UNIVERSITÀ DEGLI STUDI DI PISA

Dipartimento di Matematica
Corso di Laurea Magistrale in Matematica

16 Ottobre 2015

Tesi di Laurea Magistrale

Homological Stability for General Linear Groups

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ANNO ACCADEMICO 2014–2015

Et bientôt, machinalement, accablé par la morne journée et la perspective d'un triste lendemain, je portai à mes lèvres une cuillerée du thé où j'avais laissé s'amollir un morceau de madeleine.

Proust

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Introduction

In the last decades, homological stability and stability phenomena have played important roles in many fields of modern mathematics: from algebraic topology to algebraic and differential geometry, from functional analysis to number theory and representation theory.

This increasing interest began in the 70's with the work of D. Quillen; he defined the higher algebraic K-groups of a ring, and gave rise to a new big interdisciplinary subject in mathematics: *Algebraic K-theory.* By using homological stability arguments, Quillen proved that the K-groups of the fields $\mathbb{F}_p$ are finitely generated. Since then, inspired by his works and techniques, many mathematicians tried to get information on K-groups and limit objects by using stabilization arguments.

To better understand the great interest in Algebraic K-theory, we recall that the algebraic K-groups of a ring R, in literature denoted by $K_i(R)$, are defined as the homotopy groups of a modification of the classifying space of the infinite general linear group, namely $\mathrm{BGL}^+(R)$. Moreover, the homology groups of this complicated space coincide with those of the discrete limit group $\mathrm{GL}(R) := \cup_n \mathrm{GL}_n(R)$. Then, if one has information on the homology of the limit group $\mathrm{GL}(R)$, he can deduce properties on these K-groups. This computation is quite hard, and it involves a large knowledge of homological and homotopical properties for groups of matrices with coefficients in the ring R; homological stability is then a nice way to deal with this problems.

In the following years, the interest spread into other fields of mathematics, providing useful information on the direct limits of inductive families of topological spaces and discrete groups. Homological stability has been proved in many noteworthy examples, such as symmetric groups, braid groups, mapping class groups of a large class of manifolds, general linear groups with coefficients in finite dimensional Noetherian rings, symplectic groups, and so on. A recent application is the proof, by Madsen and Weiss, of the *Mumford Conjecture*, which states that the rational stable cohomology ring of the moduli space of Riemann surfaces is a polynomial algebra. This proof uses Harer's homological stability for mapping class groups of surfaces.

Inspired by its applications in Algebraic K-theory, in this thesis we give a complete and detailed proof of *homological stability for the family of general linear groups* $\{GL_n(R)\}_{n\in\mathbb{N}}$, with coefficients in a suitable fixed ring R, such as Noetherian commutative rings of finite dimension. This is a fairly classical result that has been proved, for different kind of rings R, by several mathematicians. Among them, there are Quillen, Maazen, Charney, van der Kallen and Suslin.

The classical proofs concerning homological stability results are designed ad hoc for the particular family considered. The aim of this work is to present a proof that does not depend on the family of groups, but just on the acyclicity of suitable spaces. This proof uses a more abstract set-up, that one can find in the beautiful article *Homological stability for automorphism groups*, by N. Wahl. She defines suitable categories in which the study of homological stability problems is natural, and then she proves a really general stability result for automorphism groups in these opportune categories, called *homogenous categories*. We study this general set-up, and then we deduce homological stability for the family $\{\mathrm{GL}_n(R)\}_{n\in\mathbb{N}}$, seen as a family of automorphisms in the category of modules. The strength of this proof, among its generality, is in the new viewpoint that it provides on the subject: it opens the way to new conjectures, relations between groups and topological spaces, and suggests new families of groups to study.

We now explain what we mean by homological stability for a family of discrete groups, and then we describe the structure of the thesis.

Definition. A family of discrete groups $\{G_n\}_{n\in\mathbb{N}}$ together with injective homomorphisms $\mathrm{stab}_n\colon G_n \hookrightarrow G_{n+1}$ satisfies *homological stability* (with integer coefficients) if the induced maps in homology

$$\mathrm{H}_i(\mathrm{stab}_n)\colon \mathrm{H}_i(G_n;\mathbb{Z}) \longrightarrow \mathrm{H}_i(G_{n+1};\mathbb{Z})$$

are isomorphisms in a range $0 \le i \le f(n)$, where $f\colon \mathbb{N} \to \mathbb{N}$ is increasing with n.

When G_n is the general linear group $\mathrm{GL}_n(R)$, the stabilization map is the upper left inclusion:

$$\mathrm{GL}_n(R) \to \mathrm{GL}_{n+1}(R)\colon A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

The family $\{\mathrm{GL}_n(R)\}_{n\in\mathbb{N}}$, with these stabilization maps, satisfies homological stability if the ring R is a Noetherian ring of finite dimension. More generally, the result is true if one considers the more general class of rings given by *rings of finite stable dimension*. This class properly includes the class of Noetherian rings of finite Krull dimension. If we denote by $\mathrm{sdim}(R)$ the stable dimension of a ring, and by $\mathrm{Kdim}(R)$ its Krull dimension, then one has $\mathrm{sdim}(R) \le \mathrm{Kdim}(R)$. The stability results is the following:

Theorem. *Let R be an associative ring with identity and suppose R is a ring of stable dimension $sdim + 2$. Then, the map*

$$H_i(GL_n(R)) \longrightarrow H_i(GL_{n+1}(R))$$

is an isomorphisms when $n \ge 2i + sdim + 2$.

The generic stability proof follows an argument due to Quillen, known as *Quillen's argument*, and it consists of three steps: first, one has to find highly connected G_n-simplicial complexes X_n, with transitive G_n-action on the set of p-simplices, for every p; then, one needs to prove that the stabilizer of a p-simplex is actually a previous group G_k in the family. Finally, by using a classical spectral sequence argument, one can deduce the isomorphisms in homology.

As a consequence, a great deal of attention is devoted to deduce homological properties of spaces with a G_n-action. In this thesis, we focus our attention on the homological properties of a particular partially ordered set on which the general linear groups act: the poset of *split unimodular sequences*. By using a technical assumption on the ring of coefficients R, namely the *stable range condition*, we prove that the simplicial complexes associated to these posets are highly acyclic.

The general idea suggested in the article *Homological stability for automorphism groups* follows the same Quillen's article. Families of automorphism groups in *homogenous categories* satisfy homological stability if the associated semi-simplicial spaces are highly acyclic. In the case of general linear groups, $\mathrm{GL}_n(R)$ is the automorphism group of the finitely generated free R-module R^n, the associated semi-simplicial space is isomorphic to the poset of split unimodular sequences, that is highly acyclic. Then, we conclude that the general linear groups satisfy homological stability. In particular we obtain the following interesting corollary on group rings:

Theorem. *Let R be a commutative Noetherian ring of finite Krull dimension. Let G be a finite group, or a group containing a finitely generated abelian group of finite index as a subgroup. Then the family of groups $GL_n(R[G])$ satisfies homological stability.*

The thesis is structured in five chapters, and developes as follows:

In the first chapter we introduce the basic definitions, and we collect the facts and results that we will need afterwards. We study the basic properties of partially ordered sets and simplicial sets, the suspension and the link of an element in a poset. Then, we study the elementary matrices and we define the first K-group of a ring. Finally, we conclude the chapter with the definition of braided monoidal categories and with a review on spectral sequences.

In the second chapter, we introduce the homological stability for a family of discrete groups. After giving the definition, we review the known results in the case of general linear groups, and then we briefly study homological stability for two easier examples: braid groups and mapping class groups.

In Chapter 3, we study a technical assumption on a given ring R: the stable range condition. Thanks to this hypothesis, we can define a notion of dimension, the *stable dimension*, for a large class of rings. This dimension, very roughly speaking, measures the average number of generators of a ring. The main result of this chapter is that a

Noetherian ring of finite dimension satisfies the stable range condition; moreover, we show that the class of rings that satisfy this condition is strictly larger.

In the fourth chapter, we study partially ordered sets of ordered sequences, and we show how the connectivity properties of a poset can be deduced from those of its subposets. In particular, we study the poset of split unimodular sequences, where we say that a sequence of elements in R^n, $(v_1, w_1, \ldots, v_k, w_k)$, is split unimodular if $(v_1, \ldots, v_k)$ is a basis of a direct summand in R^n and $w_i \cdot v_j = \delta_i^j$. We prove that this poset is highly acyclic, provided the ring R satisfies the stable range condition.

In the last chapter, we define the homogenous categories as the natural setting in which one can deduce homological stability, and prove that homogenous categories can be constructed from braided monoidal groupoids. For any pair of objects in a homogenous category, say (X, A), we define a semi-simplicial space with an $\mathrm{Aut}(A \oplus X^{\oplus n})$-action. If these semi-simplicial spaces are highly acyclic, the family of automorphism groups $G_n := \mathrm{Aut}(A \oplus X^{\oplus n})$ satisfies homological stability. By using the acyclicity property of the poset of split unimodular sequences, and by applying this general construction, we deduce that the general linear groups satisfies homological stability.

1 General facts

Contents

The goal of this thesis is to give a complete and detailed proof of homological stability for the family of general linear groups $\{\mathrm{GL}_n(R)\}_{n\in\mathbb{N}}$, the invertible matrices with coefficients in a ring R; we can think about R as a commutative Noetherian ring of finite Krull dimension. In Definition 2.1.1 we explain what we mean by *homological stability* for a family of discrete groups $\{G_n\}_{n\in\mathbb{N}}$; roughly speaking, we want to understand whether, increasing with n, the homomorphisms in homology

$$\mathrm{H}_i(\mathrm{GL}_n(R);\ \mathbb{Z}) \longrightarrow \mathrm{H}_i(\mathrm{GL}_{n+1}(R);\ \mathbb{Z})$$

are actually isomorphisms.

The first result on this kind of stability is due to D. Quillen. He proves in his unpublished MIT lectures, readable as working papers in [Qui74], that the discrete groups $\mathrm{GL}_n(\mathbb{F}_p)$ satisfy homological stability when $p \neq 2$. The general technique he uses in those lectures is nowadays generally known as *Quillen's argument*. Quillen's idea is to consider a family of simplicial complexes with a natural $\mathrm{GL}_n(\mathbb{F}_p)$-action, and with nice connectivity properties. He uses this connectivity to show that a suitable spectral sequence converges to 0 in a range; moreover, it happens that the homology groups we are interested in belong to this spectral sequence. By using the convergence to zero of the spectral sequence, Quillen argues the stability result.

The proof we are going to present in this thesis follows the classical Quillen's argument, but we need partially ordered sets and semi-simplicial spaces instead of simplicial complexes. Moreover, if we want to enlarge the class of rings that satisfy homological stability, we need to define a "stable condition" on the rings R. Finally, in Chapter 5, we analyze the spectral sequence argument in a categorical setting. Hence, to reach this goal, before going deeper in the subject, we want to collect in this first chapter the basic notions, definitions, notations, facts and results, from both abstract algebra and topology, that we need to proceed in a self-contained way.

As the bricks of our categorical construction are semi-simplicial sets, we begin with the definition of these objects, and we show how to define their geometric realization. In the second section we give the definition of a partially ordered set, and we study the topological properties that are connected to the concept of *link*. In the third section we study the relation between invertible and elementary matrices, whose properties will be really important when we will prove acyclicity results. In the last two sections we recall the definition of braided monoidal categories and we recall useful known facts on spectral sequences.

1.1 Join, suspension and semi-simplicial sets

In this section we recall the definitions of join, suspension, simplicial complexes and their generalizations, the semi-simplicial sets. We refer to [Fri08], [Hat02] and [Qui73] as general references.

Definition 1.1.1. If X, Y are topological spaces, the *join* $X * Y$ is defined as the quotient space $(X \times Y \times I)/\sim$, where I is the interval $[0,1]$, and the equivalence relation is given as follows:

- $(x, y_1, 0) \sim (x, y_2, 0)$ for every $x \in X$ and $y_1, y_2 \in Y$;
- $(x_1, y, 1) \sim (x_2, y, 1)$ for every $x_1, x_2 \in X$ and $y \in Y$.

The join construction is commutative and associative, up to isomorphism.

Definition 1.1.2. For a topological space X, the *suspension* ΣX is the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point.

Given a connected topological space X, it is easy to see that its join with the 0-sphere $\mathbb{S}^0$ is actually its suspension, i.e. $X * \mathbb{S}^0 \cong \Sigma X$; moreover, the suspension ΣX can also be seen as the space $CX \cup_X CX$, where CX is the cone over X. As a consequence of these identifications, applying the Mayer Vietoris exact sequence to $CX \cup_X CX$, and using that the cone on a space is contractible, $CX \simeq \{*\}$, we get:

$$\widetilde{\mathrm{H}}_i(\Sigma X) \cong \widetilde{\mathrm{H}}_{i-1}(X);$$

i.e. the homology of the suspension space and the homology of the space itself are isomorphic.

Since the suspension of a k-sphere is the $(k+1)$-sphere, $\Sigma(\mathbb{S}^k) = \mathbb{S}^{k+1}$, we also have:

$$\mathbb{S}^k * X = \Sigma(\mathbb{S}^{k-1}) * X = (\mathbb{S}^{k-1} * \mathbb{S}^0) * X = \mathbb{S}^{k-1} * \Sigma X = \cdots = \Sigma^{k+1} X$$

the $(k+1)$-fold suspension; this remark is of fundamental importance in studying the topological properties of partially ordered sets.

Simplicial complexes are really important in homological stability proofs; we recall the definition:

Definition 1.1.3. A *simplicial complex* X consists of a set X_0, whose elements are referred to as *vertices*, together with a collection of finite subsets of X_0 closed under taking subsets. The subsets of cardinality $p+1$ are called *p-simplices*. A *simplicial map* between two simplicial complexes X and Y is a function that maps vertices in X to vertices in Y, and such that the image of a p-simplex in X is a q-simplex in Y, $q \leq p$.

In modern algebraic topology, simplicial complexes have been generalized with the concept of semi-simplicial set:

Definition 1.1.4. A *semi-simplicial set* consists of a family of sets $\{X_n\}_{n\in\mathbb{N}}$ together with maps $d_i \colon X_p \to X_{p-1}$ for each $0 \leq i \leq p$, such that $d_i d_j = d_{j-1} d_i$ whenever $i < j$.

In analogy with simplicial complexes, elements in X_p are called p-simplices of the semi-simplicial space X, and the maps d_i are called face maps. Traditionally, our semi-simplicial sets were called Δ-sets. To follow the terminology used in [Wah14], we preferred the name *semi-simplicial set*. For a general overview on these objects, see [Fri08].

We remark that, in general, it could be better to work with semi-simplicial sets, instead of just simplicial complexes, since semi-simplicial sets have a more abstract, we will see categorical, definition, and, as a consequence, much more malleability. In fact, a simplicial complex can be viewed as a semi-simplicial set, but there are semi-simplicial sets that are not simplicial complexes: it happens because in a simplicial complex the faces of a given simplex must be unique; moreover, a collection of vertices specifies a unique simplex. This is no longer true in semi-simplicial sets.

We want to give a categorical definition of semi-simplicial sets, but we first need to define the category $\hat{\Delta}$:

Definition 1.1.5. The category $\hat{\Delta}$ has as objects the finite ordered sets $[n] := \{0, 1, \ldots, n\}$ and as morphisms the injective order-preserving maps $[m] \to [n]$.

We can now give the categorical definition of semi-simplicial sets:

Definition 1.1.6. A semi-simplicial set is a covariant functor $X \colon \hat{\Delta}^{op} \to \mathcal{S}$, where $\hat{\Delta}^{op}$ is the opposite category, and $\mathcal{S}$ is the category of sets.

Sometimes it is useful to consider also degenerate simplices; the simplicial sets encode this feature:

Definition 1.1.7. A *simplicial set* X consists of a family of sets $\{X_n\}_{n\in\mathbb{N}}$ together with maps $d_i\colon X_p \to X_{p-1}$, and $s_i\colon X_p \to X_{p+1}$, respectively called face and degeneracy maps, such that the following conditions are satisfied:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i && \text{if } i < j;\\ d_i s_j &= s_{j-1} d_i && \text{if } i < j;\\ d_j s_j &= d_{j+1} s_j = \mathrm{id};\\ d_i s_j &= s_j d_{i-1} && \text{if } i > j+1;\\ s_i s_j &= s_{j+1} s_i && \text{if } i \leq j. \end{aligned}$$

In this way, a semi-simplicial set is a simplicial set without degeneracies. As above, there is the equivalent categorical definition, but we need to change the category $\hat{\Delta}$:

Definition 1.1.8. Let Δ be the category of finite ordered sets with order-preserving maps (no longer injective) as morphisms. A simplicial set is a functor $X\colon \Delta^{op} \to \mathcal{S}$.

Remark 1.1.9. Every semi-simplicial set can be completed to a simplicial set by adjoining all possible degenerate simplices. Conversely, each simplicial set yealds a semi-simplicial set by omitting the degeracy maps.

Once we have the abstract objects to work with, we want to study their "topological" properties; but this can be done when one has a topological space associated to them. We racall that, to each simplicial complex, one can associate a topological space, its realization. A similar thing can be done in the case of simplicial sets and semi-simplicial sets or, more generally, for small categories. We recall that a category $\mathcal{C}$ is said *small* if the class of objects $\mathrm{ob}(\mathcal{C})$ is a set.

In fact, if $\mathcal{C}$ is a small category, we can define a simplicial set (hence, also a semi-simplicial set). The p-simplices are the diagrams in $\mathcal{C}$ of the form

$$X_0 \to X_1 \to \cdots \to X_p$$

The i-th face map of this simplex is obtained by omitting in the diagram the object X_i. The i-th degenerate map is obtained by replacing the object X_i with the identity map $\mathrm{id}\colon X_i \to X_i$.

Realization of a simplicial set. We briefly explain how to obtain the geometric realization for a semi-simplicial set X. Equip each set X_n with the discrete topology. Let Δ^n be the standard n-simplex with its standard topology; by labeling its vertices, say $[0,\dots,n]$, the standard n-simplex is the prototype of any object in the category $\hat{\Delta}$. Consider the morphisms induced by the inclusion maps $D_i\colon \Delta^p \to \Delta^{p+1}$, $[0,\dots,p] \mapsto$

$[0, \dots, \hat{i}, \dots, p+1]$, representing the inclusion of a face in a larger simplex. The *geometric realization* $|X|$ of the semi-simplicial set X is the topological space defined as follows:

$$|X| := (\bigsqcup_{n \in \mathbb{N}} X_n \times \Delta^n)/ \sim$$

with the equivalence relations

$$(x, D_i(y)) \sim (d_i(x), y)$$

for $x \in X_{n+1}$ and $y \in \Delta^n$. It is now clear that, if X is a simplicial set, then the realization can be defined in the same way.

We remark that this realization is a CW-complex whose p-cells are in bijection with the p-simplices that are non-degenerate. For a proof in the simplicial setting, see Theorem 1 in [Mil57].

Definition 1.1.10. If X and Y are semi-simplicial (or simplicial) sets, seen as functors, then a *simplicial morphism* $f\colon X \to Y$ is a natural transformation between the functors X and Y.

A simplicial map between semi-simplicial sets induces a continuous map between the realizations. If X and Y are semi-simplicial sets, we can construct the product $X \times Y$ that is a semi-simplicial set with p-simplices $(X \times Y)_p := X_p \times Y_p$. If we consider the projection maps $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$, we get induced maps between the realizations. The map

$$|X \times Y| \to |X| \times |Y|$$

defined by $|p| \times |q|$ is an homeomorphism when the semi-simplicial sets X and Y are countable, or when one of the two CW-complexes $|X|$ and $|Y|$ is locally finite. This is clearly also true in the case of simplicial sets, and for a proof see for example Theorem 2 in [Mil57].

1.2 Poset spaces and links

In this section, referring to [Qui78], we introduce partially ordered sets, briefly named posets.

Definition 1.2.1. A *poset* $(X, \leq_X)$ is a pair consisting of a set X endowed with a binary relation $\leq_X$, such that:

- $x \leq_X x$;
- $x \leq_X y$ and $y \leq_X x$ implies $x = y$;
- if $x \leq_X y$ and $y \leq_X z$ then $x \leq_X z$.

If the partial ordering is clear from the context, we refer to the poset just by X. A morphism of posets (or *poset map*) is a map $f\colon (X,\leq_X) \to (Y,\leq_Y)$ that respects the orderings.

Remark 1.2.2. Le X be a partially ordered set. Then, it can be viewed as a small category: the objects are the elements of the set X, and the morphisms between the objects are just the order relations. Hence, we can associate a semi-simplicial set whose p-simplices are the p-chains $x_0 \leq \cdots \leq x_p$. Moreover, this semi-simplicial set is a simplicial complex in the sense of Definition 1.1.3, with vertices the elements of X, and whose simplices are the totally ordered non-empty finite subsets of X. Finally, we associate to the poset X its geometric realization $|X|$, as the realization of this simplicial complex.

Waht we presented in the prevoios section is also true for posets and posets map. In fact, a poset map $f\colon X \to Y$ between the posets X and Y induces a continous map on the realizations $|f|\colon |X| \to |Y|$. Following the analogies in the previous section, if X and Y are two posets, we can consider the product $(X \times Y, \leq)$ where $(x, y) \leq (x', y')$ if $x \leq_X x'$ and $y \leq_Y y'$. If π_X and π_Y are the projection maps, they induce maps $|\pi_X|\colon |X \times Y| \to |X|$ and $|\pi_Y|\colon |X \times Y| \to |Y|$ between the realizations. Moreover, $|X \times Y|$ is homeomorphic to the product $|X| \times |Y|$.

We say that two posets are *homotopy equivalent* if the realizations of the associated simplicial complexes verify this property as topological spaces. Analogously, we say that a poset X is *d-connected* (or d-acyclic) if the topological realization of the simplicial complex $|X|$ is d-connected (or d-acyclic).

Remark 1.2.3. If X and Y are two posets, and $f, g\colon X \to Y$ are poset maps such that $f(x) \leq g(x)$ for all $x \in X$, then f and g are homotopic. In fact, as $f \leq g$ we can consider the poset map $\{0 < 1\} \times X \to Y$ induced by f and g; thanks to the homeomorphism between product spaces

$$|\{0 < 1\}| \times |X| \cong |\{0 < 1\} \times X|$$

we get the desired homotopy, as the realization of the poset $\{0, 1\}$ is the interval $I = [0, 1]$.

If X and Y are posets, the join $X * Y$ is defined as the poset with underlying set $X \sqcup Y$, and with the ordering $\leq_{X*Y}$ extending both the ordering of X and Y with the relation $x \leq_{X*Y} y$ for every $x \in X$ and $y \in Y$. One can see that the realization of the join of two posets as just defined coincide with the join of the associated realizations.

Remark 1.2.4. If X is a poset, we can consider the join with a single point $p \notin X$. The obtained poset is actually the cone CX with the relations $x \leq p$ for each $x \in X$.

In the same way, if X and Y are two posets, we can consider the join with a single point $p \notin X \cup Y$, $X * \{p\} * Y$. The associated simplicial complex is given by:

- the simplices $(x_1 \leq \cdots \leq x_k \leq y_1 \leq \cdots \leq y_h)$, where $x_i \in X$ and $y_j \in Y$, with possibly one of the two sequences $x_1 \leq \cdots \leq x_k$ and $y_1 \leq \cdots \leq y_h$ to be the void sequence;

- the simplices $(x_1 \leq \cdots \leq x_k \leq p \leq y_1 \leq \cdots \leq y_h)$ with possibly both the sequences to be void.

Hence, the realization $|X * \{p\} * Y|$ is actually a cone over $|X * Y|$ with top p.

One of the most important tool we will use is the *link*. Following [Maa79], Chapter 2, we want to analyze how one can use links in partially ordered sets.

Let X be a poset, let Y be a subset of X (so a subposet with the induced ordering) and let x be an element of X. We define:

$$\mathrm{Link}_Y(x) := \{y \in Y \mid y > x \text{ or } y < x\}$$

The link of an element is the union of $\mathrm{Link}^+_Y(x)$ and $\mathrm{Link}^-_Y(x)$, where $\mathrm{Link}^+_Y(x) := \{y \in Y \mid y > x\}$ and $\mathrm{Link}^-_Y(x) := \{y \in Y \mid y < x\}$, that means:

$$\mathrm{Link}_Y(x) = \mathrm{Link}^-_Y(x) * \mathrm{Link}^+_Y(x).$$

Remark 1.2.5. According to Remark 1.2.4, if X is a poset, $Y \subseteq X$ is a subposet, and x is an element of Y, then Y is obtained from $Y - \{x\}$ by attaching a cone on $\mathrm{Link}_Y(x) = \mathrm{Link}^-_Y(x) * \mathrm{Link}^+_Y(x)$ with top x.

In many proofs we will have to handle the problem to deduce connectivity results for a poset X from those of a subposet Y. With this goal in mind, it is useful to understand how the posets X and Y are related; before going further in this direction, we need another definition:

Definition 1.2.6. We say that a poset X is *discrete* if $x \leq x'$ implies $x = x'$ for all $x, x' \in X$. Equivalently, X is discrete if its realization is discrete as a topological space.

The following results can be found in [Maa79].

Proposition 1.2.7. *Consider posets $\emptyset \neq Y \subset X$ and suppose the complement poset $X \setminus Y$ is discrete. Then, the quotient $|X|/|Y|$ is given by a wedge of links, indexed by the elements in the complement of Y:*

$$|X|/|Y| \simeq \bigvee_{x \in X \setminus Y} \Sigma(|Link_Y(x)|).$$

Proof. By hypothesis, the poset $X - Y$ is discrete. Let $(x_1 \leq \cdots \leq x_k)$ be a simplex in the realization of X. Then, at most one of the x_i is an element of the complement $X \setminus Y$. This means that either $x_i \in Y$ for each i, or, if $x_i \notin Y$, then $x_j \in Y$ for each $j \neq i$.

Let $Y(x)$ be the subposet $Y \cup \{x\}$ when $x \in X \setminus Y$. Then, the poset X is the union of the posets $Y(x)$, and:

$$|X| = \bigcup_{x \in X \setminus Y} |Y(x)|;$$

if $x \neq x'$ are elements in $X \setminus Y$, then $|Y(x)| \cap |Y(x')| = |Y|$.

Collapsing $|Y|$ to a point, from the previous equality, we get:

$$|X|/|Y| \simeq \bigvee_{x \in X \setminus Y} (|Y(x)|/|Y|).$$

We have to better understand the quotient $|Y(x)|/|Y|$. The realization $|Y(x)|$ is obtained from $|Y|$ by adding the simplices $(y_1 \leq \cdots \leq y_h \leq x \leq y_{h+1} \leq \cdots \leq y_k)$; we are attaching over $|\mathrm{Link}_Y^-(x) * \mathrm{Link}_Y^+(x)|$ the space $|\mathrm{Link}_Y^-(x) * \{x\} * \mathrm{Link}_Y^+(x)|$:

$$\begin{aligned} |Y(x)| &= |Y| \bigcup_{\mathrm{Link}_Y^-(x) * \mathrm{Link}_Y^+(x)} |\mathrm{Link}_Y^-(x) * \{x\} * \mathrm{Link}_Y^+(x)| \\ &= |Y| \bigcup_{\mathrm{Link}_Y(x)} C|\mathrm{Link}_Y(x)| \end{aligned}$$

and factoring out $|Y|$, this is the same as attaching another cone over $|\mathrm{Link}_Y(x)|$. Therefore we get a suspension:

$$|Y(x)|/|Y| \simeq \Sigma|\mathrm{Link}_Y(x)|$$

and this concludes the proof. □

We conclude the section with an acyclicity result; we say that a poset X is n-acyclic if its realization is n-acyclic as a topological space, that means the i-th homology group is zero when $0 \leq i \leq n$.

Proposition 1.2.8. *Let $Y \subseteq X$ be two posets and assume the complement $X \setminus Y$ is discrete. If Y is n-acyclic and for each $x \in X \setminus Y$ the link $Link_Y(x)$ is $(n-1)$-acyclic, then X is also n-acyclic.*

Proof. Using the previous proposition we have $|X|/|Y| \simeq \vee_{x \in X - Y} \Sigma|\mathrm{Link}_Y(x)|$. Another way to look at this quotient is by attaching a cone over Y, obtaining the homotopy equivalence:

$$|X|/|Y| \simeq |X| \cup_{|Y|} C|Y|.$$

Therefore, we can use the Mayer Vietoris exact sequence associated to this union:

$$\cdots \to \widetilde{\mathrm{H}}_i(|Y|) \to \widetilde{\mathrm{H}}_i(|X|) \oplus \widetilde{\mathrm{H}}_i(C|Y|) \to \widetilde{\mathrm{H}}_i(|X|/|Y|) \to \widetilde{\mathrm{H}}_{i-1}(|Y|) \to \dots$$

The following relations

- $\widetilde{\mathrm{H}}_i(C|Y|) = 0$;
- $\widetilde{\mathrm{H}}_i(|X|/|Y|) \cong \bigoplus_{x \in X \setminus Y} \widetilde{\mathrm{H}}_i(\Sigma|\mathrm{Link}_Y(x)|)$;
- $\widetilde{\mathrm{H}}_i(\Sigma|\mathrm{Link}_Y(x)|) \cong \widetilde{\mathrm{H}}_{i-1}(|\mathrm{Link}_Y(x)|)$.

imply the statement. □

1.3 Elementary matrices and Congruence subgroups

In this section we give the definitions of elementary and congruence subgroups of $\mathrm{GL}_n(R)$, groups that we will study in Chapter 4 to get the acyclicity properties of suitable posets. We follow [Cha84] and [Vas69] for our definitions and results.

Let R be a fixed associative ring with unit and let $\mathfrak{a}$ be a two-sided ideal in R. We consider the right free R-module R^n, and we denote by $\mathrm{GL}_n(R) := \mathrm{Aut}_R(R^n)$ the automorphism group of R^n, or, once we choose a basis, the invertible matrices with entries in R. We identify the endomorphisms of R^n with the corresponding matrices.

The following notations are used:

- $\mathcal{M}_n(R)$ represents the ring of all $n \times n$ matrices over R; in the same way, $\mathcal{M}_n(\mathfrak{a})$ represents the ring of matrices with coefficients in the ideal $\mathfrak{a}$;
- I_n denotes the $n \times n$ identity matrix;
- $re_{i,j} =: e^r_{i,j}$ represents the matrix with the element $r \in R$ at the place (i, j), and zero elements in the other positions.

Definition 1.3.1. A matrix $A \in \mathcal{M}_n(R)$ is called *elementary* if it is of the form $I_n + e^r_{i,j}$, where $i \neq j$ and $r \in R$. It is called $\mathfrak{a}$-*elementary* if it is of the form $I_n + e^a_{i,j}$, where $i \neq j$ and $a \in \mathfrak{a}$.

Elementary matrices are invertible; we denote by $E_n(R)$ the subgroup of $\mathrm{GL}_n(R)$ generated by the elementary matrices, and by $E_n(R, \mathfrak{a})$ the subgroup of $E_n(R)$ generated by the $\mathfrak{a}$-elementary matrices.

Remark 1.3.2. The group of $\mathfrak{a}$-elementary matrices $E_n(R, \mathfrak{a})$ is normal in $E_n(R)$.

Definition 1.3.3. The group

$$\mathrm{GL}_n(R, \mathfrak{a}) := \ker(\mathrm{GL}_n(R) \to \mathrm{GL}_n(R/\mathfrak{a}))$$

is called *principal congruence subgroup of level* $\mathfrak{a}$. It consists of the invertible matrices of the form $I_n + M$, where $M \in \mathcal{M}_n(\mathfrak{a})$.

If Σ_n is the group generated by the permutation matrices in $\mathrm{GL}_n(R)$, we use the following notations

$$\widetilde{\mathrm{GL}}_n(R, \mathfrak{a}) \quad \text{and} \quad \widetilde{E_n}(R, \mathfrak{a})$$

for the subgroup generated by $\mathrm{GL}_n(R, \mathfrak{a})$ and the permutation group Σ_n, and for the subgroup generated by $E_n(R, \mathfrak{a})$ and Σ_n.

Let n, m be two integers with $n, m \geq 1$. The group $\mathrm{GL}_n(R)$ is a subgroup of $\mathrm{GL}_{n+m}(R)$ via the injective homomorphism:

$$\mathrm{GL}_n(R) \ni \alpha \longmapsto \alpha \oplus I_m := \begin{pmatrix} \alpha & 0 \\ 0 & I_m \end{pmatrix} \in \mathrm{GL}_{n+m}(R)$$

Then, the family of groups fits into a direct system, whose limit is the infinite general linear group:

$$\mathrm{GL}(R) := \bigcup_n \mathrm{GL}_n(R) = \mathrm{colim}\mathrm{GL}_n(R)$$

In the same way we consider the colimit of elementary matrices and congruence subgroups, with the direct system given by the upper left inclusion.

We conclude this section with the definition of $K_1(R)$. We need a technical lemma:

Lemma 1.3.4. *Let $X \in \mathcal{M}_n(\mathfrak{a})$, $Y \in \mathcal{M}_n(R)$ and suppose $I_n + XY \in GL_n(R, \mathfrak{a})$. Then:*

1. *$I_n + YX \in GL_n(R, \mathfrak{a})$;*
2. *the matrix*
$$\begin{pmatrix} I_n + XY & 0 \\ 0 & (I_n + YX)^{-1} \end{pmatrix}$$
is elementary and belongs to $E_{2n}(R, \mathfrak{a})$.
3. *$(I_n + XY)(I_n + YX)^{-1} \in E_{2n}(R, \mathfrak{a})$.*

Proof. By hypothesis, the matrix I_n+XY is invertible, so we can set $(I_n+XY)^{-1} = I_n+Z$, where Z has coefficients in $\mathfrak{a}$. Omitting the index n, we get

$$(I + Z)(I + XY) = I \Rightarrow I + XY + Z + ZXY = I \Rightarrow Z = -(X + ZX)Y,$$

$$(I + XY)(I + Z) = I \Rightarrow I + XY + Z + XYZ = I \Rightarrow Z = -X(Y + YZ).$$

1. By the relations of above, we get
$$\begin{aligned} [I - Y(X + ZX)]\,(I + YX) &= I + YX - Y(X + ZX)(I + YX) \\ &= I + YX - Y(X + ZX) - Y(X + ZX)YX \\ &= I + YX - YX - YZX + YZX = I. \end{aligned}$$
This implies that the matrix $I_n + YX$ is invertible, with inverse:
$$(I + YX)^{-1} = I - Y(X + ZX) = I - (Y + YZ)X.$$

2. First, we compute the following two products:
$$\begin{aligned} \begin{pmatrix} I & 0 \\ Y & I \end{pmatrix}\begin{pmatrix} I & -X \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ -Y & I \end{pmatrix} &= \begin{pmatrix} I & 0 \\ Y & I \end{pmatrix}\begin{pmatrix} I + XY & -X \\ -Y & I \end{pmatrix} \\ &= \begin{pmatrix} I + XY & -X \\ YXY & I - YX \end{pmatrix} \end{aligned}$$
$$\begin{pmatrix} I & X + XYX \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ YZ & I \end{pmatrix} = \begin{pmatrix} I + (X + XYX)YZ & X + XYX \\ YZ & I \end{pmatrix}$$
Because of the following relations:

- $Z = (I + XY)^{-1}$ implies
$$[I + (X + XYX)YZ]\,(I + XY) + (X + XYX)YXY = I + XY$$
- $Z = -X(YZ + Y)$ implies:
$$\begin{aligned} &-[I + (X + XYX)YZ]\,X + (X + XYX)(I - YX) \\ &= -XYZX - XYXYZX - XYXYX \\ &= -XYZX - [XYX(YZ + Y)X] = 0 \end{aligned}$$
- $Z = -(X + ZX)Y$ implies:
$$YZ(I + XY) + YXY = YZ + Y(ZX + X)Y = 0$$
- $I - (Y + YZ)X = (I + YX)^{-1}$

we get

$$\begin{pmatrix} I + (X + XYX)YZ & X + XYX \\ YZ & I \end{pmatrix} \begin{pmatrix} I + XY & -X \\ YXY & I - YX \end{pmatrix} = \begin{pmatrix} I + XY & 0 \\ 0 & (I + YX)^{-1} \end{pmatrix}$$

Observe now that the matrices in the product are elementary.

3. The matrix
$$\begin{pmatrix} I + XY & 0 \\ 0 & (I + YX)^{-1} \end{pmatrix}$$
is in $E_{2n}(R, \mathfrak{a})$ by the previous point. In the same way, by replacing X with $(I + YX)^{-1} - I$ and Y with I, the matrix
$$\begin{pmatrix} (I + YX)^{-1} & 0 \\ 0 & I + YX \end{pmatrix}$$
is elementary. The statement follows by making a product between them.

□

Proposition 1.3.5 (Whitehead's Lemma)**.** *The commutator subgroup*
$$[GL_n(R), GL_n(R, \mathfrak{a})]$$
is given by elementary matrices in $E_{2n}(R, \mathfrak{a})$*.*

Proof. By the previous lemma, the matrix $(I + XY)(I + YX)^{-1}$ is elementary. Let $A \in \mathrm{GL}_n(R)$ and $C \in \mathrm{GL}_n(R, \mathfrak{a})$ be two matrices. If we set $X = (A - I_n)C^{-1}$ and $Y = C$, then we get $[A, C] = (I_n + XY)(I_n + YX)^{-1} \in E_{2n}(R, \mathfrak{a})$. □

Lemma 1.3.6. *If $n \geq 3$ then $[E_n(R), E_n(R, \mathfrak{a})] = E_n(R, \mathfrak{a})$.*

Proof. Let $a \in \mathfrak{a}$. Then $I_n + e_{1,2}^a = [I_n + e_{1,3}, I_n + e_{3,2}^a] \in [E_n(R), E_n(R, \mathfrak{a})]$. $\square$

This lemma and Whitehead's Lemma imply, by passing to the direct limits, that:

$$E(R, \mathfrak{a}) = [E(R), E(R, \mathfrak{a})] = [\mathrm{GL}(R), \mathrm{GL}(R, \mathfrak{a})] \subseteq \mathrm{GL}(R, \mathfrak{a}).$$

This implies that $E(R, \mathfrak{a})$ is a normal subgroup of $\mathrm{GL}(R, \mathfrak{a})$. Define:

$$K_1(R, \mathfrak{a}) := \mathrm{GL}(R, \mathfrak{a})/E(R, \mathfrak{a}).$$

When $\mathfrak{a} = R$ the $K_1(R) := K_1(R, \mathfrak{a})$ is known as the first algebraic K-group of R.

1.4 Braided monoidal categories

In this section we give the definition of groupoids and braided monoidal categories, that we will use in Chapter 5. See [Mac71] for general references.

Definition 1.4.1. A *groupoid* $\mathcal{G}$ is a small category in which every arrow is invertible.

