Introduction

may my heart always be open to little
birds who are the secrets of living
whatever they sing is better than to know
and if men should not hear them men are old

may my mind stroll about hungry
and fearless and thirsty and supple
and even if it's sunday may i be wrong
for whenever men are right they are not young

and may myself do nothing usefully
and love yourself so more than truly
there's never been quite such a fool who could fail
pulling all the sky over him with one smile

E. E. Cummings

In modern algebraic geometry a relevant role is played by vanishing theorems. Very often a geometric question can be posed in terms of global sections, or more of generally cohomology of sheaves, and in these cases the vanishing of some cohomology groups can provide an answer. In the late 80's Green and Lazarsfeld proved in [13] the Generic Vanishing Theorem (GVT): it says that the cohomology of a general topologically trivial holomorphic line bundle on a compact Kähler manifold M is zero in all degree less than the Albanese dimension of M. In this thesis we reproduce in detail their proof and some important consequences, for example the inequality

$$\chi(M, \omega_M) \geq 0, \tag{1}$$

for manifolds of maximal Albanese dimension. In Chapter 1 we recall the definition and some properties of complex tori, in order to arrive at the Poincaré line bundle, that is ubiquitous in the following. Its construction is very explicit and uses the Appel-Humbert theorem and the theory of line bundles on complex tori. In Chapter 2 we explain how to associate to M two complex tori, the Picard torus and the Albanese torus, and we prove that they are dual to each other. This allows to transfer some results of

Chapter 1 about complex tori to a manifold M; in particular the existence of the Poincaré line bundle on $M \times \mathrm{Pic}^0(M)$ and the isomorphism

$$\mathrm{Pic}^0(M) \cong \mathrm{Hom}(\pi_1(M), U(1)).$$

Then we pass to the Green-Lazarsfeld first order deformation theory of cohomology groups of line bundles, that is the key argument to prove the GVT. In particular the Tangent Cone Theorem 2.24 and its corollaries are set in a general context, i.e. in such a way that it is possible to use them to derive some different results. For example in Chapter 3 we give the Green-Lazarsfeld generic version of Nakano-Kodaira vanishing theorem; here a major ingredient is also Hodge Theory. In the last Chapter 4, we include a theorem on surfaces, that was the motivation for the groundbreaking paper [13]. Then we cite, without proofs, the results of a second work of Green and Lazarsfeld on the argument, [14], where they proceed with the study of deformation theory of cohomology groups of line bundles. Hence we show how Ein and Lazarsfeld in [8] used this circle of ideas to obtain a theorem on manifolds for which the inequality (1) is an equality.

Contents

Chapter 1

Complex Tori and the Poincaré line Bundle

1.1 Complex Tori

A complex torus is a quotient $X = V/\Lambda$, where V is a n-dimensional $\mathbb{C}$-vector space and Λ is a lattice in V, i.e. $\Lambda = \oplus_{i=1}^{2n} \mathbb{Z}\lambda_i$, with $\{\lambda_i\}_i$ an $\mathbb{R}$-basis for V. If we consider the universal cover of X, $\pi: V \to X$ where π is the quotient map, we have that the deck transformations are translations by elements of Λ and so X inherits a complex structure. So a complex torus X is an n-dimensional connected, compact complex manifold, namely it's a complex Lie group, topologically equivalent to a product of S^1. The converse is also true: every connected, compact complex Lie group is a complex torus (see [15], p. 325). By definition we can identify $\pi_1(X)$ with Λ and, since Λ is abelian, $H_1(X, \mathbb{Z}) = \Lambda$. Furthermore $H^1(X, \mathbb{Z}) = \mathrm{Hom}(\Lambda, \mathbb{Z})$ thanks to the universal coefficient theorem, and we have the following

Proposition 1.1. *The canonical map $\bigwedge^n H^1(X, \mathbb{Z}) \to H^n(X, \mathbb{Z})$ induced by the cup product is an isomorphism for every $n \geq 1$.*

Proof. This is a Künneth formula generalised to n factors, see Lemma 1.3.1 in [4]. $\square$

If we define $\mathrm{Alt}^n(\Lambda, \mathbb{Z}) := \bigwedge^n \mathrm{Hom}(\Lambda, \mathbb{Z})$ the group of $\mathbb{Z}$-valued alternating n-forms on Λ, by the above proposition we have

$$H^n(X, \mathbb{Z}) = \mathrm{Alt}^n(\Lambda, \mathbb{Z})$$

Now we recall the Hodge Decomposition Theorem for general compact Kähler manifold that we will use several times in this thesis:

Theorem 1.2. *Let M be a compact Kähler manifold, then we have*

$$H^k(M, \mathbb{C}) \cong \bigoplus_{p+q=k} H^q(M, \Omega^p_M), \tag{1.1}$$

$$H^q(M,\Omega^p_M) \cong \overline{H^p(M,\Omega^q_M)}. \tag{1.2}$$

where Ω^p_M is the sheaf of holomorphic p-forms on M. For a complex torus X, we have

$$H^q(X,\Omega^p_X) \cong \bigwedge^p \Omega \otimes \bigwedge^q \overline{\Omega} \tag{1.3}$$

where $\Omega = \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})$ and $\overline{\Omega} = \mathrm{Hom}_{\overline{\mathbb{C}}}(V,\mathbb{C})$.

For the proof of all these facts we can refer to chapter 0 of [15] and Theorem 1.4.1 of [4]. So we will also assume all Hodge theory as in [15]. Anyway we describe the sheaf Ω^p_X when X is a complex torus; thanks to the additive structure of the group X we can identified every holomorphic tangent space $T_x(X)$ of X at the point $x \in X$ with V, indeed by definition of complex torus $T_0(X) = V$ and for every $x \in X$ the translation $t_{-x}: X \to X$ induces an isomorphism of vector spaces $dt_{-x}: T_x(X) \to T_0(X)$. Using the dual isomorphism $(dt_{-x})^*: \Omega^1_{X,0} = \Omega \to \Omega^1_{X,x}$ we obtain

Proposition 1.3. *Ω^p_X is a free $\mathcal{O}_X$-module of rank $\binom{n}{p}$.*

Some remarks are required:

Definition 1.4. *Let M be a complex manifold. A* (locally) free *$\mathcal{O}_M$-module of rank r is a sheaf $\mathcal{F}$ of $\mathcal{O}_M$-modules on M which is (locally) isomorphic to $\mathcal{O}_M^{\oplus r}$.*

There is a strict relation between this concept and holomorphic vector bundles, indeed it's true the following

Proposition 1.5. *The set of locally free $\mathcal{O}_M$-modules of rank r and the set of holomorphic vector bundles of rank r are in bijection.*

Proof. It suffices to associate to a holomorphic vector bundle its sheaf of sections, see Prop 2.2.19 of [18]. □

Proof of Proposition 1.3. Taken $\varphi \in \bigwedge^p \Omega$, we can define a translation invariant holomorphic p-form ω_φ on X by

$$(\omega_\varphi)_x := (\wedge^p (dt_{-x})^*)\varphi$$

So the map $\varphi \mapsto \omega_\varphi$ defines a sheaf isomorphism

$$\bigwedge^p \Omega \otimes \mathcal{O}_X \to \Omega^p_X.$$ □

1.2 Line Bundles on Complex Tori

The goal of this section is to show that we can describe the Picard group of a complex torus X in terms of group cohomology, i.e.

$$\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*) \cong H^1(\Lambda, H^0(\mathcal{O}_V^*)),$$

where the last group is the first cohomology group of the group Λ with coefficients in the Λ-module $H^0(\mathcal{O}_V^*)$. We can refer to Appendix B of [4] or to Appendix to § 2 of [23] for generalizations of this result. The action of $\Lambda = \pi_1(X)$ on $H^0(\mathcal{O}_V^*)$ is induced by that on V. This H^1 can be also seen as the group of "derivations modulo inner derivations" (see p. 195 of [17]), so let $f: \Lambda \times V \to \mathbb{C}^*$ be a function s.t. $f(\lambda, \cdot)$ is a holomorphic map for all $\lambda \in \Lambda$ and it satisfies the *cocycle relation* $f(\lambda + \mu, v) = f(\lambda, \mu + v) f(\mu, v)$ for all λ, $\mu \in \Lambda$ and $v \in V$. Such a function is called 1-cocycle of Λ with values in $H^0(\mathcal{O}_V^*)$. Under multiplication these 1-cocycles form an abelian group $Z^1(\Lambda, H^0(\mathcal{O}_V^*))$, whose elements are called *factors of automorphy*. The factors of the form $(\lambda, v) \mapsto h(\lambda + v) h(v)^{-1}$ for some $h \in H^0(\mathcal{O}_V^*)$ are called *boundaries*. They form a subgroup $B^1(\Lambda, H^0(\mathcal{O}_V^*))$ of $Z^1(\Lambda, H^0(\mathcal{O}_V^*))$, and the quotient group

$$H^1(\Lambda, H^0(\mathcal{O}_V^*)) := \frac{Z^1(\Lambda, H^0(\mathcal{O}_V^*))}{B^1(\Lambda, H^0(\mathcal{O}_V^*))}$$

is the one we are interested in. As usual let $X = V/\Lambda$ a complex torus

Theorem 1.6. *There is a canonical isomorphism*

$$\Phi: H^1(\Lambda, H^0(\mathcal{O}_V^*)) \xrightarrow{\cong} H^1(X, \mathcal{O}_X^*).$$

Proof. Let $\{U_i\}_{i \in I}$ an open cover of X s.t. for all $i \in I$ there exists a connected open set $W_i \subseteq \pi^{-1}(U_i)$ with $\pi_i := \pi|_{W_i}: W_i \to U_i$ biholomorphic. For all i, $j \in I$ there exists a unique $\lambda_{i,j} \in \Lambda$ s.t.

$$\pi_j^{-1}(x) = \pi_i^{-1}(x) + \lambda_{i,j} \quad \forall\ x \in U_i \cap U_j, \tag{1.4}$$

so we have $\lambda_{i,j} + \lambda_{j,k} = \lambda_{i,k}$ for all i, j, and $k \in I$. Now we pick f in $Z^1(\Lambda, H^0(\mathcal{O}_V^*))$, i, $j \in I$, $x \in U_i \cap U_j$ and define $g_{i,j}(x) := f(\lambda_{i,j}, \pi_i^{-1}(x))$. So $g := \{g_{i,j}\} \in Z^1(X, \mathcal{O}_X^*)$ thanks to the cocycle relation of f and (1.4), and we have a map

$$Z^1(\Lambda, H^0(\mathcal{O}_V^*)) \longrightarrow H^1(X, \mathcal{O}_X^*)$$

that is a group homomorphism. In order to define Φ we have to prove that $B^1(\Lambda, H^0(\mathcal{O}_V^*))$ goes to 0. Let h be an element of $H^0(\mathcal{O}_V^*)$, the homomorphism just defined sends $h(\lambda + v) h(v)^{-1}$ in $\{g_{i,j}\}$ with $g_{i,j}(x) = h(\lambda_{i,j} + \pi_i^{-1}(x)) h(\pi_i^{-1}(x))^{-1}$. From (1.4) we have that $g_{i,j}(x) = h(\pi_j^{-1}(x)) h(\pi_i^{-1}(x))^{-1}$ and $\{g_{i,j}\} \in B^1(X, \mathcal{O}_X^*)$. So Φ is well defined and, since its construction clearly does not depend on the choice of U_i and π_i, we have only to give the inverse map. Taken $L \in H^1(X, \mathcal{O}_X^*)$, from the $\overline{\partial}$-Poincaré lemma (see pp. 46-47 of [15]) we know that $H^1(V, \mathcal{O}_V^*) = 0$ hence $\pi^* L$ is the trivial line bundle on V. Let $\alpha: \pi^* L \to V \times \mathbb{C}$ be a trivialization. Fixed $v \in V$ and $\lambda \in \Lambda$ we have

$$\mathbb{C} \xleftarrow{\alpha_v} (\pi^* L)_v = L_{\pi(v)} = (\pi^* L)_{v+\lambda} \xrightarrow{\alpha_{v+\lambda}} \mathbb{C}$$

the composition of these two maps gives us a linear automorphism of $\mathbb{C}$, that is given by a non-zero complex number $e_\lambda(z)$. We have obtained a set of functions $\{e_\lambda \in \mathcal{O}_V^*\}_{\lambda \in \Lambda}$ that satisfies the *compatibility relation*

$$e_{\lambda+\lambda'}(v) = e_\lambda(v) e_{\lambda'}(v + \lambda), \quad \forall\ \lambda, \lambda' \in \Lambda,\ v \in V \tag{1.5}$$

Now we can define $f: \Lambda \times V \to \mathbb{C}^*$ by $(\lambda, v) \mapsto e_\lambda(v)$. We have $f \in Z^1(\Lambda, H^0(\mathcal{O}_V^*))$ thanks to the above compatibility relation. We must only show that this construction does not depend on the choice of the trivialization α. Let α' be another trivialization, there exists $h \in H^0(\mathcal{O}_V^*)$ s.t. $\alpha' \alpha^{-1}(v, z) = (v, h(v) z)$ for all $(v, z) \in V \times \mathbb{C}$, so the class $[f]$ in $H^1(\Lambda, H^0(\mathcal{O}_V^*))$ is independent of the trivialization chosen. We have constructed a map

$$H^1(X, \mathcal{O}_X^*) \longrightarrow H^1(\Lambda, H^0(\mathcal{O}_V^*))$$

and it is easy to see that this is the inverse of the previous Φ. □

With this new view on the Picard group of a complex torus we can investigate the (first) Chern class of a line bundle. By definition this is the boundary map $H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z})$, given by exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{exp} \mathcal{O}_X^* \to 0$$

where $exp(\cdot) = e^{2\pi i \cdot}$. We can prove the following

Theorem 1.7. *There is a canonical isomorphism*

$$H^2(X, \mathbb{Z}) \longrightarrow \mathrm{Alt}^2(\Lambda, \mathbb{Z})$$

that maps the Chern class $c_1(L)$ of a line bundle L on X with factor of automorphy $f = e^{2\pi i g}$ to the alternating form

$$E_L(\lambda, \mu) = g(\mu, v + \lambda) + g(\lambda, v) - g(\lambda, v + \mu) - g(\mu, v)$$

for all λ, $\mu \in \Lambda$ and $v \in V$.

As we will see E_L does not depend on the variable v. For the proof of this theorem we need the following two lemmas, where $Z^2(\Lambda, \mathbb{Z})$ is the group of 2-cocycles on Λ with values in $\mathbb{Z}$ viewed as a trivial Λ-module:

Lemma 1.8. *The map* $\alpha: Z^2(\Lambda,\mathbb{Z}) \to \mathrm{Alt}^2(\Lambda,\mathbb{Z})$ *defined by* $\alpha F(\lambda,\mu) = F(\lambda,\mu) - F(\mu,\lambda)$ *induces a canonical isomorphism*

$$\alpha: H^2(\Lambda,\mathbb{Z}) \longrightarrow \mathrm{Alt}^2(\Lambda,\mathbb{Z})$$

Proof. An $F \in Z^2(\Lambda,\mathbb{Z})$ is, by definition, a map $F: \Lambda\times\Lambda \to \mathbb{Z}$ s.t. $\partial F(\lambda,\mu,\nu) = F(\mu,\nu) + F(\lambda,\mu+\nu) - F(\lambda+\mu,\nu) - F(\lambda,\mu) = 0$ for all λ, μ, $\nu \in \Lambda$, hence $\alpha F(\lambda+\mu,\nu) - \alpha F(\lambda,\nu) - \alpha F(\mu,\nu) = \partial F(\lambda,\nu,\mu) - \partial F(\nu,\lambda,\mu) - \partial F(\lambda,\mu,\nu) = 0$, that is αF is a bilinear form and so $\alpha F \in \mathrm{Alt}^2(\Lambda,\mathbb{Z})$. α is obviously a group homomorphism and we have that $\alpha(B^2(\Lambda,\mathbb{Z})) = 0$, where $B^2(\Lambda,\mathbb{Z})$ is the subgroup of 2-coboundaries of Λ with values in $\mathbb{Z}$. Indeed the elements of $B^2(\Lambda,\mathbb{Z})$ are of the form $\partial f(\lambda,\mu) = f(\mu) - f(\lambda+\mu) + f(\lambda)$ that is symmetric in λ and μ. So we obtain a map

$$\alpha: H^2(\Lambda,\mathbb{Z}) \longrightarrow \mathrm{Alt}^2(\Lambda,\mathbb{Z})$$

α is a surjective function: the elements $f \wedge g$ generate $\mathrm{Alt}^2(\Lambda,\mathbb{Z})$ as f and g run over $\mathrm{Hom}(\Lambda,\mathbb{Z})$. So fixed such f and g, $f \otimes g$ is in $Z^2(\Lambda,\mathbb{Z})$ and $\alpha(f \otimes g)(\lambda,\mu) = f\otimes g(\lambda,\mu) - f\otimes g(\mu,\lambda) = f\wedge g(\lambda,\mu)$. Hence we have the surjectivity. Injectivity comes from the fact that every surjective homomorphism of free $\mathbb{Z}$-modules of the same rank is injective. □

From the exponential sequence we have the following exact sequence

$$0 \to H^0(V,\mathbb{Z}) = \mathbb{Z} \to H^0(V,\mathcal{O}_V) \to H^0(V,\mathcal{O}_V^*) \to 0$$

Λ acts on all these groups, hence we can consider the connecting homomorphism

$$\delta: H^1(\Lambda, H^0(\mathcal{O}_V^*)) \longrightarrow H^2(\Lambda,\mathbb{Z})$$

that, by definition, maps a 1-cocycle $f = e^{2\pi i g} \in Z^1(\Lambda, H^0(\mathcal{O}_V^*))$ to the 2-cocycle $\delta f(\lambda,\mu) = g(\mu, v+\lambda) - g(\lambda+\mu, v) + g(\lambda, v)$ for all λ, $\mu \in \Lambda$ and $v \in V$. Here δf does not depend on the variable v because of the cocycle relation $f(\lambda+\mu, v) = f(\mu, v+\lambda) f(\lambda, v)$. The second lemma we need is

