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Università degli Studi di Pisa

FACOLTÀ DI SCIENZE MATEMATICHE, FISICHE E NATURALI
Corso di Laurea Magistrale in Matematica

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Intersection Theory on Spherical Varieties

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Contents

Alla mia famiglia.

Introduction

The aim of this thesis is to do intersection theory on spherical varieties, i.e. varieties on which a reductive group G acts with an open B-orbit, where B is a Borel subgroup of G. It turns out that on these varieties intersection theory is easier (for a smooth complete spherical variety over $\mathbb{C}$ it is its simplicial cohomology), and in fact the action of the group gives us an explicit description of how the Chow ring $A^*(X)$ works.

In the first chapter we will recall what we need for the rest of the thesis about intersection theory: the definition of the intersection product and how the Chow rings are related to the cohomology, if the field is $\mathbb{C}$.

The second one is really the hearth of the thesis: we give a set of generators for the Chow groups of a spherical variety and we describe how intersection product works.

In the third chapter we study the varieties of the form G/P, where G is a semisimple group and P a parabolic subgroup of it. In this setting the Bruhat decomposition allows us to calculate the Chow group, and we will find another description of the Chow ring of G/P.

In the fourth chapter we discuss the geometry of spherical varieties: the main results will be that those varieties can be described combinatorially. In fact, we can associate to a spherical variety a fan, which can be used to describe our variety both locally and globally: we can translate in a combinatorial language questions such as when this variety is quasiprojective? When it is proper? Moreover, we will describe morphisms between spherical varieties using these fans. The main examples of spherical varieties are toric varieties, Grassmannians and complete symmetric varieties.

The fifth chapter is mostly aimed at defining the Halphen ring of a homogeneous space in a more friendly environment (which is more explicit than the general one). But what is the Halphen ring: the idea is the following. Assume that we want to do intersection theory on a homogeneous spherical variety, for example $(\mathbb{C}^*)^n$. Then the Chow ring does not give us a lot of information because it is always 0 but in degree n, so we need something else: this somethig else is the Halphen ring. The idea is that if we consider all the compactifications $p : X \to (\mathbb{C}^*)^n$ where X is a spherical variety, we can consider $\varinjlim A^*(X)$ instead of simply $A^*((\mathbb{C}^*)^n)$. This ring is much bigger than $A^*((\mathbb{C}^*)^n)$; it will describe better how "intersection theory" works, and we will give a description of this ring using the action of G. Actually this intersection theory depends on the action of the group on our spherical variety, see the examples of this section.

In the last chapter we will talk about intersection theory on another particular case of spherical varieties: toric varieties. We will find ways to compute the cap product with combinatorial methods and we will give a combinatorial description of both the Chow ring of a complete smooth toric variety, and the Halphen ring of $(\mathbb{G}_m)^n$.

Conventions

In this thesis we will assume some conventions: they will always hold, if not stated differently. In this thesis every scheme is meant to be a scheme of finite type over the field k: starting from the second chapter k will be algebraically closed. We assume also that every scheme is separated even if this is not necessary in this first chapter. When we write variety we mean an integral scheme, and a subvariety of a scheme X is a variety which is also a closed subscheme of X. A point will always be a closed point, and if a group G acts on a scheme X, X will be connected. Every map between schemes is meant to be an algebraic morphism. Every group which will appear is a connected affine algebraic group, every action is algebraic, and starting from the third chapter G will be reductive. For us a valuation over a field k will always have rational values, i.e. $v : k \to \mathbb{Q}$. Given X a scheme, $k[X]$ will be the ring of regular functions on it. When we have a cone σ in a vector space V over $\mathbb{R}$, $\mathring{\sigma}$ is the interior part of σ thought as a subspace of $< \sigma >_{\mathbb{R}}$. $\mathbb{G}_m$ will be the multiplicative group k^* with the algebraic structure of $k - \{0\}$.

Chapter 1

Intersection theory: basics

In this chapter we will introduce the terminology and the results about intersection theory (and algebraic geometry in general) that we will need in the remaining part of the thesis. We will not prove anything, the proofs can be found in the book of Fulton [7], in the first 8 chapters, in chapter 17 and 19.

Notation 1.1. *Given X a scheme, O_X will be its structure sheaf, if V is a subvariety of X, $O_{X,V}$ will be the stalk of O_X on the generic point of V. If X is a variety, $k(X)$ will be its field of rational functions.*

Definition 1.1. *Let V be a subvariety of codimension 1 of a scheme X, and let $f \in O_{X,V}$. We define $ord_V(f) := l(O_{X,V}/(f))$ where l is the length.*

Lemma 1.1. *([7] section A.3). Let X be a variety, $V \subseteq X$ a subvariety of codimension 1, and let $f = \frac{a}{b} = \frac{a'}{b'}$ be a rational function on X with $a, b, a', b' \in O_{X,V}$. Then $ord_V(a) - ord_V(b) = ord_V(a') - ord_V(b')$. Thus there is a well-defined homomorphism $k(X) \to \mathbb{Z}, \frac{a}{b} \mapsto ord_V(a) - ord_V(b)$. We call $ord_V(f)$ the order of f along V.*

This definition if X is normal or regular in codimension 1 is the usual definition of order of a rational function along a codimension 1 subvariety.

Definition 1.2. *Let X be a scheme of dimension n. $Z_k(X)$ is the free abelian group generated by the classes of the subvarieties of X of dimension k. If X is a variety and $f \in K(X)$, then we define*

$$div(f) := \sum_{V \subseteq X, codim(V)=1} ord_V(f)V \in Z_{n-1}(X)$$

where V is a subvariety of X. $R_k(X)$ is the free abelian subgroup of $Z_k(X)$ generated by $div(f)$ where f is a rational function on a subvariety $W \subseteq X$ of dimension $k+1$. $A_k(X) := Z_k(X)/R_k(X)$.

$A_k(X)$ is called the Chow group of k-cycles of X, and $\bigoplus_i A_i(X)$ will be the Chow group of X. The class of a subvariety V in $A_*(X)$ will be denoted by $[V]$.

Theorem 1.1. *([7] theorem 1.4). Given $f : X \mapsto Y$ a proper morphism of schemes, and given $V \subseteq X$ a k-dimensional subvariety of X for a certain k, let $deg(V, f(V)) = 0$ if $dim(f(V)) < dim(V)$ and $deg(V, f(V)) = [k(V) : k(f(V))]$ otherwise. We define a map $f_* : Z_k(X) \mapsto Z_k(Y)$, $V \mapsto deg(V, f(V))f(V)$: this map sends $R_k(X)$ to $R_k(Y)$, therefore gives a well-defined homomorphism $f_* : A_k(X) \to A_k(Y)$.*

This is essentially because of the following Lemma:

Lemma 1.2. *([7] proposition 1.4). Let $f : X \to Y$ a proper dominant morphism of varieties of the same dimension, and let $g \in k(X)$. Then $f_*(div(g)) = div(N(g))$ where N is the norm (which, given a finite field extension $L \subseteq K$ and given $a \in K$, is the determinant of $s \mapsto$ as thought as a L-linear map) of $k(X)/k(Y)$ and f_* is the push-forward on $Z_k(X)$ previously defined.*

Notice that this leads to another possible definition of $ord_V(f)$, where f is a rational function on a variety X and V a 1-codimensional subvariety. In fact we can take $p : \widetilde{X} \to X$ the normalization of X (which is proper) and define $div(f) = div(N(f)) = p_*(div(f))$ where the last f is meant as a rational function on $\widetilde{X}$, which has the same field of rational functions as X (thus $N(f) = f$). We can therefore define $ord_V(f) := \sum_{W \mapsto V} ord_W(f)[k(W) : k(V)]$ where the sum is over all W subvarieties of $\widetilde{X}$ of codimension 1 which map surjectively to V.

Observation 1.1. *Notice that this push-forward is functorial: if we have two proper morphisms $X \xrightarrow{f} Y$, $Y \xrightarrow{g} Z$ then $(g \circ f)_* = g_* \circ f_*$.*

Notation 1.2. *If X is complete, we can define the map $deg : A_*(X) \to A_*(spec(k)) = A_0(spec(k)) = \mathbb{Z}$ the push forward map.*

Note that this is not well-defined if X is not complete: take for example $\mathbb{A}^1 \mapsto spec(k)$. In $\mathbb{A}^1$ the class of $[0]$ is rationally equivalent to 0 but as an element in $Z_1(\mathbb{A}^1)$ it does not go to something rationally equivalent to 0 if we push it forward.

Definition 1.3. *Let $Z \subseteq X$ be a closed subscheme of X of pure dimension m, with irreducible components $\{V_i\}_{i \in I}$. Then we define $[Z] := \sum l(O_{Z,V_i})[V_i]$ where l is the length.*

Assume now that we have a flat morphism $f : X \to Y$ of relative dimension m, i.e. for all subvarieties V of Y, all irreducible components of $f^{-1}(V)$ have dimension $dim(V) + m$. An easy way to check this property is the following: if f is flat, Y is irreducible and X has pure dimension $dim(Y) + m$ then f has relative dimension m ([13] III.9.6). So if f is flat of relative dimension m, we can define a map $f^* : Z_k(Y) \to Z_{k+m}(X)$ in this way: we can send $V \subseteq Y$ to $f^{-1}(V)$. Then the result is:

Theorem 1.2. *With the notations as before, if f is flat of relative dimension m, f^* gives us a well-defined map $A_k(Y) \to A_{k+m}(X)$ for every k.*

Remark 1.1. *In this thesis every flat morphism is meant to be of a certain relative dimension.*

Observation 1.2. *([7] proposition 1.7). As in the previous observation, also the flat pull-back is functorial. Furthermore, it is true that if we have the following fiber square with X, Y, V, W schemes and f flat and g proper, then $f^*(g_*(\alpha)) = \overline{g}_*(\overline{f}^*(\alpha))$ for every $\alpha \in A_*(V)$.*

$$\begin{array}{ccc} W & \xrightarrow{\overline{f}} & V \\ {\scriptstyle \overline{g}}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X \end{array}$$

There are two important lemmas which are helpful to compute the Chow groups:

Lemma 1.3. *([7] proposition 1.8). Given $U \subseteq X$ an open subset, let $Z := X - U$, $i : Z \to X$ the closed embedding and $j : U \to X$ the open one. Then for every integer k the following sequence is exact:*

$$A_k(Z) \xrightarrow{i_*} A_k(X) \xrightarrow{j^*} A_k(U) \to 0$$

Lemma 1.4. *([7] Theorem 3.3). Consider the (flat) projection $p : X \times \mathbb{A}^1 \to X$. Then p^* is an isomorphism.*

Now we define two maps on the Chow group of X which will be very important to define a product on $A_k(X)$.

Theorem 1.3. *([7] proposition 1.10 a). Let X, Y be two schemes, there is a well-defined map $A_k(X) \otimes A_h(Y) \to A_{k+h}(X \times Y)$ which sends $[V] \otimes [W] \mapsto [V \times W]$. When we write $\alpha \otimes \beta$, with $\alpha \in A_*(X)$ and $\beta \in A_*(Y)$, we will always think at it as its image in $A_*(X \times Y)$.*
If $f : X' \to X$ and $g : Y' \to Y$ are proper, then $f \times g : X' \times Y' \to X \times Y$ is proper and $(f \times g)_*(\alpha \otimes \beta) = (f_*\alpha) \otimes (g_*\beta)$, and an analogue property holds replacing proper with flat and push forwards with pull backs.

Observation 1.3. *Notice in particular that, with the same notations as before, if $\pi : X \times Y \to Y$ is the projection, it is flat. Then we can perform the flat pull-back. Checking it on the generators, we see that $\pi^*(\alpha) = \alpha \otimes [X]$ for every $\alpha \in A_*(Y)$.*

Definition 1.4. *([17] definition 6.3.4). A morphism $f : Y \to X$ is a regular immersion if it factors through an open immersion followed by a closed immersion, and for every $p \in X$ the ideal $Ker(O_{Y,f(x)} \to O_{X,x})$ is generated by a regular sequence.*

Theorem 1.4. *([7] theorem 6.2) Let $f : Y \to X$ be a regular immersion of codimension d, let V be a scheme and let $V \to X$ be a morphism. Form then the following fiber square:*

$$\begin{array}{ccc} W & \xrightarrow{\;c\;} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ Y & \xrightarrow[\;f\;]{} & X \end{array}$$

Then there is a well-defined homomorphism, called the Gysin homomorphism, $f^! : A_k(V) \to A_{k-d}(W)$. This homomorphism satisfies the following properties: consider this diagram in which every rectangle is a fiber rectangle,

$$\begin{array}{ccc} W' & \xrightarrow{\;e\;} & V' \\ {\scriptstyle \overline{d}}\downarrow & & \downarrow{\scriptstyle d} \\ W & \xrightarrow{\;c\;} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ Y & \xrightarrow[\;f\;]{} & X \end{array}$$

Then the Gysin homomorphism satisfies:
Proper push-forward:
If d is proper then $f^!(d_*(a)) = \overline{d}_*(f^!(a))$ for every $a \in A_*(V')$.
Flat pull-back:
If d is flat with relative dimension m then $f^!(d^*(a)) = \overline{d}^*(f^!(a))$ for every $a \in A_*(V)$.
Compatibility:
If c is a regular immersion of codimension d, then $f^!(a) = c^!(a)$ for every $a \in A_*(V')$.

For other properties of this Gysin homomorphism see [7] chapter 6.

The first of these two theorems is not so difficult, whereas the other is very difficult. The first 6 chapters of [7] are mostly aimed at defining this homomorphism.

We will give an idea of what this theorem means: assume first that $V = X$ is a variety, $b = Id$ and $Y \hookrightarrow X$ is a closed embedding. Then this homomorphism sends the class of a subvariety $[Z]$ to an element $v \in A_*(X)$, and we want v to be "the class of the intersection of Z and Y".

For example, if $Y \subseteq X$ is the zero-set of a global section of $O_X(D)$ for a certain Cartier divisor D, $a_*(f^!([Z])) = c_1(O_X(D)) \cap [Z]$ where c_1 is the first Chern class ([7] example 6.3.4).

Now, this is how we will use this Gysin homomorphism: assume that we have two subvarieties V_1, V_2 of our variety X. Consider the following diagram in which every rectangle is fiber:

$$\begin{array}{ccccc} Z & \hookrightarrow & V_1 \times V_2 & \hookrightarrow & V_2 \\ \downarrow & & \downarrow & & \downarrow \\ V_1 & \hookrightarrow & X \times V_1 & \hookrightarrow & X \\ \downarrow & & \downarrow & & \\ X & \underset{diag}{\hookrightarrow} & X \times X & & \end{array}$$

Looking at the two upper-squares, we see that $Z = V_1 \times_X V_2$. Looking at the squares on the left we see that $Z = X \times_{X \times X} (V_1 \times V_2)$. Thus we can think at $V_1 \cap V_2$ in two ways: intersecting V_1 and V_2 and intersecting X embedded diagonally in $X \times X$ with $V_1 \times V_2$. The proof the latter point of view is that, provided that X is smooth, the diagonal embedding is a regular embedding ([17] Proposition 6.3.13).

Thus using these two theorems we can define a product on $A_*(X)$ when X is a smooth variety: in fact being X smooth, being the diagonal map $X \to X \times X$ a regular embedding, we define:

$$[V] \cdot [W] := diag^!([V] \otimes [W])$$

where $[V] \otimes [W] \in A_*(X \times X)$ and $diag^!([V] \otimes [W]) \in A_*(X)$.

Thanks to the Gysin homomorphism we can define an action of the Chow groups of X on the Chow groups of Y: assume that X is a smooth variety, and let $f : Y \to X$ be a morphism. Then $x \in A_*(X)$ defines a homomorphism on $A_*(Y)$: in fact, consider the following diagram:

$$\begin{array}{ccc} Y & \overset{\gamma_f}{\hookrightarrow} & X \times Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle a} \\ X & \underset{diag}{\hookrightarrow} & X \times X \end{array}$$

Then we can define the action in this way: $x \in A_*(X)$ acts sending $y \in A_*(Y)$ to $diag^!(x \otimes y)$. (In fact, even γ_f is a regular embedding of codimension n: we could replace $diag^!$ with $\gamma_f^!$). We will write $f^*(x) \cap y := diag^!(x \otimes y)$, and $f^*(x) := f^*(x) \cap [Y]$.

Observation 1.4. *([7] proposition 8.1.2 (b)) and ([7] proposition 8.1.2 (a)). If $i : V \to X$ is a regular embedding with X smooth, for every $x \in A_*(X)$*

$i^*(x) = i^!(x)$ where the last Gysin homomorphism is made with the fiber diagram obtained with $i : V \to X$ and $Id : X \to X$.

This pull-back coincides with the flat pull-back when f is flat, thus when we consider the pull-back, we will always consider the one defined above, using the Gysin homomorphism.

Notice that the first part of the previous observation agrees with our "naive" idea of pull-backs and Gysin homomorphism: assume that $x = [W]$, then $i^!([W])$ would be the "class of the intersection" of V and W, and if we consider the diagram below, we see that $i_*(i^*([W])) = i_*(diag^![V \times W]) = diag^![V \times W] = [V] \cdot [W]$ where in the third equality we have used that the Gysin homomorphism commutes with proper push-forward.

$$
\begin{array}{ccc}
V & \xrightarrow{e} & V \times X \\
\downarrow i & & \downarrow j \\
X & \xrightarrow{diag} & X \times X \\
\downarrow Id & & \downarrow Id \\
X & \xrightarrow[diag]{} & X \times X
\end{array}
$$

The main theorem is the following:

Theorem 1.5. *([7] Proposition 8.3 and proposition 17.3.2). i) If Y is a smooth variety of dimension n, then the product previously defined makes $A_*(X)$ into a commutative, associative ring with 1. If Y is smooth, we define $A^k(X) := A_{n-k}(X)$, then we get a graded ring, i.e. $\alpha \in A^a(X)$, $\beta \in A^b(X) \Longrightarrow \alpha \cdot \beta \in A^{a+b}(X)$.*

ii) The assignment $Y \rightsquigarrow A^*(Y)$ is a contravariant functor from non-singular varieties to rings, using that $f^*(\alpha \cdot \beta) = f^*(\alpha) \cdot f^*(\beta)$.

iii) If $f : X \to Y$ is a morphism from a scheme X to a nonsingular variety Y, then

$$A^p(Y) \otimes A_q(X) \xrightarrow{\cap} A_{q-p}(X)$$

makes $A_*(X)$ into an $A^*(Y)$-module.

iv) If $f : X \to Y$ is a proper morphism of nonsingular varieties, then

$$f_*(f^*y \cdot x) = y \cdot f_*(x)$$

The previous formula is called projection formula.

v) Furthermore with the notations of the point iv), given a Cartier divisor $O_Y(D)$ on Y, $f^*(c_1(O_Y(D)) \cap \alpha) = c_1(f^*O_Y(D)) \cap f^*(\alpha)$ and $f_*(c_1(f^*O_Y(D)) \cap x) = c_1(O_Y(D) \cap f_*(x)$ where c_1 is the first Chern class.

Remark 1.2. *Notice that f_* does not commute with the product: consider for example $\mathbb{P}^2 \to spec(k)$, then the product of two lines is a point which does not map to 0, but every line maps to 0 because of a dimensional reason.*

Notation 1.3. *We will write $x \cdot y$ as $x \cap y$ or $x \cup y$. We will use both notations, the second one follows from the link that there is between the cohomology and the Chow rings. In particular, follows from the fact that if X is a topological space and $x, y \in H^*(X)$, then $(x \cup y) \cap [X] = x \cap (y \cap [X])$. This last sentence and the reason why we are using $\cup$ will be clearer after last two sections of this chapter.*

1.1 How to compute the cap product

The definitions given before are quite abstract. In this section we will discuss the tools that we will use in this thesis to compute these products. First consider the following fiber diagram, with i a regular embedding of codimension d, and V a subvariety of X of dimension k, $j : V \to Y$ its associated closed embedding:

$$\begin{array}{ccc} W & \xrightarrow{f} & V \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow[i]{} & Y \end{array}$$

This is the hypothesis that we will need:

Assume that every irreducible component of W has dimension $k - d$

Definition 1.5. *When this hypothesis is satisfied we say that X and V intersect properly.*

Observation 1.5. *([7] Lemma 7.1). If we intersect two subvarieties of dimension k and d respectively with $k \geq d$, there can not be any component which has dimension less than $k - d$.*

Then if this is the case, let W_i be the irreducible components of W. Then $i^![V] = \sum a_i[W_i]$ for dimensional reasons. Therefore everything boils down to computing a_i, we will denote a_i as $i(W_i, X \cdot V; Y)$, which is the intersection multiplicity of W_i in $X \cdot V$. Thus we have:

Theorem 1.6. *([7] Proposition 7.1). Let $A := O_{V,W_i}$, then if J is the ideal of W_i we get:*

i) $1 \leq i(W_i, X \cdot V; Y) \leq l(A/J)$ where l is the length,

ii) If J is generated by a regular sequence, then $i(W_i, X \cdot V; Y) = l(A/J)$. If A is Cohen-Macaulay then the local equations for X in Y give a regular sequence generating J, thus the equality ii) holds.

This is particularly useful when $l(A/J) = 1$. We can restate this theorem in a way more useful to calculate intersection products: the first two points of the following theorem follow easily form the previous theorem, while the other requires a proof.

Theorem 1.7. *([7] proposition 8.2). Let X be a smooth variety with two subvarieties V, W which intersect properly. Let Z_i be the irreducible components of the intersection, then if $[V] \cap [W] = \sum_i a_i[Z_i]$:*

i) $1 \leq a_i \leq l(O_{W \cap V, Z_i})$;

ii) If the local ring of $V \times W$ along Z_i (diagonally embedded in it) is Cohen-Macaulay, then $a_i = l(O_{W \cap V, Z_i})$;

iii) $l(O_{W \cap V, Z_i}) = 1$ if and only if the maximal ideal of O_{X, Z_i} is the sum of the prime ideals of V and W. In this case both $O_{W,Z}$ and $O_{V,Z}$ are regular.

Definition 1.6. *Let V_1, V_2 be two subvarieties of a smooth variety X, let $p \in V_1 \cap V_2$. Then we say that V_1 intersects V_2 transversally in p if:*

i) Both V_1 and V_2 are smooth at p;

ii) If the maximal ideal of $O_{X,p}$ is m, there is a set of generators of m, $\{f_1, ..., f_r\}$, such that $r = dim O_{X,p}$, $f_1, ..., f_a$ generates the prime ideal associated to V_1 and $f_{a+1}, ..., f_b$ the one associated to V_2, for a certain couple of integers $a < b$.

We say that V_1, V_2 intersect transversally if they intersect transversally at each point $p \in V_1 \cap V_2$.

Assume that the field is algebraically closed and V_1, V_2 intersect transversally at p, and let Z_1 be the irreducible component of the intersection passing through p. Then, with the same notations as before, there is an open affine neighbourhood of p, U, such that: $f_i \in O_X(U)$ for every i; the ideal defining V_1 in U is $(f_1, ..., f_a)$ whereas the one defining V_2 is $(f_{a+1}, ..., f_b)$. Then the ideal defining $V_1 \cap V_2$ in $O_X(U)$ is $(f_1, ..., f_b)$.

$V_1 \cap V_2$ is smooth at p, the smooth locus is open, thus up to replacing U we can assume $V_1 \cap V_2 \cap U$ to be irreducible. This means that $f_1, ..., f_b$ is prime and the maximal ideal in $O_{Z,X}$ is the sum of the prime ideals defining V_1 and V_2. Therefore, using the previous theorem, we get

Observation 1.6. *If the field is algebraically closed and the intersection is transversal, then the multiplicity of each component of the intersection is 1.*

Now, the problem is that it is not clear whether we can reduce ourselves to proper intersections. Then the following theorem helps us:

Theorem 1.8. *([7] Moving Lemma section 11.4). If X is a smooth quasi-projective variety, α and β are two cycles on X, then there is a cycle α' rationally equivalent to α such that α' and β meet transversally. This means that if $\beta = \sum n_i[W_i]$, there is α' rationally equivalent to α such that $\alpha' = \sum m_j[V_j]$ and $V_j \cap W_i$ is proper for every i, j.*

1.2 Bivariant intersection theory

In what we have said before, we have restricted ourselves to smooth varieties. In this section we will extend this definition to a generic scheme. The main idea is that, as for homology and cohomology, the cohomology acts on the homology using the cap product. The same happens for the Chow ring: when X is smooth $A^*(X)$ acts on $A_*(X)$. Moreover, using both pull-back and cap product, whenever we have a map $f : Y \to X$ the cohomology of X acts on the homology of Y, and whenever we have $f : Y \to X$ with X smooth, $A^*(X)$ acts on $A_*(Y)$ (as we said before). The idea now is the following: we could try to reconstruct $A^*(X)$ from these actions. ([7] chapter 17).

Let X be a scheme with a morphism $X \xrightarrow{f} Y$, we can define $A^*(X \xrightarrow{f} Y)$ in this way:

For each $g : Y' \to Y$ consider the following fiber square:

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

A bivariant class c in $A^p(X \xrightarrow{f} Y)$ is a collection of homomorphisms

$$c_g^{(k)} : A_k(Y') \to A_{k-p}(X')$$

for evry k and g compatible with proper push-forward, flat pull back and commutes with the Gysin homomorphism, i.e.:

Proper push forward:

If $d : Y'' \to Y$ is proper and $b : Y' \to Y$ arbitrary, consider the following fiber diagram

$$\begin{array}{ccc} X'' & \xrightarrow{e} & Y'' \\ {\scriptstyle \overline{d}}\downarrow & & \downarrow{\scriptstyle d} \\ X' & \xrightarrow{c} & Y' \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Then we want that for every $\alpha \in A_k(Y'')$, $c_b^{(k)}(d_*(\alpha)) = \overline{d}_*(c_{bd}^{(k)}(\alpha))$.

