is introduced in the action, namely the θ term and the mass term, the group $H_0^{\rm loc}$ must remain untouched. However some violations of $G_0^{\rm glob}$ are allowed. To be as clear as possible we summarize the assumptions made in the last sections. The mass term $S_{\rm mass}$ explicitly breaks $G^{\rm glob}$ to a subgroup that depends on the mass matrix. If N_e flavors are of equal masses and the other are different we have a subgroup $U(N_e)_V \times [U(1)_V]^{N_f - N_e}$, if all the masses are different we have $[U(1)V]^{N_f}$. In every case the axial transformations are broken.

It is easily seen that the H_0^{loc} group remains untouched, in fact the Wilson line transforms as

$$\mathcal{P}e^{-i\int_{-\infty}^{\infty}\mathrm{d}z\,\mathcal{A}_z} \equiv \mathcal{U} \;\longrightarrow g(-\infty)\,\mathcal{U}\,g(\infty)\;. \tag{4.9.6}$$

The known feature of QCD, that states that any θ angle can be moved via a chiral rotation to the phase of the mass matrix, is reproduced as argued in (4.7.7).

This page is intentionally left blank.

Chapter 5

Holographic dual of baryons

We study the instantonic solution of the Witten–Sakai–Sugimoto model, which is interpreted as a baryon [86, 48]. We quantize the solution and compute the first corrections to the mass spectrum due to the presence of the quark masses and the θ parameter.

5.1 Baryons as D4 branes wrapped on S^4

The idea that baryons can be described holographically as D–branes wrapped on a compact manifold was proposed by Witten in [86]. In the Sakai–Sugimoto model, a baryon vertex is identified with a D4 brane wrapped on S^4 and the baryon number is defined as the charge of that D4 brane.

In presence of D4 branes, the RR fields $C_{(3)}$ and $C_{(5)}$ are not related by Hodge duality, so we have to include in the total action the term

$$\frac{1}{8\pi^2}\int_{D8} C_{(5)} \wedge \mathrm{Tr}\,\mathcal{F}^2 \ . \tag{5.1.1}$$

If $\mathcal{F}$ assumes a configuration such that $\frac{1}{8\pi^2}\int_B \mathrm{Tr}\,\mathcal{F}^2 = n_B$, where B is the space spanned by $x^{1,2,3}$, z, which is an instantonic solution, we obtain

$$n_B \int_{S^4\times x^0} C_{(5)} \ , \tag{5.1.2}$$

which is precisely a charge coupling for a D4 brane with charge n_B. An instantonic configuration on the D8 brane is thus equivalent to a charged D4 brane wrapped on S^4 with charge equal to the instanton number. To show that this setting actually describes a baryon with baryon number n_B we first need to write the action separating the abelian and the non abelian component (see (4.4.14))

$$\begin{aligned} S_{\mathrm{D8}} = &-\kappa \int \mathrm{d}^4x\mathrm{d}z \left(\frac{1}{2}h(z)\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} + k(z)\mathrm{Tr}\,F_{\mu z}F^{\mu}{}_{z}\right) + \\ &-\frac{\kappa}{2}\int \mathrm{d}^4x\mathrm{d}z \left(\frac{1}{2}h(z)\,\widehat{F}_{\mu\nu}\widehat{F}^{\mu\nu} + k(z)\widehat{F}_{\mu z}\widehat{F}^{\mu}{}_{z}\right) + \\ &+\frac{N_c}{24\pi^2}\int_{(5)}\left[\omega_5^{SU(N_f)}(A) + \frac{2}{\sqrt{2N_f}}\widehat{F}\,\mathrm{Tr}\,AF + \frac{1}{\sqrt{2N_f}}\widehat{A}\,\mathrm{Tr}\,F^2 + \right. \\ &\left. + \frac{1}{2\sqrt{2N_f}}\widehat{A}\widehat{F}^2 - \frac{i}{2\sqrt{2N_f}}\widehat{F}\mathrm{Tr}\,A^3 - \frac{i}{2\sqrt{2N_f}}\widehat{A}\mathrm{Tr}\,FA^2\right], \end{aligned} \tag{5.1.3}$$

where the wedge product symbols have been suppressed for simplicity. The $\omega_5^{SU(N_f)}$ is defined in the same way as the ω_5, but now it depends only on the non abelian components. It can be shown that some of these terms are total derivatives, namely

$$\mathrm{d}\left[\widehat{A}\,\mathrm{Tr}\left(2FA - \frac{i}{2}A^3\right)\right] = \widehat{F}\,\mathrm{Tr}\left(2FA - \frac{i}{2}A^3\right) - \widehat{A}\,\mathrm{Tr}\left(2F^2 + \frac{i}{2}FA^2\right). \tag{5.1.4}$$

Thus the action can be recast, up to total derivatives, into

$$\begin{aligned} S_{\mathrm{bulk+D8}} = &-\kappa \int \mathrm{d}^4x\mathrm{d}z \left(\frac{1}{2}h(z)\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} + k(z)\mathrm{Tr}\,F_{\mu z}F^{\mu}{}_{z}\right) + \\ &-\frac{\kappa}{2}\int \mathrm{d}^4x\mathrm{d}z \left(\frac{1}{2}h(z)\,\widehat{F}_{\mu\nu}\widehat{F}^{\mu\nu} + k(z)\widehat{F}_{\mu z}\widehat{F}^{\mu}{}_{z}\right) + \\ &+\frac{N_c}{24\pi^2}\int\left[\omega_5^{SU(N_f)}(A) + \frac{3}{\sqrt{2N_f}}\widehat{A}\,\mathrm{Tr}\,F^2 + \frac{1}{2\sqrt{2N_f}}\widehat{A}\,\widehat{F}^2\right]. \end{aligned} \tag{5.1.5}$$

It is worth noticing that $\omega_5^{SU(N_f)}$ is identically zero for $N_f = 2$. If we call $a(t) = \widehat{A}/\sqrt{2N_f}$ the abelian field and we treat it as a time dependent perturbation over the instantonic solution with baryon number n_B we obtain in the action a term

$$\frac{N_c}{8\pi^2}\int \mathrm{d}x^0\, a \int_B \mathrm{Tr}\,F^2 = n_B N_c \int \mathrm{d}x^0\, a\,. \tag{5.1.6}$$

This is a point-like particle with $U(1)_V$ charge equal to $N_c n_B$; since a baryon is composed by N_c quarks we conclude that this configuration represents a baryon with baryon number n_B. This justifies the interpretation given above.

In Section 4.6 we showed that this model reduces to the Skyrme model when $\lambda \to \infty$, we can indeed draw many analogies between the two models, as we will see in the following.

5.2 Explicit solution in the Sakai Sugimoto model

We have already seen instantons in Chapter 3, in particular we have studied and quantized the BPST instantonic solution for $N_c = 2$ in Euclidean space $x^{1,2,3,4}$. Here we will have some important differences that can easily be overcome

- This is just a conceptual but not mathematical difference, now instead of N_c we have $N_f = 2$, the instantonic solution has nothing to do with instantons in QCD.
- The space in which the instanton lives is $x^{1,2,3}, z$ so we have the non trivial warp factors $h(z)$ and $k(z)$ in the z direction.
- The Chern–Simons coupling deforms the solution.
- There is an extra $\mu = 0$ component.

For a moment let us ignore the CS terms and consider only the Yang–Mills action. Near $z \sim 0$ the space is flat and the solution is the following

$$A^{\rm cl}_M = -i f(\xi) g \partial_M g^{-1} \;, \tag{5.2.1}$$

where[1]

$$f(\xi) = \frac{\xi^2}{\xi^2 + \rho^2} \;, \quad g(x) = \frac{(z-Z)\mathbb{1} - i(\vec{x} - \vec{X}) \cdot \vec{\tau}}{\xi} \;. \tag{5.2.2}$$

If we put this in the action and try to minimize it we find that the *global* minimum $S = 8\pi^2\kappa$ is reached for $\rho = 0$: this is a singular limit called "*small instanton limit*". The introduction of the CS terms cure this problem; in fact separating x^0 and $x^{1,2,3}, z = x^M$ we find a term

$$\frac{3}{8} \varepsilon_{MNPQ} \widehat{A}_0 \mathrm{Tr}\, F_{MN} F_{PQ} \;, \tag{5.2.3}$$

that acts as a potential of a point–like charge. It is known that the energy of a small object goes as $E(\rho) \sim \rho^{-2}$ so it forces ρ to stay finite.

The solution can be found after a systematic $1/\lambda$ expansion. If the above term is included in the action we get a scaling

$$\rho \sim \frac{1}{\sqrt{\lambda}} \;, \tag{5.2.4}$$

so that the instanton will be localized in a region $x^M \sim \mathcal{O}(\lambda^{-1/2})$, $x^0 \sim \mathcal{O}(1)$. The gauge fields A_α scale in the same way as ∂_α so

$$\begin{aligned} \mathcal{A}_M &\in \mathcal{O}(\lambda^{1/2}) \;, & \mathcal{A}_0 &\in \mathcal{O}(\lambda^0) \;, \\ \mathcal{F}_{MN} &\in \mathcal{O}(\lambda) \;, & \mathcal{F}_{M0} &\in \mathcal{O}(\lambda^{1/2}) \;. \end{aligned} \tag{5.2.5}$$

[1] τ^a are the Pauli matrices.

The equations of motion read, up to $\mathcal{O}(\lambda^{-1})$

$$
\begin{aligned}
&D_M F_{MN} = 0 \ , \\
&\partial_M \widehat{F}_{MN} = 0 \ , \\
&D_M F_{0M} + \frac{N_c}{64\pi^2\kappa}\varepsilon_{MNPQ}\widehat{F}_{MN}F_{PQ} = 0 \ , \\
&\partial_M \widehat{F}_{0M} + \frac{N_c}{64\pi^2\kappa}\varepsilon_{MNPQ}\left[\mathrm{Tr}\left(F_{MN}F_{PQ}\right) + \frac{1}{2}\widehat{F}_{MN}\widehat{F}_{PQ}\right] = 0 \ .
\end{aligned}
\tag{5.2.6}
$$

At first order in $1/\lambda$ the BPST solution is not spoiled, but one has to switch on an $\widehat{A}_0$ field. Still, it is consistent to put $\widehat{A}_M = 0$ and $A_0 = 0$. The equation for $\widehat{A}_0$ in the instantonic background reads

$$
\partial_M^2 \widehat{A}_0 + \frac{3N_c}{\pi^2\kappa}\frac{\rho^4}{(\xi^2+\rho^2)^4} = 0 \ , \tag{5.2.7}
$$

from which it follows that

$$
\widehat{A}_0^{\mathrm{cl}} = \frac{N_c}{8\pi^2\kappa}\frac{1}{\xi^2}\left[1 - \frac{\rho^4}{(\rho^2+\xi^2)^2}\right] \ , \quad A_0^{\mathrm{cl}} = \widehat{A}_M = 0 \ . \tag{5.2.8}
$$

Finally we can put the whole solution in the classical action to see if it is minimized. The result upto $\mathcal{O}(\lambda^{-2})$ is[2]

$$
M_B = 8\pi^2\kappa\left[1 + \left(\frac{\rho^2}{6} + \frac{N_c^2}{320\pi^4\kappa^2}\frac{1}{\rho^2} + \frac{Z^2}{3}\right)\right] \ . \tag{5.2.9}
$$

The modulus $\vec{X}$ does not appear in the Lagrangian, so it is a genuine modulus, while Z and ρ do appear. They are classically fixed to be

$$
\rho_{\mathrm{cl}}^2 = \frac{N_c}{8\pi^2\kappa}\sqrt{\frac{6}{5}} \ , \qquad Z^{\mathrm{cl}} = 0 \ . \tag{5.2.10}
$$

Indeed we find that $\rho \sim 1/\sqrt{\lambda}$ (recall the definition of κ: $\kappa = N_c\lambda/216\pi^3$).

In the following discussion we will treat ρ and Z as approximate moduli, allowing them to fluctuate quantum mechanically around their classical value. This is not completely correct because they modify the potential energy, but it remains a good approximate description.

[2] We defined M_B = mass of the Baryon as $(S_{\mathrm{YM}} + S_{\mathrm{CS}})\big|_{\text{on shell}} = -\int \mathrm{d}t M_B$

5.3 Quantization

The solution just found represents a baryon in its ground energy state. We would like to quantize the solution and to describe also the excitations. The baryon is very heavy, its mass being given in first approximation as $8\pi^2\kappa \propto \lambda N_c$. Thus a non relativistic description is perfectly suitable (this is a general feature of baryons on large N_c QCD obviously). The program we would like to follow is:

i) Introduce another modulus so far neglected: the gauge group orientation modulus (analogous to g in (3.2.31)).

ii) Write down the quantized solution allowing the moduli to be time dependent.

iii) Find the metric in the moduli space and the Lagrangian of the moduli.

iv) Write down and solve the Schrödinger equation.

The first point is achieved by a so called "wrong" gauge transformation. The same idea will be used in Section 6.3.3 in a little more detail, so we postpone the discussion. The transformation is

$$\begin{aligned} A_0 &\longmapsto & A'_0 &= 0 \ , \\ A_M &\longmapsto & A'_M &= V A_M V^{-1} - iV\partial_M V^{-1} \ . \end{aligned} \tag{5.3.1}$$

In this section primed quantities are computed *after* the transformation by V and unprimed quantities are computed in the *classical* solution. The function $V(x)$ has a fixed boundary behaviour

$$\lim_{z\to\pm\infty} V(x) = \boldsymbol{a} \ , \tag{5.3.2}$$

where $\boldsymbol{a}$ is a $SU(2)$ matrix corresponding to the gauge group orientation modulus. Let us regard all moduli as time dependent quantities

$$X^M(t) = \{\vec{X}(t), Z(t)\} \ , \quad \rho(t) \ , \quad \boldsymbol{a}(t) \ . \tag{5.3.3}$$

This procedure is not completely harmless, we must be careful so that the equations of motion remain satisfied. The only equation to be taken care of is the *Gauss Law constraint*

$$D_M F_{0M} + \frac{N_c}{64\pi^2\kappa}\varepsilon_{MNPQ}\widehat{F}_{MN}F_{PQ} = 0 \ . \tag{5.3.4}$$

We want to solve it in the gauge $A_0 = 0$; moreover in our solution the second term vanishes reducing the equation to $D_M F_{0M} = 0$. The idea is to make use of the arbitrariness of V to solve this constraint. Calling[3] $\Phi = -iV^{-1}\dot{V}$ and using the property $D'_M X' = V(D_M X)V^{-1}$ we can recast this equation into

$$V D_M \left(\dot{X}^N \frac{\partial}{\partial X^N} A_M + \dot{\rho} \frac{\partial}{\partial \rho} A_M - D_M \Phi \right) V^{-1} = 0 \ . \tag{5.3.5}$$

Thus Φ must cancel all the other terms. The most natural thing to do is to decompose Φ as a sum, each term taking care of one specific piece: $\Phi = \Phi_X + \Phi_\rho + \Phi_{SU(2)}$

$$\begin{aligned} D_M \left(\dot{X}^N \frac{\partial}{\partial X^N} A_M - D_M \Phi_X \right) &= 0 \ , \\ D_M \left(\dot{\rho} \frac{\partial}{\partial \rho} A_M - D_M \Phi_\rho \right) &= 0 \ , \\ D_M \left(D_M \Phi_{SU(2)} \right) &= 0 \ . \end{aligned} \tag{5.3.6}$$

But we have already solved these equations! In Section 3.2 we described the zero modes of the solution. In order to define a metric in terms of them we had to impose a gauge fixing $D_M(\delta_\alpha A_N) = 0$ and we used a function Λ to enforce this gauge condition. Here we have simply $\Lambda_\alpha = -\dot{X}^\alpha \Phi_\alpha$ for $\alpha = X, \rho, SU(2)$. There is no need to redo the computation, the result is exactly the same

$$\begin{aligned} \Phi_X &= -\dot{X}_N \Lambda_N = -\dot{X}^N A_N \ , \\ \Phi_\rho &= -\dot{X}_\rho \Lambda_\rho = 0 \ , \\ \Phi^a_{SU(2)} &= \Lambda^a = f(\xi) g \frac{\tau^a}{2} g^{-1} \ , \end{aligned} \tag{5.3.7}$$

where $f(\xi)$ and g are defined in (5.2.2). We have 3 independent solutions for Φ^a, the correct one is obtained by fixing the boundary value

$$\Phi_{SU(2)} = -i f(\xi) g (\boldsymbol{a}^{-1} \dot{\boldsymbol{a}}) g^{-1} \tag{5.3.8}$$

This ends points *i)* and *ii)*. Point *iii)* is easy because we already have the metric of the moduli space

$$\mathrm{d}s^2 = g_{\alpha\beta} \mathrm{d}X^\alpha \mathrm{d}X^\beta = \frac{M_0}{2} \delta_{MN} \mathrm{d}X^M \mathrm{d}X^N + M_0 \left(\mathrm{d}\rho^2 + \rho^2 \delta_{IJ} \mathrm{d}a^I \mathrm{d}a^J \right) \ , \tag{5.3.9}$$

where $M_0 = 8\pi^2\kappa$ is the leading order expression for the baryon mass. The potential (the full baryon mass M_B) is given by

$$\mathcal{V}(X^\alpha) = M_0 \left[1 + \frac{\rho^2}{6} + \frac{N_c^2}{320\pi^4\kappa^2} \frac{1}{\rho^2} + \frac{Z^2}{3} \right] \ . \tag{5.3.10}$$

[3]The dot is a time derivative

The Lagrangian is readily written as the standard kinetic term in curved space minus the potential energy

$$\begin{aligned}\mathcal{L} &= \frac{1}{2} g_{\alpha\beta} \dot{X}^\alpha \dot{X}^\beta - \mathcal{V}(X^\alpha) = \\ &= \frac{M_0}{2}\left(\dot{\vec{X}}^2 + \dot{Z}^2 + 2(\dot{y}^I)^2\right) - M_0\left[1 + \frac{\rho^2}{6} + \frac{N_c^2}{320\pi^4\kappa^2}\frac{1}{\rho^2} + \frac{Z^2}{3}\right] ,\end{aligned} \tag{5.3.11}$$

where we have switched to cartesian coordinates for $(\rho, \boldsymbol{a})$ defining y^I as ρa^I.