A category is monoidal when it comes equipped with a "product", like the usual direct product, direct sum or tensor product. Formally:

Definition 1.4.2. A *monoidal category* is a category $\mathcal{C}$ equipped with:

(i) a bifunctor $\oplus \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, the *monoidal product*;

(ii) an object 0, the *unit object*;

(iii) for any $A, B, C \in \mathcal{C}$, a natural isomorphism $\alpha_{A,B,C} \colon (A \oplus B) \oplus C \cong A \oplus (B \oplus C)$ that encodes the associativity, and for any object $A \in \mathcal{C}$ two natural isomorphisms $\lambda_A \colon 0 \oplus A \cong A$ and $\rho_A \colon A \oplus 0 \cong A$ saying that 0 is a right and left unit for the product. The three isomorphisms fit together in the following coherence conditions:

- for any A, B, C, D, objects in $\mathcal{C}$, we have the commutative diagram:

$$\begin{array}{ccccc}
((A \oplus B) \oplus C) \oplus D & \xrightarrow{\alpha_{A,B,C} \oplus 1_D} & (A \oplus (B \oplus C)) \oplus D & \xrightarrow{\alpha_{A,B \oplus C,D} \oplus 1_D} & A \oplus ((B \oplus C)) \oplus D) \\
\Big\downarrow{\scriptstyle \alpha_{A \oplus B,C,D}} & & & & \Big\downarrow{\scriptstyle 1_A \oplus \alpha_{B,C,D}} \\
(A \oplus B) \oplus (C \oplus D) & & \xrightarrow{\qquad \alpha_{A,B,C \oplus D} \qquad} & & A \oplus (B \oplus (C \oplus D))
\end{array}$$

- for any $A, B \in \mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc}
(A \oplus 0) \oplus B & \xrightarrow{\alpha_{A,0,B}} & A \oplus (0 \oplus B) \\
& {\scriptstyle \rho_A \oplus 1_B} \searrow \quad \swarrow {\scriptstyle 1_A \oplus \lambda_B} & \\
& A \oplus B &
\end{array}$$

If the monoidal structure is clear from the context, we briefly denote a monoidal category by $\mathcal{C}$.

We say that the monoidal category is *strict* if the natural isomorphism α, λ, ρ are equalities.

Definition 1.4.3. A *braided monoidal category* is a monoidal category equipped with a *braiding*. A *braiding*, in a monoidal category, consists of a family of isomorphisms $\gamma_{A,B} \colon A \oplus B \cong B \oplus A$ natural in both A and B, such that the following diagrams are commutative:

(i) for every object A:

$$\begin{array}{ccc} A \oplus 0 & \xrightarrow{\gamma_{A,0}} & 0 \oplus A \\ & {\scriptstyle \rho_A} \searrow \quad \swarrow {\scriptstyle \lambda_A} & \\ & A & \end{array}$$

(ii) for every objects A, B, C:

$$\begin{array}{ccc} (A \oplus B) \oplus C & \xrightarrow{\gamma_{A \oplus B,C}} & C \oplus (A \oplus B) \\ \uparrow {\scriptstyle \alpha^{-1}_{A,B,C}} & & \downarrow {\scriptstyle \alpha^{-1}_{C,A,B}} \\ A \oplus (B \oplus C) & & (C \oplus A) \oplus B \\ \downarrow {\scriptstyle 1_A \oplus \gamma_{B,C}} & & \uparrow {\scriptstyle \gamma_{A,C} \oplus 1_B} \\ A \oplus (C \oplus B) & \xrightarrow{\alpha^{-1}_{A,C,B}} & (A \oplus C) \oplus B \end{array} \qquad \begin{array}{ccc} A \oplus (B \oplus C) & \xrightarrow{\gamma_{A,B \oplus C}} & (B \oplus C) \oplus A \\ \uparrow {\scriptstyle \alpha_{A,B,C}} & & \downarrow {\scriptstyle \alpha_{B,C,A}} \\ (A \oplus B) \oplus C & & B \oplus (C \oplus A) \\ \downarrow {\scriptstyle \gamma_{A,B} \oplus 1_C} & & \uparrow {\scriptstyle 1_B \oplus \gamma_{A,C}} \\ (B \oplus A) \oplus C & \xrightarrow{\alpha_{B,A,C}} & B \oplus (A \oplus C) \end{array}$$

1.5 Spectral sequences

The main computational tool in homology stability theorems is an inductive argument that involves spectral sequences. We do not prove the existence of spectral sequences and the related classical theorems, but we briefly recall them, referring to [McC00] for details.

We focus our attention on the (augmented) first quadrant bicomplexes:

Definition 1.5.1. A first quadrant (augmented) bicomplex $(C_{p,q}, d_{p,q})$ of bidegree (a, b) is a bigraded R-module $C_{p,q}$ with differentials

$$d_{p,q} \colon C_{p,q} \longrightarrow C_{p+a,q+b}$$

where $C_{p,q} = 0$ for $p < 0$ and $q < 0$ (for $p < -1$ and $q < 0$ in the augmented case).

Thanks to the differentials $d_{p,q}$, we can calculate homology, and we get a new bigraded module. If we had a new differential on the homology bigraded module, we could continue by applying the homology functor. The spectral sequences encode this feature:

Definition 1.5.2. A *spectral sequence* (of homological type) is a collection of differential bigraded modules $\{(E^r_{p,q}, d^r)\}_{r\in\mathbb{N}}$ whose differentials d^r have bidegree $(-r, r-1)$, and for every pair (p, q) the bigraded module E^{r+1} is isomorphic to the homology of the complex (E^r, d^r):

$$E^{r+1}_{p,q} \cong \mathrm{H}_q((E^r_{*,*}, d^r))$$

The bicomplexes E^r are called *pages* of the spectral sequence.

If we work with first quadrant spectral sequences (this means the pages E^r are first quadrant bicomplexes) then for each pair (p, q) all differentials from and into $E^r_{p,q}$ vanish from the $(\max\{p, q\} + 2)$-th page on. Then every module $E^r_{p,q}$ stabilizes and we call this module $E^\infty_{p,q}$.

Spectral sequences naturally arise once one has a filtered chain complex:

Definition 1.5.3. A *filtered chain complex* is a sequence of subcomplexes $(F_pC)_p$ of a chain complex C such that $F_pC \subseteq F_{p+1}C$.

To any filtered chain complex there is an associated bigraded module:

$$\mathrm{Gr}_{p,q}C := F_pC_{p+q}/F_{p-1}C_{p+q}$$

If the filtration is bounded for $p < 0$ and $q < 0$, this is a first quadrant bicomplex. In fact, the differential of C induces differentials $d^0_{p,q} \colon \mathrm{Gr}_{p,q}C \to \mathrm{Gr}_{p,q-1}C$ of bidegree $(0, -1)$.

Definition 1.5.4. A spectral sequence (E^r, d^r) *converges* to a graded module H

$$E_{p,q} \Rightarrow H_{p+q}$$

if there is a filtration F on H such that $E^\infty_{p,q} \cong \mathrm{Gr}_{p,q}H$, where $E^\infty_{p,q}$ is the limit term.

If (C, d) is a chain complex and F_pC is a filtration of C

$$\dots \subseteq F_pC \subseteq F_{p+1}C \subseteq \dots$$

then we get an induced spectral sequence $(E^r_{p,q}, d^r)$. The idea of construction is the following:

- Given the filtered chain complex, we have the associated bigraded module $\mathrm{Gr}_{p,q}C$ whose differentials are induced from C; we set $E^0 = \mathrm{Gr}C$;
- E^0 splits in vertical chain complexes, then we calculate the homology groups:

$$E^1_{p,q} := \mathrm{H}_q(\mathrm{Gr}_{p,*}C)$$

- the differential d induces now horizontal differentials, of bidegree $(-1, 0)$;
- we iterate the procedure, and at every step the differential d induces new differentials with the right bidegree.

Remark 1.5.5. A filtration on a chain complex C induces a filtration on the homology module $\mathrm{H}(C)$:

$$F_p\mathrm{H}(C) := \mathrm{Im}(\mathrm{H}_{p+q}(F_pC) \longrightarrow \mathrm{H}_{p+q}(C))$$

We can give the general result:

Theorem 1.5.6. *Let (F_pC) be a bounded filtration of a chain complex (C,d). Then there is a spectral sequence (of homological type) (E^r, d^r) converging to the homology of the complex:*

$$E_{p,q} \Rightarrow H_n(C)$$

We now specialize the result to double chain complexes. We recall that a double chain complex $(C, \partial^v, \partial^h)$ is a commutative diagram of R-modules in which every column and every row is a chain complex:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle \partial^v} & & \downarrow{\scriptstyle \partial^v} & & \\
\cdots & \xleftarrow[\partial^h]{} & C_{p-1,q} & \xleftarrow[\partial^h]{} & C_{p,q} & \xleftarrow[\partial^h]{} & \cdots \\
 & & \downarrow{\scriptstyle \partial^v} & & \downarrow{\scriptstyle \partial^v} & & \\
\cdots & \xleftarrow[\partial^h]{} & C_{p-1,q-1} & \xleftarrow[\partial^h]{} & C_{p,q-1} & \xleftarrow[\partial^h]{} & \cdots \\
 & & \downarrow{\scriptstyle \partial^v} & & \downarrow{\scriptstyle \partial^v} & & \\
 & & \vdots & & \vdots & &
\end{array}$$

this means $\partial^v \circ \partial^v = 0$, $\partial^h \circ \partial^h = 0$ and $\partial^v \circ \partial^h = \partial^h \circ \partial^v$. We consider first quadrant double chain complexes, so $C_{p,q} = 0$ if p or q are negative integers.

One can associate to a double chain complex an ordinary chain complex, called its *total complex* $\mathrm{Tot}(C)$, defined as follows:

$$\mathrm{Tot}(C)_n := \bigoplus_{p+q=n} C_{p,q}$$

the differential ∂ is defined as $\partial := \partial^h + (-1)^p \partial^v$.

There are two filtrations of $\mathrm{Tot}(C)$ one can consider. The first filtration F^I is defined by cutting the sum of above at the p-level:

$$F^I_p(\mathrm{Tot}(C))_n := \bigoplus_{i \le p} C_{i,n-i}$$

The second filtration F^{II} is the dual:

$$F^{II}_q(\mathrm{Tot}(C))_n := \bigoplus_{j \le q} C_{n-j,j}$$

In the special case of a first quadrant double chain complex, the filtrations of the induced total complex are bounded; then, by the theorem of above, both the filtrations converge to the homology of the total complex.

We remark that the 0-page of the spectral sequence arising from the first filtration is

$$IE^0_{p,q} := F^I_p(\mathrm{Tot}(C))_{p+q}/F^I_{p-1}(\mathrm{Tot}(C))_{p+q} = \oplus_{i\leq p} C_{i,p+q-i}/\oplus_{i\leq p-1} C_{i,p+q-i} = C_{p,q}$$

and for the second filtration is:

$$IIE^0_{p,q} := F^{II}_p(\mathrm{Tot}(C))_{p+q}/F^{II}_{p-1}(\mathrm{Tot}(C))_{p+q} = \oplus_{j\leq p} C_{p+q-j,j}/\oplus_{j\leq p-1} C_{p+q-j,j} = C_{q,p}$$

The differentials are respectively induced by ∂^v and by ∂^h.

The first page is now given by:

$$IE^1_{p,q} = \mathrm{H}_q(C_{p,*})$$

where the differential d^1 is ∂^h, and

$$IIE^1_{p,q} = \mathrm{H}_q(C_{*,p})$$

and the differential is ∂^v.

2

Homological stability and first examples

Contents

The aim of this chapter is to describe what we mean by *homological stability* for a family of discrete groups, and to point out what we need in the proofs. A really large class of families satisfies homological stability, for example symmetric groups (see [Nak61]), braid groups (see [Arn69]), general linear groups (see [vdK80]), mapping class groups of (non)-orientable surfaces (see [Wah11] and [Wah08]), mapping class groups of compact orientable 3-manifolds (see [HaW10]), symplectic groups (see [MavdK01]), orthogonal groups (see [Vog79]), and so on. In literature one can easily find many proofs of such results, but the idea to prove them is generally the same. Hence, to make clear this idea, we decided to study homological stability of *braid groups* as an easy example.

Then, following [Wah11], we give the general ideas about homological stability for *mapping class groups* of surfaces. It happens that a family of groups, like the family of mapping class groups, is described by more then one parameter. For these families of groups, it is not so clear how one can prove homological stability results; hence, we try to show how to proceed in those cases by looking at the ideas explained by N. Wahl.

The chapter has been structured as follows. In the first section, after giving the general definition of homological stability, we collect the principal known results for general linear groups. In the second section we study the easy example of braid groups, and in the third section we collect some useful facts to prove the stability theorem for the mapping class groups of orientable surfaces. In the last section, the acyclicity of some simplicial complexes is studied, in order to coplete the homological stability proofs.

2.1 Homological stability: an overview

Let G be a discrete group. As described in [Bro82], one can use two different definitions for the homology of G, that are equivalent for a large class of groups. In this thesis we use the algebraic one: the homology $\mathrm{H}_*(G;\ \mathbb{Z})$ of the group G with integer coefficients is the homology of the chain complex $E_*G \otimes_G \mathbb{Z}$, where E_*G is a projective resolution of $\mathbb{Z}$ in $\mathbb{Z}[G]$-modules. If H is a subgroup of G, the inclusion map $H \hookrightarrow G$ induces a map between the group rings $\mathbb{Z}[H] \hookrightarrow \mathbb{Z}[G]$. Then, if E_*G is a free resolution, E_*G becomes, by restriction, a free resolution (hence, projective) of $\mathbb{Z}$ in $\mathbb{Z}[H]$-modules, and the chain complex E_*G can be used to compute both the homologies $\mathrm{H}_*(G;\ \mathbb{Z})$ and $\mathrm{H}_*(H;\ \mathbb{Z})$.

Lemma (I.7.4) in [Bro82] yealds an augmentation-preserving $\mathbb{Z}[H]$-chain map between the two resolutions, that is well-defined up to homotopy. This chain map induces, by factoring out the tensor relations with $\mathbb{Z}[G]$-scalars, a map $E_*G \otimes_H \mathbb{Z} \to E_*G \otimes_G \mathbb{Z}$, well-defined up to homotopy, hence a map in homology:

$$\mathrm{H}_*(H;\ \mathbb{Z}) \longrightarrow \mathrm{H}_*(G;\ \mathbb{Z}).$$

It is interesting to understand whether this induced map is an isomorphism, or if there is a range $0 \leq i \leq n$ in which the maps $\mathrm{H}_i(H;\ \mathbb{Z}) \longrightarrow \mathrm{H}_i(G;\ \mathbb{Z})$ are isomorphisms. Homological stability deals with these kind of questions:

Definition 2.1.1. A family of discrete groups $\{G_n\}_{n\in\mathbb{N}}$ together with homomorphisms $\mathrm{stab}_n\colon G_n \hookrightarrow G_{n+1}$ satisfies *homological stability* (with integer coefficients) if the induced maps in homology

$$\mathrm{H}_i(\mathrm{stab}_n)\colon \mathrm{H}_i(G_n;\mathbb{Z}) \longrightarrow \mathrm{H}_i(G_{n+1};\mathbb{Z})$$

are isomorphisms for $0 \leq i \leq f(n)$, where $f\colon \mathbb{N} \to \mathbb{N}$ is a map increasing with n, i.e. $f(n+1) \geq f(n)$.

As described above, homological stability is a propriety satisfied for many families of groups, but why do we study homological stability?

The motivation arises because the homology functor commutes with the direct limit, and if a family of groups satisfies homological stability, then we get information on the homology of the colimit. Vice versa, if we know how the direct limit behaves in homology, then we can deduce homological information for the groups in the direct system. For an example of application we can cite the Mumford Conjecture, saying that the stable rational cohomology ring of the moduli space of Riemann surfaces is a polynomial algebra; in fact, this conjecture has been proved by showing that the mapping class groups satisfy homological stability. For a reference, see the original article [MaW07].

The broadest interest in homological stability arose in algebraic K-theory, because Quillen's definition of K-groups for a ring involves the knowledge of homotopy groups for the classifying space $B\mathrm{GL}(R)$. This is why we decided to give a detailed proof when the family consists of general linear groups with coefficients in a suitable ring. With this goal, it is interesting to have a short look at the various results that have been obtained in the case of general linear groups. In fact, this kind of stability has been studied by lots of mathematicians, first of all by D. Quillen, who proves in the unpublished notes [Qui74] that stability is satisfied when the coefficients belong to a field $\mathbb{F}_p \neq \mathbb{F}_2$. Borel showed in [Bor74] that we get rational stability when R is a ring of integers in a number field, but the first general published result, with a form of stability, is a theorem of Wagoner that one can find in [Wag76]:

Theorem 2.1.2 (Wagoner, 1976)**.** *Let R be any local ring and let $n \geq 0$. Then*

$$H_i(GL_{md}(R);\mathbb{Z}) \longrightarrow H_i(GL_{(m+1)d}(R);\mathbb{Z})$$

is an isomorphism provided $d \geq n+3$ and $m \geq 0$ satisfies $m > (2n+3)d$.

In Theorem 3.1 of [Cha79], she proves that the stability can be proved for Dedekind domains and principal ideal domains:

Theorem 2.1.3 (Charney, 1979)**.** *Let R be a principal ideal domain. Then*

$$H_i(GL_{n+1}(R), GL_n(R);\ \mathbb{Z}) = 0$$

for $n \geq 3i$.

Using a different approach, Maazen in [Maa79] proves the following better result for subrings of $\mathbb{Q}$:

Theorem 2.1.4 (Maazen, 1979)**.** *Let R be a local ring or a subring of $\mathbb{Q}$. Then*

$$H_i(GL_n(R);\ \mathbb{Z}) \longrightarrow H_i(GL_{n+1}(R);\ \mathbb{Z})$$

is an isomorphism for $n > 2i$ and surjective for $n = 2i$.

In [Dwy80], Theorem 2.2, we get the first result with twisted coefficients, that keeps tools from both [Cha79] and [Maa79]. He shows that for general choices of coefficient systems the homology groups stabilize. The general result that involves the definition of *stable range condition* we will give in Definition 3.1.3 is due to van der Kallen in [vdK80]. He proves the following theorem:

Theorem 2.1.5 (van der Kallen, 1980)**.** *Let R be an associative ring with identity; suppose R is a ring of stable range $sdim + 2$. Let $e := \max\{1, sdim\}$; then*

$$H_i(GL_n(R)) \longrightarrow H_i(GL_{n+1}(R))$$

is surjective for $n \geq 2i + e - 1$ and injective for $n \geq 2i + e$.

Following a different approach, Suslin proves in [Sus82] the same result as van der Kallen for rings that satisfy the stable range condition:

Theorem 2.1.6 (Suslin, 1982)**.** *Let R be a ring of stable range $sdim + 2$. Then*

$$H_i(GL_n(R)) \longrightarrow H_i(GL_{n+1}(R))$$

is surjective for $n \geq \max\{2i, sdim+i\}$ and it is bijective for $n \geq \max\{2i+1, sdim+i+1\}$.

Since the eighties many other results have been obtained for a quite large class of linear groups; an overview can be found in [Ess13] and [Knu01]. Even in recent years mathematicians keep on studying homological stability: in this thesis we follow the general result obtained by N. Wahl in [Wah14], and we deduce the stability of general linear groups from that of automorphisms of objects in a homogenous category. We describe this proof in the last chapter.

We conclude this section with the description of the general argument to prove homological stability, known as *Quillen's argument.* Given a family of groups

$$\ldots \longrightarrow G_n \longrightarrow G_{n+1} \longrightarrow \ldots$$

his general strategy to obtain stability results is to find, for each $n \in \mathbb{N}$, a simplicial complex X_n with a natural G_n-action, such that the following are satisfied:

1. X_n is a highly connected G_n-simplicial complex, i. e. $\pi_k(X_n) = 0$ in a certain range $0 \leq k \leq m$, with m increasing with n;

2. the simplicial action of G_n on X_n is transitive on the set of p-simplices, for every p;

3. the stabilizer of a p-simplex σ_p is isomorphic to G_k, with $k < n$;

Now, X_n yields an highly acyclic chain complex C_*X_n; tensoring it with a $\mathbb{Z}[G_n]$-free resolution of $\mathbb{Z}$, say E_*G_n, gives a double complex $E_*G_n \otimes_{G_n} C_*X_n$. The spectral sequence associated to this double complex converges to 0 due to the high connectivity of X_n, and analyzing this spectral sequence one obtains information on the homology groups and relates the homology of G_n with the homology of the stabilizer. Assuming other technical hypothesis, homological stability follows, as A. Hatcher and N. Wahl prove in [HaW10], Theorem 5.1.

2.2 Braid groups

In this section we show that the family of braid groups satisfies homological stability. We follow [Dam13], [FaM] and [Tra14] for definitions, theorems and further remarks.

Let β_n be the braid group of index n. It is well known that β_n has a description in terms of the mapping class group of a disc. To be more precise, we consider:

- a disc $\mathbb{D}^2 \subseteq \mathbb{R}^2$;

- a chosen point q in the boundary $\partial\mathbb{D}^2$;

- a set $S = \{p_1, \dots, p_n\}$ of n *distinct marked* points p_i in the interior of $\mathbb{D}^2$.

Let $D := \mathbb{D}^2 \setminus S$ be complement of S in $\mathbb{D}^2$. The braid group β_n is defined as the mapping class group of D, that means, as the set of isotopy classes of orientation-preserving diffeomorphisms of $\mathbb{D}^2$ that restrict to the identity on $\partial\mathbb{D}^2$ and fix set-wise the marked points; we remark that also the isotopies are supposed to fix the set of marked points set-wise.

Remark 2.2.1. Suppose we forget about the set of marked points. We can also define the mapping class group as the isotopy classes of homeomorphisms that restrict to the identity on the boundary (with isotopies being homeomorphisms at every level); when we work with surfaces, the two definitions give two groups that are isomorphic. Suppose we are considering the homoeomorphism classes. An isotopy class in β_n is represented by a homeomorphism of the disc that restricts to the identity on the boundary. Such a homeomorphism is actually isotopic to the identity, by the Alexander trick. If we follow the path of a marked point trough this isotopy we get a string, hence the usual interpretation of braid groups.

Remark 2.2.2. The set of diffeomorphisms of $\mathbb{D}^2$ that restrict to the identity on $\partial\mathbb{D}^2$ has the structure of a topological group. In fact the following:

$$d(f, g) := \sup_{p \in \mathbb{D}^2} |f(p) - g(p)| = \max_{p \in \mathbb{D}^2} |f(p) - g(p)|$$

is a metric. With the induced topology, two such diffeomorphisms are connected if and only if there exists an isotopy between them. Therefore, the isotopy classes of diffeomorphisms are the connected components of this topological group.

If we choose two different sets of points in $\mathbb{D}^2$, there is an isomorphism between the mapping class groups, and one can see that the homomorphisms $\beta_n \hookrightarrow \beta_{n+1}$ induced by adding a marked point in $\mathbb{D}^2$ are well-defined. The family of braid groups, together with these stabilization maps, satisfies homological stability. To prove it, we first need to construct β_n-simplicial complexes, the *arc complexes* as defined by Harer.

By an *arc* we mean, in this context, an embedded smooth path $a\colon I \to \mathbb{D}^2$, such that:

- $a(0) = q$;
- $a(1) \in S$;
- $a(t)$ belongs to the interior of $\mathbb{D}^2$ for every $t \in (0, 1)$;
- $a(t)$ meets the boundary at q transversely.

A collection of arcs $\{a_0, \dots, a_p\}$, is *non-intersecting* if $a_i(I) \cap a_j(I) = q$ for every $i \neq j$. More generally, we work with isotopy classes of arcs, where the isotopies are assumed to be arcs at each level. A collection of such isotopy classes of arcs is non-intersecting if there is a collection of representatives that is non-intersecting. We can define the *arc complex*:

Definition 2.2.3. The *arc complex* $\mathcal{A}_n$ is the simplicial complex defined by the following sets of simplices:

$(\mathcal{A}_n)_0$ is the set of isotopy classes of arcs;

$(\mathcal{A}_n)_p$ is the set of collections of $(p+1)$ isotopy classes of arcs that are non-intersecting.

It is useful to define an order relation on the set of arcs: if a and b are two arcs at q, we say $a \leq b$ if the tangent vectors at q are ordered counterclockwise (the arcs are thought to be transversal at q). This order relation can be extended to the isotopy classes of arcs, as the diffeomorphisms are orientation-preserving. The simplices of $\mathcal{A}_n$ have now the form $\sigma = \langle \alpha_0, \dots, \alpha_p \rangle$, with $\alpha_0 \leq \dots \leq \alpha_p$: the complex $\mathcal{A}_n$ is an *ordered* simplicial complex.

Observation 2.2.4. The complex $\mathcal{A}_n$ has dimension $n-1$.

A diffeomorphism of $\mathbb{D}^2$ that restricts to the identity on $\partial\mathbb{D}^2$ and fixes the marked points set-wise acts by composition on the set of arcs; up to isotopy classes, we get an action of the group β_n on the set of vertices $(\mathcal{A}_n)_0$. Moreover, these diffeomorphisms preserve the non-intersecting property on the collections of arcs. The previous action extends in this way to an action of β_n on the whole complex $\mathcal{A}_n$. This remark allows us to give to the complexes $\mathcal{A}_n$ the structure of β_n-complexes, with an action that is simplicial. The following theorem, that will be proved in the last section, is deduced by considerations on a bigger complex. It states the high connectivity result:

Theorem 2.2.5. *The arc complex $\mathcal{A}_n$ is $(n-2)$-connected.*

The next step is to prove that the action on the set of p-simplices is transitive. For a proof see Poposition 2.2 in [Wah11], or Lemma 2.16 in [Tra14].

Proposition 2.2.6. *The action of the braid group β_n on the set of p-simplices of the arc complex $\mathcal{A}_n$ is transitive.*

Proof. We want to show that, for each pair of p-simplices, there is a class in β_n taking the first simplex to the second one. Let $\sigma = \langle \alpha_0, \dots, \alpha_p \rangle$ and $\sigma' = \langle \alpha'_0, \dots, \alpha'_p \rangle$ be two ordered p-simplices. For any $\varepsilon > 0$, we can choose representatives $\{a_0, \dots, a_p\}$ and $\{a'_0, \dots, a'_p\}$, such that:

- every arc a_i and every arc a'_i, in the interval $[0, \varepsilon]$, is a straight path;
- each pair (a_i, a'_i) is such that $a_i|_{[0,\varepsilon]} = a'_i|_{[0,\varepsilon]}$, for every $i = 0, \dots, p$;
- the collections $\{a_0, \dots, a_p\}$ and $\{a'_0, \dots, a'_p\}$ are non-intersecting.

Consider the first collection $\{a_0, \dots, a_p\}$, and call $\tilde{a}_i$ the path in $\mathbb{D}^2$ $\tilde{a}_i(t) := a_i(t\varepsilon)$. For every $i = 0, \dots, p$ there exists a closed set D_i, homeomorphic to a disc, such that $a_i(I) \subseteq D_i$. Up to restricting these sets, we can also suppose $D_i \cap D_j = \{q\}$ for every

$i \neq j$; this can be done because, by construction, the arcs are straight paths until ε, and after this point the distance between them has to be positive.

We restrict our attention to D_i. There exists an isotopy $h_i\colon I \times I \to D_i$ between the arc a_i and the arc $\tilde{a}_i$, following a_i. By the Isotopy Extension Theorem, there exists an isotopy $\tilde{h}_i$, with compact support in D_i, that restricts to the identity on a neighbourhood of ∂D_i, and such that it extends the previous isotopy. We can extend $\tilde{h}_i$ to an isotopy H_i defined on $\mathbb{D}^2 \setminus (\cup_{j\neq i} D_j)$ by setting $H_i = 1$ outside D_i. Gluing them together we get an ambient isotopy H, taking every arc a_i to the respective $\tilde{a}_i$.

Let K be the second ambient isotopy, taking every arc a_i' to the respective arc $\tilde{a}_i'$. The composition $K_1^{-1} \circ H_1$ is a diffeomorphism that switches the two collections of arcs. The smart observation is that this diffeomorphism can move the marked points that are not in the image of $a_0, \dots, a_p$. To solve the problem, one can just complete the simplices σ and σ' to maximal simplices in $\mathcal{A}_n$ and then repeat the argument. □

We have highly connected β_n-complexes and we know the action is transitive on the set of p-simplices. What is left is to understand how the stabilizers behave. Since the action is transitive, the stabilizers are all conjugate. It follows that we can study the stabilizer of just one p-simplex. The next proposition can be found in [Tra14], Lemma 2.15.

Proposition 2.2.7. *Let σ be a p-simplex of $\mathcal{A}_n$. Then the stabilizer $St_{\beta_n}(\sigma)$ is isomorphic to β_{n-p-1}.*

Proof. Suppose the p-simplex σ is of the form $\langle \alpha_0, \dots, \alpha_p \rangle$, and consider representatives $a_0, \dots, a_p$ that are non-intersecting. Denote by H the group of diffeomorphisms of $\mathbb{D}^2$ that restrict to the identity on $\partial\mathbb{D}^2$, fix S set-wise and fix the arcs $a_0, \dots, a_p$ pointwise. We want to prove that the isotopy classes in H correspond to those in $\mathrm{St}_{\beta_n}(\sigma)$.

First we prove that, if $[\phi]$ is an isotopy class in β_n with $[\phi] \in \mathrm{St}_{\beta_n}(\sigma)$, then it has a representative $h \in H$. If this is true, the homomorphism $\pi_0(H) \to \mathrm{St}_{\beta_n}(\sigma)$ that sends the isotopy class $[h] \in H$ to the corresponding isotopy class $[h] = [\phi]$ in β_n, is surjective.

Suppose we have fixed a representative $\phi \in [\phi]$. As it is in $\mathrm{St}_{\beta_n}(\sigma)$, up to isotopy, it takes the simplex σ in itself. This means there exists a permutation θ on $p+1$ elements, and isotopies h_i's, such that $\phi(a_i) \simeq_{h_i} a_{\theta(i)}$. We want to prove that ϕ is actually isotopic to a map that fixes the arcs pointwise, and that the permutation θ is actually the trivial permutation.

First, consider the arc a_0 and the isotopy h_0 between $\phi(a_0)$ and the arc $a_{\theta(0)}$. We can extend this isotopy to an ambient isotopy, say H_0, with $H_0(0, \mathbb{D}^2) = \mathrm{id}_{\mathbb{D}^2}$ and $H_0(\phi(a_0), -) = h_0$. If we consider the composition $H_0(1, -) \circ \phi$, we get a map ϕ_1 that is isotopic to ϕ and such that $\phi_1(a_0) = a_{\theta(0)}$.

Proceed now by induction and suppose we have a map ϕ_i that is isotopic to ϕ and such that

$$\phi_i(a_j) = a_{\theta(j)} \text{ for every } j = 0, \dots, i-1.$$

There exists an isotopy h'_i such that $\phi_i(a_i)$ is isotopic to $a_{\theta(i)}$, but we can not repeat the same argument above, because if we extend the isotopy h'_i to an ambient isotopy, then we move also the arcs $a_{\theta(0)}, \dots, a_{\theta(i-1)}$ losing the equality $\phi_i(a_j) = a_{\theta(j)}$. The idea is then to make h'_i disjoint from these arcs. To do it, approximate h'_i with a transverse map respect to $a_{\theta(0)}, \dots, a_{\theta(i-1)}$, say $\tilde{h}_i$, and suppose it is disjoint to $a_{\theta(0)}, \dots, a_{\theta(k-1)}$.

Cut along the arcs $a_{\theta(0)}, \dots, a_{\theta(k-1)}$ and consider $\widetilde{h}_i^{-1}(a_{\theta(k)})$; this is a 1-submanifold of $I \times I$, so union of arcs and circles. The intersection with the 0-level and the 1-level $a_{\theta(k)} \cap (\tilde{h}_i(0,-) \cup \tilde{h}_i(1,-)) = a_{\theta(k)} \cap (a_i \cup a_{\theta(i)})$ is just the fixed point point q, as the arcs $a_0, \dots, a_p$ are disjoint in the interiors, the arcs $a_{\theta(0)}, \dots, a_{\theta(i-1)}$ coincide with a subset of $a_0, \dots, a_p$, and we are operating with isotopies, that are in particular bijections. Moreover, there are no circle intersections, as the second homotopy group of the cutted surface is trivial. Therefore, the inverse image is given by horizontal curves. An horizontal segment gives us an isotopy between $a_{\theta(k)}$ and $a_{\theta(i)}$, that can not happen as they are in different isotopy classes.

Then, $\tilde{h}_i$ can be taken disjoint from $a_{\theta(0)}, \dots, a_{\theta(k)}$ and, by repeating the argument, we make it disjoint from $a_{\theta(0)}, \dots, a_{\theta(i-1)}$. Now we can extend the isotopy $\widetilde{h}_i$ to an ambient isotopy in $S \setminus (a_{\theta(0)}, \dots, a_{\theta(i-1)})$, and then extending it to the whole S, concluding the surjectivity assumption.

If the map $\pi_0(H) \to \mathrm{St}_{\beta_n}(\sigma)$ is also injective we get the desired isomorphism with β_{n-p-1}, as cutting along σ gives us an identification of $\pi_0(H)$ with β_{n-p-1}. To check injectivity, if I is a 0-simplex, consider the fibration:

$$\mathrm{Diffeo}(\mathbb{D}^2, \mathrm{rel}\ \partial S \cup I) \longrightarrow \mathrm{Diffeo}(\mathbb{D}^2, \mathrm{rel}\ \partial S) \longrightarrow \mathrm{Emb}((I,0,1),(\mathbb{D}^2,q,n))$$

where $\mathrm{Emb}((I,0,1),(\mathbb{D}^2,q,n)$ is the space of embeddings of the interval in $\mathbb{D}^2$, having endpoints q and a marked point. The fibration is induced by restricting a diffeomorphism to the given arc.

By Theorem 5 in [Gra73], $\pi_1(\mathrm{Emb}((I,0,1),(\mathbb{D}^2,q,n)) = 0$, and injectivity follows by the induced homotopy exact sequence. □

Remark 2.2.8. For every edge in $\mathcal{A}_n$, with vertices represented by arcs a and b, there exists an element of β_n that takes a to b, and such that it commutes with all the elements of β_n leaving the edge fixed pointwise. In fact, the action of β_n that switches the arcs a and b is supported in a tubular neighbourhood of $a \cup b$, and each isotopy class in β_n that has a representative which fixes the arcs pointwise, has also a representative that fixes the tubular neighbourhood.

We have all the ingredients to prove the stability result. The spectral sequences argument is the same as in Theorem 5.2.6, so we give just the statement:

Theorem 2.2.9. *The braid groups satisfy homological stability.*

2.3 A survey on Mapping class groups

In this section we want to point out the ideas to prove homological stability for mapping class groups of orientable closed surfaces. The first proof of this result is due to Harer in [Har85], but we follow the improved and easier one, obtained by Wahl in [Wah11].

Let $S_{g,r}$ be an oriented closed surface with g holes and $r > 0$ boundary components, and denote by $\Gamma_{g,r} := \Gamma(S_{g,r})$ the mapping class group of the surface $S_{g,r}$, where the diffeomorphisms restrict to the identity on the boundary. Gluing a strip along one or two boundary components we get inclusions:

$$\alpha\colon S_{g,r+1} \hookrightarrow S_{g+1,r} \quad \beta\colon S_{g,r} \hookrightarrow S_{g,r+1}$$

By extending the diffeomorphisms to the identity on the glued strip, we get also maps on mapping class groups. The following theorem states the stability result:

Theorem 2.3.1. *Let $g \geq 0$ and $r \geq 1$. The map*

$$H_*(\Gamma(S_{g,r+1}), \mathbb{Z}) \longrightarrow H_*(\Gamma(S_{g+1,r}), \mathbb{Z})$$

is an isomorphism for $* \leq 2/3g + 1/3$, and the map

$$H_*(\Gamma(S_{g,r}), \mathbb{Z}) \longrightarrow H_*(\Gamma(S_{g,r+1}), \mathbb{Z})$$

is an isomorphism for $* \leq 2/3g$.

In fact, it implies the following one:

Corollary 2.3.2. *The following map:*

$$H_i(\Gamma(S_{g,r}), \mathbb{Z}) \longrightarrow H_i(\Gamma(S_{g+1,r}), \mathbb{Z})$$

is an isomorphism for $3i \leq 2g + 1$.

The general argument in homological stability proofs needs a complex X_n associated to each group of the family. In this case, the groups $\Gamma_{g,r}$ are indexed by two parameters, but the argument is the same. In fact, we construct a pair of simplicial complexes for each group, and then we study the relations between them. To do it, we need to generalize what we described in the previous section; we begin with the definition of an arc:

Definition 2.3.3. An *arc* in $S_{g,r}$ is an embedded arc $\alpha\colon [0,1] \to S_{g,r}$ intersecting transversely ∂S only at its endpoints.

As in the previous section, we work with isotopy classes of arcs, where the isotopies are assumed to fix the endpoints. Moreover, we choose two distinct points b_0, b_1 in the boundary of $S := S_{g,r}$, and we suppose that the arcs have endpoints $\{b_0, b_1\}$. Moreover, the endpoints have to be dinstinct, so we do not consider loops.

Definition 2.3.4. A collection of arcs with disjoint interiors $\{a_0, \ldots, a_p\}$ is called *non-separating* if its complement $S \setminus (a_0 \cup \cdots \cup a_p)$ is connected.

We remark that the surface S is oriented, hence its orientation induces an ordering on such collections of arcs at b_0 and at b_1. The complexes are defined as follows:

Definition 2.3.5. Let $\mathcal{O}(S, b_0, b_1)$ be the simplicial complex whose set of vertices is given by the isotopy classes of non-separating arcs with boundary $\{b_0, b_1\}$.

A *p-simplex* is a collection of $p+1$ distinct isotopy classes of arcs $\langle a_0, \ldots, a_p\rangle$ that can be represented by a collection of arcs with disjoint interiors, which is non-separating and such that the anticlockwise ordering of $a_0, \ldots, a_p$ at b_0 agrees with the clockwise ordering at b_1.

There are two different kind of complexes $\mathcal{O}(S, b_0, b_1)$, depending on whether b_0 and b_1 are in the same boundary component. We denote by $\mathcal{O}^1(S)$ the simplicial complex where they are in the same boundary component, and $\mathcal{O}^2(S)$ the second one. These simplicial complexes are highly connected, and this connectivity depends just on the genus g. In the last section we give a partial proof of the following:

Theorem 2.3.6. *The complexes $\mathcal{O}^i(S_{g,r})$ are $(g-2)$-connected.*

The action of the mapping class group $\Gamma(S)$ on the surface S induces an action on the simplicial complexes $\mathcal{O}^i(S, b_0, b_1)$ that is simplicial. According to Quillen's argument, following [Wah11], Proposition 2.2, we prove it is transitive on p-simplices and the stabilizers are groups of the family:

Proposition 2.3.7. *Let $i = 1$ or 2; then:*

1. *$\Gamma(S_{g,r})$ acts transitively on the set of p simplices of $\mathcal{O}^i(S_{g,r})$;*
2. *there exist isomorphisms:*

$$s_1 \colon St_{\mathcal{O}^1}(\sigma_p) \to \Gamma(S_{g-p-1,r+p+1})$$

$$s_2 \colon St_{\mathcal{O}^2}(\sigma_p) \to \Gamma(S_{g-p,r+p-1})$$

where $St_{\mathcal{O}^i}(\sigma_p)$ is the stabilizer of the p-simplex σ_p.

Proof. We study both the cases $i = 1$ and $i = 2$ and, for simplicity, we denote by $\mathcal{O}^i(S)$ the generic simplicial complex for $S_{g,r}$.

1. Let $\sigma = \langle a_0, \ldots, a_p\rangle$ be a p-simplex in $\mathcal{O}^i(S)$, represented by arcs $a_0, \ldots, a_p$ with disjoint interiors. Denote by $S \setminus \sigma := S \setminus (N_0 \cup \cdots \cup N_p)$ the cutted surface along σ, where N_j is a small open neighbourhood of the arc a_j. We remark that a cellular decomposition of $S \setminus \sigma$ is obtained from a cellular decomposition of S by doubling the arcs a_j. Hence we get $\chi(S \setminus \sigma) = \chi(S) + p + 1$.