Flat pull back:

Consider again the previous fiber diagram, with d flat instead of proper with relative dimension n. Then we want that for every $\alpha \in A_k(Y')$, $c_{bd}^{(k+n)}(d^*(\alpha)) = \overline{d}^*(c_b^{(k)}(\alpha))$.

Gysin homomorphism:
Consider the following diagram, where d is the usual closed embedding:

$$\begin{array}{ccccc}
X'' & \xrightarrow{e} & Y'' & \xrightarrow{h} & \{0\} \\
\downarrow{\scriptstyle d''} & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d} \\
X' & \xrightarrow{c} & Y' & \xrightarrow{g} & \mathbb{A}^1 \\
\downarrow{\scriptstyle a'} & & \downarrow{\scriptstyle a} & & \\
X & \xrightarrow[f]{} & Y & &
\end{array}$$

Then we want that for all $\alpha \in A_k(Y')$, $d^! c_a^{(k)}(\alpha) = c_{ad'}^{(k-1)}(d^!(\alpha))$.

Observation 1.7. *([7] theorem 17.1). We have called this third property the property of being compatible with Gysin homomorphism, but we want it to hold just for $\{0\} \to \mathbb{A}^1$. In fact it is true that if c commutes with the Gysin homomorphism which comes from the embedding $\{0\} \to \mathbb{A}^1$ (as asked before), then it commutes with any Gysin homomorphism which comes from any regular embedding $i : V \to W$.*

Definition 1.7. *Given X a scheme, $\widetilde{A}^*(X) := A^*(X \xrightarrow{Id} X)$.*

I have mentioned the general definition of $A^*(X \xrightarrow{f} Y)$ above just for completeness, we will deal just with $\widetilde{A}^*(X)$. If f is the identity the three properties above are easier to be understood.

Observation 1.8. *We can put a multiplication on $\widetilde{A}^*(X)$, which is naturally an abelian group. In fact, given $\alpha, \beta \in \widetilde{A}^k(X)$ and $\widetilde{A}^h(X)$ respectively, we define $\alpha \circ \beta$ in this way: given $g : Y \to X$ and given $y \in A_m(Y)$, $(\alpha \circ \beta)(y) := \alpha(\beta(y)) \in A_{m-k-h}(X)$*

The following theorem is what we are looking for:

Theorem 1.9. *([7] Corollary 17.4, Poincaré duality). i) If Y is smooth of dimension n then the homomorphism $\widetilde{A}^p(Y) \to A_{n-p}(Y)$ defined sending $c \to c([Y])$ is an isomorphism.*

ii) For Y smooth, the ring structure on $\widetilde{A}^*(Y)$ is compatible with that defined previously on $A^*(X)$, i.e. $c \mapsto c([Y])$ is a ring homomorphism. More generally if $f : X \to Y$ is a morphism, $\alpha \in A_*(X)$ and $\beta \in \widetilde{A}^*(Y)$, then $\beta(\alpha) = f^*(\beta([Y])) \cap \alpha$.

Therefore we will omit the tilde in $\widetilde{A}_k(X)$, i.e.:

Notation 1.4. *We will write $A^*(X)$ instead of $\widetilde{A}^*(X)$: in this way $A^*(X)$ is a ring even for singular varieties. When we write $A^*(X)$ in this thesis we will always mean $\widetilde{A}^*(X)$, idem with $A^k(X)$ and $\widetilde{A}^k(X)$.*

It is not known whether this ring remains commutative if X is not smooth.

1.3 The cl morphism

Some references for this section are [7] chapter 19 and [8] appendix B2.

Assume that the ground field is $\mathbb{C}$ and let X be a smooth variety. Then we can think at its (closed) points as a complex smooth variety and therefore we can take its singular holmology and cohomology. In this section we recall the link between Chow rings and cohomology.

Definition 1.8. *A manifold will be a real C^∞ connected oriented manifold. An n-manifold is a manifold of dimension n.*

Let M be an n-manifold and let $i : X \hookrightarrow M$ be a closed topological subspace of M, then we can define

$$H_j(i, X) := H^{n-j}(M, M - X)$$

This is something that a priori depends on the embedding i:

Theorem 1.10. *Given another closed embedding $i' : X \to M'$ where M' is another C^∞ manifold, $H_j(i, X) = H_j(i', X)$ for every integer j. Thus we can define the Borel-Moore homology of X as*

$$H_j^{BM}(X) := H_j(i, X)$$

for every integer j, for every C^∞ manifold M and every embedding $i : X \to M$.

So in particular $H_j^{BM}(M) = H^{dim(M)-j}(M, \emptyset) = H^{dim(M)-j}(M)$.

Observation 1.9. *Assume that $Z \subseteq X \subseteq M$ are closed subsets of a manifold M, then we have the following exact sequence in cohomology:*

$$... \to H^i(M, X) \to H^i(M, Z) \to H^i(X, Z) \to H^{i+1}(M, X) \to ...$$

Then in particular if $U \subseteq X$ is an open subset of X and $Y := X - U$, we have $M - X \subseteq M - Y \subseteq M$ which implies

$$\begin{gathered} ... \to H^{dim(M)-i}(M, M - Y) \to H^{dim(M)-i}(M, M - X) \\ \to H^{dim(M)-i}(M - Y, M - X) \to H^{dim(M)-i+1}(M, M - Y) \to ... \end{gathered}$$

Now, being X closed, we have that $M - X = M - Y - U$, and $M - Y$ is a manifold. Thus $H^{dim(M)-i}(M - Y, M - X) = H_i^{BM}(U)$ and we get the following exact sequence:

$$.. \to H_i^{BM}(Y) \to H_i^{BM}(X) \to H_i^{BM}(U) \to ...$$

Lemma 1.5. *Let X be a smooth algebraic variety over $\mathbb{C}$ and $Z \subseteq X$ a closed reduced subscheme of dimension k. Then $H_{2k}^{BM}(Z)$ is free with a canonical generator for every irreducible component of Z of maximal dimension, and $H_i^{BM}(Z) = 0$ for every $i > 2k$.*

Proof. If Z is smooth the theorem is true because $H_i^{BM}(Z) = H_i(Z)$. If Z has dimension 0 it is true because, being it reduced, if it has dimension 0 it is smooth. Assume that Z is not smooth, and let Z° the smooth locus of Z: it is not empty because we are assuming Z to be reduced, and it is open ([17] Lemma 4.2.21, proposition 4.2.24). Furthermore, $dim(Z - Z^\circ) < dim(Z)$ then for $Z - Z^\circ$ (with the reduced structure) the inductive hypothesis hold, therefore using the exact sequences above and inductive hypothesis we get

$$0 \to H_{dim(Z)}^{BM}(Z) \to H_{dim(Z)}^{BM}(Z^\circ) \to 0$$

This proves the lemma because Z° is smooth, and the isomorphism $H_{dim(Z)}^{BM}(Z) \to H_{dim(Z)}^{BM}(Z^\circ)$ is canonical. □

Definition 1.9. *Given $Z \subseteq X$ a closed subvariety of a smooth variety, we define η_Z the canonical generator of $H_{dim(Z)}^{BM}(Z)$, and $cl(Z)$ its image in $H_{dim(Z)}^{BM}(X)$ under the map defined in observation 1.9.*

This defines a map $Z_k(X) \to H_{2k}(X)$, the theorem now is:

Theorem 1.11. *([7] corollary 19.2). If $\alpha \in Z_*(X)$ is rationally equivalent to 0, then $cl(\alpha) = 0$: cl gives a well-defined map $A^*(X) \to H^*(X)$ using Poincaré duality. This map is a ring homomorphism, contravariant for morphism of non-singular varieties.*

Furthermore, if $f : X \to Y$ is a proper morphism of smooth varieties, then cl commutes with proper push forward ([7] lemma 19.1.2 and observations below).

Chapter 2

Intersection theory when a connected solvable group acts

In this chapter we assume that a solvable connected group B acts on our scheme X.

Remark 2.1. *From now on we assume that the ground field k is algebraically closed.*

We want mainly to present four theorems in this chapter. The main reference is the article by Fulton [11].

Definition 2.1. *Assume that B acts on a vector space V. Then we say that $v \in V$ is B-semiinvariant if it is an eigenvector for every $b \in B$. In this case we have a morphism $\lambda : B \to \mathbb{G}_m$ (which is the character of v) such that for every $b \in B$, $bv = \lambda(b)v$.*

Definition 2.2. *Let $Z_k^B(X)$ be the free $\mathbb{Z}$-module generated by the k-dimensional B-stable subvarieties of X, let $R_k^B(X)$ be the subgroup of $Z_k^B(X)$ generated by $\{div(f)\}$ such that f is a non 0 B-semiinvariant rational function on some $Y \subseteq X$ subvariety of dimension $k+1$; and let $A_k^B(X) := Z_k^B(X)/R_k^B(X)$.*

We have a graded map $I_X : A_*^B(X) \to A_*(X)$, which comes from the natural inclusion $Z_*^B(X) \subseteq Z_*(X)$.

Theorem 2.1. *([11] theorem 1). I_X is an isomorphism.*

So for example when X_Σ is a toric variety, we can describe the generators of the Chow group in terms of the cones of the fan describing X, namely $A_k(X)$ is generated by $\{V(\tau) : \tau \in \Sigma, codim(\tau) = k\}$. Assume that B is unipotent, then there are no rational eigenfunctions but the constants, so we have that the Chow group of dimension k is freely generated by the k-dimensional subvarieties. This gives for example a description of the Chow groups of G/P, with G semisimple and P parabolic. In particular when $G = SL(\mathbb{C}^n)$ and P the stabilizer of $[e_1 \wedge \cdots \wedge e_r] \in \mathbb{P}(\Lambda^r\mathbb{C}^n)$, G/P is the Grassmannian, i.e. we have a description of the Chow group for the Grassmannians. We will come back on this in chapter 3, when we will discuss the ring structure of G/B.

Theorem 2.2. *Assume that X is complete and B has finitely many orbits. Then for any scheme Y the Kunneth map $A_*(X) \otimes A_*(Y) \to A_*(X \times Y)$ is an isomorphism.*

So in particular this holds for $X \times X$: if we label the B-stable subvarieties of X with indices $\lambda \in \Sigma$ for a certain set Σ, and $\delta : X \to X \times X$ is the diagonal embedding, we can write $\delta(V(\lambda)) = \sum_{\sigma,\tau} m^{\lambda}_{\sigma,\tau}[V(\sigma)] \otimes [V(\tau)]$, where $dim(V(\sigma)) + dim(V(\tau)) = dim(V(\lambda))$.

Theorem 2.3. *Assume that X is complete and B has finitely many orbits. Then the Kronecker duality map*

$$Kr : A^k(X) \to Hom(A_k(X), \mathbb{Z}),\ a \mapsto (\alpha \mapsto deg(a \cap \alpha))$$

is an isomorphism.

This is an analogous of the map $H^i(X, \mathbb{Z}) \to Hom(H_i(X), \mathbb{Z})$, which comes from the universal coefficient theorem, and which is an isomorphism at least up to torsion. Using this theorem we can identify an element a in $A^k(X)$ with the function $Kr(a)$. So using the previous theorems, we can identify an element in $Hom(A_k(X), \mathbb{Z})$ with a function from the k-dimensional subvarieties of X to $\mathbb{Z}$ which satisfies $\sum_\sigma n_\sigma c(V(\sigma)) = 0$ for all $\sum_\sigma n_\sigma [V(\sigma)] \in R^B(X)$.

Theorem 2.4. *Assume that X is complete and B has finitely many orbits, and let $\delta : X \to X \times X$ is the diagonal embedding. Then using the previous theorem we can write $\delta(V(\lambda)) = \sum_{\sigma,\tau} m^{\lambda}_{\sigma,\tau}[V(\sigma)] \otimes [V(\tau)]$, where $dim(V(\sigma)) + dim(V(\tau)) = dim(V(\lambda))$.*

Now, let $c \in A^p(X)$, then

$$c \cap [V(\lambda)] = \sum_{\sigma,\tau} m^{\lambda}_{\sigma,\tau} Kr(c)([V(\sigma)])[V(\tau)]$$

This description will allow us to compute the Chow ring of any toric variety: we will come back on it in chapter 6.

Let now focus on the last three theorems of this chapter:

Theorem 2.2

Let X be as in the theorem, and let Y be an arbitrary scheme with the trivial action of B. Consider the following square, where f and g are the maps defined in chapter 1:

$$\begin{array}{ccc} A_*^B(X) \otimes A_*^B(Y) & \xrightarrow{\ f\ } & A_*^B(X \times Y) \\ \Big\downarrow {\scriptstyle I_X \otimes I_Y} & & \Big\downarrow {\scriptstyle I_{X \times Y}} \\ A_*(X) \otimes A_*(Y) & \xrightarrow[\ g\ ]{} & A_*(X \times Y) \end{array}$$

If we prove that f is an isomorphism, then the theorem follows from the first theorem of this chapter. The fact that f is an isomorphism follows from the following two lemmas:

Lemma 2.1. *Every B-stable subvariety of $X \times Y$ has the form $V \times W$ where V is a B-stable subvariety of X and W is a suvariety of Y.*

Proof. Let Z be a B-stable subvariety of $X \times Y$, $\pi : X \times Y \to Y$. For every orbit closure V in X, define

$$Z_V := \{y \in Y \text{ such that } V \times \{y\} \text{ is an irreducible component of } Z \cap (X \times \{y\})\}$$

Remind the following facts about constructible subsets of a Zariski topological space:
1) Let $f : X \to Y$ a morphism of finite type between noetherian schemes, then the image of any constructible subset of X is a constructible subset of Y,
2) Finite union of constructible subsets is constructible,
3) A constructible subset of a Zariski irreducible subset is dense iff it contains an open nonempty subset.
Let's show that each Z_V is constructible, by induction on the dimension of V. If V is itself an orbit in X, let $\{O_i\}_{i=1}^k$ be the orbits of X such that $V \subsetneq \overline{O_i}$, and let $A_i := p(Z \times_X O_i)$, where p is the composition of $Z \hookrightarrow X \times Y$ and $\pi : X \times Y \to Y$, and we are thinking at $Z \times_X O_i$ as its image in Z. Its points are the points of Z which map to O_i. Then $p(Z \times_X V) - \bigcup_i A_i$ is constructible and it is Z_V. In fact, let $y \in Z_V$, then $y \in p(Z \times_X V)$. If $y \in A_i$ for a certain i, $Z \cap (X \times \{y\})$ is a closed B-stable subset of X which contains a point of O_i. Therefore it contains $\overline{O_i}$ and then y does not belong to Z_V. On the other hand let $y \in p(Z \times_X V) - \bigcup_i A_i$, $(X \times \{y\}) \cap Z$ contains V because it is B-stable and contains a point which belongs to V. Let V_0 be its irreducible component which contains V, it is the closure of a B-orbit

of X, because there are only finitely many orbits of B. But then if it is not V it should be O_i for a certain i, and then $y \in A_i$ which is absurd. If V is not an orbit, we take as before O_i the B-orbits such that $V \subsetneq \overline{O_i}$, and A_i defined as before. Then $Z_V = p(Z \times_X V) - \bigcup_i A_i - \bigcup Z_{V_j}$ where V_j are the B-orbit closures such that $V_j \subsetneq V$.

Consider $\bigcup_V Z_V$. By the second point it is a constructible subset of Y contained in $A := \overline{\pi(Z)}$ which is irreducible, it contains every (closed) point of $\pi(Z)$ so it is dense and therefore there exists a V such that Z_V contains an open subset U of A. So let's consider $\overline{V \times U}$. $\overline{V \times U}$ is contained in Z and on the other hand, $Z \subseteq \overline{V \times U}$ because if not, let $W := Z - \overline{V \times U}$. It is an open nonempty subset of Z, Z is irreducible so $\overline{W} = Z$. Then choose $y \in U$ such that $W \cap (X \times \{y\})$ is not empty, we get that $\overline{W} \cap (X \times \{y\}) = \overline{W \cap (X \times \{y\})}$: it is the closure of an open subset of $Z \cap (X \times \{y\})$, open subset which does not intersect $V \times \{y\}$; the latter is an irreducible component, therefore $\overline{W \cap (X \times \{y\})}$ does not contain $V \times \{y\}$ which is absurd because $\overline{W} = Z$. Therefore $Z = \overline{V \times U} = V \times A$ □

Lemma 2.2. *For V and W as in the previous lemma, every eigenfunction for B in the field of rational functions on $V \times W$ has the form fg where f is an eigenfunction for B on $k(V)$ and g is in $k(W)$.*

Proof. Let h be such an eigenfunction, with character λ. There exists an open set $U \subseteq W$ such that for each $w \in U$ the restriction of h to $V \times \{w\}$ is a nonzero rational function h_w on V, by the previous lemma, because the zero set of h is a B-stable subvariety. h_w is an eigenfunction with the same character λ for every $w \in U$. Fix $w_0 \in U$ and set $f := h_{w_0}$. For any $w \in W$ h_w/f is an eigenfunction with trivial character, B has a dense orbit in V, and therefore such an eigenfunction is constant, $g(w)$. This is a rational function on W and thus $h = fg$. □

Corollary 2.1. *Assume moreover that X is smooth. Then $A_*(X)$ is free, and if we are working on $\mathbb{C}$, the cycle map from $A_*(X) \to H_*(X)$ is an isomorphism.*

This corollary follows from the fact that when we have X a complete smooth variety over the field $\mathbb{C}$ such that the class of the diagonal is in the image of $A_*(X) \otimes A_*(X)$, then cl is an isomorphism:

Proof. We have that $cl(diag_*[X]) = \sum cl(\alpha_i) \otimes cl(\beta_i)$. Let $X \times X \xrightarrow{p} X$ and $X \times X \xrightarrow{q} X$ be the two canonical projections. The cap and cup products in cohomology will have a subscript H, and given $\xi \in H_*(X)$, $\xi^\vee$ will be its Poincaré dual in cohomology, and similarly given $\eta \in H^*(X)$, $\eta^\vee := \eta \cap_H [X]$. Using Poincaré duality we can define a push forward in cohomology: $p_*(\alpha) := (p_*(\alpha^\vee))^\vee$. We will use this as a definition in the following proof.

Now, we have:

$\alpha \in H^*(X)$, $p^*(\alpha) \cap_H cl([diag_*([X])) =$
$= \sum p^*(\alpha) \cap_H cl(\alpha_i) \otimes cl(\beta_i) = \sum p^*(\alpha) \cap_H (p^*(cl(\alpha_i)) \cap_H q^*(cl(\beta_i))) =$
$= \sum (p^*(\alpha)) \cup_H p^*(cl(\alpha_i)) \cap_H q^*(cl(\beta_i))$. But now recall that

$$q_*(p^*(\alpha) \cap_H cl(diag_*([X]))) = \alpha \cap_H [X]$$

This follows from the definition of pull back and push forward: $p^*(\alpha) \cap_H cl(diag_*([X])) = p^*(\alpha) \cap_H diag_*[X] = diag_*(diag^*(p^*(\alpha)) \cap_H [X])$, where the first equality follows from $cl[X] = [X]$ and the fact that cl commutes with taking the push forward and the last is the projection formula. Then we get $q_*(p^*(\alpha) \cap_H cl(diag_*([X]))) = q_*(diag_*(diag^*(p^*(\alpha)) \cap_H [X]))$, but $Id = q \circ diag$ and $Id = p \circ diag$, thus $q_*(p^*(\alpha) \cap_H cl(diag_*([X]))) = diag^*(p^*(\alpha)) \cap_H [X] = \alpha \cap_H [X]$.

Thus we get, using the projection formula and functoriality,

$$\begin{aligned} \alpha = q_*(\sum (p^*(\alpha) \cup_H p^*(cl(\alpha_i)) \cap_H q^*(cl(\beta_i))) = \\ = \sum q_*(p^*(\alpha) \cup_H p^*(cl(\alpha_i))) \cap_H (cl(\beta_i)) = \\ = \sum q_*(p^*(\alpha \cap_H cl(\alpha_i))) \cap_H (cl(\beta_i)) \end{aligned}$$

But now for every $\gamma \in H_*(X)$, $q_*(p^*(\gamma))) = 0$ whenever γ has dimension greater than 0 and when it has dimension 0 then $q_*(p^*(\gamma))) = m[X]$: this follows for a dimensional reason: if γ has dimension greater than 0 in homology, then its Poincaré dual has dimension less than $dim(X)$ in cohomoloy, thus the Poincaré dual of $p^*(\gamma)$ has dimension more than $dim(X)$ in homology, thus its push forward is 0.

Thus $q_*(p^*(\alpha \cap_H cl(\alpha_i))) \cap_H (cl(\beta_i))$ is a multiple of $cl(\beta_i)$ and then

$$\alpha = \sum q_*(p^*(\alpha \cap_H cl(\alpha_i))) \cap_H (cl(\beta_i)) \in < cl(\beta_i) >_{\mathbb{Z}}:$$
cl is surjective

Now, $A^*(X)$ is free because assume that $\gamma \neq 0$ but $k\gamma = 0$ for a certain $k \neq 0$. Then proceeding in the same way as before, replacing homoogy and cohomology with $A_*(X)$ and $A^*(X)$ respectively, we see that there must exist an α_i such that $q_*(p^*(\gamma \cap \alpha_i))) = m[X] \neq 0 \implies q_*(p^*(k\gamma \cap \alpha_i))) = mk[X] = 0$, which is absurd because $A_n(X)$ is free generated by $[X]$.

This implies also that cl is injective: in fact assume that $\gamma \in Ker(cl)$, then proceeding as before there must exist an i such that $\alpha_i \cap \gamma = m[p]$ where $[p]$ is the class of a point. Then using that cl is a ring homomorphism and that $cl : A_0(X) \to H_0(X)$ is an homomorphism and $[p]$ does not go to 0, we are done. □

Theorems 2.3 and 2.4

Proposition 2.1. *Let X be a complete scheme such that the Kunneth map $A_*X \otimes A_*Y \to A_*(X \times Y)$ is an isomorphism for all schemes Y. Then the Kronecker duality maps are isomorphisms.*

This implies the third theorem.

Proof. **Surjective:**
Let ϕ be a homomorphism $A_k(X) \to \mathbb{Z}$, we want to define an element c_ϕ in $A^k(X)$ which corresponds to ϕ (chapter 1 bivariant intersection theory). Let $f : Y \to X$ be a morphism and let $m \geq k$, we must construct an homomorphism $A_m(Y) \to A_{m-k}(Y)$. This homomorphism will be the composite of $A_m(Y) \to A_m(X \times Y)$, $A_m(X \times Y) = \sum_p (A_p(X) \otimes A_{m-p}(Y)) \to A_k(X) \otimes A_{m-k}(Y)$; and $A_k(X) \otimes A_{m-k}(Y) \to \mathbb{Z} \otimes A_{m-k}(Y)$ where the first map is the inclusion induced by the graph of f (γ_f is proper because we have the following fiber square)

$$\begin{array}{ccc} Y & \xrightarrow{\gamma_f} & X \times Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times X \end{array}$$

The second map is the projection and the third one is $\phi \otimes Id$. This homomorphism must satisfy the conditions of the first chapter, i.e. it must commute with:

Proper push forward for maps $Y' \to Y \to X$ with $Y' \to Y$ proper:
Consider the following diagram:

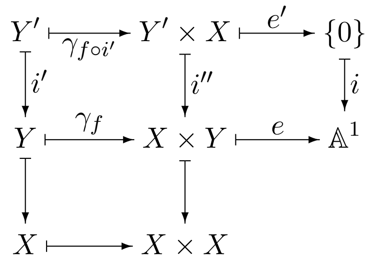
$$\begin{array}{ccc} Y' & \xrightarrow{\gamma_{f'}} & Y' \times X \\ \downarrow{\scriptstyle \overline{d}} & & \downarrow{\scriptstyle d} \\ Y & \xrightarrow{\gamma_f} & Y \times X \\ \downarrow & & \downarrow \\ X & \xrightarrow[diag]{} & X \times X \end{array}$$

We get that for every $\alpha \in A_*(Y')$, $(\gamma_{f'})_*(\alpha) = \sum x_i \otimes y'_i$ thus $c_\phi(\alpha) = \sum_{x_i \in A_k(X)} \phi(x_i) y'_i$ and $\overline{d}_*(c_\phi(\alpha)) = \sum_{x_i \in A_k(X)} \phi(x_i)\overline{d}_*(y'_i)$. On the other hand to compute $c_\phi(\overline{d}_*(\alpha))$ we need to compute $(\gamma_f)_*((\overline{d})_*(\alpha)) = d_*((\gamma_{f'})_*(\alpha)) = d_*(\sum x_i \otimes y'_i) = \sum x_i \otimes \overline{d}(y'_i)$ where we have used in the first equality the functoriality of proper push-forward and then theorem 1.3. Therefore again $c_\phi(\overline{d}_*(\alpha)) = \sum_{x_i \in A_k(X)} \phi(x_i)\overline{d}_*(y'_i)$.

Flat pull back for maps $Y' \to Y \to X$ with $Y' \to Y$ flat:
Consider again the diagram before, with $\overline{d}$ and d flat instead of proper. Then with the same computations as before, using observation 1.2 instead of the functoriality of proper push-forward, we have the thesis.