Finally point *iv)* is standard quantum mechanics, some terms are easier and some a little more complicated. First of all the Hamiltonian is easily computed as a Legendre transform

$$\mathcal{H} = M_0 + \mathcal{H}_X + \mathcal{H}_Z + \mathcal{H}_y , \tag{5.3.12}$$

where the single terms are

$$\begin{aligned}\mathcal{H}_X &= -\frac{1}{2M_0}\frac{\partial^2}{\partial X^{i2}} , \\ \mathcal{H}_Z &= -\frac{1}{2M_0}\frac{\partial^2}{\partial Z^2} + \frac{M_0}{3} Z^2 |; , \\ \mathcal{H}_y &= -\frac{1}{4M_0}\sum_{I=1}^{4}\frac{\partial^2}{\partial y^{I2}} + \frac{M_0}{6}\rho^2 + \frac{N_c^2 M_0}{320\pi^4\kappa^2}\frac{1}{\rho^2} .\end{aligned} \tag{5.3.13}$$

They respectively describe a free particle in 3 dimensions, a simple harmonic oscillator in 1 dimension and an harmonic oscillator in 4 dimensions with an extra centrifugal energy. The solution of the first two is immediate:

$$\Psi(\vec{X}) = \frac{1}{(2\pi)^{3/2}} e^{i\vec{P}\cdot\vec{X}} , \qquad E_X = \frac{\vec{P}^2}{2M_0} , \tag{5.3.14}$$

$$\Psi(Z) = H^{(n)}((\sqrt{2/3}M_0)^{1/2}\, Z) e^{-\frac{M_0}{\sqrt{6}} Z^2} , \qquad E_Z = \frac{2n_Z + 1}{\sqrt{6}} . \tag{5.3.15}$$

The last one is not too complicated if one realizes that $1/\rho^2$ is a centrifugal term, hence it is sufficient to switch to spherical coordinates in $\mathbb{R}^4$. The Laplacian decomposes as

$$\sum_{I=1}^{4}\frac{\partial^2}{\partial y^{I2}} = \frac{1}{\rho^3}\partial_\rho\left(\rho^3\partial_\rho\cdot\right) + \frac{1}{\rho^2}\nabla^2_{S^3} . \tag{5.3.16}$$

The eigenstates of $\nabla^2_{S^3}$ are the spherical harmonics, which are defined in S^3 as

$$Y^{(\ell)} = C_{I_1 \cdots I_\ell} a^{I_1} \cdots a^{I_\ell} \ , \quad C \in \boxed{1}\boxed{\cdot}\boxed{\cdot}\boxed{\cdot}\boxed{\ell} \ , \tag{5.3.17}$$

where $C_{I_1 \cdots I_\ell}$ is a totally symmetric and traceless tensor (as the Young tableaux says). The tensor C is determined in the following way: if one multiplies $Y^{(\ell)}$ by ρ^ℓ the result is an harmonic function ($\nabla^2 = 0$). The eigenvalue equation for $Y^{(\ell)}$ is

$$\nabla^2_{S^3} Y^{(\ell)} = -\ell(\ell+2) Y^{(\ell)} \ . \tag{5.3.18}$$

Here we will use these eigenfunctions to solve the problem, but the following isomorphism is very useful to gain a physical interpretation of the result:

$$SO(4) \cong SU(2) \otimes SU(2)/\mathbb{Z}_2 \ . \tag{5.3.19}$$

The irreducible representation $\boldsymbol{\ell}$, given by the single row tableaux, is mapped into the symmetric combination $\left(\frac{\boldsymbol{\ell}}{\mathbf{2}}, \frac{\boldsymbol{\ell}}{\mathbf{2}}\right)$. The two quantum numbers are interpreted as the spin and the isospin of the baryon. This picture enables us to describe only baryonic states with equal spin and isospin: this is consistent with the Skyrme model where this constraint comes directly form the "*hedgehog ansatz*" used to solve the equations.

Here we try to give a brief explanation of this physical interpretation. Recall the transformation of the Skyrmion described in Section 3.6: a *left* rotation is an isospin rotation while a *right* rotation is a global rotation. Using the isomorphism $SO(4) \cong SU(2) \otimes SU(2)$ we can interpret the two factors as *left* and *right* rotations. In fact the isomorphism is manifest in the formalism of quaternions. Let $h \in \mathbb{H}$ be

$$h = a + \underline{i}b + \underline{j}c + \underline{k}c \ . \tag{5.3.20}$$

If we regard h as a four dimensional vector, a general $SO(4)$ rotation can be parametrized by a couple of quaternions of modulus 1 q_1 and q_2

$$h \to q_1 \, h \, q_2 \ . \tag{5.3.21}$$

This is indeed a left/right rotation because the set of quaternions of modulus 1 is mapped in $SU(2)$ by identifying

$$\underline{i} = i\sigma^3 \ , \quad \underline{j} = i\sigma^2 \ , \quad \underline{k} = i\sigma^1 \ , \quad 1 = \mathbb{1}_2 \ . \tag{5.3.22}$$

We can also understand the reason of the "$/\mathbb{Z}_2$": multiplying q_1 and q_2 by -1 has no effect on (5.3.21).

The obvious ansatz for $\Psi(y^I)$ is

$$\Psi(y^I) = R(\rho) Y^{(\ell)}(a^I) \ . \tag{5.3.23}$$

The centrifugal term modifies the angular momentum

$$\ell(\ell+2) \to \ell(\ell+2) + \frac{N_c^2 M_0^2}{80\pi^4\kappa^2} \equiv \tilde{\ell}(\tilde{\ell}+2) \ . \tag{5.3.24}$$

Thus, upon substituting $\ell \to \tilde{\ell}$ we end up with a regular harmonic oscillator in 4 dimensions in spherical coordinates. The solution is

$$R(\rho) = e^{-\frac{M_0}{\sqrt{6}}\rho^2} \rho^{\tilde{\ell}} \, {}_1F_1\left(n_\rho, \tilde{\ell}+2, \sqrt{2/3} M_0 \, \rho^2\right) \ , \quad n_\rho \in \mathbb{N} \ . \tag{5.3.25}$$

The function ${}_1F_1\,(a,b,c)$ is the Confluent Hypergeometric Function. The energies are

$$E_\rho = \frac{1}{\sqrt{6}}(2n_\rho + \tilde{\ell} + 2) = \frac{2n_\rho + 1}{\sqrt{6}} + \sqrt{\frac{(\ell+1)^2}{6} + \frac{2N_c^2}{15}} \ . \tag{5.3.26}$$

We can also give some explicit examples of the angular dependence: in general the spherical harmonics $Y^{(\ell)}(a^I)$ will be polynomials in the a^I. The degree must be odd if we want to describe baryons because, since they are fermions, the wavefunction must be parity odd:

$$\Psi(-y^I) = -\Psi(y^I) \ . \tag{5.3.27}$$

In particular the states with spin s_3 and isospin I_3, $|s_3, I_3\rangle$ are

$$\begin{aligned} |s_3, I_3\rangle = \quad & |\uparrow,\uparrow\rangle \propto a_1 + i a_2 \ , \qquad && |\uparrow,\downarrow\rangle \propto a_4 + i a_3 \ , \\ & |\downarrow,\uparrow\rangle \propto a_4 - i a_3 \ , && |\downarrow,\downarrow\rangle \propto a_1 - i a_2 \ . \end{aligned} \tag{5.3.28}$$

In the general case one should apply these operators to the states to compute their quantum numbers

$$I_a = \frac{i}{2}\left(y_4 \frac{\partial}{\partial y_a} - y_a \frac{\partial}{\partial y_4} - \varepsilon_{abc} y_b \frac{\partial}{\partial y_c}\right) \ , \ J_a = \frac{i}{2}\left(-y_4 \frac{\partial}{\partial y_a} + y_a \frac{\partial}{\partial y_4} - \varepsilon_{abc} y_b \frac{\partial}{\partial y_c}\right) . \tag{5.3.29}$$

It can be easily checked that they both form a $SU(2)$ algebra, moreover they are related to the $SO(4)$ generators of rotations M_{IJ} in a simple way

$$M_{ab} = I_a + J_a \ , \qquad M_{4a} = I_a - J_a \ . \tag{5.3.30}$$

A baryon is a state $|B, s\rangle$ in the Hilbert space defined by this Hamiltonian. As said above s is the (iso)spin of the baryon, equal to $\ell/2$. The quantum numbers n_ρ and n_Z describe excited baryons and/or resonances, the case $n_\rho = n_Z = 0$ corresponds to the neutron ($I_3 = \downarrow$) and the proton ($I_3 = \uparrow$).

5.4 Quark mass and θ deformation

The mass term for the flavors gives a further contribution to the baryon mass. Let us follow the procedure of [87] in the simpler case of degenerate quark masses. First of all we compute the on shell value of the extra term in the action

$$S_{\text{mass}} = c \int \mathrm{d}^4 x \, \mathrm{Tr}\, \mathcal{P}\left[M e^{i\frac{\theta}{2}}\left(e^{-i\int_{-\infty}^{\infty} \mathrm{d}z A_z} - \mathbb{1}\right) + \text{c.c.}\right] \ . \tag{5.4.1}$$

The $\mathbb{1}$ subtraction corresponds to the subtraction of the vacuum energy (in the case of degenerate masses the minimum is for $U = \mathbb{1}$). In principle the presence of this mass term would modify the classical solution: we neglect this effect and consider the solution $A^{\rm cl}$ found before. We work in singular gauge, where the A_z field is given by

$$A_z = \left[\frac{1}{\xi^2} - \frac{1}{\xi^2+\rho^2}\right](\vec{x} - \vec{X}) \cdot \vec{\tau}\ , \tag{5.4.2}$$

which is related to the gauge used previously by the gauge transformation $A_z \to g^{-1}A_z g - i g^{-1}\partial_z g$, where g is given by (5.2.2).

> This gauge acts on the pion matrix in an easy way
>
> $$U \to g(z \to -\infty) U g^{-1}(z \to \infty) = -U\ .$$
>
> This minus sign is important to make the integral (5.4.7) converge. We can observe that this is indeed the correct prescription in the case of degenerate masses. In fact the pion matrix should approach $\mathbb{1}$ as $|\vec{x}| \to \infty$, this is what we get after computing the integral.

The pion matrix is easily computed

$$U = \mathcal{P}\exp\left[-i(\vec{x}-\vec{X})\cdot\vec{\tau}\int_{-\infty}^{\infty}\mathrm{d}z\left(\frac{1}{\xi^2} - \frac{1}{\xi^2+\rho^2}\right)\right] \tag{5.4.3}$$

We can drop the path ordering since $(x^a - X^a)\tau^a(x^b - X^b)\tau^b\cdots$ is completely symmetric so there is no ordering ambiguity; we also set $\vec{X} = 0$ without loss of generality. The integral is very easy to perform, yielding

$$U = \exp\left[-i\pi\frac{\vec{\tau}\cdot\vec{x}}{|\vec{x}|}\left(1 - \frac{1}{\sqrt{1+\rho^2/|\vec{x}|^2}}\right)\right] \equiv \exp\left[-i\frac{\vec{\tau}\cdot\vec{x}}{|\vec{x}|}\alpha(|\vec{x}|)\right]\ . \tag{5.4.4}$$

In the expansion of the exponential, every even power gives something proportional to $\mathbb{1}$, while every odd power gives $\vec{\tau}\cdot\vec{x}/|\vec{x}|$

$$U = \mathbb{1}\cos\alpha(|\vec{x}|) - i\frac{\vec{\tau}\cdot\vec{x}}{|\vec{x}|}\sin\alpha(|\vec{x}|)\ . \tag{5.4.5}$$

The shift in mass δM is given by $-\int \mathrm{d}^3x\,\mathcal{L}_{\rm mass}$, where $S_{\rm mass} = \int \mathrm{d}^4x\,\mathcal{L}_{\rm mass}$. We have

$$\delta M = -c\int \mathrm{d}^3x\,\mathrm{Tr}\left[M\left(2\mathbb{1}\cos(\varphi)(\cos\alpha - 1) - 2\frac{\vec{\tau}\cdot\vec{x}}{|\vec{x}|}\sin(\varphi)\sin\alpha\right)\right]\ , \tag{5.4.6}$$

where φ is the vacuum value of the integral of $\widehat{A}_z$ as discussed in Section 4.8. Let us now focus on the $N_f = 2$ case in the physical mass regime $m_p i \ll m_{\eta'}$, so that we can set $\varphi = -\theta/2$ up to subleading corrections in the mass ratio. From a simple parity argument ($x^3 \to -x^3$) the term proportional to $\sin(\varphi)$ vanishes. We define the integration variable $r = |\vec{x}|/\rho$ and write

$$\delta M = 4c\, m_q \cos(\theta/2) \int_0^\infty \mathrm{d}r\, 4\pi\rho^3 r^2 \left[1 + \cos\left(\frac{\pi}{\sqrt{1+r^{-2}}}\right)\right] , \tag{5.4.7}$$

where $m_q \equiv m_u = m_d$. The integral is evaluated numerically and the final result is

$$\delta M = 16\pi\rho^3 c\, m_q \cos(\theta/2) \cdot 1.104 \, . \tag{5.4.8}$$

The quantum contribution to this mass splitting, that differentiates the various species of baryons, follows in the same way as [87], so we will skip it.

A relevant result of this section is that the baryon Hamiltonian, through the mass term piece δM computed above, does not receive linear $\mathcal{O}(\theta)$ contributions at small θ. Since in the following we will be interested in the neutron electric dipole moment, which is of first order in θ, the θ-corrections to the baryon states will be neglected consistently. The mass splitting δM at $\theta = 0$ will anyway perturb some of the baryonic properties. In the semiclassical limit it will in fact affect the size of the baryon ρ which will get an $\mathcal{O}(m_q)$ correction. The latter will give subleading corrections to the NEDM, so we will discard it here.

When different quark masses are considered the result is different. First of all we should impose that the pion matrix $\mathcal{U} = e^{i\frac{\theta}{2}} U$ approaches $\mathcal{U}_0$ when $|\vec{x}| \to \infty$. The matrix $\mathcal{U}_0$ turns out to be:

$$\mathcal{U}_0 = e^{i\frac{\theta}{2}} \begin{pmatrix} e^{i\Phi} & 0 \\ 0 & e^{-i\Phi} \end{pmatrix} = e^{i\frac{\theta}{2}} U_0 \, , \tag{5.4.9}$$

$$\cos\Phi = \frac{\cos\frac{\theta}{2}}{\sqrt{\cos^2\frac{\theta}{2} + \left(\frac{m_d - m_u}{m_u + m_d}\right)^2 \sin^2\frac{\theta}{2}}} \, , \quad \sin\Phi = \frac{\frac{m_d - m_u}{m_u + m_d}\sin\frac{\theta}{2}}{\sqrt{\cos^2\frac{\theta}{2} + \left(\frac{m_d - m_u}{m_u + m_d}\right)^2 \sin^2\frac{\theta}{2}}} \, . \tag{5.4.10}$$

The classical action has to be modified as

$$S_{\text{mass}} = c \int \mathrm{d}^4x \, \mathrm{Tr}\, \mathcal{P} \left[M e^{i\frac{\theta}{2}} \left(e^{-i\int_{-\infty}^{\infty} \mathrm{d}z A_z} - U_0 \right) + \text{c.c.} \right] , \tag{5.4.11}$$

and the solution A_z must be computed after a global gauge rotation that satisfies $\lim_{|x|\to\infty} U = U_0$ (we could take for instance $g(\infty) = U_0$ and $g(-\infty) = \mathbb{1}$). The result follows easily

$$\delta M = 8\pi\rho^3 c \mathrm{Tr}\,(M U_0) \cos(\theta/2) \cdot 1.104 \, . \tag{5.4.12}$$

Another interesting difference of the non degenerate mass case is the quantization procedure. In the previous case the $SU(2)$ modulus $\boldsymbol{a}$ did not enter in the quantization because the only surviving contribution in the integral was proportional to $\mathbb{1}$. In this case the quantization gives a term

$$\delta M \propto \mathrm{Tr}\left(M\boldsymbol{a}U_0\boldsymbol{a}^{-1}\right) , \tag{5.4.13}$$

thus giving a potential mass splitting between states with different isospin. For the case of the proton and the neutron this splitting would be too small compared to the electromagnetic splitting (not included in this analysis), so we ignore this computation.

Chapter 6

Neutron electric dipole moment

In this Chapter we perform the computation of the neutron electric dipole moment in the Witten–Sakai–Sugimoto model. First we solve the supergravity equations of motion in presence of the quark mass deformation of the theory studied previously, then we quantize the solution and extract the dipole moment. In the last part we also show that the CP breaking pion–nucleon–nucleon cubic coupling is zero within the limits and approximations adopted here.

6.1 Introduction

Now we come to the main aim of this thesis: the computation of the Neutron Electric Dipole Moment (NEDM). Why is this quantity so important in the theory of Quantum Cromodynamics? As we saw in the previous chapter (Chapter 3) the theory of QCD may exhibit an explicit breaking of P and CP (or T). This breaking is induced by the presence of the θ parameter, introduced after the instantonic solution was found [3]. Effects of a non zero CP breaking parameter in QCD can be seen by focusing on observables like the NEDM and the decay rate $\eta \to \pi\,\pi$ [88]. The second observable puts an upper bound several orders of magnitude less stringent than the NEDM so we will ignore it. The NEDM receives contributions also from Electroweak interactions (the phase of the CKM matrix [89]) but its effect is much smaller than the one coming from Strong interactions, so we can keep focusing of QCD.

First of all, why is a non zero NEDM a source of CP breaking? The answer is very simple: the electric dipole couples to the electric field in the standard way $\vec{E} \cdot \vec{d}$. Since there is no preferred direction and the neutron is electrically neutral, its dipole has to be proportional to the spin, which is a pseudovector, making $\vec{E} \cdot \vec{d}$ parity odd. The magnetic moment on the other hand couples to the magnetic field which is a pseudovector too; indeed the magnetic moment of the Neutron is known and is non zero.

There are two possible ways to define a dipole:

i) Adding an external electric field to the theory, then computing the resulting term in the action to first order in the field. Everything that multiplies the term $\propto \vec{E}$ in the action, computed on shell, is the NEDM.

ii) Defining the Noether's current associated to the electromagnetic $U(1)$ symmetry, then computing the following integral

$$\vec{D}_{n,s} = \int \mathrm{d}^3x\, \vec{x}\, \langle n, s|J^0_{\mathrm{em}}|n, s\rangle \ , \tag{6.1.1}$$

where $|n, s\rangle$ is the state of the neutron in spin s and J_{em} the electromagnetic current.

It turns out that these two methods are exactly the same in our model: as we will see in the next section the definition of the chiral currents does not follow the Noether prescription[1] but uses the bulk–boundary correspondence; a field in the boundary can be seen as an external field coupled to the bulk theory, the term in the action linear in those fields is defined to be the current, the two definitions hence match.

There are many different methods to estimate the NEDM, among them Lattice QCD, Chiral Perturbation Theory and the Skyrme model (see [68] for a general review and [90, 40] for the computation in the Skyrme model for $N_f = 2, 3$). There are also other holographic computations in bottom–up models [91]. The Skyrme computation is actually very similar to ours: we can almost make a "dictionary" to translate our quantities with the ones in the Skyrme model. For instance the Skyrmion solution corresponding to a Baryon here is the Instanton $A^{\mathrm{inst}}_{\alpha}$.