To better understand the cutted surface $S \setminus \sigma$, it is convenient to count its boundary components. In addition to the $r - i$ boundary components that do not contain $\{b_0, b_1\}$, we proceed as follows. If $i = 1$, the points b_0 and b_1 belong to the same boundary component. We double the arcs a_j; let $\bar{a}_j$ be the doubling of the arc a_j, and denote by $\partial_0 S$ and $\partial_1 S$ the two arcs between b_0 and b_1 whose union is the boundary component where they sit. Then, we label the new boundary components in $S \setminus \sigma$ as $[\partial_1 S * a_0], [\bar{a}_0 * a_1], \ldots, [\bar{a}_p * \partial_0 S]$. There are $p + 2$ components.

Analogously when $i = 2$, let $\partial_0 S$ be the boundary component containing b_0, and let $\partial_1 S$ be the boundary component containing b_1. We now label the boundary components as $[\partial_0 S * a_0 * \partial_1 S * \bar{a}_p], [\bar{a}_0 * a_1], \ldots, [\bar{a}_{p-1}, a_p]$. There are $p+1$ components.

The set of curves is non-intersecting, then $S \setminus \sigma$ is connected; by the classification theorem, it has genus g_σ and r_σ boundary components. Its Eulero characteristic is $2 - 2g_\sigma - r_\sigma$, and using the relation $\chi(S \setminus \sigma) = \chi(S) + p + 1$ we get:

$$2 - 2g - r + p + 1 = 2 - 2g_\sigma - r_\sigma$$

If $i = 1$, $r_\sigma = r - 1 + p + 2 = r + p + 1$ and then $g_\sigma = g - p - 1$; if $i = 2$, $r_\sigma = r - 2 + p + 1 = r + p - 1$ and $g_\sigma = g - p$.

As the numbers g_σ and r_σ depend only on p and not on the simplex σ, the complements of any two simplices are diffeomorphic. Choosing such a diffeomorphism which is compatible with the labels we described, it glues to a diffeomorphism of S that switches the simplices, and this concludes the transitivity part of the proof.

2. Let σ be a p-simplex in $\mathcal{O}^i(S)$; we get an inclusion $S \setminus \sigma \hookrightarrow S$. This inclusion induces a map between mapping class groups, $\Gamma(S \setminus \sigma) \to \Gamma(S)$. Here we use that the diffeomorphisms we are considering restrict to the identity on the boundary. The image of this map gives a subgroup of $\mathrm{St}_{\mathcal{O}^i}(\sigma)$ and we want to prove that it is an isomorphism. As the proof of this fact is the analogous as in Proposition 2.2.7, we omit it.

□

The next goal is to find a relation between the two kinds of complexes. The smart observation is that the map α_g induces the map β_g, and vice versa. To be more precise, we observe that the maps $\alpha\colon S_{g,r+1} \hookrightarrow S_{g+1,r}$ and $\beta\colon S_{g,r} \hookrightarrow S_{g,r+1}$ gluing a strip on respectively two or one boundary components induce maps between the simplicial complexes:

$$\alpha\colon \mathcal{O}^2(S_{g,r+1}) \to \mathcal{O}^1(S_{g+1,r}) \quad \text{and} \quad \beta\colon \mathcal{O}^1(S_{g,r}) \to \mathcal{O}^2(S_{g,r+1})$$

These maps are equivariant with respect to the induced maps on mapping class groups $\alpha_g\colon \Gamma(S_{g,r+1}) \to \Gamma(S_{g+1,r})$ and $\beta_g\colon \Gamma(S_{g,r}) \to \Gamma(S_{g,r+1})$. It follows their symmetric relation (Proposition 2.3 in [Wah11]):

Proposition 2.3.8. *Given a p-simplex $\sigma \in \mathcal{O}^2(S_{g,r+1})$, the map α_g induces the map β_{g-p} on stabilizers, making the following diagram commutative:*

$$\begin{array}{ccccc} \Gamma(S_{g,r+1}) & \hookleftarrow & St_{\mathcal{O}^2}(\sigma) & \xrightarrow{s_2} & \Gamma(S_{g-p,r+p}) \\ \downarrow{\scriptstyle \alpha_g} & & \downarrow & & \downarrow{\scriptstyle \beta_{g-p}} \\ \Gamma(S_{g+1,r}) & \hookleftarrow & St_{\mathcal{O}^1}(\alpha(\sigma)) & \xrightarrow{s_1} & \Gamma(S_{g-p,r+p+1}) \end{array}$$

Analogously, if $\sigma \in \mathcal{O}^1(S_{g,r})$ then the map β_g induces the map α_{g-p-1}.

Proof. On the surface $S_{g,r+1}$, the map α defined by gluing a strip between two boundary components induces $\alpha_g \colon \Gamma(S_{g,r+1}) \to \Gamma(S_{g+1,r})$. Suppose that the points b_0 and b_1 belong to these two components, and consider the complex $\mathcal{O}^2(S, b_0, b_1)$. The p-simplex σ is chosen in this complex.

It is clear that the stabilizer $\mathrm{St}_{\mathcal{O}^2}(\sigma)$ is a subgroup of $\Gamma(S_{g,r+1})$, that $\alpha(\sigma)$ is a p-simplex in $\mathcal{O}^1(S_{g+1,r})$ and that $\mathrm{St}_{\mathcal{O}^1}(\alpha(\sigma)) < \Gamma(S_{g+1,r})$. The first diagram commutes because the map α_g restrict to a map between stabilizers.

We check the second diagram. Cut the surface $S_{g,r+1}$ along σ: in this cutted surface, the boundary components containing b_0 and b_1 gives just a new boundary component, namely $[a_0 * \partial_0 * \bar{a}_p * \partial_1]$. Gluing a strip in $S_{g,r+1}$ corresponds, in the cutted surface, to the gluing of a strip in a single boundary component. As the stabilizer of σ in $\mathcal{O}^2(S_{g,r+1})$ is isomorphic to $\Gamma(S_{g-p,r+1+p-1})$, and the stabilizer of $\alpha(\sigma)$ in $\mathcal{O}^1(S_{g+1,r})$ is isomorphic to $\Gamma(S_{g-p,r+p+1})$, then the induced map β is actually the map

$$\beta_{g-p} \colon \Gamma(S_{g-p,r+p}) \to \Gamma(S_{g-p,r+p+1})$$

on the mapping class groups, that concludes the proof. The other case is similar. □

The last result we need is a technical lemma, that we give without proof. The result is Corollary 2.6 in [Wah11]:

Lemma 2.3.9. *Let σ be a vertex in $\mathcal{O}^2(S)$. Then the map induced from the left hand square above on relative homology*

$$H_*(St_{\mathcal{O}^1}(\alpha(\sigma), St_{\mathcal{O}^2}(\sigma)) \to H_*(\Gamma(\alpha(S)), \Gamma(S))$$

is the null map.

Consider a map $f \colon X \to Y$ between simplicial complexes, equivariant with respect to a group homomorphism $\phi \colon H \to G$, where H and G act respectively on X and Y simplicially. Then we get a map of double chain complexes

$$F \colon C_*(X) \otimes_G E_*G \longrightarrow C_*(Y) \otimes_H E_*H$$

where E_*G and E_*H are free resolutions of $\mathbb{Z}$ in $\mathbb{Z}[H]$ and $\mathbb{Z}[G]$ modules. Considering the mapping cone in the q direction we get an augmented double chain complex, commutative up to sign:

$$C_{p,q} := (C_p(X) \otimes_G E_{q-1}G) \oplus (C_p(Y) \otimes_H E_q H)$$

with horizontal differential $(a \otimes b, a' \otimes b') \mapsto (\partial a \otimes b, \partial a' \otimes b')$ and vertical differential $(a \otimes b, a' \otimes b') \mapsto (a \otimes db, a' \otimes db' + F(a \otimes b))$. This double complex induces two spectral sequences, both converging to the homology of the total complex. We remark that:

- if X is $(c-1)$-connected and Y is c-connected, then the E^1 term of the horizontal spectral sequence is 0 in the range $p+q \leq c$; then also the vertical spectral sequence converges to 0 in the same range.
- if the action of G and H is transitive, then, by the relative version of Shapiro's Lemma we get at the first page of the horizontal induced spectral sequence:

$$E^1_{p,q} = H_q(C_p(Y) \otimes E_*H, C_p(X) \otimes E_*G) \cong H_q(\mathrm{St}_Y(\sigma_p), \mathrm{St}_X(\sigma_p))$$

Denote by $H_*(\alpha_g)$ the relative group $H_*(\Gamma_{g+1,r}, \Gamma_{g,r+1})$ and by $H_*(\beta_g)$ the relative group $H_*(\Gamma_{g,r+1}, \Gamma_{g,r})$. We conclude the section with the idea of the stability proof, whose detailes can be found in Theorem 3.1, [Wah11].

Theorem 2.3.10. *With the notations of above we get for every g:*

(1_g) $H_i(\alpha_g) = 0$ *for* $i \leq 2/3g + 1/3$;

(2_g) $H_i(\beta_g) = 0$ *for* $i \leq 2/3g$.

Proof. The proof proceeds by induction on the genus g. The statements (1_0), (2_0) and (2_1) are true since any group homomorphism induces an isomorphism on the 0-th homology H_0. The induction continues in two steps:

1. if $g \geq 1$, $(2_{\leq g})$ implies (1_g);
2. if $g \geq 2$, $(1_{<g})$ and (2_{g-1}) imply (2_g).

The proof of these two statements is similar, so we consider just the first one. Let X be the simplicial complex $\mathcal{O}^2(S_{g,r+1})$, Y the simplicial complex $\mathcal{O}^1(S_{g+1,r})$; X is a H-complex, where $H = \Gamma_{g,r+1}$, and Y a G-complex, where $G = \Gamma_{g+1,r}$. Then, the map $\alpha\colon X \to Y$ is equivariant with respect to the group homomorphism $\alpha_g\colon H \to G$, and as described above, we get an augmented double chain complex associated to them:

$$C_{p,q} := (C_p(\mathcal{O}^2(S_{g,r+1}) \otimes_{\Gamma_{g+1,r}} E_{q-1}(\Gamma_{g+1,r})) \oplus (C_p(\mathcal{O}^1(S_{g+1,r})) \otimes_{\Gamma_{g,r+1}} E_q(\Gamma_{g,r+1}))$$

The complex X is $(g-2)$-connected, the complex Y is $(g-1)$-connected, then the first term of the horizontal spectral sequence to 0 for $p+q \leq g-1$. As a consequence, the vertical spectral sequence converges to 0 in the same range. The first page of the vertical spectral sequence, by transitivity and Shapiro's Lemma, is:

$$E^1_{p,q} = \mathrm{H}_q(\mathrm{St}_{\mathcal{O}^1}(\alpha(\sigma_p)), \mathrm{St}_{\mathcal{O}^2}(\sigma_p))$$

As the map α on the stabilizers induce by the proposition of above the map β, we get

$$E^1_{p,q} = \mathrm{H}_q(\beta_{g-p})$$

when $p \geq 0$. When $p = -1$, it is just $E^1_{-1,q} = \mathrm{H}_q(\Gamma_{g+1,r}, \Gamma_{g,r+1}) = \mathrm{H}_q(\alpha_g)$, which are the groups we are interested in. Hence, the statement reduces to prove that the groups $E^1_{-1,q}$ are 0 when $q \leq 2/3g + 1/3$; it follows by the following remarks:

- $E^\infty_{-1,q} = 0$ for $q \leq 2/3g + 1/3$.

  The complex $\mathcal{O}(S_{g,r+1})$ is $(g-2)$-connected, and the complex $\mathcal{O}(S_{g+1,r})$ is $(g-1)$-connected. Then, the infty term $E^\infty_{p,q}$ is null in the range $p + q \leq g - 1$. In particular, when $p = -1$, we get $E^\infty_{-1,q} = 0$ for $q \leq g$. As by assumption $g \geq 1$, then $2/3g + 1/3 \leq g$.

- $E^1_{p,q} = 0$ when $p \geq 0$ and $q \leq 2/3g$, or when $0 < p$ and $q \leq 2/3g + 1/3$.

  By induction, $E^1_{p,q} = 0$ for $q \leq 2/3(g-p)$ (and $p \geq 0$). Hence, $E^1_{p,q-p} = 0$ if $q - p \leq 2/3(g-p)$, that means $3q \leq 2g + p$. The E^1-terms are 0 for any $p \geq 0$ if $q \leq 2/3g$, or for $p > 0$ when $3q \leq 2g + 1$.

- The map $d_1 \colon E^1_{0,q} \to E^1_{-1,q}$ is the 0-map.

  In fact, this is the map

  $$H_q(St_{\mathcal{O}^1}(\alpha(\sigma)), St_{\mathcal{O}^2}(\sigma)) \to H_q(\Gamma(\alpha(S)), \Gamma(S))$$

  that is the 0-map by the technical lemma.

These three properties imply the condition (1_g), as the group $E^1_{-1,q}$ must die at infinity when $q \leq 2/3g + 1/3g$, and the sources of the differentials are zeros. □

2.4 High connectivity of complexes

In this section we prove Theorem 2.2.5 and Theorem 2.3.6, the connectivity proofs being the harder part in homological stability results. We do not give the complete proofs, that one can find in [Wah11].

To prove that the complexes $\mathcal{A}_n$ in Definition 2.2.3 and $\mathcal{O}^i(S)$ in Definition 2.3.5 are highly connected, we embed them in a larger complex, that is contractible, and then we move backwards deducing their connectivity. The larger complex we consider is $\mathcal{A}(S, \Delta)$ defined below, where S is a closed oriented surface and Δ is a set of points in its boundary.

By an arc we mean a path whose endpoints (they can be the same point) belong to the set Δ. We say that such an arc is *trivial* if it separates the surface S into two connected components, one of which is a disc intersecting Δ only in the boundary of the arc. The vertices of $\mathcal{A}(S,\Delta)$ are isotopy classes of non-trivial arcs, and a collection of $p+1$ distinct isotopy classes of non-trivial arcs is a p-simplex if they are representable by arcs with disjoint interiors. We need a technical lemma, that is Lemma 4.2 in [Wah11]:

Lemma 2.4.1. *Suppose $\mathcal{A}(S,\Delta)$ is non-void. Denote by Δ' the set Δ with an extra point p' in a boundary component of S already containing a point of Δ. Then:*

$$\mathcal{A}(S,\Delta) \quad d\text{-}connected \implies \mathcal{A}(S,\Delta') \quad (d+1)\text{-}connected$$

Proof. Let $\partial_0 S$ be the boundary component of S with $p' \in \partial_0 S$, and let p be a point in $\Delta \cap \partial_0 S$ such that one of the two path components in $\partial_0 S \setminus \{p,p'\}$ has no points in Δ. If we follow the boundary component $\partial_0 S$ from p towards p', let q be the first point in $\Delta \cap \partial_0 S$ that we find. Let q' be the first point that we find in the opposite direction (we remark that q and q' can also be p or p'). Then, consider two trivial arcs connecting them: let I be an arc connecting p to q, and let I' be an arc connecting p' with q'. We get the following decomposition:

$$\mathcal{A}(S,\Delta') = \mathrm{Star}(I) \cup_{\mathrm{Link}(I)} X$$

where X is the subcomplex consisting of the simplices that do not contain the isotopy class $[I]$ as a vertex. If we prove that $\mathrm{Link}(I) \simeq \mathcal{A}(S,\Delta)$ and that X deformation retracts on $Star(I')$, then we get the statement. In fact, if d is a fixed natural and $\mathcal{A}(S,\Delta)$ is d-connected, then by Van Kampfen Theorem we get that $\mathcal{A}(S,\Delta')$ is simply connected, and by Mayer Vietoris exact sequence it is $(d+1)$-acyclic, so $(d+1)$-connected.

First, we observe that if $\sigma \in \mathrm{Link}(I)$, then it can be viewed as a simplex in $\mathcal{A}(S,\Delta)$. Vice versa, if we forget about p', then the arc I becomes trivial; to every $\sigma \in \mathcal{A}(S,\Delta)$ we can associate the same simplex in $\mathrm{Link}(I)$, as there are no arcs isotopic to I, and every such simplex can be extended by adding the class $[I]$ to a simplex in $\mathcal{A}(S,\Delta')$.

The most delicate point is to prove the retraction onto $Star(I')$. First, we observe that $Star(I')$ is given by simplices of X whose arcs do not have p as an endpoint. In fact, if a is an arc with p as an endpoint, then a must intersect I', and this does not give a simplex in $Star(I')$. The idea is now to move the curves at p to curves at p'.

Let $\sigma = \langle a_0,\dots,a_p\rangle$ be a p-simplex, and consider the arcs with an endpoint at p. Denote by $\gamma_1,\dots,\gamma_k$ these germs of arc at p, with γ_i a germ for a_{j_i}. There is a series of $(p+1)$-simplices, $r_1(\sigma),\dots,r_k(\sigma)$, associated to σ, where $r_i(\sigma)$ is obtained from σ by moving the first i germs of arcs, ordered clockwise, and leaving the last $k-i+1$ fixed. A point in the simplex σ is described by baricentric coordinates $(t_0,\dots,t_p)$. Assign to γ_i the weight $w_i := t_{j_i}/2$. Then, the retraction is defined as follows: if $\sum_{j=1}^{i-1} w_j \le s \le \sum_{j=1}^{i} w_j$ define

$$f(s,[\sigma,(t_0,\dots,t_p)]) := [r_i(\sigma),(v_0,\dots,v_{p+1})]$$

where $v_i = t_i$ except for the pair (v_{j_i}, v_{p+1}); in this case they are defined as

$$(t_{j_i} - 2(s - \sum_{j=1}^{i-1} w_j), 2(s - \sum_{j=1}^{i-1} w_j))$$

We complete the retraction by defining f to be constant when $s \geq \sum_{j=1}^{i} w_j$. □

It follows that the complex $\mathcal{A}(S, \Delta)$ is contractible (see Theorem 4.1 in [Wah11]):

Theorem 2.4.2. *The complex $\mathcal{A}(S, \Delta)$ is contractible, unless S is a disc or a cylinder with Δ included in a single boundary component, in which case is $(|\Delta| + 2r - 7)$-connected.*

Proof. The complex $\mathcal{A}(\mathbb{D}^2, \Delta)$ is non-void if $|\Delta| \geq 4$ and if C is a cylinder, $\mathcal{A}(C, \Delta)$ is non-void if $|\Delta| \geq 2$. In both these cases the complexes are (-1)-connected.

In the general case, by the previous lemma, we can suppose that every boundary component has at most one point in Δ. The complex is not empty: if Δ has at least two points, then every arc between two boundary components is not trivial (the surface is not a disc); if $|\Delta| = 1$, then S has genus at least one, or it has at least three boundary components and in both these cases the statement is true.

Fix a point $p \in \Delta$ and an arc a starting at p. We want to retract the complex onto $Star(a)$. If $\sigma \in \mathcal{A}(S, \Delta)$, we represent it by a simplex with minimal and transverse intersections at a. If there are k such intersections we construct k intermediate simplices by successively cutting the arcs at the intersections and connecting the new endpoints to p along a. If the new arc becomes trivial, we forget about it, but it turns out that both the arcs constructed from the first one can not be trivial. Formally, this can be done as in the technical lemma. □

The contractibility of $\mathcal{A}(S, \Delta)$ implies both the connectivity theorems. First, we see that the complex $\mathcal{A}_n$ is $(n-2)$-connected. To prove it, we introduce another complex:

Definition 2.4.3. The arc complex $F\mathcal{A}_n$ is the simplicial complex defined by the sets of simplices:

$(\mathcal{A}_n)_0$: as the set of isotopy classes of arcs;

$(\mathcal{A}_n)_p$: as the set of collections of $(p+1)$ isotopy classes of arcs that are intersecting only at the endpoints.

The complex $F\mathcal{A}_n$ differs from the complex $\mathcal{A}_n$ by allowing simplices to have arcs ending at the same marked point. By Theorem 2.4.2, the complex $F\mathcal{A}_n$ is contractible.

Remark 2.4.4. If $X_1, \ldots, X_k$ are simplicial complexes, and X_i is n_i-connected, then $X_1 * \cdots * X_k$ is $\left(\sum_{i=1}^{k}(n_i + 2)) - 2\right)$-connected.

The following result can be found in [Tra14], Theorem 2.12:

Theorem 2.4.5. *The complex $\mathcal{A}_n$ is $(n-2)$-connected.*

Proof. The proof proceeds by induction on n, the base case $n = 1$ being trivial because there always exists an arc to one marked point.

Suppose $n > 1$ and let $k \le n-2$. Let $f\colon \mathbb{S}^k \to \mathcal{A}_n$ be any map. By simplicial approximation, we can find a triangulation of $\mathbb{D}^{k+1}$ and of $\mathbb{S}^k$ such that the following maps are simplicial:

$$\begin{array}{ccc} \mathbb{S}^k & \xhookrightarrow{\;f\;} & \mathcal{A}_n \\ \big\downarrow & & \big\downarrow \\ \mathbb{D}^{k+1} & \overset{F}{\dashrightarrow} & F\mathcal{A}_n \end{array}$$

the latter map existing because of the contractibility of $F\mathcal{A}_n$. We want to deform F in such a way its image is contained in $\mathcal{A}_n$.

We say that a simplex in $\mathbb{D}^{k+1}$ is *bad* if there is a pair of arcs that intersect at the same marked point; we say that it is *totally bad* if for every arc there exists an other arc ending at the same marked point. Equivalently, σ is bad if $\mathbb{D}^2 \setminus F(\sigma)$ is disconnected; σ is totally disconnected if $\#\mathbb{D}^2 \setminus F(\partial_i\sigma) < \#\mathbb{D}^2 \setminus F(\sigma)$ for every boundary map, where $\#$ denotes the number of connected components.

Let σ be a totally bad simplex of maximal dimension p and let c be the number of connected components of $\mathbb{D}^2 \setminus F(\sigma)$. Let n_i be the number of marked points in the i-th connected component. We claim that F restricts on $\mathrm{Link}(\sigma)$ to a map

$$\mathrm{Link}(\sigma) \longrightarrow \mathcal{A}_{n_1} * \cdots * \mathcal{A}_{n_c}$$

In fact, let τ be a simplex in $\mathrm{Link}(\sigma)$; if its image is not in $\mathcal{A}_{n_1} * \cdots * \mathcal{A}_{n_c}$ then there exists a face $\tau' \le \tau$ that is a totally bad simplex, then it has at least dimension one. The joint $\tau' * \sigma$ is totally bad, but this is not possible as σ is maximal.

By induction hypothesis A_{n_i} is $(n_i - 2)$-connected. Then the joint $\mathcal{A}_{n_1} * \cdots * \mathcal{A}_{n_c}$ is, by Remark 2.4.4, at least $(\sum_{i=1}^{c}((n_i-2)+2)) - 2)$-connected. This sum is actually

$$\sum_{i=1}^{c}((n_i-2)+2)) - 2 = n - (p+1) + (c-1) - 2$$

because the number of left marked points is $n-(p+1)+(c-1)$. As σ is bad, the number of connected components c is at least 2; therefore $n-(p+1)+(c-1)-2 \ge n-p-2 \ge k-p$. Moreover, $\mathrm{Link}(\sigma) \cong \mathbb{S}^{k-p}$ and by this calculation there exist a $(k-p+1)$-disc K, with $\partial K = \mathrm{Link}(\sigma)$, and an extension

$$\hat{F}\colon K \longrightarrow \mathcal{A}_{n_1} * \cdots * \mathcal{A}_{n_c} \hookrightarrow \mathcal{A}_n$$

such that $\hat{F}|_{\partial K} = F$.

We can observe that $\mathrm{Star}(\sigma) = \sigma * \mathrm{Link}(\sigma)$ is a $(k+1)$-disc, whose boundary is $\partial\sigma * \mathrm{Link}(\sigma)$. In the triangulation of $\mathbb{D}^{k+1}$ we can replace this disc with the disc $\partial\sigma * K$

which has the same boundary: $\partial(\partial\sigma * K) = \partial\sigma * \partial K = \partial\sigma * \mathrm{Link}(\sigma)$. We modify F in the interior of the disc using the map:

$$F * \hat{F} \colon \partial\sigma * K \longrightarrow F\mathcal{A}_n$$

which agrees with F on $\partial(\partial\sigma * K)$. The new simplices are of the form $\alpha * \beta$, with α a $(p-1)$-face of σ and β in $\mathrm{Link}(\sigma)$; so β has image in $\mathcal{A}_n$.

Suppose $\tau \in \partial\sigma * K$ is totally bad, it must be in $\partial\sigma$, a proper face of σ. The maximal dimension of a totally bad simplex is decreased and we can repeat the argument until there are no totally bad simplices. Then there are no bad simplices so that the image of F lies in $\mathcal{A}_n$. □

The proof for $\mathcal{O}^i(S)$ is more laborious, and it involves the construction of two other complexes. Following [Wah11], first we consider a subcomplex of $\mathcal{A}(S, \Delta)$ obtained by taking a decomposition of Δ into two disjoint subsets $\Delta = \Delta_0 \cup \Delta_1$. The arcs we consider in this subcomplex have an endpoint in Δ_0 and the other in Δ_1. Call this complex $\mathcal{B}(S, \Delta_0, \Delta_1)$.

Denote by $\mathcal{B}_0(S, \Delta_0, \Delta_1)$ the subcomplex of $\mathcal{B}(S, \Delta_0, \Delta_1)$ whose vertices are, in addition, non-separating arcs.

We remark that the complexes $\mathcal{O}^i(S, b_0, b_1)$ are the subcomplexes of $\mathcal{B}_0(S, \{b_0\}, \{b_1\})$, in which the ordering at b_0 and at b_1 are opposite. We get the following chain of complexes:

$$\mathcal{O} \hookrightarrow \mathcal{B}_0 \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{A}$$

Once we get the contractibility of $\mathcal{A}(S, \Delta)$, as in the proof of above, one can deduce highly connectivity of the subcomplexes $\mathcal{B}$, $\mathcal{B}_0$ and finally $\mathcal{O}^i$. As the proofs are similar we give just the statement for the complex $\mathcal{B}_0$ and we briefly prove the result for $\mathcal{O}^i$, results that one can find in [Wah11], Section 4:

Proposition 2.4.6. *The complex $\mathcal{B}_0(S, \Delta_0, \Delta_1)$ is $(2g + r' - 3)$-connected, where r' is the number of boundary components containing points in $\Delta_0 \cup \Delta_1$.*

In the following, $\mathcal{O}$ denotes the general complex $\mathcal{O}^i(S, b_0, b_1)$. The proof is Theorem 4.9 in [Wah11], and can be also found in [RW09], Appendix A.1.

Theorem 2.4.7. *$\mathcal{O}(S, b_0, b_1)$ is $(g-2)$-connected.*

Proof. If S has genus 0 or 1, then the statement is true. Proceed by induction on g and suppose $g \geq 2$.

Let $k \leq g-2$ and consider a map $f \colon \mathbb{S}^k \to \mathcal{O}$. As $2g + r' - 3 \geq g - 2$, there exists an extension $\hat{f} \colon \mathbb{D}^{k+1} \to \mathcal{B}_0(S, \{b_0\}, \{b_1\})$. For a simplex $\sigma \in \mathcal{B}_0$, given by arcs $a_0 \leq \cdots \leq a_p$ at b_0, we write $\sigma = \sigma_g * \sigma_b$ where $\sigma_g = \langle a_0, \ldots, a_i \rangle$ is maximal in i, preserving the ordering at b_1. If $\sigma = \sigma_g$ it belongs to $\mathcal{O}$, otherwise, if σ_g is void, we call σ purely bad. We remark that the vertices are always good.

Let σ be a purely bad simplex. Then the genus of the cutted surface is at least $g - p$. In fact, if b_0 and b_1 belong to different boundary components, then cutting an arc reduces the number of boundary components by one, and then cutting the other p simplices reduces the genus of at most p. If b_0 and b_1 belong to the same boundary component, and we consider two arcs a_i and a_j with inverse order at b_1, then, by cutting the surface, the number of boundaries is preserved and thus the genus decreases by one. Now the remaining arcs reduce the genus of at most $p - 1$.

We want to remove purely bad simplices from the image of $\hat{f}$. Let σ be a maximal purely bad p-simplex. Then, $\hat{f}(\sigma)$ is a p'-simplex, with $p' \leq p$, and the cutted surface has genus at least $g - p' \geq g - p$. The link $\mathrm{Link}(\sigma)$ is a sphere of dimension $k - p$. As $k \leq g - 2$, we get $k - p \leq g - 2 - p$. As $\hat{f}(\sigma)$ is bad, $p' \geq 2$ and $S \setminus \hat{f}(\sigma)$ has genus $\tilde{g} < g$. By induction, the restriction to the link of σ extends to a map F on a $(k + 1 - p)$-disc K. Replace $\hat{f}$ on $Star(\sigma) \cong \partial\sigma * K$ by $\hat{f} * F$, then conclude by induction on dimension of σ. □

3

The Stable Range Condition

Contents

In this chapter we introduce and study the *stable range condition*. This is a technical hypothesis that we assume for the ring R when we study homological stability for the family of general linear groups $\{\mathrm{GL}_n(R)\}_{n\in\mathbb{N}}$.

Very roughly speaking, the stable range condition gives a certain control on the generators that a given ring has, and it is strictly related to its Krull dimension. To be more precise, after we explain in Definition 3.1.3 what the stable range for a ring R is, then, in Proposition 3.3.8, we show that a Noetherian ring of Krull dimension d satisfies the stable range condition with index not greater then d. The advantage of considering the stable range condition instead of the Krull dimension is that we thus obtain a different viewpoint: if in the case of Krull dimension we look at the stabilization for chains of prime ideals, in the case of the stable range condition we consider a sort of stabilization for sequences whose elements generate (as two sided-ideal) the whole ring, the so-called *unimodular sequences*. Furthermore, we will see that if a ring satisfies the stable range condition, then the associated poset of (split) unimodular sequences has interesting properties: for example, see Theorem 3.6.1, Theorem 4.3.6 and the more important Theorem 4.4.7.

The stable range condition was first introduced as an hypothesis by Bass in [Bas64], pag. 14, and it is also called *the Bass condition*. For this chapter we have followed the authors [Bas68], [EaO67], [Ohm68] and [Vas69].

The structure of the chapter is the following. In the first section we give the definition of unimodular sequences and of stable range condition for associative rings with unit. Then, we study its behaviour under some basic theoretical constructions, such as products and quotients, and we give a first example of rings satisfying the stable range condition: the semi-local rings. In the second and third sections we prove that Dedekind domains and Noetherian rings of finite Krull dimension satisfy the stable range condition, hence, the class of rings we consider is quite large. In the fourth section we generalize the stable range condition to pairs $(R, \mathfrak{a})$, where $\mathfrak{a}$ is any two-sided ideal of R, and we verify how this generalization relates to the absolute case. Finally, in the last section, we observe that, if the pair $(R, \mathfrak{a})$ satisfies the stable range condition, then the elementary matrices of Definition 1.3.1 act transitively on the set of unimodular sequences. We conclude with a technical result about this action.

3.1 Definition and first properties

In the following, unless further specified, R will denote an associative ring with identity. By an ideal $\mathfrak{a}$ of R we mean a two-sided ideal. In order to distinguish between a sequence of elements and the ideal generated by a set of elements, we shall use two different notations: $(a_1, \dots, a_n)$ indicates a sequence of elements, and $\langle a_1, \dots, a_n \rangle$ the generated two-sided ideal.

Definition 3.1.1. Let $(a_1, \dots, a_n)$ be a sequence whose elements belong to a ring R. The sequence is called *unimodular* (in R^n) if there exist elements $r_1, \dots, r_n \in R$ such that $\sum_{i=1}^{n} r_i a_i = 1$.

A sequence $(a_1, \dots, a_n)$ can also be seen as an element of the right R-module R^n. Then, saying that a sequence is unimodular is equivalent to the following: there exists an R-linear homomorphism $f \colon R^n \to R$ such that $f(a_1, \dots, a_n) = 1$. In other words, a sequence of elements of R is unimodular if the generated ideal is the whole ring.

In the following definition we clarify what we mean by stability for unimodular sequences:

Definition 3.1.2. Let $n > 1$. A unimodular sequence $(a_1, \dots, a_n)$ is called *stable* if there exist elements $b_1, \dots, b_{n-1} \in R$ such that the sequence $(a_1 + b_1 a_n, \dots, a_{n-1} + b_{n-1} a_n)$ is unimodular in R^{n-1}.

This means that, up to a linear reduction, the generated ideal has fewer generators. In a rough sense, what we are looking for is a way to understand how much we can reduce, using these reductions, a given number of generators of the whole ring; moreover, these reductions are elementary in the sense of Section 1.3. In fact, given a unimodular sequence $(a_1, \dots, a_n)$, and given elements $b_1, \dots, b_{n-1}$, the reduction is given by the following product:

$$\begin{pmatrix} 1_{n-1} & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ a_n \end{pmatrix} = \begin{pmatrix} a' \\ a_n \end{pmatrix}$$

where $b = (b_1, \ldots, b_{n-1})^T$, $a = (a_1, \ldots, a_{n-1})^T$, $a' = (a_1 + b_1 a_n, \ldots, a_{n-1} + b_{n-1} a_n)^T$, and the matrix is elementary. The definition is motivated by the Gauss algorithm: if the sequence a' is unimodular, then we can kill the element a_n.

We remark that, a priori, there is a left and a right version of stability for unimodular sequences, and as a convention we adopt the right one. Theorem 2 in [Vas71] that asserts the definition does not depend on the chosen order.

We can now define the stable range condition:

Definition 3.1.3. We say that an associative ring with identity R satisfies the *stable range condition with index* n if, for all $m \geq n$, every unimodular sequence in R^m is stable.

Rings that satisfy the stable range condition with index n will be referred to as *rings of stable range* n. It follows from the definition that, if R satisfies the stable range condition with index n, then it is a ring of stable range m, for each integer $m \geq n$.

We observe that, if one wants to prove that a ring R is a ring of stable range n, then it is not necessary to check whether any sequence $(a_1, \ldots, a_m)$ with $m \geq n$ is stable. In fact, this can be deduced by verifying the stability in the case $m = n$. To be more precise, following Theorem 1 in [Vas71], we get:

Lemma 3.1.4. *Let R be an associative ring in which every unimodular sequence in R^n is stable. Then, R is a ring of stable range n.*

Proof. Let $(a_1, \ldots, a_{n+1})$ be a fixed unimodular sequence in R^{n+1}. It is enough to prove that it is stable. By definition, there exist elements $r_1, \ldots, r_{n+1}$ in R such that $\sum_{i=1}^{n+1} r_i a_i = 1$. As a consequence, the sequence $(a_1, \ldots, a_{n-1}, r_n a_n + r_{n+1} a_{n+1})$ is unimodular in R^n, and by hypothesis, it is stable. Therefore, there exist elements $s_1, \ldots, s_{n-1} \in R$ such that the sequence

$$(a_1 + s_1(r_n a_n + r_{n+1} a_{n+1}), \ldots, a_{n-1} + s_{s-1}(r_n a_n + r_{n+1} a_{n+1}))$$

is unimodular. By adding the element a_n to this sequence, the new sequence we obtain is still unimodular. We subtract $s_i r_n a_n$ from the i-th element of the sequence, and obtain:

$$(a_1 + s_1 r_{n+1} a_{n+1}, \ldots, a_{n-1} + s_{s-1} r_{n+1} a_{n+1}, a_n)$$

that is still unimodular. This concludes the proof by setting $b_i := s_i r_{n+1}$ for $i \leq n-1$ and $b_n = 0$. □

The first example of a ring satisfying the stable range condition follows straightforward from the definition:

Example 3.1.5. Every field is a ring of stable range 2. In fact, any unimodular sequence (a_1, a_2), has a non-null element, so an invertible element. Then, the sequence is stable.

We assume the null-ring is a ring of stable range 1. Concerning non-trivial rings, we let the index n to be at least 2, and the example of above shows a large class of rings of stable range 2. Step by step, we want to enlarge this class, and we shall prove that it is rather large. We need to study the basic properties of the stable range condition; we begin with the definition of *stable dimension*. Properly speaking this is not a dimension (namely, it is not local), but it is important to point out that it is the minimum on the integers for which the stable range condition is satisfied.

Definition 3.1.6. Let R be a fixed associative ring with identity. Let S be the set of natural numbers n such that R satisfies the stable range condition with index n. Then, we define the *stable dimension* of R as the natural number

$$\operatorname{sdim}(R) := \min S - 2$$

if this minimum exists; we set $\operatorname{sdim}(R)$ to be infinite otherwise.

According to this definition, fields have stable dimension 0. We proceed now by investigating how the stable range condition behaves by making products and quotients. The first observation about products is quite trivial: if a ring R is a product of rings $R = \prod_\alpha R_\alpha$, then R is a ring of stable range n if and only if each R_α is a ring of stable range n. Therefore, the stable dimension of a product is the maximum of the stable dimensions of its factors.

The following is a lemma by Bass, see Lemma 4.1 in [Bas64].

Lemma 3.1.7. *If a ring R satisfies the stable range condition with index n, then each quotient $R/\mathfrak{q}$ is a ring of stable range n, where $\mathfrak{q}$ is any proper two-sided ideal.*

Proof. Let $(\bar{a}_1, \ldots, \bar{a}_m)$ be a unimodular sequence in $R/\mathfrak{q}$, with $m \geq n$. Then, by definition, we have a relation $\sum_{i=1}^m \bar{r}_i \bar{a}_i = \bar{1}$ for suitable $\bar{r}_i \in R/\mathfrak{q}$. Choose representatives $a_i, r_i \in R$ and consider the lifted relation $\sum_{i=1}^m r_i a_i + q = 1$, where $q \in \mathfrak{q} \subseteq R$. Then, the sequence $(a_1, \ldots, a_m, q)$ is unimodular in R^{m+1}, so stable, because the ring R satisfies the stable range condition with index $n < m + 1$. There exist elements $b_1, \ldots, b_m \in R$ such that the sequence $(a_1 + b_1 q, \ldots, a_m + b_m q)$ is unimodular in R^m. This means that, by replacing the chosen a_i's with the elements $a_i + b_i q$, we can assume that the previous lifted sequence $(a_1, \ldots, a_m)$ is unimodular.