Gysin homomorphism:
Consider s a regular function on Y and Y' its zero-locus. Let $i' : Y' \to Y$ the

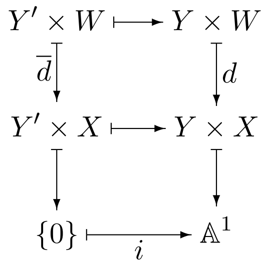
closed embedding defined by this zero-locus, consider the following diagram:



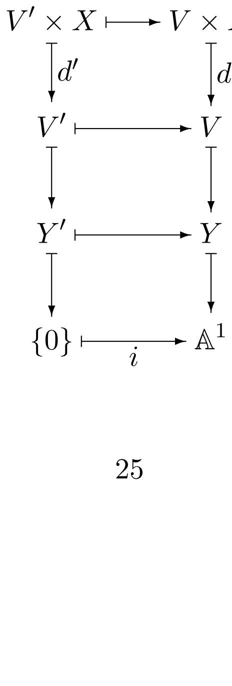
Then we have $(\gamma_{f\circ i'})_*(i^!(\alpha)) = i^!((\gamma_f)_*(\alpha))$ because the Gysin homomorphism commutes with the push forward, if $(\gamma_f)_*(\alpha) = \sum x_i \otimes y_i$ with $x_i \in A_*(X)$ and $y_i \in A_*(Y)$ then we have to show that $i^!(x_i \otimes y_i) = x_i \otimes i^!(y_i)$. In fact, if this is the case, we have

$$(\gamma_{f\circ i'})_*(i^!(\alpha)) = \sum x_i \otimes i^!(y_i) \Longrightarrow$$
$$c_\phi(i^!(\alpha)) = \sum_{x_i \in A_k(X)} \phi(x_i) \otimes i^!(y_i) = i^!(c_\phi(\alpha))$$

This is true because we can reduce ourselves to $[V] \otimes [W] = [V \times W]$ with V a subvariety of Y and W a subvariety of X by linearity and recalling theorem 1.3. We can reduce ourselves to the case $X = W$ because the Gysin homomorphism commutes with proper push-forward d in the following fiber diagram:



Finally we have the thesis because of the following fiber diagram and because the Gysin homomorphism commutes with the flat pull-backs d and d':



Thus we get $i^!([V \times X]) = i^!(d^*([V])) = d'^*(i^![V]) = i^*[V] \otimes [X]$, where the last equality follows from observation 1.3.

Therefore each of the maps above is compatible with these conditions, so it defines an element in $A^k(X)$.

Injective:

Let $\gamma_f : Y \to X \times Y$ be the graph of f and let π_1 and π_2 the two projections from $X \times Y$ to X and Y. Note that π_2 is proper. Since $\pi_1 \circ \gamma_f = f$, $\pi_2 \circ \gamma_f = Id_Y$, $f^*(c) \cap z = (\pi_2)_*(\gamma_f)_*(\gamma_f^*(\pi_1^*(c)) \cap z) = (\pi_2)_*((\pi_1^*(c)) \cap (\gamma_f)_*(z)))$ where we have used functoriality and the projection formula. Now: $(\gamma_f)_*(z) = \sum u_i \otimes v_i$, and since c commutes with flat pull back we have that $\pi_1^*(c) \cap \sum(u_i \otimes v_i) = \pi_1^*(c) \cap \sum \pi_1^*(u_i)\pi_2^*(v_i) = \sum(c \cap u_i) \otimes v_i$. So in this case we have $(\pi_2)_*(\sum(c \cap u_i) \otimes v_i) = \sum deg(c \cap u_i)v_i$ (verify this on the generators: $[V] \otimes [W] = [V \times W]$).

Observation 2.1. *This proves theorem 2.4 if $f = Id$. In fact we have shown that if $c \in A^*(X)$,*

$$(\gamma_f)_*(z) = \sum u_i \otimes v_i \Longrightarrow c \cap z = \sum deg(c \cap u_i)v_i$$

which is the thesis if $z = [V(\lambda)]$.

Note that $deg(c \cap u_i) = 0$ if u_i does not belong to $A_k(X)$. This shows that c can be recovered from the functional $deg(c \cap \cdot)$ on $A_k(X)$ applying the above sequence of three maps. Therefore it is uniquely determined by $Kr(c)$: $A^k(X) \to Hom(A_k(X), \mathbb{Z})$ is injective. □

Observation 2.2. *Note that an important equality follows from these theorems: assume that X is as in theorem 2.4, and let $c, c' \in A^p(X), A^q(X)$ respectively. Then for every $[V(\gamma)] \in A_{p+q}(X)$ the following equality holds:*

$$Kr((c \cup c'))([V(\gamma)]) = \sum_{\sigma,\tau} m^{\gamma}_{\sigma,\tau} Kr(c)([V(\sigma)]) Kr(c')([V(\tau)])$$

where the sum is over (σ, τ) such that $dim(V(\sigma)) = p$ and $dim(V(\tau)) = q$.

In fact from theorem 2.4 we have $Kr((c \cup c'))(V(\gamma)) = deg((c \cup c') \cap (V(\gamma))) = deg(c \cap (c' \cap V(\gamma))) = deg(c \cap (\sum_{\sigma,\tau} m^{\gamma}_{\sigma,\tau} Kr(c')([V(\sigma)])[V(\tau)]))$ which is equal to

$$\sum_{\sigma,\tau} m^{\gamma}_{\sigma,\tau} Kr(c)([V(\sigma)]) Kr(c')([V(\tau)])$$

Now we recall a fact that will be very useful:

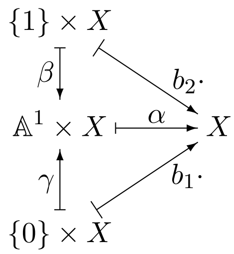
Observation 2.3. *Assume that we have a connected solvable group B which acts on a smooth variety X, and let V be a subvariety of X. Then for every $g \in B$ $[gV] = [V]$.*

Proof. We can think at B as a subgroup of $GL(W)$ for a certain W, and up to choosing the right basis of W we can assume that every matrix belonging to B is upper-triangular. Let B' be the subgroup of $GL(W)$ in which every matrix is upper-triangular in the chosen basis, let T' be the maximal torus of diagonal matrices in it and U' its unipotent radical. Then we can define a map: $\phi_1 : B' \to T'$ which sends a matrix in B to its diagonal, and a map $\phi_2 : B' \to U'$ sending $A \to \phi_1(A)^{-1}A$.

Consider now a maximal subtorus T of B. We can assume that $T' \subseteq T$ up to conjugate T' with an element of B (i.e. changing the embedding $B \hookrightarrow B'$, [21] corollary 6.3.6 (i) and [21] theorem 6.3.5). Remember that $B' = T' \ltimes U'$ where U is its unipotent radical, then consider the product map $T \times U \to B$, $(t, u) \to tu$. This map has an inverse which is $g \to (\phi_1(g), \phi_2(g))$, therefore as a variety

$$B \cong (\mathbb{G}_m)^{dim(T)} \times \mathbb{A}^{dim(U)}$$

This implies that any two points in B can be joined by a sequence of lines (curves in B isomorphic to $\mathbb{A}^1$). Consider then a line $\mathbb{A}^1 \subseteq B$, consider the two points 0 and 1 in it, which will correspond to some b_1 and b_2 in B respectively. Consider then the following diagram:



Here α is the restriction of the action $B \times X \to X$ to $\mathbb{A}^1$, β and γ are the closed embeddings which correspond to the points $\{0\}$ and $\{1\}$ and $b_i \cdot$ is the multiplication by b_i for $i = 1, 2$.

Then $b_1 \cdot$ and $b_2 \cdot$ are isomorphisms thus flat, then $(b_2 \cdot)^*[V] = [b_2^{-1}V]$ and $(b_1 \cdot)^*[V] = [b_1^{-1}V]$, furthermore $(b_2 \cdot)^* = \beta^* \circ \alpha^*$ and $(b_1 \cdot)^* = \gamma^* \circ \alpha^*$ by functoriality. Now let us observe the following fact:

Observation 2.4. *Consider the (flat) projection $p : X \times \mathbb{A}^1 \to X$, then for every subvariety $W \subseteq X$, $p^*([W]) = [W \times \mathbb{A}^1]$. Given a section $s : X \to X \times \mathbb{A}^1$, s is a regular embedding, therefore we can consider the Gysin homomorphism associated to it, with the upper fiber square of this diagram: we will call it $(s^!)^{up}$, to distinguish it from the one obtained with the lower* *square, which we will call simply* $s^!$.

$$\begin{array}{ccc} V & \longmapsto & V \times \mathbb{A}^1 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i \times Id} \\ X & \overset{s}{\longmapsto} & X \times \mathbb{A}^1 \\ {\scriptstyle Id}\downarrow & & \downarrow{\scriptstyle Id} \\ X & \underset{s}{\longmapsto} & X \times \mathbb{A}^1 \end{array}$$

Computing everything locally we see that $(s^!)^{up}[V \times \mathbb{A}^1] = [V]$ *(i.e.* $i_*(s^!)^{up}$ *is the inverse of* p^**, lemma 1.4). This implies that*

$$i_*(s^!)^{up} \text{ does not depend on the chosen section}$$

and if we consider $s^!$*, using that the Gysin homomorphism commutes with proper push forward, we have again that*

$$s^! \text{ does not depend on the chosen section}$$

Recall (observation 1.4) that $s^! = s^*$, then noticing that both β and γ are sections, we get that $\gamma^* = \beta^*$. Therefore:

$$[b_2^{-1}V] = (b_2 \cdot)^*([V]) = (\beta^* \circ \alpha^*)([V]) = \beta^*(\alpha^*([V])) = \gamma^*(\alpha^*([V])) = [b_1^{-1}V]$$

□

Assume now that the group which acts on our variety is connected but a priori not solvable, then we have again that $[gV] = [V]$ for every subvariety $[V]$ of X because of the following theorem:

Theorem 2.5. *([21] theorem 6.4.5). Assume that* G *is a connected algebraic group and let* $g \in G$*. Then there exists a Borel subgroup* B *of* G *such that* $g \in B$.

Therefore we have found that

Theorem 2.6. *Whenever a connected (algebraic affine) group acts on our variety* X*, for every subvariety* Y *of* X *and for every* $g \in G$

$$[gY] = [Y]$$

Chapter 3

The G/P case

In this chapter we will discuss the case in which our variety is a complete homogeneous variety G/P: G will be a semisimple simply connected group and P will be a parabolic subgroup. The field will be (algebraically closed and) of characteristic 0. The standard reference is [1].

In this chapter B will be Borel subgroup of G, T a maximal torus contained in B, U the unipotent radical of B and $W = N_G(T)/T$ will be the Weyl group of G. $\mathfrak{t}$ will be the Lie algebra of T, $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{b}$ and $\mathfrak{u}$ those of B and U respectively. The root decomposition of $\mathfrak{g}$ will be $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ with root system Φ, Δ will be the basis associated to B and Λ will be the character group of T. We will use the same notation also for the character group of $\mathfrak{t}$. For every Lie algebra $\mathfrak{l}$, $E(\mathfrak{l})$ will be its envelopping algebra. For every $w \in W$, $\Phi^+(w)$ will be the set of positive roots α such that $w\alpha$ is negative, and $l(w)$ will be the length of w.

A main point of this section is the Bruhat decomposition of G/P ([21] corollary 8.3.9):

Theorem 3.1. *For every $w \in W$, let $\overline{w}$ be a representative of the class of w and let $C(w) := B\overline{w}B$.*
i) G is the disjoint union of the double cosets $C(w)$.
ii) $(u, b) \to u\overline{w}b$ is an isomorphism of varieties $U_{w^{-1}} \times B \to C(w)$, where $U_w = \prod_{\alpha \in \Phi^+(w)} U_\alpha$.

Let $\mathfrak{p}$ be the Lie algebra of P, if $W_P \subseteq W$ is the Weyl group of the Levi subgroup of P, then for every $w \in W_P$ and $\alpha \in \Phi^+(w)$, $\mathfrak{g}_{w\alpha} \subseteq \mathfrak{p}$. Therefore $\forall w \in W_P$ $C(w) \subseteq P$: the variety G/P has a cellular decomposition, in which every cell is labeled by a coset of W/W_P.

This holds in particular for G/B. We will discuss the geometry of G/B and the structure of its Chow ring, and from the $P = B$ case, the description of the Chow ring of a general G/P will be clear.

3.1 The geometry of G/B

Definition 3.1. *Given $\alpha \in \Phi$, σ_α is the reflection through the hyperplane perpendicular to α.*

Definition 3.2. *(Bruhat order). Given $w_1, w_2 \in W$ and $\gamma \in \Phi^+$, we will write $w_1 \overset{\gamma}{\to} w_2$ if $l(w_2) = l(w_1) + 1$ and $w_2 = \sigma_\gamma w_1$.*
We will write $w \leq w'$ if there is a chain $w = w_1 \to w_2 \to ... \to w_k = w'$.

We will begin this section with some combinatorial results on the root system Φ.

Lemma 3.1. *Let $w = \sigma_{\alpha_1}...\sigma_{\alpha_l}$ be a reduced decomposition for w. Then $\{\sigma_{\alpha_1}...\sigma_{\alpha_{i-1}}(\alpha_i)\}_{i=1}^{l} = \Phi^+ \cap w\Phi^-$: those roots are the roots of $\Phi^+(w^{-1})$.*

This follows from a corollary in [14], pag 50.

Corollary 3.1. *Let $w = \sigma_{\alpha_1}...\sigma_{\alpha_l}$ be a reduced decomposition for w, and let $\gamma \in \Phi^+$ be a root such that $w^{-1}\gamma \in \Phi^-$. Then:*
i) for some i $\sigma_\gamma\sigma_{\alpha_1}...\sigma_{\alpha_i} = \sigma_{\alpha_1}...\sigma_{\alpha_{i-1}}$;
ii) Let $w \in W$, $\gamma \in \Phi^+$. Then $l(w) < l(\sigma_\gamma w) \iff w^{-1}\gamma \in \Phi^+$.

Proof. $i)$: From the previous Lemma we get $\gamma = \sigma_{\alpha_1}...\sigma_{\alpha_{i-1}}(\alpha_i)$ for a certain i, and from [14], pag 43, which states $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$ for $\alpha \in \Phi$ and $\sigma \in W$, get the thesis.
$ii)$: If $w^{-1}\gamma \in \Phi^-$ then by the previous point $\sigma_\gamma w = \sigma_{\alpha_1}...\sigma_{\alpha_{i-1}}\sigma_{\alpha_{i+1}}...\sigma_{\alpha_l}$, that is $l(\sigma_\gamma w) < l(w)$. Interchanging w and $\sigma_\gamma w$ we get that if $w^{-1}\gamma \in \Phi^+$ then $l(w) < l(\sigma_\gamma w)$. □

The main result on the ordering previously defined is the following:

Proposition 3.1. *Let $\dashv$ be a partial ordering $w \dashv w'$ on W with the following properties:*
i) if $\alpha \in \Delta$ and $w \in W$ with $l(\sigma_\alpha w) = l(w) + 1$, then $w \dashv \sigma_\alpha w$;
ii) if $w \dashv w'$ and $\alpha \in \Delta$, then either $\sigma_\alpha w \dashv w'$, or $\sigma_\alpha w \dashv \sigma_\alpha w'$.
Then $w \dashv w' \iff w \leq w'$.

We will not give the proof of this proposition, which can be found in [1], because it goes beyond the point of this chapter.

Theorem 3.2. *Let V be an irreducible finite dimensional representation of the Lie algebra $\mathfrak{g}$ with dominant weight λ. Assume that all the weights $w\lambda$ are distinct. Select for each $w \in W$ a non-zero vector $f_w \in V$ of weight $w\lambda$. Then*

$$w' \leq w \iff f_{w'} \in E(\mathfrak{n})f_w$$

Before starting the proof, let me recall some facts which will be used in this proof (a reference is [10]): whenever we have a representation of an algebraic group $G \subseteq GL(W)$, $\rho : G \to GL(V)$, the differential of ρ at the identity gives a Lie algebra representation $\mathfrak{g} \to \mathfrak{gl}(V) = \mathbb{M}(V)$. Whenever we have an agebraic group G, it acts on its Lie algebra $\mathfrak{g}$ differentiating the map $Ad_s^G : G \to G$, $a \mapsto sas^{-1}$ for every $s \in G$; we will call such a differential ad_s^G, if the group G is clear we will omit the apex G. Then we get for every $s \in G$ and $x \in \mathfrak{g}$, $\mathrm{d}\rho_1(Ad_s^G(x)) = Ad_{\rho(s)}^{GL(V)}\mathrm{d}\rho_1(x) = \rho(s)\mathrm{d}\rho_1(x)\rho(s)^{-1}$.

Let $w \in N_G(T)$, with it class in $N_G(T)/Z_G(T)$ denoted by $[w]$, then for each $s \in T$, $ad(w^{-1}sw)x_\gamma = \gamma([w]^{-1} \circ s)x_\gamma = (w\gamma)(t)x_\gamma$ where the action $[w]^{-1} \circ s$ is the action of W on T and the action $w\gamma$ is the action of W on Φ. But then, $ad(s)ad(w)x_\gamma = ad(w)ad(w^{-1})ad(s)ad(w)x_\gamma = ad(w)((w\gamma)(s)x_\gamma) = (w\gamma)(s)ad(w)(x_\gamma)$, i.e. $ad(w)x_\gamma \in \mathfrak{g}_{w\gamma}$. Therefore $x_{w\gamma} = k \cdot ad(w)x_\gamma$ for a suitable constant k.

Finally let me recall that when we have a unipotent group $U \subseteq GL(V)$ and we have another injective homomorphism $\rho : U \in GL(W)$, then if $\mathfrak{u}$ is the Lie algebra of U, then for every $u \in \mathfrak{u}$, $\rho(exp(u)) = exp(\mathrm{d}\rho_1 u)$.

Proof. To prove this theorem we introduce a partial ordering on W: $w' \dashv w$ if $f_{w'} \in E(\mathfrak{u})f_w$: we want to use the previous proposition.

For each $\gamma \in \Phi^+$ let $x_\gamma \in \mathfrak{g}_\gamma - \{0\}$, $y_\gamma \in \mathfrak{g}_{-\gamma} - \{0\}$ and $h_\gamma \in [\mathfrak{g}_\gamma : \mathfrak{g}_{-\gamma}]$ the $sl(2)$ triplet corresponding to γ. $\mathfrak{l}_\gamma \cong sl(2)$ will be the Lie subalgebra of $\mathfrak{g}$ generated by such a triplet. Let $w' \xrightarrow{\gamma} w$ and $\widetilde{V}$ the smallest $\mathfrak{l}_\gamma$ subspace of V containing $f_{w'}$. Using the previous corollary, $w'^{-1}\gamma \in \Phi^+ \implies 0 = x_{w'^{-1}\gamma}f_e = w'x_{w'^{-1}\gamma}f_e$. By what we have recalled before this proof we get $0 = w'x_{w'^{-1}\gamma}f_e = x_\gamma w'f_e = x_\gamma f_{w'}$: $f_{w'}$ is a vector of maximal weight for x_γ. Therefore by the theory of the representations of $sl(2)$ we get that if $n := w'\lambda(h_\gamma)$, $y_\gamma^n f_{w'}$ is a vector of weight $w\lambda$ (a multiple of f_w), and therefore $f_{w'} = cx_\gamma^n f_w$ where c is a non zero constant: $f_{w'} \in E(\mathfrak{u})f_w$.

Let $w \dashv w'$. We choose $\alpha \in \Delta$ such that $w \xrightarrow{\alpha} \sigma_\alpha w$. Replacing w' by $\sigma_\alpha w'$ if necessary, we may assume that $\sigma_\alpha w' \to w'$, and we want to prove that $\sigma_\alpha w \dashv w'$. If we have $w' \to \sigma_\alpha w'$ then $w \to \sigma_\alpha w'$ and then $\sigma_\alpha\sigma_\alpha w' \to \sigma_\alpha w'$. We would prove instead that $\sigma_\alpha w \to \sigma_\alpha w'$, which is OK as well. From what we have done before and from the theory of the representations of $sl(2)$, $f_{\sigma_\alpha w} = cy_\alpha^n f_w$. Let $\mathfrak{p}_\alpha$ be the Lie algebra generated by $\mathfrak{b}, \mathfrak{l}_\alpha$. Since $w \dashv w'$, $f_w \in E(\mathfrak{u})f_{w'}$ and so $f_{\sigma_\alpha w} = cy_\alpha^n f_w = Xf_{w'}$, where $X \in E(\mathfrak{p}_\alpha)$. But now, every element of $U(\mathfrak{p}_\alpha)$ can be represented as $X = \sum Y_i Y_i' + \widetilde{Y}y_\alpha$ where $Y_i \in E(\mathfrak{u})$, $Y_i' \in E(\mathfrak{t})$ and $\widetilde{Y} \in E(\mathfrak{p}_\alpha)$ (here we are using $\alpha \in \Delta$). But again from the representations of $sl(2)$, $y_\alpha f_{w'} = 0$; therefore $\sum Y_i Y_i' + \widetilde{Y}y_\alpha f_{w'} = \sum Y_i Y_i' f_{w'}$ which is the proof. □

Definition 3.3. *Let $\pi : G \to G/B$, given $w \in W$ we will call $X_w = \pi(C(w))$ the Schubert cell relative to w.*

We want to use the preceding theorem to describe the mutual disposition of the Schubert cells. Given λ a strictly dominant weight, we think at $X := G/B$ as imbedded as a closed subscheme of $\mathbb{P}(V_\lambda)$. For each $w \in W$ choose a representative of $\widetilde{w} \in w$ and f_λ a vector of weight λ. Consider then the functional $\phi_w \in V_\lambda^*$ which is 1 on $f_w := \widetilde{w} f_\lambda$ and 0 on a T-stable complement of $< f_w > \subseteq V_\lambda$. Let $V := V_\lambda$.

Lemma 3.2. *Let $f \in V - \{0\}$ be such that $[f] \in X$. Then $[f] \in X_w \Longleftrightarrow f \in E(\mathfrak{u}) f_w$ and $\phi_w(f) \neq 0$*

Proof. Let $[f] = [g f_e] \in X_w$, i.e. $g \in UwB$. Then $f = c_1 exp(Y) w f_e$ for some $Y \in \mathfrak{u}$. Therefore $f \in E(\mathfrak{u}) f_w$ and $\phi_w(f) \neq 0$. On the other hand, for each $f \in V$ there is at most one $w \in W$ such that $f \in E(\mathfrak{u}) f_w$ and $\phi_w(f) \neq 0$. In fact, if we choose a basis of V of eigenvectors for T, the lowest weight for which a coordinate of f is non zero is less then or equal to $w\lambda$ (this is the condition $f \in E(\mathfrak{u}) f_w$) and in particular it is equal to $w\lambda$ (the condition $\phi_w(f) \neq 0$). □

Theorem 3.3. *Let $w \in W$, then $X_{w'} \subseteq \overline{X_w} \Longleftrightarrow w' \leq w$.*

Proof. Let $X_{w'} \subseteq \overline{X_w}$, then $[f_{w'}] \in \overline{X_w}$ because the condition $\phi_{w'}(f) \neq 0$ is an open condition, and $f_{w'} \in E(\mathfrak{u}) f_w$, which implies $w' \leq w$ by the previous theorem.
To prove the other inclusion it is enough to consider the case $w' \xrightarrow{\gamma} w$. We will use the same notations as the ones in the previous theorem: $n := w\lambda(h_\gamma)$, then as in the proof of theorem 3.2 we get $x_\gamma^n f_w = c f_{w'}$ and $0 = x_\gamma^{n+1} f_w = c f_{w'}$. Therefore $\lim_{t \to \infty} t^{-n} exp(t x_\gamma) f_w = \dfrac{c}{n!} f_{w'}$ which is the proof of our theorem. □

3.2 The ring of polynomials on Λ

Definition 3.4. *Let $R := Sym^*(\Lambda_\mathbb{Q})$ the graded ring of polynomials on the lattice Λ of characters of $\mathfrak{t}$. W acts on Λ and therefore it acts on R, and let J the ideal of R generated by $\{f$ such that $\forall w \in W \ wf = f$ and $f(0) = 0\}$.*

Definition 3.5. *Let $\overline{R} := R/I$.*

The reason why we are introducing these rings is that they will give a description of the Chow ring of G/B.

Definition 3.6. *For every $\gamma \in \Phi(\subseteq \Lambda)$, we define an operator on R,*

$$A_\gamma : R \to R, \ f \mapsto \tfrac{f - \sigma_\gamma f}{\gamma}$$

Observation 3.1. *Note that $A_\gamma(f)$ is actually a polynomial, i.e. $\gamma | f - \sigma_\gamma f$: if $s \in \mathfrak{t}$ is such that $\gamma(s) = 0$ then for every $\alpha \in \Lambda$ $(\sigma_\gamma(\alpha))(s) = (\alpha - < \alpha, \gamma > \gamma)(s) = \alpha(s)$. Therefore $\gamma(s) = 0 \Longrightarrow s - \sigma_\gamma(s) = 0 \Longrightarrow \gamma | s - \sigma_\gamma(s)$.*

Observation 3.2. *Given $f \in R$ homogeneous, $deg(A_\gamma(f)) = deg(f) - 1$ if $A_\gamma(f) \neq 0$.*

Observation 3.3. *$A_\gamma(J) \subseteq J$: in fact given $f = f_1 f_2$ with $wf_1 = f_1$ and $f_2 \in R$, $\frac{f_1 f_2 - \sigma_\gamma f_1 f_2}{\gamma} = \frac{f_1 f_2 - f_1 \sigma_\gamma f_2}{\gamma} = f_1 \frac{f_2 - \sigma_\gamma f_2}{\gamma} \in J$.*

The following one is the main theorem about those operators A_γ, we will not report the proof here. A proof can be found in [1].

Theorem 3.4. *Let $\alpha_1, ..., \alpha_l \in \Delta$, and let $w := \sigma_{\alpha_1} ... \sigma_{\alpha_l}$. Then:*
i) If $l(w) < l$ then $A_{\alpha_1} \cdot ... \cdot A_{\alpha_l} = 0$ ($\cdot$ stands for the composition).
ii) If $l(w) = l$ then $A_{\alpha_1} \cdot ... \cdot A_{\alpha_l}$ depends just on w and not on the set $(\alpha_1, ..., \alpha_l)$. In this case we will denote $A_w := A_{\alpha_1} \cdot ... \cdot A_{\alpha_l}$.
iii) For every $\chi \in \Lambda_{\mathbb{Q}}$ the commutator of A_w with the operator of multiplication by χ satisfies the following relation:

$$[w^{-1} A_w, \chi] = \sum_{w' \xrightarrow{\gamma} w} w' \chi(h_\gamma) w^{-1} A_{w'}$$

where h_γ are defined as in the beginning of theorem 3.2.