In Table 6.1 we summarize the estimates of the NEDM coming from different approaches. A first rough estimate of the relevant order of magnitude can be given as follows. Chiral symmetry breaking and the CP breaking are both responsible for the NEDM to be non zero: the term in the effective action that causes the breaking is $\sim \theta m_\pi^2$. Hence taking as an energy scale the nucleon mass M_N, on dimensional grounds one gets

$$d_n \sim |\theta|\, e\, m_\pi^2/M_N^3 \sim 10^{-16}\, |\theta|\, e \cdot \mathrm{cm}\ . \tag{6.1.2}$$

[1] In [83] the currents are computed via the Noether prescription, however they have some problems: first of all they are not gauge invariant, second the transformation parameter necessary for the Noether theorem $\zeta(z)$ is an unspecified function of z, so there is a further ambiguity in the definition.

As one can see from Table 6.1 the calculated values agree with this estimate within an order 10 but the results are still very different from one another and there is no agreement on the sign.

Year	Approach/model	$c_n = d_n/(\theta \cdot 10^{-16} e \cdot \mathrm{cm})$
1979 [34]	bag model	2.7
1980	ChPT	3.6
1981	ChPT	1
1981	ChPT	5.5
1982	ChPT	20
1984	chiral bag model	3.0
1984	soft pion Skyrme model	1.2
1984	single nucleon contribution	11
1990 [40]	Skyrme model $N_f = 3$	2
1991 [90]	Skyrme model $N_f = 2$	1.4
1991	ChPT	3.3(1.8)
1991	ChPT	4.8
1992	ChPT	$-7.2, -3.9$
1999	sum rules	2.4(1.0)
2000	heavy baryon ChPT	7.5(3.2)
2004	instanton liquid	10(4)
2007	holographic QCD “hard–wall”	1.08
2007 [36]	Lattice QCD	$-27.6(7.2)$
2015 [35]	Lattice QCD	$-3.9(2)(9)$

Table 6.1: Here the theoretical values for $d_n = \theta\, c_n\, 10^{-16}\, e \cdot \mathrm{cm}$ are summarized, the table is partially taken from [68], where the original references are indicated. For ChPT we mean Chiral Perturbation Theory.

Experimentally the NEDM is measured [31] using ultra cold neutrons trapped in a potential and exposed to an electro–magnetic field $\vec{E}$, $\vec{B}$. The Larmor frequency is easily computed to be (μ_n is the magnetic moment)

$$h\nu_L = |2\mu_n \vec{B} + 2d_n \vec{E}| \ . \tag{6.1.3}$$

When $\vec{E}$ is parallel to $\vec{B}$ we get the maximum value, the contrary happens if they are antiparallel, so the aim is to measure any shift in the Larmor frequency when the electric and magnetic field are flipped from the parallel configuration to the antiparallel one. The result is an upper bound

$$|d_n| < 2.9 \cdot 10^{-26} e \cdot \mathrm{cm} \quad (90\%\ \mathrm{CL}) \ . \tag{6.1.4}$$

From this it follows a bound $|\theta| \lesssim 10^{-10}$. The reason why θ is so small (or even zero) in nature is an open problem in modern Physics that goes under the name of

"Strong CP problem". A famous solution involves a new particle, the Axion; the corresponding model exhibits a new spontaneously broken symmetry at very high energies (of which the Axion is a Goldstone boson) [33] and θ becomes dynamical, allowing it to relax at the minimum of the potential.

6.2 Chiral currents

Here we will give a brief summary of the chiral currents in the Sakai–Sugimoto model, as they have been derived in [49]. We use the bulk–boundary correspondence to define the currents. Let us introduce an external field in the theory by switching on a non–normalizable mode for the gauge fields $\mathcal{A}_\mu$, so that

$$\lim_{z\to\pm\infty} \mathcal{A}_\mu(x^\mu, z) = \mathcal{A}_{\mu\, L(R)}(x^\mu) \, . \tag{6.2.1}$$

These modes can be seen as perturbations over the background (that approaches zero at infinity), whose boundary value is kept fixed. The theory is now modified and we expect an additional term in the action

$$S\big|_{\mathcal{A}_{L(R)}} = -2 \int \mathrm{d}^4x \,\mathrm{Tr}\, \left(\mathcal{A}_{\mu\, L}\mathcal{J}^\mu_L + \mathcal{A}_{\mu\, R}\mathcal{J}^\mu_R\right) \, , \tag{6.2.2}$$

which is a source–current coupling. This term defines the chiral currents $\mathcal{J}^\mu_{L(R)}$. The result is the following

$$\begin{aligned} \mathcal{J}_{\mu\, L} &= -\kappa[k(z)\mathcal{F}_{\mu z}]_{z\to\infty} \, , \\ \mathcal{J}_{\mu\, R} &= +\kappa[k(z)\mathcal{F}_{\mu z}]_{z\to-\infty} \, . \end{aligned} \tag{6.2.3}$$

We can split the abelian and non abelian part of the current as in (4.4.14) to get the isoscalar and isovector contribution; moreover we can define the axial and vector current in the following way ($\psi_0 = \frac{2}{\pi}\arctan z$)

$$\begin{aligned} \mathcal{J}_{\mu\, V} &= -\kappa[k(z)\mathcal{F}_{\mu z}]^{z\to\infty}_{z\to-\infty} \, , \\ \mathcal{J}_{\mu\, A} &= -\kappa[\psi_0(z)k(z)\mathcal{F}_{\mu z}]^{z\to\infty}_{z\to-\infty} \, , \end{aligned} \tag{6.2.4}$$

associated to the vector $(+)$ and axial $(-)$ current

$$\mathcal{V}^{(+)}_\mu = \frac{1}{2}(\mathcal{A}_{\mu\, L} + \mathcal{A}_{\mu\, R}) \, , \quad \mathcal{V}^{(-)}_\mu = \frac{1}{2}(\mathcal{A}_{\mu\, L} - \mathcal{A}_{\mu\, R}) \, . \tag{6.2.5}$$

We can use this prescription to calculate observables like form factors. Let us take the mode expansion of the mesonic tower reviewed in Section 4.6. The source–current term reads

$$S\big|_{\mathcal{A}_{L(R)}} = 2 \int \mathrm{d}^4x \,\mathrm{Tr}\, \left[\mathcal{V}^{(+)}_\mu \sum_{n=1} g_{v^n} v^n_\mu + \mathcal{V}^{(-)}_\mu \left(\sum_{n=1}^{\infty} g_{a^n} a^n_\mu + f_\pi \partial_\mu \Pi\right)\right] \, , \tag{6.2.6}$$

where a_μ^n and v_μ^n are, respectively the pseudovector and vector mesons while Π is the pion (non abelian part) and the η' (abelian part)[2]. The decay constants g_{v^n} and g_{a^n} are given by the z integral of the eigenfunctions ψ_n

$$g_{v^n} = \lambda_{2n-1}\kappa \int \mathrm{d}z\, h(z)\psi_{2n-1}(z) \;, \quad g_{a^n} = \lambda_{2n}\kappa \int \mathrm{d}z\, h(z)\psi_{2n}(z)\psi_0(z) \;. \quad (6.2.7)$$

These integrals are actually only a boundary term thanks to the eigenvalue equations (4.6.5)

$$g_{v^n} = -\kappa[k(z)\partial_z\psi_{2n-1}(z)]_{z\to-\infty}^{z\to\infty} \;, \quad g_{a^n} = -\kappa[k(z)\partial_z\psi_{2n}(z)\psi_0(z)]_{z\to-\infty}^{z\to\infty} \;. \quad (6.2.8)$$

Finally f_π has the following expression

$$f_\pi = 2\sqrt{\frac{\kappa}{\pi}} \;. \quad (6.2.9)$$

From this interaction action we can study how the mesons interact with an external field; this is very useful if we want to compute form factors.

If we insert in the definition (6.2.4) the instantonic solution, the result, after quantization, will give us the chiral current in presence of baryons. This is not as simple as that, since the solution found in [48] (and reviewed in Chapter 5.1) is valid only around $\xi \ll 1$, while here we need to take a limit $z \to \pm\infty$. In order to get a sensible answer we need to find the asymptotic solution $\xi \gg 1$. This solution was found in [49] using the following method: fist notice that there is a large region of overlapping between the asymptotic solution and the near $\xi \sim 0$ solution. This is the zone $\rho \ll \xi \ll 1$. One then proceeds to find the asymptotic form for $\rho \ll \xi \ll 1$ of the known solution and then try to match this behaviour to the one for $\xi \gg 1$.

Far from the origin the warp factors $k(z)$ and $h(z)$ cannot be neglected anymore, so we have more complicated functions. Here is the asymptotic solution[3] in the so called singular gauge

$$\begin{aligned}
&\widehat{A}_0 = -\frac{N_c}{2\kappa\lambda} G(\vec{x}, z, \vec{X}, Z) \;, \\
&\widehat{A}_i = \widehat{A}_z = 0 \;, \\
&A_0 = 0 \;, \\
&A_i = -2\pi^2\rho^2\tau^a \left(\varepsilon^{iaj}\frac{\partial}{\partial X^j} - \delta^{ia}\frac{\partial}{\partial Z}\right) G(\vec{x}, z, \vec{X}, Z) \;, \\
&A_z = -2\pi^2\rho^2\tau^a \frac{\partial}{\partial X^a} H(\vec{x}, z, \vec{X}, Z) \;,
\end{aligned} \quad (6.2.10)$$

[2]Actually the pions and the η' mix when the masses are non zero, we should call it the "singlet".

[3]This is a little bit different from the one in [49] because we have not considered the gauge group orientation moduli yet, this will be done in the following section, moreover here all the moduli of the solution are time independent.

where

$$\begin{aligned}
G(\vec{x}, z, \vec{X}, Z) &= -\kappa \sum_{n=1}^{\infty} \psi_n(z)\psi_n(Z) \frac{e^{-\sqrt{\lambda_n} r}}{4\pi\, r} \ , \quad r = |\vec{x} - \vec{X}| \ , \\
H(\vec{x}, z, \vec{X}, Z) &= -\kappa \sum_{n=0}^{\infty} \phi_n(z)\phi_n(Z) \frac{e^{-\sqrt{\lambda_n} r}}{4\pi\, r} \ .
\end{aligned} \tag{6.2.11}$$

Notice that the currents are gauge invariant under H_0^{loc} as one would require (see Section 4.9). This solution has to be quantized and the standard way to do it is to perform a rotation in the group (not a gauge transformation, it has to be a physical deformation of the solution) and to introduce a time dependence on the moduli $X, Z, \rho, \boldsymbol{a}$ where $\boldsymbol{a}$ is the parameter of the gauge group orientation. After doing this we can quantize the moduli by means of the Hamiltonian discussed in Chapter 5.1 and then compute the expectation values $\langle B', s' | \mathcal{J}_\mu^{A,V} | B, s \rangle$. All the interesting static properties of baryons can be derived with this formalism.

In the following we will study how this solution is deformed by the presence of a mass and a θ term. We will then compute the additional contributions to the chiral currents and finally the neutron electric dipole moment according to the definition (6.1.1).

6.3 Computation on the Sakai–Sugimoto model

To perform the computation of the NEDM we will need to solve the equations of motion of the gauge fields, modified by the mass term. This will be done at leading order both in m_q and θ. For simplicity we will focus on the case of two degenerate masses $m_u = m_d$. The equations are

$$-\kappa \left(h(z)\partial_\nu \widehat{F}^{\mu\nu} + \partial_z (k(z) \widehat{F}^{\mu z}) \right) + \frac{N_c}{128\pi^2} \varepsilon^{\mu\alpha\beta\gamma\delta} \left(F^a_{\alpha\beta} F^a_{\gamma\delta} + \widehat{F}_{\alpha\beta} \widehat{F}_{\gamma\delta} \right) = 0 \ , \tag{6.3.1}$$

$$-\kappa \left(h(z) D_\nu F^{\mu\nu} + D_z (k(z) F^{\mu z}) \right)^a + \frac{N_c}{64\pi^2} \varepsilon^{\mu\alpha\beta\gamma\delta} F^a_{\alpha\beta} \widehat{F}_{\gamma\delta} = 0 \ , \tag{6.3.2}$$

$$\begin{aligned}
&-\kappa\, k(z) \partial_\nu \widehat{F}^{z\nu} + \frac{N_c}{128\pi^2} \varepsilon^{z\mu\nu\rho\sigma} \left(F^a_{\mu\nu} F^a_{\rho\sigma} + \widehat{F}_{\mu\nu} \widehat{F}_{\rho\sigma} \right) = \\
&\quad = -\frac{4\pi}{3} \sqrt{\frac{N_f}{2}} [\mathrm{d}C_{(7)}]_{0123} + ic \,\mathrm{Tr} \left[\frac{M}{\sqrt{2N_f}} \left(\mathcal{P} e^{-i\int_{-\infty}^{\infty} \mathcal{A}_z \mathrm{d}z} - \text{c.c.} \right) \right] \ ,
\end{aligned} \tag{6.3.3}$$

$$\begin{aligned}
&-\kappa\, k(z) (D_\nu F^{z\nu})^a + \frac{N_c}{64\pi^2} \varepsilon^{z\mu\nu\rho\sigma} F^a_{\mu\nu} \widehat{F}_{\rho\sigma} = \\
&\quad = ic \,\mathrm{Tr}\, \mathcal{P} \left[M \frac{\tau^a}{2} \left(e^{-i\int_{-\infty}^{\infty} \mathcal{A}_z \mathrm{d}z} - \text{c.c.} \right) \right] \ .
\end{aligned} \tag{6.3.4}$$

The factors N_f are displayed explicitly but will soon be substituted by "2".

The solution will be decomposed in three different contributions: $\mathcal{A}^{\text{vac}}$, $\mathcal{A}^{\text{inst}}$ and $\mathcal{A}^{\text{mass}}$. The first one is the vacuum solution found in Section 4.8, the second one is the BPST instantonic solution in singular gauge given below and the last one is the perturbation due to the presence of the mass term that we wish to compute.

$$\begin{aligned}
&A_z^{\text{vac}} = \mathcal{A}_\mu^{\text{vac}} = 0 \,, && \int \mathrm{d}z\, \widehat{A}_z^{\text{vac}} = 2\,\varphi^{\text{vac}} \,, \\
&A_M^{\text{inst}} = -i(1-f(\xi))g^{-1}\partial_M g \,, && \widehat{A}_0^{\text{inst}} = \frac{N_c}{8\pi^2\kappa}\frac{1}{\xi^2}\left[1-\frac{\rho^4}{(\rho^2+\xi^2)^2}\right] \,, \\
&A_0^{\text{inst}} = \widehat{A}_M^{\text{inst}} = 0 \,,
\end{aligned} \tag{6.3.5}$$

where φ^{vac} is the solution to (4.8.10). We also have the vacuum for $C_{(7)}$

$$[\mathrm{d}C_{(7)}]_{0123} = \frac{3cm_q}{2\pi}\sin\varphi^{\text{vac}} \tag{6.3.6}$$

This is the vacuum solution but it still holds in the baryon background since all the fluctuations are required to vanish in the limit $|\vec{x}| \to \infty$, hence the boundary condition (4.7.5) is left unchanged. In the following we will focus on the limit in which the η' is much more massive than the pion (the one we called " case *ii)*" in Section 4.8), at the same time we must recall that both masses must be much smaller than M_{KK}. With this choice we can set $\varphi^{\text{vac}} = -\theta/2$ up to subleading corrections in the mass ratio.

To determine the perturbation $\mathcal{A}^{\text{mass}}$, we expand the equations of motion to first order in m_q (regarding $\mathcal{A}^{\text{mass}} \in \mathcal{O}(m_q)$); the terms that are already $\mathcal{O}(m_q)$ will be calculated in the backround $\mathcal{A}^{\text{vac}} + \mathcal{A}^{\text{inst}}$. The resulting equations for the mass perturbation will be mixed by the presence of the Chern–Simons terms, making very difficult to find a solution. The following argument will enable us to simplify the problem.

There are 3 different regions in which we can divide the space: $\xi \ll \rho$, $\rho \ll \xi \ll 1$ and $1 \ll \xi$. We will call them respectively the flat, the overlapping and the asymptotic regions. The flat region describes the near-core instanton solution. Since $\rho \sim 1/\sqrt{\lambda}$, this is the region $\xi \to 0$. Taking into account the instantonic solution (5.2.1) we have a behaviour of the type

$$\begin{aligned}
&\mathcal{A}_M(x) \in \mathcal{O}(\lambda^{1/2}) \,, && \mathcal{A}_0(x) \in \mathcal{O}(\lambda^0) \,, \\
&\mathcal{F}_{MN}(x) \in \mathcal{O}(\lambda) \,, && \mathcal{F}_{0M}(x) \in \mathcal{O}(\lambda^{1/2}) \,.
\end{aligned} \tag{6.3.7}$$

In the asymptotic region (see (6.2.10)) we have again a suppression of an overall λ for each field, but also the functions $G(\vec{x}, z, \vec{X}, Z)$ and $H(\vec{x}, z, \vec{X}, Z)$ are of order $\sim e^{-r}$ in r, $\sim 1/z$ in z and $\sim 1/r$ in r, $\sim 1/z^2$ in z respectively, as one can check from the definitions. In the overlapping region the solution is again

(6.2.10) but with the functions G and H replaced by the flat Green's function $G^{\text{flat}} = -1/4\pi^2\xi^2$; the maximum value of the fields is reached when ξ approaches ρ, so the scaling is precisely (6.3.7), but here this behaviour is reached as an upper limit (see Table 6.2).

	Flat	Overlapping	Asymptotic
Region	$\xi \ll \rho$	$\rho \ll \xi \ll 1$	$1 \ll \xi$
Solution	BPST instanton	function G^{flat}	functions G and H
Scaling	λ scaling	λ scaling (limit)	z and r scaling

Table 6.2: This table summarizes the previous discussion.

With this in mind let us look at the Chern–Simons terms; in the asymptotic region all of them will be negligible as they are quadratic in the fields, in the other two regions however some of them have to be considered. If we look at (6.3.7) we conclude that, whenever an $\mathcal{A}_0$ is present in the CS, its λ scaling is lowered, so the leading ones will be those with $\mu = 0$. In fact in the equations for $\mu = 0$ all terms are of the same order in λ, while in $\mu = i$ or z the CS terms happen to be suppressed as $1/\lambda$ with respect to the YM terms, hence we will drop them in the following.