Applying again the stable range condition for R, as $m \geq n$, we get there exist elements $c_1, \ldots, c_{m-1}$, $c_i \in R$, such that the sequence $(a_1 + c_1 a_m, \ldots, a_{m-1} + c_{m-1} a_m)$ is unimodular. Therefore, the choice of the classes $\bar{c}_1, \ldots, \bar{c}_{m-1} \in R/\mathfrak{q}$, makes the sequence $(\bar{a}_1 + \bar{c}_1 \bar{a}_m, \ldots, \bar{a}_{m-1} + \bar{c}_{m-1} \bar{a}_m)$ unimodular. □

This lemma implies that the stable range condition is satisfied by passing to quotients. The converse is also true: if all proper quotients have stable dimension n, then the ring

R has stable dimension n. To get this result, it is enough to check the stability range of $R/\mathfrak{J}(R)$, where $\mathfrak{J}(R)$ is its Jacobson ideal. We recall that the Jacobson ideal is defined as the intersection of all the maximal ideals in R. A useful result is the following (Prop. 1.9, [AaM69]):

Lemma 3.1.8. *Let $\mathfrak{J}(R)$ be the Jacobson ideal. Then $r \in \mathfrak{J}(R)$ if and only if $1 - rs$ is invertible, for every $s \in R$.*

Thanks to this lemma, we easily get the following (see Lemma 3 in [Vas71] for the non-commutative result):

Lemma 3.1.9. *Let R be a commutative ring. Then it satisfies the stable range condition with index n if and only if $R/\mathfrak{J}(R)$ does.*

Proof. One direction is given by the previous lemma. Suppose the quotient $R/\mathfrak{J}(R)$ is a ring of stable range n, and fix an integer $m \geq n$. Let $(a_1, \dots, a_m)$ be a unimodular sequence in R^m. As $\sum_{i=1}^m r_i a_i = 1$ for suitable $r_i \in R$, then we get the same relation in $R/\mathfrak{J}(R)$ which means that the sequence of equivalence classes $(\bar{a}_1, \dots, \bar{a}_m)$ is unimodular in $(R/\mathfrak{J}(R))^m$, so stable. As a consequence, there exist elements $\bar{b}_i \in R/\mathfrak{J}(R)$ such that $\langle \bar{a}_1 + \bar{b}_1 \bar{a}_m, \dots, \bar{a}_{m-1} + \bar{b}_{m-1} \bar{a}_m \rangle = R/\mathfrak{J}(R)$. Lifting the relation back to R, we have:

$$\sum_{i=1}^{m} s_i(a_i + b_i a_{m+1}) + j = 1$$

where the elements s_i's belong to R and $j \in \mathfrak{J}(R)$. But $j \in \mathfrak{J}(R)$ implies that $1 - j$ is a unit of R. This proves the sequence is unimodular, concluding the proof. □

We can now show that a semi-local ring has stable dimension 0.

Example 3.1.10. Let R be a semi-local commutative ring. Then, there are finitely many maximal ideals, and the Jacobson radical is a finite intersection. By Chinese Remainder Theorem we get:

$$R/\mathfrak{J}(R) = R/(\cap_i \mathfrak{m}_i) \cong \prod_i R/\mathfrak{m}_i;$$

It has stable dimension 0 because every field has stable dimension 0.

3.2 Dedekind domains

The examples of fields and semi-local rings we have just shown are quite simple. The next goal is to broaden the class of rings of finite stable dimension. As the hypothesis of stable range condition is related to the generators of a ring, the larger class we can investigate is that of principal ideal domains. In fact, one can prove that principal ideal domains are rings of stable range three; but with almost the same proof we can obtain the same result in the case of Dedekind domains.

We recall that a *Dedekind domain* D is a Noetherian integral domain of Krull dimension one, for which one of the following (equivalent) conditions is satisfied (see [AaM69], Theorem 9.3):

i. D is integrally closed;

ii. every primary ideal in D is a prime power;

iii. every local ring $D_{\mathfrak{p}}$, for any prime $\mathfrak{p} \neq (0)$, is a discrete valuation ring.

In a Dedekind domain every non-zero ideal has a unique factorization as a product of prime ideals (see for instance, Corollary 9.4 of [AaM69]). Moreover, any ideal $\mathfrak{a} \subseteq D$ is contained in finitely many maximal ideals.

Example 3.1.10 shows that a ring with a finite number of maximal ideals has finite stable dimension. In general, rings do not have a finite number of maximal ideals; but, as in Dedekind domains, if any ideal is contained in finitely many maximal ones, then we get finite stable dimension (three). We follow [JaS14], and in particular Satz K.13.

Proposition 3.2.1. *Let D be a Dedekind domain. Then, D is a ring of stable range* 3.

Proof. By Lemma 3.1.4 we can just consider unimodular sequences in D^3. Therefore, let (a_1, a_2, a_3) be a fixed unimodular sequence. If $a_1 = 0$, the sequence is clearly stable. Then we can suppose, without loss of generality, $a_1 \neq 0$.

Consider the ideal generated by the element a_1, say $\mathfrak{a} := \langle a_1 \rangle$. If it is the whole ring D, then the sequence (a_1, a_2, a_3) is stable. If it is not, then it is a proper ideal, and there exist finitely many maximal ideals, say $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$, containing it. By the Chinese remainder Theorem, there exists an element $\varepsilon \in D$, such that:

$$\varepsilon \equiv 0 \mod \mathfrak{m}_i \quad \text{if } a_2 \notin \mathfrak{m}_i$$

$$\varepsilon \equiv 1 \mod \mathfrak{m}_i \quad \text{if } a_2 \in \mathfrak{m}_i$$

Given such an element ε, we get $a_2 + \varepsilon a_3 \notin \mathfrak{m}_i$ for each i. In fact, it is not possible to have both a_3 and a_2 in the same maximal ideal $\mathfrak{m}_i$ because then $\langle a_1, a_2, a_3 \rangle \subseteq \mathfrak{m}_i$; so, when a_2 belongs to $\mathfrak{m}_i$, a_3 does not.

Now we want to prove that the sequence $(a_1, a_2 + \varepsilon a_3)$ is unimodular. If it is not unimodular, there exists a maximal ideal $\mathfrak{m}$ containing the generated ideal, and so containing $\mathfrak{a}$. This means that $\mathfrak{m}$ has to coincide with $\mathfrak{m}_i$ for some i. But this can not happen as the element $a_2 + \varepsilon a_3$ does not belong to m_i for every i. Therefore the sequence $(a_1, a_2 + \varepsilon a_3)$ was unimodular. □

Corollary 3.2.2. *A principal ideal domain is a ring of stable range three.*

As a concrete example, the ring of integers $\mathbb{Z}$ has stable dimension 1.

3.3 Noetherian rings of finite dimension

An important feature of Dedekind domains is that they have Krull dimension one. In this section we generalize the previous results by considering the larger class of Noetherian rings of finite Krull dimension, or, more generally, the class of rings in which the combinatorial dimension of the maximal spectrum is finite. To do this, we follow [EaO67]. In the whole section, R is assumed to be commutative.

First, we recall some basic facts from algebraic geometry. If R is a commutative ring with identity, we can consider its *spectrum* $\mathrm{Spec}(R)$, i.e., the set of all prime ideals of R, endowed with the Zariski topology. To be more precise, its closed sets are all the subsets of the type:

$$\mathcal{V}(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \supseteq \mathfrak{a}\}$$

where $\mathfrak{a}$ is any ideal of R. Moreover, to each closed set we can associate an ideal: if $X \subseteq \mathrm{Spec}(R)$ is a subset, $\mathcal{I}(X) := \bigcap_{\mathfrak{p}\in X} \mathfrak{p}$ is the associated ideal, that is radical as an intersection of prime ideals.

Remark 3.3.1. Observe that $\mathcal{V}$ gives a one-to-one correspondence between radical ideals of R and closed sets of the spectrum.

A closed set is said to be *irreducible* if it is not a union of two non-empty, closed, proper, subsets. It is well-known that the closed set $\mathcal{V}(\mathfrak{a})$ is irreducible if and only if the radical of $\mathfrak{a}$ is prime. In particular, the correspondence $\mathfrak{p} \mapsto \mathcal{V}(\mathfrak{p})$ is a bijection between the prime ideals of R and the irreducible closed subsets of $\mathrm{Spec}(R)$.

Definition 3.3.2. The *maximal spectrum* of the ring R is the set

$$\mathrm{MaxSpec}(R) := \{\mathfrak{m} \subseteq R \mid \mathfrak{m} \text{ is maximal}\} \subseteq \mathrm{Spec}(R)$$

endowed with the subspace Zariski topology.

As in $\mathrm{Spec}(R)$, we can simply describe the topology by specifying the closed sets:

$$\mathcal{V}_M(\mathfrak{a}) := \{\mathfrak{m} \in \mathrm{MaxSpec}(R) \mid \mathfrak{m} \supseteq \mathfrak{a}\}$$

The ideal we can associate to a subset $Y \subseteq \mathrm{MaxSpec}(R)$ is now $\mathcal{I}_M(Y) := \bigcap_{\mathfrak{m}\in Y} \mathfrak{m}$. This is no longer the radical ideal associated to $\mathfrak{a}$ because we are intersecting a priori fewer prime ideals (just the maximal ones), but we can carry on the analogy by using the Jacobson ideals:

Definition 3.3.3. If $\mathfrak{a}$ is an ideal of R, then the *Jacobson radical* of $\mathfrak{a}$, or J-radical of $\mathfrak{a}$, denoted by $\mathfrak{J}(\mathfrak{a})$, is the intersection of all the maximal ideals containing $\mathfrak{a}$. We say that $\mathfrak{a}$ is Jacobson radical, or simpler J-radical, if $\mathfrak{a}$ coincides with its Jacobson radical.

As a consequence of the definition, J-radical ideals correspond to closed sets in MaxSpec(R), with J-radical prime ideals corresponding to the irreducible closed sets. This means that chains of closed irreducible subsets in MaxSpec(R) correspond to chains of prime J-radical ideals in R.

Recall that the *combinatorial dimension* of a topological space is defined as the supremum of the lengths over all strictly ascending chains of closed irreducible subsets. Thanks to the correspondence just observed, the combinatorial dimension of MaxSpec(R) is the supremum of the lengths of chains of prime Jacobson-radical ideals. Moreover, its dimension is bounded from above by the Krull dimension of the ring.

A ring in which the set of J-radical ideals satisfies a.c.c. is called *Jacobson-noetherian*, or simpler J-noetherian. Equivalently, R is said to be J-noetherian, if its maximal spectrum is noetherian as a topological space. We conclude this overview, with the following:

Definition 3.3.4. We say that a prime Jacobson-radical ideal $\mathfrak{p}$ is a *component* of the ideal $\mathfrak{a}$, if it is minimal among all the prime Jacobson-radical ideals containing $\mathfrak{a}$.

Thus, a component of $\mathfrak{a}$ corresponds to an irreducible component of $\mathcal{V}_M(\mathfrak{a})$. Observe that if R is J-noetherian, then every ideal $\mathfrak{a}$ has finitely many components.

Observation 3.3.5. By Hilbert's Basis Theorem, if the ring R is noetherian then the polynomial extension $R[X]$ is noetherian. But if the ring R is J-noetherian, then the extension $R[X]$ is possibly not J-noetherian. The implication is true, if we additionally ask the R-algebra to be finite (i.e. finitely generated as an R-module): see for example Theorem 3.6 in [Ohm68].

Let $\mathfrak{p}_1, \dots, \mathfrak{p}_h$ and $\mathfrak{q}_1, \dots, \mathfrak{q}_k$ be prime ideals of the ring R, such that $\mathfrak{q}_i \not\subseteq \mathfrak{p}_j$ for each i and j. Using repeatedly Proposition 1.11 of [AaM69] we get:

$$\mathfrak{q}_1 \cdots \mathfrak{q}_n \subseteq \bigcap_{i=1}^{k} \mathfrak{q}_i \not\subseteq \bigcup_{j=1}^{h} \mathfrak{p}_j$$

and there exist elements in $\mathfrak{q}_1 \cdot \ldots \cdot \mathfrak{q}_k$ that do not belong to $\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_h$. Now let $a \in R$ be fixed, with the property that $a \in \mathfrak{p}_1, \dots, \mathfrak{p}_h$ but $a \notin \mathfrak{q}_1, \dots, \mathfrak{q}_k$. By the previous remark, we can consider an element a' of the form $a' = a + br$, where $b \in \mathfrak{q}_1, \dots, \mathfrak{q}_k$, $b \notin \mathfrak{p}_1, \dots, \mathfrak{p}_h$, and $r \in R$ is not contained in any $\mathfrak{p}_i$ or $\mathfrak{q}_j$. Then:

- if $a' \in \mathfrak{p}_i$ for some index i, the element $br = a' - a$ is in $\mathfrak{p}_i$ and this is not possible as $\mathfrak{p}_i$ is prime;
- if $a' \in \mathfrak{q}_j$ for some index j, then $a = a' - br$ belongs to $\mathfrak{q}_j$ too, that is absurd.

Therefore, if the element r is not contained in any prime, the same is true for a'. With the same argument, we get the following technical result:

Lemma 3.3.6. *Let R be a commutative ring with identity. Let $a_0, \ldots, a_{s+1} \in R$ and consider an element $a_{s+2} \notin \mathfrak{p}_1, \ldots, \mathfrak{p}_t$, where the $\mathfrak{p}_i$'s are prime ideals of R with the property that $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$ for any $i \neq j$. Then, for any $j = 1, \ldots, s+1$, there exists $b_j \in R$ such that, if $a'_j := a_j + b_j a_{s+2}$, then $a'_j \notin \mathfrak{p}_1, \ldots, \mathfrak{p}_t$.*

Proof. For every $j = 1, \ldots, s+1$, the element a_j belongs, up to a reenumeration of the prime ideals, to $\mathfrak{p}_1, \ldots, \mathfrak{p}_h$, but not in $\mathfrak{p}_{i+1}, \ldots, \mathfrak{p}_t$. Now, repeat the argument of above. □

Corollary 3.3.7. *Let R be a Jacobson-noetherian ring and consider elements $a_0, \ldots, a_{s+2}$. Then there exist elements $b_1, \ldots, b_{s+1} \in R$ such that, for $1 \leq i \leq s+1$, any component of $\langle a_0, a_1 + b_1 a_{s+2}, \ldots, a_{i-1} + b_{i-1} a_{s+2} \rangle$ which contains $a_i + b_i a_{s+2}$, contains also the element a_{s+2}.*

Proof. The ring R is J-noetherian, then every ideal has finitely many components. Now apply the previous Lemma for any $i = 1, \ldots, s+1$. □

We can finally give the result:

Proposition 3.3.8. *Let R be a commutative ring whose maximal spectrum is a Noetherian space of dimension s. Then, R satisfies the stable range condition with index $s+2$.*

Proof. We have to prove that any unimodular sequence $(a_1, \ldots, a_{s+2})$ is stable. Set $a_0 := 0$, for $i = 1, \ldots, s+1$ choose elements b_i's as in the previous corollary, and set $a'_j = a_j + b_j a_{s+2}$. Suppose with this choice that the sequence $(a_0, a'_1, \ldots, a'_{s+1})$ is not unimodular. This implies that the generated ideal $\langle a_0, a'_1, \ldots, a'_{s+1} \rangle$ is proper, and there exists a component $\mathfrak{p}$ containing it.

Since $\langle a_0, a'_1, \ldots, a'_{s+1}, a_{s+2} \rangle = \langle a_1, \ldots, a_{s+2} \rangle = R$, we get $a_{s+2} \notin \mathfrak{p}$. Consider now the chain of ideals

$$\langle a_0, a'_1, \ldots, a'_{s+1} \rangle \supseteq \langle a_0, a'_1, \ldots, a'_s \rangle \supseteq \cdots \supseteq \langle a_0 \rangle;$$

this chain gives rise to a chain of prime J-radical ideals:

$$\mathfrak{p} \supseteq \mathfrak{p}_s \supseteq \cdots \supseteq \mathfrak{p}_0$$

where $\mathfrak{p}_i$ is a component of $\langle a_0, a'_1, \ldots, a'_i \rangle$. But a_{s+2} does not belong to any $\mathfrak{p}_i$'s, and by the previous corollary, these inclusions are all proper inclusions. What we get is a proper chain of prime J-radical ideals of length $s+1$. This contradicts the hypothesis on dimension. So $(a'_1, \ldots, a'_{s+1})$ is unimodular. □

Corollary 3.3.9. *A commutative Noetherian ring of Krull dimension d has stable dimension not greater then d.*

Let R be a commutative Noetherian ring of finite Krull dimension; then, the finitely generated commutative algebra $R[X_1, \dots, X_n]$ is still a commutative ring of finite Krull dimension, hence it has finite stable dimension. This is also true if the R-algebra is not commutative, but we need the R-algebra is also finitely generated as an R-module; for a proof see Theorem 3.5 in [Bas68]:

Theorem 3.3.10. *Let R be a commutative ring whose maximal spectrum is a noetherian space of finite dimension d. Let R' be an R-algebra that is finitely generated as an R-module; then, R' satisfies the stable range condition with index $d+2$.*

3.4 An example: group rings

Let R be a commutative Noetherian ring of finite Krull dimension d. By Theorem 3.3.10 any finite R-algebra satisfies the stable range condition with index $d+2$, and we want to apply this result to group rings. With this goal in mind, let G be a finite group, and consider the group ring $R[G]$.

We first look at the easy example of finite cyclic groups. Let C_p be the finite cyclic group of order p, then the group ring $R[C_p]$ is actually the ring $R[t]/(1-t^p)$, a finite commutative algebra over R with generators $1, t, \dots, t^p$. Hence, applying Corollary 3.3.9 or Theorem 3.3.10, we obtain that the group ring $R[C_p]$ is a ring of finite stable dimension.

Let G be any finite group. The group ring $R[G]$ is still a finite algebra, but it is not commutative. It has the elements of G as generators, and we apply Theorem 3.3.10 to see that the group ring $R[G]$ is a ring that satisfies the stable range condition with index $d+2$.

We treat now the case of infinite cyclic groups.

Example 3.4.1. Let C be the cyclic infinite group and consider the group ring $\mathbb{Z}[C]$. Then, $\mathbb{Z}[C]$ is isomorphic to $\mathbb{Z}[x, x^{-1}]$, that is a Noetherian domain of dimension two. In fact, $\mathbb{Z}[x, x^{-1}]$ can be obtained from $\mathbb{Z}[x]$ by inverting the element x; topologically, this corresponds to focus on the open set of $\operatorname{Spec}(\mathbb{Z}[x])$ given by $U_x := \{\mathfrak{p} \in \operatorname{Spec}(\mathbb{Z}[x]) \mid x \notin \mathfrak{p}\}$. As $\operatorname{spec}(\mathbb{Z}[x])$ is irreducible, every open set has the same total dimension, that is two. Then the group ring $\mathbb{Z}[C]$ has stable dimension not greater then two.

We can further generalize the previous argument to any commutative Noetherian ring R of finite Krull dimension. In fact, if R is such a ring, the group ring $R[C]$ is isomorphic to the localization $(R[x])_x = R[x, x^{-1}]$ in $R[x]$; if we denote the Krull dimension by Kdim, we know that $\operatorname{Kdim}(R[x]) = \operatorname{Kdim}(R) + 1$, hence

$$\operatorname{Kdim}(R[C]) = \operatorname{Kdim}(R[x, x^{-1}]) \leq \operatorname{Kdim}(R[x]) = \operatorname{Kdim}(R) + 1$$

as the chains of prime ideals in a localization are subchains of prime ideals in the whole ring. Then:

Proposition 3.4.2. *Let R be a commutative Noetherian ring of finite Krull dimension. Then, the group ring $R[C]$ is a ring of finite stable dimension.*

Remark 3.4.3. Observe that $R[G \times H] \cong (R[G])[H]$.

By the Classification Theorem for Finitely Generated Abelian Groups, if G is a finitely generated abelian group, then $G \cong \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_r} \oplus \mathbb{Z}^n$. Assuming the ring R is a commutative Noetherian ring of finite dimension, using the previous remark we get $R[G]$ satisfies the stable range condition with finite index, whenever G is a finitely generated abelian group.

We can merge these two results:

Proposition 3.4.4. *Let R be a commutative Noetherian ring of finite Krull dimension. Let G be a group containing a finitely generated abelian group of finite index. Then the group ring $R[G]$ is a ring of finite stable dimension.*

Proof. Let A be the finitely generated abelian group of finite index. Then, the group ring $R[A]$ is a commutative Noetherian ring of finite Krull dimension.

Let $T = \{x_1, \dots, x_n\}$ be a complete set of representatives of left cosets of A in G. Then, the group ring $R[G]$ is a left $R[A]$-module that is free over $R[H]$ with basis T. As R is commutative, and $A < G$ is a subgroup, the induced map between group rings $R[A] \to R[G]$ is an injective homomorphism of R-algebras (and $R[A]$ is a subring of $R[G]$). Therefore, $R[G]$ is a finite algebra over $R[A]$. Now apply Theorem 3.3.10 to get the result. □

3.5 Stable range condition for pairs

Let R be an associative ring with unit and let $\mathfrak{a} \subseteq$ be a two-sided ideal. We can extend the stable range condition of Definition 3.1.3 to pairs $(R, \mathfrak{a})$. First, the unimodular sequences we are going to use have to be related to the ideal $\mathfrak{a}$:

Definition 3.5.1. A unimodular sequence $(a_1, \dots, a_n)$ is called $\mathfrak{a}$-*unimodular in* R^n if the elements $a_1 - 1, a_2, \dots, a_n$ belong to $\mathfrak{a}$.

If $\mathfrak{a} = R$ this definition coincides with the previous one. There are different characterizations for $\mathfrak{a}$-unimodular sequences. The following lemma gives us some equivalences:

Lemma 3.5.2. *Let R be an associative ring with unit and let the elements $a_1 - 1, a_2, \dots, a_n$ belong to a two-sided ideal $\mathfrak{a}$. The following are equivalent:*

(i) the sequence $(a_1, \dots, a_n)$ is $\mathfrak{a}$-unimodular in R^n;

(ii) $\sum_{i=1}^n \mathfrak{a} a_i = \mathfrak{a}$;

(iii) there exist elements $r_1, \dots, r_n \in R$, with $r_1 - 1, r_2, \dots, r_n \in \mathfrak{a}$, such that $\sum_{i=1}^n r_i a_i$ is equal to 1.

Proof. Assume (i) and suppose that the sequence $(a_1, \dots, a_n)$ is $\mathfrak{a}$-unimodular; fix an element $a \in \mathfrak{a}$. As $\sum_{i=1}^n r_i a_i = 1$ for some $r_1, \dots, r_n \in R$, we have $a = \sum_{i=1}^n a r_i a_i$, and the elements ar_i's belong to $\mathfrak{a}$.

If we assume (ii), then, as $1 - a_1 \in \mathfrak{a}$, there exist elements $u_i \in \mathfrak{a}$ such that $\sum_{i=1}^n u_i a_i$ is equal to $1 - a_1$. Set $r_1 := u_1 + 1$ and $r_i := u_i$. Then $\sum_{i=1}^n r_i a_i = 1$ and $(a_1, \dots, a_n)$ is $\mathfrak{a}$-unimodular.

Finally, in the implication $(iii) \Rightarrow (i)$ there is nothing to prove. □

The stability condition on $\mathfrak{a}$-unimodular sequences is easily generalized:

Definition 3.5.3. An $\mathfrak{a}$-unimodular sequence $(a_1, \dots, a_m)$ is $\mathfrak{a}$-*stable* if there exist elements $b_1, \dots, b_{m-1} \in \mathfrak{a}$, such that the sequence $(a_1 + b_1 a_m, \dots, a_{m-1} + b_{m-1} a_m)$ is unimodular in R^{m-1} (and so, also $\mathfrak{a}$-unimodular).

We can now give the definition of stable range condition for pairs:

Definition 3.5.4. Let R be an associative ring with unit, and let $\mathfrak{a}$ be a proper two-sided ideal. We say that the pair $(R, \mathfrak{a})$ satisfies the stable range condition with index n if, for all $m \geq n$, every $\mathfrak{a}$-unimodular sequence in R^m is $\mathfrak{a}$-stable.

Notation. We write $\mathrm{SR}_n(R, \mathfrak{a})$ to say that the pair $(R, \mathfrak{a})$ satisfies the stable range condition with index n.

In the following result, we show that the stable range condition for pairs is also satisfied by passing to sub-ideals:

Proposition 3.5.5. *Let $\mathfrak{a} \subseteq \mathfrak{b} \subseteq R$ be two-sided ideals of R. Then*

$$SR_n(R, \mathfrak{b}) \Rightarrow SR_n(R, \mathfrak{a}).$$

Proof. Let $m \geq n$ and fix an $\mathfrak{a}$-unimodular sequence $(a_1, \dots, a_m)$. It is also a $\mathfrak{b}$-unimodular sequence, but if we apply directly the stable range condition to this sequence, we get the existence of elements b_i's in $\mathfrak{b}$, a priory not in $\mathfrak{a}$. To avoid this problem, write the element a_m as a sum $a_m = \sum_{i=1}^m r_i a_i a_m$, with $r_1, \dots, r_m \in R$. In this way, we get that a_m belongs to the ideal $\langle a_1, \dots, a_{m-1}, r_m a_m^2 \rangle$.

Moreover, the sequence $(a_1, \dots, a_{m-1}, r_m a_m^2)$ is $\mathfrak{b}$-unimodular. We can now apply the stable range condition, getting the existence of elements b_i's in $\mathfrak{b}$, such that the sequence $(a_1 + b_1 r_m a_m a_m, \dots, a_{m-1} + b_{m-1} r_m a_m a_m)$ is unimodular. In fact, this proves the statement as the elements $b_i r_m a_m$ belong to $\mathfrak{a}$. □

As a consequence, we can deduce a relation between the two definitions we gave:

Corollary 3.5.6. *If the ring R satisfies the stable range condition with index n, then we get $SR_n(R, \mathfrak{a})$ for any two-sided ideal $\mathfrak{a} \subseteq R$.*

3.6 Transitive action on unimodular sequences

We conclude this chapter with two results we will use in Proposition 4.3.5. In Definition 1.3.1 we introduced the elementary matrices. If the stable range condition is satisfied, this group acts transitively on the set of unimodular sequences:

Theorem 3.6.1. *Suppose the pair $(R, \mathfrak{a})$ satisfies the stable range condition with index n, and fix an integer $m \geq n$. Then, the group $E_m(R, \mathfrak{a})$ acts transitively on the set of $\mathfrak{a}$-unimodular sequences in R^m.*

Proof. Fix an $\mathfrak{a}$-unimodular sequence in R^m, say $(a_1, \dots, a_m)$. By the stable range condition 3.5.4, there exist coefficients b_i's, in the ideal $\mathfrak{a}$, such that the sequence $(a_1 + b_1 a_m, \dots, a_{m-1} + b_{m-1} a_m)$ is $\mathfrak{a}$-unimodular.

Using the equivalences (ii) of Lemma 3.5.2, there exist elements $u_i \in \mathfrak{a}$ such that:

$$\sum_{i=1}^{m-1} u_i(a_i + b_i a_m) = a_1 - 1 - a_m + b_1 a_m \in \mathfrak{a}.$$

For simplicity, set $\alpha = (a_1, \dots, a_m)^T$, $\beta = (b_1, \dots, b_{m-1})^T$ and $u = (u_1, \dots, u_{m-1})$. Let γ be the sequence given by:

$$\gamma^T := \begin{pmatrix} I_{m-1} & 0 \\ u & 1 \end{pmatrix} \begin{pmatrix} I_{m-1} & \beta \\ 0 & 1 \end{pmatrix} \alpha$$

Therefore, $\gamma = (a_1 + b_1 a_m, \dots, a_{m-1} + b_{m-1} a_m, a_1 + b_1 a_m - 1)$. Moreover, the two matrices above are $\mathfrak{a}$-elementary, as u and b have elements in $\mathfrak{a}$.

The sequence γ is useful because $\gamma_m - \gamma_1 = 1$: by applying the matrix $(I_m + e_{1,m})^{-1} = (I_m - e_{1,m}) \in E_m(R, \mathfrak{a})$ to γ we get the same sequence γ, but with first coordinate 1. We can use then the first element in γ to nullify the other ones:

$$(I_m + e_{1,m}) \prod_{i=2}^{m} (I_m - \gamma_i e_{i,1})(I_m + e_{1,m})^{-1} \gamma = e_m$$

The matrices $I_m - \gamma_i e_{i,1}$ belong to $E_m(R, \mathfrak{a})$; but the matrix $(I_m + e_{1,m})^{-1}$ is not in $E_m(R, \mathfrak{a})$, but we can conclude by using the normality of $\mathfrak{a}$-elementary matrices observed in Remark 1.3.2 (this is why we have also $(I_m + e_{1,m})$ in the product). □

We conclude the section with a technical result.

Proposition 3.6.2. *Suppose the pair $(R, \mathfrak{a})$ satisfies the stable range condition with index n. Then, for any $\mathfrak{a}$-unimodular sequence $(a_1, \dots, a_{n+1})$, there exist elements $v_1, \dots, v_{n-1}$ in $\mathfrak{a}$, such that the sequence $(a_1 + v_1 a_{n+1}, \dots, a_{n-1} + v_{n-1} a_{n+1}, a_n)$ is $\mathfrak{a}$-unimodular.*

Proof. Let $(a_1, \dots, a_{n+1})$ be an $\mathfrak{a}$-unimodular sequence in R^{n+1}. By applying twice the stable range condition, we get there exist elements $b_1, \dots, b_n, c_1, \dots, c_{n-1} \in \mathfrak{a}$ such that the sequence:

$$((a_1 + b_1 a_{n+1}) + c_1(a_n + b_n a_{n+1}), \dots, (a_{n-1} + b_{n-1} a_{n+1}) + c_{n-1}(a_n + b_n a_{n+1}))$$

is $\mathfrak{a}$-unimodular in R^{n-1}.

Clearly, if we add to the sequence a last coordinate a_n, the sequence remains $\mathfrak{a}$-unimodular, and this is also true if we subtract $c_i a_n$ from the i-th element. Therefore, the sequence

$$\alpha := ((a_1 + b_1 a_{n+1}) + c_1 b_n a_{n+1}, \dots, (a_{n-1} + b_{n-1} a_{n+1}) + c_{n-1} b_n a_{n+1}, a_n)$$

is $\mathfrak{a}$-unimodular in R^n. But α can be viewed as $(a_1 + v_1 a_{n+1}, \dots, a_n + v_n a_{n+1})$, by setting $v_i := b_i + c_i b_n$ for $i = 1, \dots, n-1$ and $v_n = 0$. This concludes the proof. □

Remark 3.6.3. The matrix operations we have just used are elementary. To be more precise, starting with the sequence $a := (a_1, \dots, a_{n+1})$, let a' be the vector $(a_1, \dots, a_n)$ and let w be the vector $(b_1 + c_1 b_n, \dots, b_{n-1} + c_{n-1} b_n, 0)$. Then:

$$\begin{pmatrix} I_n & w^T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a' \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} \alpha \\ a_{n+1} \end{pmatrix}$$

and in the proposition we proved that α is unimodular.

By repeating the argument, we get the following:

Corollary 3.6.4. *Suppose the pair $(R, \mathfrak{a})$ satisfies the stable range condition with index n, and fix an integer $m \geq n$. Then, for any $\mathfrak{a}$-unimodular sequence in R^{m+1}, say $(a_1, \dots, a_{m+1})$, there exist elements $v_i \in \mathfrak{a}$ with $v_i = 0$ for $i \geq n$, such that the sequence $(a_1 + v_1 a_{m+1}, \dots a_m + v_m a_{m+1})$ is $\mathfrak{a}$-unimodular in R^m.*

Remark 3.6.5. We remark that also in this case the matrix operation is of a special type:

$$\begin{pmatrix} I_n & * \\ 0 & I_{m-n+1} \end{pmatrix}$$

and the corollary says that the first m coordinates are unimodular.

4

Posets of sequences

Contents

The aim of the following chapter is to present a family of $\mathrm{GL}_n(R)$-posets, namely the posets of *split unimodular sequences*, and to study their acyclicity properties. These posets will be of fundamental importance in Theorem 5.3.2, where we will prove the homological stability result for general linear groups.

Theorem 3.6.1, the group of elementary matrices acts transitively on the set of unimodular sequences. As a consequence, the whole general linear group $\mathrm{GL}_n(R)$ acts transitively on this set (we assume the stable range condition is satisfied). According to Quillen's argument (see Chapter 2), if we want homological stability holds, we need a family of simplicial complexes with transitive $\mathrm{GL}_n(R)$-actions. Then, we have a problem: the set of unimodular sequences, as in Definition 3.5.1, is not a simplicial complex. To solve it, we consider a partial ordering on the family of unimodular sequences, getting a poset. Hence, by Remark 1.2.2, we obtain a simplicial complex.

We need a simplicial complex for each $n \in \mathbb{N}$. We will deal with this second problem by generalizing the definition of unimodular sequences. In Definition 4.3.1, we will define unimodular sequences in R^n as *k-frames*, i.e. sequences of vectors in R^n that are basis of direct summands. The poset of split unimodular sequences is then studied, in order to apply the categorical machinery in the next chapter.

The chapter is structured as follows. In the first section we study the poset of ordered sequences $\mathcal{O}(X)$ with elements in a set X; then, Proposition 4.1.7 shows that the link of a sequence in such posets has a nice topological interpretation. In the second section we study the acyclicity properties for a special kind of subposets of $\mathcal{O}(X)$. In particular, in Proposition 4.2.7, we show how the connectivity properties of a poset are related to those of its subposets. In Section 3 we give the generalization of unimodular sequences we need, and Proposition 4.3.5 proves that the elementary matrices act transitively also on the set of k-frames (when the stable range condition is satisfied). We conclude the section with Corollary 4.3.7, in which we prove the poset of unimodular sequences is highly connected. In the last section, we define the poset of split unimodular sequences. The transitive action is preserved, and Theorem 4.4.7 shows that these complexes are highly acyclic.

We follow the article [Cha84], and we will use the same notations that we introduced in the first chapter.

4.1 Posets of ordered sequences

In the following, X will be a set, without further structures. Denote by $\mathcal{O}(X)$ the poset of ordered sequences $\mathbf{v} = (v_1, \dots, v_k)$ where $v_1, \dots, v_k$ are *distinct* elements of X. The partial ordering on $\mathcal{O}(X)$ is defined by refinements, i.e. $\mathbf{v} \leq \mathbf{w}$ if and only if $\mathbf{v}$ is an *ordered* subsequence of $\mathbf{w}$. Equivalently, $(v_1, \dots, v_k) \leq (w_1, \dots, w_h)$ if and only if there is a strictly increasing map $\phi\colon \{1, \dots, k\} \to \{1, \dots, h\}$ such that $v_i = w_{\phi(i)}$. We will use the following notations:

- $|\mathbf{v}|$ is the *length* of the sequence $\mathbf{v}$;
- if $\mathbf{v} = (v_1, \dots, v_k)$ and $\mathbf{w} = (w_1, \dots, w_j)$ are two ordered sequences whose elements are all distinct, we denote by $\mathbf{vw}$ the ordered sequence given by the elements of $\mathbf{v}$ and then by the elements of $\mathbf{w}$:
$$\mathbf{vw} := (v_1, \dots, v_k, w_1, \dots, w_j)$$
- if $\mathbf{w} < \mathbf{v}$, $\mathbf{v} - \mathbf{w}$ is the complement of $\mathbf{w}$ in $\mathbf{v}$.

We will not consider every subset of $\mathcal{O}(X)$, but we will focus on a class of subposets, closed under refinements and permutations:

Definition 4.1.1. We say that a subset $F \subseteq \mathcal{O}(X)$ satisfies the *chain condition* if:

1. $\mathbf{v} \in F$, $\mathbf{w} \leq \mathbf{v} \Rightarrow \mathbf{w} \in F$;
2. $\mathbf{v} \in F \Rightarrow \sigma(\mathbf{v}) := (v_{\sigma(1)}, \dots, v_{\sigma(|\mathbf{v}|)}) \in F$, where $\sigma \in \mathcal{S}_{|\mathbf{v}|}$ is any permutation of the set of $|\mathbf{v}|$ elements.

Considering such particular subsets of $\mathcal{O}(X)$ we get interesting consequences on the structure of the link of a sequence. As a first observation, suppose that $F \subseteq \mathcal{O}(X)$ satisfies the chain condition, and fix a sequence $\mathbf{v} \in F$. The negative link
$$\mathrm{Link}_F^-(\mathbf{v}) = \{\mathbf{w} \in F \mid \mathbf{w} < \mathbf{v}\}$$

is given by *all* the proper ordered subsequences of $\mathbf{v}$. We remark that this is not true for general subsets of $\mathcal{O}(X)$. Moreover, the realization of this poset is homeomorphic to the sphere $\mathbb{S}^{|\mathbf{v}|-2}$:

Remark 4.1.2. Suppose $F \subseteq \mathcal{O}(X)$ satisfies the chain condition and $\mathbf{v} = (v_1, \dots, v_k) \in F$. Without loss of generality, by the chain condition, we can assume $F = \mathcal{O}(\{1, \dots, k\})$, and $\mathbf{v} = (1, \dots, k)$. We can view $\mathbf{v}$ itself as the simplex $\Delta^{k-1} = 1 \leq \dots \leq k$, whose vertices are the numbers $1, \dots, k$, and a subsequence $(i_1, \dots, i_k)$ as the face $i_1 \leq \dots \leq i_k$. We know that the $\mathrm{Link}_F^-(\mathbf{v})$ is given by all the subsequences $(i_1 \leq \dots \leq i_k)$ where $i_j \in \{1, \dots, k\}$; the relation $(i_1, \dots, i_k) \leq (i'_1, \dots, i'_h)$ reflects in Δ^{k-1} to the fact that $i_1 \leq \dots \leq i_k$ is a face of $i'_1 \leq \dots \leq i'_h$, therefore the elements of $\mathrm{Link}_F^-\mathbf{v}$ correspond actually to all the faces of Δ^{k-1}, so the realization of the link is just $\mathbb{S}^{k-2}$.

Furthermore, if F satisfies the chain condition, we also get an interesting description of the positive link

$$\mathrm{Link}_F^+(\mathbf{v}) = \{\mathbf{w} \in F \mid \mathbf{w} > \mathbf{v}\}.$$

This is not so straightforward, then, we proceed step by step. First of all, by definition, a sequence $\mathbf{w} \in \mathrm{Link}_F^+(\mathbf{v})$ is a sequence in F, and has to contain $\mathbf{v}$ as an ordered subsequence (observe that if $\mathrm{Link}_F^+(\mathbf{v})$ is not void, then the sequence $\mathbf{v}$ is also in F). By the chain condition, the complement $\mathbf{w} - \mathbf{v}$ has to be in F, as $\mathbf{w} - \mathbf{v} \leq \mathbf{w}$. There exists a permutation $\sigma \in \Sigma_{|\mathbf{w}|}$ such that the sequence $(\mathbf{w} - \mathbf{v})\mathbf{v}$ is $\sigma(\mathbf{w})$; then, by the permutation property, $(\mathbf{w} - \mathbf{v})\mathbf{v}$ belongs to F; briefly:

$$\mathbf{w} \in \mathrm{Link}_F^+(\mathbf{v}) \Rightarrow \mathbf{w} \in F \Rightarrow \mathbf{w} - \mathbf{v} \in F \Rightarrow (\mathbf{w} - \mathbf{v})\mathbf{v} \in F.$$

Vice versa, if $\mathbf{z}$ is a sequence such that $\mathbf{zv} \in F$, the permutations $\sigma(\mathbf{zv})$ are then in F; the permutations that fix the order of $\mathbf{v}$ give elements $\sigma(\mathbf{zv}) \in \mathrm{Link}_F^+(\mathbf{v})$. This leads us to the following definition, that will play a crucial role in our discussion:

Definition 4.1.3. If $F \subseteq \mathcal{O}(X)$ and $\mathbf{v} \in \mathcal{O}(X)$, define the poset of transverse sequences to $\mathbf{v}$ as the subposet of $\mathcal{O}(X)$ $F_{\mathbf{v}} := \{\mathbf{w} \in F \mid \mathbf{wv} \in F\}$.

Remark 4.1.4. Observe that in the previous definition we do not assume that F satisfies the chain condition (in such a case the constrain $\mathbf{w} \in F$ is vacuous). The following associative property is satisfied: $(F_{\mathbf{v}})_{\mathbf{w}} = F_{\mathbf{vw}}$.