As pointed out before, R is a graded ring, where the grading is given by the degree: $R = \bigoplus R^i$.

Definition 3.7. *Let S_i be the dual of R^i, $S := \bigoplus S_i$ and F_γ the dual of A_γ. We will denote as $(\cdot, \cdot)$ the pairing between R and S.*

Then similar theorems hold for F_γ:

Theorem 3.5. *Let $\alpha_1, ..., \alpha_l \in \Delta$, and let $w := \sigma_{\alpha_1} ... \sigma_{\alpha_l}$. Then:*
i) If $l(w) < l$ then $F_{\alpha_l} \cdot ... \cdot F_{\alpha_1} = 0$ ($\cdot$ stands for the composition).
ii) If $l(w) = w$ then $F_{\alpha_l} \cdot ... \cdot F_{\alpha_1}$ depends just on w and not on the set $(\alpha_1, ..., \alpha_l)$. In this case we will denote $F_w := F_{\alpha_1} \cdot ... \cdot F_{\alpha_l}$.
iii) For every $\chi \in \Lambda_{\mathbb{Q}}$ the commutator of F_w with the dual of the operator of multiplication by χ satisfies the following relation:

$$[\chi^*, F_w w] = \sum_{w' \xrightarrow{\gamma} w} w' \chi(h_\gamma) F_{w'} w$$

Definition 3.8. *For every $w \in W$ we set $D_w := F_w(1)$.*

In the following theorem we will recall some properties of D_w for $w \in W$, which follow from the definition of D_w and the previous theorem:

Theorem 3.6. *i) $D_w \in S_{l(w)}$;*
ii) Let $\chi \in \Lambda_{\mathbb{Q}}$, then $\chi^*(D_w) = \sum_{w' \xrightarrow{\gamma} w} w' \chi(h_\gamma) D_{w'}$.
iii) Let $w \in W$ with $l(w) = l$, $\chi_1, ..., \chi_l \in \Lambda_{\mathbb{Q}}$. Then

$$(D_w, \chi_1 ... \chi_l) = \sum \chi_1(h_{\gamma_1})...\chi_l(h_{\gamma_l})$$

where the summation extends over all chains $e \xrightarrow{\gamma_1} w_1 \xrightarrow{\gamma_2} ... \xrightarrow{\gamma_l} w_l = w^{-1}$

Proof. The first follows from the definition of F_w and the observation 3.2, and the second one as well ($\chi^*(1) = 0$). We will focus now on the third one. The point is that if we put $\widetilde{D}_w := D_{w^{-1}}$ then

$$\chi^*(\widetilde{D}_w) = \sum_{w'' \xrightarrow{\alpha} w} \chi(h_\alpha)\widetilde{D}_{w''}$$

Once this equality is proved, we have the thesis by induction. As regards this equality, we are essentially rewriting the equality of the second point, in fact:

$$w'\chi(h_\gamma) = \chi(w'^{-1} \cdot h_\gamma) = \chi(h_{w'^{-1}(\gamma)});$$

$$\sigma_\gamma w' = w \Longrightarrow w^{-1} = w'^{-1}\sigma_\gamma = \sigma_{w'^{-1}\gamma}w'^{-1};$$

therefore if $\alpha := w'^{-1}\gamma$ we are done. □

We will reinterpret all these combinatorial results in a geometric way once the main theorem of this chapter is proved. In particular, the last point of this theorem will give us a way to compute the intersection of a Schubert cell and a product of Chern classes of line bundles (the line bundles on G/B which on the fiber of $[B]$ are representations of weight χ_i).

Definition 3.9. *Let H be the subspace of S orthogonal to J.*

Observation 3.4. *$1 \in H$, using observation 3.3 H is F_γ-invariant, therefore $\{D_w\}_{w \in W} \subseteq H$.*

Theorem 3.7. *The functionals D_w, $w \in W$ form a basis for H.*

Proof. **Linearly independent:**
First of all notice that if w_0 is the element of maximal length in W, $r := l(w_0)$ and δ is the semisum of all the positive roots, then using the previous theorem, $D_{w_0}(\delta^r) = \sum \delta(h_{\gamma_1})...\delta(h_{\gamma_r})$. But this quantity is positive because δ is strictly dominant. Now, let $0 = \sum c_w D_w$, and let $\widetilde{w}$ be an element of minimal length l such that $c_{\widetilde{w}} \neq 0$. There is a sequence $\alpha_1, ..., \alpha_{r-l}$ such that $\widetilde{w}\sigma_{\alpha_1}...\sigma_{\alpha_{r-l}} = w_0$. Then from theorem 3.4 follows that $F_{\alpha_{r-l}}...F_{\alpha_1}D_{\widetilde{w}} = D_{w_0}$ and $F_{\alpha_{r-l}}...F_{\alpha_1}D_w = 0$ for any other w such that $c_w \neq 0$ and $\widetilde{w}\sigma_{\alpha_1}...\sigma_{\alpha_{r-l}} \neq w_0$, i.e. for any w such that $c_w \neq 0$ and $w \neq \widetilde{w}$. Therefore $0 = F_{\alpha_{r-l}}...F_{\alpha_1}(\sum c_w D_w) = c_{\widetilde{w}} D_s \neq 0$ which is absurd.
Generators:
It is enough to prove that if $f \in R$ and $(D_w, f) = 0$ for all $w \in W$, then $f \in J$. We can assume f to be homogeneous of degree k, and do it by induction on k. If $k = 0$ OK, let $k > 0$. Then for every $\alpha \in \Delta$ and $w \in W$ we

get $(D_w, A_\alpha(f)) = (F_\alpha D_w, f) = 0$ from theorem 3.4 (remind how we have chosen f!). But then by inductive hypothesis $A_\alpha(f) \in J \Longrightarrow f - \sigma_\alpha f \in J$. Therefore for every $w \in W$ $f \equiv wf (\text{mod } J) \Longrightarrow f \equiv \dfrac{\sum_{w\in W} wf}{|W|} (\text{mod } J)$. But the left hand side is W-invariant and therefore it belongs to J: this proves the theorem. □

Definition 3.10. *$\{P_w\}$ will be the basis of R/J dual to $\{D_w\}$, i.e. $(D_w, P_{w'}) = 1$ iff $w = w'$, otherwise it is 0.*

3.3 Description of the Chow ring $A^*(G/B)$

In this section we will discuss the main theorem of this chapter.

Theorem 3.8. *A basis for the Chow group $A_*(G/B)$ is given by the classes of the closures of the Schubert cells $\{[\overline{X_w}]\}_{w\in W}$. We will denote $s_w := [\overline{X_w}]$.*

Proof. This follows from the first theorem of this chapter and what we have said after theorem 2.1 in the second chapter. □

From now on, we will assume that the Chow group has coefficients in $\mathbb{Q}$ (i.e. we will forget the subscript $\mathbb{Q}$ in $A^*(X)_\mathbb{Q}$). We are working with a group which is simply connected, which means that every $\lambda \in \Lambda$ gives rise to a line bundle $E_\lambda \to G/B$. Therefore we can define a map $\Lambda \ni \lambda \mapsto c_1(E_\lambda)$ where $c_1(\cdot)$ is the first Chern class. Using the fact that the Chow ring is commutative we get a map $\alpha : R \to A^*(G/B)$, which extends $\lambda \mapsto c_1(E_\lambda)$. Thus for every $w \in W$ we get a map $R \to \mathbb{Q}$ which sends $p \mapsto deg(\alpha(p) \cap s_w)$, this map is linear therefore we get a map $\beta : A_*(X) \to S$.

Theorem 3.9. *$\beta(s_w) = D_w$.*

Before giving with the proof of this theorem, we will make some observations which follow from the theorem:

Observation 3.5. *β is surjective: both H and $A^*(G/B)$ have dimension $|W|$.*

Observation 3.6. *$deg(\alpha(p) \cap s_w) = (D_w, p)$: this follows from the definitions.*

Observation 3.7. *$p \in J \Longrightarrow \alpha(p) = 0$, in fact from the previous observation there are no elements $x \in A^*(X)$ such that $deg(\alpha(p) \cap x) \neq 0$, and the Kronecker duality map is an isomorphism (theorem 2.3).*

Observation 3.8. *$R/J \to A^*(X)$ is an isomorphism: both rings have the same dimension, and using the previous observations, it is injective.*

Observation 3.9. *If we come back to Chow rings with integral coefficients we have* $A_*(X) =< \{s_w\}_{w\in W} >$, *and again using the Kronecker duality map (theorem 2.3) and the previous observations we get that a dual (integral) basis for* $A^*(X)$ *is given by* $\{\alpha(P_w)\}_{w\in W}$.

We are now ready to prove the theorem:

Proof. We want to prove the thesis by induction on the length of w: if $w = 1$ the thesis is clear. For the inductive hypothesis, it is enough to show that $\beta(s_w)$ satisfies the following equality:

$$\chi^*(\beta(s_w)) = \sum_{w' \xrightarrow{\gamma} w} w'\chi(h_\gamma)\beta(s_{w'}) \qquad (*)$$

Geometrically, we have to show that

$$\alpha(\chi) \cap s_w = c_1(E_\chi) \cap s_w = \sum_{w' \xrightarrow{\gamma} w} w'\chi(h_\gamma)s_{w'}$$

In fact, from the previous observations and the associativity of $\cap$ in $A^*(G/B)$, we get $(\chi^*(\beta(s_w)), p) = (\beta(s_w), \chi p) = deg((c_1(E_\chi) \cap s_w) \cap \alpha(p))$. Using the linearity of c_1, we can assume χ to be a regular dominant weight. Therefore we have the following situation: if i is the composition $\overline{X_w} \hookrightarrow /B \hookrightarrow \mathbb{P}(V_\chi)$, then $c_1(E_\chi) \cap s_w$ is the divisor associated to a section of $i^*O_{\mathbb{P}(V_\chi)}(1)$. From the description of the geometry of G/B and using the fact that the Chow ring of G/B is generated by B-orbits closures (i.e. s_w) we get $c_1(i^*O_{\mathbb{P}(V_\chi)}(1)) \cap s_w = \sum_{w' \xrightarrow{\gamma} w} a_{w'}s_{w'}$. The point of the theorem is to show that $a_{w'} = w'\chi(h_\gamma)$.

For a root $\gamma \in \Sigma^+$ we construct a three dimensional Lie subalgebra $\mathfrak{l}_\gamma \subseteq \mathfrak{g}$ as in the proof of theorem 3.2. This corresponds to a group homomorphism $i : SL(2) \to G$ which sends the subgroup B' of upper-triangular matrices of $SL(2)$ in our Borel subgroup B, sending the diagonal matrices in $SL(2)$ in the torus. Let $w' \xrightarrow{\gamma} w$, and let $\widetilde{V}$ be the smallest $\mathfrak{l}_\gamma$-invariant subspace of V containing $f_{w'}$. Then $\widetilde{V}$ is $SL(2)$-invariant and the stabilizer of the point $[f_{w'}]$ is B'. But $\delta : SL(2)/B' \cong \mathbb{P}^1$, therefore we have a map $\mathbb{P}^1 \to X$. We identify the points $0 \in \mathbb{P}^1$ and $\infty \in \mathbb{P}^1$ as the classes of the identity and of $\sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ respectively. Notice that σ goes in a representative of $\sigma_\gamma \in W$, we will call it w_γ. If U is the unipotent radical of B and U^- is its opposite, let $U_{w'} := w'U^-(w')^{-1} \cap U$ (its Lie algebra is corresponds to the sum of root spaces $g_{w'(\alpha)}$ such that α is negative $w'(\alpha)$ is positive). We can define a map $\xi : U_{w'} \times \mathbb{P}^1 \to X$, $(x, z) \mapsto x\delta(z)$. Then ξ has the following properties:

i) $\xi(U_{w'} \times \{0\}) = X_{w'}$,

ii) $\xi(U_{w'} \times (\mathbb{P}^1 - 0)) \subseteq X_w$,

iii) the restriction of ξ to $U_{w'} \times (\mathbb{P}^1 - \infty)$ is an isomorphism onto a certain open subset of $\overline{X_w}$, we will call it U_0.
The first assertion follows from the definition of X_w, and as regards the second assertion it is enough to show that $\delta(z) \in X_w$ for $z \in \mathbb{P}^1 - 0$. So, given $h \in SL(2)$ an inverse image of z, we can write h as $h = b_1\sigma b_2$ with $b_i \in B$. Then: $\delta(b_1\sigma b_2) = i(b_1)w_\gamma i(b_2)f_{w'} = i(b_1)w_\gamma c f_{w'} = i(b_2)c' f_w = c'' f_w$ where c, c', c'' are constants. Therefore $\delta(x) \in X_w$.

In order to prove the last assertion, let us remind the following result:

Proposition 3.2. *Let B be a solvable connected group with a maximal torus T and unipotent radical U. Then $G \cong T \ltimes U$ and $Lie(G) = Lie(T) \oplus Lie(U)$. Assume that $Lie(U) = \bigoplus_{\alpha \in \phi(U)} g_\alpha$ with $dim(g_\alpha) = 1$, and assume that there is $U_\alpha < U$ such that $Lie(U_\alpha) = g_\alpha$. Then, chosen any order on $\phi(U) = \{\alpha_1, ..., \alpha_n\}$, the multiplication $m : U_{\alpha_1} \times ... \times U_{\alpha_n} \to U$ is an isomorphism.*

Consider now the mapping $w'^{-1} \circ \xi : U_{w'} \times (\mathbb{P}^1 - \infty) \to X$. The space $\mathbb{P}^1 - \infty$ is isomorphic to the one one parameter subgroup $U'_- := \{\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}\}$ and the mapping $U_{w'} \times U'_- \to X$ is given by the rule $\xi(n, n_1) = ni(n_1)[f_{w'}]$ where $n \in U$ and $n_1 \in U'_-$. Therefore $w'^{-1} \circ \xi(n, n_1) = w'^{-1} n w' w'^{-1} i(n_1) w'[f_e]$. Now we will use the previous preposition: in fact $w'^{-1} i(U'_-) w' \subseteq U_-$, and $w'^{-1} U_{w'} w' \subseteq U_-$. The first follows from corollary 3.1 whereas the second by the definition of $U_{w'}$. The map $U_- \to X$ which maps $n \mapsto n[f_e]$ is an inclusion of an open subset of X, and the image of $w'^{-1} \circ \xi$ is contained in this open subset. So we get a map $j : U_{w'} \times U'_- \to U_-$: using the previous proposition with this map we are done, because the Lie algebra of $U_{w'}$ by the map $d_1(j|_{U_{w'} \times \{1\}})$ is mapped to the sum of root spaces g_α such that α is negative and $w'(\alpha)$ is positive, and this space has 0 intersection with $g_{w'^{-1}(-\gamma)}$ because $-\gamma = w' w'^{-1}(-\gamma)$ is still negative.

Now, consider the global section of $\phi_w \in O_{\mathbb{P}(V_\chi)}(1)(\mathbb{P}(V_\chi))$ previously defined: we want to show that $\mu := (\phi_w)|_{\overline{X}_w}$ has divisor $\sum_{w' \xrightarrow{\gamma} w} w'\chi(h_\gamma) s_{w'}$.
By theorem 3.8 $div(\mu) = \sum_{w' \xrightarrow{\gamma} w} a_\gamma s_{w'}$, we want to evaluate a_γ. Using the maps previously defined, we get that the coefficient a_γ equals the multiplicity at 0 of the section $\delta^*(\mu)$. In fact, by *iii*), the open subset U_0 of $\overline{X}_w$ is isomorphic to $\mathbb{A}^n \times \mathbb{A}^1$ and μ vanishes on $\mathbb{A}^n \times \{0\}$, which maps to $X_{w'}$. If c is the multiplicity of the divisor of $\mathbb{A}^n \times \{0\}$ in μ, then c is also the multiplicity of 0 in the divisor $\delta^*(\mu)$. This is the multiplicity of 0 of the function $\phi_w(exp(tx_{-\gamma})[f_{w'}])$, but again from the theory of the representations of $sl(2)$ we get, as in the proof of theorem 3.3, that this multiplicity is $n = w'\chi(h_\gamma)$, which proves the theorem. □

3.4 The G/P-case

Let $I \subseteq \Delta$ such that the levi subgroup of P has Weyl group W_I generated by σ_α such that $\alpha \in I$. Let $\pi_I : G \to G/P_I$ and $\pi : G \to G/B$ be the projections.

Lemma 3.3. *Let $W_I^1 := \{w \in W$ such that $wI \subseteq \Sigma^+\}$. Then each coset of W/W_I contains exactly one element of W_I^1. Furthermore, the element $w \in W_I^1$ is characterized by the fact that its length is less than that of any other element in the coset wW_I.*

We will not prove it, a proof can be found in [21].

Using what we have said at the beginning of this chapter, we get that the cells $\pi_I(C(w)), w \in W_I^1$ give a cellular decomposition of G/P. Therefore, as for G/P, we get:

Theorem 3.10. *The classes $\overline{\pi_I(C(w))}$, with $w \in W_I^1$, form a free basis of $A^*(G/P)$.*

Therefore we are almost done: in fact if $p : G/B \to G/P$ is the projection, $p_* : A^*(G/B) \to A^*(G/P)$ is surjective, then using the projection formula p^* is injective.

p_* is surjectve because if $w \in W_I^1$, then $dim(\pi(C(w))) = |\Phi^+(w^{-1})| = |\{\alpha \in \phi^+ : \mathfrak{g}_{w^{-1}\alpha} \cap \mathfrak{p} = \{0\}\}| = dim(\pi_I(C(w)))$ (so $p_*(s_w) = [\overline{\pi_I(C(w))}]$). In fact $\geq$ is clear, assume that there is no equality then there exists an $\alpha \in \Phi^+$ such that $\mathfrak{g}_{w^{-1}\alpha} \subseteq \mathfrak{p}$ but $w^{-1}\alpha < 0$. Then $w^{-1}\alpha = -\sum n_i\alpha_i$ with $\alpha_i \in I$, therefore $\alpha = -\sum n_i w\alpha_i < 0$ by definition of W_I^1 which is absurd.

In the following theorem we will characterize the image of p^*:

Theorem 3.11. *$P_w \in Im(p^*)$ for $w \in W_I^1$ and $\{(p^*)^{-1}P_w\}_{w\in W_I^1}$ is the basis in $A^*(G/P)$ dual to the basis $\{p_*(s_w)\}_{w\in W_I^1}$.*

Proof. This follows from the projection formula and from the previous theorem. ☐

Chapter 4

Spherical varieties

4.1 Spherical varieties: basic definitions

In this section we will introduce the basic notations and properties of a spherical variety. In this chapter the field k will be of characteristic 0, if not stated differently all groups G will be reductive and connected, $B < G$ will be a Borel subgroup of G with maximal torus T.

Definition 4.1. *Given a vector space V with an action of B, $V^{(B)}$ will be the set of B-semiinvariant vectors, i.e. the vectors $v \in V$ such that there exists a character λ of B such that $bv = \lambda(b)v$.*

The main reference for this chapter is [16]

Definition 4.2. *A spherical homogeneous variety is a homogeneous variety G/H such that G/H has an open dense $B-orbit$.*
A spherical variety X is a (connected) normal variety with an action of G such that X has an open orbit $O \subseteq X$ and a point $p \in O$ such that $G/Stab(p)$ is a spherical homogeneous variety. We will say that X is a $G/Stab(p)$-embedding or a spherical embedding of $G/Stab(p)$.

Observation 4.1. *There is a more direct way to define what a spherical variety is: it is a normal variety X with an action of a reductive group G such that X has an open B-orbit.*

Toric varieties (chapter 6), flag varieties (chapter 3), and symmetric varieties (chapter 5) are the examples that we will mainly discuss in this thesis. The main observation when we are dealing with open orbits is the following one:

Observation 4.2. *Let X be a variety on which a group G acts with an open orbit. Then any G-invariant function is constant, because it is constant on this open orbit.*

This ensures us that when we have two B-semiinvariant rational functions, with the same character, then they are multiple one of the other.

It is very important to deal with normal varieties, because we want to work with valuations of $k(G/H)$.

Definition 4.3. *$\mathbf{V}_{G/H}$ is the set of G-invariant valuations of $k(G/H)$. Let X be a spherical variety, $\mathbf{D}(X)$ is the set of $B-$stable irreducible divisors of X. We will omit X if the total space is clear.*

Definition 4.4. *Let X be a spherical variety and let $Y \subseteq X$ be an orbit of G, $\mathbf{D}_Y(X) := \{B-$ stable divisors D of X such that $Y \subseteq D\}$.*

Notice that being X normal, every such a divisor defines a valuaion on the field of rational functions:

Definition 4.5. *Given a normal variety X with an irreducible divisor D, we will call v_D the valuation associated to this divisor. Moreover if v is a valuation on X, assume that there is a closed subset $D \subseteq X$ such that, if we perform the blow-up of X along D and D' is the exceptional divisor, $v = v_{D'}$, then we will say that D is the center of v.*

The following two theorems are two basic results of this section, we will not give a proof of them because it goes beyond the aim of this thesis. A reference is [16]: the first one is Theorem 1.3 in [16] whereas the other is Corollary 1.7 always in [16].

Theorem 4.1. *Let G be a connected reductive group acting on a normal variety X, let $Y \subseteq X$ be an orbit. Then there is a B-stable affine open subset $U \subseteq X$ such that $U \cap Y \neq \emptyset$*

Theorem 4.2. *Let X be a spherical embedding of G/H and x be a point in the open B-orbit. Let $f \in k[Bx]$, $v_0 \in \mathbf{V}$; then there is $f' \in k(G/H)^{(B)}$ such that $v_0(f') = v_0(f)$, $v(f') \geq v(f)$ for every $v \in \mathbf{V}$, and $v_D(f') \geq v_D(f)$ for every $D \in \mathbf{D}(G/H)$.*

In the first theorem we say that there is an open affine B-stable set which meets Y, now we will show that there is even a canonical one:

Definition 4.6. *Given Y an orbit of G, let $X_0^Y := X - \bigcup_{D \in \mathbf{D} - \mathbf{D}_Y(X)} D$. If X has a unique closed orbit we will say that X is simple and we will denote $X_0 := X_0^Y$ where Y is this closed orbit.*

Observation 4.3. *Notice that for X simple, the B-stable divisors which intersect X_0 are exactly those which contain Y.*

Theorem 4.3. *Given Y an orbit of G, we have:*
i) X_0^Y is affine B-stable and open;
ii) Y is the only closed orbit of GX_0^Y;
iii) $X_0^Y \cap Y$ is a B-orbit.

Proof. The orbit Y is fixed, thus for simplicity we will drop the apex Y. $i)$ Let U be the affine set of theorem 4.1, and let f be a function which vanishes on all $D \in \mathbf{D}(X) - \mathbf{D}_Y(X)$ which meet U but does not vanish on Y (this is possible because U is affine). Let v_0 be the valuation associated to the variety Y (i.e. the one of the exceptional divisor of the blow up of X on Y), then using the previous theorem with such a v_0 we get a B-semistable regular function f' on $K[U]$. This function does not vanish on Y and $U_{f'}$ satisfies our requirements.
$ii)$ Assume that there is another closed orbit Z, then $Z \cap X_0 \neq \emptyset$. Therefore in X_0 there are two B-stable subsets (namely $Y \cap X_0$ and $Z \cap X_0$) which do not intersect, thus there are two closed B-orbits in X_0. Then the thesis follows from a general fact: let X be an affine variety with an action of a solvable group B with an open orbit. Then X has just one closed orbit. In fact, assume the contrary. Then there are two closed orbits with ideals I_1 and I_2. Two orbits do not intersect therefore $I_1 + I_2 = A$ so $A/(I_1 + I_2) \to A/I_1 \times A/I_2$ is an isomorphism, i.e. $p : A \to A/I_1 \times A/I_2$ is surjective. Furthermore, I_j is B-stable therefore p is B-equivariant. Let $f_1 \in A$ which maps to $(1, 0)$ and f_2 which maps to $(0, 1)$, then there are functions of the shape $f_i + h_i$ with $h_i \in I_i$ which are B-invariant for every i. This is absurd because X has an open orbit.
$iii)$ Assume the contrary, and let O be a dense B-orbit in Y. Then $Y \cap X_0$ is B-stable and then it contains a closed B-orbit. Let O' be such a closed orbit, then using theorem 4.2 we can find a B-semiinvariant regular function on U which vanishes on O' but not on Y. This is absurd because the divisor of f is a sum of B-stable divisors, and every B-stable divisor of X_0 contains $Y \cap X_0$. □

Corollary 4.1. *The number of G-orbits is finite and each orbit is spherical*

Proof. There are only finitely many possibilities for X_0 and Y is uniquely determined by X_0. By the third point of the previous theorem every orbit Y is spherical. □

Even more is true:

Theorem 4.4. *Let X be a spherical variety. Then X has finitely many B-orbits.*

Proof. Using the fact that it has finitely many G-orbits, it is enough to prove this result for $X = G/H$ a homogeneous spherical variety.

Case 1: assume $G = SL(2)$.
Then $X = G/H$, and X has an open B-orbit. Thus $dim H \geq 1$, which implies that H contains a torus of positive dimension or that H contains a unipotent subgroup. If H contains a unipotent subgroup, then we have the thesis because of the Bruhat decomposition of G (theorem 3.1). If instead H contains a torus T', we can assume that it is a subtorus of T: the diagonal

matrices in $SL(2)$. T has finitely many orbits in $G/B \cong \mathbb{P}^1$ therefore every element x^{-1} in $G = SL(2)$ can be written uniquely as tgb with $t \in T$, g in a finite subset of G and $b \in B$, where B are the upper triangular matrices. Therefore x can be written uniquely as $x = b^{-1}g^{-1}t^{-1}$: B has finitely many orbits in G/T. Using the fact that T' has finite index in T we have again the thesis.