Now we are ready to write down the equations for the mass perturbation (gauge fields without superscript are $\mathcal{A}^{\text{inst}}$, our convention is $\varepsilon_{0123z} = -\varepsilon^{0123z} = 1$):

$$-\kappa\left(h(z)\partial_\nu \widehat{F}^{0\nu}_{\text{mass}} + \partial_z(k(z)\widehat{F}^{0z}_{\text{mass}})\right) - \frac{N_c}{128\pi^2}\varepsilon^{ijk}\left(4F^a_{ij}F^{a,\,\text{mass}}_{kz} + 4F^a_{iz}F^{a,\,\text{mass}}_{jk}\right) = 0, \tag{6.3.8}$$

$$-\kappa\left(h(z)\partial_\nu \widehat{F}^{i\nu}_{\text{mass}} + \partial_z(k(z)\widehat{F}^{iz}_{\text{mass}})\right) + (\text{subleading}) = 0\,, \tag{6.3.9}$$

$$-\kappa\left(h(z)D_\nu F^{0\nu} + D_z(k(z)F^{0z})\right)^a\Big|_{\text{mass}} - \frac{N_c}{64\pi^2}\varepsilon^{ijk}\left(2F^a_{ij}\widehat{F}^{\text{mass}}_{kz} + 2F^a_{iz}\widehat{F}^{\text{mass}}_{jk}\right) = 0, \tag{6.3.10}$$

$$-\kappa\left(h(z)D_\nu F^{i\nu} + D_z(k(z)F^{iz})\right)^a\Big|_{\text{mass}} + (\text{subleading}) = 0\,, \tag{6.3.11}$$

$$-\kappa\, k(z)\partial_\nu \widehat{F}^{z\nu}_{\text{mass}} + (\text{subleading}) = (\cdots)\mathrm{d}C_{(7)} + ic\,\mathrm{Tr}\left[M(\cdots)\right]\,, \tag{6.3.12}$$

$$-\kappa\, k(z)(D_\nu F^{z\nu})^a\Big|_{\text{mass}} + (\text{subleading}) = ic\,\mathrm{Tr}\left[M\tau^a(\cdots)\right]\,, \tag{6.3.13}$$

The notation $\big|_{\text{mass}}$ means "pick up the linear contribution in m_q", the dots in the last two equations can be read from (6.3.3) and (6.3.4). For now we work in the static gauge and we admit no time dependence for $\mathcal{A}_{\text{mass}}$ (so the indices "ν" in the equations above become "j").

This system of equation can be divided into four parts

i) Abelian space equations (6.3.9), (6.3.12).

ii) Non abelian space equations (6.3.11), (6.3.13).

iii) Non abelian time equation (6.3.10).

iv) Abelian time equation (6.3.8).

Our plan will be to solve *i)* and *iii)* in the next two paragraphs. The solution to *ii)* will be presented in the Appendix A: we will show only the algebraic structure without solving numerically the partial differential equations (PDE). In the final remarks we will show that both *iv)* and the Gauss law constraint introduced by the quantization process do not contribute to the NEDM.

After the complete solution has been found we can proceed with the computation of the current and then the NEDM.

6.3.1 Equations for the abelian component

A solution to the set *i)* can be found with the ansatz $\widehat{A}_i^{\mathrm{mass}} = 0$. We will verify in the end this assumption.

Here follows the solution of (6.3.12), we will focus on the $N_f = 2$ mass degenerate case. To have a more compact notation we define $\alpha \equiv \pi/\sqrt{1+\rho^2/r^2}$. After calculating the right hand side on the background and using the value (6.3.6) for $\mathrm{d}C_{(7)}$, we see that the equation of motion reads

$$\kappa\, k(z)\partial_i\partial^i \widehat{A}_z^{\mathrm{mass}} = -2cm_q \sin\varphi^{\mathrm{vac}}(\cos\alpha + 1)\ . \tag{6.3.14}$$

The z dependence of this equation can be factorized by setting

$$\widehat{A}_z^{\mathrm{mass}} = \frac{1}{1+z^2}u(r)\ , \tag{6.3.15}$$

yielding an ordinary differential equation for $u(r)$

$$\frac{1}{r^2}\partial_r(r^2\partial_r u(r)) = \frac{2cm_q}{\kappa}\sin\frac{\theta}{2}(\cos\alpha + 1)\ , \tag{6.3.16}$$

where we have substituted φ^{vac} by $-\theta/2$ as discussed previously. When $r \to \infty$ the function α approaches to a constant $\alpha \to \pi$, so the source term vanishes. The standard way to solve this equation is to use the Green function:

$$u_G(r, r') = \begin{cases} -\, r' & r < r' \\ -\, r'\left(\dfrac{r'}{r}\right) & r > r' \end{cases}\ . \tag{6.3.17}$$

The solution is given by the following integral

$$u(r) = \frac{2cm_q}{\kappa}\sin\frac{\theta}{2}\int_0^\infty \mathrm{d}r'\, u_G(r, r')\left(1 + \cos\frac{\pi}{\sqrt{1+\rho^2/r'^2}}\right)\ . \tag{6.3.18}$$

It can be easily checked that the solution $\widehat{A}_z^{\text{mass}}$ modifies only the axial current $\mathcal{J}_A$ and leaves untouched the vector current $\mathcal{J}_V$: in fact it is an even function in z. We will only need the vector current for our purposes so we will not write down this modification. Moreover this particular z dependence is sufficient to identically solve equation (6.3.9), hence we can put $\widehat{A}_i^{\text{mass}}$ to zero: the ansatz claimed at the beginning was correct.

It may be interesting to see the asymptotic solution for big λ. Changing variables $r' = \rho y$ (6.3.18) becomes

$$\begin{aligned} u(r) = & \frac{c m_q \theta}{\kappa} \int_{r/\rho}^{\infty} \mathrm{d}y \, (-y\rho^2) \left(1 + \cos \frac{\pi}{\sqrt{1 + 1/y^2}}\right) + \\ & + \frac{c m_q \theta}{\kappa} \int_0^{r/\rho} \mathrm{d}y \, \frac{-y^2 \rho^3}{r} \left(1 + \cos \frac{\pi}{\sqrt{1 + 1/y^2}}\right) . \end{aligned} \tag{6.3.19}$$

Since ρ approaches zero and we are away from $r = 0$, the solution can be approximated by

$$u(r) = \frac{2 c m_q}{\kappa} \sin \frac{\theta}{2} \frac{\rho^3}{r} \int_0^{\infty} \mathrm{d}y (-y^2) \left(1 + \cos \frac{\pi}{\sqrt{1 + 1/y^2}}\right) . \tag{6.3.20}$$

The integral in $\mathrm{d}y$ is a constant that evaluates to -1.104.

6.3.2 Equation for the non abelian time component

Let us now look at equation (6.3.10). To first order in m_q the equation for the perturbation is the following

$$\begin{aligned} & h(z) D_\nu^{\text{inst}} \left(-\partial^\nu A^0_{\text{mass}} + i[A^0_{\text{mass}}, A^\nu_{\text{inst}}]\right) + \\ & + D_z^{\text{inst}} \left(-k(z) \partial^z A^0_{\text{mass}} + i k(z) [A^0_{\text{mass}}, A^z_{\text{inst}}]\right) = \\ & = -\frac{N_c}{8\pi^2 \kappa} \frac{\rho^2}{(\xi^2 + \rho^2)^2} \frac{u'(r)}{1 + z^2} \frac{\vec{x} - \vec{X}}{r} \cdot \vec{\tau} \, . \end{aligned} \tag{6.3.21}$$

In the static gauge ($\partial_0 = 0$) the only field excited by this perturbation is A_0^{mass}. Let us consider the following ansatz

$$A^0_{\text{mass}} = W(r, z)(\vec{x} - \vec{X}) \cdot \vec{\tau} \, . \tag{6.3.22}$$

When plugging this ansatz into the equations, the $(\vec{x} - \vec{X}) \cdot \vec{\tau}$ factorizes and we are left with a partial differential equation for W.

$$\begin{aligned} & h(z) \left(\partial_r^2 W(r, z) + \frac{4}{r} \partial_r W(r, z) + \frac{8\rho^2}{(\xi^2 + \rho^2)^2} W(r, z)\right) + \partial_z (k(z) \partial_z W(r, z)) = \\ & = \frac{27\pi}{\lambda} \frac{\rho^2}{(\xi^2 + \rho^2)^2} \frac{1}{r} \frac{u'(r)}{1 + z^2} \equiv \mathscr{F}(r, z) \, . \end{aligned} \tag{6.3.23}$$

There are two possible approaches that can be used to solve this equation: *i)* numerical PDE analysis; *ii)* expansion in the eigenfunctions ψ_n. The latter, which we are going to describe in Appendix B, provides us interesting insights about the physical content of our result. The results of the numerical analysis will be used in the calculation of the NEDM.

6.3.3 Quantization and the dipole moment

The classical soliton solution we have found has to be quantized. Both the mass term and the θ parameter could in principle give corrections to the moduli space Hamiltonian. If this is so, the eigenstates found in in Section 5.3 have to be modified accordingly.

Crucially, however, we have found that the corrections to the Hamiltonian (i.e. those to the baryon mass formula (5.4.8)) are of order θ^2 for small θ: thus, at first order in θ we can forget about this issue and keep using the baryon eigenstates already found at $\theta = 0$. Moreover, the mass term just gives rise to a $\mathcal{O}(m_q)$ correction to the instanton size ρ. We will neglect this correction since it will give rise to a subleading (in m_q) contribution to the NEDM.

In order to compute the electromagnetic current we need to switch on the moduli of the gauge group orientations. We would also have to consider the time dependence of the moduli $X^I = \{\vec{X}, Z, \rho\}$, but this is a subleading effect and so we substitute them with their classical values. The reason for this is the following: $\vec{X}$ disappears from everywhere since in the end we must integrate in d^3x to obtain the NEDM; translational invariance allows us to put $\vec{X} = 0$, moreover $\dot{\vec{X}} = \vec{P}$ is the momentum of the baryon, which is of course zero because we are in the rest frame[4]. The other two moduli instead are actually *approximate moduli*: the rest energy of the un-quantized instanton depends on them. At lowest order they can be neglected as opposed to the modulus $\boldsymbol{a}$ which is a genuine excitation.

The gauge group orientation is a little bit tricky: we need to add a degree of freedom to the solution by making a rotation in the gauge group. The latter however should not be a gauge transformation, otherwise we end up with the same solution. Some authors work in the $A_0 = 0$ gauge and make a “wrong” gauge transformation involving only the remaining components. Here we would like to maintain $A_0 \neq 0$ so we use this kind of transformation instead ($M = 1, 2, 3, z$)

$$\begin{aligned} A_0 &\longmapsto \quad A_0' = V A_0 V^{-1} \,, \\ A_M &\longmapsto \quad A_M' = V A_M V^{-1} - iV\partial_M V^{-1} \,, \end{aligned} \tag{6.3.24}$$

for a certain $V(x)$ group valued function whose requirements are $V \to \boldsymbol{a}$ as

[4]Clearly for $\vec{P} \neq 0$ we have a non zero electric dipole moment, but it would be just a magnetic moment observed from a boosted frame.

$z \to \pm\infty$ and and imposing that the transformed field still solves the equations of motion.

Another possible choice, which is gauge equivalent to ours, is the following [82].

$$\begin{aligned} A_0 &\longmapsto \quad A_0' = W(t)A_0W(t)^{-1} + \Delta(x,t) \\ A_M &\longmapsto \quad A_M' = W(t)A_MW(t)^{-1} \ , \end{aligned} \tag{6.3.25}$$

where the function $\Delta(x,t)$ is now necessary to solve the EOM also in the non stationary case. Defining Y so that $-iY^{-1}\dot{Y} = \Delta(x,t)$ and making the gauge transformation with parameter Y allows us to find exactly (6.3.24) with $V(x,t) = W(t)Y(x,t)$.

Of course many other choices are possible, non necessarily related by gauge transformations; the only important thing is that the EOM should remain satisfied.

After the transformation (6.3.24) the M components of the equation remain untouched, while (6.3.10) changes. The resulting equation is the *Gauss law constraint*

$$\begin{aligned} &-\kappa\left(h(z)D_\nu F^{0\nu} + D_z(k(z)F^{0z})\right)^a\big|_{\text{mass}} + (\text{CS terms}) + \\ &+\kappa\left(h(z)D_\nu D^\nu\Phi + D_z(k(z)D_z\Phi)\right) = 0 \ . \end{aligned} \tag{6.3.26}$$

The second row has been obtained using the identities $V\dot{V}^{-1} = -\dot{V}V^{-1}$ and $\dot{V}^{-1} = -V^{-1}\dot{V}V^{-1}$ and some variants of these, where we have defined

$$\Phi = -iV^{-1}\dot{V} \ , \tag{6.3.27}$$

and the time dependence of the moduli ρ, Z and $\vec{X}$ has been neglected for the reasons discussed in the beginning of the paragraph. The first row is already zero if we substitute A_0^{mass} found previously; thus we have to solve

$$D_\nu D^\nu\Phi + h(z)^{-1}(D_z(k(z)D_z\Phi)) = 0 \ . \tag{6.3.28}$$

Since we are interested in the asymptotic behaviour for $z \to \infty$ we could look at the linearized equation

$$\partial_\nu\partial^\nu\Phi + h(z)^{-1}(\partial_z(k(z)\partial_z\Phi) = 0 \ . \tag{6.3.29}$$

The second term suggests to expand Φ in a series in ψ_n

$$\Phi(r,z) = \sum_{n=1}^{\infty}\frac{\tau^a}{2}\,c_n^a(r)\psi_n(z) \ . \tag{6.3.30}$$

The functions c_n^a are not just arbitrary: they should enforce the following requirements for Φ:

- When z is not going to ∞ the function Φ should solve the full equation (i.e. the one where all the covariant derivatives are explicitly evaluated).
- When $z \to \infty$ we have a boundary condition $\Phi \to -i\boldsymbol{a}^{-1}\dot{\boldsymbol{a}}$.

The solution at finite z will be very complicated algebraically and there is no reason why we should expect it to factorize as $-i\boldsymbol{a}^{-1}\dot{\boldsymbol{a}}$ times some function of r and z; in other words $SU(2)$ and space indices will mix. However on the boundary (which is the region we are interested in) such a factorization is necessary for consistency, because the boundary limit cannot depend the direction $\vec{x}$ along which we approach; equivalently, $\boldsymbol{a}$ is a global parameter in the boundary theory and not a local one. Thus we can write

$$\Phi(r,z) \underset{z\gg 1}{\sim} \sum_{n=1}^{\infty} -i\boldsymbol{a}^{-1}\dot{\boldsymbol{a}}\, c_n(r)\psi_n(z)\ . \tag{6.3.31}$$

Here, again, the whole sum must be independent of r when the $z \to \infty$ limit is reached, but it is not necessary for the present discussion to impose this requirement explicitly. The functions $c_n(r)$ contain all the information about the near-core behaviour of the instanton and of course they depend on the mass. At $m_q = 0$ the solution can be found explicitly and reads (reintroducing the Z modulus dependence only for now)

$$c_n(r) = \pi\kappa\rho^2 \frac{e^{-\sqrt{\lambda_n}r}}{r}\psi_n(Z)\ . \tag{6.3.32}$$

This just implies that $\Phi \propto G(\vec{x}, z)$ as defined in (6.2.11).

To compute the current J_0 we need the field strength F_{0z}, which after the transformation (6.3.24) becomes

$$F'_{0z} = -V(D_z\Phi)V^{-1} - V(D_zA_0)V^{-1}\ . \tag{6.3.33}$$

As argued before $\dot{X}^I$ has been neglected. At first sight both A_0^{mass} and Φ may contribute to the NEDM. The current is easily computed from the definition (6.2.4)

$$J_V^0 = \kappa\left[k(z)V(\partial_z A^0_{\text{mass}} + \partial_z\Phi)V^{-1}\right]^{z\to\infty}_{z\to-\infty}\ , \tag{6.3.34}$$

where the covariant derivatives have been replaced by ordinary derivatives because when $z \to \infty$ the fields A^{inst} and A^{mass} are suppressed by powers of z^{-1}, so the commutators disappear when the limit is taken. The gauge structure is very simple: we have

$$\begin{aligned} &\text{for } A^0_{\text{mass}}: \quad V(\vec{x}-\vec{X})\cdot\vec{\tau}V^{-1} \underset{z\to\pm\infty}{\longrightarrow} (x^j - X^j)\,\boldsymbol{a}\,\tau^j\,\boldsymbol{a}^{-1}\ , \\ &\text{for } D_z\Phi: \quad V\boldsymbol{a}^{-1}\dot{\boldsymbol{a}}V^{-1} \underset{z\to\pm\infty}{\longrightarrow} -\boldsymbol{a}\dot{\boldsymbol{a}}^{-1}\ . \end{aligned} \tag{6.3.35}$$

The electric dipole is computed using the definition (6.1.1), with $|B, s\rangle$ being a generic baryon in the spin state s.

$$\vec{\mathcal{D}}_{B,s} = \int \mathrm{d}^3x\, \vec{x}\, \langle B, s| \left(J_V^{0,\, a=3} + \frac{1}{N_c} \widehat{J}_V^0 \right) |B, s\rangle \,. \tag{6.3.36}$$

At this point is rather obvious that Φ cannot contribute to the NEDM: the form (6.3.31) depends only on r, so the integral is odd in $\vec{x}$ hence it is vanishing due to the antisymmetry $\vec{x} \to -\vec{x}$. Also the abelian contribution vanishes for the same reason (this is explained in a little more detail in the final remarks). The matrix element is evaluated using the identity [83]

$$\langle B', s'| \mathrm{Tr}\, (\boldsymbol{a}\tau^i \boldsymbol{a}^{-1} \tau^a) |B, s\rangle = -\frac{2}{3} (\sigma^i)_{s's} (\tau^3)_{I_3' I_3} \,, \tag{6.3.37}$$

where σ and τ are Pauli matrices for spin and isospin respectively and the subscript indicate the matrix elements in the standard representation. The final result is[5]

$$\vec{\mathcal{D}}_{n,s} = \frac{8\pi}{9} \int_0^\infty \mathrm{d}r\, r^4\, \kappa [k(z)\, \partial_z W(r, z)]_{z\to-\infty}^{z\to\infty} \langle s|\vec{\sigma}|s\rangle = -\vec{\mathcal{D}}_{p,s} \,. \tag{6.3.38}$$

As we can see, the dipole moment is proportional to the spin of the particle, as one would expect, and the dipole moment of the neutron has an opposite sign w.r.t the dipole moment of the proton. Factorizing the tensorial structure we define the NEDM d_n as

$$d_n = \frac{8\pi}{9} \int_0^\infty \mathrm{d}r\, r^4\, \kappa [k(z)\, \partial_z W(r, z)]_{z\to-\infty}^{z\to\infty} \,. \tag{6.3.39}$$

Below we show and discuss the numerical analysis of the quantity d_n as a function of λ for $N_c = 3$.

6.3.4 Final remarks

As claimed at the beginning of the computation, we want to show now that the abelian field $\widehat{A}_0^{\mathrm{mass}}$ does not enter in the NEDM, even if it is included in the definition of the electromagnetic current. First of all let us observe that the solutions $A_{i,z}^{\mathrm{mass}}$ enter only in the equation for $\widehat{A}_0^{\mathrm{mass}}$ and not in his non abelian counterpart, as one can see from the CS terms of (6.3.8) and (6.3.10), so the following proof will also explain why we do not need to write those solution explicitly[6].

[5] The neutron is the state with Isospin $-1/2$ and the proton with $+1/2$

[6] In principle one could think that the field A_z enters directly into the definition of the current $\propto F_{0z}$, however in the static gauge ($\mathrm{d}/\mathrm{d}t = 0$) it does not.