Lemma 1.9. *The following diagram is commutative*

$$\begin{array}{ccc}
H^1(\Lambda, H^0(\mathcal{O}_V^*)) & \xrightarrow{\delta} & H^2(\Lambda,\mathbb{Z}) \\
\Phi\downarrow & & \downarrow\xi \\
H^1(X,\mathcal{O}_X^*) & \xrightarrow[c_1]{} & H^2(X,\mathbb{Z})
\end{array}$$

where ξ *is an isomorphism.*

Proof. The function ξ is defined similarly to the Theorem 1.6, for details see Lemma 2.1.4 of [4]. □

Proof of Theorem 1.7. We take the composition $\alpha\xi^{-1}: H^2(X,\mathbb{Z}) \to \mathrm{Alt}^2(\Lambda,\mathbb{Z})$. This is the canonical isomorphism we are interested in, indeed by Lemma 1.9, taken $f \in H^1(\Lambda, H^0(\mathcal{O}_V^*))$, the cocycle δf represents the element $\xi^{-1}c_1(L)$, where $L = \Phi(f)$. So it is easy to see that $\alpha\xi^{-1}c_1(L) = \alpha\delta f = E_L$ □

Now the inverse isomorphism $\mathrm{Alt}^2(\Lambda,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ extends to an isomorphism

$$\beta: \mathrm{Alt}^2(\Lambda,\mathbb{Z}) \otimes \mathbb{C} = \mathrm{Alt}^2(\Lambda,\mathbb{C}) = \mathrm{Alt}^2_{\mathbb{R}}(V,\mathbb{C}) \longrightarrow H^2(X,\mathbb{C})$$

where $\mathrm{Alt}^2_{\mathbb{R}}(V,\mathbb{C})$ is the group of $\mathbb{R}$-linear alternating 2-forms on V with values in $\mathbb{C}$. Both domain and codomain of β have a decomposition, indeed $\mathrm{Alt}^2_{\mathbb{R}}(V,\mathbb{C})$ can be written as $\bigwedge^2 \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) = \bigwedge^2(\Omega \oplus \overline{\Omega}) = \bigwedge^2 \Omega \oplus \Omega \otimes \overline{\Omega} \oplus \bigwedge^2 \overline{\Omega}$, and we have the Hodge decomposition for $H^2(X,\mathbb{C})$. As expected, β respects these decompositions (see p. 27 of [4]). So we can characterize all alternating forms that come from line bundles with the following

Proposition 1.10. *Let $E: V \times V \to \mathbb{R}$ be an alternating form. The following conditions are equivalent:*

i) $\exists\ L \in \mathrm{Pic}(X)$ *s.t. E represents $c_1(L)$,*

ii) $E(\Lambda,\Lambda) \subseteq \mathbb{Z}$ *and* $E(iv,iw) = E(v,w) \quad \forall v,w \in V$

Proof. Consider the diagram

$$\begin{tikzcd}
H^1(\mathcal{O}_X^*) \arrow[r, "c_1"] & H^2(X,\mathbb{Z}) \arrow[r] \arrow[d, hook] & H^2(\mathcal{O}_X) \arrow[d, equal] \\
& H^2(X,\mathbb{C}) \arrow[r, "p"] \arrow[d, "\beta^{-1}"] & H^{0,2}_{\bar\partial}(X) \arrow[d, "\cong"] \\
& \bigwedge^2 \Omega \oplus \Omega \otimes \overline{\Omega} \oplus \bigwedge^2 \overline{\Omega} \arrow[r, "p"] & \bigwedge^2 \overline{\Omega}
\end{tikzcd}$$

where the first line is exact and p is the projection. The diagram commutes for what we have seen and for the following

Lemma 1.11. *Let M be a compact Kähler manifold. The map $H^k(M,\mathbb{C}) \to H^k(M,\mathcal{O}_M)$, induced by the inclusion $\mathbb{C} \subseteq \mathcal{O}_M$ and the projection $H^k(M,\mathbb{C}) \to H^{0,k}_{\bar\partial}(M)$ coincide.*

Let L be an element of $H^1(\mathcal{O}_X^*)$, we have $\beta^{-1}c_1(L) = E = E_1 + E_2 + E_3$ where $E_1 \in \bigwedge^2 \Omega$, $E_2 \in \Omega \otimes \overline{\Omega}$ and $E_3 \in \bigwedge^2 \overline{\Omega}$. But $E_1 = \overline{E_3}$ because E is an $\mathbb{R}$-valued form and moreover $E_3 = 0$ by the above diagram. So we have $E = E_2$, hence *ii*). The other implication follows also from the diagram. □

Finally Lemma 1.11 remains to be proven

Proof of Lemma 1.11. Let $\mathcal{A}^r(M)$ be the sheaf of germs of complex r-forms on M. We know that

$$\mathcal{A}^r(M) = \bigoplus_{p+q=r} \mathcal{A}^{p,q}(M)$$

where on the right side we have the sum over the (p, q)-forms $\mathcal{A}^{p,q}(M)$. Now consider the following diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[r] \arrow[d, hook] & \mathcal{A}^0(M) \arrow[r, "d"] \arrow[d, equal] & \mathcal{A}^1(M) \arrow[r, "d"] \arrow[d, "\pi^{0,1}"] & \mathcal{A}^2(M) \arrow[r] \arrow[d, "\pi^{0,2}"] & \dots \\
\mathcal{O}_M \arrow[r] & \mathcal{A}^0(M) \arrow[r, "\bar\partial"] & \mathcal{A}^{0,1}(M) \arrow[r, "\bar\partial"] & \mathcal{A}^{0,2}(M) \arrow[r] & \dots
\end{tikzcd}$$

where the two lines are acyclic resolutions of $\mathbb{C}$ and $\mathcal{O}_M$ and the map $\pi^{0,q}$ is the projection of a q-form on its $(0, q)$-component. Using the commutativity of the diagram and the Abstract de Rham Theorem (see Theorem 4.1 of [5]) we have done. ☐

1.3 Appell-Humbert Theorem

We define the Néron-Severi group of a complex manifold

Definition 1.12. *Let M be a compact complex manifold, the* Néron-Severi group $\mathrm{NS}(M)$ *of M is the image of first Chern class map $c_1 : H^1(M, \mathcal{O}^*_M) \to H^2(M, \mathbb{Z})$.*

As usual we denote by $\Re z$ and $\Im z$ the real and imaginary part of a complex number z.

Remark 1.13. *If X is a complex torus, we can consider $\mathrm{NS}(X)$ either as the group of $\mathbb{R}$-valued alternating forms E on V satisfying $E(\Lambda, \Lambda) \subseteq \mathbb{Z}$ and $E(iv, iw) = E(v, w)$ for all v and $w \in V$, or as the group of Hermitian forms H on V with $\Im H(\Lambda, \Lambda) \subseteq \mathbb{Z}$.*

Indeed we have the following

Lemma 1.14. *There is a bijection between the set of Hermitian forms H on V and the set of alternating forms E on V with values in $\mathbb{R}$, satisfying $E(iv, iw) = E(v, w)$, given by $E(v, w) = \Im H(v, w)$ and $H(v, w) = E(iv, w) + iE(v, w)$, for all v, $w \in V$.*

Proof. Given H, the form $E = \Im H$ is alternating and $E(iv, iw) = \Im H(iv, iw) = \Im H(v, w) = E(v, w)$. Conversely given E, the form H is Hermitian, indeed $H(v, w) = E(iv, w) + iE(v, w) = -E(iw, -v) - iE(w, v) = \overline{H(w, v)}$. ☐

So we see $\mathrm{NS}(X) = \{H: V \times V \to \mathbb{C} \mid \Im H(\Lambda, \Lambda) \subseteq \mathbb{Z}\}$ and we denote with $S^1 \subset \mathbb{C}$ the unitary circle in the complex plane. A *semicharacter* for $H \in \mathrm{NS}(X)$ is a map $\chi: \Lambda \to S^1$ s.t.

$$\begin{aligned} \chi(\lambda + \mu) &= \chi(\lambda)\chi(\mu)e^{\pi i \Im H(\lambda,\mu)} \\ &= (-1)^{\Im H(\lambda,\mu)}\chi(\lambda)\chi(\mu), \quad \forall\ \lambda, \mu \in \Lambda \end{aligned} \tag{1.6}$$

we denote by $\mathcal{C}(\Lambda)$ the set of couples (H, χ), where $H \in \mathrm{NS}(X)$ and χ is a semicharacter for H. Clearly $\mathcal{C}(\Lambda)$ is a group with respect to the product $(H_1, \chi_1)(H_2, \chi_2) = (H_1 + H_2, \chi_1\chi_2)$ and we have the following exact sequence

$$0 \to \mathrm{Hom}(\Lambda, S^1) \xrightarrow{i} \mathcal{C}(\Lambda) \xrightarrow{p} \mathrm{NS}(X)$$

where $i(\chi) = (0, \chi)$ and $p(H, \chi) = H$. We want to show that p is surjective. If $\{\lambda_i\}$ is a base of Λ, we fix $\chi(\lambda_i) \in S^1$ arbitrarily. Then for every element $\sum n_i \lambda_i$ of Λ there is a unique choice of the sign in

$$\chi(\sum n_i \lambda_i) = \pm \prod \chi(\lambda_i)^{n_i}$$

s.t. (1.6) is verified. Let's consider now the map $\mathcal{C}(\Lambda) \to \mathrm{Pic}(X)$ defined in the following way: taken $(H, \chi) \in \mathcal{C}(\Lambda)$ we have a cocycle $a_{(H,\chi)} = a \in Z^1(\Lambda, H^0(\mathcal{O}_V^*))$ given by

$$a(\lambda, v) := \chi(\lambda)e^{\pi H(v,\lambda) + \frac{\pi}{2}H(\lambda,\lambda)}$$

indeed this satisfies the cocycle relation

$$\begin{aligned} a(\lambda + \mu, v) &= \chi(\lambda)\chi(\mu)e^{\pi i \Im H(\lambda,\mu) + \pi H(v,\lambda+\mu) + \frac{\pi}{2}H(\lambda+\mu,\lambda+\mu)} \\ &= \chi(\lambda)\chi(\mu)e^{\pi i \Im H(\mu,\lambda) + \pi H(v,\lambda) + \pi H(v,\mu) + \frac{\pi}{2}H(\lambda,\lambda) + \frac{\pi}{2}H(\mu,\mu)} \\ &= \chi(\lambda)e^{\pi H(v+\mu,\lambda) + \frac{\pi}{2}H(\lambda,\lambda)}\chi(\mu)e^{\pi H(v,\mu) + \frac{\pi}{2}H(\mu,\mu)} \\ &= a(\lambda, v + \mu)a(\mu, v), \quad \forall\ v \in V,\ \lambda, \mu \in \Lambda. \end{aligned}$$

So we can define the map $\mathcal{C}(\Lambda) \to \mathrm{Pic}(X)$ sending a couple (H, χ) to the line bundle $L_{(H,\chi)}$ given by the 1-cocycle a. We have

Lemma 1.15. *The function just defined is a group homomorphism and the following diagram commutes*

$$\begin{tikzcd} \mathcal{C}(\Lambda) \arrow[r, "p"] \arrow[dr] & \mathrm{NS}(X) \\ & \mathrm{Pic}(X) \arrow[u, "c_1"'] \end{tikzcd}$$

Proof. It can be easily proved that $a_{(H_1+H_2,\chi_1\chi_2)} = a_{(H_1,\chi_1)}a_{(H_2,\chi_2)}$, therefore the map is a homomorphism. In order to conclude we have to show that $c_1(L_{(H,\chi)}) = H$. We write $\chi(\lambda) = e^{2\pi i\varphi(\lambda)}$, so we have $a_{(H,\chi)}(\lambda, v) = e^{2\pi i g(\lambda,v)}$, where $g(\lambda, v) = \varphi(\lambda) - \frac{i}{2}H(v, \lambda) - \frac{i}{4}H(\lambda, \lambda)$. By the Theorem 1.7 the imaginary part of the Hermitian form $c_1(L_{(H,\chi)})$ is the alternating form

$$\begin{aligned} E_{L_{(H,\chi)}}(\lambda, \mu) &= g(\mu, v + \lambda) + g(\lambda, v) - g(\lambda, v + \mu) - g(\mu, v) \\ &= \frac{1}{2i}(H(\lambda, \mu) - \overline{H(\lambda, \mu)}) \\ &= \Im H(\lambda, \mu), \end{aligned}$$

and by Remark 1.13 we finish. $\square$

Now we would show that $\mathcal{C}(\Lambda) \to \mathrm{Pic}(X)$ is an isomorphism. Consider its restriction to $\mathrm{Hom}(\Lambda, S^1)$. Let as usual $\mathrm{Pic}^0(X)$ be the kernel of c_1, we have

Proposition 1.16. *The homomorphism* $\mathcal{C}(\Lambda) \to \mathrm{Pic}(X)$ *induces an isomorphism*

$$\mathrm{Hom}(\Lambda, S^1) \longrightarrow \mathrm{Pic}^0(X)$$

Proof. We take the following commutative diagram

$$\begin{tikzcd} H^1(X,\mathbb{Z}) \arrow[r] & H^1(\mathcal{O}_X) \arrow[r] & H^1(\mathcal{O}_X^*) \arrow[r, "c_1"] & H^2(X,\mathbb{Z}) \\ & H^1(X,\mathbb{C}) \arrow[u, twoheadrightarrow, "p"] \arrow[r, "\epsilon"] & H^1(\mathcal{O}_X^*) \arrow[u, equal] & \end{tikzcd}$$

where the first line is exact, p is the projection associated to the Hodge decomposition and ϵ is the map induced by the restricted exponential function $\mathbb{C} \xrightarrow{exp} \mathbb{C}^* \subseteq \mathcal{O}_X^*$. Hence $\mathrm{Pic}^0(X) = \mathrm{Im}(\epsilon)$, i.e. every line bundle with null Chern class is cohomologous in $H^1(X, \mathcal{O}_X^*)$ to a cocycle with constant coefficients, that is the element corresponding via the isomorphism $H^1(\mathcal{O}_X^*) \xrightarrow{\cong} H^1(\Lambda, H^0(\mathcal{O}_V^*))$ is given by a factor of automorphy $f\colon \Lambda \times V \to \mathbb{C}^*$ independent of the second variable. So we have that $f\colon \Lambda \to \mathbb{C}^*$ is a homomorphism and $f(\lambda) = e^{2\pi i g(\lambda)}$ for a certain function g. Now

$$g(\lambda + \mu) \equiv g(\lambda) + g(\mu) \quad mod\ \mathbb{Z}, \quad \forall\ \lambda, \mu \in \Lambda$$

hence $\Im g\colon \Lambda \to \mathbb{R}$ is a homomorphism that $\mathbb{R}$-linearly extends to a function still denoted by $\Im g\colon V \to \mathbb{R}$. So we define a $\mathbb{C}$-linear form $l\colon V \to \mathbb{C}$ by sending $v \mapsto \Im g(iv) + i\Im g(v)$. At this point $\chi_L(\lambda, v) := f(\lambda)e^{2\pi i l(v) - 2\pi i l(v+\lambda)} \in Z^1(\Lambda, H^0(\mathcal{O}_V^*))$, since $e^{2\pi i l} \in H^0(\mathcal{O}_V^*)$, and it is cohomologous to f by definition. Moreover χ_L does not depend on the variable v and it takes values in S^1, indeed

$$\begin{aligned} \chi_L(\lambda, v) &= e^{2\pi i g(\lambda) - 2\pi i l(\lambda)} \\ &= e^{2\pi i(\Re g(\lambda) - \Im g(i\lambda))}, \end{aligned}$$

and $\Re g(\lambda) - \Im g(i\lambda) \in \mathbb{R}$. So we have that $\chi_L \in \mathrm{Hom}(\Lambda, S^1)$, because f and $e^{2\pi i l}$ are homomorphism. Hence χ_L is a semicharacter for $0 \in \mathrm{NS}(X)$ and $L \cong L_{(0,\chi_L)}$. This shows that $\mathrm{Hom}(\Lambda, S^1) \to \mathrm{Pic}^0(X)$ is surjective. In order to show the injectivity we suppose that there is another $\chi \in \mathrm{Hom}(\Lambda, S^1)$ s.t. $L \cong L_{(0,\chi)}$, then $a_{(0,\chi_L)}(\lambda, v) = \chi_L(\lambda)$ and $a_{(0,\chi)}(\lambda, v) = \chi(\lambda)$ are cohomologous in $H^1(\Lambda, H^0(\mathcal{O}_V^*))$, i.e. there exists a map $h \in H^0(\mathcal{O}_V^*)$ s.t. $\chi_L(\lambda) = \chi_L(\lambda)h(v+\lambda)h(v)^{-1}$ for all $v \in V$ and $\lambda \in \Lambda$. Now $|\chi_L| = |\chi| = 1$ implies $|h(v+\lambda)| = |h(v)|$ for all $v \in V$ and $\lambda \in \Lambda$, hence h is bounded on V. Now by the Maximum Modulus Principle we conclude that h is constant and $\chi_L = \chi$. □

Therefore we have obtained the following commutative diagram with exact lines

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\Lambda, S^1) \arrow[r, "i"] \arrow[d, "\cong"] & \mathcal{C}(\Lambda) \arrow[r, "p"] \arrow[d] & \mathrm{NS}(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] & \mathrm{Pic}(X) \arrow[r, "c_1"] & \mathrm{NS}(X) \arrow[r] & 0
\end{tikzcd}$$

and in particular, using the five lemma, we have

Theorem 1.17 (Appel-Humbert)**.**

$$\mathcal{C}(\Lambda) \xrightarrow{\cong} \mathrm{Pic}(X).$$

Moreover from the above diagram we also derive a canonical way to associate a factor of automorphy to every line bundle on X. Given $L \in \mathrm{Pic}(X)$ we consider $c_1(L) \in \mathrm{NS}(X)$ like an Hermitian form H. Now we know that there is a semicharacter χ for H s.t. $L_{(H,\chi)} = L$. The corresponding 1-cocycle in $Z^1(\Lambda, H^0(\mathcal{O}_V^*))$

$$a_L(\lambda, v) := a_{L_{(H,\chi)}}(\lambda, v) = \chi(\lambda)e^{\pi H(v,\lambda) + \frac{\pi}{2}H(\lambda,\lambda)}$$

is called *canonical factor of automorphy for L*.