General case.
It is enough to show that each irreducible B-stable subvariety Y contains an open B-orbit. We argue by induction on the codimension n of Y.

If $n = 0$ we have the thesis by the definition of spherical variety. Otherwise: being X a homogeneous space, $GY = X$; then G is generated by its minimal parabolic subgroups which contain B. Every such a parabolic subgroup has its Levi subgroup S which is either isomorphic to $PSL(2)$ or to $SL(2)$, and there must exist a minimal parabolic subgroup P such that $Y \subsetneq PY =: Z$. Now, for Z the inductive hypothesis holds, thus Z contains an open B-orbit Z_0. Thus PZ_0 is a homogeneous variety for P, contained in Z. Therefore Z_0 is an open B-orbit for PZ_0. Consider now $(PZ_0)/R(P)$ where $R(P)$ is the unipotent radical of P. It is a homogeneous variety for the Levi subgroup S, because $P = SR(P)$ is the Levi decomposition of P, and it contains $Z_0/R(P)$ as an open orbit for $B/R(P)$. Thus it is a spherical homogeneous space for S: it contains finitely many $B/R(P)$ orbits, thus every element of PZ_0 can be written uniquely as b_1gb_2 with $b_1 \in B$, g in a finite subset of P and $b_2 \in R(P)$. But $R(P)$ is normal, thus $b_1gb_2 = b_1gb_2g^{-1}g$, and $gb_2g^{-1} \in R(P)$: this implies that PZ_0 contains only finitely many orbits of B.

Assume now that $PZ_0 \cap Y \neq \emptyset$, then it has to be an open B-stable subset of Y with finitely many orbits, thus it contains an open orbit and therefore Y contains an open orbit. So it in enough to show that $PZ_0 \cap Y \neq \emptyset$. This follows from the fact that PZ_0 is an open subset of $PY = Z$: given $z_0 \in PZ_0$, there is $p \in P$ and $y \in Y$ such that $z_0 = py \Longrightarrow p^{-1}z_0 = y$. □

The previous theorem is what we need more about spherical varieties to do intersection theory on them. In fact all the theorems of the previous chapter but the first one require that we have finitely many B orbits.

Definition 4.7. *Let Y be an orbit on X a G/H-embedding, then we can define*

$$B_Y(X) := \{v_D \in \mathbf{V} \text{ such that } D \in \mathbf{D}_Y(X) \text{ is } G\text{-stable } \}$$

$$F_Y(X) := \{D \in \mathbf{D}_Y(X) \text{ such that } D \cap G/H \neq \emptyset\}.$$

Theorem 4.5. *Let X be a simple G/H embedding with closed orbit Y. Then X is uniquely determined by the pair $(B_Y(X), F_Y(X))$.*

Proof. Let X' be another simple embedding with the same data, and let X'_0 be an analogous to X_0. Then, if $X_1 = G/H - \bigcup_{D \in F_Y(X)^c} D$, then using again normality we get

$$k[X_0] = \{f \in k[X_1] \text{ such that } v(f) \geq 0 \text{ for all } v \in B_Y(X)\} = k[X'_0]$$

Therefore $X_0 = X'_0$, but $X = GX_0$, so the isomorphism $X_0 \to X'_0$ extends to X. □

Now, we have the following exact sequence:

$$0 \longrightarrow k^* \longrightarrow k(G/H)^{(B)} \longrightarrow \Lambda \longrightarrow 0$$

where Λ is the coker and $k^* \longrightarrow k(G/H)^{(B)}$ is the inclusion. We will denote the map $k(G/H)^{(B)} \longrightarrow \Lambda$ by χ. Notice that we can think at Λ as a subset of $\chi(B)$, the characters of B. In fact, we can associate to every B-semiinvariant rational function f its character, and this map is injective because of observation 4.2.

Definition 4.8. *$Q_{G/H} := Hom(\Lambda, \mathbb{Z})$. If the quotient G/H is clear, we will omit the subscript.*

Given X a simple embedding with closed orbit Y, every element D of $F_Y(X)$ gives a valuation v_D and an element $\rho(v_D)$ in $Q, f \to v_D(f)$, and the same holds for $B_Y(X)$. Furthermore, given $v \in B_Y(X) - \{0\}$, the corresponding element in Q is not 0. This is not always the case for a valuation in $F_Y(X)$.

Definition 4.9. *Let $C_Y(X)$ be the cone in $Q_{\mathbb{Q}}$ generated by the elements of $B_Y(X)$ and $\rho(v)$ for $v \in F_Y(X)$.*

Notation 4.1. *If the space X is clear, we will omit (X) in the definitions of $B_Y(X)$, $F_Y(X)$ and $C_Y(X)$.*

Furthermore, assume that we have two $\mathbb{Q}$-vector spaces V_1 and V_2 of finite dimension, assume that $(\cdot, \cdot)$ is a pairing between them and let $\sigma \subseteq V_1$ be a cone in V_1. Then $\sigma^\vee$ will be the dual cone of σ, i.e. $\sigma^\vee := \{v \in V_2 : (v, w) \geq 0$ for every $w \in \sigma\}$.

Definition 4.10. *Let G/H be a spherical homogeneous space. A colored cone is a pair (C, F) with $C \subseteq Q$ and $F \subseteq \mathbf{D}(G/H)$ such that:*

i) C is a cone generated by $\rho(F)$ and finitely many elements of $\mathbf{V}$

ii) $C^\circ \cap \mathbf{V}$ is not empty, where C° is the interior part of C in the vector space generated by C.

A colored cone (C, F) is strictly convex if C is strictly convex and 0 does not belong to $\rho(F)$.

If (C, F) is a colored cone, F is called the set of colors.

Theorem 4.6. *i) For every simple spherical embedding X of G/H with closed orbit Y, (C_Y, F_Y) is a strictly convex colored cone.*
ii) For every strictly convex colored cone (C, F) (with $C \subseteq (Q_{G/H})_{\mathbb{Q}}$, i.e. we have chosen G/H), there is a simple spherical embedding of G/H with closed orbit Y such that $(C, F) = (C_Y, F_Y)$.

We will just sketch some parts of the proof. The first point is true because we can observe that

$$k[X_0]^{(B)} = \{f \in k(G/H)^{(B)} \text{ such that } \chi(f) \in C_Y(X)^{\vee}\} \qquad (4.1)$$

Now, let v be the valuation associated to Y, if v did not belong to C°, then there would exists a non-zero f with $\chi(f) \in C_Y(X)^{\vee}$ which vanishes on v but is not 0 on $C_Y(X)$, therefore there would exists a non-invertible B-invariant function on $k[X_0]^{(B)}$ which do not vanish on Y. But this is absurd: the set of B-stable divisors which intersect X_0 is the set of B-stable divisors which contain Y. The cone is strictly convex because we can consider a regular function on the open set which vanishes on Y: this cone is contained in $\{v : \chi(f)(v) \geq 0\}$ and has intersection 0 with $\{v : \chi(f)(v) = 0\}$.

One direction of the second point is true because we can reconstruct $B_Y(X)$ from the data (C_Y, F_Y). The second direction of the second point is non-trivial. We will not prove it here, a reference is [16] theorem 3.1.

Observation 4.4. *([16] theorem 3.1). Assume that G is a torus. Then those simple embeddings are the affine toric embeddings, the theorem in the case of a toric variety simply states that a simple toric variety is uniquely determined by its cone, in fact for a toric variety there are no B-stable divisors in G/H because $G = B$.*

Now, as for toric varieties, we want to describe how these simple spherical embeddings glue together:

Definition 4.11. *We will denote a colored cone as C^c. A pair (C_0, F_0) is called a face of the colored cone (C, F) if C_0 is a face of C, $C_0^{\circ} \cap \mathbf{V}$ is not empty and $F_0 = F \cap \rho^{-1}(C_0)$.*

Note that, as for the toric varieties, the faces of a colored cone give us an information about the action of the group G:

Proposition 4.1. *If X is a simple spherical embedding for G/H with closed orbit Y, there is a bijection between faces of the colored cone $C^c = (C_Y, F_Y)$ and G-orbits Z.*

Proof. In fact, let (C, F) be a face of C^c. Then by definition there is a $v \in \mathbf{V} \cap C^{\circ}$. $v \in C_Y(X)$ then by equation (4.1) it has a center because it is non trivial on $k[X_0]$ (remember that if X' is an affine variety and v is a valutation on $k(X')$, then v has a center on X' iff it is nonnegative on $k[X']$, and in that case the ideal of the center of this valutation is $\{f \in k[X'] : v(f) > 0\}$). If $\overline{Z}$ is its center, it contains an open G-orbit Z: we consider (C_Z, F_Z). We want to prove that $(C_Z, F_Z) = (C, F)$. F is uniquely determined by C, thus it is enough to prove that $C_Z = C$.

Let $f \in k[X_0]^{(B)}$ which does not vanish on Z but vanishes on all B-stable divisors in X_0 containing Y which do not contain Z. Then $\chi(f)$ vanishes on C because it vanishes on v, and by equation (4.1) $f \in C_Y^\vee$.

Let $C' := C_Y \cap \{\chi(f) = 0\}$, it is a face of C_Y and $C \subseteq C'$. If D is a B-stable divisor in X_0 which does not contain Z, f vanishes on it, i.e. $0 \neq v_D(f) = \chi(f)(v_D)$, therefore C' is generated by $B_Z(X)$ and $\rho(F_Z(X))$, i.e. $C' = C_Z$. But $v_0 \in C_Z^\circ \cap C^\circ \Longrightarrow C_Z = C$.

On the other hand, let $Z \subseteq X$ be an orbit, then there exists a $f \in k[X_0]^{(B)}$ with $f|_Z \neq 0$ but vanishes on all B-stable divisors in X_0 containing Y which do not contain Z. This implies that $C_Z^c(X)$ is the face defined by χ_f. □

Again as for the toric varieties, we can glue those simple varieties together:

Definition 4.12. *A colored fan is a finite nonempty set $\overline{F}$ of colored cones with the following properties:*

i) every face of $C^c \in \overline{F}$ belongs to $\overline{F}$;

ii) for every $v \in \mathbf{V}$ there is at most one $(C, F) \in \overline{F}$ with $v \in C^\circ$.

A colored fan $\overline{F}$ is strictly convex if $(0, \emptyset) \in \overline{F}$, or equivalently if all the elements of $\overline{F}$ are strictly convex.

Definition 4.13. *For an embedding X denote $\overline{F}(X) := \{C_Y^c(X)$ such that $Y \subseteq X$ is a G-orbit $\}$.*

Theorem 4.7. *The map $X \mapsto \overline{F}(X)$ gives a bijection between isomorphism classes of embeddings and strictly convex colored fans.*

This is essentially because for each colored cone $C \in \overline{F}$ we have the simple G/H embedding X associated to this cone (theorem 4.6) and we glue them on the open sets defined by the faces (to each face is associated a G-orbit Z, and we consider GX_0^Z). The content of the theorem is that what we obtain is separated.

4.2 Morphisms between spherical varieties

As for the toric varieties, we can define morphisms just using our combinatorial data, i.e. the colored fans. Let G/H and G/H' be two spherical homogeneous varieties with $H \subseteq H'$. Then there is a projection $p : G/H \to G/H'$, assume that we have two spherical varieties $G/H \subseteq X$ and $G/H' \subseteq X'$, we want to understand whether p extends to a map $X \to X'$.

First of all notice that p induces a map $p_* : Q_{G/H} \to Q_{G/H'}$. In fact, p induces a map $k(G/H') \to k(G/H)$, therefore a map $k(G/H')^{(B)} \to k(G/H)^{(B)}$ and then a map $Q_{G/H} \to Q_{G/H'}$. It is a theorem proved in [16] that $p_*(\mathbf{V}_{G/H}) = \mathbf{V}_{G/H'}$ (even more is true: for every valuation $v \in \mathbf{V}_{G/H}$ we can lift it to a valuation $v \in \mathbf{V}_G$) and notice that if $F_p := \{D$ such that D is a B-stable divisor on G/H which p maps dominantly on $G/H'\}$, then we get a map $p_* : F_p^c \to \mathbf{D}(G/H')$, where F_p^c is the complement of F_p.

Definition 4.14. *Let (C, F) and (C', F') be colored cones for G/H and G/H' respectively. We say that (C, F) maps to (C', F') if the following conditions hold:*
i) $p_*(C^\circ) \subseteq C'^\circ$;
ii) $p_*(F_p^c) \subseteq F'$.

Definition 4.15. *Let F, $\overline{F}$ be two colored fans for G/H and G/H', we say that F maps to $\overline{F}$ if every element of F maps to some element of $\overline{F}$*

Theorem 4.8. *With the same notations as before, p extends to a morphism $X \to X'$ if and only if $\overline{F}(X)$ maps to $\overline{F}(X')$.*

Observation 4.5. *Notice that this morphism if exists it is unique. In fact, it is uniquely determined in the nonempty open set G/H.*

We now prove the previous theorem:

Proof. Assume first that both X and X' are simple, with closed orbits Y and Y'.

If p extends to $\overline{p} : X \to X'$, then for every $D \in F$ colour of the fan $\overline{F}(X)$, $Y' \subseteq \overline{\overline{p}(D)}$ because $\overline{p}(Y) \subseteq \overline{\overline{p}(D)}$ and any G-orbit closure contains Y': the second condition in definition 4.14 is satisfied. Now, let $f \in k[X_0']^{(B)}$, $\overline{p}^*(f)$ is a rational function on X which has not poles along any B-stable divisor which contains Y. Therefore, being it B-semiinvariant and being $Y \cap X_0$ the only colsed B-orbit of X_0, $\overline{p}^* f \in k[X_0]$. Thus we get the first point of definition 4.14 using equation 3.1 and the fact that $\cdot^\vee$ inverts the inclusions.

As regards the other arrow: assume that $C^c(X)$ maps to $C^c(X')$. Let $X_1 := X_0 \cap G/H$ and $X_1' := X_0' \cap G/H'$: then the second condition of definition 4.14 implies that

$$p(X_1) \subseteq X_1'$$

In fact if this is not the case, then $p(X_1)$ would intersect a B-stable divisor D in $\mathbf{D} - \mathbf{D}_{Y'}$ (remember the definition of X_0). Then p^*D would be a B-stable divisor in X_0 and thus it would be a colour: this contradicts the second point of definition 4.14.

From the first condition and equation 3.1 follows that $p^*(k[X_0']^{(B)}) \subseteq k[X_0]^{(B)}$: this implies that $k[X_0'] \xrightarrow{p^*} k[X_0]$. In fact given a regular function f on X_0', $p^* f$ does not belong to $k[X_0]$ if and only if there is a divisor D in

X_0 such that $v_D(f) < 0$. But from what we have said before p^*f belongs to $k[X_1]$, therefore this divisor must be a B stable divisor. Every B-stable divisor of X_0 contains Y, thus using theorem 4.2 with v_0 the valuation associated to Y, we can assume f to be B-semiinvariant, which contradicts $p^*(k[X_0']^{(B)}) \subseteq k[X_0]^{(B)}$. Therefore the rational morphism $X \dashrightarrow X'$ is defined on X_0, thus it is defined on $GX_0 = X$.

For the general case (i.e. X and X' can also be not simple), for one arrow ($\overline{p}$ exists as hypothesis) we have the thesis restricting $\overline{p}$ to the open subsets which correspond to the maximal cones of $\overline{F}(X)$. For the other arrow we can construct $\overline{p}$ on every open subset corresponding to a cone, and notice that all these maps glue together because they coincide on the open set G/H. □

Definition 4.16. *Given a spherical variety X, the support of $\overline{F}(X)$ is*

$$Supp(\overline{F}(X)) := \mathbf{V} \cap (\bigcup_{(C,F)\in\overline{F}} C)$$

Theorem 4.9. *With the same notations as above, the extension of p is proper iff $Supp(\overline{F}(X)) = p_*^{-1}(Supp(\overline{F}(X')))$.*

Proof. For the first arrow notice that $Supp(\overline{F}(X))$ is the set of G-invatiant valuations which have a center on X. In fact if v has a center on X, let Y be its center. Consider then X_0^Y, for every $f \in k[X_0^Y]^{(B)}$ f vanishes on Y thus $v \in C$ where C is the cone C_Y. If $v \in Supp(\overline{F}(X))$ there must exist a cone C (associated to the orbit Y) such that $v \in C^\circ$, this implies that $\overline{Y}$ is the center of v. Therefore if the morphism is proper then follows using the valutative criterion of properness ([13] theorem 2.4.7) we get $Supp(\overline{F}(X)) = p_*^{-1}(Supp(\overline{F}(X')))$.

As regards the other arrow, let v be such that it does not belong to $Supp(\overline{F}(X))$ but $p^*(v) \in Supp(\overline{F}(X'))$. Then we can consider X^* the embedding associated to the fan $\overline{F}(X) \cup \{\mathbb{Q}^+v, \emptyset\}$. Then there exists a morphism $X^* \to X'$, X is an open subset (check it locally) and p factors through $X \to X^*$ and $X^* \to X'$: p can not be proper. □

In particular this holds for $p : G/H \to G/G = spec(k)$, i.e. a spherical variety is complete iff $Supp(\overline{F}(X)) = \mathbf{V}_{G/H}$.

Observation 4.6. *Every spherical homogeneous variety admits a projective normal embedding.*

Proof. We will use the following two facts:

G/H is quasi-projective ([21] corollary 5.5.6)

which implies that there is an ample sheaf on G/H. Then we use that given L a line bundle on a normal G-variety, there is $n \in \mathbb{N}$ such that $L^{\otimes n}$ is G-linearized ([20], introduction)

This implies that we can find a G-representation V such that G/H is a closed embedding in an open subset of $\mathbb{P}(V)$, and the map $G/H \to \mathbb{P}(V)$ is equivariant. Take X, the closure of the image. If it is normal, we are done, otherwise take its normalizaion X'. Using the universal property of normalizations, G acts on X' because it acts on X. If $p : X' \to X$ is the normalization morphism, it is an isomorphism on G/H because G/H is smooth thus it is a spherical embedding. The normalization is again projective because p is finite thus proper and X was proper, and furthermore the pull-back of an ample line bundle through a finite morphism is still ample, thus we have an ample sheaf on a proper variety: it is projective. □

Definition 4.17. *A spherical variety is toroidal if every cone of its fan has the form $(C, \emptyset)$, i.e. has no color.*

Proposition 4.2. *Given G/H a spherical homogeneous variety, there exists a complete toroidal G/H-embedding*

Proof. Consider any complete G/H embedding and let $\overline{F}$ its fan. Remove the colors from $\overline{F}$ and consider the corresponding embedding of G/H: this is toroidal and complete. □

In particular for every G/H spherical homogeneous variety, $\mathbf{V}_{G/H}$ is a union of finitely generated cones, and therefore we get:

Observation 4.7. *For every spherical homogeneous space G/H and every X embedding of G/H, there exists $\overline{X}$ a toroidal embedding of G/H and a G-equivariant morphism $\overline{X} \to X$ with $\overline{X}$ toroidal.*

In fact it is enough first to delete the colors from the fan of X. The previous observation is what we mainly use about toroidal varieties.

The geometry of these spherical varieties is very similar to the one of toric varieties, the reason is the following proposition:

Proposition 4.3. *([22], theorem 29.1). Let G/H be a spherical homogeneous space and let X a toroidal embedding of G/H. Let $\Delta_X := \bigcup_{D \in \mathbf{D}(G/H)} D$, P_X the parabolic subgroup of G which stabilizes Δ_X and $P_X = LP_u$ its Levi decomposition. Then there exists a closed subvariety Z of $X - \Delta_X$ which is stable for the Levi subgroup of P_X, such that the map $P_u \times Z \to X - \Delta_X, (p, z) \mapsto pz$ is an isomorphism. Furthermore, $[L : L]$ acts trivially on Z, Z is a toric variety for $L/[L : L]$ defined by the same fan of X, and there is a well defined map which is a bijection*

$$\{ \text{ Orbits of } G \text{ in } X \} \longrightarrow \{ \text{ Orbits of } L/[L:L] \text{ in } Z \}, \; O \longrightarrow O \cap Z$$

We will not prove this, in chapter 5 we will prove a special case of this result. Notice that X is smooth if and only if the corresponding toric variety is smooth. Furthermore, let X be a G/H toroidal embedding. As for toric varieties, we can perform blow ups along G-orbits closures, obtaining again a toroidal embedding of G/H. In fact, given an orbit closure $\overline{O}$, $\overline{O} \cap Z$ is a $L/[L:L]$-orbit closure. Therefore if we consider the fan of the toric variety obtained by blowing up such an orbit, the toroidal variety that has this fan is the blow up of X along $\overline{O}$.

Definition 4.18. *A toroidal smooth projective embedding of G/H is also called a regular embedding of G/H.*

Furthermore, when we are dealing with complete smooth toroidal varieties, it is possible to characterize the property of being projective in terms of the fan. The theorem is the following:

Theorem 4.10. *([2], theorem 3.1 and 3.3). i) Let X be a spherical variety. Then any Cartier divisor can be written as*

$$\delta = \sum_{v \in \mathbf{V}_X} l(v) D_v + \sum_{D \in A} l(v_D) D + \sum_{D \in \mathbf{D}(X) - A} n_D D$$

where $A := \bigcup_{Y \subseteq X} \mathbf{D}_Y(X)$ where Y is a G-orbit l is a continuous function which is linear on each maximal cone and has integral value on $\mathbf{V}_X$ and on A.
ii) if l is strictly convex ($l(v+w) \geq l(v) + l(w)$ with $=$ if and only if v and w belong to the same cone) and $n_D > l(v_D)$ for every $D \in \mathbf{D}(X) - A$ then δ is ample.

If a toroidal variety is smooth then the condition $n_D > l(v_D)$ for every $D \in \mathbf{D} - A$ can always be fulfilled up to adding some $D \in \mathbf{D}(X) - A$ which are Cartier. Thus we have just to take care of the existence of a strictly convex function on our fan. This follows from the toric Chow's lemma:

Theorem 4.11. *([4] theorem 6.1.18). Let X be a toric variety, which is a spherical embedding of a torus T, with fan Σ. We can find another fan, $\widetilde{\Sigma}$, which refines Σ, and such that there exists a strictly convex piecewise linear function, linear on each cone of $\widetilde{\Sigma}$.*

This implies that the toric variety defined by $\widetilde{\Sigma}$ is projective.

Proof. Let n be the dimension of X, let σ be a $(n-1)$-dimensional cone in Σ and let m_σ be a primitive element generating $\sigma^\perp$. The arrangiament of all the hyperplanes defined by $\{m_\sigma = 0\}$ defines a fan $\widetilde{\Sigma}$ in N which is a subdivision of Σ and thus gives us a refinement of Σ. We can define a function

$$\psi : N \to \mathbb{Q}, u \mapsto -\textstyle\sum_{\sigma \in \Sigma(n-1)} |(u, m_\sigma)|$$

where $\Sigma(n-1)$ are the $(n-1)$-dimensional cones of Σ. Now ψ is strictly convex, in fact it is convex because by the triangular inequality $\psi(u+v) \geq \psi(u) + \psi(v)$, with equality if and only if for every cone $\sigma \in \Sigma(n-1)$, u and v belongs to the same half-space defined by $\{m_\sigma = 0\}$, i.e. if and only if they belong to the same cone σ. □

We can apply it in this way: consider the fan of our complete toroidal variety X, consider then the fan of the toric variety associated to it, complete the latter fan to obtain a fan of a complete toric variety and then use the previous theorem to get a toric projective toric variety $\widetilde{X}$. A subset of the fan of $\widetilde{X}$ refines the fan of X, consider the toroidal variety X' associated to this fan. X' is projective: $\widetilde{X}$ is projective, thus there is a piecewise linear function on its fan which is strictly convex, and its restriction to this function to the fan of X' is still strictly convex.

The fact that our piecewise linear function ψ must have integral value on $\mathbf{V}_X$ is easy to solve: in fact we can to all the theory working on $\mathbb{Q}$, thus if ψ does not meet our requirements, we can take a multiple of it.

Therefore we have:

Corollary 4.2. *Every complete embedding of G/H is dominated by a projective toroidal embedding of G/H.*

Chapter 5

Symmetric varieties and Halphen ring

In this chapter we will work on the field of complex numbers. We will present the content of the article by De Concini and Procesi Complete symmetric varieties 2 [6].

5.1 Regular configurations

Definition 5.1. *Given X a smooth variety, a finite family of hypersurfaces $S := \{S_i\}_{i\in I}$ is a regular configuration if the following properties are satisfied:*
a) Each S_i is smooth;
b) If the intersection of $S_{i_1}, ..., S_{i_r}$ is not empty, then it is transversal.

Furthermore, given a regular configuration and $J \subseteq I$, $S_J := \bigcap_{j\in J} S_j$ is called a coordinate variety.

Let us give some examples of a regular configuration:
1) If S is a regular configuration, then any subset $S' \subseteq S$ is a regular configuration.
2) If X is a smooth torus embedding we can take as configurations the family of closures of codimension 1 orbits.
3) Let Y be a variety with a principal T-bundle P for a torus T; and let X be a smooth torus embedding of a quotient of T. Then $P \times^T X$ has a regular configuration: $\{P \times^T S_i\}$ where $\{S_i\}$ is the regular configuration of X.
4) If X has a regular configuration $\{S_i\}$ and $Y \subseteq X$ is a smooth subvariety such that, if $S'_i := S_i \cap Y$; S'_i is not empty, the intersection of Y with any coordinate variety S_J is transversal; then $\{S'_i\}$ is a regular configuration.