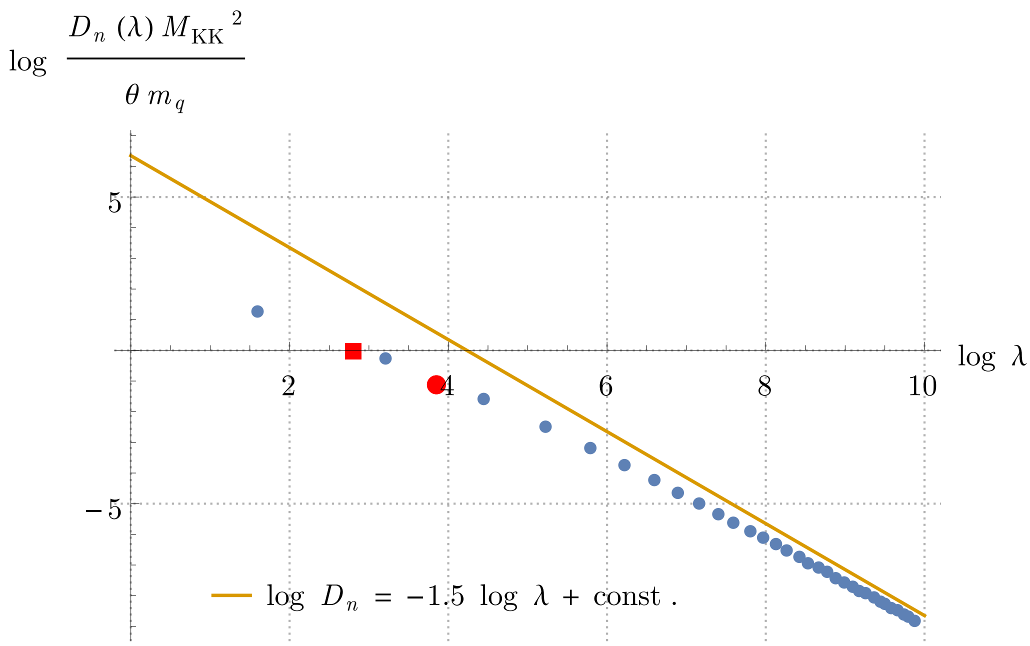


Figure 6.1: Logarithmic plot of the NEDM in units of $M_{\mathrm{KK}}^2/\theta m_q$ as a function of λ. The two marked points are the results (6.3.40) (■) and (6.3.43) (●).

Our baryonic solution, which acts as both a source and a background for the mass perturbation, satisfies a symmetry with combined spin and isospin. This is analogous to what is imposed in the Skyrme ansatz (the so–called "hedgehog" ansatz). This invariance implies that the solutions must be a radial function $f(r)$ times an isorotationally invariant object $(\vec{x} - \vec{X}) \cdot \vec{\tau}$ (times a z dependence which will not be relevant in this discussion). The field A_0^{mass} indeed satisfies this behaviour, but the abelian field can obviously be only a radial function. Thus the integral $\int \mathrm{d}^3x\, \vec{x}\, \widehat{A}_0^{\mathrm{mass}}(r)$ must give zero.

6.3.5 Numerical results

The NEDM obtained performing the numerical analysis for $\lambda = 16.632$, $M_{\mathrm{KK}} = 949$ MeV and $m_q = 2.92$ MeV, yields

$$d_n = 0.79 \cdot 10^{-16}\, \theta\, e \cdot \mathrm{cm}\ . \tag{6.3.40}$$

These parameters are fitted with the experimental observables $f_\pi = 92$ MeV, $m_\rho = 776$ MeV and $m_\pi = 135$ MeV, where ρ is the lightest vector meson. The parameter m_q has been chosen to reproduce correctly the pion mass via the Gell-Mann–Oakes–Renner relation

$$4cm_q = f_\pi^2 m_\pi^2\ . \tag{6.3.41}$$

It is a physically acceptable value being in between the up and the down mass. The mass of the ρ meson is computed in Section 4.6

$$m_\rho^2 \equiv m_1^2 = \sqrt{\lambda_1} M_{\rm KK} \simeq \sqrt{0.67} M_{\rm KK} \; . \qquad (6.3.42)$$

We could also propose a slightly different estimate by fitting the mass of the nucleon $M_0 = 8\pi^2\kappa = 940$ MeV instead of the mass of the ρ meson. In this case the parameters are found to be: $\lambda = 47.599$, $M_{\rm KK} = 558.4$ MeV and $m_q = 2.93$ MeV. This gives the value

$$d_n = 0.74 \cdot 10^{-16}\, \theta \; e \cdot {\rm cm} \; . \qquad (6.3.43)$$

For other values of λ we observe a power law for $d_n(\lambda) \sim \lambda^{-3/2}$ reached at very high values of λ: the fitted values can be seen in the plot in Figure 6.1. Both the equations for $u(r)$ and $W(r,z)$ have been solved via standard methods of numerical integration, using the software *Mathematica*.

6.4 Coupling with the pions

There are essentially two different approaches to compute the NEDM, the one followed by us, inspired to the work of Dixon et al [40], and the chiral perturbative approach in [37]. This last method involves the computation of the CP breaking cubic coupling between baryons and pions $\overline{g}_{\pi NN}$. As we will show in the following, within the limiting regimes where the above computations have been performed, this coupling turns out to be zero, at leading order in $1/N_c$ in the Sakai–Sugimoto model.

We will give two different proofs of this claim, the first one needs the Form Factor formalism so we will dedicate the next few pages to introduce it.

When we compute a matrix element of a Noether current between two states, Lorentz covariance and current conservation put very strong constraints on the specific form of the final result, namely, for an axial current

$$\langle \vec{p}', B', s' | \mathcal{J}_A^{\mu,C}(0) | \vec{p}, B, s \rangle = (2\pi)^{-3} \frac{(\tau^C)_{I_3' I_3}}{2} \overline{u}(p', s') \Gamma_\mu^{(C)}(\vec{p}, \vec{p}') u(\vec{p}, s) \; , \qquad (6.4.1)$$

where $C = 0,1,2,3$ and $\tau^0 = \mathbb{1}_2$, $\overline{u}$,u are Dirac spinors normalized to

$$\overline{u}(\vec{p}, s') u(\vec{p}, s) = \delta_{s's} \frac{m_B}{p^0} \; . \qquad (6.4.2)$$

Lorentz covariance and C, P conservation combined with the Dirac equation reduces $\Gamma_\mu^{(C)}$ to the following form (define $k^\mu = p^\mu - p'^\mu$)

$$\Gamma_\mu^{(C)}(\vec{p}, \vec{p}') = i\gamma_5 \gamma^\mu g_A^{(C)}(k^2) + \frac{1}{2m_B} k^\mu \gamma_5 g_P^{(C)}(k^2) \; . \qquad (6.4.3)$$

The Form Factors $g_{A,P}^{(C)}(k^2)$ are not independent in the massless theory because current conservation imposes

$$g_P^{(C)} = \frac{4m_B^2}{k^2} g_A^{(C)} \ . \tag{6.4.4}$$

However when the quark masses are non zero $\partial_\mu \mathcal{J}_A^\mu \neq 0$ and this relation no longer holds. When we allow a strong CP violation also other terms may arise, the same as these ones, without the γ_5 insertion. In the non-relativistic limit (where the Baryons are quantized via standard quantum mechanics)

$$\begin{aligned}
\overline{u}(\vec{p}', s')\gamma_5 u(\vec{p}, s) &= \frac{1}{2m_B} k_a (\sigma^a)_{s's} + \mathcal{O}(m_B^{-2}) \ , \\
\overline{u}(\vec{p}', s')\gamma_5 \gamma^0 u(\vec{p}, s) &= \frac{i}{2m_B}(p+p')_a (\sigma^a)_{s's} + \mathcal{O}(m_B^{-2}) \ , \\
\overline{u}(\vec{p}', s')\gamma_5 \gamma^j u(\vec{p}, s) &= i(\sigma^j)_{s's} + \mathcal{O}(m_B^{-2}) \ ,
\end{aligned} \tag{6.4.5}$$

and the Dirac spinor is simply

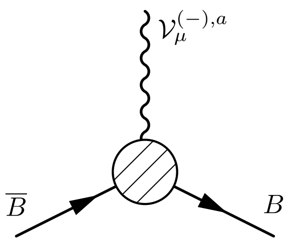
$$u(\vec{p}, s) = \begin{bmatrix} \chi_{(s)} \\ \frac{1}{2m_B}\vec{p}\cdot\vec{\sigma}\,\chi_{(s)} \end{bmatrix} + \mathcal{O}(m_B^{-2}) \ . \tag{6.4.6}$$

In this representation the γ matrices are given by

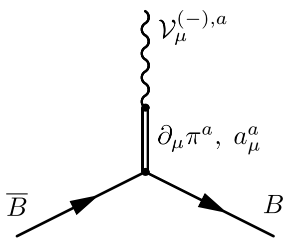
$$\gamma^0 = -i \begin{bmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{bmatrix} \ , \quad \gamma^j = -i \begin{bmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{bmatrix} \ , \quad \gamma_5 = \begin{bmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{bmatrix} \ . \tag{6.4.7}$$

Comparing the actual result obtained for the matrix element of $\mathcal{J}_A$ with the general form, one can compute the factors $g_{A,P}$.

Where do cubic couplings arise in this formalism? The answer is simple: recall that the matrix element of a current describes a cubic interaction between the two states $|B, s\rangle$ and the external source coupled to the field ($\mathcal{V}_\mu^{(-)}$ in this case), so we are actually computing diagrams of the type in Figure 6.2(a). However we can imagine that the mesons are mediating this interaction, indeed we already know their coupling with the external field (6.2.6). Hence we find something of the form shown in Figure 6.2(b)



(a) General form



(b) With mesons mediating

Figure 6.2: Cubic vertex corresponding to the matrix element $\langle B', s'|\mathcal{J}_A^\mu|B, s\rangle$

Of course, if we want to compute diagrams such as Figure 6.2(b) we need to introduce in the theory an effective coupling between mesons and baryons, which is precisely what we wanted. The coupling constants of this effective theory will be fixed by matching the two different computations for the matrix element $\langle B', s'|\mathcal{J}^\mu_A|B,s\rangle$. Let us write down the effective theory

$$\begin{aligned}\mathcal{L}_{\text{eff}} = \sum_{n\geq 1}&\left(\widehat{g}_{a^n BB}\widehat{a}^n_\mu \overline{B} i\gamma_5\gamma^\mu \frac{\mathbb{1}_2}{2}B + g_{a^n BB} a^{n\,c}_\mu \overline{B} i\gamma_5\gamma^\mu\frac{\tau^c}{2}B\right) + \\ &+ 2i\left(\widehat{g}_{\pi BB}\widehat{\pi}\overline{B}\gamma_5\frac{\mathbb{1}_2}{2}B + g_{\pi BB}\pi^c\overline{B}\gamma_5\frac{\tau^c}{2}B\right)\ .\end{aligned} \tag{6.4.8}$$

This is only the CP conserving part: the CP breaking one is the same but without $i\gamma_5$. For example the coupling $\overline{g}_{\pi BB}$ appears as

$$\mathcal{L}_{\text{eff}\cancel{CP}} = 2\left(\widehat{\overline{g}}_{\pi BB}\widehat{\pi}\overline{B}\frac{\mathbb{1}_2}{2}B + \overline{g}_{\pi BB}\pi^c\overline{B}\frac{\tau^c}{2}B\right) + (\text{vector mesons})\ . \tag{6.4.9}$$

Since the η' is very massive we expect that the low energy physics is dominated by the isovector coupling $\overline{g}_{\pi BB}$.

Let us proceed to write down the amplitude of Figure 6.2(b) retaining only the CP conserving terms of $\mathcal{L}_{\text{eff}}$ plus the CP breaking $\overline{g}_{\pi BB}$. The propagators can be read from the kinetic terms for the mesons (4.6.9), namely a Proca propagator and a scalar propagator (not massless in this case because the pions acquire a mass)

$$\langle B', s'|\mathcal{J}^{\mu\,C}_A|B,s\rangle = \sqrt{2p^0}\sqrt{2p'^0}\,\overline{u}(\vec{p}', s')\ \left[\mathscr{A}^{\mu\,C}\right]\ u(\vec{p},s) \tag{6.4.10}$$

Where

$$\begin{aligned}\mathscr{A}^{\mu\,C} = &\left[\frac{\delta^{C0}\delta_{I_3' I_3}}{2}\left(i\gamma_5\gamma_\nu\sum_{n\geq 1}\frac{\eta^{\mu\nu}+k^\mu k^\nu/\lambda_{2n}}{k^2+\lambda_{2n}}g_{a^n}\widehat{g}_{a^n BB} + 2k^\mu f_\pi(\gamma_5\widehat{g}_{\pi BB} - i\widehat{\overline{g}}_{\pi BB})\frac{1}{k^2+m_\pi^2}\right)\right.\\ &\left.+\frac{\delta^{Ca}(\tau^a)_{I_3' I_3}}{2}\left(i\gamma_5\gamma_\nu\sum_{n\geq 1}\frac{\eta^{\mu\nu}+k^\mu k^\nu/\lambda_{2n}}{k^2+\lambda_{2n}}g_{a^n}g_{a^n BB} + 2k^\mu f_\pi(\gamma_5 g_{\pi BB} - i\overline{g}_{\pi BB})\frac{1}{k^2+m_\pi^2}\right)\right]\\ &+(\cancel{CP}\ \text{vector mesons})\ .\end{aligned} \tag{6.4.11}$$

The Lorentz tensor structure can be readily compared with the general Form Factor: take for instance only the isoscalar CP conserving part

$$\begin{aligned}\widehat{g}_A(k^2) &= \sum_{n\geq 1}\frac{g_{a^n}\widehat{g}_{a^n BB}}{k^2+\lambda_{2n}}\ ,\\ \widehat{g}_P(k^2) &= 2m_B\frac{2f_\pi\widehat{g}_{\pi BB}}{k^2+m_\pi^2} - 4m_B^2\sum_{n\geq 1}\frac{g_{a^n}\widehat{g}_{a^n BB}}{\lambda_{2n}}\frac{1}{k^2+\lambda_{2n}}\ .\end{aligned} \tag{6.4.12}$$

It is worth noticing the following feature: the relation (6.4.4), that holds only when $m_\pi^2 = 0$, implies that residue at the pole of g_P in $k^2 = 0$ is proportional to g_A, more precisely

$$g_A(0) = \frac{f_\pi g_{\pi BB}}{m_B} \ . \tag{6.4.13}$$

This is known as the Goldberg–Treitman relation. However when the pion is massive the pole of g_P is displaced and the conservation of the axial current is broken also at the classical level, so this relation no longer holds.

6.4.1 Proof 1

We see that, in order to have a non zero $\overline{g}_{\pi BB}$ in the theory, we need a term in the Form Factor proportional to (contract the two spinors $\overline{u}u$)

$$(\tau^a)_{I_3' I_3} \delta_{s's} k^\mu \ , \tag{6.4.14}$$

which means in the current a term like

$$J_V^{\mu\, a} = I^a \frac{\partial}{\partial X^\mu} f(Z, \vec{x} - \vec{X}) \ , \quad \langle n_z, n_\rho | f(Z, \vec{x} - \vec{X}) | n_z, n_\rho \rangle \neq 0 \ . \tag{6.4.15}$$

The derivative with respect to X^μ can be traded for a derivative with respect to x^μ, which in Fourier transform yields k^μ. The Isospin operator is given by

$$I^a = \frac{i}{2}\left(a_4 \frac{\partial}{\partial a_a} - a_a \frac{\partial}{\partial a_4} - \varepsilon_{abc} a_b \frac{\partial}{\partial a_c} \right) = -4i\pi^2 \kappa \rho^2 \mathrm{Tr}\left(\tau^a \boldsymbol{a}\, \dot{\boldsymbol{a}}^{-1} \right) . \tag{6.4.16}$$

Clearly this term, which contains an $\dot{\boldsymbol{a}}$, can only appear in the field F_{0z}^a, as a result of the Gauss Law constraint. Indeed we have[7]

$$F_{0z}' = -V(D_z \Phi) V^{-1} - V(D_z A_0) V^{-1} \ . \tag{6.4.17}$$

The relevant term is the first one, indeed

$$V \partial_z \Phi V^{-1} \underset{z \gg 1}{\sim} i \boldsymbol{a} \dot{\boldsymbol{a}}^{-1} \sum_{n=1}^{\infty} c_n(r) \partial_z \psi_n(z) \ , \tag{6.4.18}$$

as we argued in (6.3.31). In the axial current, as it is easy to see from the definition, only the terms with *even* n contribute. Clearly c_n for even n has to vanish for $\theta = 0$, as a consequence of CP conservation and it can also be inferred by (6.3.32) computed at $Z = 0$. Now the argument is simple: since c_n solves an equation $D_M^2 \Phi = 0$, it gets contribution *only* from the non abelian fields, the CP breaking part of those is proportional to $\cos\frac{\theta}{2}$. The only way for c_n to vanish at $\theta \to 0$ is to be identically zero.

[7] Recall: the primed quantities are the *quantized* ones

6.4.2 Proof 2

This proof is somewhat simpler, but it contains the same idea. As in [92] let us notice that a possible way to define the $g_{\pi NN}$ coupling is to take the large r behaviour of the pion expectation value in a nucleon state

$$\langle N|\pi^a|N\rangle = -\frac{g_{\pi NN}}{8\pi m_B}\frac{m_\pi x^i}{r^2}e^{-m_\pi r}\langle\sigma^i\tau^a\rangle \ . \tag{6.4.19}$$

This is easy to see: it just suffices to insert iS_{eff} in the expectation value, to take the Wick contractions

$$\langle N|\pi^a iS_{\text{eff}}|N\rangle = g_{\pi NN}\langle N|\pi^a\pi^c\overline{N}\gamma^5\tau^c N|N\rangle \ , \tag{6.4.20}$$

and to use the expression of the massive propagator for π^a and the scalar product $\overline{u}\gamma_5 u$. In the same fashion the CP breaking contribution will be

$$\langle N|\pi^a|N\rangle = -\frac{g_{\pi NN}}{8\pi m_B}\frac{m_\pi x^i}{r^2}e^{-m_\pi r}\langle\sigma^i\tau^a\rangle - \frac{\overline{g}_{\pi NN}}{8\pi}\frac{m_\pi}{r^2}e^{-m_\pi r}\langle\tau^a\rangle \ . \tag{6.4.21}$$

In our model this expectation value becomes

$$\langle N|\boldsymbol{\pi}|N\rangle = \langle N|\int \mathrm{d}z\, A'_z(x^M;\boldsymbol{a})|N\rangle \ , \tag{6.4.22}$$

Where the moduli dependence has been explicitly indicated. There are essentially two reasons why this does not give a CP breaking contribution to the πNN coupling. The first is analogous to the previous one: A_z, being a non abelian field, contains CP breaking contributions proportional to $\cos\frac{\theta}{2}$, which cannot automatically vanish in the limit $\theta \to 0$ unless $\overline{g}_{\pi NN}$ is identically zero. Secondly, we would expect a precise moduli dependence from A_z, namely

$$A_{z,\not{CP}} \supset \boldsymbol{a}\dot{\boldsymbol{a}}^{-1} \ . \tag{6.4.23}$$

On the contrary we have a dependence

$$A_{z,\not{CP}} \supset \boldsymbol{a}(\vec{x}\cdot\vec{\tau})\boldsymbol{a}^{-1} \ . \tag{6.4.24}$$

This can be explicitly checked by the solution given in Appendix A, but there is no need since it is the only combination compatible with the spin–isospin symmetry with no time derivatives. This dependence gives precisely the CP conserving behaviour $\langle\sigma^i\tau^a\rangle$.