1.4 The Poincaré line Bundle

The previous theorem tells us that $\mathrm{Pic}^0(X) \cong \mathrm{Hom}(\Lambda, S^1)$. In this section we give another description of this group, showing that $\mathrm{Pic}^0(X)$ is still a complex torus if X is. We have an isomorphism of $\mathbb{R}$-vector spaces $\overline{\Omega} \to \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$, defined by sending $l \mapsto k = \Im l$ and $k \mapsto l(v) = -k(iv) + ik(v)$. Hence we obtain the canonical non-degenerate $\mathbb{R}$-bilinear form (it is the evaluation map)

$$\langle \cdot, \cdot \rangle \colon \overline{\Omega} \times V \longrightarrow \mathbb{R}, \quad \langle l, v \rangle := \Im l(v).$$

We note that

$$\widehat{\Lambda} := \{l \in \overline{\Omega} \mid \langle l, \Lambda \rangle \subseteq \mathbb{Z}\}$$

is a lattice in $\overline{\Omega}$, called the *dual lattice* of Λ. So the quotient $\widehat{X} := \overline{\Omega}/\widehat{\Lambda}$ is an n-dimensional complex torus, called the *dual complex torus*. Identifying V with the space of $\mathbb{C}$-antilinear form $\overline{\Omega} \to \mathbb{C}$ by double antiduality we have that the dual lattice of $\widehat{\Lambda}$ in V is Λ. So $\widehat{\widehat{X}} = X$, i.e. to take the dual complex torus is an involution.

Proposition 1.18. *The canonical homomorphism* $\overline{\Omega} \to \mathrm{Hom}(\Lambda, S^1)$, $l \mapsto e^{2\pi i \langle l, \cdot \rangle}$ *induces an isomorphism*

$$\widehat{X} \xrightarrow{\cong} \mathrm{Pic}^0(X)$$

Proof. The non-degeneracy of $\langle \cdot, \cdot \rangle$ implies that the homomorphism $\overline{\Omega} \to \mathrm{Hom}(\Lambda, S^1)$ is surjective and $\widehat{\Lambda}$ is exactly its kernel. □

Hence we can see $\mathrm{Pic}^0(X)$ as a complex torus and the points of $\widehat{X}$ parameterize the isomorphism classes of line bundles in $\mathrm{Pic}^0(X)$. We will denote by L_y the line bundle corresponding to the point $y \in \widehat{X}$. So we can now give the definition of the Poincaré line bundle:

Definition 1.19. *A line bundle* $\mathcal{P}$ *on the product* $X \times \widehat{X}$ *is called a* Poincaré line bundle *if it satisfies:*

i) $\mathcal{P}|_{X \times \{y\}} \cong L_y \quad \forall\ y \in \widehat{X}$,

ii) $\mathcal{P}|_{\{0\} \times \widehat{X}} \cong \mathcal{O}_{\widehat{X}}$.

Now the question is whether there is such a line bundle. We can answer with the following

Theorem 1.20. *There exists a Poincaré line bundle on* $X \times \widehat{X}$*, uniquely determined up to isomorphism.*

Proof. First of all we note that $X \times \widehat{X} = V \times \overline{\Omega}/\Lambda \times \widehat{\Lambda}$ is again a complex torus. We want to apply Appel-Humbert Theorem 1.17. Let $H \colon (V \times \overline{\Omega}) \times (V \times \overline{\Omega}) \to \mathbb{C}$ be the Hermitian form defined by

$$H((v_1, l_1), (v_2, l_2)) = l_1(v_2) + \overline{l_2(v_1)}.$$

According to the Remark 1.13 $H \in \mathrm{NS}(X \times \widehat{X})$, since $\Im H(\Lambda \times \widehat{\Lambda}, \Lambda \times \widehat{\Lambda}) \subseteq \mathbb{Z}$. Hence the form H is the first Chern class of a certain line bundle on $X \times \widehat{X}$. We define the map $\chi \colon \Lambda \times \widehat{\Lambda} \to S^1$ via

$$\chi(\lambda, l_0) := e^{\pi i \Im l_0(\lambda)}$$

χ is a semicharacter for H, hence the couple (H, χ) gives us a line bundle $\mathcal{P}$ on $X \times \widehat{X}$ with canonical factor $a_{\mathcal{P}}: (\Lambda \times \widehat{\Lambda}) \times (V \times \overline{\Omega}) \to \mathbb{C}^*$

$$a_{\mathcal{P}}((\lambda, l_0), (v, l)) = \chi(\lambda, l_0) e^{\pi H((v,l),(\lambda,l_0)) + \frac{\pi}{2} H((\lambda,l_0),(\lambda,l_0))}$$

Now we have to check the two properties required:

i) Fixed $y \in \widehat{X}$, there is an $l \in \overline{\Omega}$ s.t. $L_y = L_{(0, e^{2\pi i \Im l})}$. The restriction $\mathcal{P}|_{X \times \{y\}}$ is given by the factor $a_{\mathcal{P}}|_{\Lambda \times \{0\} \times V \times \{l\}}$, but

$$\begin{aligned} a_{\mathcal{P}}((\lambda, 0), (v, l)) &= \chi(\lambda, 0) e^{\pi H((v,l),(\lambda,0)) + \frac{\pi}{2} H((\lambda,0),(\lambda,0))} \\ &= e^{\pi l(\lambda)}, \end{aligned}$$

that is cohomologous to

$$a_{\mathcal{P}}((\lambda, 0), (v, l)) e^{\pi \overline{l(v)}} (e^{\pi \overline{l(v+\lambda)}})^{-1} = e^{2\pi i \Im l(\lambda)}$$

and this last is the canonical factor of L_y. So we conclude using Theorem 1.6.

ii) As in the previous case the restriction $\mathcal{P}|_{\{0\} \times \widehat{X}}$ has the canonical factor $a_{\mathcal{P}}((0, l_0), (0, l)) = 1$ where $l_0 \in \widehat{\Lambda}$ and $l \in \overline{\Omega}$. But 1 is also the factor of automorphy of the trivial line bundle on $\{0\} \times \widehat{X}$. Now we use again Theorem 1.6.

Hence we have proved the existence of the Poincaré line bundle. Now suppose that $\mathcal{P}'$ is another Poincaré line bundle on $X \times \widehat{X}$, i.e. it satisfies conditions $i)$ and $ii)$ of the definition 1.19. Then $\mathcal{Q} := \mathcal{P}' \otimes \mathcal{P}^*$ is a line bundle on $X \times \widehat{X}$ s.t.

$$\mathcal{Q}|_{X \times \{y\}} = \mathcal{O}_X \quad \forall\ y \in \widehat{X}$$

and

$$\mathcal{Q}|_{\{0\} \times \widehat{X}} = \mathcal{O}_{\widehat{X}}.$$

Thanks to the Seesaw Theorem (see Appendix), we know that there exists a line bundle $\mathcal{R}$ on $\widehat{X}$ s.t. $p_2^* \mathcal{R} = \mathcal{Q}$, where $p_2 : X \times \widehat{X} \to \widehat{X}$ is the second projection. Now the uniqueness follows easily, indeed

$$\mathcal{O}_{\widehat{X}} = \mathcal{Q}|_{\{0\} \times \widehat{X}} = p_2^* \mathcal{R}|_{\{0\} \times \widehat{X}} = \mathcal{R}$$

and so $\mathcal{Q} = \mathcal{O}_{X \times \widehat{X}}$, that is to say $\mathcal{P} \cong \mathcal{P}'$. □

Chapter 2

The Generic Vanishing Theorem of Green and Lazarsfeld

In this chapter we prove Green-Lazarsfeld generic vanishing theorem and we show some of its immediate corollaries. At first, we give several necessary definitions and we cite two general theorems; later, we study infinitesimal deformations of cohomology groups as in [13]. In what follows M will denote a compact, connected Kähler manifold of complex dimension n.

2.1 Albanese and Picard Tori

We can associate to M two complex tori, the Albanese torus $\mathrm{Alb}(M)$ and the Picard torus $\mathrm{Pic}^0(M)$. We have already seen the second, that is $\mathrm{Pic}^0(M) := \mathrm{Ker}\left[c_1 : \mathrm{Pic}(M) \to H^2(M, \mathbb{Z})\right]$. We have the following

Proposition 2.1. $\mathrm{Pic}^0(M)$ *is in a natural way a complex torus of dimension* $b_1(M)/2$*, where* b_1 *is the first Betti number.*

We note that $b_1(M)$ is even because M is Kähler.

Proof. From the exponential sequence we know that

$$\mathrm{Pic}^0(M) \cong \frac{H^1(M, \mathcal{O}_M)}{H^1(M, \mathbb{Z})}$$

because the map $H^1(M, \mathbb{Z}) \to H^1(M, \mathcal{O}_M)$ is injective for compact manifolds. By Hodge theorem we have $H^1(M, \mathbb{C}) = H^0(\Omega^1_M) \oplus H^1(\mathcal{O}_M)$ and $\overline{H^1(\mathcal{O}_M)} = H^0(\Omega^1_M)$. Hence $\mathrm{Rank}(H^1(M, \mathbb{Z})) := \dim_{\mathbb{C}} H^1(M, \mathbb{C})$ is equal to $2\ \dim_{\mathbb{C}} H^1(\mathcal{O}_M)$ and the map

$$H^1(M, \mathbb{Z}) \hookrightarrow H^1(M, \mathbb{C}) \to H^1(M, \mathcal{O}_M)$$

is injective, with discrete image that generates $H^1(M, \mathcal{O}_M)$ as a real vector space, thanks to the Lemma 1.11. $\square$

We call $q(M) := h^0(\Omega^1_M)$ the *irregularity* of M. By (1.1) holomorphic forms are closed and so the Stokes' Theorem implies that the map $H_1(M, \mathbb{Z}) \to H^0(\Omega^1_M)^*$, $[\gamma] \longmapsto (\omega \mapsto \int_\gamma \omega)$ is well defined. We have that the image of this map is a lattice in $H^0(\Omega^1_M)^*$, again by Hodge theory. Indeed if $\omega_1, \ldots, \omega_q$ is a base of $H^0(\Omega^1_M)$, we know that $\omega_1, \ldots, \omega_q, \overline{\omega_1}, \ldots, \overline{\omega_q}$ is a base of $H^1(M, \mathbb{C})$. Let $[\gamma_1], \ldots, [\gamma_{2q}]$ be a "base" of $H_1(M, \mathbb{Z})/$ torsion, it is easy to see that the vectors

$$\left(\int_{\gamma_j} \omega_1 \ , \ldots, \int_{\gamma_j} \omega_q\right) \in \mathbb{C}^q \quad j = 1, \ldots, 2q$$

are linearly independent over $\mathbb{R}$. If they were not the $2q$ vectors of $\mathbb{R}^{2q}$

$$\left(\int_{\gamma_j} \omega_1 + \overline{\omega_1} \ , \ldots, \int_{\gamma_j} \omega_q + \overline{\omega_q} \ , \ i\int_{\gamma_j} \omega_1 - \overline{\omega_1} \ , \ldots, \ i\int_{\gamma_j} \omega_q - \overline{\omega_q}\right)$$

would be linearly dependent. So by duality there would be real numbers $\alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_q$, not all zero, s.t. $\alpha_1(\omega_1 + \overline{\omega_1}) + \ldots + \alpha_q(\omega_q + \overline{\omega_q}) + i\beta_1(\omega_1 - \overline{\omega_1}) + \ldots + i\beta_q(\omega_q - \overline{\omega_q})$ would be cohomologous to 0. But this would imply that $\omega_1, \ldots, \omega_q, \overline{\omega_1}, \ldots, \overline{\omega_q}$ would be dependent over $\mathbb{C}$, that is absurd. Hence the quotient

$$\mathrm{Alb}(M) := \frac{H^0(\Omega^1_M)^*}{H_1(M, \mathbb{Z})}$$

is a complex torus, called the *Albanese torus* of M. We make the following

Remark 2.2. *If $M = X = V/\Lambda$ is a complex torus, we have that $H^0(\Omega^1_X)^* = V$ and $H_1(M, \mathbb{Z}) = \Lambda$, so $\mathrm{Alb}(X) = X$.*

Remark 2.3. *If $M = C$ is an algebraic curve, i.e. $n = 1$, the Albanese torus is also called* Jacobian variety *and it is denoted by* $\mathrm{Jac}(C)$. *In this case Abel's Theorem gives us an isomorphism between Jacobian variety and Picard torus.*

In general this two tori are distinct, however there is the following useful relation

Proposition 2.4.

$$\mathrm{Pic}^0(M) \cong \widehat{\mathrm{Alb}(M)}$$

Before proving this duality we introduce the Albanese map: let p_0 be a point in M, the *Albanese map* a of M with base point p_0 is

$$a = a_{p_0} \colon M \to \mathrm{Alb}(M), \quad p \mapsto \left(\omega \mapsto \int_{p_0}^p \omega\right). \tag{2.1}$$

The Albanese map is well defined because of the definition of $\mathrm{Alb}(M)$ and it is clearly a holomorphic map. Moreover a change of the base point p_0 coincides with a translation in the Albanese torus. We have the following simple but useful

Lemma 2.5. *The differential da_p of the Albanese map at a point $p \in M$ is the linear function $T_pM \to H^0(\Omega^1_M)^*$, given by sending $v \mapsto (\omega \mapsto \omega_p(v))$. So the codifferential $da^* : H^0(\Omega^1_M) \otimes \mathcal{O}_M \to \Omega^1_M$ is just the evaluation morphism.*

Proof. It is enough to differentiate the integral (2.1) in the definition of the Albanese map in order to have the first assertion. The second follows from the definition of the transpose map. □

Now we arrived to the

Proof of Proposition 2.4. Let $a : M \to \mathrm{Alb}(M)$ be the Albanese map of M with some base point $p_0 \in M$. From Proposition 1.18 we know that $\widehat{\mathrm{Alb}(M)} \cong \mathrm{Pic}^0(\mathrm{Alb}(M))$, so it suffices to see that $\mathrm{Pic}^0(\mathrm{Alb}(M)) \cong \mathrm{Pic}^0(M)$. By the above lemma we have that the pullback

$$a^* \colon H^0(\Omega^1_{\mathrm{Alb}(M)}) \longrightarrow H^0(\Omega^1_M)$$

is an isomorphism, because it is injective and (1.3). Then taking the complex coniugation

$$\overline{a^*} \colon H^1(\mathcal{O}_{\mathrm{Alb}(M)}) \longrightarrow H^1(\mathcal{O}_M)$$

we have an isomorphism s.t. $\overline{a^*}\left(H^1(\mathrm{Alb}(M), \mathbb{Z})\right) = H^1(M, \mathbb{Z})$. The desired relation follows. □

Corollary 2.6. *There exists a Poincaré line bundle $\mathcal{P}$ on $M \times \mathrm{Pic}^0(M)$, i.e. a line bundle s.t. $\mathcal{P}|_{M \times \{y\}} \cong L_y$, for all $y \in \mathrm{Pic}^0(M)$.*

Proof. Let's denote $\mathrm{Alb}(M) = X$. From Theorem 1.20 we know that there is a Poincaré line bundle $\mathcal{P}'$ on $X \times \widehat{X}$. Let θ be the isomorphism between $\mathrm{Pic}^0(M)$ and $\widehat{X}$, we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{P} & \longrightarrow & \mathcal{P}' \\ \downarrow & & \downarrow \\ M \times \mathrm{Pic}^0(M) & \xrightarrow{a \times \theta} & X \times \widehat{X} \end{array}$$

where $\mathcal{P}$ is the pullback line bundle. It is easy to verify that this $\mathcal{P}$ is the required line bundle. □

Corollary 2.7. *Let $S^1 \subseteq \mathbb{C}$ be the unitary circle, then*

$$\mathrm{Pic}^0(M) \cong \mathrm{Hom}(\pi_1(M), S^1).$$

Proof. As above, denote $\mathrm{Alb}(M) = X$. Using propositions 2.4 and 1.18, we have

$$\mathrm{Pic}^0(M) \cong \widehat{X} \cong \mathrm{Pic}^0(X).$$

Now the Proposition 1.16 and the definition of $\mathrm{Alb}(M)$ give

$$\mathrm{Pic}^0(X) \cong \mathrm{Hom}(H_1(M,\mathbb{Z}), S^1) = \mathrm{Hom}(\pi_1(M), S^1).$$

□

2.2 The Generic Vanishing Theorem: statement and corollaries

Let us denote by $\mathcal{P}$ the Poincaré line bundle on $M \times \mathrm{Pic}^0(M)$, as in the previous section. For given integers $i, m \geq 0$ we consider the subset

$$S^i_m(M) := \left\{ y \in \mathrm{Pic}^0(M) \mid h^i(M, L_y) \geq m \right\}.$$

In particular, when $m = 1$, we write

$$S^i(M) := S^i_1(M) = \left\{ y \in \mathrm{Pic}^0(M) \mid H^i(M, L_y) \neq 0 \right\}.$$

We want to state, without proof, the semicontinuity theorem in order to apply it to $\mathcal{P}$. Let Y, Z be complex manifolds of dimension p and q respectively, with $p \geq q$ and let $f : Y \to Z$ be a proper, regular (i.e. with Jacobian matrix of maximal rank q in every point) holomorphic function of Y onto Z. We know that $Y_z := f^{-1}(z)$ is a compact complex submanifold of Y of dimension $d = p - q$, for all $z \in Z$. Such a f is called a *regular family.* Let E be a holomorphic vector bundle of rank r on Y; we denote the restriction to Y_z by $E_z := E|_{Y_z}$. The semicontinuity theorem gives us information about the behavior of $H^i(Y_z, E_z)$ as a function of $z \in Z$.

Theorem 2.8 (Semicontinuity)**.** *Let $f : Y \to Z$ and E be defined as above. Then for all $m \geq 0$ the set*

$$\mathcal{A}_{i,m} := \left\{ z \in Z \mid \dim_{\mathbb{C}} H^i(Y_z, E_z) \geq m \right\}$$

is an analytic subvariety of Z.