If F satisfies the chain condition, we get a map:

$$\mathrm{Link}_F^+(\mathbf{v}) \to F_{\mathbf{v}}, \quad \mathbf{w} \longmapsto \mathbf{w} - \mathbf{v}$$

that is a well-defined poset map. Moreover, it is a surjective morphism, but it is not injective, so we can think about it as a projection. We can get back $\mathrm{Link}_F^+(\mathbf{v})$ from the poset $F_{\mathbf{v}}$ by inserting $\mathbf{v}$ with the correct order in the sequences of $F_{\mathbf{v}}$; and each fixed way of inserting $\mathbf{v}$ will give us a section.

What we have done, roughly speaking, is deleting the v_i's to get the projection map and then adding the v_i's in a nice way to get a section. A generalization of this process can be done by considering abstract elements instead of the elements v_i's. This generalization can be described by the following Z^n*-construction*, that we give without any assumptions on $F \subseteq \mathcal{O}(X)$:

Definition 4.1.5. Fix $z_1, \ldots, z_n \notin X$ (formally we can think we are taking these elements in $\mathbb{N}$) and suppose they are all distinct. Define:

$$Z^n F := \{\mathbf{w} \in \mathcal{O}(X \cup \{z_1, \ldots, z_n\}) \mid \mathbf{z} = (z_1, \ldots, z_n) < \mathbf{w} \text{ and } \mathbf{w} - \mathbf{z} \in F\}$$

There is a projection $Z^n F \to F$ obtained by considering the complements $\mathbf{w} - \mathbf{z}$, so by deleting the z_i's, and a noteworthy section $l_n \colon F \to Z^n F$ defined by $\mathbf{v} \to \mathbf{vz}$, so by adding the z_i's.

Thanks to this definition, we btain the isomorphism we were looking for:

Proposition 4.1.6. *Suppose the poset $F \subseteq \mathcal{O}(X)$ satisfies the chain condition. Then the map*

$$\psi \colon \mathit{Link}_F^+(\mathbf{v}) \longrightarrow Z^{|v|}(F_{\mathbf{v}}), \quad \mathbf{w} \mapsto \mathbf{w}' := \psi(\mathbf{w})$$

defined by sending the elements v_i's to the elements z_i's is an isomorphism.

Proof. This is a well-defined map because of the following facts:

- by the definitions, $\mathbf{v} \notin \mathrm{Link}_F^+(\mathbf{v})$ and $\mathbf{z} \notin Z^n F$;
- if $\mathbf{w} \in \mathrm{Link}_F^+(\mathbf{v})$ then $\mathbf{w} \in F$, $\mathbf{v} < \mathbf{w}$ and by refinement $\mathbf{w} - \mathbf{v} \in F$; then $w' = \psi(\mathbf{w})$ is such that $\mathbf{w}' \in \mathcal{O}(X \cup \{z_1, \ldots, z_{|\mathbf{v}|}\}$, $\mathbf{z} < \mathbf{w}'$ and $\mathbf{w}' - \mathbf{z} \in F$.
- $(\mathbf{w} - \mathbf{v})\mathbf{v} = \sigma(\mathbf{w}) \in F$, where σ is the permutation that sends the elements v_i's in the last coordinates; then $(\mathbf{w}' - \mathbf{z})\mathbf{v} \in F$ and $\mathbf{w}' - \mathbf{z} \in F_{\mathbf{v}}$.

The map ψ is a poset map, and vice versa, the map $z_i \mapsto v_i$ is its inverse poset map. □

As the link is the join of the positive and negative ones, using this result together with the previous identification of the negative link with the sphere, we get the useful and interesting homeomorphism:

Proposition 4.1.7. *Let $F \subseteq \mathcal{O}(X)$ be a subposet satisfying the chain condition. Let $\mathbf{v} \in F$ be a sequence of length k, then:*

$$|\mathit{Link}_F(\mathbf{v})| \simeq \mathbb{S}^{k-2} * |Z^k F_{\mathbf{v}}| \simeq \Sigma^{k-1} |Z^k F_{\mathbf{v}}|.$$

We conclude the section with the description of a filtration for the Z^n-poset:

Remark 4.1.8. If $\mathbf{w}$ is a sequence in Z^nF, namely $\mathbf{w} = (w_1, \dots, w_k)$, then $\mathbf{z} < \mathbf{w}$ and there is a map $\phi\colon \{1, \dots, n\} \to \{1, \dots, k\}$ such that $w_{\phi(i)} = z_i$. We can define the position of z_1 in $\mathbf{w}$ as $\mathrm{pos}_{z_1}(\mathbf{w}) := \phi(1)$. If there is an index $i > \phi(1)$ such that $w_i \in X$ then we call $\mathbf{w}$ of *general type*. We call it of *special type* otherwise. The sequence $\mathbf{w}$ is special if and only if it lies in the image of the section $l_n\colon F \to Z^nF$ sending $\mathbf{v}$ to $\mathbf{vz}$.

As Z^nF is the union of general and special sequences, starting with the set of general sequences and adding step by step the special ones we get a filtration: let X_0 be the subposet consisting of general vectors, then define inductively

$$X_i := X_{i-1} \cup \{\mathbf{w} \in l_n(F) \mid |\mathbf{w} - \mathbf{z}| = i\} = X_{i-1} \cup \{l_n(\mathbf{v}) \mid |\mathbf{v}| = i\}$$

Clearly $X_{i-1} \subseteq X_i$, $Z^nF = \cup_i X_i$, and Z^nF is filtered by the X_i's.

4.2 Acyclicity properties

An important feature of the Z^n-construction is given by the acyclicity properties of the section map $l_n\colon F \to Z^nF$:

Definition 4.2.1. Let $f\colon F \hookrightarrow G$ be an injective poset map. We say that f is 0-*connected* (or 0-acyclic) if the induced map $f_*\colon \pi_0(|F|) \twoheadrightarrow \pi_0(|G|)$ is surjective; if $d > 0$ we say that f is d-*connected* (respectively d-*acyclic*) if f is 0-connected and $\pi_q(|G|, |f(F)|) = 0$ (respectively $H_q(|G|, |f(F)|) = 0$) for each $0 < q \leq d$.

Remark 4.2.2. If $f\colon F \hookrightarrow G$ is d-acyclic (d-connected) and F is d-acyclic (d-connected) then, by the long exact sequence of the pair (in homotopy), G is also d-acyclic (d-connected).

To study these acyclicity properties it will be useful to understand, given two subposets $G \subseteq F \subseteq \mathcal{O}(X)$, whenever they are homotopic equivalent. Recall that for arbitrary subsets S of a poset X, a *supremum* for S is an element $x \in X$ such that $s \leq x$ for each $s \in S$, and for any $y \in X$ with this property, it happens that $x \leq y$. We will use the Remark 1.2.3, i.e. if $f, g\colon X \to Y$ are poset maps with $f \leq g$ then f and g are homotopic.

Lemma 4.2.3. *Let $G \subseteq F \subseteq \mathcal{O}(X)$.*

1. *Suppose there exists $\mathbf{v} \in F$ such that $G \subseteq F_{\mathbf{v}}$. Then $G \hookrightarrow F$ is null homotopic;*

2. *Suppose that for all $\mathbf{v} \in F$ the set $G^-(\mathbf{v}) := \{\mathbf{w} \in G \mid \mathbf{w} \leq \mathbf{v}\}$ has supremum in itself. Then G is a deformation retract of F.*

Proof. 1. Consider the inclusion map $\iota\colon G \hookrightarrow F$ and the map $\mathbf{v}_*\colon G \to F$ defined by $\mathbf{w} \mapsto \mathbf{wv}$. As $G \subseteq F_{\mathbf{v}} \subseteq F$, these maps are well defined; but $\mathbf{w} \leq \mathbf{wv}$ for each $\mathbf{w} \in G$, therefore the maps ι and v_* are homotopic. We have also $\mathbf{v} \leq \mathbf{wv}$, so the map $\mathbf{v}_*$ is homotopic to the constant map $\mathbf{v} \mapsto \mathbf{v}$. Then ι is null-homotopic.

2. Define $f\colon F \to G$ as $f(\mathbf{v}) = \sup G^{-}(\mathbf{v})$; then $f(\iota(\mathbf{w})) = \mathbf{w}$ for each $\mathbf{w} \in G$ and $\iota(f(\mathbf{v})) = \sup G^{-}(\mathbf{v})) \leq \mathbf{v}$ for each $\mathbf{v} \in F$, that means $\iota \circ f$ is homotopic to the constant map. This shows G is a deformation retract of F.

□

In the following proposition we prove that the map l_n is highly acyclic if we assume $F_{\mathbf{v}}$ highly acyclic:

Proposition 4.2.4. *Fix an integer $d \in \mathbb{Z}$ and suppose the poset $F \subseteq \mathcal{O}(X)$ satisfies the chain condition. Assume that $F_{\mathbf{v}}$ is $(d - |\mathbf{v}|)$-acyclic for each $\mathbf{v}$ in F. Then the poset map*

$$l_n \colon F \hookrightarrow Z^n F, \quad \mathbf{v} \mapsto \mathbf{vz}$$

id d-acyclic.

Proof. We prove the statement by induction on n.

If $n = 0$, the map $l_0 \colon F \to Z^0 F$ is the identity map. It is d-acyclic.

If $n \neq 0$, suppose the statement is true for the maps l_i's, with $i < n$. Then the map $l_{n-1} \colon F \to Z^{n-1}F$ is d-acyclic.

Adding z_n to a sequence $\mathbf{v} = (v_1, \ldots, v_k) \in Z^{n-1}F$ gives rise to an isomorphism map:

$$Z^{n-1}F \longrightarrow P_0 \subseteq Z^n F$$

onto the poset $P_0 := \{(w_1, \ldots, w_h) \in Z^n F \mid w_h = z_n\}$. In fact, the inverse map $P_0 \to Z^{n-1}F$ is the morphism that deletes the last coordinate z_n in the sequence $\mathbf{w} \in P_0$. Both the maps are poset maps.

To deduce the acyclicity properties we want to filter $Z^n F$ with simpler posets and simpler maps. We use the same idea as in Remark 4.1.8. Define first the section

$$r \colon F \hookrightarrow Z^n F \quad (v_1, \ldots, v_k) \mapsto (z_1, \ldots, z_n, v_1, \ldots, v_k).$$

Then define the poset $P_1 := Z^n F - r(F)$. We have the inclusions $P_0 \subseteq P_1 \subseteq Z^n F$ as an element of P_0 is not in the image of r. Moreover, if $\mathbf{v} \in P_1$ then $P_0^{-}(\mathbf{v})$ has supremum in itself. In fact, if $\mathrm{pos}_{z_n}(\mathbf{v}) = j$, let $\mathbf{w}$ be the ordered sequence $\mathbf{w} := (v_1, \ldots, v_j)$. As $\mathbf{v} \in P_1 = Z^n F - r(F)$, $\mathbf{v} \notin r(F)$ and then $\mathbf{w} \neq (z_1, \ldots, z_n)$. The sequence $\mathbf{w}$ belongs to P_0 as $v_j = z_n$, $\mathbf{w}$ is a subsequence of $\mathbf{v}$ and it is the supremum for $P_0^{-}(\mathbf{v})$. By the previous Lemma 4.2.3, P_0 is a deformation retract for P_1. As a remark, observe that, to get this deformation, we had to factor out all the image of r from $Z^n F$, in such a way we don't have $\mathbf{w} = \mathbf{z}$, that is not in $Z^n F$.

We proceed by filtering $Z^n F$ with base $Q_0 := P_1$, adding step by step the image of r:

$$Q_i := Q_{i-1} \cup \{\mathbf{zv} \in r(F) \mid |v| = i\}.$$

We get the map l_n decomposes in the following way:

$$l_n \colon F \to Z^{n-1}F \to P_0 \hookrightarrow P_1 = Q_0 \hookrightarrow Q_1 \hookrightarrow \ldots \hookrightarrow \bigcup_i Q_i = Z^n F$$

Observe that $Q_i - Q_{i-1}$ is a discrete poset; as we described in the first Chapter, $|Q_i|$ is obtained from $|Q_{i-1}|$ by attaching a cone over each $\mathrm{Link}_{Q_i} r(\mathbf{v})$, with $|\mathbf{v}| = i$. If we prove that these links are $(d-1)$-acyclic, then by using the Mayer-Vietoris argument (as in Proposition 1.2.8) and the long exact sequence in homology, we get the map $Q_{i-1} \hookrightarrow Q_i$ is d-acyclic. Because the composition $F \hookrightarrow P_1$ is also d-acyclic (as composition of d-acyclic maps), this implies also the map l_n is d-acyclic.

The last part of the proof is to prove that the links $\mathrm{Link}_{Q_i} r(\mathbf{v})$ are $(d-1)$-acyclic. We study separately the positive and negative links.

Let $\mathbf{v} \in F$ be a sequence of length i, and consider $\mathbf{w} \in \mathrm{Link}^-_F(\mathbf{v})$. The sequence $\mathbf{w}$ belongs to F and $\mathbf{w} < \mathbf{v}$, then $|\mathbf{w}| < i$; moreover, $r(\mathbf{w}) = \mathbf{zw} \in Q_i$ and $\mathbf{zw} < \mathbf{zv}$, that means $r(\mathbf{w}) = \mathbf{zw} \in \mathrm{Link}^-_{Q_i}(\mathbf{zv})$. On the other hand, suppose $\mathbf{w} \in \mathrm{Link}^-_{Q_i}(\mathbf{zv})$. As $\mathbf{w} \in Q_i \subseteq Z^n F$ we get $\mathbf{z} < \mathbf{w} < \mathbf{zv}$. Then, $\mathbf{w} \in Q_{i-1}$ and $|\mathbf{w}| < n + i$. Therefore, $\mathbf{w} - \mathbf{z} < \mathbf{v}$ and it belongs to $\mathrm{Link}^-_F(\mathbf{v})$. This implies the map r induces an isomorphism

$$\mathrm{Link}^-_F(\mathbf{v}) \longrightarrow \mathrm{Link}^-_{Q_i}(r(\mathbf{v})).$$

But F satisfies the chain condition, and then the link $\mathrm{Link}_{Q_i}(r(\mathbf{v}))$ is homotopic to the suspension $\Sigma^{i-1}|\mathrm{Link}^+_{Q_i}(\mathbf{zv})|$.

Consider the positive link; the map:

$$Z^{n-1}(F_{\mathbf{v}}) \to \mathrm{Link}^+_{Q_i}\mathbf{zv}, \quad \mathbf{s} = (s_1, \ldots, s_k) \mapsto (s_1, \ldots, s_k, z_n, v_1, \ldots, v_i)$$

is a homotopy equivalence. First of all, this map is a well defined poset map: if $\mathbf{s} \in Z^{n-1}F_{\mathbf{v}}$ then $(\mathbf{z} - (z_n)) < \mathbf{s}$, and $\mathbf{s} - (\mathbf{z} - (z_n)) \in F_{\mathbf{v}}$; therefore $(\mathbf{s}, z_n, \mathbf{v}) \in Z^n F$, and actually it belongs to P_1, as it can not be in the image of r (the string $(z_1, \ldots, z_{n-1} \notin Z^{n-1}(F_{\mathbf{v}}))$.

As an homotopic inverse define the poset map:

$$\mathrm{Link}^+_{Q_i}\mathbf{zv} \to Z^{n-1}F_{\mathbf{v}}, \quad (t_1, \ldots, t_k) \mapsto (t_1, \ldots, t_j) \text{ where } j < k, t_{j+1} = z_n.$$

If $(t_1, \ldots, t_k)$ is an element of $\mathrm{Link}^+_{Q_i}\mathbf{zv} \subseteq Z^n F$ then $\mathbf{zv} < (t_1, \ldots, t_k)$ and $\mathbf{t} - \mathbf{z} \in F$; as a consequence $(t_1, \ldots, t_j)$ has $(z_1, \ldots, z_{n-1})$ as a refinement, and by the chain condition $((t_1, \ldots, t_j) - (z_1, \ldots, z_{n-1}))$ is in $F_{\mathbf{v}}$.

Finally, observe that the composition $\mathbf{s} \mapsto \mathbf{s}z_n\mathbf{v} \mapsto \mathbf{s}$ is the identity map, and that the composition $\mathbf{t} \mapsto (t_1, \ldots, t_j) \mapsto (t_1, \ldots, t_j, z_n, v_1, \ldots, v_i) \leq \mathbf{t}$ is homotopic to the constant map $\mathbf{t} \mapsto \mathbf{t}$. Therefore the two posets of above are homotopy equivalent.

By induction, as $(F_{\mathbf{v}})_{\mathbf{w}} = F_{\mathbf{vw}}$, $F_{\mathbf{v}} \hookrightarrow Z^{n-1}F_{\mathbf{v}}$ is d-acyclic; by hypothesis $F_{\mathbf{v}}$ is $(d-i)$-acyclic, so $Z^{n-1}F_{\mathbf{v}}$ is $(d-i)$-acyclic. Using the homotopy equivalence of above and applying the $(i-1)$-fold suspension map, we get the link $\mathrm{Link}_{Q_i} r(\mathbf{v}) \simeq \Sigma^{i-1}|Z^{n-1}F_{\mathbf{v}}|$ is $(d-1)$-acyclic, that concludes the proof. □

Corollary 4.2.5. *With the same assumptions of the previous Proposition, suppose in addiction F δ-connected, with $\delta = \min(1, d-1)$. Then the map $l_n \colon F \to Z^n F$ is also d-connected.*

Proof. If $d \leq 0$, we have nothing to prove, as d-acyclic is the same as d-connected.

Suppose $d \geq 1$. By induction, we can suppose the maps l_i are d-connected for $i < n$. As in the previous Proposition, we simplify the map l_n through the compositions:

$$l_n \colon F \to Z^{n-1}F \to P_0 \hookrightarrow P_1 = Q_0 \hookrightarrow Q_1 \hookrightarrow \ldots \hookrightarrow \bigcup_i Q_i = Z^n F$$

The map $F \hookrightarrow Q_0$ is d-acyclic and d-connected because of the induction hypothesis and the retraction. We have to prove that every inclusion map $Q_{i-1} \hookrightarrow Q_i$ is also d-connected.

By hypothesis, F is δ-connected, then Q_0 is δ-connected. Proceed supposing Q_{i-1} is δ-connected. Then Q_{i-1} is at least connected; as $|Q_i|$ is obtained from $|Q_{i-1}|$ by attaching cones, Q_i is 0-connected.

Applying Van Kampen Theorem, we get the map $\pi_1(|Q_{i-1}|) \twoheadrightarrow \pi_1(|Q_i|)$ is a surjection. In fact, as we have seen in the previous proposition, to get Q_i, we attach cones over $(d-1)$-connected spaces, that are at least connected. Let Z be the space obtained from $|Q_i|$ by attaching a strip for each cone, this strip starting at a fixed based point and connecting it with the cone. Clearly Z deformation retracts over $|Q_i|$; let A be the space $Z - \{\mathbf{zv} \mid |\mathbf{v}| = i\}$ (we suppose the chosen based point is not of this form), and B be the space $Z - |Q_{i-1}|$. Then A is connected and deformation retracts over $|Q_{i-1}|$ and B is connected and contractible. The intersection $A \cap B$ deformation retracts over a connected graph, and the union $A \cup B$ is the whole Z. Then, by Van Kampen Theorem, the map $\pi_1(|Q_{i-1}|) \simeq \pi_1(A) \to \pi_1(Z) \simeq \pi_1(|Q_i|)$ is surjective.

Using the long exact sequence of homotopy groups, $\pi_1(|Q_i|, |Q_{i-1}|) = 0$ and the map $Q_{i-1} \hookrightarrow Q_i$ is 1-connected; this prove the statement in the case $d = 1$.

If $d \geq 2$, F is also simply connected, then $\pi_1(|Q_i|) = \pi_1(|Q_{i-1}|) = 0$. Using step by step, until d, the relative Hurewicz Theorem, we get $\pi_j(|Q_i|, |Q_{i-1}|)$ is the trivial group. Therefore $Q_{i-1} \hookrightarrow Q_i$ is d-connected and Q_i is d-connected, that concludes the induction argument. □

Remark 4.2.6. Suppose $F \subseteq \mathcal{O}(X)$ satisfies the chain condition and $F_{\mathbf{v}}$ is $(d - |\mathbf{v}|)$-acyclic for each $\mathbf{v} \in F$. If F is d-acyclic (d-connected), then so is $Z^n F$.

It will be important, in the next sections, to deduce properties of F by properties of its subsets. The next proposition makes clear this relation:

Proposition 4.2.7. *Let $X \subseteq Y$ be sets and $F \subseteq \mathcal{O}(Y)$. Set $F_0 = F \cap \mathcal{O}(X)$ and choose $d \in \mathbb{Z}$. Suppose F satisfies the chain condition and $\mathcal{O}(X) \cap F_{\mathbf{v}}$ is $(d - |\mathbf{v}|)$-connected for each $\mathbf{v} \in F - F_0$. Then:*

1. *F_0 $(d-1)$-connected implies $F_0 \hookrightarrow F$ d-connected;*

2. *F_0 d-connected implies F d-connected.*

Proof. The second statement is a direct implication of the first one.

Suppose F_0 is $(d-1)$-connected and filter F by adding step by step the elements in $\mathcal{O}(Y-X)$ (in such a way we can simplify the map $F \cap \mathcal{O}(X) = F_0 \hookrightarrow F$):

$$P_0 := \{(v_1, \dots, v_k) \in F \mid \text{ some } v_j \in X\}; \quad P_i := P_{i-1} \cup \{(v_1, \dots, v_i) \in F \cap \mathcal{O}(Y-X)\}.$$

Clearly $F_0 \subseteq P_0$ and $F_0^-(\mathbf{v}) = \{\mathbf{w} \in F \cap \mathcal{O}(X) \mid \mathbf{w} \leq \mathbf{v}\}$ has supremum given by the ordered subsequence of $\mathbf{v}$ consisting of all the elements v_j such that $v_j \in X$. By Lemma 4.2.3, F_0 is a deformation retract of P_0, therefore P_0 is $(d-1)$-connected too. Proceed now by induction, supposing P_{i-1} is $(d-1)$-connected.

If $d < 0$ the statement is always true. If $d = 0$ the statement is true because attaching cones does not increase the number of connected components; $\pi_0(P_{i-1}) \to \pi_0(P_i)$ is surjective.

Suppose $d \geq 1$. If the links $\mathrm{Link}_{P_i}\mathbf{v}$ are $(d-1)$-connected, we can repeat the arguments in Proposition 4.2.4 and Corollary 4.2.5 to get the result.

The poset $P_i \subseteq \mathcal{O}(Y)$ satisfies the chain condition; if $\mathbf{v} \in P_i - P_{i-1}$ has length $|\mathbf{v}| = i$, by applying Proposition 4.1.7 we get $|\mathrm{Link}_{P_i}\mathbf{v}| \simeq \Sigma^{i-1}|Z^i(P_i)_{\mathbf{v}}|$. Let $\mathbf{w}$ be a sequence in $(P_i)_{\mathbf{v}}$; then $\mathbf{w} \in P_i$ and $\mathbf{wv} \in P_i$. But $|\mathbf{v}| = i$, therefore $\mathbf{w}$ is in P_0. For the same reason $\mathbf{wv}$ has to be in P_0 as well. This implies that:

$$\Sigma^{i-1}|Z^i(P_i)_{\mathbf{v}}| = \Sigma^{i-1}|Z^i(P_0)_{\mathbf{v}}|.$$

Moreover, $Z^i(P_0)_{\mathbf{v}}$ and $Z^i(F_0 \cap F_{\mathbf{v}})$ are homotopic equivalent. In fact, $F_0 \cap F_{\mathbf{v}} \subseteq (P_0)_{\mathbf{v}}$, and the supremum for $(F_0 \cap F_{\mathbf{v}})^-(\mathbf{w})$ is the ordered subsequence of $\mathbf{w}$ whose elements are given by all the elements of $\mathbf{w}$ that are in X. What we get is:

$$|\mathrm{Link}_{P_i}\mathbf{v}| \simeq \Sigma^{i-1}|Z^i(F_0 \cap F_{\mathbf{v}})| = \Sigma^{i-1}|Z^i(\mathcal{O}(X) \cap F_{\mathbf{v}})|$$

By hypothesis F satisfies the chain condition, then $\mathcal{O}(X) \cap F_{\mathbf{v}}$ satisfies the chain condition as well. As $\mathbf{v}$ does not belong to F_0, by hypothesis, $\mathcal{O}(X) \cap F_{\mathbf{v}}$ is $(d-i)$-connected. By the chain property, $(\mathcal{O}(X) \cap F_{\mathbf{v}})_{\mathbf{w}} = \mathcal{O}(X) \cap F_{\mathbf{vw}}$, and it is $(d-i-|\mathbf{w}|)$-connected. By Remark 4.2.6, $Z^i(F_0 \cap F_{\mathbf{v}})$ is $(d-i)$-connected, and as a consequence, the links $\mathrm{Link}_{P_i}\mathbf{v}$ are $(d-i+i-1)$-connected, that concludes the proof. □

Let $F \subseteq \mathcal{O}(X)$, $\mathbf{v} \in \mathcal{O}(X)$ and let S be any set. We can define an other useful construction, whose properties are quite similar to those of the Z^n-construction:

Definition 4.2.8. The S-construction over F is the poset $F < S >$ defined as

$$F < S > := \{\mathbf{w} \in \mathcal{O}(X \times S) \mid \mathbf{w} = ((v_1, s_1), \dots, (v_k, s_k)) \text{ and } (v_1, \dots, v_k) \in F\}$$

We want to show that this construction has the same connectivity properties of the Z^n-construction. First of all, chosen an element $s_0 \in S$ we can define the injective poset map:

$$s_{0*} \colon F \to F < S >, \quad (v_1, \dots, v_k) \mapsto ((v_1, s_0), \dots, (v_k, s_0))$$

As it is injective, we can wonder about its homotopy properties; we get the following result:

Proposition 4.2.9. *Let $d \in \mathbb{Z}$ and suppose that the poset $F \subseteq \mathcal{O}(X)$ satisfies the chain condition. Suppose moreover that $F_{\mathbf{v}}$ is $(d - |\mathbf{v}|)$-acyclic for every $\mathbf{v} \in F$. Then the poset maps*

1. $s_{0*} \colon F \to F < S >$;
2. $l_n \circ s_{0*} \colon F \to Z^n(F < S >)$.

are d-acyclic.

If, in addition, F is δ-connected, with $\delta = \min(1, d-1)$, then the above maps are d-connected.

Proof. 1. The proof is quite similar to the other ones, so we follow the same ideas. As a filtration for $F < S >$ we consider the following one:

$$Q_0 := \{((v_1, s_1), \dots, (v_k, s_k)) \in F < S >|\ \exists j : s_j = s_0\}$$

$$Q_i := Q_{i-1} \cup \{((v_1, s_1), \dots, (v_i, s_i)) \in F < S >|\ s_j \neq s_0\ \forall j \leq i\}$$

The base Q_0 deformation retracts on the poset $F \simeq s_{0*}(F)$; the supremum is given by the subsequence of $((v_1, s_1), \dots, (v_k, s_k))$ where all the s_j's are equal to s_0. We have to prove the inclusions $Q_{i-1} \hookrightarrow Q_i$ are d-acyclic (d-connected). Proceeding as in Proposition 4.2.4 and Corollary 4.2.5, we should prove the links are $(d-1)$-acyclic.

As Q_i satisfies the chain condition we get:

$$|\mathrm{Link}_{Q_i}((v_1, s_1), \dots, (v_i, s_i))| \simeq \Sigma^{i-1}|Z^i Q_{i((v_1,s_1),\dots,(v_i,s_i))}|$$

We want an homotopy equivalence between $|Z^i Q_{i((v_1,s_1),\dots,(v_i,s_i))}|$ and $Z^i F_{(v_1,\dots,v_i)}$. To do it, consider the links $\mathrm{Link}^+_{Q_i}((v_1, s_1), \dots, (v_i, s_i))$ and $\mathrm{Link}^+_F(v_1, \dots, v_i)$, and observe that $\mathrm{Link}^+_{Q_i}((v_1, s_1), \dots, (v_i, s_i))$ is the same as $\mathrm{Link}^+_{Q_0}((v_1, s_1), \dots, (v_i, s_i))$. Define the morphism:

$$\sigma \colon \mathrm{Link}^+_F(v_1, \dots, v_i) \longrightarrow \mathrm{Link}^+_{Q_0}((v_1, s_1), \dots, (v_i, s_i))$$

defined in the following way: if $(w_1, \dots, w_k) \in \mathrm{Link}^+_F(v_1, \dots, v_i)$ then there exists a morphism $\phi \colon \{1, \dots, i\} \to \{1, \dots, k\}$ such that $w_{\phi(j)} = v_j$; if j is not in the image of ϕ we will set $(\sigma(\mathbf{w}))_j := (w_j, s_0)$. In the other case we set $(\sigma(\mathbf{w}))_{\phi(j)} := (w_j, s_j) = (v_j, s_j)$. As an inverse we consider

$$\pi \colon \mathrm{Link}^+_{Q_0}((v_1, s_1), \dots, (v_i, s_i)) \longrightarrow \mathrm{Link}^+_F(v_1, \dots, v_i)$$

such that $\pi((w_1, t_1), \dots, (w_k, t_k))$ is the subsequence of $(w_1, \dots, w_k)$ consisting by either $w_j = v_k$ for some k or $t_j = s_0$. Now, the composition $\pi \circ \sigma$ is the identity map. In the other way, $\sigma \circ \pi((w_1, t_1), \dots, (w_k, t_k)) \leq ((w_1, t_1), \dots, (w_k, t_k))$ and the composition is homotopic to the constant map.

As a consequence,

$$|Z^i Q_{i((v_1,s_1),\dots,(v_i,s_i))}| \simeq |Z^i F_{(v_1,\dots,v_i)}|$$

and by the Proposition 4.2.4, they are $(d-i)$-connected. This completes in the usual way the proof of the first point.

2. The map is obtained by composing the maps l_n and s_{0*}. We have just to verify the poset $F < S >$ satisfies the assumptions of Proposition 4.2.4.

   As F satisfies the chain condition, $F < S >$ does. Fix $((v_1,s_1),\dots,(v_k,s_k))$ in $F < S >$ and consider $(w_1,t_1),\dots,(w_j,t_j))$ in $F < S >_{((v_1,s_1),\dots,(v_k,s_k))}$; then $(w_1,\dots,w_j,v_1,\dots,v_k)$ belongs to F and we get:

   $$F < S >_{((v_1,s_1),\dots,(v_k,s_k))} = F_{(v_1,\dots,v_k)} < S > .$$

   But this latter poset is $(d-k)$-acyclic by the previous point; we apply now Proposition 4.2.4.

□

Remark 4.2.10. In the hypothesis of the previous Proposition, if F is d-acyclic (d-connected) then $F < S >$ and $Z^n(F < S >)$ are d-acyclic (d-connected).

4.3 Posets of unimodular sequences

In this section we focus our attention on posets of unimodular sequences. As we did in Chapter 3, let R be a fixed associative ring with unit and let $\mathfrak{a}$ be a *proper* two-sided ideal. We have already given, step by step, the definition of unimodular sequences (as sequences of elements of R generating the whole ring), of $\mathfrak{a}$-unimodular sequences, of stable range condition for the ring R and for the pair $(R,\mathfrak{a})$, and we showed some examples and their properties.

In this section we want to further generalize the definition of unimodular sequences: what we need now are sequences of vectors $v_i \in R^n$, and not just of elements of R. As a possible generalization, we can consider sequences whose vectors split a direct summand in R^n (in the previous definition the elements generate R). If we use such a generalization, we get that a unimodular sequence of length one is actually a unimodular sequence as in Definition 3.1.1, and there will be no confusion. We can give the definition:

Definition 4.3.1. A sequence of vectors $(v_1,\dots,v_k)$, with $v_i \in R^n$, is called *unimodular* in R^n if it forms a basis for a direct summand of R^n. We will call such sequences also k-frames.

Equivalently, the map $R^k \to R^n$ defined by mapping the i-th generator to v_i is split injective. We can observe that, if $m > n$, and if $\mathbf{v}$ is unimodular in R^n, then it is unimodular in R^m.

Remark 4.3.2. The property to be unimodular in R^n is closed up to refinements and permutations of indices. This will be useful when we will need subsets of unimodular sequences that satisfy the chain condition.

Notation. Denote by R^∞ the direct limit, over n, of R^n, i.e. $R^\infty := \cup_n R^n$.

We will not deal just with the poset of unimodular sequences, but we are interested at its subsets. Let $X \subseteq R^\infty$ be any subset; we identify a sequence in R^n with its image in R^∞. We give some general notations:

- $\mathcal{O}_\mathfrak{a}(X) := \{(v_1, \ldots, v_k) \in \mathcal{O}(X) \mid \forall i\ \exists j_i \text{ such that } v_i \equiv e_{j_i}(\mathfrak{a})\}$ is the subposet of $\mathcal{O}(X)$ consisting of the canonical vectors in $\mathcal{O}(X)$, up to $\mathfrak{a}$-congruence;
- $U := \{\mathbf{v} = (v_1, \ldots, v_k) \in \mathcal{O}(R^\infty) \mid \mathbf{v} \text{ is unimodular}\}$ is the poset of unimodular sequences in R^k, for any k;
- $U_\mathfrak{a} := U \cap \mathcal{O}_\mathfrak{a}(R^\infty)$ is the poset of unimodular sequences, given by canonical vectors up to $\mathfrak{a}$-congruence;
- $U_\mathfrak{a}(X) := U \cap \mathcal{O}_\mathfrak{a}(X)$.

As in the previous chapter, once we have the notion of unimodular sequences we proceed by giving the definition of stable range condition. The definition we are going to give is nothing more then Definition 3.5.4, but this is a formulation that will be useful to argue connectivity properties.

Definition 4.3.3. Let $n \geq 2$. We say that the pair $(R, \mathfrak{a})$ satisfies the *stable range condition* with index n, and we denote this property by $SR_n(R, \mathfrak{a})$, if for every $m \geq n$ the following property is satisfied:

$$\forall (v) \in U_\mathfrak{a}(R^m) \text{ with } v \equiv e_1(\mathfrak{a}) \quad \exists\, w \in R^{m-1} + e_m \text{ such that } (v, w) \in U_\mathfrak{a}(R^m).$$

If we use the terminology of Definition 3.5.1, this condition means that for every $\mathfrak{a}$-unimodular vector $v = (v_1, \ldots, v_n)$ in R^n there exist an other vector $w = (w_1, \ldots, w_{n-1}, 1)$, with elements w_i's that belong to $\mathfrak{a}$, such that the couple (v, w) is a basis for a direct summand. The two definitions are related by the transformations:

$$\begin{pmatrix} v_1 & w_1 \\ v_2 & \vdots \\ \vdots & w_{n-1} \\ v_n & 1 \end{pmatrix} \longleftrightarrow \begin{pmatrix} v_1 - w_1 v_n & w_1 \\ \vdots & \vdots \\ v_{n-1} - w_{n-1} v_n & w_{n-1} \\ 0 & 1 \end{pmatrix} \longleftrightarrow \begin{pmatrix} v_1 - w_1 v_n & 0 \\ \vdots & \vdots \\ v_{n-1} - w_{n-1} v_n & 0 \\ 0 & 1 \end{pmatrix}$$

that means: the couple (v, w) is a basis for a direct summand if and only if the sequence $(v_1 + w_1 v_n, \ldots, v_{n-1} + w_{n-1} v_n)$ is $\mathfrak{a}$-unimodular. Therefore, the two definitions are equivalent.

Remark 4.3.4. For simplicity, we assume that the couple $(R, \mathfrak{a})$ satisfies the stable range condition $SR_{\text{sdim}+2}(R, \mathfrak{a})$ for some fixed sdim ≥ 0. If R is a commutative Noetherian ring of Krull dimension d, then we can let sdim be equal to d, according to Corollary 3.3.9. If R has stable dimension n then we set sdim $= n$.

We need the last notation:

Notation. We denote by $R^{n,m}$ the sum $R^n + \mathfrak{a}e_{n+1} + \cdots + \mathfrak{a}e_{n+m}$ and by $R^{n,\infty}$ the direct limit over m of $R^{n,m}$.

As the two definition of stable range condition are equivalent, we can use the results we get in the previous chapter. To be more precise, we extend here the transitive action of the elementary matrices on the set of unimodular sequences. Recall that $\widetilde{E}_n(R, \mathfrak{a})$ indicates the group generated by permutations and the elementary matrices.

Proposition 4.3.5. *Suppose the pair $(R, \mathfrak{a})$ satisfies the stable range condition with index sdim + 2.*

1. *If $n \geq$ sdim $+ k + 1$, $\widetilde{E}_n(R, \mathfrak{a})$ acts transitively on length k sequences $(v_1, \ldots, v_k)$ in $U_{\mathfrak{a}}(R^n)$.*
2. *If $n \geq$ sdim $+ 2$ and $(v) \in U_{\mathfrak{a}}(R^{n,\infty})$, then there exists a matrix g of the form*

$$\begin{pmatrix} \widetilde{E}_n(R, \mathfrak{a}) & * \\ 0 & I \end{pmatrix} \subseteq \widetilde{E}(R, \mathfrak{a})$$

   such that gv has first coordinate equal to one.
3. *If $n \geq$ sdim $+ k$ and $(v_1, \ldots, v_k) \in U_{\mathfrak{a}}(R^{n,\infty})$ then there exists a $w \in R^n + e_{n+1}$ such that $(w, v_1, \ldots, v_k) \in U_{\mathfrak{a}}$.*

Proof. 1. If $k = 1$, then the statement is proved in Theorem 3.6.1: the difference is that we consider now vectors $v = (v_1, \ldots, v_n)$ which are $\mathfrak{a}$-congruent to e_j, where j can be different from 1. Therefore, to get the action to be transitive on $U_{\mathfrak{a}}(R^n)$, we have to use also permutation matrices.

Proceed now by induction and suppose the statement is true for sequences of length $k-1$. Then, let $(v_1, \ldots, v_k)$ be a sequence of length k; by operating with permutation matrices we can suppose $v_i \equiv e_i(\mathfrak{a})$. Let $n \geq \text{sdim}+k+1$. By induction, there exists an elementary matrix M of order n such that $M(v_1, \ldots, v_k) = (e_1, \ldots, e_{k-1}, Mv_k)$. By using the first $k-1$ canonical vectors, we can suppose Mv_k has the first $k-1$ coordinates equal to zero. What is left is to operate on the last $n-k+1$ coordinates. But $n - k + 1 \geq \text{sdim} + k + 1 - k + 1 = \text{sdim} + 2$: we can apply Theorem 3.6.1 again.

2. Proceed step by step. Suppose (v) in $U_{\mathfrak{a}}(R^{n,1})$; then, up to a permutation of the elements, v is a vector of the form $(a_1, \ldots, a_{n+1})$, $\mathfrak{a}$-congruent to e_1. It is also unimodular, and we can apply Proposition 3.6.2, getting the existence of elements $v_1, \ldots, v_{n-1}$ such that the sequence $((a_1 + v_1 a_{n+1}, \ldots, a_{n-1} + v_{n-1} a_{n+1}, a_n)) \in U_{\mathfrak{a}}(R^n)$. By Remark 3.6.3, this operation is of the form

$$\begin{pmatrix} I_n & * \\ 0 & 1 \end{pmatrix}$$

Now, as $n = \text{sdim} + 2$ and (v) is in $U_{\mathfrak{a}}(R^n)$, the elementary matrices act transitively, giving a matrix g of the form

$$\begin{pmatrix} \widetilde{E}_n(R, \mathfrak{a}) & * \\ 0 & 1 \end{pmatrix}$$

such that the first coordinate of $g(a_1, \ldots, a_{n+1})$ is 1. As a remark, observe that we operate by permutation matrices before applying Proposition 3.6.2; but this does not change the result, as $a_{n+1} \in \mathfrak{a}$ and the permutation matrix operates just on the first $n \times n$ block.