Chapter 7

Conclusions

In this Thesis work we have studied topological effects in a large N QCD-like model introduced by Witten, Sakai and Sugimoto (WSS). The model shares with real world QCD all the striking non perturbative low energy features, like confinement, chiral symmetry breaking, formation of a mass gap and so on. Remarkably, since at strong coupling the model has a dual holographic description in terms of a classical theory of gravity with sources, all of these features can be simply analysed without the need of numerical simulations. Thus for example chiral symmetry breaking is realized in a simple intuitive geometrical way. The hadronic sector in the model has a dual description in terms of a five-dimensional $U(N_f)$ Yang-Mills theory with Chern-Simons terms. Remarkably this automatically includes, once rewritten in terms of the 4d fields, both the chiral QCD Lagrangian including the Skyrme term and the effective Lagrangian describing massive (axial) vector mesons. All the coefficients appearing in the 4d effective Lagrangian are analytically given in terms of the few bare parameters of the model.

The general focus of this Thesis work has been on the θ–dependence of the QFT observables, where θ is the coefficient of the (instanton–driven) topological term. As a relevant output of our analysis, we have computed the Neutron Electric Dipole Moment (NEDM) at leading order in θ and in the pion mass, for the case of $N_f = 2$ degenerate flavors. The relevance of our computation is that it is complementary to those already existing in QCD literature. Differently from the Lattice approach the holographic setup does not present any technical ("sign") problem at finite θ. As a complement to the chiral Lagrangian and the Skyrme models, it automatically takes into account the contribution of the whole mesonic tower (and not just of the pion) to the NEDM. Moreover it offers the possibility to take the chiral and the large N limit in different orders providing physically reasonable results in both cases.

The limitations of the model stay in the large N limit and in the fact that the QCD sector is coupled with spurious Kaluza-Klein modes. This coupling is weighed by the parameter λ which has to be taken much larger than one in the dual classical gravity regime. Going to small λ is technically very difficult at the moment so that a dual string description of just (large N) QCD is not available. However, the fact that the model has a precise UV completion allows in principle to take into account the systematic errors ($\mathcal{O}(1/N^{\#})$ and $\mathcal{O}(1/\lambda^{\#})$ subleading corrections) related to our computations.

In addition to the NEDM, we have computed the θ-dependent ground state energy density and the topological susceptibility, finding agreement with the chiral Lagrangian approach. Moreover, we have also computed the CP–breaking pion–nucleon coupling $\overline{g}_{\pi NN}$ finding that it is zero at leading order in the large N (and large λ) limit. This result helps clarifying the issue of the large N behaviour of this coupling, about which some contradictory claims in the literature emerged in the past.

In the holographic model the baryons are instantons of the 5d action describing the mesonic sector of the QCD-like theory. The computation of the NEDM has led us to find a novel instanton solution describing the baryons at finite θ and with a non-zero mass term for the flavors. Using this solution it will be possible to compute further corrections to the baryons mass spectrum, with or without θ (here we just retraced the computation of [87] in presence of a θ term).

There are a few generalizations of this computation that could be interesting both from a theoretical and phenomenological point of view: one of them would be computing the $m_u - m_d$ corrections to this solution; the isospin symmetry would have to be broken and the non abelian equation could turn out to be rather intricate. The other one is the $N_f = 3$ extension, in the simple "$2+1$" case ($m_u = m_d = m$, $m_s > m$). In this case the baryonic solution at $\theta = 0$ is already known [82]. We should thus extend it to the $\theta \neq 0$ case. The advantage of the "2+1" case is that the isospin remains unbroken, but there are some technical differences from the 2 flavors case. First of all we should expect to have a non trivial solution for the η meson. Secondly there is a subtle issue with the ω_5 form coming from the Chern–Simons term in the D8 branes action. In fact using the action adopted so far the constraint $J_8 = N_c/2\sqrt{3}$, necessary to describe correctly the baryon states, fails to be satisfied. This constraint is a consequence of a gauge invariance which leaves the isospin invariant. In [93] this feature is nicely explained. In the work [82] it is observed that when Baryons are present $\mathrm{Tr}\,\mathcal{F}^3$ is not *globally* $\mathrm{d}\omega_5$ anymore, so in the action the six dimensional term $\int \mathrm{Tr}\,\mathcal{F}^3$ cannot be integrated by parts. This problem does not arise with 2 flavors because $\omega_5^{SU(2)} = 0$.

Appendix A
Non abelian equations

A.1 Choice of the ansatze

We are seeking solutions for the non abelian equations, these are Yang–Mills equations coupled with an external source, namely:

$$
\begin{aligned}
&D_M F^a_{Mi} = 0 \;, \\
&D_M F^a_{Mz} = \frac{2cm_q}{\kappa} \cos\frac{\theta}{2} \frac{(x-X)^a}{r} \sin\left(\frac{\pi}{\sqrt{1+\rho^2/r^2}}\right) \;.
\end{aligned}
\tag{A.1.1}
$$

As in the main body we look for solutions which are expressed as a sum of the instantonic background and a perturbation $\sim \mathcal{O}(m_q)$. We will this solve the equations perturbatively to first order in m_q. Let us first rewrite the background instanton field as

$$
A^{a,\,\mathrm{inst}}_M = -\eta^a_{MN} \partial_N \log f_0(\xi) \;, \quad f_0(\xi) = 1 + \frac{\xi^2}{\rho^2} \;, \tag{A.1.2}
$$

where the η^a_{MN} are the 't Hooft symbols, which constitute a basis for the self dual tensors. The above solution represents an instanton with insanton number +1. The anti–instanton is given by the same expression with η replaced by $\overline{\eta}$

$$
\begin{aligned}
\eta^a_{MN} &= \varepsilon_{aMNz} + \delta_{aM}\delta_{Nz} - \delta_{aN}\delta_{Mz} \;, \\
\overline{\eta}^a_{MN} &= \varepsilon_{aMNz} - \delta_{aM}\delta_{Nz} + \delta_{aN}\delta_{Mz} \;.
\end{aligned}
\tag{A.1.3}
$$

Our ansatz will be composed by two functions, one modifies the f_0 and the other will be an extra contribution to A_z

$$
A^a_M = -\eta^a_{MN} \partial_N (\log f_0(\xi) + \phi(r,z)) + \delta_{Mz} \partial_a \psi(r) \;. \tag{A.1.4}
$$

Note the different arguments in $\phi(r,z)$ and $\psi(r)$: we will see later that this is the correct assumption. These two functions have to be regarded as $\mathcal{O}(m_q)$, so the resulting equations will be linear in them (of course the zeroth order is already satisfied by f_0).

Here we compare our parametrization with the one usually found in literature, in particular we refer to [48]. The solution found by them relates to (A.1.2) with

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\log f_0(\xi) = \frac{2}{\xi} f(\xi) , \tag{A.1.5}$$

with the only difference that their solution has the anti–self dual 't Hooft symbols $\overline{\eta}^a_{MN}$. In fact we have these correspondences (see the appendix B of [49] and compare it to (A.1.3))

$$\eta^a_{MN}\frac{(x-X)^N}{\xi^2}\tau^a = i\, g^{-1}\partial_M g , \qquad \overline{\eta}^a_{MN}\frac{(x-X)^N}{\xi^2}\tau^a = i\, g\partial_M g^{-1} . \tag{A.1.6}$$

Hence we have this identification for, respectively, the self dual and the anti–self dual case:

$$\begin{aligned} A^{\text{self}}_M \equiv A^{\text{self},\,a}_M\frac{\tau^a}{2} &= -\frac{1}{2}\,\eta^a_{MN}\tau^a\partial_N \log f_0(\xi) = -i\, f(\xi)\, g^{-1}\partial_M g , \\ A^{\text{ant-self}}_M \equiv A^{\text{ant-self},\,a}_M\frac{\tau^a}{2} &= -\frac{1}{2}\,\overline{\eta}^a_{MN}\tau^a\partial_N \log f_0(\xi) = -i\, f(\xi)\, g\partial_M g^{-1} . \end{aligned} \tag{A.1.7}$$

The field strength is easily written as

$$F^{\text{self},\,a}_{MN} = \frac{4\rho^2}{(\xi^2+\rho^2)^2}\eta^a_{MN} , \qquad F^{\text{ant-self},\,a}_{MN} = \frac{4\rho^2}{(\xi^2+\rho^2)^2}\overline{\eta}^a_{MN} . \tag{A.1.8}$$

So the (anti)–self duality of the gauge field strength is directly inherited by the gauge vector, the properties satisfied by the 't Hooft symbols being

$$\frac{1}{2}\varepsilon_{MNPQ}\eta^a_{PQ} = \eta^a_{MN} , \qquad \frac{1}{2}\varepsilon_{MNPQ}\overline{\eta}^a_{PQ} = -\overline{\eta}^a_{MN} , \tag{A.1.9}$$

with $\varepsilon_{123z} = +1$.

A.2 Solution

The most boring part now consists in putting this ansatz into the equations (A.1.1) and write down the equations for ϕ and ψ. Let us first focus on the tensor structure

$$\begin{aligned} &\text{With the ansatz } \phi(r,z) && \text{With the ansatz } \psi(r) \\ &D_M F^a_{Mi} = -\varepsilon_{aij}x^j\big(\phi \text{ eqn.}\big) , && D_M F^a_{Mi} = \varepsilon_{aij}x^j\big(\psi \text{ radial eqn.}\big) , \\ &D_M F^a_{Mz} = x^a\big(\phi \text{ eqn.}\big) , && D_M F^a_{Mz} = x^a\big(\psi \text{ zeta eqn.}\big) . \end{aligned} \tag{A.2.1}$$

As we can see the structure is very simple; moreover we have three different parenthesis, the ones with ϕ (they are identical) and the two different ones with ψ. Down here we will write only the ψ equation for brevity, the ϕ one can be found below.

$$\begin{aligned}
\big(\psi \text{ radial eqn.}\big) &\equiv -\frac{8\rho^2}{r(\xi^2+\rho^2)^2}\psi'(r) \ , \\
\big(\psi \text{ zeta eqn.}\big) &\equiv -\frac{2(\rho^4+\xi^4-2\rho^2 r^2+2\rho^2 z^2)}{r^3(\xi^2+\rho^2)^2}\psi'(r)+\frac{2}{r^2}\psi''(r)+\frac{1}{r}\psi'''(r) \ .
\end{aligned} \tag{A.2.2}$$

The third derivative comes from the fact that in our definition of A_M only the derivatives of ϕ and ψ enter. Let us define for brevity $\Psi \equiv \psi'(r)$. The equations we were looking for finally read

$$\begin{aligned}
&-\big(\phi \text{ eqn.}\big)+\big(\psi \text{ radial eqn.}\big)=0 \ , \\
&\big(\phi \text{ eqn.}\big)+\big(\psi \text{ zeta eqn.}\big)=\frac{2cm_q}{\kappa}\cos\frac{\theta}{2}\frac{1}{r}\sin\left(\frac{\pi}{\sqrt{1+\rho^2/r^2}}\right) \ .
\end{aligned} \tag{A.2.3}$$

These equations are easily decoupled: it suffices to sum them, first solve for Ψ and then for ϕ.

$$\begin{aligned}
&-\frac{2(\rho^4+\xi^4-2\rho^2 r^2+2\rho^2 z^2)}{r^3(\xi^2+\rho^2)^2}\Psi(r)+\frac{2}{r^2}\Psi'(r)+\frac{1}{r}\Psi''(r)-\frac{8\rho^2}{r(\xi^2+\rho^2)^2}\Psi(r)= \\
&=\frac{2cm_q}{\kappa}\cos\frac{\theta}{2}\frac{1}{r}\sin\left(\frac{\pi}{\sqrt{1+\rho^2/r^2}}\right) \ , \\
&\big(\phi \text{ eqn.}\big)=-\frac{8\rho^2}{r(\xi^2+\rho^2)^2}\Psi(r) \ .
\end{aligned} \tag{A.2.4}$$

In the first one the ξ dependence completely disappears, yielding an equation for $\Psi(r)$

$$-\frac{2}{r^2}\Psi(r)+\frac{2}{r}\Psi'(r)+\Psi''(r)=\frac{2cm_q}{\kappa}\cos\frac{\theta}{2}\sin\left(\frac{\pi}{\sqrt{1+\rho^2/r^2}}\right) \ . \tag{A.2.5}$$

This is an ODE that can be easily integrated numerically. While for the ϕ equation we must consider the dependence on both r and z.

$$\begin{aligned}
\big(\phi \text{ eqn.}\big) \equiv\ & \frac{\phi^{(1,2)}+\phi^{(3,0)}}{r}+\frac{2}{r^2}\phi^{(2,0)}+\frac{\phi^{(1,0)}\left(4r^2(z^2-r^2+5\rho^2)-2(\xi^2+\rho^2)^2\right)}{r^3\left(\xi^2+\rho^2\right)^2}+ \\
&+\frac{4\left(r\phi^{(0,2)}-z\phi^{(1,1)}\right)}{r(\xi^2+\rho^2)}-\frac{8z\phi^{(0,1)}}{(\xi^2+\rho^2)^2} \ ,
\end{aligned} \tag{A.2.6}$$

where for $\phi^{(i,j)}$ we mean $\partial_r^i\partial_z^j\phi(r,z)$. The final equation to be solved is

$$\big(\phi \text{ eqn.}\big)=-\frac{8\rho^2}{(\xi^2+\rho^2)^2}\frac{\Psi(r)}{r} \ , \tag{A.2.7}$$

where $\Psi(r)$ is substituted by the solution found above. This equation can be integrated via numerical methods, even though now we are dealing with a PDE which is certainly more challenging. We will not show the numerical results here because the only purpose of this Appendix is to show what is the correct tensor structure of the solution and how to get it.

Appendix B

Solution using the expansion in eigenfunctions

B.1 Homogeneous equation

The equation for A_0^{mass} is given by (6.3.23). Let us forget for the moment about the source term $\mathscr{F}$. The last term in the *l.h.s.*, being the eigenvalue equation for the ψ_n, suggests an expansion of the form

$$W(r,z) = \sum_{n=1}^{\infty} R_n(r)\psi_n(z) \ . \tag{B.1.1}$$

The eigenfunctions $\psi_n(z)$ are defined as the solution of the following system (we have already encountered them in Chapter 4.6)

$$\begin{aligned} &-h^{-1}(z)\partial_z(k(z)\partial_z\psi(z)) = \lambda_n\psi_n(z) \ , \\ &\kappa\int \mathrm{d}z\, h(z)\psi_n(z)\psi_m(z) = \delta_{mn} \ . \end{aligned} \tag{B.1.2}$$

These functions constitute a complete set[1], so our solution will be completely determined by the coefficients $R_n(r)$. The eigenvalues λ_n have been determined in [43] up to the fourth one. We extended their analysis to:

$$\{\lambda_n\} = \{0.69,\ 1.70,\ 3.16,\ 5.15,\ 7.67,\ 10.72,\ 14.31,\ 18.44\} \tag{B.1.3}$$

[1]This is a general result in mathematics concerning this kind of equations, see Sturm–Liouville theory.

Inserting the expansion into the equation and using the eigenvalue equation for ψ_n we have

$$\sum_{n=1}^{\infty}\left(\partial_r^2 R_n(r)+\frac{4}{r}\partial_r R_n(r)-\lambda_n R_n(r)\right)\psi_n(z)+\frac{8\rho^2}{(\xi^2+\rho^2)^2}\sum_{n=1}^{\infty}R_n(r)\psi_n(z)=0\ . \tag{B.1.4}$$

Let us now project this equation with $\kappa\int \mathrm{d}z\, h(z)\psi_m(z)$. Obviously the analogy with quantum mechanics is evident, the $\psi_n(z)$ in fact form an Hilbert space. Using a Dirac notation we can write

$$\psi_n(z)\mapsto |n\rangle\ ,\quad \kappa\int \mathrm{d}z\, h(z)\psi_n(z)A(z)\psi_m(z)\mapsto \langle n|\hat{A}|m\rangle\ . \tag{B.1.5}$$

We thus obtain

$$\partial_r^2 R_m(r)+\frac{4}{r}\partial_r R_m(r)-\lambda_m R_m(r)+\sum_{n=1}^{\infty}\left\langle m\left|\frac{8\rho^2}{(\xi^2+\rho^2)^2}\right|n\right\rangle R_n(r)=0\ . \tag{B.1.6}$$

The matrix defined above, which explicitly reads

$$\left\langle m\left|\frac{8\rho^2}{(\xi^2+\rho^2)^2}\right|n\right\rangle\equiv\kappa\int \mathrm{d}z\, h(z)\psi_n(z)\psi_m(z)\frac{8\rho^2}{(\xi^2+\rho^2)^2}\ , \tag{B.1.7}$$

is a symmetric matrix. Let us call it C_{mn}. In order to include also the other term it is better to define

$$\mathcal{C}_{mn}=-\lambda_m\delta_{mn}+C_{mn}\ . \tag{B.1.8}$$

The equation is, sadly, non diagonal, but since $\mathcal{C}$ is symmetric it can always be diagonalized by means of an orthogonal transformation. In the exact case this matrix will be infinite dimensional, but in a practical computation the Hilbert space of the ψ_n must be truncated, so it is a K by K matrix where K is the number if eigenfunctions kept in the numerical analysis. Denoting by $\mathcal{O}$ the orthogonal matrix we have

$$\mathcal{D}_{mn}\equiv\mathcal{D}_m\delta_{mn}=\sum_{k,l=1}^{\infty}\mathcal{O}_{nk}\mathcal{C}_{kl}[\mathcal{O}^{-1}]_{lm}\ . \tag{B.1.9}$$

At the same time define the $R_n(r)$ in the new basis

$$\mathcal{R}_n(r)=\sum_{m=1}^{\infty}\mathcal{O}_{nm}R_m(r)\ . \tag{B.1.10}$$

The (B.1.6) can now be multiplied by $\mathcal{O}$. The result is the diagonalized equation

$$\partial_r^2\mathcal{R}_m(r)+\frac{4}{r}\partial_r\mathcal{R}_m(r)+\mathcal{D}_m(r)\mathcal{R}_m(r)=0\ , \tag{B.1.11}$$

which can be solved by standard methods. Beware that the $\mathcal{D}_m$ is actually a function of r, so the solution cannot be expressed in terms of elementary functions like exponentials, as opposed to the case for $\mathcal{D}(r) = -\nu^2$ where the solution is very simple.