A proof of this theorem can be read in chapter 10 §5 of [12], or chapter 3.4 of [1] and, in a more algebraic version, in chapter III.12 of [16]. Applying this result to

$$\mathcal{P} \to M \times \mathrm{Pic}^0(M) \xrightarrow{p_2} \mathrm{Pic}^0(M)$$

where p_2 is the second projection, we get that the locus $S^i_m(M)$ is an analytic subvariety of $\mathrm{Pic}^0(M)$. One of the central theorems of [13] is the following

Theorem 2.9.

$$\operatorname{codim}(S^i(M), \operatorname{Pic}^0(M)) \geq \dim a(M) - i$$

where $a : M \to \operatorname{Alb}(M)$ *is the Albanese map of* M*.*

We can observe that a is a proper map and, thanks to the proper mapping theorem ([15], p. 395), the dimension of $a(M)$ is a well defined number called the *Albanese dimension*. Now we prefer to see several corollaries of this theorem and, in the following section, we will introduce the machinery needed for the proofs.

Corollary 2.10 (Generic Vanishing Theorem)**.** *If* $L \in \operatorname{Pic}^0(M)$ *is a generic line bundle, then* $H^i(M, L) = 0$ *for all* $i < \dim a(M)$*.*

Here "generic" means that the property is true on the complement of a proper analytic subvariety of $\operatorname{Pic}^0(M)$.

Definition 2.11. *M has* maximal Albanese dimension *if* $\dim a(M) = n$*, i.e. the Albanese map is generically finite over its image.*

The generic vanishing theorem is especially useful when M has maximal Albanese dimension, indeed in this case we have that $H^i(M, L) = 0$ for general $L \in \operatorname{Pic}^0(M)$ and $i < n$.

Corollary 2.12. *If* M *has maximal Albanese dimension, then*

$$\chi(M, \omega_M) \geq 0,$$

where $\omega_M := \det \Omega_M$ *is the canonical line bundle.*

Proof. The Hirzebruch-Riemann-Roch theorem ([18], p. 232) implies that

$$\chi(M, \omega_M \otimes L) = \chi(M, \omega_M) \quad \forall\ L \in \operatorname{Pic}^0(M),$$

and so, using Serre duality ([15], p. 153), we have

$$\begin{aligned}
\chi(M, \omega_M) &= \chi(M, \omega_M \otimes L) \\
&= \sum_{i=0}^{n} (-1)^i \dim H^i(M, \omega_M \otimes L) \\
&= \sum_{i=0}^{n} (-1)^i \dim H^{n-i}(M, L^*) \\
&= \dim H^n(M, L^*) \\
&= \dim H^0(M, \omega_M \otimes L) \geq 0
\end{aligned}$$

for a sufficiently generic line bundle $L \in \operatorname{Pic}^0(M)$. □

Now if V is an analytic subvariety of M, after fixing a point p in V, we denote by $\underline{m}_p$ the maximal ideal of the local ring $\mathcal{O}_{M,p}$. Recall that the *local dimension* of V at p, $\dim_p V$, is the maximum of the dimensions of the irreducible components of V containing p. We give the following

Definition 2.13. *Let $\mathcal{I}$ be the ideal defining V in $\mathcal{O}_M$, we set*

$$J_k := \frac{\underline{m}_p^k \cap \mathcal{I}}{\underline{m}_p^{k+1} \cap \mathcal{I}}$$

$$J := \bigoplus_{k\geq 0} J_k \subseteq \bigoplus_{k\geq 0} \underline{m}_p^k/\underline{m}_p^{k+1} = \bigoplus_{k\geq 0} \mathrm{Sym}^k(T_pM^*) = \mathrm{Sym}(T_pM^*)$$

where $T_pM := (\underline{m}_p/\underline{m}_p^2)^*$ is the Zariski tangent space to M at p. We can define the tangent cone *to V at p, $TC_p(V) \subseteq T_pM$, as the cone given by the ideal J, i.e. the common zero locus of the polynomials in J.*

An important property of the tangent cone, that we will use later, is that its dimension is equal to the local dimension of V at p ([24], Corollary of p. 162). During the process which will lead to the proof of the Theorem 2.9, we will also prove the following

Proposition 2.14. *Fix a point $y \in \mathrm{Pic}^0(M)$. Then*

a) *If $m := h^i(M, L_y)$, we have*

$$\dim_y S_m^i(M) \leq \dim N$$

where $N \subseteq H^0(M, \Omega_M^1)$ is the subspace consisting of all 1-forms ω s.t. $\omega \wedge \alpha = \omega \wedge \beta = 0$ for all $\alpha \in H^0(M, \Omega_M^{i-1} \otimes L_y^*)$ and $\beta \in H^0(M, \Omega_M^i \otimes L_y^*)$.

b) *Let m be a positive integer. If the sequence*

$$H^0(\Omega_M^{i-1} \otimes L_y^*) \xrightarrow{\wedge\omega} H^0(\Omega_M^i \otimes L_y^*) \xrightarrow{\wedge\omega} H^0(\Omega_M^{i+1} \otimes L_y^*) \qquad (2.2)$$

is exact for some holomorphic 1-form ω on M, then $S_m^i(M)$ is a proper subvariety of $\mathrm{Pic}^0(M)$.

c) *Let m be a positive integer. If $y \in S_m^i(M)$ and the sequence (2.2) is exact for every non-zero $\omega \in H^0(M, \Omega_M^1)$, then y is an isolated point of $S_m^i(M)$.*

2.3 First order deformations of cohomology groups

We start with some definitions:

Definition 2.15. *Let* $(X, \mathcal{O}_X)$ *be a ringed topological space (see p. 37 of [20]) and let* $\mathcal{S}$ *be an* $\mathcal{O}_X$*-sheaf (i.e. a sheaf of modules over* $\mathcal{O}_X$*). We say that* $\mathcal{S}$ *is* of finite type at $x \in X$ *if there is an open neighborhood* U *of* x *and sections* $s_1, \dots, s_p \in \mathcal{S}(U)$ *s.t.*

$$\sigma: \mathcal{O}^p(U) \to \mathcal{S}(U), \quad (a_{1x}, \dots, a_{px}) \mapsto \sum_{i=1}^{p} a_{ix} s_{ix}, \ x \in U$$

is surjective. An $\mathcal{O}$*-sheaf* $\mathcal{S}$ *is called* of finite type on X, *if it is of finite type at all points of* X.

Definition 2.16. *An* $\mathcal{O}$*-sheaf* $\mathcal{S}$ *is* of relation finite type at $x \in X$ *if for all sections* $s_1, \dots, s_p \in \mathcal{S}(U)$*, where* U *is an open neighborhood of* x*, the* sheaf of relations of $s_1, \dots, s_p$

$$\text{Ker } \sigma = \bigcup_{y \in U} \left\{ (a_{1y}, \dots, a_{py}) \in \mathcal{O}_y^p \mid \sum a_{iy} s_{iy} = 0 \right\}$$

is of finite type at x*. An* $\mathcal{O}$*-sheaf* $\mathcal{S}$ *is* of relation finite type on X, *if it is of relation finite type at all points of* X.

Definition 2.17. *An* $\mathcal{O}$*-sheaf* $\mathcal{S}$ *on* X *is called* $(\mathcal{O}_X$-)coherent *if it is of finite and of relation finite type on* X.

The famous Oka's Coherence Theorem (chapter 2 § 5.2 of [12]) says that the structure sheaf $\mathcal{O}_M$ of a complex manifold M is a coherent sheaf, so it follows that all locally free sheaves of finite rank on M are coherent sheaves (see p. 238 of [12] or Proposition 1.11, p. 161 of [20]). In particular the Poincaré line bundle $\mathcal{P}$ on $M \times \mathrm{Pic}^0(M)$ is a coherent sheaf (remember the Proposition 1.5). Now let $(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed topological spaces (see [20] p. 37). By definition

$$f^{\#}: \mathcal{O}_Y \to f_*(\mathcal{O}_X)$$

where f_* is the direct image functor. So every stalk $\mathcal{S}_x$ has a structure of $\mathcal{O}_{Y,f(x)}$-module induced by the canonical homomorphism $f_x^{\#} : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. We give the following

Definition 2.18. *An* $\mathcal{O}$*-sheaf* $\mathcal{S}$ *on* X *is* f-flat *if* $\mathcal{S}_x$ *is flat as* $\mathcal{O}_{Y,f(x)}$*-module, for all* $x \in X$.

Definition 2.19. *Let* $\mathcal{G}$ *be an* $\mathcal{O}_Y$*-sheaf on* Y*, we define the* pullback sheaf

$$f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

where f^{-1} *is the inverse image functor (see p. 37 of [20]). In this way* $f^*\mathcal{G}$ *is an* $\mathcal{O}_X$*-sheaf on* X*, indeed the map* $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ *induces a morphism of sheaves* $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ *and so we can consider* $\mathcal{O}_X$ *as an* $f^{-1}\mathcal{O}_Y$*-sheaf.*

In the above definition we use the fact that f^{-1} is a covariant functor and the canonical morphism of sheaves $f^{-1}f_*\mathcal{O}_X \to \mathcal{O}_X$, see exercise 1.18, p. 68 of [16]. Finally we give

Definition 2.20. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{F}$ be a sheaf (of abelian groups) on X, then for all $i \geq 0$ the* higher direct image sheaf on Y $R^i f_*(\mathcal{F})$ *is the sheaf associated to the presheaf*

$$V \longmapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$$

on Y.

$R^i f_*$ can also be defined as the right derived functors of the direct image functor f_* (see Proposition 8.1, Chapter III.8 of [16]), from which the notation results. Now we can enounce the theorem that justifies the next abstract investigations. For a proof we refer to chapter 3, Theorem 4.1 of [1], or chapter III, Theorem 12.11 of [16] for an algebraic version.

Theorem 2.21. *Let $\mathcal{S}$ be a coherent, p-flat sheaf on $M \times \mathrm{Pic}^0(M)$, where $p : M \times \mathrm{Pic}^0(M) \to \mathrm{Pic}^0(M)$ is the second projection. Let $y \in \mathrm{Pic}^0(M)$ be a point. Then there exists an open neighborhood V of y and a finite complex $E^\bullet$ of vector bundles on V, with the property that for every coherent sheaf $\mathcal{F}$ on V, there is an isomorphism*

$$R^i p_*(\mathcal{S} \otimes p^*\mathcal{F}) = H^i(E^\bullet \otimes \mathcal{F}). \tag{2.3}$$

Moreover this isomorphism is compatible with short exact sequences of $\mathcal{O}_V$-modules, i.e. it is an isomorphism of δ-functors in the variable $\mathcal{F}$.

We point out that the flatness hypothesis on $\mathcal{S}$ implies the finiteness of the complex $E^\bullet$. Applying now the previous theorem with $\mathcal{S} = \mathcal{P}$ (the Poincaré line bundle on $M \times \mathrm{Pic}^0(M)$) and $y \in \mathrm{Pic}^0(M)$ a fixed point, we have the neighborhood V and the complex $E^\bullet$. If $\underline{m}_y \subseteq \mathcal{O}_V$ is the maximal ideal at y, we denote by $\mathbb{C}(y) := \mathcal{O}_V/\underline{m}_y$ the skyscraper sheaf supported at y with stalk $\mathbb{C}$. Taking $\mathcal{F} = \mathbb{C}(y)$, the isomorphism (2.3) becomes

$$\begin{aligned} H^i(E^\bullet(y)) &:= H^i(E^\bullet \otimes \mathbb{C}(y)) \\ &= R^i p_*(\mathcal{P} \otimes p^*(\mathbb{C}(y))) \\ &= H^i(M, L_y), \end{aligned}$$

where the last equality follows from the definition of Poincaré line bundle and Lemma 1.3, p. 106 of [1]. Using this approach we can study the cohomology groups $H^i(E^\bullet(y))$ abstractly, in order to derive information about the locus $S^i(M)$. Hence let $E^\bullet = \{E^i, d^i\}$, $0 \leq i \leq N$, be a finite complex of locally free sheaves on M, with $\mathrm{Rank}(E^i) = e_i$. Given a point $y \in M$ we denote by $\underline{m}_y \subseteq \mathcal{O}_M$ the maximal ideal at y and by $E^\bullet(y)$ the complex of vector spaces at y determined by the fibres of $E^\bullet$, i.e. $E^\bullet(y) := E^\bullet \otimes \mathbb{C}(y) = E^\bullet/\underline{m}_y E^\bullet$, where $\mathbb{C}(y) := \mathcal{O}_M/\underline{m}_y$. We want to study how the cohomology groups of this complex of vector spaces depend on $y \in M$. So we consider the *cohomological support loci* of the complex $E^\bullet$

$$S^i_m(E^\bullet) := \left\{ y \in M \mid h^i(E^\bullet(y)) \geq m \right\}$$

As before we write $S^i(E^\bullet) := S^i_1(E^\bullet) = \left\{ y \in M \mid H^i(E^\bullet(y)) \neq 0 \right\}$. We have that

Proposition 2.22. *Each $S^i_m(E^\bullet)$ is an analytic subvariety of M.*

Proof. We observe that

$$\begin{aligned} \dim H^i(E^\bullet(y)) \geq m &\Leftrightarrow e_i - \operatorname{Rank} d^i(y) - \operatorname{Rank} d^{i-1}(y) \geq m \\ &\Leftrightarrow \operatorname{Rank} d^{i-1}(y) + \operatorname{Rank} d^i(y) \leq e_i - m, \end{aligned} \tag{2.4}$$

where $d^{-1} = d^N = 0$ by convention. Therefore we have

$$S^i_m(E^\bullet) = \bigcap_{\substack{a+b=e_i-m+1 \\ a,b\ \geq\ 0}} \left\{ y \in M | \operatorname{Rank} d^{i-1}(y) \leq a-1 \text{ or } \operatorname{Rank} d^i(y) \leq b-1 \right\}$$

Let $u : E \to F$ be a morphism of locally free sheaves on M, with $\operatorname{Rank}(E) = e$ and $\operatorname{Rank}(F) = f$. For any integer $j \geq 0$ we denote by $\mathcal{I}_j(u)$ the ideal sheaf on M locally generated by the determinants of the $j \times j$ minors of u, with the conventions that $\mathcal{I}_0(u) = \mathcal{O}_M$ and $\mathcal{I}_j(u) = 0$ if $j > min\{e, f\}$. This makes sense because, after fixing local trivializations for E and F, u is given by a matrix of holomorphic functions and the sheaf is defined on a basis for the topology of M. So we may take the ideal sheaf of $S^i_m(E^\bullet)$ to be

$$\mathcal{I}(S^i_m(E^\bullet)) = \sum_{\substack{a+b=e_i-m+1 \\ a,b\ \geq\ 0}} \mathcal{I}_a(d^{i-1}) \cdot \mathcal{I}_b(d^i)$$

□

Given a point $y \in M$, for simplicity's sake we denote by $T = T_y M$ the holomorphic tangent space to M at y, by $\mathcal{O} = \mathcal{O}_{M,y}$ the local ring of M at y, and by $\underline{m} = \underline{m}_y \subseteq \mathcal{O}$ its maximal ideal. We recall that T is dual to $\underline{m}/\underline{m}^2$. The exact sequence of complexes

$$\begin{tikzcd} 0 \arrow[r] & \underline{m}E^\bullet/\underline{m}^2E^\bullet \arrow[r] \arrow[d, equal] & E^\bullet/\underline{m}^2E^\bullet \arrow[r] & E^\bullet/\underline{m}E^\bullet \arrow[r] \arrow[d, equal] & 0 \\ & E^\bullet(y) \otimes T^* & & E^\bullet(y) & \end{tikzcd}$$

gives rise to the connecting homomorphism

$$D(d^i(y)) \colon H^i(E^\bullet(y)) \longrightarrow H^{i+1}(E^\bullet(y)) \otimes T^*$$

In particular, for every tangent vector $v \in T$, we obtain a homomorphism

$$D_v(d^i(y)) \colon H^i(E^\bullet(y)) \longrightarrow H^{i+1}(E^\bullet(y))$$

that varies linearly with v. We have the following

Proposition 2.23.

$$D_v(d^i(y)) \circ D_v(d^{i-1}(y)) = 0$$

Proof. The filtered complex

$$\frac{\underline{m}^2 E^\bullet}{\underline{m}^3 E^\bullet} \subseteq \frac{\underline{m} E^\bullet}{\underline{m}^3 E^\bullet} \subseteq \frac{E^\bullet}{\underline{m}^3 E^\bullet}$$

gives us two exact sequences

$$0 \to \underline{m}E^\bullet/\underline{m}^3 E^\bullet \to E^\bullet/\underline{m}^3 E^\bullet \to E^\bullet/\underline{m}E^\bullet \to 0, \tag{2.5}$$

$$0 \to \underline{m}^2 E^\bullet/\underline{m}^3 E^\bullet \to \underline{m}E^\bullet/\underline{m}^3 E^\bullet \to \underline{m}E^\bullet/\underline{m}^2 E^\bullet \to 0. \tag{2.6}$$

From the first we obtain a connecting homomorphism

$$H^{i-1}(E^\bullet(y)) \to H^i(\underline{m}E^\bullet/\underline{m}^3 E^\bullet)$$

that, after projecting to $H^i(\underline{m}E^\bullet/\underline{m}^2 E^\bullet)$, gives $D(d^{i-1}(y))$. Moreover the exact sequence (2.6) allows to write the following commutative diagram

$$\begin{tikzcd}
H^{i-1}(E^\bullet(y)) \arrow[d] \arrow[dr, "D(d^{i-1}(y))"] & & \\
H^i(\underline{m}E^\bullet/\underline{m}^3 E^\bullet) \arrow[r] & H^i(\underline{m}E^\bullet/\underline{m}^2 E^\bullet) \arrow[r, "\zeta"] \arrow[d, equal] & H^{i+1}(\underline{m}^2 E^\bullet/\underline{m}^3 E^\bullet) \arrow[d, equal] \\
& H^i(E^\bullet(y)) \otimes T^* & H^{i+1}(E^\bullet(y)) \otimes \mathrm{Sym}^2 T^*
\end{tikzcd}$$

where the vertical equality on the right follows from $\mathrm{Sym}^2 T^* = \underline{m}^2/\underline{m}^3$, $H^{i+1}(E^\bullet(y)) \otimes \mathrm{Sym}^2 T^* \subseteq H^{i+1}(E^\bullet(y)) \otimes T^* \otimes T^*$ and the map $\zeta = D(d^i(y)) \otimes id$. Hence we have that $D_v(d^i(y)) \circ D_v(d^{i-1}(y)) = 0$ by exactness of the central row. □