If $(v) \in U_{\mathfrak{a}}(R^{n,m})$ we apply in the same way Corollary 3.6.4 and Remark 3.6.5.

The general case $n \geq \text{sdim} + 2$ and $(v) \in U_{\mathfrak{a}}(R^{n,\infty})$ is then a consequence, as $SR_{\text{sdim}+2}(R, \mathfrak{a})$ implies $SR_n(R, \mathfrak{a})$.

3. Proceed by induction on k. If $k = 1$, then the statement is actually the definition of stable range condition with index sdim + 2; in fact, the condition $(v_1) \in U_{\mathfrak{a}}(R^{n,\infty})$ tells us that v_1 has at least $n + 1 (\geq \text{sdim} + 2)$ coordinates, and the condition is satisfied.

Suppose $k > 0$, so $n \geq \text{sdim} + 2$. By the above point (2), there exists a matrix g such that gv_1 has first coordinate equal to 1. Apply the matrix g also at the vectors $v_2, \ldots, v_k$. Let β_i be the first coordinate of gv_i, and observe that $\beta_i \in \mathfrak{a}$. Let V be the subspace of R^∞ generated by the vectors whose first coordinate is 0. Then the sequence $(g(v_2 - \beta_2 v_1), \ldots, g(v_k - \beta_k v_1))$ is a unimodular sequence in $U_{\mathfrak{a}}(V \cap R^{n,\infty})$.

By induction there exists $u \in V \cap (R^n + e_{n+1})$ such that

$$(g(v_2 - \beta_2 v_1), \ldots, g(v_k - \beta_k v_1), u) \in U_{\mathfrak{a}}(V)$$

Hence $(gv_1, gv_2, \ldots, gv_k, u) \in U_{\mathfrak{a}}$, i. e. $(v_1, \ldots, v_k, g^{-1}u) \in U_{\mathfrak{a}}$. The statement is so proved, by setting $w := g^{-1}u$.

□

We can finally prove connectivity results for subposets of the unimodular sequences $U_{\mathfrak{a}}$; as convention we let the 0-length sequence be the void string, and in such a case we set $F_{\mathbf{v}} := F$.

Theorem 4.3.6. *Suppose the pair $(R, \mathfrak{a})$ satisfies the stable range condition with index sdim + 2. Let I be a set such that $0 \in I \subseteq R^{0,\infty}$.*

If $\mathbf{v} = (v_1, \dots, v_k) \in \mathcal{O}(R^\infty)$, where $k \geq 0$, then for each $n \geq 0$:

$A(n)$: *$\mathcal{O}_\mathfrak{a}(R^n + I) \cap U_\mathbf{v}$ is $(n - k - sdim - 2)$-connected if $k = 0$ or $\mathbf{v} \in U_\mathfrak{a}(R^{n,\infty})$;*

$B(n)$: *$\mathcal{O}_\mathfrak{a}((R^n + I) \cup (R^n + e_{n+1})) \cap U_\mathbf{v}$ is $(n - k - sdim - 1)$-connected if $k = 0$ or $\mathbf{v} \in U_\mathfrak{a}(R^{n,\infty})$;*

$B'(n)$: *$B(n)$ also holds if $\mathbf{v} \in U_\mathfrak{a}(R^{n+1,\infty})$ providing that some v_i has $(n+1)$-th coordinate equal to 1, or $\mathfrak{a}e_{n+1} = I$.*

Proof. Proceed by induction on n. If $n = 0$ the condition $A(0)$ is vacuous; the conditions $B(0)$ and $B'(0)$ are also vacuous for $k \neq 0$. If $k = 0$ (and sdim = 0), by convention, $\mathbf{v}$ is the void string and $U_\mathbf{v} = U$; in such a case the set $\mathcal{O}_\mathfrak{a}(I + e_{n+1}) \cap U$ certainly contains the sequence (e_{n+1}). Therefore we can suppose by induction $A(i)$, $B(i)$ and $B'(i)$ hold for $0 \leq i < n$. We proceed now by proving separately the three conditions.

$A(n)$: We want to prove a connectivity result about the poset $F := \mathcal{O}_\mathfrak{a}(R^n + I) \cap U_\mathbf{v} \subseteq \mathcal{O}(R^\infty)$; the idea is then to deduce the connectivity properties from those of some subposet, applying Proposition 4.2.7. We want to use also the induction hypothesis; to do it, consider the set $X = (R^{n-1} + \mathfrak{a}e_n) \cup (R^{n-1} + e_n))$. With the same notations of Proposition 4.2.7, set $F_0 := F \cap \mathcal{O}_X$ and $d := (n - \text{sdim} - k - 2)$. By Remark 4.3.2, F satisfies the chain condition.

We need connectivity properties for $\mathcal{O}(X) \cap F_\mathbf{w}$, where $\mathbf{w} \in F - F_0 \subseteq F$. Let $\mathbf{w}$ be of the form $(w_1, \dots, w_j)$, with $j \geq 0$. Then:

$$\mathcal{O}(X) \cap F_{(w_1 \dots, w_j)} = \mathcal{O}_\mathfrak{a}(X) \cap U_{(v_1, \dots, v_k, w_1, \dots, w_j)}.$$

In fact, if $\mathbf{z}$ is in $\mathcal{O}(X) \cap F_\mathbf{w}$ then $\mathbf{z} \in F \cap \mathcal{O}(X)$ and $\mathbf{zw} \in F = \mathcal{O}_\mathfrak{a}(R^n + I) \cap U_\mathbf{v}$, so $\mathbf{zwv}$ is unimodular and each component is $\mathfrak{a}$-congruent to some e_h; in particular each z_i is $\mathfrak{a}$-congruent to some e_h. By the condition $B'(n-1)$, $\mathcal{O}_\mathfrak{a}(X) \cap U_{(v_1, \dots, v_k, w_1, \dots, w_j)}$ is $(d - j)$-connected. Applying Proposition 4.2.7, as F_0 is d-connected, F is d-connected. This concludes the case $\mathbf{v} \in U_\mathfrak{a}(R^{n,\infty})$; the case $k = 0$ follows by $B(n-1)$.

$B(n)$, $I = \{0\}$ To prove $B(n)$ we distinguish the case $I = \{0\}$ from the general case. As above, the idea is to merge the induction hypothesis and Proposition 4.2.7. Let

$$\begin{aligned} F &:= \mathcal{O}_\mathfrak{a}(R^n \cup (R^n + e_{n+1})) \cap U_\mathbf{v} \\ X &:= R^n \\ F_0 &:= F \cap \mathcal{O}(X) \\ d &:= n - \text{sdim} - k - 1 \end{aligned}$$

If $d = -1$ and $\mathbf{v} \in U_{\mathfrak{a}}(R^{n,\infty})$, we get $n = \mathrm{sdim} + k$; by Proposition 4.3.5, there exists an element $w \in R^n + e_{n+1}$ such that $(w, v_1, \ldots, v_k) \in U_{\mathfrak{a}}$, then $(w) \in F$. The case $k = 0$ is obtained using the element $(e_n) \in F$.

Suppose $d \geq 0$ and fix a sequence $\mathbf{w} \in F - F_0$, $\mathbf{w} = (w_1, \ldots, w_j)$. As the sequence $\mathbf{w}$ is not in F_0, there exists an index i such that the vector $w_i \in R^n + e_{n+1}$; actually, this is the only index with this property (as the ideal $\mathfrak{a}$ is proper, $\mathbf{w}$ has to be unimodular, and the vectors w_h have to be congruent to some canonical vector). Up to a permutation of the indices, we can suppose $i = 1$. Consider the poset $\mathcal{O}(X) \cap F_{\mathbf{w}}$: if $\mathbf{z} \in \mathcal{O}(X) \cap F_{\mathbf{w}}$, then the vectors z_i's are in R^n, there exist indices j_i's such that $z_i \equiv e_{j_i}(\mathfrak{a})$, and $\mathbf{zwv} \in U$. This gets to the following equality:

$$\mathcal{O}(X) \cap F_{\mathbf{w}} = \mathcal{O}_{\mathfrak{a}}(X) \cap U_{\mathbf{wv}}$$

As the vectors v_i's are in $R^{n,\infty}$, they have $(n+1)$-th coordinate in $\mathfrak{a}$, say α_i. For each i let v'_i be the vector $v_i - \alpha_i w_1$. Then $\mathcal{O}_{\mathfrak{a}}(X) \cap U_{\mathbf{wv}} = \mathcal{O}_{\mathfrak{a}}(X) \cap U_{\mathbf{wv'}}$, as we are just giving a different presentation of the basis $\{w_1, \ldots, w_j, v_1, \ldots, v_k\}$. Finally,

$$\mathcal{O}_{\mathfrak{a}}(X) \cap U_{\mathbf{wv'}} = \mathcal{O}_{\mathfrak{a}}(X) \cap U_{(w_2, \ldots, w_j)\mathbf{v'}}$$

In fact, the sequences $\mathbf{z}$ in these two posets are, in particular, in $\mathcal{O}_{\mathfrak{a}}(R^n)$; then $\mathbf{z}$ is unimodular, with $\mathbf{zwv'}$ unimodular, if and only if $\mathbf{z}(w_2, \ldots, w_j)\mathbf{v'}$ is unimodular (these elements have to give a basis for a direct summand, but w_1 is the only vector with non-null $(n+1)$-th component). Applying the property $A(n)$, we get $\mathcal{O}(X) \cap F_{\mathbf{w}}$ is $(d-j)$-connected and F_0 is $(d-1)$-connected; by Proposition 4.2.7 the inclusion map $F_0 \hookrightarrow F$ is d-connected.

The difference with the case $A(n)$ is that we know F_0 is $(d-1)$-connected, but not d-connected. The long exact sequence of homotopy groups gives isomorphisms $\pi_i(F) \simeq \pi_i(F_0) = 0$ for $i < d$ and a surjection $\pi_d(F_0) \twoheadrightarrow \pi_d(F)$. The next step is to show that this is the 0-map, that concludes the argument.

$\pi_d(F_0) \twoheadrightarrow \pi_d(F)$ If $k = 0$ we get $F = \mathcal{O}_{\mathfrak{a}}(R^n \cup (\mathbb{R}^n + e_{n+1})) \cap U$; $\mathbf{z} \in F_0$ implies that $\mathbf{z}$ is unimodular, the vectors z_i's are in R^n, and they are $\mathfrak{a}$-congruent to e_{j_i}. As $\mathbf{z}(e_{n+1})$ is unimodular with components in $\mathcal{O}_{\mathfrak{a}}$, we have the inclusions $F_0 \subseteq F_{(e_{n+1})} \subseteq F$. By Lemma 4.2.3 the inclusion map $F_0 \hookrightarrow F$ is null-homotopic.

Suppose $k \neq 0$; then $n \geq \mathrm{sdim} + 2$, and by Proposition 4.3.5, there exists a matrix g in $\widetilde{E}(R, \mathfrak{a})$, that operates in the first n coordinates as $\widetilde{E}_n(R, \mathfrak{a})$, such that gv_1 has first coordinate equal to 1. The action of g on the poset F is, up to $\mathfrak{a}$-congruence, a permutation. As F satisfies the chain condition, the posets F and gF are equivalent, and we can suppose w.l.o.g. that v_1 has first coordinate 1; moreover, we can suppose v_i, for $i \neq 1$, with first coordinate equal to 0.

The idea is now to factor the inclusion

$$F_0 = \mathcal{O}_{\mathfrak{a}}(R^n) \cap U_{\mathbf{v}} \hookrightarrow \mathcal{O}_{\mathfrak{a}}(R^n \cup (R^n + e_{n+1})) \cap U_{\mathbf{v}} = F$$

trough a d-connected space. As subspaces of F_0 and F we could consider sets of the form $R^n + \mathfrak{a}u$ and $(R^n + \mathfrak{a}u) \cup (R^n + e_{n+1})$ (to which we can apply the inductive hypothesis), with u some nice vector; as the only element of $\mathbf{v}$ with first non-null coordinate is v_1, we also restrict to vectors with first coordinate 0:

$$X_u := \{x \in (R^n + u\mathfrak{a}) \mid x^1 = 0\}, \quad Y_u := \{x \in (R^n + u\mathfrak{a}) \cup (R^n + e_{n+1}) \mid x^1 = 0\}.$$

A nice choice for the vector u is $u := v_1 - e_1$; to avoid the components greater then n, we will use the ideal:

$$\mathfrak{b} := \{a \in \mathfrak{a} \mid ua \in R^n\} = \{a \in \mathfrak{a} \mid v_1 a \in R^n\}.$$

Suppose we have the commutative diagram:

$$\begin{array}{ccc} \mathcal{O}(X) \cap F = F_0 & \hookrightarrow & F \\ {\scriptstyle (\psi|_{X_u})_*} \uparrow & & \uparrow {\scriptstyle \psi_*} \\ (\mathcal{O}_{\mathfrak{a}}(X_u) \cap U_{\mathbf{v}}) < \mathfrak{b} > & \hookrightarrow & (\mathcal{O}_{\mathfrak{a}}(Y_u) \cap U_{\mathbf{v}}) < \mathfrak{b} > \end{array}$$

where $(\psi|_{X_u})_*$ is an isomorphism. The poset $\mathcal{O}_{\mathfrak{a}}(Y_u) \cap U_{\mathbf{v}}$ is the same as the poset $\mathcal{O}_{\mathfrak{a}}(Y_u) \cap U_{(v_2,\dots,v_k)}$, as the elements in Y_u have first coordinate 0, and the same is for v_i, $i \neq 1$. If σ is the permutation matrix that shifts all the first n coordinates up by one:

$$\sigma = \begin{pmatrix} 0 & I_{n-1} \\ 1 & 0 \end{pmatrix}$$

we get $\mathcal{O}_{\mathfrak{a}}(Y_u) \cap U_{(v_2,\dots,v_k)} \simeq \mathcal{O}_{\mathfrak{a}}((R^{n-1} + (\sigma u)\mathfrak{a}) \cup (R^{n-1} + e_n)) \cap U_{(\sigma v_2,\dots,\sigma v_k)}$. By Remark 4.2.10, $\mathcal{O}_{\mathfrak{a}}(Y_u) \cap U_{\mathbf{v}} < \mathfrak{b} >$ is then d-connected if $\mathcal{O}_{\mathfrak{a}}(Y_u) \cap U_{\mathbf{v}}$ is d-connected, and for all $\mathbf{w} \in \mathcal{O}_{\mathfrak{a}}(Y_u) \cap U_{\mathbf{v}}$, $(\mathcal{O}_{\mathfrak{a}}(Y_u) \cap U_{\mathbf{v}})_{\mathbf{w}}$ is $(d - |\mathbf{w}|)$-connected.

As $\mathbf{v} \in R^{n,\infty}$, then the vector $(\sigma v_2, \dots, \sigma v_k)$ belongs to $R^{n-1,\infty}$ and we can apply $B(n-1)$ to $\mathcal{O}_{\mathfrak{a}}((R^{n-1} + (\sigma u)\mathfrak{a}) \cup (R^{n-1} + e_n)) \cap U_{(\sigma v_2,\dots,\sigma v_k)}$ to get that the poset $\mathcal{O}_{\mathfrak{a}}(Y_u) \cap U_{\mathbf{v}}$ is d-connected. If we take $\mathbf{w} \in \mathcal{O}_{\mathfrak{a}}(Y_u) \cap U_{\mathbf{v}}$, it could happen either $w_i \in R^n + u\mathfrak{a}$ for each i or, for some index i, $w_i \in R^n + e_{n+1}$. Observing that

$$(\mathcal{O}_{\mathfrak{a}}(Y_u) \cap U_{\mathbf{v}})_{\mathbf{w}} = \mathcal{O}_{\mathfrak{a}}(Y_u) \cap U_{\mathbf{vw}},$$

in the first case $\sigma\mathbf{w}$ belongs to $R^{n-1,\infty}$ and we can apply $B(n-1)$ as above, to get the $(d - |\mathbf{w}|)$-connectivity. In the second case, there exists a component of $\mathbf{w}$ belonging to $R^n + e_{n+1}$; after the shift of σ we get therefore $\sigma\mathbf{w} \in R^{n,\infty}$, but we have a component whose n-th factor is 1. We apply $B'(n-1)$ to get the statement also in this case.

Diagram: It remains to construct the diagram of above. We first observe that if $x \in X_u$, by definition of X_u, we can choose $v_x \in R^n$, $\alpha_x \in \mathfrak{a}$ such that $x = v_x + u\alpha_x$.

As a remark, observe that we want the image of x to be in R^n, and to do it we should annul the coordinates x^i, for $i \geq n+1$; the idea is then to subtract to $x = v_x + u\alpha_x$, α_x times the vector v_1. This would be enough to built a poset map $\psi\colon \mathcal{O}_{\mathfrak{a}}(Y_u) \to F$, with ψ injective if restricted to X_u; but we need also this restriction to be surjective. This is the reason why we introduced the technical set $\mathfrak{b}$.

Define first the set map:

$$\psi\colon Y_u \times \mathfrak{b} \to R^n \cup (R^n + e_{n+1})$$

as:

$$\psi(x,\beta) = \begin{cases} x & \text{if } x \in R^n + e_{n+1} = Y_u - X_u \\ x + v_1(\beta - \alpha_x) & \text{if } x \in X_u \end{cases}$$

where we suppose a decomposition $x = v_x + u\alpha_x$ for each $x \in X_u$ has been chosen. Observe that, if $v_x + u\alpha_x = x \in X_u$,

$$\psi(x,\beta) = v_x + u\alpha_x + v_1(\beta - \alpha_x) = v_x + v_1\beta - \alpha_x e_1 \in R^n.$$

Moreover, the restriction to X_u is injective: if $\psi(x,\beta) = \psi(y,\gamma)$, then $x - y$ is a multiple of v_1, but $v_1^1 = 1$ and $(x-y)^1 = 0$ so $x = y$.

As the map ψ adds to x a multiple of v_1, it preserves the unimodular property: $\psi(x) \in U_{\mathbf{v}}$. Therefore ψ can be extended to a poset map

$$\psi_*\colon (\mathcal{O}_{\mathfrak{a}}(Y_u) \cap U_{\mathbf{v}}) < \mathfrak{b} > \to F$$

that restricts to an injective map on $(\mathcal{O}_{\mathfrak{a}}(X_u) \cap U_{\mathbf{v}}) < \mathfrak{b} >$.

If we prove the surjectivity holds, we conclude this proof. Let $\mathbf{w}$ be a sequence in $\mathcal{O}(R^n) \cap F$. By definition of F the sequence $\mathbf{wv}$ is unimodular with component vectors in $\mathcal{O}_{\mathfrak{a}}$. As $v_1^1 = 1$, $\mathbf{w}$ has first coordinates congruent to 0 modulo $\mathfrak{a}$, say $w_i^1 = \alpha_i \in \mathfrak{a}$. Thus $x_i := w_i - v_1\alpha_i = (w_i - w_1\alpha_i) - \alpha_i u \in X_u$. Observe that if $v_x + \alpha_x u = v + \alpha u$ are two decomposition for $x \in X_u$ then $(\alpha_x - \alpha) \in \mathfrak{b}$, therefore, saying that our chosen decomposition for x_i was $v_{x_i} + \alpha_{x_i} u$, then $\beta_i := \alpha_{x_i} + \alpha_i$ belongs to $\mathfrak{b}$. Finally, applying ψ_* at the element $((x_1,\beta_1),\ldots,(x_j,\beta_j)) \in (\mathcal{O}_{\mathfrak{a}}(X_u) \cap U_{\mathbf{v}}) < \mathfrak{b} >$ we obtain the sequence $\mathbf{w}$.

$B(n)$, $I \neq \{0\}$ In the general case $0 \in I \subseteq R^{0,\infty}$. As above let:

$$\begin{aligned} F &:= \mathcal{O}_{\mathfrak{a}}((R^n + I) \cup (R^n + e_{n+1})) \cap U_{(v_1,\ldots,v_k)} \\ X &:= R^n \cup (R^n + e_{n+1}) \\ F_0 &:= F \cap \mathcal{O}(X) \\ d &:= n - \mathrm{sdim} - k - 1 \end{aligned}$$

and apply Proposition 4.2.7 to prove that F is d-connected. First of all, observe that F_0 is actually the case $I = \{0\}$. Then F_0 is d-connected. Suppose $\mathbf{w} \in F$; if $\mathbf{w} \in \mathcal{O}(R^n + I)$, then the case $I = \{0\}$ also shows

$$\mathcal{O}(X) \cap F_{\mathbf{w}} = \mathcal{O}_{\mathfrak{a}}(R^n \cup (R^n + e_{n+1})) \cap U_{\mathbf{vw}}$$

is $(d - |\mathbf{w}|)$-connected.

If $\mathbf{w}$ is not in $\mathcal{O}(R^n + I)$ there exists an index i, that we suppose to be 1, such that $w_i \in R^n + e_{n+1}$. As in the case $I = \{0\}$, we get

$$\mathcal{O}(X) \cap F_{\mathbf{w}} = \mathcal{O}_{\mathfrak{a}}(R^n) \cap U_{(w'_2,\dots,w'_j,v'_1,\dots,v'_k)}$$

where every $(n+1)$-th coordinate is 0 (we can do it because this last coordinates are in $\mathfrak{a}$ and so, up to $\mathfrak{a}$-congruence, this does not change the basis). By the condition $A(n)$, we get the $(d - |\mathbf{w}|)$-connectivity that we need.

$B'(n)$ The statement is the same as in $B(n)$, but now $\mathbf{v} \in U_{\mathfrak{a}}(R^{n+1,\infty})$. W.l.o.g. we can suppose v_1 is $\mathfrak{a}$-congruent to e_{n+1} and $v_i \equiv e_{j_i}(\mathfrak{a})$, for $i \neq 1$ and $j_i \leq n$. This means that:

$$\mathcal{O}_{\mathfrak{a}}((R^n + I) \cup (R^n + e_{n+1})) \cap U_{(v_1,\dots,v_k)} = \mathcal{O}_{\mathfrak{a}}(R^n + I) \cap U_{\mathbf{v}} =: F$$

Let d be equal to $n - \text{sdim} - k - 1$ and distinguish the two cases.

$B'(n), (i)$ Suppose first $v_1^{n+1} = 1$. We can assume, up to subtracting multiples of v_1, that $v_i^{n+1} = 0$ for $i \neq 1$. If $I = \{0\}$, the poset F is of the form $\mathcal{O}_{\mathfrak{a}}(R^n) \cap U_{\mathbf{v}}$, then of the form $\mathcal{O}_{\mathfrak{a}}(R^n) \cap U_{(v_2,\dots,v_k)}$, which is d-connected by $A(n)$. If I is not the set $\{0\}$, let X be equal to R^n, and apply Proposition 4.2.7 to prove that F is d-connected (the argument is the same).

$B'(n), (ii)$ Suppose $\mathfrak{a}e_{n+1} = I$. If $d \leq -2$ the condition is void, so we can assume $n \geq \text{sdim} + 1$ (as $k \geq 1$, $n \leq \text{sdim} \Rightarrow d \leq -2$). By Proposition 4.3.5 there exists a matrix g such that (gv_1) has $(n+1)$-th coordinate equal to 1; $g \in E(R, \mathfrak{a})$ since we don't have to permute the coordinates of v_1. As $I = \mathfrak{a}e_{n+1}$, g preserves $R^n + I$; therefore, up to changing F the equivalent gF, we reduced to the case (i) of above.

□

As a consequence, we get the important:

Corollary 4.3.7. *If $(R, \mathfrak{a})$ satisfies the stable range condition $SR_{sdim+2}(R, \mathfrak{a})$ then $U_{\mathfrak{a}}(R^n)$ and $U_{\mathfrak{a}}(R^{n,m})$ are $(n - sdim - 2)$-connected.*

Proof. Let $\mathbf{v}$ be the void sequence, then $U_{\mathbf{v}} = U$ and $k = 0$. The condition $A(n)$ is satisfied and $\mathcal{O}_{\mathfrak{a}}(R^n + I) \cap U$ is $(n - \text{sdim} - 2)$-connected. If $I = \{0\}$, the poset reduces to $\mathcal{O}_{\mathfrak{a}}(R^n) \cap U = U_{\mathfrak{a}}(R^n)$. If $I = \mathfrak{a}e_{n+1} + \cdots + \mathfrak{a}e_{n+m}$ then the poset is $\mathcal{O}_{\mathfrak{a}}(R^{n,m}) \cap U$ that is $U_{\mathfrak{a}}(R^{n,m})$. □

4.4 Posets of split unimodular sequences

In this section we define the posets of split unimodular sequences, which plays a fundamental role in the stability result of $\mathrm{GL}_n(R)$. Let $\mathbf{v}$ be a unimodular sequence $\mathbf{v} = (v_1, \dots, v_k) \in U_{\mathfrak{a}}$.

Definition 4.4.1. The poset of *split unimodular sequences* is the following set of sequences $((v_1, w_1), \dots, (v_k, w_k)) \in \mathcal{O}(R^\infty \times R^\infty)$:

$$SU_{\mathfrak{a}} := \{((v_1, w_1), \dots, (v_k, w_k)) \mid \mathbf{v} \in U_{\mathfrak{a}}, w_i \equiv v_i(\mathfrak{a}) \text{ and } w_i \cdot v_j = \delta_i^j \ \forall i, j\}$$

endowed with the induced ordering. The dot-product is the standard scalar product $w_i^T v_j$.

For any subset $X \subseteq R^\infty$ we define:

$$SU_{\mathfrak{a}}(X) := SU_{\mathfrak{a}} \cap \mathcal{O}(X \times X).$$

We can not deduce homotopical properties for $SU_{\mathfrak{a}}$ directly from those of $U_{\mathfrak{a}}$; we need an intermediate poset:

Definition 4.4.2. Let $\mathbf{v} = (v_1, \dots, v_k)$ be a unimodular sequence in $U_{\mathfrak{a}}$ and consider the set of pairs $((v_1, w_1), \dots, (v_k, w_k)) \in \mathcal{O}(R^\infty \times R^\infty)$. We define

$$MU_{\mathfrak{a}} := \{((v_1, w_1), \dots, (v_k, w_k)) \mid \text{ either } w_i = 0, \text{ or } w_i \equiv v_i(\mathfrak{a}) \text{ and } w_i v_j = \delta_i^j \ \forall j\}$$

where the ordering is obtained by restriction as well.

As above, if $X \subseteq R^\infty$ is any subset, we use define:

$$MU_{\mathfrak{a}}(X) := MU_{\mathfrak{a}} \cap \mathcal{O}(X \times X).$$

Notation. Denote by $(\mathbf{v}, \mathbf{w})$ the sequence $((v_1, w_1), \dots, (v_k, w_k)) \in MU_{\mathfrak{a}}$; let $|(\mathbf{v}, \mathbf{w})|_0$ be the number of vectors w_i's such that $w_i = 0$. In the following we represent a pair $(\mathbf{v}, \mathbf{w})$ simply by $\mathbf{v}$.

With this notation in mind, we can give another definition for $SU_{\mathfrak{a}}$:

$$SU_{\mathfrak{a}} = \{(\mathbf{v}, \mathbf{w}) \in MU_{\mathfrak{a}} \mid |(\mathbf{v}, \mathbf{w})|_0 = 0\} = MU_{\mathfrak{a}} \cap \mathcal{O}(R^\infty \times (R^\infty - \{0\}))$$

Moreover, we can identify also the poset of unimodular sequences $U_{\mathfrak{a}}$ with a subposet of $MU_{\mathfrak{a}}$:

$$\begin{aligned} U_{\mathfrak{a}} &= \{(\mathbf{v}, \mathbf{w}) \in MU_{\mathfrak{a}} \mid |(\mathbf{v}, \mathbf{w})|_0 = |((v_1, w_1), \dots, (v_k.w_k))| = k\} \\ &= MU_{\mathfrak{a}} \cap \mathcal{O}(R^\infty \times \{0\}). \end{aligned}$$

Remark 4.4.3. There is a natural action of the congruence group $\widetilde{GL}(R,\mathfrak{a})$ on the posets $SU_{\mathfrak{a}}$ and $MU_{\mathfrak{a}}$: on a pair $(\mathbf{v},\mathbf{w})$ the action of an element g is:

$$g\cdot(\mathbf{v},\mathbf{w}) := (g\mathbf{v}, g^*\mathbf{w}) \quad \text{where } g^* = (g^T)^{-1}$$

As g belongs to $\ker(GL_n(R)\to GL_n(R/\mathfrak{a}))\times\Sigma_n$ for an opportune n, it does not change nor the unimodular properties of $\mathbf{v}$, neither the $\mathfrak{a}$-congruences. Moreover, the scalar product gives $(g^*\mathbf{w})^T g\mathbf{v} = \mathbf{w}^T g^{-1} g\mathbf{v}$ and the products are so preserved.

As a first result, we extend to the poset of split unimodular sequences $SU_{\mathfrak{a}}$ the transitive action we proved in Proposition 4.3.5. If $S\subseteq\mathbb{N}^+$ is any subset of natural numbers, we define the poset:

$$MU^S_{\mathfrak{a}}(X) := \{((v_1,w_1),\dots,(v_k,w_k))\in MU_{\mathfrak{a}}(X)\mid w_i = 0 \text{ iff } i\in S\}$$

and we observe that the action of $\widetilde{GL}$ preserves also these posets. We get for them a transitive action:

Proposition 4.4.4. *Suppose the pair $(R,\mathfrak{a})$ satisfies the stable range condition with index $sdim+2$ and let $S\subseteq\mathbb{N}^+$ be any subset. If $m\geq sdim+k+1$, then $\widetilde{E}_m(R,\mathfrak{a})$ acts transitively on the set of length k sequences of $MU^S_{\mathfrak{a}}(R^m)$.*

Proof. Up to a permutation of the vectors, we suppose $S=\{s+1,\dots,m\}$, with $s\leq m$. By Proposition 4.3.5, as $m\geq \mathrm{sdim}+k+1$, $\widetilde{E}_m(R,\mathfrak{a})$ acts transitively on length k sequences in $U_{\mathfrak{a}}(R^m)$. Therefore, if $((v_1,w_1),\dots,(v_k,w_k))\in MU^S_{\mathfrak{a}}$, there exists a matrix $g\in\widetilde{E}_m(R,\mathfrak{a})$ such that $gv_i=e_i$; define $g^*w_i =: w_i'$.

If $i\leq s$, by definition of the poset $MU_{\mathfrak{a}}(R^m)$, we get $v_i\equiv w_i'(\mathfrak{a})$ and $w_i'v_j=\delta_i^j$ for each $j=1,\dots,k$. As a consequence, $w_i' = e_i + r_i$, where with $r_i\equiv 0(\mathfrak{a})$ and $r_ie_j=0$ for all $j=1,\dots,k$. If we suppose also $s\leq k\leq m$, the vectors w_i' are of the form:

$$(w_i')^T = (e_i,0,\alpha) \text{ where } e_i\in R^s,\ 0\in R^{k-s},\ \alpha\in R^{0,m-k}$$

Consider the matrix

$$h\in\begin{pmatrix} I_s & 0 & 0\\ 0 & I_{k-s} & 0\\ * & 0 & I_{m-k}\end{pmatrix}\subseteq E_m(R,\mathfrak{a})$$

where the first s columns are the vectors w_i'. Then $he_i = w_i'$, for $i\leq s$, and

$$(h^T)^* w_i' = h^{-1}w_i' = e_i$$

This means:

$$(h^Tg)(v_i,w_i) = h^T(e_i,w_i') = (h^Te_i,(h^T)^*w_i') = \begin{cases}(e_i,e_i) & \text{if } 1\leq i\leq s\\ (e_i,0) & \text{if } s<i\leq k\end{cases}$$

and this shows the action is transitive. ☐

If we suppose in the proof of above the vectors v_i to be $\mathfrak{a}$-congruent to e_i, then we don't need the permutation matrices to fix the order. We can take g in $E_m(R,\mathfrak{a})$ instead of $\widetilde{E}_m(R,\mathfrak{a})$, and so $h^T g$ in $E_m(R,\mathfrak{a})$. If $\mathbf{v} \in MU_{\mathfrak{a}}(R^{n,m'})$, where $n+m'=m$, and we recall that $R^{n,m'} = R^n + \mathfrak{a}e_{n+1} + \cdots + \mathfrak{a}e_{n+m'}$, we get that the matrix $h^T g$ belongs to the group of matrices

$$\left\{ A \in \widetilde{E}_m(R,\mathfrak{a}) : A \equiv \begin{pmatrix} \tau & 0 \\ 0 & I_{m'} \end{pmatrix} \quad \text{mod } \mathfrak{a}, \text{ and } \tau \in \Sigma_n \right\}$$

as we don't have to move the last m' coordinates. This is a useful observation in the next lemma:

Lemma 4.4.5. *Let S be a subset of the positive natural numbers. Suppose we have $\mathbf{v} \in MU_{\mathfrak{a}}(R^{n,m})$, $|\mathbf{v}| = k$, $|\mathbf{v}|_0 = r$, and $n \geq sdim + k + 1$. Then there is an isomorphism:*

$$MU^S_{\mathfrak{a}}(R^{n,m})_{\mathbf{v}} \simeq MU^S_{\mathfrak{a}}(R^{n-k,m}) < R^r >$$

Proof. We distinguish three cases:

Case 1 Suppose first the sequence $\mathbf{v}$ is in $U_{\mathfrak{a}}$; then $r = k$, and $v_i \equiv e_{j_i}(\mathfrak{a})$. Because of the transitive action, there exist a matrix $h^T g$ such that $h^T g \cdot \mathbf{v} = ((e_1,0),\ldots,(e_k,0))$. As v_i belongs to $R^{n,m}$, the matrix permutes just the first n coordinates, as explained above. The set $R^{n,m}$ is so preserved by $h^T g$. As a consequence, the following is an isomorphism:

$$MU^S_{\mathfrak{a}}(R^{n,m})_{\mathbf{v}} \longrightarrow MU^S_{\mathfrak{a}}(R^{n,m})_{((e_1,0),\ldots,(e_k,0))}$$

This latter poset is better described as the set:

$$\{((x_1,y_1),\ldots,(x_k,y_k)) \in MU^S_{\mathfrak{a}}(R^{n,m}) \text{ such that } ((x_1,y_1),\ldots,(x_k,y_k),(e_1,0),\ldots,(e_k,0)) \in MU^S_{\mathfrak{a}}(R^{n,m})\}$$

it means that the sequence $(x_1,\ldots,x_k,e_1,\ldots,e_k)$ belongs to $U_{\mathfrak{a}}(R^{n,m})$, and that $y_h e_t$ is 0 for each $t = 1,\ldots,k$, then the first k coordinates of y_h are all 0.

Consider the isomorphism:

$$(\pi_1,\pi_2)\colon R^{n+m} \longrightarrow R^k \oplus R^{n+m-k}$$

where π_1 is the projection on the first k coordinates and π_2 on the last $n+m-k$ coordinates. As $k \leq n$ it induces an isomorphism:

$$R^{n,m} \longrightarrow R^k \oplus R^{n-k,m}$$

The sequence $(x_1,\ldots,x_k,e_1,\ldots,e_k)$ is in $U_{\mathfrak{a}}(R^{n,m})$, so it is a basis for a direct summand in R^{n+m}; this happens if and only if $x_i \equiv e_h(\mathfrak{a})$ with $h > k$, if and only if $(\pi_2(x_1),\ldots,\pi_2(x_j))$ is a basis of a direct summand in R^{n+m-k}:

$$(x_1,\ldots,x_k,e_1,\ldots,e_k) \in U_{\mathfrak{a}}(R^{n+m}) \iff (\pi_2(x_1),\ldots,\pi_2(x_j)) \in U_{\mathfrak{a}}(R^{n+m-k})$$

Therefore the map

$$MU^S_{\mathfrak{a}}(R^{n,m})_{((e_1,0),\ldots,(e_k,0))} \longrightarrow MU^S_{\mathfrak{a}}(R^{n-k,m}) < R^r >$$

where

$$((x_1,y_1),\ldots,(x_k,y_k)) \mapsto ((\pi_2(x_1),\pi_2(y_1),\pi_1(x_1)),\ldots,(\pi_2(x_j),\pi_2(y_j),\pi_1(x_j))$$

is an isomorphism (as the first k coordinates of y_h are 0).

Case 2 Suppose now $\mathbf{v} \in SU_{\mathfrak{a}}(R^{n,m})$, and so $r = 0$. As above, we get:

$$MU^S_{\mathfrak{a}}(R^{n,m})_{\mathbf{v}} \simeq MU^S_{\mathfrak{a}}(R^{n,m})_{((e_1,e_1),\ldots,(e_k,e_k))}$$

This is the set

$$\{((x_1,y_1),\ldots,(x_j,y_j) \in MU^S_{\mathfrak{a}}(R^{n,m}) \mid \text{ all the first } k \text{ components are } 0\}$$

as now we have also $e_h x_t = 0$. Because of the isomorphism (π_1,π_2) of above, this set is actually $MU^S_{\mathfrak{a}}(R^{n-k,m}) = MU^S_{\mathfrak{a}}(R^{n-k,m}) < R^0 >$.

Case 3 The proof in the general case is obtained from those of above, by using a permutation $\sigma \in \mathcal{S}_k$ such that $\sigma \cdot \mathbf{v} = \mathbf{wz}$, with $\mathbf{z} \in U_{\mathfrak{a}}(R^{n,m})$ and $\mathbf{w} \in SU_{\mathfrak{a}}(R^{n,m})$ (σ shifts the vectors $(v_i, 0)$ in the last coordinates). Then $|\mathbf{z}| = r$ and $|\mathbf{w}| = k - r$.

Look at the poset $MU^S_{\mathfrak{a}}(R^{n,m})_{\mathbf{v}}$ as union of the subsets $MU^S_{\mathfrak{a}}(R^{n,m})^h_{\mathbf{v}}$ given by the vectors $((x_1,y_1),\ldots,(x_h,y_h)) \in MU^S_{\mathfrak{a}}(R^{n,m})_{\mathbf{v}}$ of length h. These sets are in bijection with the sets $MU^{S'}_{\mathfrak{a}}(R^{n,m})^h_{\mathbf{wz}}$, where S' is the set obtained by S by applying the same permutation σ to the set of numbers $\{h+1,\ldots,h+k\}$. In this sense, a permutation of $\mathbf{v}$ does not affect $MU^S_{\mathfrak{a}}(R^{n,m})$, and we can restrict to vectors $\mathbf{v} = \mathbf{wz}$ as above. We get the statement by the following equivalences:

$$\begin{aligned} MU^S_{\mathfrak{a}}(R^{n,m})_{\mathbf{wz}} \simeq & MU^S_{\mathfrak{a}}(R^{n-k+r,m})_{\mathbf{z}'}, \quad \mathbf{z}' = h^T_{\mathbf{w}} g_{\mathbf{w}} \cdot \mathbf{z} \in U_{\mathfrak{a}}(R^{n-k+r+m}) \\ \simeq & MU^S_{\mathfrak{a}}(R^{n-k,m}) < R^r > \end{aligned}$$ □

Theorem 4.4.6. *Suppose the pair $(R,\mathfrak{a})$ satisfies the stable range condition with index sdim + 2. Then the poset $MU_{\mathfrak{a}}(R^{n,m})$ is $(n - sdim - 2)$-connected.*

Proof. By Corollary 4.3.7, the poset $U_{\mathfrak{a}}(R^{n,m})$ is d-connected, where $d = n - \text{sdim} - 2$. Let X be $R^{n,m} \times \{0\}$; then

$$U_{\mathfrak{a}}(R^{n,m}) = \mathcal{O}(X) \cap MU_{\mathfrak{a}}(R^{n,m})$$

As $(d-k) \geq -1$ implies $n \geq \text{sdim} + k + 1$, we can apply the previous lemma. If $\mathbf{v} \in MU_{\mathfrak{a}}(R^{n,m})$, with $|\mathbf{v}| = k, |\mathbf{v}|_0 = r$, then:

$$\mathcal{O}(X) \cap MU_{\mathfrak{a}}(R^{n,m})_{\mathbf{v}} \simeq U_{\mathfrak{a}}(R^{n-k,m}) < R^k >$$

by Corollary 4.3.7 and Remark 4.2.10 it is $(d-k)$-connected. By Proposition 4.2.7, $MU_{\mathfrak{a}}(R^{n,m})$ is d-connected. □

Finally, we prove the acyclicity of the poset of split unimodular sequences. Suppose the property $SR_{\mathrm{sdim}+2}(R, \mathfrak{a})$ is satisfied.