B.2 Inhomogeneous modification

Introducing the inhomogeneous term is not so difficult. Let us first define

$$\mathscr{F}_n = \langle n|h^{-1}\,\mathscr{F}\rangle \equiv \kappa \int \mathrm{d}z\, \psi_n(z)\mathscr{F}(r,z) \ . \tag{B.2.1}$$

Tracing back our derivation in the previous section, the final equation is given by

$$\partial_r^2 \mathcal{R}_m(r) + \frac{4}{r}\partial_r \mathcal{R}_m(r) + \mathcal{D}_m(r)\mathcal{R}_m(r) = \sum_{m=1}^{\infty} \mathcal{O}_{mn}\mathscr{F}_n \ . \tag{B.2.2}$$

As usual, we can solve this equation using the Green's function method. The Green's function is defined as the solution of the equation

$$\partial_r^2 \mathcal{R}_{G,m}(r,r') + \frac{4}{r}\partial_r \mathcal{R}_{G,m}(r,r') + \mathcal{D}_m(r)\mathcal{R}_{G,m}(r,r') = \delta(r-r') \ . \tag{B.2.3}$$

Let us now integrate both sides in dr along a very short segment $[r'-\varepsilon, r'+\varepsilon]$. We need to require some regularity from the solution $\mathcal{R}$ and the function $\mathcal{D}$. If we assume that $\mathcal{D}$ is continuous[2], it is sufficient to require that $\mathcal{R}$ and $\partial_r\mathcal{R}$ are in $L^1(\mathbb{R}^+)$. With this assumption the last two terms in the *l.h.s.* vanish when $\varepsilon \to 0$ and we are left with

$$\partial_r \mathcal{R}_{G,m}(r,r')\Big|_{r'-\varepsilon}^{r'+\varepsilon} = 1 \tag{B.2.4}$$

The Green's function can be defined as a piecewise function, for $r > r'$ and for $r < r'$. In both regions we have to solve the equation with appropriate boundary conditions, namely regularity in $r = 0$ for $r < r'$ and regularity at infinity for $r > r'$. Then we attach these two solutions (which will depend on some arbitrary coefficients even after the imposition of the regularity) requiring continuity and a jump of 1 in the first derivative at $r = r'$. When the Green's function has been calculated the general solution is expressed as an integral

$$\mathcal{R}_m(r) = \int_0^\infty \mathrm{d}r'\, R_{G,m}(r,r') \left(\sum_{n=1}^{\infty} \mathcal{O}_{mn}\mathscr{F}_n \right) \ . \tag{B.2.5}$$

[2]This passage is not so obvious, since $\mathcal{D}$ is the result of a diagonalization; even if the function we start with, $\mathcal{C}$, is continuous, some algebraic manipulation might change this property. When restricting to the finite case, $n \leq K$, the result holds, in fact the eigenvalues of the matrix $\mathcal{C}$ are the solutions of a polynomial of degree K whose coefficients are just products of the entries of $\mathcal{C}$, thus well defined and continuous.

B.3 Dipole moment

With this solution the dipole moment can be expressed as an infinite sum. As found previously

$$J_V^0 = \kappa \left[k(z) V \partial_z A^0_{\text{mass}} V^{-1} \right]^{z\to\infty}_{z\to-\infty} \ . \tag{B.3.1}$$

Let us now use the solution found above $A^0_{\text{mass}} = \sum_n R_n(r)\psi_n(z)(\vec{x} - \vec{X}) \cdot \vec{\tau}$ and the relations (6.2.8): the CP violating part of the non abelian vector current reads

$$J^0_{V\cancel{CP}} = -\sum_{n=1}^{\infty} g_{v^n} R_{2n-1}(r)(x^j - X^j)\, \boldsymbol{a}\, \tau^j\, \boldsymbol{a}^{-1} \ . \tag{B.3.2}$$

The electric dipole is computed using the definition (6.1.1)

$$\vec{\mathcal{D}}_{n,s} = -\frac{8\pi}{9} \sum_{n=1}^{\infty} g_{v^n} \int_0^\infty \mathrm{d}r\, r^4 R_{2n-1}(r)\, \langle s|\vec{\sigma}|s\rangle = -\vec{\mathcal{D}}_{p,s} \ . \tag{B.3.3}$$

Clearly the tensor structure is the same but the dipole is now expressed as a sum on the radial functions R_{2n-1}. The dipole reads

$$d_n = -\frac{8\pi}{9} \sum_{n=1}^{\infty} g_{v^n} \int_0^\infty \mathrm{d}r\, r^4 R_{2n-1}(r) \ . \tag{B.3.4}$$

The only problem with this method is that it is numerically very unstable, it requires a pointwise diagonalization of the matrix $\mathcal{C}_{mn}$, as well as a truncation in the space of ψ_ns. The numerical errors are far bigger than the ones made with the standard numerical integration performed in the previous Chapter. Despite being useless for the purpose of the numerical result, this expansion can be an intuitive way to understand the order of magnitude of the solution. On general grounds we know that the R_n must be limited functions decaying at infinity, with a mean radius of order $1/\sqrt{\lambda}$. The height of the function can be estimated by solving the equation above with $\mathcal{C}_{mn} = 0$ (this cannot give the correct answer but the order of magnitude is right). We get something like Figure B.1. We can brutally approximate $R_n(r)$ with a step function: 10^{-2} for $r < 1/\sqrt{\lambda}$ and zero otherwise. We can also retain only the first term of the sum; g_{v^1} is found to be roughly $1.9\sqrt{\kappa}$. Putting all the overall factors from the source $\mathscr{F}$, the source of $u(r)$ and the definition of the NEDM and computing the elementary integral we get

$$d_n \simeq \frac{8\pi}{9}\frac{27\pi}{\lambda}\frac{c m_q \theta}{\kappa} 1.9\sqrt{\kappa}\frac{10^{-2}}{5}\left(\frac{1}{\sqrt{\lambda}}\right)^5 \frac{\hbar c_{\text{light}}}{M_{\text{KK}}} \sim 0.7 \cdot 10^{-16}\, \theta\, e \cdot \text{cm} \ . \tag{B.3.5}$$

This result is remarkably close to the value found before, obviously only the orders of magnitude should be compared.

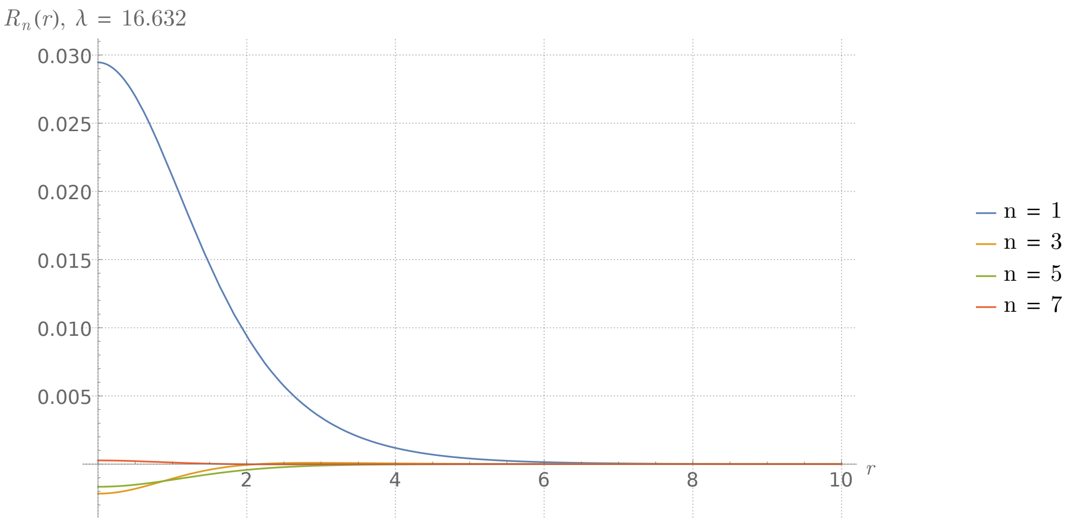


Figure B.1: *Qualitative* behaviour of the functions R_n obtained by putting $\mathcal{C}_{mn} = 0$.

This page is intentionally left blank.

Appendix C

Hořava–Witten ansatz

C.1 Equations of motion

We use the notation

$$A, B, .. = 0123zy\ ,\quad M, N, .. = 0123z\ ,\quad \mu, \nu, .. = 0123\ ,\quad i, j = 123\ . \tag{C.1.1}$$

Assume that $\widetilde{F}_{(2)}$ has a form

$$\begin{aligned}\widetilde{F}_{Ny} &= f_{Ny}\ ,\\ \widetilde{F}_{MN} &= \sqrt{\frac{N_f}{2}}\,\widehat{F}_{MN}\,\Theta(y) + f_{MN}\ ,\end{aligned} \tag{C.1.2}$$

The Bianchi identity is satisfied provided $\mathrm{d}f = 0$, hence one can always put

$$f_{AB} = \partial_A g_B - \partial_B g_A\ . \tag{C.1.3}$$

The Hodge dual is given by

$$\begin{aligned}{}^\star\widetilde{F}_{(2)} &= \frac{\sqrt{g}}{2}\frac{1}{(4!)^2}\varepsilon_{M_1\ldots M_4\psi_1\ldots\psi_4 NP}\, g^{NN'}g^{PP'}\widetilde{F}_{N'P'}\,\mathrm{d}x^{M_{1,2,3,4}}\wedge\mathrm{d}\psi_{1,2,3,4} =\\ &= \frac{1}{2}\frac{1}{(4!)^2}\sqrt{\det\Omega_4}\,\mathrm{d}\psi_{1,2,3,4}\wedge\Bigg(\frac{3}{2}\varepsilon_{\mu_1\ldots\mu_4}\,U^6\,2\widetilde{F}_{zy}\,\mathrm{d}x^{\mu_{1,2,3,4}} +\\ &\quad + \frac{8}{27}\varepsilon_{zy\mu_1\mu_2\mu\nu}\,U^5\,\eta^{\mu\mu'}\eta^{\nu\nu'}\widetilde{F}_{\mu'\nu'}\,12\,\mathrm{d}z\wedge\mathrm{d}y\wedge\mathrm{d}x^{\mu_{1,2}}\\ &\quad + \frac{4}{9}\varepsilon_{\mu_1\mu_2\mu_3 z\nu y}\,U^{7/2}\,\eta^{\nu\nu'}(g^{yy}\widetilde{F}_{\nu'y} + 2g^{yz}\widetilde{F}_{\nu'z})\,4\,\mathrm{d}x^{\mu_{1,2,3}}\wedge\mathrm{d}z\\ &\quad + \frac{4}{9}\varepsilon_{\mu_1\mu_2\mu_3 y\nu z}\,U^{7/2}\,\eta^{\nu\nu'}(g^{zy}\widetilde{F}_{\nu'y} + 2g^{zz}\widetilde{F}_{\nu'z})\,4\,\mathrm{d}x^{\mu_{1,2,3}}\wedge\mathrm{d}y\Bigg)\ .\end{aligned} \tag{C.1.4}$$

And thus the equation of motion $\mathrm{d}\,{}^\star\widetilde{F}_{(2)} = 0$ reads

- Component $\mathrm{d}x^{\mu_{1,2,3,4}} \wedge \mathrm{d}y$

$$\frac{3}{2}\partial_y\left(U^6 f_{zy}\right) - \frac{4}{9}U^{7/2}\left(g^{zy}\partial\cdot f_y + g^{zz}\sqrt{\frac{N_f}{2}}\Theta(y)\partial^\nu \widehat{F}_{\nu z} + g^{zz}\partial^\nu f_{\nu z}\right) = 0 \tag{C.1.5}$$

Where here and in the following $\partial \cdot f_A = \partial_\mu f_{\nu A}\eta^{\mu\nu}$ and $\partial^\nu = \eta^{\nu\mu}\partial_\mu$.

- Component $\mathrm{d}x^{\mu_{1,2,3,4}} \wedge \mathrm{d}z$

$$\frac{3}{2}\partial_z\left(U^6 f_{zy}\right) + \frac{4}{9}U^{7/2}\left(g^{yy}\partial\cdot f_y + g^{yz}\sqrt{\frac{N_f}{2}}\Theta(y)\partial^\nu \widehat{F}_{\nu z} + g^{yz}\partial^\nu f_{\nu z}\right) = 0 \tag{C.1.6}$$

- Component $\mathrm{d}x^{\mu_{1,2,3}} \wedge \mathrm{d}z \wedge \mathrm{d}y$

$$\begin{aligned}
&\partial_y\left[U^{7/2}\left(g^{yy}f_{\nu y} + g^{yz}\sqrt{\frac{N_f}{2}}\Theta(y)\widehat{F}_{\nu z} + g^{yz}f_{\nu z}\right)\right] + \\
&+\partial_z\left[U^{7/2}\left(g^{zy}f_{\nu y} + g^{zz}\sqrt{\frac{N_f}{2}}\Theta(y)\widehat{F}_{\nu z} + g^{zz}f_{\nu z}\right)\right] + \\
&+\frac{2}{3}U^5\left(\sqrt{\frac{N_f}{2}}\Theta(y)\partial^\mu \widehat{F}_{\mu\nu} + \partial\cdot f_\nu\right) = 0
\end{aligned} \tag{C.1.7}$$

The equations of motions written above, together with the Bianchi identity automatically imposed from the beginning, will be referred to as Maxwell/Bianchi system of equations.

C.2 Definition of θ

Let us suppose that there exists a solution to the above system of equations, i.e. a solution for f_{AB}. The existence of such solution will be addressed in the next Section, let us focus on the uniqueness. It can be shown that f_{AB} is *not* unique because we can find the zero mode

$$f^{(0)}_{zy} = \frac{C}{U^6}\,, \quad f^{(0)}_{AB\neq zy} = 0\,. \tag{C.2.1}$$

This mode satisfies $\mathrm{d}f^{(0)} = 0$ and $\mathrm{d}\star f^{(0)} = 0$, hence C is a degree of freedom of the solution. Some observations follow:

- Whatever the solution f_{AB} is, also $f_{AB} + f^{(0)}_{AB}$ is a solution.

- Perhaps C has a physical meaning. Fixing this constant by means of some boundary condition may correspond to choosing a value θ.

We propose the following boundary condition

$$\lim_{|\vec{x}|\to\infty} \int \mathrm{d}z\mathrm{d}y\, \widetilde{F}_{zy} = \theta + \lim_{|\vec{x}|\to\infty} \sqrt{\frac{N_f}{2}} \int \mathrm{d}z\, \widehat{A}_z \ , \tag{C.2.2}$$

where $\vec{x} \equiv x^i$ is the spatial component. Three steps need to be addressed in order to understand this boundary condition:

- It should be actually able to fix the value of C, thus fixing the arbitrariness of the solution f_{AB}.
- It should be motivated physically that this is a reasonable holographic definition of the θ term in QCD.
- There should be a solution to the Maxwell/Bianchi system of equation of the form
$$f_{AB} = f^{(0)}_{AB} + f^{(1)}_{AB} \ , \tag{C.2.3}$$
where $f^{(0)}$ is the zero mode in (C.2.1) with
$$C = \frac{1}{\pi}\left(\theta + \lim_{|\vec{x}|\to\infty} \sqrt{\frac{N_f}{2}} \int \mathrm{d}z\, \widehat{A}_z\right) \ , \tag{C.2.4}$$
while $f^{(1)}$ satisfies the Bianchi/Maxwell system with
$$\lim_{|\vec{x}|\to\infty} \int \mathrm{d}z\mathrm{d}y\, f^{(1)}_{zy} = 0 \ . \tag{C.2.5}$$

For the first point, from (C.2.4) we see that the boundary condition above is able to fix C. The physical interpretation of the θ appearing in (C.2.2) is not completely clear from the Supergravity point of view because there is no $C_{(1)}$ anymore; but the analogy with the Chiral effective Lagrangian should be strong enough to justify the claim (see Section 4.8). The last point is the most subtle one. It is addressed in the following Section.

C.3 Existence of a solution

The field $f^{(1)}_{AB}$ should solve the Maxwell/Bianchi system with vanishing boundary condition (C.2.5). First of all we argue that the boundary condition is consistent with the equations. This is a consequence of

Claim 1 The field $\widehat{F}_{MN}$ satisfies the boundary condition

$$\lim_{|\vec{x}|\to\infty} \widehat{F}_{MN} = 0 \,, \tag{C.3.1}$$

i.e. it has to vanish at spatial infinity, as one would expect.

From this claim it follows that the limit $|\vec{x}| \to \infty$ of the Maxwell/Bianchi system is linear in $f^{(1)}_{AB}$, hence we can find solutions that vanish at spatial infinity[1].

Secondly we show that, whatever is the explicit form of $f^{(1)}_{AB}$, it does not mix with the equations of motion of the gauge fields $\widehat{A}$. This follows from

Claim 2 The Hořava–Witten solution is antisymmetric under $y \to -y$. Since $f^{(1)}_{AB}$ is smooth in y it must be

$$f^{(1)}_{AB}\big|_{y=0} = 0 \,. \tag{C.3.2}$$

Recalling that the mixing between the equations is given by (schematically[2])

$$\frac{\delta S_{\text{DBI+CS+mass}}}{\delta \widehat{A}} = (\text{const.})\, \delta(y)\, U^{\#} \widetilde{F}_{(2)} \,, \tag{C.3.3}$$

we see that y has to be zero, hence only the zero mode $f^{(0)}_{AB}$ can contribute.

Lastly we want to show that there is indeed an explicit solution for $f^{(1)}_{AB}$, even though we will not need it. According to **Claim 2** the solution is antisymmetric in y, we can thus solve the Bianchi/Maxwell system for $y > 0$ and continue the solution for negative values. This means that we can substitute the $\Theta(y)$ by "1" everywhere. At this point the existence of the solution is obtained by a counting: there are 3 independent equations, while the unknowns are the $g^{(1)}_A$ defined by

$$f^{(1)}_{AB} = \partial_A g^{(1)}_B - \partial_B g^{(1)}_A \,, \tag{C.3.4}$$

analogously to (C.1.3). The independent components are 3 because the Lorentz symmetry relates the μ indices, hence we have only z, y and μ. The system is solvable having the same number of components and unknowns.

[1] We are implicitly exchanging $\lim_{|\vec{x}|\to\infty}$ and $\int \mathrm{d}z\mathrm{d}y$

[2] The constant does not matter in this discussion. # is 6 for the z component and 7/2 for the μ component.

References

[1] D. J. Gross and F. Wilczek, "Ultraviolet Behavior of Non-Abelian Gauge Theories", *Phys. Rev. Lett.* **30** no. 26, (Jun, 1973) 1343–1346.

[2] H. D. Politzer, "Reliable Perturbative Results for Strong Interactions?", *Phys. Rev. Lett.* **30** no. 26, (Jun, 1973) 1346–1349.