Equivalently we can argue in the following way:

Proof. 2 of Proposition 2.23. After fixing local trivializations of the E^j near y for all j, d^i is given by a matrix of holomorphic functions. If we differentiate this matrix at y in the direction v, then we get a linear map

$$E^i(y) \xrightarrow{\delta_{i,y,v}} E^{i+1}(y),$$

that we denote simply by δ_i. By hypothesis, we know that $d^i d^{i-1} = 0$, so

$$\delta_i d^{i-1}(y) + d^i(y)\delta_{i-1} = 0 \tag{2.7}$$

and differentiating again

$$\delta_i^2 d^{i-1}(y) + 2\delta_i\delta_{i-1} + d^i(y)\delta_{i-1}^2 = 0 \tag{2.8}$$

Now (2.7) implies that δ_i passes to cohomology and, by definition, it coincides with $D_v(d^i(y))$. Moreover (2.8) becomes $D_v(d^i(y))D_v(d^{i-1}(y)) = 0$, in cohomology. □

This gives a complex of vector spaces

$$D_v(E^\bullet, y) = \left[\ldots \to H^{i-1}(E^\bullet(y)) \to H^i(E^\bullet(y)) \to H^{i+1}(E^\bullet(y)) \to \ldots\right]$$

that we call the *derivative complex of $E^\bullet$ at y in the direction v*. From all these complexes we can build a complex of trivial vector bundles on T, the *derivative complex of $E^\bullet$ at y*

$$\ldots \to H^{i-1}(E^\bullet(y)) \otimes \mathcal{O}_T \to H^i(E^\bullet(y)) \otimes \mathcal{O}_T \to H^{i+1}(E^\bullet(y)) \otimes \mathcal{O}_T \to \ldots$$

that we denote by $D(E^\bullet, y)$ with differentials $D(d^i(y))$. Hence, starting from a complex on M, we arrive to a complex on T and we can take the relative cohomological support loci

$$S^i_m(D(E^\bullet, y)) = \left\{v \in T \mid h^i(D_v(E^\bullet, y)) \geq m\right\}$$

We observe that the linearity of the differentials $D_v(d^i(y))$ in the variable $v \in T$ implies that the loci $S^i_m(D(E^\bullet, y))$ are cones in T. Now we can prove the following fundamental

Theorem 2.24 (Tangent Cone Theorem)**.** *Let y be a point of $S^i_m(E^\bullet)$. Then*

$$TC_y(S^i_m(E^\bullet)) \subseteq S^i_m(D(E^\bullet, y)).$$

Proof. First of all we fix some notations: given vector spaces V and W, a linear map $\delta : V \to W \otimes T^*$ can be viewed as a matrix with entries in T^*, after choosing some basis. We denote by $J_j(\delta)$ the homogeneous ideal of $\mathrm{Sym}\, T^*$ generated by the determinants of the $j \times j$ minors of δ, that are elements of $\mathrm{Sym}^j T^* = \underline{m}^j / \underline{m}^{j+1}$. We denote by $J_j(\delta)_k \subseteq \underline{m}^k / \underline{m}^{k+1}$ the degree k piece of the ideal $J_j(\delta)$. Moreover, if $\mathcal{I}$ is an ideal sheaf of $\mathcal{O}_M$, we set $Gr_k(\mathcal{I}) := \mathcal{I} \cap \underline{m}^k / \mathcal{I} \cap \underline{m}^{k+1}$ and $Gr(\mathcal{I}) := \bigoplus Gr_k(\mathcal{I})$. The second is clearly a homogeneous ideal of $Gr(\mathcal{O}_M) = \bigoplus \underline{m}^k / \underline{m}^{k+1} = \mathrm{Sym}\, T^*$. Now let $d : E \to F$

be a morphism of holomorphic vector bundles on M, using again the snake lemma applied to the diagram

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] \\ E(y)\otimes T^* \arrow[r, "d(y)\otimes id"] \arrow[d] & F(y)\otimes T^* \arrow[d] \\ E/\underline{m}^2E \arrow[r] \arrow[d] & F/\underline{m}^2F \arrow[d] \\ E(y) \arrow[r, "d(y)"] \arrow[d] & F(y) \arrow[d] \\ 0 & 0 \end{tikzcd}$$

we obtain the derivative homomorphism

$$\delta : \operatorname{Ker}(d(y)) \to \operatorname{Coker}(d(y)) \otimes T^*.$$

We have the following

Lemma 2.25. *Let $r = \operatorname{Rank} d(y)$. Then for $k \geq r$:*

i) $\mathcal{I}_k(d) \subseteq \underline{m}^{k-r}$

ii) $Gr_{k-r}(\mathcal{I}_k(d)) = J_{k-r}(\delta)_{k-r}$.

The verification of the above lemma will be provided later on. Now we simply use it. The differential $d^i : E^i \to E^{i+1}$ defines the commutative diagram

$$\begin{tikzcd} \operatorname{Ker}(d^i(y)) \arrow[r, "\delta^i"] \arrow[d, two heads] & \operatorname{Coker}(d^i(y)) \otimes T^* \\ H^i(E^\bullet(y)) \arrow[r, "D(d^i(y))"] & H^{i+1}(E^\bullet(y)) \otimes T^* \arrow[u, hook] \end{tikzcd}$$

where the top horizontal row is constructed as before. We have that $J_k(D(d^i(y)) = J_k(\delta^i)$ and so

$$Gr_{k-r_i}(\mathcal{I}_k(d^i)) = J_{k-r_i}(D(d^i(y)))_{k-r_i} \quad \forall\ k \geq r_i := \operatorname{Rank} d^i(y). \qquad (2.9)$$

Let $h := \dim H^i(E^\bullet(y))$. Then the Proposition 2.22 says that $S^i_m(D(E^\bullet, y))$ is defined in T by the homogeneous ideal

$$J = \sum_{\substack{a+b=h-m+1 \\ a,b\ \geq\ 0}} J_a(D(d^{i-1}(y))) \cdot J_b(D(d^i(y))) \quad \subseteq \ \operatorname{Sym} T^*$$

Now, if

$$\mathcal{I} = \sum_{\substack{a+b=e_i-m+1 \\ a,b \geq 0}} \mathcal{I}_a(d^{i-1}) \cdot \mathcal{I}_b(d^i)$$

denotes the ideal sheaf defining $S^i_m(E^\bullet)$ in M (2.22 again), then the tangent cone $TC_y(S^i_m(E^\bullet))$ is defined in T by the homogeneous ideal $Gr(\mathcal{I}) \subseteq \mathrm{Sym}\, T^*$. Since J is generated by polynomials of degree $h-m+1$, it suffices to show that

$$J_{h-m+1} \subseteq Gr_{h-m+1}(\mathcal{I}) \tag{2.10}$$

to prove the theorem. To this end, set $r_k := \mathrm{Rank}\, d^k(y)$, and fix integers $a, b \geq 0$ s.t. $a+b = h-m+1$. By (2.9) we have that

$$J_a(D(d^{i-1}(y)))_a = Gr_a(\mathcal{I}_{a+r_{i-1}}(d^{i-1}))$$

and

$$J_b(D(d^i(y)))_b = Gr_b(\mathcal{I}_{b+r_i}(d^i))$$

Hence

$$\left(J_a(D(d^{i-1}(y))) \cdot J_b(D(d^i(y)))\right)_{a+b} \subseteq Gr_{a+b}(\mathcal{I}_{a+r_{i-1}}(d^{i-1}) \cdot \mathcal{I}_{b+r_i}(d^i))$$

But $r_{i-1} + r_i = e_i - h$ thanks to (2.4) and so $a + r_{i-1} + b + r_i = e_i - m + 1$. Therefore the product $\mathcal{I}_{a+r_{i-1}}(d^{i-1}) \cdot \mathcal{I}_{b+r_i}(d^i)$ is contained in $\mathcal{I}$, and (2.10) follows. □

Finally we give the

Proof of Lemma 2.25. We may assume that E and F are trivial, because the question is local near y. So we choose frames $v_1, \dots, v_n$ for E and $w_1, \dots, w_m$ for F s.t.

$$d(v_i) \equiv w_i \ (mod\ \underline{m}) \quad 1 \leq i \leq r,$$

and

$$d(v_i) \equiv \delta(v_i) \in \underline{m}F \ (mod\ (w_1, \dots, w_r) + \underline{m}^2 F) \quad i > r.$$

For $i_1 < i_2 < \dots < i_k$ and $l_1 < l_2 < \dots < l_k$ we denote by $\Delta^{l_1,\dots,l_k}_{i_1,\dots,i_k}$ the determinant of the $k \times k$ minor corresponding to the indicated indexes of a given matrix. So we have

$$\Delta^{l_1,\dots,l_k}_{i_1,\dots,i_k}(d) \equiv \Delta^{l_{r+1},\dots,l_k}_{i_{r+1},\dots,i_k}(\delta) \ (mod\ \underline{m}^{k-r+1})$$

if $i_1 = l_1 = 1, \dots, i_r = l_r = r$, and

$$\Delta^{l_1,\dots,l_k}_{i_1,\dots,i_k}(d) \equiv 0 \ (mod\ \underline{m}^{k-r+1})$$

otherwise. This implies the Lemma. □

The fact that $\dim TC_y(S^i_m(E^\bullet)) = \dim_y S^i_m(E^\bullet)$ gives the following simple results, that we will use to prove the Theorem 2.9.

Corollary 2.26. *Set* $m = \dim H^i(E^\bullet(y))$*. Then*

$$\dim_y S^i_m(E^\bullet) \le \dim \left\{ v \in T \mid D_v(d^{i-1}(y)) = D_v(d^i(y)) = 0 \right\}$$

In particular, if either $D_v(d^{i-1}(y)) \ne 0$ *or* $D_v(d^i(y)) \ne 0$ *for some tangent vector* $v \in T$*, then* $S^i_m(E^\bullet)$ *is a proper subvariety of* M*.*

Proof. From the tangent cone theorem we get

$$\begin{aligned}\dim_y S^i_m(E^\bullet) &\le \dim \ S^i_m(D(E^\bullet, y)) \\ &= \dim \left\{ v \in T \mid \dim H^i(D_v(E^\bullet, y)) \ge m \right\},\end{aligned}$$

but $m = \dim H^i(E^\bullet(y))$ and

$$\dim H^i(D_v(E^\bullet, y)) = \dim \ \frac{\text{Ker } \ D_v(d^i(y))}{\text{Im } \ D_v(d^{i-1}(y))}$$

is $\ge \dim \ H^i(E^\bullet(y))$ iff $v \in T$ is s.t. $D_v(d^i(y)) = 0$ and $D_v(d^{i-1}(y)) = 0$. □

We can also derive the following condition for $S^i_m(E^\bullet)$ to be a proper subvariety of M.

Corollary 2.27. *If* $H^i(D_v(E^\bullet, y)) = 0$ *for some point* $y \in M$ *and some tangent vector* $v \in T$*, then* $S^i_m(E^\bullet) \subset M$ *properly for all* $m > 0$*.*

Proof. We use again the inequality

$$\begin{aligned}\dim_y S^i_m(E^\bullet) &\le \dim S^i_m(D(E^\bullet, y)) \\ &= \dim \left\{ v \in T \mid \dim H^i(D_v(E^\bullet, y)) \ge m > 0 \right\}.\end{aligned}$$

By hypothesis there exists a $v \in T$ s.t. $H^i(D_v(E^\bullet, y)) = 0$ and hence $S^i_m(E^\bullet)$ is contained in M properly. □

Moreover we have a sufficient condition to detect the isolated points of $S^i_m(E^\bullet)$.

Corollary 2.28. *If* $y \in S^i_m(E^\bullet)$ *for some* $m > 0$*, and if* $H^i(D_v(E^\bullet, y)) = 0$ *for every non-zero tangent vector* $v \in T$*, then* y *is an isolated point of* $S^i_m(E^\bullet)$*.*

Proof. As in the previous corollary, with these hypothesis we have that $\dim_y S^i_m(E^\bullet) = 0$, and this means that y is an isolated point of $S^i_m(E^\bullet)$, by definition of the local dimension. □

2.4 Proof of the Generic Vanishing Theorem

If we use the notations of the above section, given a point $y \in \mathrm{Pic}^0(M)$, we can fix a neighborhood $V \subseteq \mathrm{Pic}^0(M)$ of y and a complex $E^\bullet$ on V for which

$$S^i_m(M) \cap V = S^i_m(E^\bullet).$$

We also recall that $H^i(E^\bullet(y)) = H^i(M, L_y)$. In this situation, thanks to the canonical identification $T_y \mathrm{Pic}^0(M) = H^1(M, \mathcal{O}_M)$, the derivative complex $D_v(E^\bullet, y)$ takes the following simple form

Lemma 2.29. *Let $v \in H^1(M, \mathcal{O}_M)$ be a tangent vector to $\mathrm{Pic}^0(M)$ at y, then the derivative complex $D_v(E^\bullet, y)$ of $E^\bullet$ at y in the direction v is identified with the complex*

$$\ldots \to H^{i-1}(M, L_y) \to H^i(M, L_y) \to H^{i+1}(M, L_y) \to \ldots$$

where differentials are given by cup product with v.

Proof. We recall that the differentials in the derivative complex $D_v(E^\bullet, y)$ are constructed by taking the short exact sequence

$$0 \to \underline{m}/\underline{m}^2 \to \mathcal{O}/\underline{m}^2 \to \mathcal{O}/\underline{m} \to 0 \qquad (2.11)$$

where $\underline{m}$ is the ideal sheaf of the point $y \in \mathrm{Pic}^0(M)$. After tensoring (2.11) with $E^\bullet$, we consider the connecting homomorphism of the cohomological long exact sequence. Fixing now a tangent vector $v \in H^1(M, \mathcal{O}_M) = H^1(M, L_y^* \otimes L_y) = \mathrm{Ext}^1(L_y, L_y)$, we obtain an extension of L_y by L_y (see [15], p. 706). The corresponding homomorphism

$$\delta \colon H^i(M, L_y) \to H^{i+1}(M, L_y)$$

is cup product with v by Theorem 7.1, Chapter II of [5]. So it suffices to recall that the Theorem 2.21 gives an isomorphism of δ-functors in the variable $\mathcal{F}$. $\square$

Now, if we apply the Hodge theory again, we can pass from cohomology groups of line bundles to global holomorphic forms. Indeed using Corollary 2.7 and the isomorphism of vector spaces

$$\overline{H^i(M, L_y)} \cong H^0(M, \Omega^i_M \otimes L_y^*) \qquad (2.12)$$

given by coniugation of harmonic forms ([3], pp. 35, 36), we have a commutative diagram

$$\begin{array}{ccc} \overline{H^i(M, L_y)} & \xrightarrow{\smile v} & \overline{H^{i+1}(M, L_y)} \\ \| & & \| \\ H^0(M, \Omega^i_M \otimes L_y^*) & \xrightarrow{\wedge \omega} & H^0(M, \Omega^{i+1}_M \otimes L_y^*) \end{array}$$

where $\omega \in H^0(M, \Omega^1_M)$ is the holomorphic 1-form conjugate to $v \in H^1(M, \mathcal{O}_M)$. We point out that the Hodge symmetry (2.12) occurs with vector bundles on M given by a unitary representation of $\pi_1(M)$ ([3], p. 19), and here we use the Corollary 2.7. So, coniugating the complex of the Lemma 2.29, we obtain the complex

$$\ldots \to H^0(M, \Omega^{i-1}_M \otimes L^*_y) \to H^0(M, \Omega^i_M \otimes L^*_y) \to H^0(M, \Omega^{i+1}_M \otimes L^*_y) \to \ldots$$

where the differentials are given by wedge product with $\omega \in H^0(M, \Omega^1_M)$. From this Hodge-theoretical reinterpretation of the derivative complex, we obtain the claimed Proposition 2.14 as a consequence of corollaries 2.26 - 2.28. Finally we are set to prove the Theorem 2.9 :

Proof of Theorem 2.9. We recall that $a : M \to \mathrm{Alb}(M)$ is the Albanese map for some choice of the base point $p_0 \in M$. Let Y be an irreducible component of $S^i(M)$ and fix a point $y_0 \in Y$ at which $\dim H^i(M, L_{y_0})$ is as small as possible. Setting $m = \dim H^i(M, L_{y_0}) \geq 1$, we have $y_0 \in Y \subseteq S^i_m(M)$ and so it is enough to show that

$$\dim \mathrm{Pic}^0(M) - \dim_{y_0} S^i_m(M) \geq \dim a(M) - i.$$

Since $m = \dim H^0(M, \Omega^i_M \otimes L^*_{y_0}) \geq 1$ it is possible to fix a non-zero section $\beta \in H^0(M, \Omega^i_M \otimes L^*_{y_0})$. By Proposition 2.14 a) we have that

$$\dim_{y_0} S^i_m(M) \leq \dim \left\{\omega \in H^0(M, \Omega^1_M) \mid \omega \wedge \beta = 0\right\}$$

and so it suffices to prove that $W := \left\{\omega \in H^0(M, \Omega^1_M) \mid \omega \wedge \beta = 0\right\}$ has codimension $\geq \dim a(M) - i$ in $H^0(M, \Omega^1_M)$, that is the cotangent space to $\mathrm{Alb}(M)$. For any $x \in M$ we consider the evaluation morphism

$$e(x) \colon H^0(M, \Omega^1_M) \to T_x M^*.$$

If we set $W(x) = \{v \in T_x M^* \mid v \wedge \beta(x) = 0\}$ we have

$$\dim W \leq \dim \mathrm{Ker}(e(x)) + \dim W(x)$$

because $e(x)(\omega) = \omega(x) \in W(x)$, if $\omega \in W$. By Lemma 2.5 $e(x)$ is the codifferential of the Albanese map at the point x and so, at a sufficiently general point x, the rank of $e(x)$ is equal to $\dim a(M)$, i.e. $\dim \mathrm{Ker}(e(x)) = \dim H^0(M, \Omega^1_M) - \mathrm{Rank}\, e(x) = \dim \mathrm{Pic}^0(M) - \dim a(M)$. Therefore we are reduced to proving that

$$\dim W(x) \leq i \tag{2.13}$$

for a sufficiently general point x. So we may assume that $\beta(x) \neq 0$. Now (2.13), and hence also the Theorem 2.9, is a consequence of the following elementary

Lemma 2.30. *Let V a finite dimensional vector space, let $\beta \in \bigwedge^i V$ and let $e_1, \dots, e_k \in V$ linearly independent vectors s.t. $e_j \wedge \beta = 0$ for $1 \leq j \leq k$. If $k > i$, then $\beta = 0$.*

Proof. Let $n = \dim V$ and complete $e_1, \dots, e_k$ to a basis $e_1, \dots, e_n$ of V. If $\beta \neq 0$, then there exists an element $\alpha \in \bigwedge^{n-i} V$ s.t. $\alpha \wedge \beta \neq 0$. Since $n - i > n - k$, every term of α must involve one of the e_j, with $1 \leq j \leq k$. Therefore $\alpha \wedge \beta = 0$, that is a contradiction. □

□

Chapter 3

Nakano-Kodaira-type Generic Vanishing Theorem

In all this Chapter we still denote by M a compact, connected Kähler manifold of complex dimension n. We continue to follow Green-Lazarsfeld's paper [13].