Theorem 4.4.7. *If $n \geq 2q + sdim + 3$, then the poset $SU_{\mathfrak{a}}(R^{n,m})$ is q-acyclic.*

Proof. We proceed by induction on q. If $q = -1$, then $n \geq 1$ and $SU_{\mathfrak{a}}(R^{n,m})$ is not void because $((e_1, e_1))$ is split unimodular.

Let $q \geq 0$. By induction, $SU_{\mathfrak{a}}(R^{n,m})$ is $(q-1)$-acyclic. To prove it is also q-acyclic, we construct a surjection on $H_q(SU_{\mathfrak{a}}(R^{n,m}))$ that is the 0-map.

Step 1. We first show that, if $i \geq 1$, then the map

$$l_n \circ (s_0)_* \colon SU_{\mathfrak{a}}(R^{n-i,m}) \longrightarrow Z^i SU_{\mathfrak{a}}(R^{n-i,m}) < R^i >$$

is $(q - i + 1)$-acyclic.

Let $d := q - i + 1$. As the poset $SU_{\mathfrak{a}}(R^{n-i,m})$ satisfies the chain condition, if we prove that $SU_{\mathfrak{a}}(R^{n,m})_{\mathbf{v}}$ is $(d - |\mathbf{v}|)$-acyclic for each $\mathbf{v}$, using Proposition 4.2.9 we get the map $l_n \circ (s_0)_*$ is d-acyclic.

Suppose $|\mathbf{v}| = j \geq 1$; if $d - j < -1$ we have nothing to prove. Suppose $d - j \geq -1$; as $i, j \geq 1$, then $2(d - j) = 2q - 2i - 2j + 2 \leq 2q - i - j$. This means $n - i - j \geq 2(d - j) + \mathrm{sdim} + 3 \geq \mathrm{sdim} + 1$. As $\mathbf{v}$ is split unimodular (then $|\mathbf{v}|_0 = 0$) and $n - i \geq \mathrm{sdim} + |\mathbf{v}| + 1$, we can apply Lemma 4.4.5, getting:

$$SU_{\mathfrak{a}}(R^{n-i,m})_{\mathbf{v}} \simeq SU_{\mathfrak{a}}(R^{n-i-j,m})$$

By induction, it is $(d - j)$-acyclic (as $d - j = q - i - j + 1 \leq q - 1$). This concludes the first step.

In the special case of $i = 1$, we get $d = q$ and as a consequence we have proved that the following map is a surjection:

$$H_q(SU_{\mathfrak{a}}(R^{n-1,m})) \longrightarrow H_q(Z^1 SU_{\mathfrak{a}}(R^{n-1,m}) < R >)$$

Step 2. Consider the filtration of the poset $MU_{\mathfrak{a}}(R^{n,m})$ defined by adding step by step the unimodular sequences of length i:

$$F_0 := MU_{\mathfrak{a}}(R^{n,m}) - U_{\mathfrak{a}}(R^{n,m})$$
$$F_i := F_{i-1} \cup \{\mathbf{v} \in U_{\mathfrak{a}}(R^{n,m}) \mid |\mathbf{v}| = i\}$$

As in the other proofs, what we are going to do is the studying of $\mathrm{Link}_{F_i}(\mathbf{v})$; but in such a case, the poset F_i does not satisfy the chain condition. In fact, it could be possible to choose a sequence in F_i, not unimodular, say $((v_1, w_1), \ldots, (v_k, w_k))$, with $w_{j_1} = \cdots = w_{j_{i+1}} = 0$; then the subsequence $((v_{j_1}, 0), \ldots, (v_{j_{i+1}}, 0))$ is not in F_i anymore. We cannot directly apply Proposition 4.1.7.

Let $\mathbf{v}$ be a sequence in $F_i - F_{i-1}$. The example of above shows us how the poset F_i fails to satisfy the chain condition: it could fail if we consider subsequences of vectors in F_0, as the permutation property is satisfied. It is now clear that the negative link

$$\mathrm{Link}^-_{F_i}(\mathbf{v}) = \{\mathbf{z} \in F_i \mid \mathbf{z} < \mathbf{v}\}$$

is given by all the subsequences of $\mathbf{v}$, exactly as in the case the chain condition is satisfied. Let ψ be the map:

$$\psi\colon \mathrm{Link}^+_{F_i}(\mathbf{v}) \longrightarrow Z^i(F_i)_{\mathbf{v}}, \quad v_i \mapsto z_i$$

Notice that a sequence $\mathbf{v} < \mathbf{t} \in F_i$ cannot be unimodular of length $|\mathbf{t}| \leq i$, therefore it has some pairs (t_j, s_j) with $s_j \neq 0$. The same happens for $\mathbf{t} - \mathbf{v}$ (with the same pair) and $\mathbf{t} - \mathbf{v} \in F_i$. The sequence $\psi(\mathbf{t}) =: t'$ has then the property that $\mathbf{z} < \mathbf{t}'$ and $\mathbf{t}' - \mathbf{z} \in F_i$. Moreover, $(\mathbf{t}' - \mathbf{z})\mathbf{v}$ belongs to $(F_i)_{\mathbf{v}}$. The inverse map is $z_i \mapsto v_i$. Then we get:

$$|\mathrm{Link}_{F_i}(\mathbf{v})| \simeq \Sigma^{i-1}|Z^i(F_i)_{\mathbf{v}}|$$

Step 3. We claim the map $F_{i-1} \hookrightarrow F_i$ is $(q+\delta)$-acyclic, where $\delta := \min\{1, i-1\}$ (which implies $2\delta \leq i$). To prove the claim, by Mayer Vietoris exact sequence, it suffices to show the links are $(q+\delta-1)$-acyclic.

First, observe that the poset $SU_{\mathfrak{a}}(R^{n,m}) \subseteq F_0$ is a deformation retract of F_0. The supremum in $SU_{\mathfrak{a}}(R^{n,m})^-(\mathbf{v})$ is the subsequence whose second components w_i are non zero.

If $q+\delta < i-1$ then, by Step 2, the link is $(q+\delta-1)$-acyclic. Suppose $q+\delta \geq i-1$, then

$$n - i \geq 2q + \mathrm{sdim} + 3 + 2\delta - 2i \geq \mathrm{sdim} + 1$$

Now, $(F_i)_{\mathbf{v}} = (F_0)_{\mathbf{v}}$, as an element in $(F_i)_{\mathbf{v}}$ can't be unimodular. But F_0 deformation retracts on $SU_{\mathfrak{a}}(R^{n,m})$. We want to apply Lemma 4.4.5; with this aim, using the retraction, we have:

$$Z^i(F_i)_{\mathbf{v}} \simeq Z^i(SU_{\mathfrak{a}} \cap MU_{\mathfrak{a}}(R^{n,m})_{\mathbf{v}}) \simeq Z^i(SU_{\mathfrak{a}}(R^{n-i,m}) < R^i >)$$

where we have applied the lemma.

In Step 1, we have proved the map

$$l_n \circ (s_0)_*\colon SU_{\mathfrak{a}}(R^{n-i,m}) \longrightarrow Z^i SU_{\mathfrak{a}}(R^{n-i,m}) < R^i >$$

is $(q-i+1)$-acyclic. But $q-i+\delta \leq q-i+1$, and as $n-i \geq 2(q-i+\delta) + \mathrm{sdim}+3$, by induction $SU_{\mathfrak{a}}(R^{n-i,m})$ is $(q-i+\delta)$-acyclic. Then the same holds for $Z^i SU_{\mathfrak{a}}(R^{n-i,m}) < R^i >$; as a consequence $\mathrm{Link}_{F_i}(\mathbf{v})$ is $(q+\delta-1)$-acyclic.

Step 4. The maps $F_{i-1} \hookrightarrow F_i$ are $(q+\delta)$-acyclic, then we get the isomorphism in homology:

$$H_q(F_1) \simeq H_q(MU_{\mathfrak{a}}(R^{n,m}))$$

the latter being 0 by Theorem 4.4.6. Therefore, in the long exact sequence in homology, we get the surjection $\partial\colon H_{q+1}(F_1 \setminus F_0) \to H_q(F_0)$; as in Proposition 1.2.8, we also have:

$$H_{q+1}(F_1 \setminus F_0) \simeq \bigoplus_{\mathbf{v} \in F_1 - F_0} H_{q+1}(\Sigma \mathrm{Link}_{F_i}(\mathbf{v})) \simeq \bigoplus_{\mathbf{v} \in F_1 - F_0} H_q(\mathrm{Link}_{F_i}(\mathbf{v}))$$

Using now also the deformation retraction between $SU_{\mathfrak{a}}(R^{n,m})$ and F_0, we get the commutative diagram:

$$\begin{array}{ccc}
H_{q+1}(F_1 \setminus F_0) & \xrightarrow{\ \partial\ } \!\!\!\!\!\rightarrow & H_q(F_0) \\
\downarrow & & \downarrow \\
\bigoplus_{\mathbf{v} \in F_1 - F_0} H_q(\mathrm{Link}_{F_i}(\mathbf{v})) & \cdots\cdots\rightarrow & H_q(SU_{\mathfrak{a}}(R^{n,m}))
\end{array}$$

By Step 2, and using the surjection in homology of Step 1, we obtain the surjection:

$$\begin{array}{ccc}
\bigoplus_{\mathbf{v} \in F_1 - F_0} H_q(SU_{\mathfrak{a}}(R^{n-1,m})) & \longrightarrow\!\!\!\!\!\rightarrow & \bigoplus_{\mathbf{v} \in F_1 - F_0} H_q(Z^1 SU_{\mathfrak{a}}(R^{n-1,m}) < R >) \\
 & \overset{\Phi}{\searrow\!\!\searrow} & \downarrow\!\!\downarrow \\
 & & H_q(SU_{\mathfrak{a}}(R^{n,m})).
\end{array}$$

Step 5. If we show the composite map Φ of above is the 0-map, we get the statement. To prove this fact, we let $\mathbf{v}$ be a fixed element in $F_1 - F_0$ and, looking at the factor corresponding to $\mathbf{v}$ in the composite map Φ, we show that the composition

$$SU_{\mathfrak{a}}(R^{n-1,m}) \longrightarrow Z^1 SU_{\mathfrak{a}}(R^{n-1,m}) < R > \longrightarrow \mathrm{Link}_{F_1}(\mathbf{v}) \longrightarrow SU_{\mathfrak{a}}(R^{n,m})$$

is null-homotopic, where we identify $R^{n-1,m}$ with the subset of $R^{n,m}$ of vectors with first null coordinate.

First, suppose $\mathbf{v} = ((e_1, 0))$ and let $((v_1, w_1), \ldots, (v_k, w_k))$ be in $SU_{\mathfrak{a}}(R^{n-1,m})$. Let s_0 be equal to $0 \in R$; then, applying the map $l_1 \circ (s_0)_*$, we get the sequence $((v_1, w_1, 0), \ldots, (v_k, w_k, 0), z_1)$. The map $Z^1 SU_{\mathfrak{a}}(R^{n-1,m}) < R > \to \mathrm{Link}_{F_1}(\mathbf{v})$ sends z_1 in $\mathbf{v}$, and we get $((v_1, w_1, 0), \ldots, (v_k, w_k, 0), (e_1, 0))$. The last map just gets rid of the pair $(e_1, 0)$. The composition map

$$((v_1, w_1), \ldots, (v_k, w_k)) \mapsto ((v_1, w_1), \ldots, (v_k, w_k))$$

is the inclusion map $SU_{\mathfrak{a}}(R^{n-1,m}) \to SU_{\mathfrak{a}}(R^{n,m})$. But this inclusion factors through the poset $SU_{\mathfrak{a}}(R^{n,m})_{((e_1,e_1))}$, as we identify $SU_{\mathfrak{a}}(R^{n-1,m})$ with the vectors with first coordinate equal to 0. Finally Lemma 4.2.3 implies the inclusion map is null-homotopic.

Consider the general case $\mathbf{v} \in F_1 - F_0$, say $\mathbf{v} = (v, 0)$. The isomorphism

$$Z^1 SU_{\mathfrak{a}}(R^{n-1,m}) < R > \longrightarrow \mathrm{Link}_{F_1}(\mathbf{v})$$

is now obtained by choosing a matrix g such that $g(e_1, 0) = (v, 0)$ and identifying $\mathrm{Link}_{F_1}((e_1, 0))$ with $\mathrm{Link}_{F_1}(\mathbf{v})$ via the g-action (as explained in Lemma 4.4.5, Case 1). Then, the composition map is just the inclusion map

$$SU_{\mathfrak{a}}(R^{n-1,m}) \longrightarrow SU_{\mathfrak{a}}(R^{n,m})$$

followed by the g-action. Hence it is null-homotopic. □

5 The categorical viewpoint

Contents

The goal of this last chapter is to prove Theorem 5.3.2. It states that the family of general linear groups, $\{\mathrm{GL}_n(R)\}_{n\in\mathbb{N}}$, satisfy homological stability when the coefficients belong to a ring of stable dimension $\mathrm{sdim}+2$; moreover, the stability range is given by $0 \leq i \leq (n - \mathrm{sdim} - 2)/2$. As a corollary, namely Corollary 5.3.4, we see that the family $\mathrm{GL}_n(R[G])$ satisfies homological stability when the ring R is a commutative Noetherian ring of finite Krull dimension d, and the group G contains a finitely generated abelian subgroup of finite index.

To prove Theorem 5.3.2, we need an acyclicity result for posets of split unimodular sequences, namely Theorem 4.4.7. We can not prove the homological stability theorem in complete generality, for every ring, because we need some kind of stability condition, such as the Krull dimension or the stable range condition. In literature, one can find many other proofs, with both constant and twisted coefficients; for a list, see Chapter 2.

The proof we are going to present is not straghtforward, but it works in a categorical set-up: it as an application of a general result that has been proved by N. Wahl, in the beautiful article [Wah14]. Following this paper, first we define suitable abstract categories, called *homogenous categories*, that encode the "transitivity" and the "stabilizer" properties in the Quillen's argument (see Chapter 2). Then, to any pair of objects (X, A) in a homogenous category, we associate a family of semi-simplicial sets, namely $\{W_n(X, A)\}_{n\in\mathbb{N}}$, and we show that the automorphism groups $G_n := \mathrm{Aut}(A \oplus X^{\oplus n})$ act "nicely" on them. We also have stabilization maps $G_n \hookrightarrow G_{n+1}$. As a consequence, in Theorem 5.2.6, we prove that, if the semi-simplicial sets $W_n(X, A)$ are $(n-2)/2$-acyclic, then the family of automorphism groups $\{G_n\}_{n\in\mathbb{N}}$ satisfies homological stability.

The theorem for general linear groups is now obtained by observing that $\mathrm{GL}_n(R)$ is the automorphism group of the R-module R^n, that is an object in the category of finitely generated free R-modules. This idea is suggested by N. Wahl; in [Wah14], she proves this stability theorem when the coefficient ring is the ring of integers $\mathbb{Z}$, and she gives the ideas to generalize to any ring of finite stable dimension the same result.

The chapter is structured as follows. In the first section we define homogenous categories, and in Theorem 5.1.4 we show that they can build up from braided monoidal categories. We show that the category of finite sets and injection, and the category of finite dimensional vector spaces, are examples of homogenous categories. In the second section we define, for any pair of objects X and A, the family of semi-simplicial sets $W_n(X, A)$, and we prove that the automorphism groups homologically stabilize. In the third section, we finally prove the general linear groups satisfy homological stability, concluding this thesis.

5.1 Homogenous categories

The categories in which we study homological stability properties are called *homogenous categories*. Before giving their definition and going deeper in this abstract set-up, we want to show how they arise and why they are somehow natural.

We begin by observing that there are lots of families of groups satisfying homological stability. If one wants to think about families without this property, it is enough to consider non-isomorphic abelian groups. For example, let $G_n := \mathbb{Z}^n$, and consider the family $\{\mathbb{Z}^n\}_{n\in\mathbb{N}}$ of finitely generated free abelian groups. As the first homology group $\mathrm{H}_1(G;\mathbb{Z})$ is the abelianization of G, we get $\mathrm{H}_1(\mathbb{Z}^n;\mathbb{Z}) = \mathbb{Z}^n$, and the homological stability is not satisfied. Otherwise, braid groups, symmetric groups and general linear groups, satisfy this property. We want to analyze whether there are differences between the groups $\mathbb{Z}^n$ and these "automorphism groups".

First, consider a family of groups and stabilization maps:

$$\ldots \longrightarrow G_{n-1} \longrightarrow G_n \longrightarrow G_{n+1} \longrightarrow \ldots$$

Define the groupoid $\mathcal{G}$ whose objects are the natural numbers, with morphisms $\mathrm{Aut}(0) = \{\mathrm{id}\}$ and $\mathrm{Aut}(n) := G_n$ when $n > 0$. Hence, the groups G_n can be viewed as automorphism groups in this groupoid. We want to remark that symmetric groups are the automorphism groups of n points, braid groups the automorphism group of n strips, the general linear groups $\mathrm{GL}_n(\mathbb{K})$ are the automorphism groups of the finitely generated vector spaces $\mathbb{K}^n$; when G_n is the abelian group $\mathbb{Z}^n$, then, it can be viewed as the automorphism group of the n-th lattice. Therefore, being a family of automorphism groups does not imply, in general, homological stability.

Sometimes the groupoid above has an additional structure, that is a "block sum" operation:

$$G_n \oplus G_m \longrightarrow G_{n+m}$$

For braid groups and symmetric groups, this operation corresponds to operate on different strips or points. In the case of general linear groups, this operation can be seen as the map $\mathrm{GL}_n(\mathbb{K}) \oplus \mathrm{GL}_m(\mathbb{K}) \longrightarrow \mathrm{GL}_{n+m}(\mathbb{K})$ defined as:

$$(A, B) \longmapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

A similar block sum operation is satisfied for $\mathbb{Z}^n$: the block sum between two vectors $u \in \mathbb{Z}^n$ and $v \in \mathbb{Z}^m$ is the vector $(u, v)^T$ in $\mathbb{Z}^{n+m}$.

We are associating to a family of groups a groupoid that comes equipped with a monoidal structure; moreover, in this category, the groups G_n are exactly the automorphism groups of the category. The turning point is given by a smart observation: symmetric groups, braid groups and general linear groups satisfy some kind of commutativity, i.e. a braiding $G_n \oplus G_m \cong G_m \oplus G_n$. This is not clear for the family $\{\mathbb{Z}^n\}_{n\in\mathbb{N}}$, seen as automorphism groups, and indeed, this could be the difference we were looking for. In fact, in Theorem 5.1.4, we will see that one can build up homogenous categories from *braided* monoidal groupoids, groupoids that have a braiding, and we will see that this construction that does not work when the groupoid does not come equipped with a braiding. We now proceed with their definition.

Let $(\mathcal{C}, \oplus, 0)$ be a monoidal category where the unit 0 is a unit for the monoidal structure, and it is also initial. In the following, the category $\mathcal{C}$ is a *strict* monoidal category (see Definition 1.4.2), but the same results can also be proved if $\mathcal{C}$ is not strict. Let $\iota_A \colon 0 \to A$ be the unique morphism from 0 to A; we define the set of automorphisms in $\mathcal{C}$ that fix a given object B:

$$\mathrm{Fix}(A, A \oplus B) := \{\phi \in \mathrm{Aut}(A \oplus B) \mid \phi \circ (\iota_A \oplus 1_B) = (\iota_A \oplus 1_B)\}$$

that means, the following diagram commutes:

$$\begin{array}{ccc} A \oplus B & \xrightarrow{\ \phi\ } & A \oplus B \\ {\scriptstyle \iota_A \oplus 1_B} \uparrow & \nearrow {\scriptstyle \iota_A \oplus \mathrm{id}_B} & \\ 0 \oplus B & & \end{array}$$

The automorphisms in $\mathrm{Fix}(A, A \oplus B)$ are just the morphisms in the category that restrict to the identity on B. Hence, a more suggestive notation is $\mathrm{Fix}(B)$, a notation we are going to use whenever it is clear from the context who the object A is.

Definition 5.1.1. Let $(\mathcal{C}, \oplus, 0)$ be a monoidal category, where the unit 0 is initial. The category $(\mathcal{C}, \oplus, 0)$, or briefly $\mathcal{C}$, is called *homogenous* if, for any pair of objects A and B in $\mathcal{C}$, the following axioms are satisfied:

(H1) $\mathrm{Hom}(A, B)$ is a transitive $\mathrm{Aut}(B)$-set under composition;

(H2) the map $\mathrm{Aut}(A) \to \mathrm{Aut}(A \oplus B)$ taking f to $f \oplus 1_B$ is injective with image $\mathrm{Fix}(B)$.

We remark that, in these categories, *every* element in $\mathrm{Fix}(B)$ is taken in a injective way, via the map $f \mapsto f \oplus 1_B$. This allows us to apply the orbit stabilizer Theorem.

In fact, let $X := \mathrm{Hom}(B, A \oplus B)$ be a set of homomorphisms in a homogenous category. By axiom $(H1)$, this is a transitive $\mathrm{Aut}(A \oplus B)$-set. As the action is transitive, the orbit of an element $x \in X$ is the whole X, and by the orbit stabilizer theorem, the set X is in bijection with the set $\mathrm{Aut}(A \oplus B)/\mathrm{St}(x)$, where $\mathrm{St}(x)$ is the stabilizer of x. The stabilizers are all conjugate, so we can choose w.l.o.g. the element $x = \iota_A \oplus 1_B \in X$; we choose this elements because the stabilizer $\mathrm{St}(\iota_A \oplus 1_B)$ is given by the automorphisms in $\mathrm{Aut}(A \oplus B)$ that fix the map $\iota_A \oplus 1_B$, that means $\mathrm{Fix}(B)$. By axiom $(H2)$, this is the injective image of $\mathrm{Aut}(A)$ in $\mathrm{Aut}(A \oplus B)$, via the map $f \mapsto f \oplus 1_B$. Therefore, X is in bijection with the cosets of $\mathrm{Aut}(A)$ in $\mathrm{Aut}(A \oplus B)$, where $\mathrm{Aut}(A)$ is seen as a subgroup of $\mathrm{Aut}(A \oplus B)$ via the map $\mathrm{Aut}(A) \hookrightarrow \mathrm{Aut}(A \oplus B)$, and the quotient is taken by precomposition. Briefly, we get the following relation:

$$\mathrm{Hom}(B, A \oplus B) \cong \mathrm{Aut}(A \oplus B)/\mathrm{Aut}(A) \tag{5.1.2}$$

In particular, when the object A is the unit 0, we get $\mathrm{Hom}(B, B) \cong \mathrm{Aut}(B)$: the endomorphisms are isomorphisms, and this fact suggests that maybe homogenous categories can be build up from groupoids. Indeed, this is true, as we show below.

It turns out that having a monoidal groupoid is not enough (the family of groups $\{\mathbb{Z}^n\}$ gives just a monoidal groupoid). What we need is a braiding.

Definition 5.1.3. Let $(\mathcal{C}, \oplus, 0)$ be a monoidal groupoid where 0 is initial. We say that the category $\mathcal{C}$ is *pre-braided* if its underlying groupoid is braided and for each pair of objects A, B in $\mathcal{C}$, the braiding $\gamma_{A,B} \colon A \oplus B \to B \oplus A$ makes the following diagram commutative:

$$\begin{array}{ccc} 0 \oplus A = A = A \oplus 0 & \xrightarrow{1_A \oplus \iota_B} & A \oplus B \\ & \searrow^{\iota_B \oplus 1_A} & \downarrow \gamma_{A,B} \\ & & B \oplus A \end{array}$$

that means $\gamma_{A,B} \circ (1_A \oplus \iota_B) = \iota_B \oplus 1_A \colon A \to B \oplus A$.

A pre-braiding does not imply the existence of a braiding; in fact, the pre-braiding can not satisfy the naturality property. We proceed with the general construction of homogenous categories:

Theorem 5.1.4. *Let $(\mathcal{G}, \oplus, 0)$ be a braided monoidal groupoid such that:*

(i) $Aut(0) = \{id\}$;

(ii) $Aut(A) \to Aut(A \oplus B)$ taking f to $f \oplus 1_B$ is injective for every A, B in $\mathcal{G}$.

Then, there exists a pre-braided homogenous category $(U(\mathcal{G}), \oplus, 0)$*, which has* $(\mathcal{G}, \oplus, 0)$ *as its underlying braided monoidal groupoid.*

Proof. As we have seen above, a homogenous category is, in particular, a monoidal category where the endomorphisms are isomorphisms; the idea to construct homogenous categories is then to start from the monoidal groupoid, and then to add the relation 5.1.2. This leads us to the following definition: the category $U(\mathcal{G})$ has the same objects of the groupoid $\mathcal{G}$; if A and H are any two objects, we set

$$\mathrm{Hom}_{U(\mathcal{G})}(A, H \oplus A) := \mathrm{Aut}_{\mathcal{G}}(H \oplus A)/\mathrm{Aut}_{\mathcal{G}}(H)$$

where $\mathrm{Aut}_{\mathcal{G}}(H)$ is seen as a subgroup of $\mathrm{Aut}_{\mathcal{G}}(H \oplus A)$ via the map $f \mapsto f \oplus 1_A$. This map is injective by hypothesis, and the quotient is defined by precomposition, i.e. $f \sim g$ if and only if there exists a map $\varphi \in \mathrm{Aut}_{\mathcal{G}}(H)$ such that $g = f \circ (\varphi \oplus 1_A)$. If $B \in \mathcal{G}$ is any object such that $B \cong H \oplus A$, we choose an isomorphism $\phi\colon H \oplus A \to B$, and we use it to identify the morphism set $\mathrm{Hom}_{U(\mathcal{G})}(A, B)$ with $\mathrm{Hom}_{U(\mathcal{G})}(A, H \oplus A)$. Finally, we say that $\mathrm{Hom}_{U(\mathcal{G})}(A, B)$ is empty unless there exists H such that $B \cong H \oplus A$.

Let $[f] \in \mathrm{Hom}_{U(\mathcal{G})}(A, H \oplus A)$ and $[g] \in \mathrm{Hom}_{U(\mathcal{G})}(H \oplus A, K \oplus (H \oplus A))$ be two morphisms, and let f, g be respectively two representatives; we define the composition law $[g] \circ [f]$ as the equivalence class of the map $g \circ (1_K \oplus f) \in \mathrm{Aut}_{\mathcal{G}}(K \oplus H \oplus A)$ in $\mathrm{Hom}_{U(\mathcal{G})}(A, (K \oplus H) \oplus A)$. If we consider f' and g' such that $f' = f \circ (\phi \oplus 1_A)$ and $g' = g \circ (\psi \oplus 1_{H \oplus A})$, where $\phi \in \mathrm{Aut}_{\mathcal{G}}(H)$ and $\psi \in \mathrm{Aut}_{\mathcal{G}}(K)$, then

$$\begin{aligned} g' \circ (1_K \oplus f') &= (g \circ (\psi \oplus 1_{H \oplus A})) \circ (1_K \oplus (f \circ (\phi \oplus 1_A))) \\ &= g \circ (\psi \oplus 1_{H \oplus A}) \circ (1_K \oplus f) \circ (1_K \oplus (\phi \oplus 1_A)) \\ &= g \circ (1_K \oplus f) \circ (\psi \oplus 1_{H \oplus A}) \circ (1_K \oplus (\phi \oplus 1_A)) \\ &= g \circ (1_K \oplus f) \circ \Phi \end{aligned}$$

where $\Phi \in \mathrm{Aut}_{\mathcal{G}}(K \oplus H)$, so the composition law is well-defined for the chain of objects A, $H \oplus A$, $K \oplus H \oplus A$. In the general case, one uses the chosen isomorphisms to extend the composition law. It is now easy to check that $U(\mathcal{G})$ is indeed a category, by using that also $\mathcal{G}$ is a category.

We remark that $U(\mathcal{G})$ has $\mathcal{G}$ as its underlying groupoid. In fact,

$$\mathrm{Aut}_{U(\mathcal{G})}(A) = \mathrm{Aut}_{\mathcal{G}}(0 \oplus A)/\mathrm{Aut}_{\mathcal{G}}(0)$$

and, by hypothesis, $\mathrm{Aut}_{\mathcal{G}}(0) = \{\mathrm{id}\}$. Therefore, the label $\mathcal{G}$ or $U(\mathcal{G})$ is useless when we speak of automorphisms, and we can obmit it. We also remark that 0 is initial, as the morphism set $\mathrm{Hom}_{U(\mathcal{G})}(0, A)$ is actually $\mathrm{Aut}(A \oplus 0)/\mathrm{Aut}(A)$, that consists of a unique element.

Now we check that the axioms of homogenous category are satisfied. By definition, $\mathrm{Aut}(B)$ acts transitively on $\mathrm{Hom}_{U(\mathcal{G})}(A, B)$, and, by hypothesis, the map $\mathrm{Aut}(A) \to \mathrm{Aut}(A \oplus B)$ is injective. We have to pay attention with axiom $(H2)$; although we don't have at the moment a monoidal structure on $U(\mathcal{G})$, we want the preferred map $A \to H \oplus A$ to be of type $\iota_H \oplus 1_A$, where the $\oplus$ is a monoidal product on $U(\mathcal{G})$. Therefore, we first define the map $\iota_H \oplus 1_A$ as the equivalence class of $1_{H\oplus A}$ in $\mathrm{Hom}_{U(\mathcal{G})}(A, H \oplus A)$; with this definition, we extend the monoidal structure in $\mathcal{G}$ to the maps ι and id, and we get that $\mathrm{Fix}(A)$ is the whole $\mathrm{Aut}(H)$.

It is left to extend the monoidal structure of $\mathcal{G}$ to $U(\mathcal{G})$; we define the new product to be that of $\mathcal{G}$ on objects and automorphisms. Moreover, we want that this new product $\iota \oplus \mathrm{id}$ is the equivalence class of the map id defined above. The definition of the monoidal product for morphisms is given by the chain of the following maps:

$$\begin{aligned}
&\mathrm{Hom}(B, A \oplus B) \times \mathrm{Hom}(D, C \oplus D) = \mathrm{Aut}(A \oplus B)/\mathrm{Aut}(A) \times \mathrm{Aut}(C \oplus D)/\mathrm{Aut}(C)\\
&\quad\longrightarrow \mathrm{Aut}(A \oplus B \oplus C \oplus D)/\mathrm{Aut}(A) \times \mathrm{Aut}(C) \quad \text{using the product } \oplus \text{ in } \mathcal{G}\\
&\quad\longrightarrow \mathrm{Aut}(A \oplus C \oplus B \oplus D)/\mathrm{Aut}(A) \times \mathrm{Aut}(C) \quad \text{conjugating by the braiding } \gamma_{B,C}\\
&\quad\longrightarrow \mathrm{Aut}(A \oplus C \oplus B \oplus D)/\mathrm{Aut}(A) \oplus \mathrm{Aut}(C) \quad \text{surjectively}\\
&\quad\longrightarrow \mathrm{Aut}(A \oplus B \oplus C \oplus D)/\mathrm{Aut}(A) \oplus \mathrm{Aut}(C) \quad \text{applying the isomorphism } \gamma_{B,C}^{-1}\\
&\quad= \mathrm{Hom}(B \oplus D, A \oplus B \oplus C \oplus D)
\end{aligned}$$

To be more precise, we know that the group $\mathrm{Aut}(A\oplus B)$ acts transitively on $\mathrm{Hom}(B, A\oplus B)$; then, we can write any element $f \in \mathrm{Hom}(B, A \oplus B)$ as $f = \bar{f} \circ (\iota_A \oplus 1_B)$, where $\bar{f} \in \mathrm{Aut}(A \oplus B)$ and $\iota_A \oplus 1_B$ is, as above, the equivalence class of $1_{A\oplus B}$. If g is $\bar{g} \circ (\iota_C \oplus 1_D) \in \mathrm{Hom}(D, C \oplus D)$, then

$$f \oplus g := (\bar{f} \oplus \bar{g}) \circ \gamma_{B.C}^{-1} \circ (\iota_{A\oplus C} \oplus 1_{B\oplus D})$$

In particular, if ι_H is the unique map in $\mathrm{Hom}(0, H \oplus 0)$, that we can also write as $1_{H\oplus 0} \circ \iota_H$, and if 1_A is the map $1_{0\oplus A} \circ (\iota_0 \oplus 1_A) \in \mathrm{Hom}(A, 0 \oplus A)$, we have

$$\iota_H \oplus 1_A = (1_{H\oplus 0} \oplus 1_{0\oplus A}) \circ \gamma_{0,0}^{-1} \circ (\iota_{H\oplus 0} \oplus 1_{0\oplus A})$$

and the new monoidal structure extends the previous one also on the maps $\iota \oplus \mathrm{id}$.

One checks that this is indeed a monoidal structure on $U(\mathcal{G})$, because $\mathcal{G}$ is braided. To check it is also pre-braided, one sees that

$$1_A \oplus \iota_H = (1_{0\oplus A} \oplus 1_{H\oplus 0}) \circ \gamma_{A,H}^{-1} \circ (\iota_{0\oplus H} \oplus 1_{A\oplus 0}) = \gamma_{A,H}^{-1} \circ (\iota_H \oplus 1_A).$$

This concludes the proof. □

The category $U(\mathcal{G})$ is the "smallest" homogenous category with $\mathcal{G}$ as its underlying braided monoidal groupoid. In fact, $U(\mathcal{G})$ satisfies the following universal property: if $(\mathcal{C}, \oplus, 0)$ is another such homogenous category, then there is a unique monoidal functor $F\colon U(\mathcal{G}) \longrightarrow \mathcal{C}$ which restricts to the identity on the braided monoidal groupoid $\mathcal{G}$. The functor F is defined as the identity on objects and automorphisms; if f is a map in

$\mathrm{Hom}_{U(\mathcal{G})}(A, H \oplus A)$, we can write it as $f = \phi \circ \iota_A$. The map $F(f) := F(\phi) \circ F(\iota_A)$ is well-defined as 0 is initial in both the categories, and F is already defined on automorphisms. Finally, the monoidal structure we defined in the theorem above is the only pre-braided monoidal structure extending that of $\mathcal{G}$: if we write, as above, $f = \bar{f} \circ (\iota_A \oplus 1_B)$ and $g = \bar{g} \circ (\iota_C \oplus 1_D)$, then we must have

$$\begin{aligned} f \oplus g &= (\bar{f} \circ (\iota_A \oplus 1_B)) \oplus (\bar{g} \circ (\iota_C \oplus 1_D)) \\ &= (\bar{f} \oplus \bar{g}) \circ ((\iota_A \oplus 1_B) \oplus (\iota_C \oplus 1_D)) \\ &= (\bar{f} \oplus \bar{g}) \circ (\iota_A \oplus (\gamma_{B,C})^{-1} \circ (\iota_C \oplus 1_B)) \oplus 1_D) \\ &= (\bar{f} \oplus \bar{g}) \circ \gamma_{B,C}^{-1} \circ (\iota_{A \oplus C} \oplus 1_{B \oplus D}) \end{aligned}$$

Hence, it follows that the functor F respects also the monoidal structure.

There are many examples of homogenous categories; we give two concrete ones:

Example 5.1.5. The most basic example is given by the category $(FI, \sqcup, \emptyset)$ of finite sets and injections, where the strict monoidal structure has the disjoint union as a product and the empty set as a unit element. The automorphism group of an object B in FI is the permutation group of its elements $\Sigma_{|B|}$. It is easy to verify the two axioms:

H1: the homomorphisms in FI are injections, so $\mathrm{Hom}(A, B) = \mathrm{Inj}(A, B)$. If we have two injections $f, g \colon A \hookrightarrow B$, then they are the same up to a permutation of B;

H2: the map $f \mapsto f \sqcup 1_B$ is clearly injective and its image is given by all the permutations that fix the injection $B \hookrightarrow A \sqcup B$.

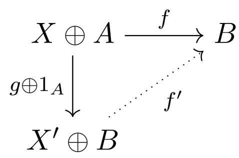
Example 5.1.6. The second example is given by the category $(\mathcal{V}, \oplus, 0)$ of finite dimensional $\mathbb{K}$-vector spaces and split injective linear maps: a morphism $V \to W$ is a pair (f, H) where $f \colon V \hookrightarrow W$ is an injective linear homomorphism and $H \leq W$ is a subspace such that $W = H \oplus f(V)$. The monoidal product is the direct sum of vector spaces and the unit element is the 0-dimensional vector space.

The automorphism group of an object $V \in \mathcal{V}$ is the general linear group $\mathrm{GL}(V)$. If the pairs $(f, H), (g, K)$ belong to $\mathrm{Hom}(V, W)$ then $W = H \oplus f(V) = K \oplus g(V)$. As f, g are split injective homomorphisms then $f(V) \cong g(V)$ and $H \cong K$. There exists an automorphism $\phi \in \mathrm{GL}(W)$ such that $\phi(H) = K$ and $\phi(f(V)) = \phi(g(V))$, and the action on $\mathrm{Hom}(V, W)$ is transitive.

The map $\mathrm{GL}(V) \longrightarrow \mathrm{GL}(V \oplus W)$ is clearly injective. Let $\phi \in \mathrm{Fix}(W) = \mathrm{Fix}(W, V \oplus W)$ be an automorphism of $V \oplus W$ that fixes the map $\iota_V \oplus 1_W$; observe that this latter map is just the pair $(i_W, V) \in \mathrm{Hom}(W, V)$ where $i_W \colon W \hookrightarrow V \oplus W$ is the canonical inclusion and V is the chosen complement. Then ϕ fixes V setwise and W pointwise, and thus it has to come from an automorphism of V.