[3] A. Belavin, A. Polyakov, A. Schwartz, and Y. Tyupkin, "Pseudoparticle solutions of the Yang–Mills equations", *Physics Letters* **B59** no. 1, (1975) 85–87.

[4] G. 't Hooft, "Computation of the quantum effects due to a four-dimensional pseudoparticle", *Phys. Rev. D* **14** no. 12, (Dec, 1976) 3432–3450.

[5] H. J. Rothe, *Lattice Gauge Theories: An Introduction.* World Scientific, 2012.

[6] S. Weinberg, *The Quantum Theory of Fields, Volume II Modern Applications.* Cambridge University Press, 1996.

[7] S. Weinberg, "Dynamical approach to Current Algebra", *Phys. Rev. Lett.* **18** no. 5, (Jan, 1967) 188–191.

[8] H. Leutwyler, "Principles of Chiral Perturbation Theory", hep-ph/9406283.

[9] G. 't Hooft, "A planar diagram theory for strong interactions", *Nuclear Physics B* **72** no. 3, (Apr, 1974) 461–473.

[10] S. Weinberg, "The U(1) problem", *Phys. Rev. D* **11** no. 12, (Jun, 1975) 3583–3593.

[11] E. Witten, "Current algebra theorems for the U(1) "Goldstone boson"", *Nucl. Phys. B* **156** no. 2, (Sep, 1979) 269–283.

[12] G. Veneziano, "U(1) without instantons", *Nucl. Phys. B* **159** no. 1-2, (Nov, 1979) 213–224.

[13] E. Witten, "Large N chiral dynamics", *Annals of Physics* **128** no. 2, (Sep, 1980) 363–375.

[14] I. Zahed and G. Brown, "The Skyrme Model", *Physics Reports* **142** no. 1&2, (Mar, 1986) 1–102.

[15] J. Maldacena, "The Large N Limit of Superconformal Field Theories and Supergravity", *International Journal of Theoretical Physics* **38** no. 4, (Apr, 1999) 1113–1133, hep-th/9711200v3. [Adv. Theor. Math. Phys.2,231(1998)].

[16] E. Witten, "Anti de Sitter space and Holography", *Adv. Theor. Math. Phys.* **2** no. 2, (Apr, 1998) 253–291, hep-th/9802150.

[17] S. Gubser, I. Klebanov, and A. Polyakov, "Gauge Theory Correlators from Non-Critical String Theory", *Phys. Lett. B* **428** no. 1-2, (1998) 105–114, hep-th/9802109.

[18] J. Harvey, "Magnetic Monopoles, Duality, and Supersymmetry", hep-th/9603086.

[19] J. Bekenstein, "Universal upper bound on the entropy-to-energy ratio for bounded systems", *Phys. Rev. D* **23** no. 2, (Jan, 1981) 287–298.

[20] J. Bekenstein, "Black Holes and Entropy", *Phys. Rev. D* **7** no. 8, (Apr, 1973) 2333–2346.

[21] S. W. Hawking, "Black Holes and thermodynamics", *Phys. Rev. D* **13** no. 5, (Jan, 1976) 191–197.

[22] D. Mateos, "String Theory and Quantum Chromodynamics", *Class. and Quantum Grav.* **24** no. 21, (Oct, 2007) S713, 0709.1523.

[23] S. Hartnoll, "Horizons, holography and condensed matter", 1106.4324.

[24] J. McGreevy, "Holographic duality with a view toward many-body physics", *Advances in High Energy Physics* **2010** no. ID 723105, (May, 2010) 54, 0909.0518.

[25] D. Son and A. Starinets, "Viscosity, Black Holes, and Quantum Field Theory", *Nuclear and Particle Science* **57** no. 1, (Jul, 2007) 95–118, 0704.0240.

[26] T. Nishioka, S. Ryu, and T. Takayanagi, "Holographic Entanglement Entropy: An Overview", *J. Phys. A: Math. Theor.* **42** no. 50, (Jun, 2009) 504008, 0905.0932.

[27] A. Ramallo, "Introduction to the AdS/CFT correspondence", *Springer Proc. Phys.* **161** (2015) 411–474, 1310.4319.

[28] A. Zaffaroni, "Introduction to the AdS/CFT correspondence". lezionilosanna.pdf, 2008. Lectures given at EPFL in Lausanne for the Troisième cycle de la physique en Suisse romande.

[29] O. Aharony, S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, "Large N Field Theories, String Theory and Gravity", *Physics Reports* **323** no. 3-4, (Jan, 2000) 183–386, hep-th/9905111v3.

[30] E. Purcell and N. Ramsey, "On the Possibility of Electric Dipole Moments for Elementary Particles and Nuclei", *Phys. Rev.* **78** no. 6, (Jun, 1950) 807.

[31] C. A. Baker, D. D. Doyle, P. Geltenbort, K. Green, M. G. D. Van der Grinten, P. G. Harris, P. Iaydjiev, S. N. Ivanov, D. J. R. May, J. M. Pendlebury, J. D. Richardson, D. Shiers, and K. F. Smith, "An Improved Experimental Limit on the Electric Dipole Moment of the Neutron", *Phys. Rev. Lett.* **97** no. 13, (Sep, 2006) 131801, hep-ex/0602020.

[32] J. Pendlebury, S. Afach, N. Ayres, C. Baker, G. Ban, G. Bison, K. Bodek, M. Burghoff, P. Geltenbort, K. Green, W. Griffith, M. van der Grinten, Z. Grujic, P. Harris, V. Helaine, P. Iaydjiev, S. Ivanov, M. Kasprzak, Y. Kermaidic, K. Kirch, H. Koch, S. Komposch, A. Kozela, J. Krempel, B. Lauss, T. Lefort, Y. Lemiere, D. May, M. Musgrave, O. Naviliat-Cuncic, F. Piegsa, G. Pignol, P. Prashanth, G. Quemener, M. Rawlik, D. Rebreyend, J. Richardson, D. Ries, S. Roccia, D. Rozpedzik, A. Schnabel, P. Schmidt-Wellenburg, N. Severijns, D. Shiers, J. Thorne, A. Weis, O. Winston, E. Wursten, J. Zejma, and G. Zsigmond, "Revised experimental upper limit on the electric dipole moment of the neutron", *Phys. Rev. D* **92** no. 9, (Nov, 2015) 092003, 1509.04411.

[33] R. D. Peccei and H. R. Quinn, "CP Conservation in the presence of Pseudoparticles", *Phys. rev. Lett.* **38** no. 25, (Jun, 1977) 1440–1443.

[34] V. Baluni, "CP-nonconserving effects in quantum chromodynamics", *Phys. Rev. D* **19** no. 7, (Apr, 1979) 2227–2230.

[35] F. Guo, R. Horsley, U. Meissner, Y. Nakamura, H. Perlt, P. Rakow, G. Schierholz, A. Schiller, and J. Zanotti, "The electric dipole moment of the neutron from 2+1 flavor lattice QCD", *Phys. Rev. Lett.* **115** no. 6, (Aug, 2015) 062001, 1502.02295.

[36] E. Shintani, S. Aoki, N. Ishizuka, K. Kanaya, Y. Kikukawa, Y. Kuramashi, M. Okawa, A. Ukawa, and T. Yoshié, "Neutron electric dipole moment with external electric field method in lattice QCD", *Phys. Rev. D* **75** no. 3, (Feb, 2007) 034507, hep-lat/0611032.

[37] R. Crewther, P. Di Vecchia, G. Veneziano, and E. Witten, "Chiral estimate of the electric dipole moment of the neutron in QCD", *Physics Letters B* **88** no. 1&2, (Dec, 1979) 123–127.

[38] R. Crewther, P. Di Vecchia, G. Veneziano, and E. Witten, "Chiral estimate of the electric dipole moment of the neutron in QCD - Errata", *Physics Letters B* **91** no. 3&4, (Apr, 1980) 487.

[39] G. Adkins, C. Nappi, and E. Witten, "Static properties of nucleons in the Skyrme model", *Nuclear Physics B* **228** no. 3, (Dec, 1983) 552–566.

[40] L. Dixon, A. Langnau, Y. Nir, and B. Warr, "The Electric Dipole Moment of the Neutron in the Skyrme Model", *Physics Letters B* **253** no. 3-4, (Jan, 1991) 459–464.

[41] H. Riggs and H. Schnitzer, "CP-violating Yukawa couplings in the Skyrme model and the neutron electric dipole moment", *Physics Letters B* **305** no. 3, (May, 1993) 252–260, hep-ph/9212273.

[42] E. Witten, "Anti-de Sitter Space, Thermal Phase Transition, And Confinement In Gauge Theories", *Adv.Theor.Math.Phys* **1998** no. 2, (Mar, 1998) 505–532, hep-th/9803131v2.

[43] T. Sakai and S. Sugimoto, "Low Energy Hadron Physics in Holographic QCD", *Prog. Theor. Phys.* **113** no. 4, (Dec, 2005) 843–882, hep-th/0412141v5.

[44] T. Sakai and S. Sugimoto, "More on a Holographic Dual of QCD", *Prog. Theor. Phys.* **114** no. 5, (Aug, 2005) 1083–1118, hep-th/0507073v4.

[45] O. Aharony and D. Kutasov, "Holographic Duals of Long Open Strings", *Phys. Rev. D* **78** no. 2, (Jul, 2008) 026005, 0803.3547v2.

[46] K. Hashimoto, T. Hirayama, F. Lin, and H. Yee, "Quark Mass Deformation of Holographic Massless QCD", *JHEP* **2008** no. 7, (Jul, 2008) 089, 0803.4192v2.

[47] E. Witten, "Theta Dependence In The Large N Limit Of Four-Dimensional Gauge Theories", *Phys. Rev. Lett.* **81** no. 14, (Oct, 1998) 2862, hep-th/9807109.

[48] H. Hata, T. Sakai, S. Sugimoto, and S. Yamato, "Baryons from instantons in holographic QCD", *Prog. Theor. Phys.* **117** no. 6, (Mar, 2007) 1157–1180, hep-th/0701280v3.

[49] K. Hashimoto, T. Sakai, and S. Sugimoto, "Holographic Baryons : Static Properties and Form Factors from Gauge/String Duality", *Prog. Theor. Phys.* **120** no. 6, (Jul, 2008) 1093–1137, 0806.3122v4.

[50] J. Polchinski, *String Theory, Volume 1*. Cambridge University Press, 2005.

[51] M. Green, J. Schwarz, and E. Witten, *Superstring Theory, Volume 1*. Cambridge University Press, 1988.

[52] C. Johnson, *D-Branes*. Cambridge University Press, 2003.

[53] M. Ammond and J. Erdmenger, *Gauge/Gravity duality - Foundations and applications*. Cambridge University Press, 2015.

[54] F. Gliozzi, J. Sherk, and D. Olive, "Supersymmetry, supergravity theories and the dual spinor model", *Nuclear Physics B* **122** no. 2, (Apr, 1977) 253–290.

[55] S. Weinberg, *The Quantum Theory of Fields, Volume I Foundations*. Cambridge University Press, 1995.

[56] M. Green, J. Harvey, and G. Moore, "I-Brane Inflow and Anomalous Couplings on D-Branes", *Classical and Quantum Gravity* **14** no. 1, (1997) 47, hep-th/9605033.

[57] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory*. Springer-Verlag, 1997.

[58] D. Simons Duffin, "TASI Lectures on the Conformal Bootstrap", 1602.07982.

[59] S. Rychkov, "EPFL Lectures on Conformal Field Theory in $D \geq 3$ Dimensions", 1601.05000.

[60] A. Smilga, *Lectures on Quantum Chromodynamics*. World Scientific, 2001.

[61] H. Georgi, *Lie Algebras in Particle Physics: from Isospin To Unified Theories*. Westview Press, 1999.

[62] P. Breitenlohner and D. Freedman, "Stability in gauged extended supergravity", *Ann. Phys.* **144** no. 2, (Dec, 1982) 249–281.

[63] M. Bianchi, D. Z. Freedman, and K. Skenderis, "Holographic Renormalization", *Nucl. Phys. B* **631** no. 1-2, (Jun, 2002) 159–194, hep-th/0112119.

[64] J. Maldacena, "Wilson loops in large N field theories", *Phys.Rev.Lett* **80** no. 22, (Jun, 1998) 4859, hep-th/9803002v3.

[65] F. Giani and M. Pernici, "$N = 2$ supergravity in ten dimensions", *Phys. Rev. D* **30** no. 2, (Jul, 1984) 325–333.

[66] M. Shifman, *Advanced Topics in Quantum Field Theory*. Cambridge University Press, 2012.

[67] P. Di Vecchia, "Large N gauge theories and AdS/CFT correspondence", hep-th/9908148v1.

[68] E. Vicari and H. Panagopoulos, "Theta dependence of SU(N) gauge theories in the presence of a topological term", *Physics Reports* **470** no. 3-4, (Jan, 2009) 93–150, 0803.1593v5.

[69] T. Muta, *Foundations of Quantum Chromodynamics*. World Scientific, 1997.

[70] A. Belitsky, S. Vandoren, and P. van Nieuwenhuizen, "Yang–Mills and D–instantons", *Classical and Quantum Gravity* **17** no. 17, (2000) 3521, hep-th/0004186.

[71] M. Nakahara, *Geometry, Topology and Physics*. Institute of Physics Publishing, 2003.

[72] D. Tong, "TASI Lectures on Solitons", hep-th/0509216.

[73] M. Laidlaw and C. DeWitt, "Feynman functional integrals for systems of indistinguishable particles", *Phys. Rev. D* **3** no. 6, (Mar, 1971) 1375–1378.

[74] F. Suzuki, "Homotopy and Path Integrals", 1107.1459.

[75] S. Adler, "Axial-Vector Vertex in Spinor Electrodynamics", *Phys. Rev.* **177** no. 5, (Jan, 1969) 2426–2438.

[76] J. Bell and R. Jakiw, "A PCAC puzzle: $\pi^0 \to \gamma\gamma$ in the σ–model", *Nuovo Cimento* **60** no. 1, (Mar, 1969) 47–61.

[77] E. Witten, "Baryons in the $1/N$ expansion", *Nuclear Physics B* **160** no. 1, (Nov, 1979) 57–115.

[78] G. Derrick, "Comments on nonlinear wave equations as models for elementary particles", *J. Math. Phys.* **5** no. 9, (1964) 1252–1254.

[79] S. Hawking and D. N. Page, "Thermodynamics of Black Holes in Anti–de Sitter Space", *Comm. Math. Phys.* **87** no. 4, (1982) 577–588.

[80] F. Bigazzi and A. Cotrone, "Holographic QCD with dynamical flavors", *JHEP* **2015** no. 1, (Jan, 2015) 1–43, 1410.2443.

[81] S. Sugimoto and K. Takahashi, "QED and String Theory", *JHEP* **2004** no. 04, (May, 2004) 051, hep-th/0403247.

[82] H. Hata and M. Murata, "Baryons and the Chern-Simons term in holographic QCD with three flavors", *Prog. Theor. Phys.* **119** no. 3, (Nov, 2008) 461–490, 0710.2579v2.

[83] H. Hata, M. Murata, and S. Yamato, "Chiral currents and static properties of nucleons in holographic QCD", *Phys. Rev. D* **78** no. 8, (Oct, 2008) 086006, 0803.0180v2.

[84] E. Witten and P. Hořava, "Eleven-Dimensional Supergravity on a Manifold with Boundary", *Adv.Theor.Math.Phys* **475** no. 1-2, (Sep, 1996) 94–114, hep-th/9603142.

[85] F. Bigazzi, A. Cotrone, and R. Sisca, "Notes on Theta Dependence in Holographic Yang-Mills", *JHEP* **2015** no. 8, (Aug, 2015) 1–31, 1506.03826.

[86] E. Witten, "Baryons And Branes In Anti de Sitter Space", *JHEP* **1998** no. 07, (Sep, 1998) 006, hep-th/9805112.

[87] K. Hashimoto, T. Hirayama, and D. Hong, "Quark mass dependence of hadron spectrum in holographic QCD", *Phys. Rev. D* **81** no. 4, (Feb, 2010) 045016, 0906.0402v2.

[88] KLOE collaboration, "Upper limit on the $\eta \to \pi^+\pi^-$ branching ratio with the KLOE experiment", *Phys. Lett. B* **606** no. 3-4, (Jan, 2005) 276–280, hep-ex/0411030.

[89] D. V. Nanopoulos, A. Yildiz, and P. H. Cox, "CP violation in gauge theories and the electric dipole moment of the neutron", *Ann. Phys.* **127** no. 1, (Jun, 1980) 126–148.

[90] P. Salomonson, B. Skagerstam, and A. Stern, "Electric Dipole Moment of the Neutron in the two–flavor Skyrme model", *Mod. Phys. Lett. A* **06** no. 39, (Dec, 1991) 3647.

[91] D. K. Hong, H.-C. Kim, S. Siwach, and H.-U. Yee, "The Electric Dipole Moment of the Nucleons in Holographic QCD", *JHEP* **2007** no. 11, (Nov, 2007) 036, 0709.0314.

[92] G. Adkins and C. Nappi, "The Skyrme model with pion masses", *Nuclear Physics B* **233** no. 1, (Feb, 1984) 109–115.

[93] E. Guadagnini, “Baryons as solitons and mass formulae”, *Nuclear Physics B* **236** no. 1, (Apr, 1984) 35–47.

[94] A. Sagnotti and C. Angelantonj, “Open Strings”, *Physics Reports* **371** no. 1-2, (Nov, 2002) 1–150, hep-th/0204089.

[95] S. Bolognesi and P. Sutcliffe, “The Sakai-Sugimoto soliton”, *JHEP* **2014** no. 1, (2014) 1–32, 1309.1396.

[96] J. de Vries, E. Mereghetti, and A. Walker-Loud, “Baryon mass splittings and strong CP violation in $SU(3)$ Chiral Perturbation Theory”, *Phys. Rev. C* **92** no. 4, (Oct, 2015) 045201, 1506.06247.

[97] M. Atiyah and N. Manton, “Skyrmions from instantons”, *Physics Letters B* **222** no. 3-4, (May, 1989) 438–442.

[98] D. Hong, T. Inami, and H. Yee, “Baryons in AdS/QCD”, *Physics Letters B* **646** no. 4, (Mar, 2007) 165–171, hep-ph/0609270v2.

[99] D. Hong, M. Rho, H. Yee, and P. Yi, “Chiral Dynamics of Baryons from String Theory”, *Phys. Rev. D* **76** no. 6, (Sep, 2007) 061901, hep-th/0701276.

[100] F. Ruffino, “Topics on the geometry of D-brane charges and Ramond-Ramond fields”, *JHEP* **2009** no. 11, (Nov, 2009) 012, 0909.0689v3.

[101] N. Manton and P. Sutcliffe, *Topological Solitons*. Cambridge University Press, 2004.

[102] D. Freedman and A. Van Proeyen, *Supergravity*. Cambridge University Press, 2012.

[103] R. Sisca, “Theta Dependence in Holographic Yang-Mills”, Master’s thesis, University of Pisa, 2015.