3.1 Nakano-Kodaira vanishing

The most famous vanishing theorem is Nakano-Kodaira's one (see Chapter 1.2 of [15] for a proof), that we mention for the sake of completeness:

Theorem 3.1. *Let L be an ample line bundle on M, then*

$$H^p(M, \Omega^q_M \otimes L) = 0 \quad \forall\ p + q > n.$$

Equivalently, we have the following dual version

Theorem 3.2. *Let L be an anti-ample line bundle on M, then*

$$H^p(M, \Omega^q_M \otimes L) = 0 \quad \forall\ p + q < n.$$

The $q = 0$ case of Theorem 3.2 was proved by Kodaira and Corollary 2.10 is its generic analogue, due to Green and Lazarsfeld, where the complex dimension n of M is replaced by its Albanese dimension. Because of Serre duality, we have the following generic acyclicity theorem for $K_M \otimes L$

Proposition 3.3. *Let M be of maximal Albanese dimension, then*

$$H^p(M, K_M \otimes L) = 0 \quad \forall\ p > 0,$$

where $L \in \mathrm{Pic}^0(M)$ is a generic line bundle.

Proof. We have

$$H^p(M, K_M \otimes L) \cong (H^{n-p}(M, L^*))^* = 0$$

if $n - p < n$, thanks to the generic vanishing theorem 2.10. □

However the general behavior is more complicated. In this chapter we will use the first order deformation theory previously developed, to obtain a Kodaira-Nakano-type generic vanishing theorem. Precisely, we will prove the following

Theorem 3.4. *Assume M has a holomorphic 1-form ω whose zero locus $Z(\omega)$ has codimension k in M. Then for generic $y \in \mathrm{Pic}^0(M)$*

$$H^p(M, \Omega^q_M \otimes L_y) = 0 \quad \forall\ p + q < k,$$

and if $Z(\omega)$ is empty, the above equality is satisfied with $k = \infty$.

As before the word "generic" means that the property is true for all points in the complement of a proper analytic subvariety of $\mathrm{Pic}^0(M)$. This generic version of Theorem 3.2 may come as a surprise: indeed we obtain a similar, although weaker result, even if we assume very different hypotheses. On one hand we take anti-ample line bundle, on the other we move inside another ambient, $\mathrm{Pic}^0(M)$. The replacement of the Albanese dimension by the codimension of the zero locus of a holomorphic 1-form is due to the fact that we will use a spectral sequence argument, as in [6], to prove Theorem 3.4. However, an analogous statement is not true under the hypothesis $p + q < \dim a(M)$ (look the Remark in [13], p. 401). We start studying the analytic subvariety

$$S^i(M, \Omega^q_M) := \left\{ y \in \mathrm{Pic}^0(M) \mid H^i(M, \Omega^q_M \otimes L_y) \neq 0 \right\}$$

in a neighborhood of 0 in $\mathrm{Pic}^0(M)$. After fixing an integer q, we apply Theorem 2.21 with $\mathcal{S} = \mathcal{P} \otimes p_1^* \Omega^q_M$, where $\mathcal{P}$ is the Poincaré line bundle on $M \times \mathrm{Pic}^0(M)$ as usual, and $p_1 : M \times \mathrm{Pic}^0(M) \to M$ is the first projection. So we obtain an open neighborhood V of 0 in $\mathrm{Pic}^0(M)$ and a finite complex $E^\bullet$ of vector bundles on V s.t. $H^i(M, \Omega^q_M \otimes L_y) = H^i(E^\bullet_y)$ and

$$S^i(M, \Omega^q_M) \cap V = S^i(E^\bullet).$$

As in Lemma 2.29, given a tangent vector $v \in H^1(M, \mathcal{O}_M)$, one shows that the derivative complex $D_v(E^\bullet, 0)$ is identified with

$$\ldots \to H^{i-1}(M, \Omega^q_M) \xrightarrow{\smile v} H^i(M, \Omega^q_M) \xrightarrow{\smile v} H^{i+1}(M, \Omega^q_M) \to \ldots$$

that becomes

$$\ldots \to H^q(M, \Omega^{i-1}_M) \xrightarrow{\wedge \omega} H^q(M, \Omega^i_M) \xrightarrow{\wedge \omega} H^q(M, \Omega^{i+1}_M) \to \ldots$$

where $\omega \in H^0(M, \Omega^1_M)$ is the holomorphic 1-form coniugate to v, using the Hodge duality ([15], p. 116)

$$\overline{H^i(M, \Omega^q_M)} \cong H^q(M, \Omega^i_M).$$

Now Proposition 2.14 b) allows to reduce the proof of Theorem 3.4 to the following

Proposition 3.5. *Let $\omega \in H^0(M, \Omega^1_M)$ be a holomorphic 1-form whose zero locus $Z(\omega)$ has codimension k in M. Then the sequence*

$$H^q(M, \Omega^{i-1}_M) \xrightarrow{\wedge\omega} H^q(M, \Omega^i_M) \xrightarrow{\wedge\omega} H^q(M, \Omega^{i+1}_M) \tag{3.1}$$

is exact whenever $i + q < k$.

3.2 Spectral sequences and Hypercohomology

First of all, we recall the following

Definition 3.6. *A (first-quadrant)* spectral sequence *is a sequence $\{E_r, d_r\}$, $r \geq 0$, of bigraded groups*

$$E_r = \bigoplus_{p,q \geq 0} E^{p,q}_r$$

together with differentials

$$d_r \colon E^{p,q}_r \to E^{p+r,q-r+1}_r, \quad d^2_r = 0$$

such that

$$E^{p,q}_{r+1} = H^\bullet(E^{p,q}_r).$$

Since there are, by definition, non zero groups only in the first quadrant of each page, it is well defined the limiting page for the spectral sequence, whose elements we denote by $E^{p,q}_\infty$. Given a double complex of abelian groups $(K^{\bullet,\bullet}, d, \delta)$, where

$$d \colon K^{p,q} \to K^{p+1,q},$$
$$\delta \colon K^{p,q} \to K^{p,q+1},$$

with $d^2 = \delta^2 = 0$ and $d\delta + \delta d = 0$, we can consider the associated single complex $(K^\bullet, D)$, where

$$K^n = \bigoplus_{p+q=n} K^{p,q}, \quad D = d + \delta.$$

There exist two filtrations of $(K^\bullet, D)$

$$'F^p K^n := \bigoplus_{\substack{p'+q=n \\ p' \geq p}} K^{p',q}$$

$$''F^q K^n := \bigoplus_{\substack{p+q''=n \\ q'' \geq q}} K^{p,q''}$$

from which we get two spectral sequences $'E_r$ and $''E_r$ (see [15], p. 443), both abutting to the cohomology $H^\bullet(K^\bullet)$ of the associated single complex, i.e. we have two filtrations

$$0 \subseteq \ldots \subseteq {'F}^{p+1}H^{p+q}(K^\bullet) \subseteq {'F}^{p}H^{p+q}(K^\bullet) \subseteq \ldots \subseteq H^{p+q}(K^\bullet),$$
$$0 \subseteq \ldots \subseteq {''F}^{p+1}H^{p+q}(K^\bullet) \subseteq {''F}^{p}H^{p+q}(K^\bullet) \subseteq \ldots \subseteq H^{p+q}(K^\bullet),$$

such that

$$'E_\infty^{p,q} = \frac{'F^pH^{p+q}(K^\bullet)}{'F^{p+1}H^{p+q}(K^\bullet)},$$
$$''E_\infty^{p,q} = \frac{''F^pH^{p+q}(K^\bullet)}{''F^{p+1}H^{p+q}(K^\bullet)}.$$

This abutting property is compactly written $'E_r$, $''E_r \Rightarrow H^\bullet(K^\bullet)$. Moreover we have that

$$'E_1^{p,q} = H_\delta^q(K^{p,\bullet}), \qquad 'E_2^{p,q} = H_d^p(H_\delta^q(K^{\bullet,\bullet})),$$
$$''E_1^{q,p} = H_d^p(K^{\bullet,q}), \qquad ''E_2^{q,p} = H_\delta^q(H_d^p(K^{\bullet,\bullet})),$$

In fact, for example, the differential $'d_1$ is computed from $D = d + \delta$ on $'E_1$, hence $'d_1 = d$ because $\delta = 0$ on $'E_1$ by definition. Now let $(\mathcal{K}^\bullet, \delta)$ be a complex of sheaves on M, we define the *cohomology sheaf* $\mathcal{H}^q(\mathcal{K}^\bullet)$ as the sheaf associated to the presheaf

$$U \longmapsto \frac{\operatorname{Ker}\left[\delta : \mathcal{K}^q(U) \to \mathcal{K}^{q+1}(U)\right]}{\delta\left(\mathcal{K}^{q-1}(U)\right)}$$

on M. Let $\mathcal{U} = \{U_\alpha\}$ be an open covering of M and $C^p(\mathcal{U}, \mathcal{K}^q)$ the Čech cochains of degree p with values in $\mathcal{K}^q$. The two maps

$$C^p(\mathcal{U}, \mathcal{K}^q) \xrightarrow{d} C^{p+1}(\mathcal{U}, \mathcal{K}^q),$$
$$C^p(\mathcal{U}, \mathcal{K}^q) \xrightarrow{\delta} C^{p}(\mathcal{U}, \mathcal{K}^{q+1})$$

give a double complex $(C^p(\mathcal{U}, \mathcal{K}^q), d, \delta)$, whose associated single complex is denoted by $(C^\bullet(\mathcal{U}), D)$. A refinement $\mathcal{U}'$ of $\mathcal{U}$ induces a map

$$C^p(\mathcal{U}, \mathcal{K}^q) \to C^p(\mathcal{U}', \mathcal{K}^q)$$

and hence

$$H^\bullet(C^\bullet(\mathcal{U})) \to H^\bullet(C^\bullet(\mathcal{U}')).$$

Finally we define the *hypercohomology* of M with values in the complex $\mathcal{K}^\bullet$

$$\mathbb{H}^\bullet(M, \mathcal{K}^\bullet) := \varinjlim_{\mathcal{U}} H^\bullet(C^\bullet(\mathcal{U})).$$

The two spectral sequences $'E_r(\mathcal{U})$ and $''E_r(\mathcal{U})$ associated to the double complex $(C^p(\mathcal{U},\mathcal{K}^q),d,\delta)$ behave well with respect to refinements of the covering (in the first sequence $'E_r$ taking $r\geq 2$), so passing to the direct limit we obtain two spectral sequences abutting to $\mathbb{H}^\bullet(M,\mathcal{K}^\bullet)$ s.t.

$$'E_1^{p,q}(\mathcal{U}) = H^q_\delta(C^p(\mathcal{U},\mathcal{K}^\bullet)),$$

$$'E_2^{p,q} = \varinjlim_{\mathcal{U}} H^p_d(H^q_\delta(C^\bullet(\mathcal{U},\mathcal{K}^\bullet))) = H^p(M,\mathcal{H}^q(\mathcal{K}^\bullet)),$$

and

$$''E_1^{q,p} = H^p(M,\mathcal{K}^q), \qquad\qquad ''E_2^{q,p} = H^q_\delta(H^p(M,\mathcal{K}^\bullet)),$$

where $H^p(M,\mathcal{H}^q(\mathcal{K}^\bullet))$ and $H^p(M,\mathcal{K}^\bullet)$ are the p-cohomology groups of M with values in the sheaves $\mathcal{H}^q(\mathcal{K}^\bullet)$ and $\mathcal{K}^\bullet$ respectively, and $H^\bullet_\delta(H^p(M,\mathcal{K}^\bullet))$ is the cohomology of the complex

$$\ldots \to H^p(M,\mathcal{K}^{q-1}) \xrightarrow{\delta} H^p(M,\mathcal{K}^q) \xrightarrow{\delta} H^p(M,\mathcal{K}^{q+1}) \to \ldots\ .$$

3.3 Proof of the Proposition 3.5 and a Corollary

A holomorphic 1-form ω gives a complex of vector bundles on M

$$\mathcal{K}^\bullet : 0 \to \mathcal{O}_M \xrightarrow{\wedge\omega} \Omega^1_M \xrightarrow{\wedge\omega} \ldots \xrightarrow{\wedge\omega} \Omega^n_M \to 0\ .$$

If we set $\mathcal{K}^i = \Omega^i_M$, by the above paragraph we have two spectral sequences $'E$, $''E$ s.t.

$$'E_2^{i,q} = H^i(M,\mathcal{H}^q(\mathcal{K}^\bullet))$$

and

$$''E_1^{i,q} = H^q(M,\mathcal{K}^i),$$

both abutting to the hypercohomology $\mathbb{H}^\bullet(M,\mathcal{K}^\bullet)$. We note that the second page of $''E$ is exactly the cohomology of the sequence (3.1), so we are led to study the behavior of this spectral sequence. We recall the classical

Proposition 3.7 ($\partial\bar\partial$-lemma). *Let $a\in\mathcal{A}^{i,q}(M)$ be an (i,q)-form on M which is d-closed. The following condition are equivalent:*

i) *a is d-exact,*

ii) *a is ∂-exact,*

iii) *a is $\bar\partial$-exact,*

iv) *a is $\partial\bar\partial$-exact.*

Proof. See [18], pp. 128-129. □

Using Hodge theory and the $\partial\bar{\partial}$-lemma we can prove the following

Proposition 3.8. *The spectral sequence* $''E$ *degenerates at* $''E_2$*, i.e.*

$$''E_2 = ''E_\infty.$$

Proof. We set $E_r^{i,q} = ''E_r^{i,q}$ and denote by d_r the differentials. First we show that $d_2 = 0$. An element of $E_2^{i,q}$ is represented by a harmonic form $a \in H^{i,q}(M)$ whose wedge product with ω is zero in $H^{i+1,q}(M)$, i.e. $a \wedge \omega$ is $\bar{\partial}$-exact. Since a and ω are harmonic (see [15], p. 117) and hence d-closed, by the $\partial\bar{\partial}$-lemma 3.7 there exists $c_1 \in \mathcal{A}^{i,q-1}(M)$ s.t.

$$a \wedge \omega = \bar{\partial}\partial c_1. \tag{3.2}$$

Now

$$d_2\colon E_2^{i,q} \to E_2^{i+2,q-1} = \frac{\operatorname{Ker}\left[H^{q-1}(M,\Omega_M^{i+2}) \xrightarrow{\wedge\omega} H^{q-1}(M,\Omega_M^{i+3})\right]}{\operatorname{Im}\left[H^{q-1}(M,\Omega_M^{i+1}) \xrightarrow{\wedge\omega} H^{q-1}(M,\Omega_M^{i+2})\right]}$$

and $d_2([a]) = [\partial c_1 \wedge \omega]$ (see [15], p. 441). Moreover

$$\partial c_1 \wedge \omega = \partial(c_1 \wedge \omega) \tag{3.3}$$

and in particular $\partial c_1 \wedge \omega$ is ∂-closed. But (3.2) implies that $\partial c_1 \wedge \omega$ is also $\bar{\partial}$-closed and thanks to (3.3) we can apply the $\partial\bar{\partial}$-lemma 3.7 and obtain

$$\partial c_1 \wedge \omega = \bar{\partial}\partial c_2$$

for some $c_2 \in \mathcal{A}^{i+1,q-2}(M)$. In particular $\partial c_1 \wedge \omega$ is $\bar{\partial}$-exact and $d_2 = 0$. Now, proceeding by induction, we suppose that $d_k([a]) = 0$ for $2 \le k < r$ and that

$$\partial c_{k-1} \wedge \omega = \bar{\partial}\partial c_k$$

where $c_k \in \mathcal{A}^{i-1+k,q-k}(M)$ and $2 \le k < r$. Then $d_r([a]) = [\partial c_{r-1} \wedge \omega] \in E_r^{i+r,q-r+1} = E_2^{i+r,q-r+1}$ and $\partial c_{r-1} \wedge \omega = \partial(c_{r-1} \wedge \omega)$, so $\partial c_{r-1} \wedge \omega$ is both ∂- and $\bar{\partial}$-closed, because $\bar{\partial}(\partial c_{r-1} \wedge \omega) = \bar{\partial}\partial c_{r-1} \wedge \omega = \partial c_{r-2} \wedge \omega \wedge \omega = 0$. The $\partial\bar{\partial}$-lemma 3.7 gives

$$\partial c_{r-1} \wedge \omega = \bar{\partial}\partial c_r, \quad c_r \in \mathcal{A}^{i-1+r,q-r}(M).$$

In particular, $[\partial c_{r-1} \wedge \omega] = 0$ and this concludes. □

Using Proposition 3.8, the Proposition 3.5 is an easy consequence of the following

Lemma 3.9. *Let F be a holomorphic vector bundle of rank n on M and let $s \in H^0(M, F)$ be a global section whose zero locus Z has codimension $\geq k$ in M. Denote by $\mathcal{K}^\bullet$ the complex*

$$0 \to \mathcal{O}_M \to F \to \bigwedge^2 F \to \ldots \to \bigwedge^n F \to 0$$

constructed from $s : \mathcal{O}_M \to F$ and indexed so that $\mathcal{K}^i = \wedge^i F$. Then

$$\mathcal{H}^q(\mathcal{K}^\bullet) = 0$$

for $q < k$.