This example is more suggestive then the first one, because it shows another way to construct homogenous categories, a way that is known as *Quillen's construction*:

Quillen's Construction. Let $(\mathcal{G}, \oplus, 0)$ be a monoidal groupoid. The category $\langle \mathcal{G}, \mathcal{G} \rangle$ has the same objects as $\mathcal{G}$; given two objects A and B, a morphism in $\langle \mathcal{G}, \mathcal{G} \rangle$ from A to B is an equivalence class of pairs (X, f), where X is an object of $\mathcal{G}$ and $f\colon X \oplus A \to B$ is a morphism in G, then an isomorphism. We say that two pairs (X, f) and (X', f') are equivalent if there exists an isomorphism $g\colon X \to X'$ in $\mathcal{G}$ such that $f' \circ (g \oplus 1_A) = f$, that means:



We give an easy example of this construction:

Example 5.1.7. If we consider the category of finite sets and bijections, this is a monoidal groupoid with the sum induced by the disjoint union of sets. The category we get has as objects the finite sets; a morphism $A \to B$ is an equivalence class of pairs $[(X, f)]$, where X is a finite set and $f\colon X \sqcup A \to B$ a bijection. The pairs (X, f), (X', f') are equivalent if there exists a bijection between X and X' such that $f = f' \circ (g \sqcup 1_A)\colon X \sqcup A \to X' \sqcup A \to B$; in particular we have $f|_A = f'|_A$ and actually the pair $[(X, f)]$ is completely determined by $f|_A$. This means that the new category is just the category $(FI, \sqcup, \emptyset)$ of finite sets and injections, that means an homogenous category.

A feature of the category of finite sets and bijections is the following property: if $A \sqcup C$ is in bijection with $B \sqcup C$, then we get a bijection between A and B. This is a property that is true in general for many groupoids, then we give the definition in the general setting:

Definition 5.1.8. A monoidal groupoid $(\mathcal{G}, \oplus, 0)$ satisfies the (right) *cancellation property* if, for every objects A, B, C in $\mathcal{G}$, we have that:

$$A \oplus C \cong B \oplus C \Rightarrow A \cong B$$

This property is really important because it implies an equivalence of categories between $U(\mathcal{G})$ and $\langle \mathcal{G}, \mathcal{G} \rangle$. In fact we have the following result:

Proposition 5.1.9. *Let $(\mathcal{G}, \oplus, 0)$ be a braided monoidal groupoid that satisfies the right cancellation property. Then, if $\mathcal{G}$ satisfies also the hypothesis of Theorem 5.1.4, we get the following isomorphism:*

$$U(\mathcal{G}) \cong \langle \mathcal{G}, \mathcal{G} \rangle$$

Proof. By definition of the category $\langle \mathcal{G}, \mathcal{G} \rangle$, we get that the morphism set $\mathrm{Hom}_{\langle \mathcal{G}, \mathcal{G} \rangle}(A, B)$ is void unless there exists an object H in $\mathcal{G}$ such that $B \cong H \oplus A$. In this case, we have that

$$\mathrm{Hom}_{\langle \mathcal{G}, \mathcal{G} \rangle}(A, B) = \{[(X, f)] \mid f\colon X \oplus A \to H \oplus A, \text{ where } f \text{ is an isomorphism}\}$$

By the cancellation property, we get an isomorphism $X \cong H$, so

$$\mathrm{Hom}_{\langle \mathcal{G},\mathcal{G}\rangle}(A,B) = \{[f] \mid f\colon H \oplus A \to H \oplus A, \text{ where } f \text{ is an isomorphism}\}$$

that says $\mathrm{Hom}_{\langle \mathcal{G},\mathcal{G}\rangle}(A,B)$ is a quotient of $\mathrm{Aut}(H \oplus A)$. Two isomorphisms f, f' are equivalent if there exist an isomorphism of the form $g \oplus 1_A$ such that $f' = f \circ (g \oplus 1_A)$. Therefore, the quotient is by pre-composition factoring out the subgroup $\mathrm{Aut}(H)$, where $\mathrm{Aut}(H) \hookrightarrow \mathrm{Aut}(H \oplus A)$ via the map $g \mapsto g \oplus 1_A$. Hence we have an isomorphism $\mathrm{Hom}_{\langle \mathcal{G},\mathcal{G}\rangle}(A, H \oplus A) \cong \mathrm{Hom}_{U(\mathcal{G})}(A, H \oplus A)$, that leads to an isomorphism of categories $U(\mathcal{G}) \cong \langle \mathcal{G},\mathcal{G}\rangle$. □

In the case of braided monoidal groupoids that satisfy the cancellation property, the two constructions $U(\mathcal{G})$ and $\langle \mathcal{G},\mathcal{G}\rangle$ give the same category; but there are also homogenous categories $U(\mathcal{G})$ that do not come from the Quillen's construction.

Remark 5.1.10. If $(\mathcal{G}, \oplus, 0)$ is the braided monoidal groupoid of finite dimensional $\mathbb{K}$-vector spaces and linear isomorphisms, where $\oplus$ is the direct sum and 0 is the 0-dimensional vector space, we have:

- $\mathrm{Aut}(0) = \{1\}$;
- $\mathrm{Aut}(A) \longrightarrow \mathrm{Aut}(A \oplus B)$ taking f to $f \oplus 1_B$ is injective for every A, B in $\mathcal{G}$, as we proved in Example 5.1.6;
- the cancellation property is satisfied;

then, the Quillen's construction, applied to $\mathcal{G}$, yields the homogenous category $(\mathcal{V}, \oplus, 0)$ described in Example 5.1.6.

In the same way, the Quillen's construction, applied to the braided monoidal groupoid of free finitely generated R-modules over some ring R, yields the category of free finitely generated R-modules and split injections.

5.2 Homological stability for automorphism groups

Let $(\mathcal{C}, \oplus, 0)$ be a homogenous category and consider the automorphism groups

$$G_n := \mathrm{Aut}(A \oplus X^{\oplus n})$$

where A and X are two any objects in $\mathcal{C}$. We have canonical maps $\Sigma_X^n \colon G_n \longrightarrow G_{n+1}$ taking the automorphism $f \in \mathrm{Aut}(A \oplus X^{\oplus n})$ to the automorphism $f \oplus 1_X \in \mathrm{Aut}((A \oplus X^{\oplus n}) \oplus X)$ where we use the identification coming from the monoidal structure:

$$\mathrm{Aut}(A \oplus X^{\oplus n+1}) = \mathrm{Aut}((A \oplus X^{\oplus n}) \oplus X)$$

By axiom $(H2)$, the maps Σ_X^n are injective, and these maps fit into a chain of inclusions

$$\ldots \longrightarrow G_{n-1} \longrightarrow G_n \longrightarrow G_{n+1} \longrightarrow \ldots$$

whose homological stability can now be studied.

As described in Chapter 2, and following Quillen's argument, if we want that the family $\{G_n\}_{n\in\mathbb{N}}$ satisfies homological stability, we also need a family of G_n-spaces, where the G_n-action is transitive on p-simplices and the stabilizers are of the same form G_k. We now see that the definition of homogenous category encodes these two requests.

First, by axiom $(H1)$, the set $\mathrm{Hom}(A, B)$ is a transitive $\mathrm{Aut}(B)$-set. Therefore, the idea is to consider the sets $\mathrm{Hom}(-, A \oplus X^{\oplus n})$, that are $\mathrm{Aut}(A \oplus X^{\oplus n})$-sets for any pair of objects in $\mathcal{C}$. To have also a dependence on p (the p of p-simplices) we consider the sets $\mathrm{Hom}(X^{\oplus p+1}, A \oplus X^{\oplus n})$. This leads us to the following definition of a G_n-semi simplicial set (see Definition 1.1.4):

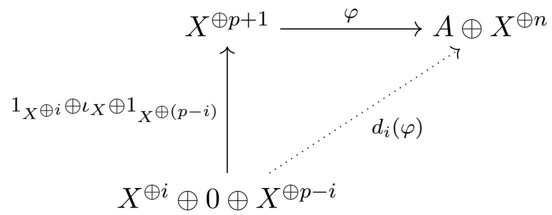
Definition 5.2.1. Let $(\mathcal{C}, \oplus, 0)$ be a homogenous category and let X, A be two any objects in $\mathcal{C}$. Define $W_n(X, A)$ as the semi-simplicial set with set of p-simplices:

$$(W_n(X, A))_p := \mathrm{Hom}(X^{\oplus(p+1)}, A \oplus X^{\oplus n})$$

and with face maps

$$d_i\colon \mathrm{Hom}(X^{\oplus(p+1)}, A \oplus X^{\oplus n}) \longrightarrow \mathrm{Hom}(X^{\oplus p}, A \oplus X^{\oplus n})$$

defined as $d_i(\varphi) := \varphi \circ (1_{X^{\oplus i}} \oplus \iota_X \oplus 1_{X^{\oplus p-i}})$:



By construction, the automorphism group G_n acts transitively on the set of p-simplices, where the action is by post-composition. Moreover, for each p, the stabilizer of each simplex fixes the simplex pointwise. In fact, if $f \in \mathrm{Hom}(X^{\oplus(p+1)}, A \oplus X^{\oplus n})$, and $\phi \in \mathrm{St}(f)$ is an element of its stabilizer, then $\phi \circ f = f$ and $\phi \circ (f \circ i_j) = f \circ i_j$, where i_j is the inclusion of X in $X^{\oplus(p+1)}$ at the j-th place. This means that ϕ fixes its vertices, so it fixes the whole simplex. We get:

Lemma 5.2.2. *The group $Aut(A \oplus X^{\oplus n})$ acts transitively on the set $(W_n(X, A))_p$ for each p, and the stabilizer of each simplex fixes the simplex pointwise.*

Observation 5.2.3. The subgroup of G_n fixing a p-simplex pointwise is isomorphic to G_{n-p-1}. In fact, by transitivity, we choose to fix the simplex

$$i := \iota_{A\oplus X^{\oplus n-p-1}} \oplus 1_{X^{\oplus p+1}} : 0 \oplus X^{\oplus p+1} \cong X^{\oplus p+1} \to (A \oplus X^{\oplus n-p-1}) \oplus X^{\oplus p+1}$$

By axiom $(H2)$, the subgroup that fixes i is isomorphic to G_{n-p-1}. As a consequence, the stabilizer of a p-simplex is conjugate to the group $G_{n-p-1} \leq G_n$.

The last ingredient in Quillen's argument is the aciclicity property for these semi-simplicial sets. As this condition is not a property that we can deduce from the category (and it is not true in general), we need a third axiom:

Definition 5.2.4. Let $(\mathcal{C}, \oplus, 0)$ be a homogenous category and let X, A, be two objects in $\mathcal{C}$. We define the *connectivity property*, with slope $k \geq 2$ as:

(H3) For all $n \geq 1$, $W_n(X, A)$ is $((n-2)/k)$-acyclic.

Remark 5.2.5. If $a > 2$ and $W_n(X, A)$ is $((n-a)/k)$-acyclic, then the semi-simplicial set $W_n(X, A \oplus X^{\oplus(a-2)})$ is $((n-2)/k)$-acyclic. In fact,

$$(W_n(X, A \oplus X^{\oplus(a-2)}))_p = \mathrm{Hom}(X^{\oplus(p+1)}, A \oplus X^{\oplus(n+a-2)}) = (W_{n+a-2}(X, A))_p$$

which is $((n+a-2-a)/k)$-acyclic.

Axioms $(H1)$, $(H2)$ and $(H3)$ encode all the properties we need to prove homological stability. In fact, we get the following theorem:

Theorem 5.2.6. *Suppose that $(\mathcal{C}, \oplus, 0)$ is a homogenous category satisfying the connectivity property $(H3)$ for a pair of objects (X, A), with slope $k \geq 2$. Then the map*

$$H_i(Aut(A \oplus X^{\oplus n});\ \mathbb{Z}) \longrightarrow H_i(Aut(A \oplus X^{\oplus n+1});\ \mathbb{Z})$$

is an isomorphism if $i \leq (n-1)/k$ and a surjection if $i \leq n/k$.

Proof. To prove the theorem we suppose it is true until $n-1$, and we show that the map $G_n \hookrightarrow G_{n+1}$ induces isomorphisms in the stated range. We use the spectral sequence associated to the double chain complex

$$K_{p,q} := E_p G_{n+1} \otimes_{G_{n+1}} \widetilde{C}_q(W_{n+1})$$

where W_{n+1} denotes, for simplicity, the semi-simplicial set $W_{n+1}(X, A)$, $E_* G_{n+1}$ is a free resolution in $\mathbb{Z}[G_{n+1}]$-modules of the trivial $\mathbb{Z}[G_{n+1}]$-module $\mathbb{Z}$, and $\widetilde{C}_*(W_{n+1})$ is the augmented chain complex associated to the semi-simplicial space W_{n+1}.

This double complex gives two spectral sequences; the spectral sequence related to the first filtration has 0-term:

$$\begin{array}{ccc}
\cdots & \cdots & IE^0_{*,*} \\
\big\downarrow \partial^v & \big\downarrow \partial^v & \\
E_{p-1}G_{n+1} \otimes_{G_{n+1}} \widetilde{C}_q(W_{n+1}) & E_p G_{n+1} \otimes_{G_{n+1}} \widetilde{C}_q(W_{n+1}) & \\
\big\downarrow \partial^v & \big\downarrow \partial^v & \\
E_{p-1}G_{n+1} \otimes_{G_{n+1}} \widetilde{C}_{q-1}(W_{n+1}) & E_p G_{n+1} \otimes_{G_{n+1}} \widetilde{C}_{q-1}(W_{n+1}) & \\
\big\downarrow \partial^v & \big\downarrow \partial^v & \\
\cdots & \cdots &
\end{array}$$

whose homology is the homology of the complexes $K_{p,*}$. As the module $E_p(G_{n+1})$ is a free G_{n+1}-module, the homology group $\mathrm{H}_q(K_{p,*})$ is actually $E_pG_{n+1}\otimes_{G_{n+1}}\mathrm{H}_q(\widetilde{C}(W_{n+1}))$ that is 0 when $q\leq(n-1)/k$ due to the acyclicity of W_{n+1}. Then, the infty term $IE^{\infty}_{p,q}$ is 0 for $q\leq(n-1)/k$. As the diagonals are null for $p+q\leq(n-1)/k$, by using the convergence to the homology of the total complex, we get that also the second spectral sequence converges to 0 for $p+q\leq(n-1)/k$.

We denote by $E^r_{p,q}$ this second spectral sequence. Its columns are given by the complexes $K_{*,p}$, and the elements in the first page are of the form:

$$E^1_{p,q}=\mathrm{H}_q(K_{*,p})=\mathrm{H}_q(E_*G_{n+1}\otimes_{G_{n+1}}\widetilde{C}_p(W_{n+1}))=\mathrm{H}_q(G_{n+1};\widetilde{C}_p(W_{n+1}))$$

To compute this homology group, for $p\geq 0$, we use the orbit stabilizer theorem. Let σ_p be a *chosen* p-simplex in W_{n+1}, and let $\mathrm{St}(\sigma_p)\leq G_{n+1}$ be its stabilizer. As G_{n+1} acts transitively on W_{n+1}, then

$$\widetilde{C}_p(W_{n+1})=\bigoplus_{x\in(W_{n+1})_p}\mathbb{Z}x=\bigoplus_{g\in G_{n+1}/\mathrm{St}(\sigma_p)}\mathbb{Z}(g\cdot\sigma_p)$$

By Lemma 5.2.2, the stabilizer $\mathrm{St}(\sigma_p)$ acts trivially on σ_p. This implies that the set of p-simplices $\widetilde{C}_p(W_{n+1})$ can be described as $\mathbb{Z}[G/\mathrm{St}(\sigma_p)]$, with a $\mathbb{Z}$-copy for each representative element. The induced module $\mathrm{Ind}^{G_{n+1}}_{\mathrm{St}(\sigma_p)}\mathbb{Z}$ is isomorphic to $\mathbb{Z}[G/\mathrm{St}(\sigma_p)]$, and by Shapiro's Lemma we get:

$$\mathrm{H}_q(G_{n+1};\ \widetilde{C}_p(W_{n+1}))\cong\mathrm{H}_q(G_{n+1};\ \mathrm{Ind}^{G_{n+1}}_{\mathrm{St}(\sigma_p)}\mathbb{Z})\cong\mathrm{H}_q(\mathrm{St}(\sigma_p);\ \mathbb{Z})$$

If $p=-1$, then the group $E^1_{-1,q}=\mathrm{H}_q(G_{n+1};\ \widetilde{C}_{-1}(W_{n+1}))$ is actually the homology of G_{n+1}, $\mathrm{H}_q(G_{n+1};\ \mathbb{Z})$. If $p=0$ and σ_0 is a vertex, according to Observation 5.2.3, the stabilizer $\mathrm{St}(\sigma_0)$ is isomorphic to G_n, therefore $E^1_{0,q}\cong\mathrm{H}_q(G_n;\ \mathbb{Z})$. The differentials are induced by the vertical boundary map in the double complex $K_{*,*}$, then by the alternating sum of face maps d_i. By Observation 5.2.3, every group $\mathrm{St}(d_i(\sigma_p))$ is conjugated to the stabilizer of the chosen σ_{p-1} simplex. Then, if c_i is this conjugation map, we get that the differentials are induced by the alternating sum of maps:

$$\mathrm{H}_q(\mathrm{St}(\sigma_p))\xrightarrow{\ d_i\ }\mathrm{H}_q(\mathrm{St}(d_i(\sigma_p)))\xrightarrow{\ c_i\ }\mathrm{H}_q(\mathrm{St}(\sigma_{p-1}))$$

In particular, at the 0-level we have differentials $d^1\colon\mathrm{H}_q(G_n)\to\mathrm{H}_q(G_{n+1})$. We want to see that these differentials are induced by the inclusion $G_n\hookrightarrow G_{n+1}$. As conjugation maps do not give information in homology (see to [Bro82], Proposition 6.2), it is enough to verify that the map $\mathrm{H}_q(\mathrm{St}(\sigma_0))\to\mathrm{H}_q(G_{n+1})$ is induced by the inclusion $\mathrm{St}(\sigma_0)\hookrightarrow G_{n+1}$. First, we remark that if E_*G_{n+1} is a free resolution in $\mathbb{Z}[G_{n+1}]$-modules then, by restriction, this is a projective (in general non-free) resolution of in $\mathbb{Z}[\mathrm{St}(\sigma_0)]$-modules. The inclusion $i\colon\mathrm{St}(\sigma_0)\hookrightarrow G_{n+1}$ induces a chain map $i_*\colon E_*G_{n+1}\otimes_{\mathrm{St}(\sigma_0)}\mathbb{Z}\to E_*G_{n+1}\otimes_{G_{n+1}}\mathbb{Z}$. We want to see that the induced map in homology, that we also call i_* coincide with the differential d^1. This can be deduced from the following commutative diagram of chain complexes:

$$\begin{array}{ccc} E_*G_{n+1} \otimes_{G_{n+1}} \widetilde{C}_0(W_{n+1}) & \xrightarrow{\cong} & E_*G_{n+1} \otimes_{\mathrm{St}(\sigma_0)} \mathbb{Z} \\ \big\downarrow d^1 & & \big\downarrow i_* \\ E_*G_{n+1} \otimes_{G_{n+1}} \widetilde{C}_{-1}(W_{n+1}) & \xrightarrow{\cong} & E_*G_{n+1} \otimes_{G_{n+1}} \mathbb{Z} \end{array}$$

where the first isomorphism if given by:

$$\begin{aligned} E_*G_{n+1} \otimes_{G_{n+1}} \widetilde{C}_0(W_{n+1}) &\cong E_*G_{n+1} \otimes_{G_{n+1}} \mathbb{Z}[G_{n+1}/\mathrm{St}(\sigma_0)] \\ &\cong E_*G_{n+1} \otimes_{G_{n+1}} (\mathbb{Z}[G_{n+1}] \otimes_{\mathrm{St}(\sigma_0)} \mathbb{Z}) \\ &\cong (E_*G_{n+1} \otimes_{G_{n+1}} \mathbb{Z}[G_{n+1}]) \otimes_{\mathrm{St}(\sigma_0)} \mathbb{Z} \\ &\cong E_*G_{n+1} \otimes_{\mathrm{St}(\sigma_0)} \mathbb{Z} \end{aligned}$$

The second isomorphism sends the generator of $\widetilde{C}_{-1}(W_{n+1})$ in $1 \in \mathbb{Z}$, i is the inclusion $\mathrm{St}(\sigma_0) \hookrightarrow G_{n+1}$ and i_* is the induced map.

The picture of the first page is now:

$$\mathrm{H}_1(G_{n+1}) \xleftarrow{i_*} \mathrm{H}_1(G_n) \xleftarrow{d^1} \mathrm{H}_1(\mathrm{St}(\sigma_1)) \xleftarrow{d^1} \dots$$

$$\mathrm{H}_0(G_{n+1}) \xleftarrow{i_*} \mathrm{H}_0(G_n) \xleftarrow{d^1} \mathrm{H}_0(\mathrm{St}(\sigma_1)) \xleftarrow{d^1} \dots$$

and the proof reduces to prove that the differential

$$i_* = d^1 \colon E^1_{0,i} = \mathrm{H}_i(G_n) \to \mathrm{H}_i(G_{n+1}) = E^1_{-1,i}$$

is surjective if $i \leq n/k$ and injective if $i \leq (n-1)/k$. The proof is an induction on i, the base $i = 0$ being trivial because the 0-homology of a group is $\mathbb{Z}$.

Surjectivity: Suppose $i \geq 1$ and $i \leq n/k$. The surjectivity follows if we prove:

(i) $E^\infty_{-1,i} = 0$;
(ii) $E^2_{p,q} = 0$ for $p + q = i$ with $q < i$.

Indeed, by *(i)*, $E^1_{-1,i}$ has to be killed at infinity, and, by *(ii)*, d^1 has to kill it, that means d_1 must be surjective. We observe that the condition *(i)* is satisfied because $E^\infty_{p,q} = 0$ when $p+q \leq (n-1)/k$; but $p+q = i-1 \leq n/k-1 = (n-k)/k \leq (n-1)/k$ as $k \geq 2$.

We want to prove $E^2_{p,q}$ is zero. To do it, we first construct a chain map between the complex $\mathrm{H}_q(E_*G_{n+1} \otimes_{G_{n+1}} \widetilde{C}_*(W_{n+1}))$ and the augmented singular chain complex of a point with constant coefficients $\mathrm{H}_q(G_{n+1})$. We have a preferred map $\widetilde{C}_p(W_{n+1}) \to \mathbb{Z}$ sending each p-simplex in $1 \in \mathbb{Z}$; then the following diagram commutes:

$$\begin{array}{ccc} E_*G_{n+1} \otimes_{G_{n+1}} \widetilde{C}_p(W_{n+1}) & \longrightarrow & E_*G_{n+1} \otimes_{G_{n+1}} \mathbb{Z} \\ \Big\downarrow{\scriptstyle \mathrm{id}\otimes_{G_{n+1}} d_i} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ E_*G_{n+1} \otimes_{G_{n+1}} \widetilde{C}_{p-1}(W_{n+1}) & \longrightarrow & E_*G_{n+1} \otimes_{G_{n+1}} \mathbb{Z} \end{array}$$

where d_i is the i-th face map. The differential is defined as the alternating sum of the face maps, then we get the following induced commutative diagram in homology:

$$\begin{array}{ccc} \mathrm{H}_q(E_*G_{n+1} \otimes_{G_{n+1}} \widetilde{C}_p(W_{n+1})) & \longrightarrow & \mathrm{H}_q(E_*G_{n+1} \otimes_{G_{n+1}} \mathbb{Z}) \\ \Big\downarrow{\scriptstyle (\mathrm{id}\otimes_{G_{n+1}}\sum_i(-1)^i d_i)_*} & & \Big\downarrow{\scriptstyle \sum_i(-1)^i \mathrm{id}} \\ \mathrm{H}_q(E_*G_{n+1} \otimes_{G_{n+1}} \widetilde{C}_{p-1}(W_{n+1})) & \longrightarrow & \mathrm{H}_q(E_*G_{n+1} \otimes_{G_{n+1}} \mathbb{Z}) \end{array}$$

We get the desired chain map by observing that $\mathrm{H}_q(E_*G_{n+1} \otimes_{G_{n+1}} \mathbb{Z}) = \mathrm{H}_q(G_{n+1})$. We use now the identification between $\mathrm{H}_q(E_*G_{n+1} \otimes_{G_{n+1}} \widetilde{C}_p(W_{n+1}))$ and $\mathrm{H}_q(\mathrm{St}(\sigma_p))$ to get a chain map between $E^1_{*,q}$ and the augmented singular chain complex of a point with constant coefficients $\mathrm{H}_q(G_{n+1})$ (we use again the fact that conjugation does not affect the homology). If we had isomorphisms $E^1_{p,q} = \mathrm{H}_q(\mathrm{St}(\sigma_p)) \to \mathrm{H}_q(G_{n+1})$ in the range $p + q \leq i$ and a surjection when $p + q = i + 1$, then (ii) is proved:

$$\begin{array}{ccccccccc} \cdots & \xleftarrow{d^1} & E^1_{p-1,q} & \xleftarrow{d^1} & E^1_{p,q} & \xleftarrow{d^1} & E^1_{p+1,q} & \xleftarrow{d^1} & \cdots \\ & & \Big\downarrow{\scriptstyle\cong} & & \Big\downarrow{\scriptstyle\cong} & & \Big\downarrow & & \\ \cdots & \longleftarrow & \mathrm{H}_q(G_{n+1}) & \longleftarrow & \mathrm{H}_q(G_{n+1}) & \longleftarrow & \mathrm{H}_q(G_{n+1}) & \longleftarrow & \cdots \end{array}$$

the homology at (p, q) is the p-homology of a point $\widetilde{\mathrm{H}}_p(\{*\};\ \mathrm{H}_q(G)) = 0$.

It remains to prove that the map $\mathrm{H}_q(\mathrm{St}(\sigma_p)) \to \mathrm{H}_q(G_{n+1})$ is an isomorphism when $p+q \leq i$ and a surjection when $p+q = i+1$, providing $q < i$. We first observe that the stabilizer $\mathrm{St}(\sigma_p)$ is conjugated, in G_{n+1}, to G_{n-p} (by Observation 5.2.3). As $q < i$, we can use the inductive hypothesis on i; by combining it with the induction hypothesis on n, we get for $n - p - 1 \geq qk$ an isomorphism $\mathrm{H}_q(G_{n-p}) \to \mathrm{H}_q(G_{n+1})$, induced by the inclusion $G_{n-p} \hookrightarrow G_{n+1}$, and a surjection for $n - p \geq kq$. We check the relations $p + kq \leq n - 1$ and $p + kq \leq n$ are actually satisfied respectively for $p + q \leq i$ and $p + q = i + 1$:

- if $p+q \leq i-1$, as $k \geq 2$, we get $p+kq \leq p+k(i-1-p) = ki-(k-1)p-k \leq n-p-2$. Then the inequality is satisfied as $p \geq -1$;

- suppose $p+q=i\geq 1$; then $p+kq\leq n-p$, and the inequality follows because $p\geq 1$;
- if $p+q=i+1$, $p+kq=ki+k-(k-1)p\leq n+k-2(k-1)=n-k+2\leq n$ as $p\geq 2$ and $k\geq 2$.

Injectivity: To prove the map $d^1\colon E^1_{0,i}\to E^1_{-1,i}$ is injective in the range $i\leq (n-1)/k$ we prove that:

(i) $E^{\infty}_{0,i}=0$;

(ii) $E^2_{p,q}=0$ for $p+q=i+1$ and $q<i$;

(iii) $d^1\colon E^1_{1,i}\to E^1_{0,i}$ is the zero-map.

If these facts are satisfied, then the term $E^1_{0,i}$ has to be killed at infinity; by (ii), it can not be killed by the terms $E_{p,q}$, $p+q=i+1$ and $q<i$. This means that, by (iii), the only way to kill it is the differential at $E^1_{0,i}$ is injective.

The first condition is satisfied by assumption, as $ki\leq n-1$, and $E^{\infty}_{p,q}=0$ if $p+q\leq (n-1)/k$. The second condition is proved as in the surjectivity part, with indices shifted by 1: we have to prove the map $E^1_{p,q}\to \mathrm{H}_q(G_{n+1})$ is an isomorphism when $p+q\leq i+1$ and a surjection when $p+q=i+2$. Then we need, respectively, the inequalities $p+kq\leq n-1$ and $p+kq\leq n$ are satisfied, for $q<i$. Now, we have $p+kq\leq ki+k-(k-1)p\leq n-1+1+(k-1)(1-p)\leq n-p+1$, providing $p+q\leq i+1$; and $n-p+1\leq n-1$ is satisfied if $p+q=i+1$ ad in such a case $p\geq 2$. If $p+q\leq i$, then the inequalities of above are strict, then $p+kq\leq n-p$, that is satisfied because in such a case $p\geq 1$. Finally, if $p+q=i+2$ we get the chain $kp+q\leq ki+2k-(k-1)p\leq n+(2k-1)+(k-1)p$. As $p\geq 3$ and $k\geq 2$, $n+(2k-1)+(k-1)p\leq n-k+2\leq n$, proving (ii).

For condition (iii) we need that, for each edge f of W_{n+1}, there exists an element g in G_{n+1} that takes $d_1 f$ to $d_0 f$, d_i being the i-th face map, and such that g commutes with any element in $\mathrm{St}(f)$. Suppose this is true, we will prove in Lemma 5.2.7. The differential $d^1\colon E^1_{1,i}\to E^1_{0,i}$ is defined as $(c_1\circ d_1)-(c_0\circ d_0)$, where c_i is the conjugation by the element of G_{n+1} sending $\mathrm{St}(d_i\sigma_1)$ in $\mathrm{St}(\sigma_0)$. We can suppose, without loss of generality, that the chosen 0-simplex is $d_0(\sigma_1)$; the conjugation c_0 is then the identity map. About c_1 we can choose it as the element h in G_{n+1} taking $d_1(\sigma_1)$ in $d_0(\sigma_1)$ and commuting with any element in $\mathrm{St}(\sigma_1)$; the map c_1 is then the conjugation by h. Now, if we take the inclusions $\mathrm{St}(\sigma_1)\hookrightarrow \mathrm{St}(d_1(\sigma_1))$ and $\mathrm{St}(\sigma_1)\hookrightarrow \mathrm{St}(d_1(\sigma_0))$ induced by d_1 and d_0, we observe that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{St}(\sigma_1) & \xrightarrow{d_1} & \mathrm{St}(d_1(\sigma_1)) \\ & \searrow^{d_0=c_0\circ d_0} & \downarrow c_1 \\ & & \mathrm{St}(d_0(\sigma_1)) \end{array}$$

as h commutes with any element in $\mathrm{St}(\sigma_1)$. Then the maps $(c_1 \circ d_1)$ and $(c_0 \circ d_0)$ are the same. This concludes the proof.

□

Lemma 5.2.7. *For each edge f in $W_n(X, A)$, there exists an element g in G_n that takes $d_1 f$ to $d_0 f$, d_i being the i-th face map, and such that g commutes with any element in St(f).*

Proof. By transitivity, we suppose f is the map

$$i_{n-1,n} := \iota_{A\oplus X^{\oplus(n-2)}} \oplus 1_{X^{\oplus 2}} \colon X^{\oplus 2} \longrightarrow A \oplus X^{\oplus n}$$

The face maps are $d_1(f) = f \circ (1_X \oplus \iota_X)$ and $d_0(f) = f \circ (\iota_X \oplus 1_X)$. By $(H1)$, the exists an element $g \in \mathrm{Aut}(X \oplus X)$ such that $g \circ (1_X \oplus \iota_X) = \iota_X \oplus 1_X$:

$$\begin{array}{ccc} X \oplus X & \xrightarrow{\;g\;} & X \oplus X \\ {\scriptstyle \_X\oplus\iota_X}\uparrow & \nearrow{\scriptstyle \iota_X\oplus 1_X} & \\ X & & \end{array}$$

Let $\hat{g}$ be the map $1_{A\oplus X^{\oplus(n-2)}} \oplus g \in G_n$; then $\hat{g} \circ d_1(f) = d_0(f)$.

We now check $\hat{g}$ commutes with every element in $\mathrm{St}(f)$. The set $\mathrm{Fix}(f)$ is actually the set fixing both the maps i_n and i_{n-1}. By $(H2)$, this set is G_{n-2}, and we have the commutativity because $\hat{g}$ acts as the identity on $A \oplus X^{\oplus(n-2)}$. □

Remark 5.2.8. The proof of above does not give a better range if the slope is better then 2, that means if $k < 2$.

The condition $(H3)$ can be enlarged:

Definition 5.2.9. Let $(\mathcal{C}, \oplus, 0)$ be a homogenous category and let X, A, be two objects in $\mathcal{C}$. We define the *connectivity property*, with slope $k \geq 2$ and index $a \geq 2$ as:

$(H'3)$ For all $n \geq 1$, $W_n(X, A)$ is $((n - a)/k)$-acyclic.

Suppose the pair (X, A) satisfies the condition $(H'3)$ for a given integer $a \geq 2$. Then, with a shift of indices in the proof of above, we get the following:

Corollary 5.2.10. *Suppose that $(\mathcal{C}, \oplus, 0)$ is a homogenous category satisfying the connectivity property $(H'3)$ with slope k and index a for a pair of objects (X, A). Then the map*

$$H_i(Aut(A \oplus X^{\oplus n});\ \mathbb{Z}) \longrightarrow H_i(Aut(A \oplus X^{\oplus n+1});\ \mathbb{Z})$$

is an isomorphism if $n \geq ki + (a - 1)$ and a surjection if $n \geq ki + a - 2$.

Proof. If $W_n(X, A)$ is $((n-a)/k)$-acyclic, then $W_n(X, A \oplus X^{\oplus(a-2)})$ satisfies $(H3)$. By the stability theorem the maps

$$\mathrm{H}_i(\mathrm{Aut}(A \oplus X^{\oplus(n+a-2)})) \longrightarrow \mathrm{H}_i(\mathrm{Aut}(A \oplus X^{\oplus(n+a-1)}))$$

is an isomorphism when $ki \leq n-1$ and a surjection when $ki \leq n$. With a shift of indices, this means that

$$\mathrm{H}_i(\mathrm{Aut}(A \oplus X^{\oplus n})) \longrightarrow \mathrm{H}_i(\mathrm{Aut}(A \oplus X^{\oplus(n+1)}))$$

is an isomorphism when $ki \leq n-a+2-1$ and a surjection when $ki \leq n-a+2$. $\square$

5.3 Stability for general linear groups

Let R be an associative ring with identity. Let $(\mathcal{FM}, \oplus, 0)$ be the monoidal groupoid consisting of finitely generated free right R-modules, with 0 as a unit and the direct sum as a product. Then

- if M and N are free modules, then $M \oplus N \cong N \oplus M$, and the groupoid is braided;
- $\mathrm{Aut}(0) = \{\mathrm{id}\}$;
- the map $\mathrm{Aut}(M) \to \mathrm{Aut} M \oplus N$ extending the automorphisms with the identity is injective;
- if M, N, P are free modules, then the cancellation property is satisfied:

$$M \oplus N \cong P \oplus N \Rightarrow M \cong P$$

Therefore, we can apply Theorem 5.1.4, or the Quillen's construction, to obtain a homogenous category $U(\mathcal{FM})$ with underlying groupoid $(\mathcal{FM}, \oplus, 0)$.

This homogenous category is the category of finitely generated R-modules and split injections. It has the same objects as the category $(\mathcal{FM}, \oplus, 0)$, namely the modules R^n, and morphisms $R^m \to R^n$ given by pairs (f, M) where $f\colon R^m \to R^n$ is an injective homomorphism and $M \subseteq R^n$ is a submodule such that $R^n = M \oplus f(R^m)$. We want to find a stability result for the family of general linear groups with coefficients in R: $\mathrm{GL}(n, R) = \mathrm{Aut}(R^n)$, and to do it we apply Theorem 5.2.6.

As a pair in the category $(U(\mathcal{FM}), \oplus, 0)$ we choose $(X, A) = (R, 0)$. We observe that we can associate to any semi-simplicial set the partially ordered set consisting of its simplices. Then:

Lemma 5.3.1. *The poset associated to the semi-simplicial set $W_n(R, 0)$ is isomorphic to the poset of split unimodular sequences $SU_{R^n}(R^n)$.*

Proof. Let (f, M) be a morphism in $\mathrm{Hom}(R^k, R^n)$, that means $R^n = M \oplus f(R^k)$. Let $\{e_i\}$ be the canonical basis in R^k; then we define the vectors v_i as $v_i := f(e_i)$. As vectors w_i we consider the unique vector orthogonal to M and to every v_j for $j \neq i$ such that $v_i \cdot w_i = 1$. We show this vector exists. As M is a complement for R^k in R^n, then there exist vectors $y_1, \dots, y_{n-k}$ such that $\{v_1, \dots, v_k, y_1, \dots, y_{n-k}\}$ is a basis for R^n and the matrix $A = (v_1|\dots|v_k|y_1|\dots|y_{n-k})$ is in $\mathrm{GL}_n(R)$. The vector we are looking for has to be such that $w_i A = e_i$; but A is invertible, then $w_i := e_i A^{-1}$. The sequence $((v_1, w_1), \dots, (v_k, w_k))$ is then split unimodular.

Vice versa let $((v_1, w_1), \dots, (v_k, w_k))$ be a split unimodular sequence. We define the injective map f as $f = (f_1, \dots, f_k) \colon R^k \to R^n$, where $f(e_i) = v_i$. The complement M is given by $w_1^\perp \cap \dots \cap w_k^\perp$. To check it is a complement, we show that $w_i^\perp$ is a complement for f_i. Let x be a vector in $\mathbb{Z}^n$; then we consider the vector $y \in R^n$ such that $y = x - (x \cdot w_i)v_i$. We get $y \in w_i^\perp$ as $y \cdot w_i = (x \cdot w_i) - (x \cdot w_i)(w_i \cdot v_i) = 0$ and this implies $R^n = w_i^\perp \oplus f(e_i)$.

Finally, these maps respect the orderings and give an isomorphism between the two posets. □

Thanks to this lemma, we can check the axiom $(H3)$ by looking at the connectivity of the poset of split unimodular sequences. By Theorem 4.4.7, if R is a ring of stable range sdim + 2 then the poset $SU_{R^n}(R^n)$ is q-acyclic, for $q \leq (n - \mathrm{sdim} - 3)/2$. This implies the semi-simplicial set $W_n(R, 0)$ is $(n - \mathrm{sdim} - 3)/2$-acyclic, and the pair $(R, 0)$ satisfies the axiom $(H'3)$ with slope $k = 2$ and index $a = \mathrm{sdim} + 3$. By Corollary 5.2.10, the inclusions

$$\mathrm{GL}_n(R) = \mathrm{Aut}(R^n) \hookrightarrow \mathrm{Aut}(R^{n+1}) = \mathrm{GL}_{n+1}(R)$$

induce isomorphisms in homology when $n \geq 2i + \mathrm{sdim} + 2$. The general liner groups with coefficients in a ring of finite stable dimension satisfies homological stability:

Theorem 5.3.2. *Let R be an associate ring with identity and suppose R is a ring of stable dimension $sdim + 2$. Then the map*

$$H_i(GL_n(R)) \longrightarrow H_i(GL_{n+1}(R))$$

is an isomorphisms when $n \geq 2i + sdim + 2$.

As $\mathbb{Z}$ is a ring of stable dimension $\mathrm{sdim}(\mathbb{Z}) = 1$, we have the result for the general linear group with integer coefficients:

Corollary 5.3.3. *The family of groups $\{GL_n(\mathbb{Z})\}$ satisfies homological stability and the maps*

$$H_i(GL_n(\mathbb{Z})) \longrightarrow H_i(GL_{n+1}(\mathbb{Z}))$$

are isomorphisms for $i \leq (n - 3)/2$.

By Proposition 3.4.4, we get the following interesting corollary on homological stability for group rings:

Corollary 5.3.4. *Let R be a commutative Noetherian ring of finite Krull dimension. Let G be a finite group, or a group containing a finitely generated abelian group of finite index as a subgroup. Then the family of groups $GL_n(R[G])$ satisfies homological stability.*

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