Theorem 5.1. *Any regular configuration can be obtained through the previous examples.*

Proof. Let $\{S_i\}_{i\in I}$ be a regular configuration in X, and let $n := |I|$. For each i we can consider $O_X(S_i)$ the line bundle with a section s_i with divisor S_i. Let $F := \bigoplus O_X(S_i)$. Let U be an open subset on which the vector bundle is trivial, then $F|_U \cong U \times \mathbb{A}^n$ and we have a natural action of the torus $R := (\mathbb{C}^*)^n$ on $F|_U$, which is $(t_1, ..., t_n)\cdot(u, (a_1, ..., a_n)) := (u, (t_1a_1, ..., t_na_n))$. The change of cohordinates maps are equivariant, because our vector bundle is the sum of line bundles, therefore this action comes from a global action of $(\mathbb{C}^*)^n$ on F. The set locally described as $(u, (a_1, ..., a_n))$ with a_i not 0 for each i is our principal R-bundle P. Given $v = (a_1, ..., a_n) \in \mathbb{A}^n$, let $J_v := \{i : a_i = 0\}$. Define Y as the smooth torus embedding $\{v \in \mathbb{A}^n$ such that $S_{J_v} \neq \emptyset\} \subseteq \mathbb{A}^n$. Then the fiber bundle $P \times^R Y$ is the open subset of F described fiberwise in the same way: given a $p = (u, v) \in F$, $p \in P \times^R Y \Leftrightarrow v \in Y$. The regular configuration in $Y \subseteq \mathbb{A}^n$ is given from the one of the hyperplanes $x_i = 0$ in $\mathbb{A}^n$. Therefore: given a smooth variety Y with a regular configuration, we obtain a principal bundle P and a regular configuration $\Sigma := \{P \times^R Y \cap (\bigoplus_{j\neq i} O_X(S_i))\}$ on $P \times^R Y$. Let $j := s_1 \oplus ... \oplus s_n$. Now it is clear by our definition that:
1) j defines an embedding $X \hookrightarrow P \times^R Y$;
2) $\Sigma_J = \emptyset \Longleftrightarrow S_J = \emptyset$;
3) $j(X)$ meets Σ_J transversally in S_J; because all these properties are of local nature. This is the proof of the theorem. □

Remark 5.1. *Given a smooth variety X with a regular configuration $S := \{S_i\}_{i=1}^n$, we can associate to this configuration some geometric objets, as in the proof of the previous theorem:*

$R_{X;S} := (\mathbb{C}^*)^n$ a torus of dimension n (R in the proof above),
$F_{X;S}$, a vector bundle on X (F in the proof above),
$Y_{X;S}$, an open $R_{X;S}$-stable subvariety of $\mathbb{A}^n$ (Y in the proof above),
$P_{X;S}$, a principal $R_{X;S}$-bundle on X (P in the proof above),
A closed embedding $j : X \to P_{X;S} \times^{R_{X;S}} Y_{X;S}$.

We will omit the subscript $X;S$ if the space X and the regular configuration S are clear.

Let X be a smooth variety with a regular configuration $S = \{S_i\}$ as above. Assume that we have a smooth torus embedding of R, Z, with a torus morphism $\pi : Z \to Y_{X,S}$. Then we can construct the map between fiber bundles $\overline{\pi} : P \times^R Z \to P \times^R Y$, and we can consider the variety $X_Z := \overline{\pi}^{-1}(j(X))$.

Definition 5.2. *With the previous notations, we will call X_Z the variety obtained from X, the regular configuration $\{S_i\}$ and π.*

The map π and the associated map $X_Z \to X$ share some properties, in fact using that being proper and birational are stable under base change, we get:

Proposition 5.1. *i) the projection $X_Z \to X = j(X)$ is birational;*
ii) π proper implies $\overline{\pi}$ proper.

Observation 5.1. *Assume, with the previous notations, that Z is obtained from Y blowing up C: a T-stable subvariety of codimension k. Then also $P \times^R Z$ is obtained from $P \times^R Y$ blowing up $P \times^R C$.*

Proof. Let U be an open subset of X such that $F = P \times^R \mathbb{A}^n$ is trivial over it: $F|_U \cong U \times \mathbb{A}^n$, and consider the fiber square

$$\begin{array}{ccc} U \times Z & \longmapsto & U \times Y \\ \downarrow & & \downarrow p \\ Z & \underset{\text{blow up}}{\longmapsto} & Y \end{array}$$

$p : U \times Y \to U$ is flat, and therefore if I is the sheaf of ideals defining the blow up in U then $p^{-1}(I) = p^*(I) \Rightarrow Proj(Sym(p^{-1}(I))) = Proj(Sym(p^*(I))) = U \times Z$: $U \times Z$ is the blow up of $U \times Y$ along $p^{-1}(C)$. $\square$

Now, since $j(X)$ is transversal to our regular configuration, $X_Z := \overline{\pi}^{-1}(j(X))$ is obtained blowing up X along $X \cap P \times^R C$. In fact the following is true:

Observation 5.2. *Let X be a smooth variety and let $Y \subseteq X$, $Z \subseteq X$ be two closed smooth subvarieties of X, which intersect transversally. Let $B_Z(X)$ be the blow up of X along Z and $B_{Z\cap Y}(Y)$ the blow up of Y along $Z \cap Y$. Then the following diagram is a fiber diagram:*

$$\begin{array}{ccc} B_{Z\cap Y}(Y) & \longmapsto & B_Z(X) \\ \downarrow & & \downarrow \\ Y & \underset{i}{\longmapsto} & X \end{array}$$

Proof. This statement is of local nature, i.e. we can assume $X = spec(A)$, p_1 and p_2 the prime ideals defining Z and Y. As before, it is enough to show that $i^*I = i^{-1}I$ which means that we want the following sequence to be exact:

$$0 \to p_2 \otimes_A A/p_1 \to A/p_1 \to A/(p_1 + p_2) \to 0$$

This sequence is exact if and only if it is exact locally for each maximal ideal m containing $p_1 + p_2$, so we can assume A local with maximal ideal $m = (f_1, ..., f_n)$. The intersection is transversal therefore we can assume $p_1 = (f_1, ..., f_r)$ and $p_2 = (f_{r+1}, ..., f_s)$. We prove the thesis by induction on $a = r - s$. If $a = 1$ the thesis is true because f_{r+1} is not a 0-divisor in A/p_1. Assume now the thesis true for $s = r + k - 1$, then $Tor_1^A(A/p_1, A/(f_{r+1}, ..., f_{r+k-1})) = 0$, but Tor is a bifunctor so

$Tor_1^A(A/(f_{r+1},...,f_{r+k-1}), A/p_1) = 0$ as well and therefore the following sequence is exact:

$$0 \to p_1 \otimes_A A/(f_{r+1},...,f_{r+k-1}) \to A/(f_{r+1},...,f_{r+k-1}) \to \\ A/(p_1, f_{r+1},...,f_{r+k-1}) \to 0$$

Let $R := A/(f_{r+1},...,f_{r+k-1})$. The previous sequence becomes:

$$0 \to p_1 \otimes_A R \to R \to R/p_1R \to 0$$

Now, the maximal ideal of R is $(f_1,...,f_r,f_s,f_{s+1},...,f_n)$ and we know that the thesis is true for $a = 1$, i.e. $Tor_1^R(R/p_1, R/f_s) = 0$, so tensoring the previous sequence by $R/f_s = A/p_2$ we are done. □

So summarizing we get the following theorem:

Theorem 5.2. *Given a regular configuration $\{S_i\}_{i=1}^n$ in X we can construct a smooth torus embedding Y of $T = (\mathbb{C}^*)^n$, a principal bundle P on X and an embedding $j : X \to P \times^R Y$ such that:*
i) for any smooth torus embedding $f : Z \to Y$ for the torus R, the fiber product of the following diagram is birational over X and proper over X if Z is proper over Y.

$$\begin{array}{ccc} X_Z & \longmapsto & P \times^R Z \\ \downarrow & & \downarrow \overline{\pi} \\ X & \longmapsto & P \times^R Y \end{array}$$

ii) If Z is obtained by a sequence of blow ups of orbit closures, X_Z is obtained from X by a sequence of blow ups of coordinate varieties. In particular, it is smooth.

5.2 Complete symmetric varieties

Recalling some facts about symmetric varieties

G will be a semisimple simply connected group and H will be the fixed subgroup of an automorphism $\sigma : G \to G$ such that $\sigma^2 = Id$. In [5] the authors give a natural compactification of the symmetric variety G/H, called wonderful. Let me remind the basic properties of this compactification. When we have such an automorphism, we can choose a maximal torus T such that $\sigma(T) = T$. For such a maximal torus, σ acts on its Lie algebra $\mathfrak{t}$, which decomposes as $\mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{t}_0$ where $\mathfrak{t}_1$ is the -1 eigenspace, whereas $\mathfrak{t}_0$ is the 1-eigenspace. Let's choose a σ-stable torus which maximizes the dimension of its $\mathfrak{t}_1$, and let T_1 the connected component of the identity of the maximal subtorus of T such that for each $s \in T_1$ $\sigma(s) = s^{-1}$. We can decompose Φ as $\Phi = \Phi_0 \sqcup \Phi_1$ where $\Phi_0 = \{\alpha \in \Phi : \sigma(\alpha) = \alpha\}$ and Φ_1 is its complement. For such a T it is proved in [5] that we can choose a Borel subgroup $T \subseteq B$ such that the corresponding set of positive roots Φ^+ satisfies $(\Phi_1^+)^\sigma \subseteq \Phi^-$. Let $\Delta = \Delta_0 \sqcup \Delta_1$ the set of simple roots, where $\Delta_i \subseteq \Phi_i$. Let $\{\alpha_1, ..., \alpha_l\} := \Delta_1$ and let ω_i the fundamental weight dual to α_i. Then ([5]) $\omega_i^\sigma = \omega_{\tilde{\sigma}(i)}$ for a certain $\tilde{\sigma} \in S_l$. We define a dominant weight special if it is of the form $\sum n_i \omega_i$ with $n_i = n_{\tilde{\sigma}(i)}$ and special regular if none of the n_i is 0.

Proposition 5.2. *If λ is a special weight, there exists an H-invariant vector in $V_{2\lambda} - \{0\}$ unique up to scalar multiplication, we will call it h_λ.*

Definition 5.3. *Let λ be a special regular weight, we define $X_\lambda := \overline{G[h_\lambda]} \subseteq \mathbb{P}(V_{2\lambda})$ a compactification of G/H.*

Theorem 5.3. *If λ, μ are two special regular weights, then there is an equivariant isomorphism $X_\lambda \to X_\mu$ which maps $[h_\lambda] \to [h_\mu]$: this compactification is unique up to G-isomorphism, in this chapter we will call it X.*

So summarizing, let me remind how to construct X: first of all we have to choose a regular special weight λ, then pick the unique H-invariant point $[h_\lambda] \in \mathbb{P}(V_{2\lambda})$, finally $X = \overline{G[h_\lambda]}$.

Theorem 5.4. *Let λ be a regular special weight and let $i : G \to X, g \mapsto g[h_\lambda] \in \mathbb{P}(V_{2\lambda})$. Let A be the open subset of $\mathbb{P}(V_{2\lambda})$ where the coordinate relative to the weight 2λ does not vanish. Then there is a T_1-equivariant isomorphism $i : \mathbb{A}^l \to (\overline{i(T_1)})_A$, where the action on the coordinates of $\mathbb{A}^l$ is with the characters $-2\alpha_i|_{T_1/T_1 \cap H}$.*

Theorem 5.5. *Let λ be a special regular weight and let U be the unipotent radical of the parabolic subgroup which fixes $[v_{-\lambda}] \in \mathbb{P}(V_{-w_0\lambda})$ where w_0 is the longest element in the Weyl group of G. Then the map $\phi : U \times \mathbb{A}^l \to X, (u, r) \mapsto u \cdot i(r)$ is an isomorphism with its image, which is an open subset of X, we will call this image V. In particular X has an open B^--orbit.*

Theorem 5.6. *The complement of the open B^--orbit of X is the union of l smooth divisors $S_i := \overline{\phi(U \times H_i)}$ where $H_i := \{x_i = 0\}$ and $\mathbb{A}^l$ has ring of regular functions $k[x_1, ..., x_l]$. They intersect with transversal intersection and $S_1 \cap ... \cap S_l$ is the unique closed orbit, isomorphic to G/Q where Q is the parabolic subgroup associated to the set Δ_0. Furthermore X is smooth because $\bigcup_{g \in G} g\phi(U \times \mathbb{A}^l) = X$*

Theorem 5.7. *Every line bundle on X admits a G-linearization.*

The category Sym

By theorem 5.6, we have a regular configuration on X, therefore we can apply what we have previously done. We will use the notations recalled in remark 5.1. The last theorem holds for $O_X(S_i)$, so this holds for F: we obtain an action of G on F which commutes with the action of R.

Observation 5.3. *s_i is G-equivariant.*

This implies in particular that $j : X \to P \times^R \mathbb{A}^l$ is a G-equivariant embedding.

Proof. In fact for each S_i and $g \in G$, the section $s_i \in O_X(S_i)$ with divisor S_i has the same divisor of gs_i. Then, restricting these sections to $U \times \mathbb{A}^l \cong \mathbb{A}^m$ we get that they are multiples. But G is semisimple, so it does not have any character but the trivial one, so $gs_i = s_i$, i.e., s_i is equivariant. □

Assume now that we have a toric morphism $f : Z \to \mathbb{A}^l$ for the torus R, where Z is a smooth toric variety for $R \cong \mathbb{C}^l$. Then we can consider X_Z the variety obtained from X, the regular configuration $\{S_i\}$, and f; let $\psi : X_Z \to X$ be the induced morphism. Notice that G acts also on $P \times^R Z$ because it acts on P, and X_Z is G-invariant because $j(X)$ is G-invariant.

Theorem 5.8. *i) X_Z is smooth*
ii)The G-orbits in X_Z are in one to one correspondence with the R-orbits in Z.

Proof. Let us consider V, the open set of X previously defined. The closure in V of the open T-orbit is isomorphic to $\mathbb{A}^l$ with the action given by the characters $-2\alpha_i|_{T_1/T_1 \cap H}$ and the regular configuration of X restricted to V is given by the hypersurfaces $U \times H_i$, where H_i are the hyperplanes $x_i = 0$ in $\mathbb{A}^l$. Let $V_Z := \psi^{-1}(V)$, we can think at V_Z also as the variety obtained from V, the regular configuration $\{U \times H_i\}$ and the toric morphism $Z \to \mathbb{A}^l$. Now, the map $j : U \times \mathbb{A}^l \to U \times \mathbb{A}^l \times \mathbb{A}^l = F|_V$ is a map between affine varieties, so it is uniquely determined by the associated map between regular functions on them. Looking at j at the ring-level, we note that the section $j : U \times \mathbb{A}^l \to U \times \mathbb{A}^l \times \mathbb{A}^l$ is the map obtained using the universal property of fiber products with the maps $Id : U \to U$ and $diag : \mathbb{A}^l \to \mathbb{A}^l \times \mathbb{A}^l$, so considering the following two fiber diagrams

$$\begin{array}{ccc} (U \times \mathbb{A}^l)_Z & \longmapsto & U \times \mathbb{A}^l \times Z \\ \downarrow & & \downarrow \pi \\ U \times \mathbb{A}^l & \longmapsto & U \times \mathbb{A}^l \times \mathbb{A}^l \end{array}$$

and

$$\begin{array}{ccc} (\mathbb{A}^l)_Z & \overset{\alpha}{\longmapsto} & \mathbb{A}^l \times Z \\ \gamma \downarrow & & \downarrow \pi \\ \mathbb{A}^l & \underset{\beta}{\longmapsto} & \mathbb{A}^l \times \mathbb{A}^l \end{array}$$

we get that $(U \times \mathbb{A}^l)_Z \cong U \times (\mathbb{A}^l)_Z$. Then to check the smoothness it is enough to look at $(\mathbb{A}^l)_Z$, but looking again at the second diagram we note that it is a piece of a bigger fiber diagram

$$\begin{array}{ccccc} (\mathbb{A}^l)_Z & \xrightarrow{\alpha} & \mathbb{A}^l \times Z & \longrightarrow & Z \\ \gamma \downarrow & & \downarrow \overline{\pi} & & \downarrow \\ \mathbb{A}^l & \xrightarrow[\beta]{} & \mathbb{A}^l \times \mathbb{A}^l & \longrightarrow & \mathbb{A}^l \end{array}$$

and therefore $(\mathbb{A}^l)_Z \cong Z$ (looking at α on the ring of regular functions we get that it is the graph of f). The first point is done.

Observation 5.4. *Note furthermore that we have shown that $\psi^{-1}(\mathbb{A}^l) \cong Z$, and $\psi^{-1}(V) \cong U \times Z$.*

Since $(T_1/T_1 \cap H)^\circ$ is identified with $(\mathbb{C}^*)^l \subseteq \mathbb{A}^l$, we can canonically identify it with R. In fact the actions on Z coincide, because they coincide on an open subset.

Every G orbit of X intersects $\mathbb{A}^l$, thus every G-orbit of X_Z intersects $(\mathbb{A}^l)_Z$ and the intersection is a union of R-orbits. We want to prove that it is exactly one R orbit. The main point is that G is irreducible, and therefore every orbit of G is irreducible.

Let W_i be the closures of the orbits of codimension 1 in $(\mathbb{A}^l)_Z$, and let O be an orbit of G. If $O = G/H$ then $O \cap (\mathbb{A}^l)_Z$ is the open orbit, otherwise $O \cap (\mathbb{A}^l)_Z$ is irreducible and contained in $(\mathbb{A}^l)_Z - R = \bigcup_{i=1}^m W_i$. Let I be the biggest subset of $\{1, ..., m\}$ such that $O \subseteq \bigcap_{i \in I} W_i$. $\bigcap_{i \in I} W_i$ has an open R-orbit, U_O. Then O contains a point of this orbit U_O, because Z is smooth and in a smooth toric variety every orbit closure is the intersection of codimension 1 orbits which contain this orbit closure; and because I is minimal. But then we get a map

$$\{ \text{ Orbits of } G \text{ in } X_Z \} \longrightarrow \{ \text{ Orbits of } R \text{ in } Z \}, \; O \longrightarrow U_O$$

It is surjective because for each R-orbit o we can take $G \cdot o$. In fact, $U \times o \subseteq V$ is an orbit for $U \times T_1$, which is dense in G/H. Therefore $dim(Go) = dim(U \times o) \Longrightarrow Go \cap Z \subseteq \overline{o}$. It is injective because any two orbits can not intersect. □

Corollary 5.1. *$j(X_Z)$ is transversal to the configuration of $P \times^R Z$*

Proof. In fact the regular configuration is given by the closures of the G-orbits of codimension 1, the statement is true in the open set V_Z, and using theorem 5.6 every point of X is conjugate to a point in V_Z. □

Definition 5.4. *Let Sym be the category in which the objects are the smooth varieties Y such that G acts on Y with an open orbit isomorphic to G/H and* *there is an equivariant morphism $\psi_Y : Y \to X$. The maps are G-equivariant X-morphisms.*

Observation 5.5. *If X_1, X_2 belong to Sym, then $X_1 \times_X X_2$ belongs to Sym as well: the objects of Sym are a direct family.*

Observation 5.6. *Assume more generally that we have two varieties X_1, X_2 with an open subset isomorphic to Y: $i_1 : Y \hookrightarrow X_1$, $i_2 : Y \hookrightarrow X_2$ and two maps $f : X_1 \to X_2$ and $g : X_2 \to X_1$ such that $f \circ i_1 = i_2$ and $g \circ i_2 = i_1$, then f and g are isomorphism, because they are inverse one of the other on the open set Y*

Let Y_1, Y_2 be two object of *Sym*. Then we can perform the fiber product $Y_1 \times_X Y_2$, which is a new embedding of G/H.

Lemma 5.1. *There exists a map from $Y_1 \to Y_2$ if and only if the canonical projection $Y_1 \times_X Y_2 \to Y_1$ is an isomorphism.*

Proof. If $Y_1 \times_X Y_2 \to Y_1$ is an isomorphism we can take the projection $Y_1 \times_X Y_2 \to Y_2$. As regards the other direction, if we have $Y_1 \to Y_2$ then we have $f : Y_2 \to Y_2 \times_X Y_1$ the map induced from the universal property and $g : Y_2 \times_X Y_1 \to Y_2$ the projection. They satisfy the condition of the previous observation so we are done. □

Lemma 5.2. *Let Y_1, Y_2 be objects of Sym and $f : Y_1 \to Y_2$ a morphism in Sym. Assume that $f|_{\psi_{Y_1}^{-1}(V)} : \psi_{Y_1}^{-1}(V) \to \psi_{Y_2}^{-1}(V)$ (where V is as in theorem 5.5) is an isomorphism, then f is an isomorphism.*

Proof. The main point is to use Theorem 5.6.

f is injective:
In fact let $x_1, x_2 \in Y_1$ such that $f(x_1) = f(x_2)$. Let $g \in G$ such that $g\psi_{Y_1} \circ f(x_1) \in V$, then $gx_1, gx_2 \in f^{-1} \circ \psi_{Y_1}^{-1}(V)$. But $f|_{\psi_{Y_1}^{-1}(V)} : \psi_{Y_1}^{-1}(V) \to \psi_{Y_2}^{-1}(V)$ is an isomorphism, so $gx_2 = gx_1 \Rightarrow x_2 = x_1$.

f is surjective:
It is essentially the same.
Finally it is an isomorphism on the level of rings because it is a local property and locally it is true.

□

The following one is the main theorem of this section:

Theorem 5.9. *Let Y an object of Sym with the map $\psi_Y : Y \to X$, and let $p \in Y$ be the unique point which maps to $(0, (1, ..., 1)) \in U \times \mathbb{A}^l$:*
i)$\psi_Y^{-1}(\mathbb{A}^l)$ is the closure in $(\psi_Y)^{-1}(V)$ of the orbit of p under T_1
ii) $\psi_Y^{-1}(V) \cong U \times \psi_Y^{-1}(\mathbb{A}^l)$;
iii) The map $f : Z \to \mathbb{A}^l$ gives an equivalence between the category of embeddings of G/H over X and the category of embeddings of $(T_1/(T_1 \cap H))^\circ$ over $\mathbb{A}^l$

Proof. **Proof of i) and ii):**

Let us consider the composite map $\phi : \psi_Y^{-1}(V) \to U \times \mathbb{A}^l \to U$, and let $\xi : \psi_Y^{-1}(V) \to \psi_Y^{-1}(V)$, $x \mapsto (\phi(x))^{-1} \cdot x$. Let N be the closure of the T_1-orbit of p in $\psi_Y^{-1}(V)$, then N is contained in $\psi_Y^{-1}(\mathbb{A}^l)$. We claim that $\xi(x) \in N$ for every $x \in \psi_Y^{-1}(V)$. This holds because it is true in $x \in G/H$, and G/H is open and dense, therefore it is true in V. Consider now the following two maps:
i) $U \times N \to \psi_Y^{-1}(V)$ given by the action of U;
ii) $\psi_Y^{-1}(V) \to U \times N$ given by $x \mapsto (\phi(x), \xi(x))$.
Those maps are one the inverse of the other on $G/H \cap \psi_Y^{-1}(V)$, but this is an open dense subset, so they are inverse one of the other: this proves the first statement. The second statement is clear as well.
Proof of iii): We must show that $Y \cong X_Z$ where $Z := \psi_Y^{-1}(\mathbb{A}^l)$. Let $\psi : X_Z \to X$, first of all note that $\psi^{-1}(V) \cong \psi_Y^{-1}(V)$. So, using again lemma 5.2, in order to get an isomorphism it is enough to produce a G-equivariant map $Y \to X_Z$. We want to use lemma 5.1. Consider the fiber product $Y \times_X X_Z$. The projection $Y \times_X X_Z \to Y$ is an isomorphism if and only if it is an isomorphism on the preimage of V, but consider the following diagram:

$$\begin{array}{ccc} \psi^{-1}(V) & \xrightarrow{Id} & \psi^{-1}(V) \\ \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \psi} \\ \psi^{-1}(V) & \xrightarrow[\psi]{} & V \end{array}$$

it is a fiber diagram. In fact the diagram

$$\begin{array}{ccc} Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

is a fiber diagram because of the lemma 5.1, and if we restrict it on the open set V we get the previous one. Therefore $\psi_Y^{-1}(V) = \psi_Y^{-1}(V) \times_V \psi_Y^{-1}(V)$. But then if we restrict to V the following one

$$\begin{array}{ccc} X_Z \times_X Y & \xrightarrow{\tau} & Y \\ \downarrow & & \downarrow \\ X_Z & \longrightarrow & X \end{array}$$

we obtain again the first one, i.e. $\tau^{-1}(\psi_Y^{-1}(V)) \cong \psi_Y^{-1}(V)$ which proves the theorem. □

Observation 5.7. *Therefore any object of Sym is of the form* X_Z *for a suitable* Z *which maps to* $\mathbb{A}^l$ *(actually, Sym is the category of the toroidal compactifications of* G/H*).*

Let us recall a theorem about 2-blow ups on a toric variety, where a 2-blow up is a blow up on the closure of an orbit of codimension 2.

Theorem 5.10. *([6] Theorem 2.4). Given a fan* $\Sigma = \{\sigma_\alpha\}$ *in which each* σ_α *is non singular, and another fan* $\Sigma' = \{\tau_\beta\}$*, we can perform on* X_Σ *a sequence of 2-blow ups so that the resulting fan* $\Sigma'' = \{\sigma''_\alpha\}$ *has the property that for each* γ*,* σ''_γ *is either contained in one of the* $\tau'_\beta s$ *or is in the complement of* $\bigcup_\beta \tau_\beta$*.*

Therefore given a toric variety Z for $(\mathbb{C}^*)^l$, and a proper toric morphism $p : Z \to \mathbb{A}^l$, we can perform a sequence of 2-blow ups of $\mathbb{A}^l$, $\pi : Z' \to \mathbb{A}^l$, in such a way that there exists a map $f : Z' \to Z$ such that $\pi = p \circ f$.