In fact, in our situation, we have $\mathcal{H}^q(\mathcal{K}^\bullet) = 0$ for $q < k$ by the preceding lemma 3.9 and ${}'E_2^{i,q} = H^i(M, \mathcal{H}^q(\mathcal{K}^\bullet)) \Rightarrow \mathbb{H}^{i+q}(M, \mathcal{K}^\bullet)$. So we deduce that $\mathbb{H}^m(M, \mathcal{K}^\bullet) = 0$ for $m < k$. Moreover ${}''E_2 = {}''E_\infty$, hence ${}''E_2^{i,q} = 0$ for $i + q < k$, and this means precisely that the sequence

$$H^q(M, \Omega_M^{i-1}) \xrightarrow{\wedge\omega} H^q(M, \Omega_M^i) \xrightarrow{\wedge\omega} H^q(M, \Omega_M^{i+1})$$

is exact whenever $i + q < k$. It remains to give the

Proof of Lemma 3.9. It suffices to show that the stalks

$$\mathcal{H}^j(\mathcal{K}^\bullet)_x = \varinjlim_{x \in U} \frac{\operatorname{Ker}\left[\wedge^j F(U) \to \wedge^{j+1} F(U)\right]}{\operatorname{Im}\left[\wedge^{j-1} F(U) \to \wedge^j F(U)\right]} = 0$$

for $j < k$ and $x \in Z$, because the sheaves $\mathcal{H}^j(\mathcal{K}^\bullet)$ are supported on $Z = Z(s)$. So pick a point $x \in Z$. Let $\mathcal{O}_x$ be the local ring of M at x, and $\underline{m}$ its maximal ideal. After choosing a local trivialization for F near x, the section s is given by n holomorphic functions $f_1, \ldots, f_n \in \underline{m}$ that generate the ideal of Z in $\mathcal{O}_x$. Since $\dim Z \leq n - k$, we may assume that $f_1, \ldots, f_k$ cut out near x a subvariety of dimension $n-k$. But this algebraically means that $f_1, \ldots, f_k$ is a regular sequence in the regular local ring $\mathcal{O}_x$ (see Appendix and Theorem 17.4 of [21]) and so

$$H^j(K(f_1, \ldots, f_n)) = 0 \quad \forall\ j < k,$$

where $K(f_1, \ldots, f_n)$ is the Koszul complex constructed from the sequence $f_1, \ldots, f_n \in \mathcal{O}_x$. To conclude we note that the complex of $\mathcal{O}_x$-modules determined stalkwise by $\mathcal{K}^\bullet$ is isomorphic to the Koszul complex $K(f_1, \ldots, f_n)$. $\square$

Using similar techniques, we can now prove the following

Corollary 3.10. *Let $m(M)$ be the least codimension of the zero locus of a non zero holomorphic 1-form on M. Then $S^i(M, \Omega_M^q)$ consists of finitely many isolated points whenever $i + q < m(M)$, if it is non empty.*

Proof. If $y \in \mathrm{Pic}^0(M)$ and ω is a non zero holomorphic 1-form coniugate to a non zero tangent vector to $\mathrm{Pic}^0(M)$ at y, the complex

$$\dots \to H^{i-1}(M, \Omega^q_M \otimes L^*_y) \xrightarrow{\wedge\omega} H^i(M, \Omega^q_M \otimes L^*_y) \xrightarrow{\wedge\omega} H^{i+1}(M, \Omega^q_M \otimes L^*_y) \to \dots$$

is related to the locus $S^i(M, \Omega^q_M)$, as showed in the first section of this chapter. We note that the Proposition 3.8 and the Lemma 3.9 are also true for the complex

$$\mathcal{K}^\bullet \otimes L^*_y \;:\; 0 \to L^*_y \xrightarrow{\wedge\omega} \Omega^1_M \otimes L^*_y \xrightarrow{\wedge\omega} \dots \xrightarrow{\wedge\omega} \Omega^n_M \otimes L^*_y \to 0$$

and so $\mathcal{H}^j(\mathcal{K}^\bullet \otimes L^*_y) = 0$ for $j < m(M)$. Now applying Corollary 2.28 and the compactness of M, we obtain the result. $\square$

Chapter 4

Some applications of Deformation Theory of cohomology groups

In this last chapter we will see some applications of Green-Lazarsfeld deformation theory of cohomology groups of line bundles: first of all a theorem on surfaces, that was the motivation for [13]; then, using without proofs the results of a second paper of Green and Lazarsfeld, [14], we prove the Ein-Lazarsfeld theorem on manifolds of maximal Albanese dimension with $\chi(M, \omega_M) = 0$.

4.1 Surfaces

Let S be a compact, connected Kähler manifold of complex dimension 2, i.e. a surface. An *irrational pencil* of genus g on S is a surjective holomorphic map with connected fibres $f : S \to C$, where C is a algebraic curve of genus $g \geq 1$. We recall that the irregularity of S is $q(S) = h^0(\Omega^1_S) = h^1(\mathcal{O}_S)$ and we set $p_g(S) = h^0(\omega_S)$, which is called *geometric genus* of S. A consequence of Green-Lazarsfeld deformation theory is the solution of a problem raised by Enriques. Let $\mathrm{Pic}^{[\omega]}(S)$ be the component of $\mathrm{Pic}(S)$ containing the point $[\omega_S] \in \mathrm{Pic}(S)$, i.e. $\mathrm{Pic}^{[\omega]}(S) = c_1^{-1}(\alpha_S)$, where $c_1 : \mathrm{Pic}(S) \twoheadrightarrow \mathrm{NS}(S)$ is the first Chern class map and $\alpha_S = c_1([\omega_S]) \in \mathrm{NS}(S)$. If $\mathrm{Div}^{[\omega]}(S)$ denotes the space of all effective divisors on S associated to a line bundle in $\mathrm{Pic}^{[\omega]}(S)$ (see Lectures 12 and 15 of [22]), we have a natural map

$$p : \mathrm{Div}^{[\omega]}(S) \to \mathrm{Pic}^{[\omega]}(S).$$

Definition 4.1. *The* paracanonical system $\{K_S\}$ *of* S *is the union of those irreducible components of* $\mathrm{Div}^{[\omega]}(S)$ *which contain the complete linear system* $p^{-1}([\omega_S]) = |K_S|$ *of all canonical divisors on* S.

In [10], pp. 354-357 Enriques asks for criterion under which $\dim\{K_S\} \le p_g(S)$; Green and Lazarsfeld give an answer in [13], § 4.

Theorem 4.2. *Assume that S does not carry any irrational pencils of genus $g \ge 2$, and let $Z \subseteq \mathrm{Div}^{[\omega]}(S)$ be an irreducible family of curves on S which contains at least one canonical divisor. Then*

$$\dim Z \le p_g(S).$$

In particular, $\dim\{K_S\} \le p_g(S)$.

This theorem was conjectured by Catanese ([7], p. 103) and Green and Lazarsfeld proved it by using the following

Theorem 4.3. *Assume that $q(S) > 0$, and that S does not carry any irrational pencils of genus ≥ 2. Then the trivial bundle $\mathcal{O}_S$ is an isolated point of*

$$S^1(S) = \left\{y \in \mathrm{Pic}^0(M) \mid H^1(S, L_y) \ne 0\right\}.$$

Proof. By hypothesis S is an irregular surface, i.e. $q(S) > 0$, so $\mathcal{O}_S \in S^1(S)$. Using Proposition 2.14 c), it is enough to show that the sequence

$$H^0(S, \mathcal{O}_S) \xrightarrow{\wedge\omega} H^0(S, \Omega^1_S) \xrightarrow{\wedge\omega} H^0(S, \omega_S)$$

is exact for any non-zero $\omega \in H^0(S, \Omega^1_S)$. By contradiction let η be a holomorphic 1-form on S s.t. $\eta \wedge \omega = 0$, but $\eta \ne \lambda \cdot \omega$ for all $\lambda \in \mathbb{C}^*$. Since ω and η are linearly independent, we conclude thanks to the classical theorem (see [2], Proposition IV.5.1 for a proof)

Theorem 4.4 (Castelnuovo - de Franchis)**.** *Let η, ω be linearly independent holomorphic 1-forms on S s.t. $\eta \wedge \omega = 0$. Then there exist an irrational pencil $f : S \to C$ of genus $g \ge 2$ and 1-forms α, $\beta \in H^0(C, \Omega^1_C)$ s.t. $\eta = f^*\alpha$, $\omega = f^*\beta$.*

□

We note that if S admits an irrational pencil $f : S \to C$ of genus g, then S is irregular. Indeed the Riemann-Roch theorem implies that $\dim H^1(\mathcal{O}_C) = g \ge 1$ and one can lift a non-zero 1-form on C to a non-zero 1-form on S, thanks to Sard's theorem. Moreover $f_*\mathcal{O}_S = \mathcal{O}_C$ and the beginning of the Leray spectral sequence ([11], Chapitre II) gives the inclusion $H^1(\mathcal{O}_C) \subseteq H^1(\mathcal{O}_S)$. Now we turn to the

Proof of Theorem 4.2. Given $z \in Z$, we denote by $D_z \subseteq S$ the corresponding effective divisor on S and by $p : Z \to \mathrm{Pic}^{[\omega]}(S)$ the natural map, with a slight abuse of notation. We hypothesize that $[\omega_S] \in p(Z)$. Now, if $Z \subseteq |K_S|$, then

$$\begin{aligned}\dim Z &\le \dim \mathbb{P}(H^0(S, \omega_S)) \\ &= p_g(S) - 1.\end{aligned}$$

So we suppose that Z is not contained in $|K_S|$, i.e. there exists some $z \in Z$ s.t. $p(z) \neq [\omega_S]$. For such a z we have that $\omega_S(-D_z) \in \mathrm{Pic}^0(S)$ and it is different from $\mathcal{O}_S$, so

$$H^0(S, \omega_S(-D_z)) = 0 = H^2(S, \mathcal{O}_S(D_z)),$$

where we use the fact that a non trivial global section of a line bundle $L \in \mathrm{Pic}^0(S)$ must be everywhere non-zero (see [15], p. 413), and the last equality follows from Serre duality. Moreover

$$H^1(S, \mathcal{O}_S(D_z)) = H^1(S, \omega_S(-D_z)) = 0$$

for a general point $z \in Z$, applying Theorem 4.3 and Serre duality again. Hence for a general $z \in Z$:

$$\begin{aligned} h^0(S, \mathcal{O}_S(D_z)) &= \chi(S, \mathcal{O}_S(D_z)) \\ &= \chi(S, \omega_S) \\ &= h^0(\omega_S) - h^1(\omega_S) + h^2(\omega_S) \\ &= p_g(S) - q(S) + 1, \end{aligned}$$

where we use the Hirzebruch-Riemann-Roch theorem and Serre duality. But for any $z \in Z$, one has

$$\begin{aligned} \dim p^{-1}(p(z)) &= \dim |D_z| \\ &= \dim H^0(S, \mathcal{O}_S(D_z)) - 1 \end{aligned}$$

and since $\dim \mathrm{Pic}^{[\omega]}(S) = \dim \mathrm{Pic}^0(S) = q(S)$, it follows that

$$\begin{aligned} \dim Z &\leq \dim p^{-1}(p(z)) + q(S) \\ &= \dim H^0(S, \mathcal{O}_S(D_z)) - 1 + q(S) \\ &= p_g(S), \end{aligned}$$

from the fibre dimension theorem ([25], Chapter I, § 6.3), with $z \in Z$ a general point. □

4.2 The structure of the locus $S^i_m(M)$

Let M be a compact, connected Kähler manifold of complex dimension n as usual, and let $S^i_m(M)$ be defined as in Chapter 2.2. In [14] Green and Lazarsfeld, proceeding with the study of deformations of cohomology groups, arrive at the following result, among other things

Theorem 4.5. *Let Z be an irreducible component of $S^i_m(M)$, then Z is a translate of a subtorus of* $\mathrm{Pic}^0(M)$.

Some years later Ein and Lazarsfeld return to the subject in [8], and prove the following proposition, essentially due to Green:

Proposition 4.6 (M. Green)**.** *Let M be of maximal Albanese dimension, then*

$$\{\mathcal{O}_M\} = S^0(M) \subseteq S^1(M) \subseteq \ldots \subseteq S^n(M) \subseteq \mathrm{Pic}^0(M).$$

Proof. Suppose that $y \in S^i(M)$, so that $H^i(M, L_y) \neq 0$. This means that there exists a non-zero form

$$\alpha \in H^0(M, \Omega^i_M \otimes L^*_y) \cong \overline{H^i(M, L_y)}$$

and it is enough to show that $\omega \wedge \alpha \neq 0 \in H^0(M, \Omega^{i+1}_M \otimes L^*_y)$, for some $\omega \in H^0(M, \Omega^1_M)$. Since M has maximal Albanese dimension, we can take a general point $x \in M$ at which $\alpha(x) \neq 0$ and s.t. the evaluation morphism

$$e(x)\colon H^0(M, \Omega^1_M) \to T_x M^*$$

is surjective, i.e. we can find n holomorphic 1-forms $\omega_1, \ldots, \omega_n \in H^0(M, \Omega^1_M)$ s.t. the vectors $\omega_1(x), \ldots, \omega_n(x)$ form a basis of the holomorphic cotangent space $T_x M^*$. So we have that

$$\alpha(x) \wedge \omega_j(x) \neq 0$$

for some index $j = 1, \ldots, n$. □

The above Proposition has a number of consequences, for example we refind a special case of Theorem 3.5.5, [4]:

Corollary 4.7. *Let X be a compact complex torus, then*

$$H^p(X, \Omega^q_X \otimes L_y) \neq 0 \iff L_y \cong \mathcal{O}_X$$

for any pair of integers p, q s.t. $0 \leq p, q \leq \dim X$ and for any $y \in \mathrm{Pic}^0(X)$.

Proof. Thanks to the Proposition 1.3, we reduce us to the case $q = 0$. If we show that $S^{\dim X}(X) = \{\mathcal{O}_X\}$, we conclude using Proposition 4.6 and (1.3). But since $\omega_X \cong \mathcal{O}_X$, we have that

$$H^n(X, L_y) \cong (H^0(X, L^*_y))^*$$

by Serre duality, and we know that $H^0(X, L^*_y) \neq 0$ iff $L^*_y \cong \mathcal{O}_X$. □

4.3 Holomorphic Euler characteristic

Let M be of maximal Albanese dimension, we have seen that $\chi(M, \omega_M) \geq 0$ (Corollary 2.12). Here we prove a theorem, due to Ein and Lazarsfeld [8], concerning the boundary case $\chi(M, \omega_M) = 0$; precisely we have

Theorem 4.8. *Let M be of maximal Albanese dimension. If $\chi(M, \omega_M) = 0$, then the Albanese image*

$$a(M) \subseteq Alb(M)$$

of M is ruled by tori.

Before starting the proof of this theorem, we give some new definitions:

Definition 4.9. *The m-th* plurigenerus *of M is*

$$P_m(M) := h^0(M, \omega_M^{\otimes m}).$$

Remark 4.10. *We note that*

$$P_1(M) = p_g(M) = h^{n,0}(M).$$

Let L be a line bundle on M. If $L^{\otimes m}$ has a nontrivial global section, then it defines a rational map

$$\phi_m = \phi_{|L^{\otimes m}|} : M \dashrightarrow \mathbb{P}(H^0(M, L^{\otimes m})) = \mathbb{P}.$$

We denote by $\phi_m(M)$ the closure of its image where defined, and observe that it is an algebraic subvariety of $\mathbb{P}$ (see [15], p. 493). The *Iitaka dimension* of L is defined as

$$k(M, L) = \max_{\substack{m>0, \\ H^0(M, L^{\otimes m}) \neq 0}} \dim\ \phi_m(M)$$

if $H^0(M, L^{\otimes m}) \neq 0$ for at least one $m > 0$. Otherwise we put $k(M, L) = -\infty$.

Definition 4.11. *The* Kodaira dimension *$k(M)$ of M is the Iitaka dimension of ω_M, i.e.*

$$k(M) := k(M, \omega_M).$$

In this case the map $\phi_{|\omega_M^{\otimes m}|}$ is called the *m-canonical map.* We observe that $k(M) \leq \dim M$. There are other equivalent definitions of the Kodaira dimension, see for example [18] pp. 73-74. The Kodaira dimension gives informations about the behavior of the plurigenera $P_m(M)$ for $m \to +\infty$, in fact we have the following

Theorem 4.12. *Let M be a compact complex manifold, then:*

1. *$k(M) = -\infty$ iff $P_m(M) = 0$ for all $m \geq 1$;*

2. $k(M) = 0$ *iff* $P_m(M) = 0$ *or* 1 *for* $m \geq 1$, *but not always* 0;

3. $k(M) = k$, $1 \leq k \leq \dim M$ *iff there are real constants* α, $\beta > 0$ *s.t.* $\alpha m^k \leq P_m(M) \leq \beta m^k$, *for* m *large enough.*

Proof. See [26], p. 86. □

Proposition 4.13. *If* M_1, M_2 *are two compact complex manifolds, then*

$$P_m(M_1 \times M_2) = P_m(M_1) \cdot P_m(M_2),$$

and

$$k(M_1 \times M_2) = k(M_1) + k(M_2).$$

Proof. See [26], p. 69. □

Now we give some easy calculations

Example 4.14.

$$k(\mathbb{P}^n) = -\infty.$$

Proof. From the Euler sequence we know that $\omega_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$ and

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = 0$$

if $m < 0$. □

Example 4.15. *Let* X *be a compact complex torus, then* $k(X) = 0$.