Furthermore, in observation 5.2, we have pointed out that if we have obtained Z from Y by a sequence of blow ups of codimension-2 orbits closures, then the same holds for X_Z and X_Y. Therefore by this observation and the correspondence between G-orbits and R-orbits of theorem 5.8, we get:

Observation 5.8. *Given a smooth complete variety* X' *which is an object of Sym, we can perform a sequence of blow ups of codimension-2 orbits in* X*,* $X'' \to X$*, such that we have a morphism in Sym* $X'' \to X'$*.*

Therefore, whenever we have to perform a direct limit on the objects of *Sym*, we can do this limit just on the varieties obtained by a sequence of 2-blow ups. This last sentence will be clearer in the next section.

The Halphen ring, or the ring of conditions

We start this section reminding two theorems. The first one is due to Kleiman, [15], corollary 8; while the other one to De Concini and Procesi [6] Theorem 4.7 and proposition 4.2. We will just sketch the proof of the first one, because the second goes beyond the aim of this thesis.

Theorem 5.11. *Let* G *be an algebraic group,* X *an integral scheme with a transitive action of* G*, let* Y_1, Y_2 *be two subvarieties of* X*. Then* $Y_1 \cap gY_2$ *is a proper intersection with multiplicity 1 in each component for* g *belonging to a non empty open set of* G*. If* $codim(Y_1) + codim(Y_2) > dim X$*,* $Y_1 \cap gY_2$ *is empty for* g *belonging to a non empty open set of* G*.*

The main point is to prove this result:

Proposition 5.3. *Let* X *be a homogeneous space for an algebraic group* G*. Let* $f : Y \to X$ *and* $g : Z \to X$ *be two morphisms of nonsingular varieties to* X*. Then for any* $\sigma \in G$*, we consider* $Y^\sigma := \sigma(f(Y))$*: there is a nonempty* *open subset of* G, V, *such that for every* $\sigma \in V$, $Y^\sigma \times_X Z$ *is smooth in every irreducible component and it is either empty or it has dimension exactly* $dim(Y) + dim(Z) - dim(X)$.

Proof. Consider the map $h : G \times Y \to X$ obtained composing f with the action $G \times X \to X$. Y and G are smooth, thus $Y \times G$ is smooth; the field is algebraically closed of characteristic 0 thus there is an open subset U of X such that $h : h^{-1}(U) \to U$ is smooth. But now G acts on both $G \times Y$ and X, and h is equivariant thus h is smooth everywhere, because X is a homogeneous space. Consider now $W := (G \times Y) \times_X Z$, with projections $h' : W \to Z$ and $g' : W \to G \times Y$. h is smooth, thus h' is smooth by base change, and since Z is smooth also W is smooth. Therefore we can again apply the criterion of generic smoothness to the map $q := \pi \circ g'$ where $g' : W \to G \times Y$ and $\pi : G \times Y \to G$: we get that there is $q : V \subseteq G$ such that $q : q^{-1}(V) \to V$ is smooth.

So, let $\sigma \in V$, then the fiber of q in σ, W_σ, is smooth, but this fiber is $Y^\sigma \times Z$: this proves the smoothness part of the theorem.

As regards the dimension, h is smooth of relative dimension $dim(G) + dim(Y) - dim(X)$ thus also h' has the same relative dimension ([13] III.10.1). Moreover, $dim(W) = dim(G) + dim(Y) - dim(X) + dim(Z)$.

Thus if W is nonempty, then q on $q^{-1}(V)$ has relative dimension equal to $dim(W) - dim(G)$, which means that $dimW_\sigma = dimW - dimG = dim(Y) - dim(X) + dim(Z)$. □

Now using this proposition and using that the smooth locus is open it is easy to prove (by induction) the thesis of the dimensional part of the theorem. As regards the multiplicity 1 of the intersection, we have to use again the proposition, the fact that the smooth locus is open, and theorem 1.7.

Definition 5.5. *Let* X *be a smooth scheme with a regular configuration* $S = \{S_i\}_{i=1}^n$, *and let* S_I *be a coordinate subvariety. Then if we consider* X' *the blow up of* X *along* S_I, *the proper transforms of the* S_i *and the exceptional divisor are a regular configuration (check it locally). We will call it the regular configuration associated to the configuration* S *and the blow up along* S_I.

Definition 5.6. *Let* X *be a smooth variety with a regular configuration* $S = \{S_i\}$. *Then we say that* $Y \subseteq X$ *is transversal to* S *in a point* p *if the following property is satisfied. Let* $S_1, ..., S_n$ *be the hypersurfaces of* S *passing through* p *and* $x_1, ..., x_n$ *their local equations in a neighborhood of* p, *then* $x_1, ..., x_n$ *is a regular configuration in the local ring* $O_{Y,p}$ *of* Y *in* p.

This implies that, when Y is a subvariety, the intersection of Y with every coordinate subvariety S_J is proper.

Theorem 5.12. *Let X be a smooth variety with a regular configuration S, and let $Y \subseteq X$ be a subscheme. Then there is a sequence of blow ups $X' = X_m \to X_{m-1} \to \ldots \to X$ where $X_r \to X_{r-1}$ is a blow up along a codimension-2 coordinate subvarieties of X_{r-1}, such that the proper transform $Y' \subseteq X'$ is transversal to S', where S' is the regular configuration in X' associated to the configuration S and the sequence of blow ups. If Y is already transversal to the configuration S, then Y' is transversal to S' and $Y' = \pi^{-1}(Y)$.*

Note that when we have more than one Y, $Y_1, \ldots, Y_n$, we make all of them transversal to the configuration. In fact once we have performed a sequence of blow ups such that $\widetilde{Y_1}$ is transversal to the configuration, we can work on $\widetilde{Y_2}$ knowing that any successive blow up will not affect the property of being transversal to the configuration of $\widetilde{Y_1}$, and once $\widetilde{Y_1}, \widetilde{Y_2}$ are transversal we can work on $\widetilde{Y_3}$ and so on.

Observation 5.9. *Let X be a smooth variety, V_1, V_2 two subvarieties of X. Assume that there are open subsets $U_i \subseteq V_i$ such that for every $x \in U_1 \cap U_2$ the intersection $U_1 \cap U_2$ is transversal in x, and such that $U_1 \cap U_2$ is dense in $V_1 \cap V_2$. Then the intersection $V_1 \cap V_2$ is proper and $[V_1] \cap [V_2] = \sum [W_i]$, where W_i are the irreducible components of $V_1 \cap V_2$: we are saying that every irreducible component has multiplicity 1.*

This follows from the fact that the Gysin homomorphism commutes with flat pull back, the following diagram, and the fact that for a scheme X of pure dimension m with a dense open subset $i : U \hookrightarrow X$, $i^* : A^m(X) \to A^m(U)$ is an isomorphism.

$$\begin{array}{ccc} U_1 \cap U_2 & \longmapsto & U_1 \times U_2 \\ \downarrow & & \downarrow \\ V_1 \cap V_2 & \longmapsto & V_1 \times V_2 \\ \downarrow & & \downarrow \\ X & \xmapsto[diag]{} & X \times X \end{array}$$

Proposition 5.4. *Let X be a smooth variety over which G acts with a finite number of orbits $\{O_i\}_{i=0}^m$, where O_0 is the open orbit. Let $Z := \bigcup_{i \geq 1} O_i$. If Y_1 and Y_2 are two subvarieties which have proper intersection with every orbit closure $\overline{O_j}$, then there is an open set $U \subseteq G$ such that $gY_1 \cap Y_2$ is proper with multiplicity 1 in each component for every $g \in U$; and $gY_1 \cap Y_2$ has proper intersection with each orbit closure $\overline{O_j}$ for $g \in U$.*

Proof. The intersection of Y_i with O_j is proper for each i, then $codim_{O_j} Y_i \cap O_j = codim_X Y_i$ if $Y_i \cap O_j$ is not empty. We apply Kleiman transversality lemma to each orbit O_j, and we can find a unique U open subset of G such that for each O_j, $gY_1 \cap Y_2 \cap O_j$ is proper with multiplicity 1. Therefore for every orbit O_j which is not the open one, $dim(gY_1 \cap Y_2 \cap O_j) < dim(gY_1 \cap Y_2)$ and for the open one, there is an equality. Then if W_i are the irreducible components of $gY_1 \cap Y_2 \cap O_0$, $gY_1 \cap Y_2 = \bigcup_i \overline{W_i}$, the multiplicity of each component is 1 because it is 1 on the open set O_0 (we are using the previous observtion). As regards the intersection with the orbit closures, it follows from $codim_{O_j}(gY_1 \cap Y_2 \cap O_j) = codim_{O_j}(Y_1 \cap O_j) + codim_{O_j}(Y_2 \cap O_j) = codim(Y_1) + codim(Y_2)$. □

Corollary 5.2. *If Y_1, Y_2 are irreducible varieties of complementary codimension, for g in a open subset of G, $gY_1 \cap Y_2 \subseteq O_0$*

Observation 5.10. *Assume furthermore that X is complete and smooth, then we can compute the number of points given by the previous corollary using the Chow groups. In fact, this number is $deg([gY_1] \cap [Y_2]) = deg([Y_1] \cap [Y_2])$.*

Proposition 5.5. *If G is connected and Y_1, Y_2 are irreducible closed subschemes of a variety X of complementary codimension, by the proof of Kleiman transversality lemma, the number of points in $gY_1 \cap Y_2$ is not only finite, but constant, for g belonging to an open subset $U \subseteq G$.*

Therefore using this proposition we can define a pairing $Z^k(X) \times Z^{n-k}(X) \to \mathbb{Z}$:

Definition 5.7. *Let X be a variety of dimension n on which a connected algebraic group G acts transitively. Then we can define a pairing $Z^k(X) \times Z^{n-k}(X) \to \mathbb{Z}$, defined on the generators in this way: if Y_1, Y_2 are irreducible subvarieties of codimension k and $n-k$ respectively, then the number of points of $gY_1 \cap Y_2$ for g belonging to an open subset of G is constant, and we define (Y_1, Y_2) to be this number.*

Definition 5.8. *Let X and G be as before, then $B^r(X) := \{x \in Z^k(X)$ such that $\forall y \in Z^{n-k}(X), (x, y) = 0\}$. Let $C^r(X) := Z^r(X)/B^r(X)$. If Y is a cycle we will denote its class by $\{Y\}$.*

A question that could come to our minds is whether this definition depends just on the geometry of our homogeneous space or the action of G is essential.

Example 5.1.

If V is a 2-dimensional vector space, it is an homogeneous space for its own action, by translation, and for $GL(V) \ltimes V$. The groups $B^*(V)$ depends on the group chosen: for the group V two lines are equivalent iff they are parallel, for the group $GL(V) \ltimes V$ any two lines are equivalent.

Now we want to define a compatible product on $C^*(X)$. Given g, h in an open subset of G such that $gY_1 \cap Y_2$ and $hY_1 \cap Y_2$ is a proper intesection with multiplicity 1 in each component (proposition 5.4) with $Z_1, ..., Z_r$ the irreducible components of $gY_1 \cap Y_2$ and $W_1, ..., W_s$ the ones of $hY_1 \cap Y_2$, we would like that $\sum\{Z_i\} = \sum\{W_j\}$.

Definition 5.9. *With the previous notations, if* $\sum\{Z_i\} = \sum\{W_j\}$*, we will define a product on* $C^*(G/H)$ *in this way:* $\{Y_i\}\{Y_2\} := \{gY_1 \cap Y_2\} := \sum\{Z_i\}$.

The problem is that this is not always well-defined:

Example 5.2.

Let $G := V$ be a 3-dimensional vector space acting on itself by translation. It is clear that two lines (resp. planes) are equivalent if an only if they are parallel. Consider the quadric $xy - z$: we want to intersect it with a generic translate of the plane $x = 0$, i.e. $x = t$. We get $x = t$, $yt = z$ i.e. $x = t, z/t = y$: a family of inequivalent lines.

But if we restrict our attention to homogeneous spaces of the form G/H with G semisimple simply connected and H fixed by an automorphism of G of order 2, then we can actually define a product:

Theorem 5.13. *i) There is a canonical isomorphism of graded vector spaces* $\phi : \varinjlim A^*(X_Z) \to C^*(G/H)$*, where the limit is taken over the class of the smooth complete* X_Z*.*
ii) Given two cycles Y_1, Y_2 *in* G/H*, the class* $\{gY_1 \cap Y_2\}$ *is constant for* g *in a nonempty open set of* G*. We have* $\phi^{-1}(\{gY_1 \cap Y_2\}) = \phi^{-1}(\{Y_1\}) \cap \phi^{-1}(\{Y_2\})$.
iii) Given $a \in C^k(G/H), b \in C^{n-k}(G/H)$*, the value* (a, b) *is* $deg(\phi^{-1}(a) \cap \phi^{-1}(b))$.

Definition 5.10. *The ring we obtain is* $C^*(G/H)$ *and it is called the Halphen ring of* G/H*, or its ring of conditions.*

Proof. Using the observation 5.2 it is enough to do the limit on the X_Z's which are projective. Let $a \in A^k(X_Z)$ for a certain X_Z. Then using the Chow moving lemma, a can be represented by $\sum n_i Y_i$, with Y_i which intersects the regular configuration properly, for every i. We define $\phi(a) := \sum n_i\{Y_i \cap G/H\}$.
Well defined:
One important point is the following observation:

Observation 5.11. *([7], chapter 6 corollary 6.7.2). In general, let* X *be a smooth variety with two closed subvarieties* Y*,* V *with* Y *smooth. Then if the intersection of* V *and* Y *is proper and* $B_Y(X)$ *is the blow up of* X *along* Y *with* $\pi : B_Y(X) \to X$ *the projection, we get that* $[\widetilde{V}] = \pi^*[V]$ *where* $\widetilde{V}$ *is the proper transform of* V*.*

Using this observation, if $\pi : X_{Z'} \to X_Z$ is a blow-up along an orbit closure, then $\pi^*([Y_i]) = (\overline{Y_i \cap G/H})_{X'_Z}$. Therefore $[Y_i] \in A^*(X_Z)$ and $\pi^*([Y_i]) \in A^*(X_{Z'})$ represent the same element in $\varinjlim A^*(X_Z)$.

Let $\sum m_i W_i$ be another representative for a in $A^k(X_Z)$, and let $D \in Z^{n-k}(G/H)$: we want that the following equality holds: $(\sum m_i(W_i \cap G/H), D) = (\sum n_i(Y_i \cap G/H), D)$.

By theorem 5.12 we can find a sequence of blow ups $\pi : X_{Z'} \to X_Z$ such that the closure $\overline{D} = (\overline{D})_{X_Z}$ of D, W_i and Y_i are transversal to the regular configuration: in fact thanks to theorem 5.12 the property of being transversal to a configuration is preserved under blow-ups along coordinate subvarieties. Therefore we may assume that Y_i, $\overline{D}$ and W_j have all proper intersection with the regular configuration. Let $[\overline{D}]$ be the class of $\overline{D}$ in $A^*(X_{Z'})$, then for every g belonging to an open subset $U \subseteq G$, for every orbit $O_i \subseteq X_{Z'}$, $gY_i \cap \overline{D} \cap O_i$ and $gW_j \cap \overline{D} \cap O_i$ have codimension n for every i, j. Therefore $gY_i \cap \overline{D} \subseteq G/H$ and the number of points with multiplicity of this intersection is $deg([gY_i] \cap [\overline{D}]) = (\{Y_i \cap G/H\}, D)$ which implies $(\sum n_i\{Y_i \cap G/H\}, D) = deg(\sum n_i[Y_i] \cap [\overline{D}]) = deg(\sum m_i[W_i] \cap [\overline{D}]) = (\sum m_i\{W_i \cap G/H\}, D)$. In particular, given $a \in C^k(G/H), b \in C^{n-k}(G/H)$, the value (a, b) is $deg(\phi^{-1}(a) \cap \phi^{-1}(b))$, which is the proof of *iii*).

Injective:
Using theorem 2.3 the Kronecker duality map is an isomorphism, then for every $a \in A^k(X_Z)$ there is a $b \in A^{n-k}(X_Z)$ such that $deg(a \cap b) \neq 0$. Therefore if $a = \sum n_i[Y_i]$ and $b = \sum m_i[D_i]$, proceeding as before we get $(\phi(a), \phi(b)) = deg(a \cap b) \neq 0$. So, assume that $\phi(a) = 0$, this would imply that $(\phi(a), \phi(b)) = 0$.

Surjective:
Let $Y \in Z^m(G/H)$. There is an X_Z such that $\overline{Y}$ has proper intersection with the regular configuration of X_Z, therefore $\phi([\overline{Y}]) = \{Y\}$.

The *ii*) point follows from the fact that ϕ is well-defined: in fact given two sbvarieties Y_1, Y_2 of X with intersection with multiplicity 1 in each component, then $[Y_1] \cap [Y_2] = \sum [Z_i]$ where Z_i are the irreducible components of the intersection. □

5.3 Halphen ring in general

We want to generalize what we have done in the specific case of the varieties in *Sym* with spherical varieties. A reference is [22]. The main idea is that we can perform the limit on toroidal embeddings which, as in the *Sym* case, have a geometry which is not so far from the geometry of a toric variety. In this section G will be a reductive connected group over an algebraically closed field of characteristic 0. The main proposition that we will use is proposition 4.3, which is an analogous of theorems 5.8 and 5.9. In fact, the varieties defined in *Sym* are toroidal varieties.

Theorem 5.14. *Let $O := G/H$ a spherical homogeneous space, then $C^*(O)$ is a ring with the product $\{Z\} \cdot \{Z'\} := \{gZ \cap Z'\}$ with g belonging to an open subset of G. Furthermore, $C^*(O) = \varinjlim A^*(X)$ where the limit is over all smooth complete G/H-embeddings.*

Proof. We have just to follow what we have said in the previous proof, and notice that we can follow very closely the proof. The required observations are

We can perform the limit on projective smooth toroidal embeddings.

This is because what we have said in chapter 3 (it is important to work with projective varieties because we want to use the Chow moving lemma). In this case the G-orbits closures of codimension one are a regular configuration (follows from proposition 4.3).

We can perform blow-ups along G-orbit closures, the result is still a toroidal variety

This ensures us that we can use theorem 5.12 remaining in the category of toroidal embeddings. Then the proof is the same as before. □

Observation 5.12. *The Halphen ring of G/P coincides with the Chow ring.*

The reason is that when we perform the limit of theorem 5.14, this limit is trivial because there are no complete G/P embeddings but G/P. We can check it explicitly, the points which we will use are the following:

i) The Kronecker duality map is an isomorphism (theorem 2.3);

ii) Given $Y \subseteq G/P$ a subvariety of G/P, $g \in G$, $[gY] = [Y]$ in $A^*(G/P)$ (theorem 2.6);

iii) $A^0(G/P) \cong \mathbb{Z}$ and an isomorphism is deg.

In fact, consider the map $f : Z^*(G/P) \to C^*(G/P)$, $Z \mapsto \{Z\}$. If $z := \sum n_i Z_i \in Z_k(G/P)$ with $\sum n_i[Z_i] = 0$ in $A^*(G/P)$, then $f(z) = 0$. In fact given $a = \sum m_j\{Y_j\} \in C^k(G/P)$, if $(a, z) \neq 0$ then by definition there is an open subset U of G such that for every $g \in U$, $gY_i \cap Z_j$ is proper with multiplicity 1 for every i, j. Therefore $(a, z) = \sum_{i,j} n_i m_j |Z_i \cap gY_j| = \sum_{i,j} n_i m_j deg([Z_i] \cap [gY_j] = deg((\sum n_i[Z_i]) \cap (\sum m_j[Y_j])$ where the last equality follows from *ii*). In this equality the latter is 0 because $z = 0$, which contraddicts $(a, z) \neq 0$.

Thus we get a map $A^*(G/P) \to C^*(G/P)$ which is surjective and, using *i*) and *iii*) it is injective.

A particular case is the one of toric varieties. We will study the toric case more in detail in the last part of this thesis.

5.4 An example

The idea behind the Halphen ring:

What is this Halphen ring: the idea is that we want to find a ring such that it codifies properly how two subvarieties of a variety intersect. The properties that we want are the following two:

i) when two subvarieties intersect in some points, we want to be able to count how many are these points (here the usual Chow ring fails).

ii) if we want to count the points of intersection of two subvarieties, the answer must be "generic" (something similar to theorem 2.6).

To explain better what does generic mean, think at the example of two lines l_1, l_2 in $\mathbb{C}^2$. In how many points do they intersect? Well, they could intersect in 1 point, they could be the same line and they could not intersect at all (if they are parallel). But, in general, they intersect in 1 point: this means that up to changing l_1 with gl_1 for g belonging to an open subset of $Aff(\mathbb{C}^2)$ (the group of affine transformations of $\mathbb{C}^2$), the intersection is one point, so we want a theory that answers us one point: to codify what generic menas, we want an action of a gropu G on our variety (in this case $G = Aff(\mathbb{C}^2)$), and we want that this action is transitive.

Intersecting quadrics:

Consider a complex vector space $V \cong \mathbb{C}^{n+1}$ of dimension $n+1$, let $\mathbb{M}_{n+1}(V)$ be the matrices on it and let S be the set of symmetric matrices $S := \{A \in \mathbb{M}_{n+1}(V)$ such that A si symmetric$\}$. Then any quadric can be identified with a symmetric matrix up to scalars, and we can associate to every symmetric matrix (up to scalars) a quadric, i.e. $M := \mathbb{P}(S)$ is a moduli space for the quadrics. We will call $Q(A)$ the quadric associated to the matrix A and $A(Q)$ will be a matrix associated to the quadric Q.

Notation 5.1. *In this section G will be $PSL(n+1)$.*

Now, G acts on M: given $[B] \in M$ and given $A \in PSL(n+1)$, the action sends $(A, [B]) \mapsto [(A^{-1})^t BA^{-1}]$. Consider the map $\sigma : PSL(n+1) \to PSL(n+1)$, $A \mapsto (A^{-1})^t$ where t is the transpose. This is an involution, let H be the fixed subgroup. The stabilizer of $Id \in M$ (thought as a symmetric matrix) is again H, and the orbit of 1 are all the non degenerate symmetric matrices (i.e. the nonsingular quadric). Therefore M is a compactification of the nonsingular quadric, $PSL(n+1)/H$.

Observation 5.13. *We can think at non-singular quadric as points of G/H.*

We want to solve the following problem: find the number of non-degenerate quadrics in $\mathbb{P}^3$ tangent to 9 quadrics in general position. We will not solve the problem completely (for this see [5] section 10), we will just find the right compactification. First notice the following proposition: we will prove it in detail later.

Proposition 5.6. *Given a nondegenerate quadric Q, the set of quadrics which are tangent to Q are a closed subset D of codimension 1 in $PSL(n+1)/H$.*

So for our problem of the 9 quadrics, we want to calculate $\{D\}^9$ (In our problem M has dimension 9): this gives a certain number of points, which is the number of non-degenerate quadrics tangent 9 general quadrics.

Using the previous theorem, first we want to find a compactification, and this compatification should be such that $\overline{D}$ intersects every orbit properly, i.e. it has codimension 1 in every orbit.

Assume that we find a simple compactification, then being X smooth it is enough to see that it intersects just the closed orbit properly.

Why M does not work:

First, notice that an orbit decomposition for M is given by the rank of the matrices. Then if we consider just the matrices with rank one, they are tangent to every nonsingular matrix: this compactification does not work, D does not have proper intersection with the closed orbit.

Another compactification:

Notation 5.2. *Fix the canonical basis for $\mathbb{C}^{n+1} \cong V$, $e_1, ..., e_{n+1}$, and the corresponding dual basis for $V^\vee$: $x_1, ..., x_{n+1}$.*

Let T be the maximal torus of diagonal matrices of $PSL(n+1)$ with respect to the choosen basis $e_1, ..., e_{n+1}$.

For every $1 \leq i \leq n$ consider ω_i, the character on T which sends $(t_1, ..., t_{n+1}) \to t_1 \cdot ... \cdot t_i$ (we will use the same notations also for $SL(n+1)$).

Notice that T is the maximal torus mentioned at the beginning of the section “complete symmetric varieties”, and the weights ω_i are special weights, i.e. with the notations of the beginning of the section “complete symmetric varieties”, $\widetilde{\sigma} = Id$.

Notation 5.3. *For every character λ, V_λ will be the representation of $SL(n+1)$ with maximal weight λ*

Then V_{ω_i} is a subrepresentation of $\Lambda^i V$, and $V_{2\omega_i}$ is a subrepresentation of $S_2(\Lambda^i V)$ where $S_j(V)$ is the j-th symmetric power of V.

Observation 5.14. *Thank to proposition 5.2, we have an unique H-fixed point in $\mathbb{P}(V_{2\omega_i})$ for every i, $[h_{2\omega_i}]$.*

Definition 5.11. *Let X' be the closure of the orbit of $([h_{2\omega_1}], ..., [h_{2\omega_n}])$ in $\prod_{i=1}^n \mathbb{P}(V_{2\omega_i})$.*

Theorem 5.15. *([5] section 4, the lemma). X' is isomorphic to X, the complete symmetric variety defined in theorem 5.3.*

So in particular it is simple. We have reintroduced X in this way because it is more helpful for our purpose.

Now, fix a flag in $\mathbb{P}^n$: $\pi_0 \subseteq \pi_1 \subseteq .... \subseteq \pi_{n-1}$ where π_i has dimension i (for example we fix the flag given by $\pi_i =< e_1, ..., e_{i+1} >_k$). We want to consider, for every i, the set of nondegenerate quadrics D_i tangent to π_i.