Proof. The group structure implies that $\omega_X = \mathcal{O}_X$, hence

$$P_m(X) = h^0(X, \omega_X^{\otimes m}) = h^0(X, \mathcal{O}_X) = 1$$

for all $m \geq 1$, and we conclude using Theorem 4.12. □

The Kodaira dimension allows to distinguish all compact algebraic manifolds of complex dimension n, even if in a coarse way; it puts them in $n+2$ birationally invariant classes, indeed we have the following

Proposition 4.16. *The space of global sections of any controvariant holomorphic tensor bundle on a compact algebraic manifold is a birational invariant.*

Corollary 4.17. *The Hodge numbers* $h^{0,q}(M)$ *are birational invariants, in particular the irregularity* $q(M)$ *is.*

Corollary 4.18. *The plurigenera* $P_m(M)$ *and the Kodaira dimension* $k(M)$ *are birational invariants.*

Proof of the Proposition 4.16. Let $f : M \dashrightarrow N$ be a rational map between compact algebraic manifolds, we know that it is defined on the complement of an analytic subvariety V of $\operatorname{codim}_M V \geq 2$, i.e.

$$f\colon M - V \longrightarrow N.$$

Let $E_M \to M$ be a controvariant holomorphic tensor bundle, for example Ω^p_M. If η is a holomorphic section of E_N on N, the pullback $f^*\eta$ is a section on $M-V$ and extends uniquely to a section on all of M by Hartogs' extension theorem ([15], p. 7), so we have a map

$$f^*\colon H^0(N, \mathcal{O}(E_N)) \to H^0(M, \mathcal{O}(E_M)).$$

In particular, if f is birational, then this function f^* is an isomorphism. □

Definition 4.19. *A compact complex manifold M is* of general type *if $k(M) = \dim M$, i.e. if it has maximal Kodaira dimension.*

Now we return to the holomorphic Euler characteristic. Let M be a compact algebraic manifold; if M is birational to a product of a torus X and some other manifold N then clearly $\chi(M, \omega_M) = 0$, because $\chi(\omega_{X\times N}) = \chi(\omega_X)\cdot\chi(\omega_N)$ and

Remark 4.20. *If X is a compact complex torus, then*

$$\chi(X, \omega_X) = 0.$$

Proof. Denote by t the dimension of X. By (1.3) we have

$$\begin{aligned}\chi(X, \omega_X) &= h^0(\Omega^t_X) - h^1(\Omega^t_X) + \ldots + (-1)^t h^t(\Omega^t_X)\\ &= \binom{t}{0} - \binom{t}{1} + \ldots + (-1)^t\binom{t}{t}\\ &= 0.\end{aligned}$$

□

As a consequence of Theorem 4.8, we have that if M is of maximal Albanese dimension and of general type and it is birationally a subvariety of $\mathrm{Alb}(M)$, then $\chi(M, \omega_M) > 0$; precisely

Corollary 4.21. *Let M be a compact algebraic variety of maximal Albanese dimension, and suppose that $\chi(M, \omega_M) = 0$. If*

$$a : M \to \mathrm{Alb}(M)$$

is birational onto its image, then M is not of general type.

The same statement, without the assumption on the birationality of M, was conjectured by Kollár ([19]), but Ein and Lazarsfeld give a counterexample in [8].

Proof of Corollary 4.21. Suppose that $a : M \to a(M)$ is birational. By Theorem 4.8 we have a holomorphic map

$$h : a(M) \to W$$

where W is a complex space and all fibres are isomorphic to a connected complex subtorus of $\mathrm{Pic}^0(M)$. Using Theorem 6.12 of [26] we have the following inequality:

$$k(M) = k(a(M)) \leq k(h^{-1}(w)) + \dim W,$$

for some point $w \in W$, that in our case becomes

$$k(M) \leq \dim W$$

thanks to the example 4.15. If M is of general type, then

$$n = \dim W,$$

and hence $\dim h^{-1}(w) = 0$, absurd. □

Finally we turn to the

Proof of Theorem 4.8. We will use an irreducible component of a suitable cohomology support locus to construct the required fibration. Assume that $\chi(M, \omega_M) = 0$. Since $\chi(M, \omega_M) = h^0(M, \omega_M \otimes L_y)$ for a general $y \in \mathrm{Pic}^0(M)$, we have that $S^n(M)$ is a proper subvariety of $\mathrm{Pic}^0(M)$ by Serre duality. Now fix an irreducible component $Z \subseteq S^n(M)$ and put

$$k = \mathrm{codim}_{\mathrm{Pic}^0(M)} Z \geq 1$$

We note that Z cannot be contained in any $S^i(M)$ with $i < n - k$, because $\mathrm{codim}_{\mathrm{Pic}^0(M)} S^i(M) \geq n - i$ (Theorem 2.9) and we want to prove that Z is an irreducible component of $S^{n-k}(M)$. Take a general point $y \in Z$ s.t. y is a smooth point at which the function $h^i(M, L_y)$ is as small as possible for all $i = 0, \dots, n$. Of course $h^i(M, L_y) = 0$ for $i < n - k$. If $h^i(M, L_y) \neq 0$ for some $n - k \leq i \leq n$, then $Z \subseteq S^i(M)$ and in fact Z is an irreducible component of $S^i(M)$, thanks to the Proposition 4.6. We claim that

$$h^{n-k}(M, L_y) \neq 0.$$

In order to prove it we need the following

Lemma 4.22 ([14])**.** *Fix any irreducible component*

$$Z \subseteq S^i(M)$$

and let $y \in Z$ be a general point, i.e. a smooth point at which the function $h^i(M, L_y)$ assumes its generic value on Z. If $0 \neq v \in H^1(M, \mathcal{O}_M)$ is tangent to Z, then the maps in

$$H^{i-1}(M, L_y) \xrightarrow{\smile v} H^i(M, L_y) \xrightarrow{\smile v} H^{i+1}(M, L_y) \tag{4.1}$$

vanish, whereas if v is not tangent to Z then (4.1) *is exact.*

Sketch of proof of Lemma 4.22. Take $0 \neq v \in H^1(M, \mathcal{O}_M)$ and consider the map

$$\exp_v(y) \colon \mathbb{C} \longrightarrow \operatorname{Pic}^0(M), \quad t \longmapsto y + t\tau(v)$$

where $\tau : H^1(M, \mathcal{O}_M) \to \operatorname{Pic}^0(M)$ is given by the exponential sequence. Let $\Delta_v(y) \subseteq \operatorname{Pic}^0(M)$ denote the image of a small disk centered at 0 under $\exp_v(y)$. Since Z is a translate of a subtorus of $\operatorname{Pic}^0(M)$, $\Delta_v(y) \subseteq Z$ if v is tangent to Z and

$$h^i(M, L_t) = h^i(M, L_y) \tag{4.2}$$

for all $t \in \Delta_v(y)$, because $h^i(M, L_y)$ assumes its generic value at y. Corollary 3.3 of [14] says that there exists a punctured neighborhood U of y in $\Delta_v(y)$ s.t. $H^i(M, L_t) \cong$ homology of (4.1), for all $t \in U - \{y\}$. It then follows from (4.2) that the maps in (4.1) must vanish. If $\Delta_v(y)$ is not contained in Z, then $H^i(M, L_t)$ vanishes for generic t thanks to the fact that Z is an irreducible component of $S^i(M)$, and so (4.1) is exact. □

Now we can prove the claim: let

$$V \subseteq H^1(M, \mathcal{O}_M) = T_y \operatorname{Pic}^0(M)$$

be a k-dimensional subspace complementary to $T_y Z \subseteq T_y \operatorname{Pic}^0(M)$. By Lemma 4.22 we have that

$$0 \to H^{n-k}(L_y) \xrightarrow{\smile v} H^{n-k+1}(L_y) \xrightarrow{\smile v} \ldots \xrightarrow{\smile v} H^n(L_y) \to 0 \tag{4.3}$$

is exact for each $0 \neq v \in V$. Let $\mathbb{P} := \mathbb{P}(V^*)$. The isomorphism ([15], p. 165)

$$\operatorname{Sym}^d(V) \cong H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d)), \quad d > 0$$

becomes, in our case $d = 1$,

$$V \cong H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)),$$

so we may assemble the exact sequence (4.3) determined by $0 \neq v \in V$ into a complex of vector bundles on $\mathbb{P}$:

$$0 \to H^{n-k}(L_y) \otimes \mathcal{O}_{\mathbb{P}}(-k) \to H^{n-k+1}(L_y) \otimes \mathcal{O}_{\mathbb{P}}(-k+1) \to \ldots$$
$$\ldots \to H^n(L_y) \otimes \mathcal{O}_{\mathbb{P}} \to 0,$$

that is exact as a complex of sheaves on $\mathbb{P}$, because the point-wise complexes are exact by the choice of V. The spectral sequence

$$\begin{aligned} E_1^{p,q} &= H^q(\mathbb{P}, H^{n-k+p}(L_y) \otimes \mathcal{O}_{\mathbb{P}}(-k+p)) \\ &= H^{n-k+p}(L_y) \otimes H^q(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-k+p)), \quad 0 \le p \le k \end{aligned}$$

converges to the hypercohomology of the complex, which is zero because of the exactness. Now we use the following theorem on the cohomology of line bundles on projective spaces ([18], example 5.2.5)

Theorem 4.23.

$$\begin{aligned} \dim H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) &= \binom{n+m}{m}, \quad m \ge 0 \\ \dim H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) &= \binom{-m-1}{-n-m-1}, \quad m \le -n-1 \end{aligned}$$

and in all other cases, the dimension is zero.

Remembering that $\dim \mathbb{P} = k-1$, we have

$$E_1^{p,q} = 0, \quad 0 < p < k$$

hence the differentials of the spectral sequence $d_1 = \ldots = d_{k-1} = 0$ and d_k has to be an isomorphism, because the limiting page is zero. In particular

$$d_k^{0,k-1} \colon H^{n-k}(L_y) \otimes H^{k-1}(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-k)) \xrightarrow{\cong} H^n(L_y) \otimes H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}).$$

But $\dim H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) = \dim H^{k-1}(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-k)) = 1$, again by Theorem 4.23, and so

$$H^{n-k}(M, L_y) \cong H^n(M, L_y) \neq 0,$$

because $y \in Z$. Until now we have proved that Z is an irreducible component of $S^{n-k}(M)$. By Theorem 4.5 we know that Z is a translate of a subtorus T of $\mathrm{Pic}^0(M)$. Let $\widehat{T}$ be the dual torus. Since $A := \mathrm{Alb}(M)$ is the dual of $\mathrm{Pic}^0(M)$, the inclusion $T \hookrightarrow \mathrm{Pic}^0(M)$ gives a surjective holomorphic map

$$\pi : A \to \widehat{T}$$

whose fibres are translates of the k-dimensional subtorus

$$B := \mathrm{Ker}(\pi) \subseteq A,$$

(here we use Propositions 2.4.2 and 1.2.4 of [4]). We are going to show that the Albanese image $a(M) \subseteq A$ of M is fibred by translates of B. Let

$$\begin{aligned} h &: \ a(M) \subseteq A \to \widehat{T}, \\ g &: M \to a(M) \subseteq A \to \widehat{T}, \end{aligned}$$

denote $\pi|_{a(M)}$ and $\pi|_{a(M)} \circ a$, respectively. Grant for the moment that

$$\dim g(M) \leq n - k, \tag{4.4}$$

we conclude in the following way: since M is of maximal Albanese dimension, it follows that

$$\dim h^{-1}(t) \geq n - \dim g(M) \geq k,$$

for all $t \in h(a(M))$. But these fibres are contained in translates of the k-dimensional torus B, so the fibres of $a(M) \to h(a(M))$ fill up the fibres of $A \to \widehat{T}$ over $h(a(M)) \subseteq \widehat{T}$ and we have that $a(M)$ is ruled by tori, i.e. there exists a holomorphic map from $a(M)$ to a complex space of dimension $n-k$, whose fibres are translates of B. It remains only to prove (4.4), and here we use the fact that a translate of T is contained in $S^{n-k}(M)$. Let

$$v_1, \ldots, v_{q-k} \in T_y Z \subseteq H^1(M, \mathcal{O}_M)$$

be a basis for the tangent space to Z at y, where $q := q(M) = \dim \operatorname{Pic}^0(M)$, and let $\omega_i = \overline{v_i} \in H^0(M, \Omega^1_M)$ be the conjugate holomorphic 1-forms. The map

$$g : M \to a(M) \to \widehat{T}$$

arises by integrating the ω_i, and so

$$\dim g(M) = \dim \operatorname{Span}\left\{\omega_1(x), \ldots, \omega_{q-k}(x)\right\} \subseteq T_x M^*$$

for a general point $x \in M$. Since v_i is a tangent vector to Z for all i, it follows that each of the maps

$$H^0(M, \Omega^{n-k}_M \otimes L^*_y) \xrightarrow{\wedge \omega_i} H^0(M, \Omega^{n-k+1}_M \otimes L^*_y)$$

vanishes. But, as we have seen, $H^0(M, \Omega^{n-k}_M \otimes L^*_y) \neq 0$. So there exists an element $0 \neq \alpha \in H^0(M, \Omega^{n-k}_M \otimes L^*_y)$ s.t.

$$\alpha \wedge \omega_i = 0 \quad \forall\ i = 1, \ldots, q-k$$

and we conclude using the elementary Lemma 2.30:

$$\dim g(M) \leq \dim \widehat{T} = q - k \leq n - k.$$

□

Chapter 5

Appendix

5.1 Seesaw Theorem

During the proof of the uniqueness of the Poincaré line bundle we invoke the so-called Seesaw Theorem; here we give a precise statement and a proof of it.

Theorem 5.1. *Let X, Y be connected compact complex manifolds and $\mathcal{L}$ a holomorphic line bundle on $X \times Y$, then*

a)

$$Y_0 := \left\{ y \in Y \mid \mathcal{L}|_{X\times\{y\}} \cong \mathcal{O}_X \right\}$$

is an analytic subvariety of Y;

b) If $q : X \times Y_0 \to Y_0$ denotes the projection map, then there is a holomorphic line bundle $\mathcal{M}$ on Y_0 s.t.

$$\mathcal{L}|_{X\times Y_0} \cong q^*\mathcal{M}.$$

Proof. a): A holomorphic line bundle $\mathcal{N}$ on a compact complex manifold is trivial iff $h^0(\mathcal{N}) > 0$ and $h^0(\mathcal{N}^*) > 0$, hence

$$Y_0 = \left\{ y \in Y \mid h^0(\mathcal{L}|_{X\times\{y\}}) > 0,\ h^0(\mathcal{L}|^*_{X\times\{y\}}) > 0 \right\}$$

and we conclude using the Semicontinuity Theorem 2.8.

b): Thanks to the Base Change Theorem ([23], p. 51) the sheaf

$$\mathcal{M} := q_*(\mathcal{L}|_{X\times Y_0})$$

is invertible on Y_0, and

$$\mathcal{M}_y \cong H^0(\mathcal{L}|_{X\times\{y\}}) \cong \mathbb{C},$$

for all $y \in Y_0$. Since $\mathcal{L}|_{X\times\{y\}}$ is trivial for all $y \in Y_0$, the natural map

$$q^*\mathcal{M} \to \mathcal{L}|_{X\times Y_0}$$

is an isomorphism. □

5.2 Koszul complex

Here we give a quick introduction to Koszul complex, referring to Chapter 17 of Eisenbud's book [9] for a complete treatment. In what follows A denotes a Noetherian commutative ring with unity.

Definition 5.2. *Let M be an A-module. A sequence of elements $a_1, \dots, a_n \in A$ is called an M-*regular sequence *if:*

1. *For $i = 1, \dots, n$, a_i is a non-zero-divisor on $M/\sum_{j=1}^{i-1} a_j M$, i.e. $a_i x \neq 0$ for all $x \neq 0$ in $M/\sum_{j=1}^{i-1} a_j M$;*
2. *$M/\sum_{j=1}^{n} a_j M \neq 0$.*

If N is a free A-module, we denote by $x = (x_1, \dots, x_n)$ its elements, via the natural isomorphism $N \cong A^n$. We define the *Koszul complex*

$$K(x_1, \dots, x_n) = K(x) : 0 \to A \to N \to \bigwedge^2 N \to \dots \to \bigwedge^n N \to 0,$$

where

$$d_x : \bigwedge^i N \to \bigwedge^{i+1} N$$

maps an element a to the element $a \wedge x$.

Remark 5.3. *It is easy to see that*

$$H^n(K(x_1, \dots, x_n)) \cong A/(x_1, \dots, x_n)$$

The relation between Koszul complex and regular sequences is given by the following

Theorem 5.4. *Suppose M is a finitely generated A-module. Let $a_1, \dots, a_n$ be an M-regular sequence, then*

$$H^j(M \otimes K(a_1, \dots, a_n)) = 0 \quad \forall\ j < n$$

and

$$H^n(M \otimes K(a_1, \dots, a_n)) = M/\sum_{j=1}^{n} a_j M.$$

In the local case we have the stronger

Theorem 5.5. *Let (A, m) be a local ring, $a_1, \dots, a_n \in m$ and M a finitely generated A-module. If*

$$H^k(M \otimes K(a_1, \dots, a_n)) = 0$$

for some k, then

$$H^j(M \otimes K(a_1, \dots, a_n)) = 0 \quad \forall\ j \leq k.$$

Moreover if $H^{n-1}(M \otimes K(a_1, \dots, a_n)) = 0$, then $a_1, \dots, a_n$ is an M-regular sequence.

If we have two complexes of A-modules $K^\bullet$ and $L^\bullet$, the *tensor product* $K^\bullet \otimes_A L^\bullet$ is the single complex associated to the double complex $W^{\bullet,\bullet}$, where $W^{i,j} = K^i \otimes_A L^j$, with differentials given by

$$d(x \otimes y) = d^K(x) \otimes y + (-1)^i x \otimes d^L(y).$$

Proposition 5.6. *If $N = N' \oplus N''$ and $x = (x', x'') \in N$, then*

$$K(x) \cong K(x') \otimes K(x'')$$

as complexes.

Using the tensor product of complexes it is possible to be more precise about Theorem 5.4:

Theorem 5.7. *If $a_1, \ldots, a_k$ is an M-regular sequence in the ideal $(a_1, \ldots, a_n)$, then*

$$H^j(M \otimes K(a_1, \ldots, a_n)) = 0 \quad \forall\ j < k.$$

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