Observation 5.15. *The condition of being tangent to π_i is an algebraic condition, in particular D_i is a divisor.*

Proof. Consider the set of nondegenerate quadrics as an open subset of M. Then a quadric Q is tangent to the subspace π_i if and only if there is a point $p \in \pi_i$ such that, if this point p has coordinates $[v_1, ..., v_{n+1}]$, $p \in \pi_i$ and, if the matrix of the qudaric is $A := A(Q)$, for every $q = [(w_1, ..., w_{n+1})] \in \pi_i$, $(w_1, ..., w_{n+1})A(v_1, ..., v_{n+1}) = 0$. But this is the same as requiring that $Q_{|\pi_i}$ is degenerate: this condition is a divisor because it is the condition of vanishing of the determinant a certain minor in the matrix A. $\square$

Observation 5.16. *D_i is the divisor of the pull-back of a section of $O_{\mathbb{P}(V_{2\omega_{n-i+1}})}(1)$, where $p : X \to \mathbb{P}(V_{2\omega_{n-i+1}})$ is the projection.*

This is why we have reintroduced X as a subvariety of $\prod_{i=1}^{n} \mathbb{P}(V_{2\omega_i})$.

Proof. First recall that a maximal weight of $\Lambda^i V_{\omega_n}$ is ω_{n-i+1} for the action of $SL(n + 1)$.

In fact it is enough to notice that:

i) The element w of the Weyl group sending the diagonal matrix $diag(t_1, ..., t_{n+1}) \mapsto diag(t_{n+1}, ..., t_1)$ is a root of maximal length;

ii) Using that the maximal weight of the dual of V_{ω_i} is $-w\omega_i$, we see that V_{ω_i} is the dual of $V_{\omega_{n+1-i}}$;

iii) Finding a root-decomposition for $V^\vee$ we see that in $\Lambda^i V_{\omega_n}$ there is just one maximal vector of weight ω_{n+1-i}.

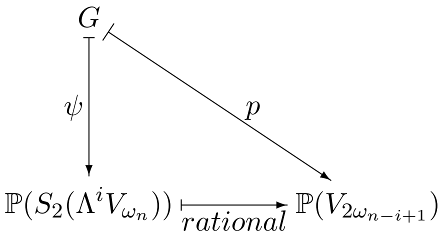
Furthermore, notice that the action of $A \in SL(n+1)$ on $V^\vee$ is this one: A acts in the basis $x_1, ..., x_{n+1}$ through the matrix $(A^{-1})^t$. Thus in the basis of $\Lambda^i V^\vee$ given by the vectors $x_{j_1} \wedge ... \wedge x_{j_i}$, the action is given by the matrix $\Lambda^i(A^t)^{-1}$.

$SL(n + 1)$ acts on the symmetric matrices over $\Lambda^i V_{\omega_n}$ in this way: B sends the matrix $A \mapsto (\Lambda^i B^{-1})^t A \Lambda^i B^{-1}$, and checking with coordinates we see that the symmetric matrices with this action are isomorphic to $S_2(\Lambda^i V_{\omega_n})$ (the isomorphism sends $S_2(\Lambda^i V_{\omega_n}) \ni e_i e_j \mapsto \frac{1}{2}(e_{i,j} + e_{j,i})$ where $e_{i,j}$ has 1 in the coordinate (i, j) and 0 otherwhere).

Furthermore, looking at the weights of $S_2(\Lambda^i V_{\omega_n})$, we see that it has a (unique) subrepresentation isomorphic to $V_{2\omega_{n-i+1}}$, thus there is a $SL(n+1)$-equivariant surjective homomorphism $S_2(\Lambda^i V_{\omega_n}) \to V_{2\omega_{n-i+1}}$.

Consider the identity $Id \in S_2(\Lambda^i V_{\omega_n})$ thinked as a symmetric matrix on $\Lambda^i V_{\omega_n}$. Looking at the coordinates we see that it does not go to 0 if we project it to $V_{2\omega_{n-i+1}}$ (for example look at the coordinate relative to the maximal weight $2\omega_{n-i+1}$), and $< Id >_k$ is H-invariant, thus if we consider the rational map given by this projection $\mathbb{P}(S_2(\Lambda^i V_{\omega_n})) \dashrightarrow \mathbb{P}(V_{2\omega_{n-i+1}})$, we see that it is defined at $[Id]$ and $[Id]$ goes to $[h_{2\omega_{n-i+1}}]$.

So we have the commutative diagram below, where $\psi(A) = (\Lambda^i A^t)^{-1} Id(\Lambda^i A)^{-1}$:



Now, looking at the equation of the map $\mathbb{P}(S_2(\Lambda^i V_{\omega_n})) \dashrightarrow \mathbb{P}(V_{2\omega_{n-i+1}})$ we see that $p^*(O_{\mathbb{P}(V_{2\omega_{n-i+1}})}(1)) \cong \psi^*(O_{\mathbb{P}(S_2(\Lambda^i V_{\omega_n}))}(1))$. So it is enough to check the thesis for the latter line bundle.

But now the thesis is true because $(\Lambda^i A^t)^{-1} Id(\Lambda^i A)^{-1} = \Lambda^i((A^t)^{-1}A^{-1})$ and the condition that a quadric passes through a point is a linear condition on $\mathbb{P}(S_2(\Lambda^i V_{\omega_n}))$ (look at the equations).

Now notice that the condition that $\Lambda^i((A^t)^{-1}A^{-1})$ passes through $x_1 \wedge ... \wedge x_i$ is the same as requiring that the quadric $Q := Q((A^t)^{-1}A^{-1})$ has the determinant relative to the minor given by the first i columns and i rows is 0: as pointed out in the previous observation, this is the same as requiring that the quadric Q has a point $p \in \pi_i$ such that the plane tangent to Q at p contains π_i. □

Observation 5.17. *Notice that $V^\vee$ is isomorphic, as a $SL(n+1)$ representation, to $\Lambda^n V$. Thus $\mathbb{P}(V^\vee)$ is isomorphic to $\mathbb{P}(\Lambda^n V)$ and the isomorphism is $PSL(n+1)$-equivariant. Looking closely, we see that this isomorphism sends an element $[\phi]$ to the hyperplane $\{v : \phi(v) = 0\}$, we will call $\psi : \mathbb{P}(V^\vee) \to \mathbb{P}(\Lambda^n V)$ this isomorphism.*

Definition 5.12. *Let $Y_1 := \{(Q;p) \in G/H \times \mathbb{P}(V)$ such that $Q(p) = 0\}$, where we think at G/H as a moduli space for the non-degenerate quadrics. Notice that $dim(Y_1) = dimX + n - 1$ (look at the equations!).*

Observation 5.18. *With the action previously defined on the quadrics, Y_1 is G-invariant: $(Q,[v]) \in Y_1 \iff v^t A(Q)v = 0 \iff v^t g^t (g^t)^{-1} A(Q) g^{-1} g v = 0$ which is $(gQ, g[v]) \in Y_1$. Therefore we have an action of G on Y_1.*

Consider the map $\phi : Y_1 \to \mathbb{P}(V^\vee)$ which sends $(Q;[v]) \mapsto [A(Q)v]$ where $A(Q)v$ is written in the coordinates $x_0, ..., x_n$.

Observation 5.19. *ϕ is G-equivariant. In fact $0 = \phi((Q;[v]))[w] \implies \phi((gQ, g[v]))(g[w]) = w^t g^t (g^t)^{-1} A(Q) g^{-1} g v = 0$.*

Consider then the graph of ϕ as a subspace of $G/H \times \mathbb{P}(V) \times \mathbb{P}(\Lambda^n V)$. It is contained in $G/H \times G/P_0$ where P_0 is the parabolic subgroup defined below. This follows from the fact that $Q([v]) = 0 \implies v^t A(Q)v = 0$ which implies that $[v]$ belongs to the tangent plane through $[v]$.

$$P_0 := \begin{bmatrix} * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$$

Thus Y_1 is a closed subset in $G/H \times G/P_0$ which is an open subset in $X \times G/P_0$.

Definition 5.13. *Let* $Y := (\overline{Y_1})_{X \times G/P_0}$.

Observation 5.20. *G/P_0 has dimension $2n-1$. Therefore in particular*

$$A^n(G/P_0) \times A^{n-1}(G/P_0) \to \mathbb{Z} \text{ sending } \alpha, \beta \mapsto deg(\alpha \cap \beta)$$

is a perfect pairing (theorem 2.3).

Proposition 5.7. *Let G be a reductive group with a parabolic subgroup P, Y a variety with a G action on it, and let $f : Y \to G/P$ a G-equivariant morphism. Then if F is the fiber of f at $[1] \in G/P$, $Y \cong F \times^P G$. In particular, being locally a product, it is flat.*

Proof. Consider the map $\alpha : G \times^P F \to Y$ sending $(g, f) \mapsto gf$. First notice that it is equivariant. Moreover, it is surjective being f equivariant, and it is injective because if $g_1 f_1 = g_2 f_2 \implies g_2^{-1} g_1 f_1 = f_2 \implies g_2^{-1} g_1 \in P$ which implies $(g_1, f_1) = (g_1(g_1^{-1} g_2), g_2^{-1} g_1 f_1) = (g_2, f_2)$. Thus to show that it is an isomorphism is enough to prove it locally.

If U is the unipotent subgroup of G opposite to the unipotent radical of P and $\pi : G \to G/P$ is the projection, $\pi_{|U} : U \to \pi(U)$ is an isomorphism. Let $\phi : U \to \pi(U)$ be the restriction of π to U and let $p : G \times^P F \to G/P$ be the projection , sending $(g, f) \to \pi(g)$. Then $p = f \circ \alpha$.

We want to prove that α is an isomorphism if restricted to $p^{-1}(\phi(U))$.

$p^{-1}(\phi(U)) \cong F \times U$ because $P \cap U = \{1\}$, thus an inverse for $\alpha_{|p^{-1}(\phi(U))}$ sends $x \to (\phi^{-1}(f(x)), (\phi^{-1}(f(x)))^{-1}x)$, which is algebraic. Using the fact that α is G-equivariant and $Gf^{-1}(\phi(U)) = Y$, we are done.

□

Before going on, we need some facts about Grassmanians.

More on $A^*(G/P)$:

We will use the notations and the results of the section "the G/P case" of chapter 3.

Observation 5.21. *If we think at the Weyl group $W = N_G(T)/T$ acting on the diagonal matrices, we see that $W = S_{n+1}$. Moreover, if the matrix $e_{i,i+1}$ in the Lie algebra of $PSL(n+1)$ is a generator for the root space relative to α_i, then our parabolic subgroup P_0 is associated to a certain subset I of the basis of our root sistem: $I = \{\alpha_2, ..., \alpha_{n-1}\}$. Thus the Weyl group W_I is the group generated by $(2,3), (3,4), ..., (n-1,n)$ in S_{n+1}.*

In order to find a basis for the Chow ring of G/P_0, we need to find a family of representatives for the cosets of S_{n+1}/W_I: this family is

$$\{(1, m_1)(m_2, n+1)\}$$

In fact, given $\sigma \in S_{n+1}$, in order to understand to which coset does it belong is it enough to look where σ sends $\{1, n+1\}$. Thus a basis for $A^*(G/P_0)$ is given by $\overline{U^-[(1, m_1)(m_2, n+1)]}$ where U^- is the maximal unipotent subgroup of the lower-triangular matrices in $PSL(n+1)$.

Consider now the two parabolic subgroups made in this way:

$$P_1 := \begin{bmatrix} * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \end{bmatrix}$$

$$P_2 := \begin{bmatrix} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$$

Then $G/P_1 \cong \mathbb{P}^n$ whereas $G/P_2 \cong \mathbb{P}(\Lambda^n V)$, and $G/P_0 \hookrightarrow G/P_1 \times G/P_2$ is a closed subscheme.

Observation 5.22. *The parabolic subgroup associated to $I_2 = \{\alpha_1, ..., \alpha_{n-1}\}$ is P_2, whereas P_1 is associated to $I_1 = \{\alpha_2, ..., \alpha_n\}$.*

Tthe Weyl subgroup of P_2 is $W_{I_2} :=< (1,2), ..., (n-1,n) >$ whereas the one of P_1 is $W_{I_1} :=< (2,3), ..., (n, n+1) >$. Thus the cosets of S_{n+1}/W_{I_1} are $\{[(1,m)]\}$ whereas the ones of S_{n+1}/W_{I_2} are $\{[(m,n)]\}$.

Example 5.3.

Assume that $n+1=7$, then $U^{-}Id$ are the matrices which are made as the ones below

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 & 0 & 0 \\ * & * & * & 1 & 0 & 0 & 0 \\ * & * & * & * & 1 & 0 & 0 \\ * & * & * & * & * & 1 & 0 \\ * & * & * & * & * & * & 1 \end{bmatrix}$$

These matrices in $\mathbb{P}(\Lambda^n V)$ are the vectors of the shape $[ue_1 \wedge ... \wedge ue_6]$ with $u \in U^-$: these are the hyperplanes generated by the first n columns of the previous matrix.

But then notice that for every $a_i \in k$,

$$[ue_1 \wedge ... \wedge ue_6] = [ue_1 \wedge ... \wedge ue_i - a_1 ue_1 - a_2 ue_2 - ... - a_6 ue_6 \wedge ... \wedge ue_6]$$

This menas that, to understand which are the planes corresponding to the previous matrix, we can perform the Gauss operations on the columns. So in particular we can put a 0 "beside" every 1 in the first 6 columns, i.e. the matrix we need to consider instead of all the ones above, are the ones below (for $U^{-}Id$). The planes spanned by the first 6 columns are in one to one correspondence with points $(a_1, ..., a_6) \in \mathbb{C}^n$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & 1 \end{bmatrix}$$

If we do the same calculations with $(6, 7)$ we get:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 & 0 & 0 \\ * & * & * & 1 & 0 & 0 & 0 \\ * & * & * & * & 1 & 0 & 0 \\ * & * & * & * & * & 0 & 1 \\ * & * & * & * & * & 1 & * \end{bmatrix}$$

and we have to consider just the one below:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Therefore, using that $G/P_0 \hookrightarrow G/P_1 \times G/P_2$ is a closed subset, we see that, for example a set of representatives for $U^-(6,7)$ are the following matrices:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ b_2 & 0 & 1 & 0 & 0 & 0 & 0 \\ b_3 & 0 & 0 & 1 & 0 & 0 & 0 \\ b_4 & 0 & 0 & 0 & 1 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 & 0 & 1 \\ b_5 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Observation 5.23. *We want to find a basis for $A^n(G/P_0)$: proceding as in the previous example we see that the classes*

$$Y_i := \{U^-[(1,j)(j+1,n+1)]\}_{1 \leq j \leq n}$$

form a basis for $A^n(G/P_0)$.

Example 5.4.

If $n+1=7$ then the following ones are some classes which give rise to a basis of $A_6(G/P_0)$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_2 & 1 & 0 & 0 & 0 & 0 & 1 \\ b_3 & 0 & 1 & 0 & 0 & 0 & 0 \\ b_4 & 0 & 0 & 1 & 0 & 0 & 0 \\ b_5 & 0 & 0 & 0 & 1 & 0 & 0 \\ b_6 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ b_2 & b_1 & 0 & 0 & 0 & 0 & 0 \\ b_3 & 0 & 1 & 0 & 0 & 0 & 0 \\ b_4 & 0 & 0 & 1 & 0 & 0 & 0 \\ b_5 & 0 & 0 & 0 & 1 & 0 & 0 \\ b_6 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_3 & b_1 & b_2 & 0 & 0 & 0 & 1 \\ b_4 & 0 & 0 & 1 & 0 & 0 & 0 \\ b_5 & 0 & 0 & 0 & 1 & 0 & 0 \\ b_6 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Observation 5.24. *Assume that $P = P_1$ or $P = P_2$. Then given σ in the Weyl group, the cell dual to $\overline{U^-[\sigma]}$ is $\overline{U^+[\sigma]}$, and $\overline{U^-[\sigma]} \cap \overline{U^+[\sigma]} = [\sigma]$.*

Proof. **Case 1: $P = P_1$.**
Then we see that if $[\sigma] = (1, m)$ then $\overline{U^-[\sigma]} = \bigcap_{j=0}^{m-1}\{x_j = 0\}$, whereas $\overline{U^+[\sigma]} = \bigcap_{j=m+1}^{n}\{x_j = 0\}$. This is clear looking at the matrices below: the points corresponding to $U^-[\sigma]$ are the ones in the first matrix whereas the ones corresponding to $U^+[\sigma]$ are the ones in the last one.

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 1 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ a_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ a_4 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} b_2 & 0 & 1 & 0 & 0 & 0 & 0 \\ b_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore $|\overline{U^+[\sigma]} \cap \overline{U^-[\sigma]}| = 1$ (counted with multiplicity) $\Longrightarrow [\overline{U^+[\sigma]}] \cap [\overline{U^-[\sigma]}] = [p]$ where p is a point.

Case 2: $P = P_2$. Also in that case it is useful to look at the problem using matrices. In fact in this way we see that if $w = (m, n+1)$ then a set of representatives for the classes in $\overline{U^-[\sigma]}$ is

$$\{[w] = [v_1 \wedge ... \wedge v_n] \text{ such that } w \wedge e_i = 0 \text{ for every } i \geq m+1\}$$

For example, if $n+1 = 7$ and $m = 5$, U^-w can be represented by the matrix below:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

whereas $U^+[\sigma]$ by this other matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_1 & b_2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Again we see that a set of representatives for the classes in $\overline{U^+[\sigma]}$ are

$$\{[w] = [v_1 \wedge ... \wedge v_n] \text{ such that } w \wedge e_i = 0 \text{ for every } i \leq m-1\}$$

Therefore again $|\overline{U^+[\sigma]} \cap \overline{U^-[\sigma]}| = 1 \Longrightarrow [\overline{U^+[\sigma]}] \cap [\overline{U^-[\sigma]}] = [p]$ where p is a point. □

Corollary 5.3. *For our parabolic subgroup P_0,*

$$|\overline{U^-[(1,m_1)(m_2,n+1)]} \cap \overline{U^+[(1,m_1)(m_2,n+1)]}| = 1$$

Proof. The idea is that $G/P_0 \hookrightarrow G/P_1 \times G/P_2$ is injective.

For simplicity, in this proof we will omit the overline to indicate the closure of a subset: whenever we write $U^+[\sigma]$ or $U^-[\sigma]$ we always mean its closure.

For every $g \in G$, $[g]$ will be the class of g in G/P_0, whereas $[g]_i$ will be its class in G/P_i for $i = 1, 2$.

Now, if $p_i : G/P_0 \to G/P_i$ is the canonical projection, $p_1(U^-[(1,m_1)(m_2,n+1)]) = U^-[(1,m_1)(m_2,n+1)]_1 = U^-[(1,m_1)]_1$; $p_1(U^+[(1,m_1)(m_2,n+1)]) = U^+[(1,m_1)(m_2,n+1)]_1 = U^+[(1,m_1)]_1$ and similarly, $p_2(U^-[(1,m_1)(m_2,n+1)]) = U^-[(1,m_1)(m_2,n+1)]_2 = U^-[(m_2,n+1)]_2$; $p_2(U^+[(1,m_1)(m_2,n+1)]) = U^+[(1,m_1)(m_2,n+1)]_2 = U^+[(m_2,n+1)]_2$.

Now, $[(1,m_1)(m_2,n+1)] \in U^-[(1,m_1)(m_2,n+1)] \cap U^+[(1,m_1)(m_2,n+1)]$ thus this intersection is not empty.

Furthermore, $p_1(U^-[(1,m_1)(m_2,n+1)] \cap U^+[(1,m_1)(m_2,n+1)]) \subseteq U^-[(1,m_1)]_1 \cap U^+[(1,m_1)]_1 = [(1,m_1)]_1$ and
$p_2(U^-[(1,m_1)(m_2,n+1)] \cap U^+[(1,m_1)(m_2,n+1)]) \subseteq U^-[(m_2,n+1)]_2 \cap U^+[(m_2,n+1)]_2 = [(m_2,n+1)]_2$.

Therefore $U^-[(1,m_1)(m_2,n+1)] \cap U^+[(1,m_1)(m_2,n+1)] \subseteq$
$p_2^{-1}(p_2(U^-[(1,m_1)(m_2,n+1)] \cap U^+[(1,m_1)(m_2,n+1)])) \cap$

$p_1^{-1}(p_1(U^-[(1,m_1)(m_2,n+1)] \cap U^+[(1,m_1)(m_2,n+1)])$ which has cardinality 1. Therefore $U^-[(1,m_1)(m_2,n+1)] \cap U^+[(1,m_1)(m_2,n+1)] = [(1,m_1)(m_2,n+1)]$. □

Coming back to the example of the quadrics:

Fix the nondegenerate quadric $Q = \{x_1^2 + ... + x_{n+1}^2 = 0\}$ and consider $(\{Q\} \times G/P) \cap Y$. This is the set $(Q,(p,\pi))$ with Q passing through p with tangent plane π in p. Being f proper, we can perform the proper push-forward on the Chow groups: consider $F_Q := f((\{Q\} \times G/P) \cap Y)$. This is the set of couples (p,π) with $p \in Q$ and with π the tangent plane of Q at p: we want to calculate $[F_Q]$.

Observation 5.25. *The dimension of F_Q is $n-1$.*

Therefore to calculate $[F_Q]$ it is enough to calculate $[F] \cap [Y_i] = [F] \cap [\overline{U^-[(1,j)(j+1,n+1)]}$ for every $1 \le j \le n$ (observation 5.20).

Calculating $F \cap Y_i$:

First consider the set theoretic intersection. The points we have to look for are the ones which belongs to $\bigcap_{k<j}\{x_k = 0\}$ and to Q.

Moreover, these points must also belong to $\bigcap_{j+2 \le k}\{x_k = 0\}$ because its tangent space must contain $[e_i]$ for $j+2 \le i$ and the quadric is $\{x_1^2 + ... + x_{n+1}^2 = 0\}$.

So we are looking at the points on the line $\mathbb{P}(< e_j, e_{j+1} >)$ which vanish on the quadric $x_j^2 + x_{j+1}^2$. There are just two distinct points, with multiplicity 1. So we get that $[F_Q] \cap [Y_i] = 2$:

Observation 5.26. *Using observation 5.20,*

$$[F_Q] = \sum 2[Y_i]^\vee$$

From what we have said before $[Y_j]^\vee = \overline{U^+[(1,j)(j+1,n+1)]}$

Definition 5.14. *Let* $[Z_j] := \overline{U^+[(1,j)(j+1,n+1)]}$

Observation 5.27. *Let Q' be another nondegenerate quadric. Then there is a $g \in PSL(n+1)$ such that $g \cdot Q' = Q$. Thus if we define $F_{Q'}$ as above, we see that $F_{Q'} = g \cdot F_Q$, therefore using theorem 2.6, $[F_{Q'}] = [F_Q]$.*

Now we want to understand what is $g_* f^*[Z_j]$.

First notice that being f flat and g proper, $g_* f^*[Z_j] = [g(f^{-1}(Z_j))]$. A quadric $Q \in g(f^{-1}(Z_j))$ is such that there is a point $[w] \in \mathbb{P}(< e_1, ..., e_j >)$ such that the tangent space of Q at $[w]$ contains $\mathbb{P}(< e_1, ..., e_j >)$. Moreover, given any quadric such that there is a point $[w] \in \mathbb{P}(< e_1, ..., e_j >)$ with the tangent space π of Q at p containing $\mathbb{P}(< e_1, ..., e_j >)$, then (p,π) belongs to Z_i: this implies that

$g(f^{-1}(Z_j))$ is the closure of the set of nondegenerate quadrics with the determinant of the minor corresponding to the first j coordinates 0

This implies, recalling observation 5.15,

$$g_* f^*[Z_j] = [D_j]$$

So $g_* f^*[F_Q] = \sum_i 2[D_j]$, and $g(f^{-1}(F_Q)) \cap G/H$ is the set of quadrics tangent to the quadric Q, we will call this divisor $[D] \in A^1(X)$.

Proposition 5.8. *D_j does not contain the closed orbit.*

Proof. For every i, consider the projection map $p : X \to \mathbb{P}(V_{2\omega_i})$. p is closed being X complete, and in $\mathbb{P}(V_{2\omega_i})$ there just one closed orbit G/P_I for a certain parabolic subgroup I. Thus the image of the closed orbit in X is G/P_I, because it is an orbit and it is closed.

Consider the divisor $\{a_1 x_1 + ... + a_{n+1} x_{n+1} = 0\}$. Being $V_{2\omega_i}$ irreducible, the closed orbit is not contained in this divisor, thus given a section of the line bundle $O_{\mathbb{P}(V_{2\omega_i})}(1)$, it does not vanish on the closed orbit, and therefore the pull-back if any section of $O_{\mathbb{P}(V_{2\omega_i})}(1)$ is a section of $p^*(O_{\mathbb{P}(V_{2\omega_{n-i+1}})}(1))$ which does not vanish on the closed orbit. But we have proved that D_i is the divisor of the pull-back of a section of $O_{\mathbb{P}(V_{2\omega_{n-i+1}})}(1)$, so we are done. □

Definition 5.15. *Let s_i be the section of $p^*(O_{\mathbb{P}(V_{2\omega_i})}(1))$ corresponding to D_j.*

Then if we consider $s_1^{\otimes 2} \otimes s_2^{\otimes 2} \otimes ... \otimes s_n^{\otimes 2}$, this is a section of $\bigotimes p^*(O_{\mathbb{P}(V_{2\omega_i})}(1))^{\otimes 2}$ with divisor D' with $[D'] = [D]$, therefore

$$D' \text{ does not contain the closed orbit}$$

In fact this closed orbit is irreducible, for every i s_i does not vanish on it thus $s_1^{\otimes 2} \otimes s_2^{\otimes 2} \otimes ... \otimes s_n^{\otimes 2}$ does not vanish on it.

Chapter 6

The toric case

In this chapter we will study intersection theory on a toric variety.

Definition 6.1. *X is a toric variety for a torus T if it is a spherical embedding of $T = T/\{1\}$, or equivalently X has an action of T with an open T-orbit.*

The two main references for the geometry of these varieties are [4], [9].

Notation 6.1. *Whenever we have a toric variety X, we will consider the torus T as an open subset of X: i.e. for us a toric variety is the data of a variety X with a point $p \in X$ such that X is a T-embedding with the open T-orbit O and a point $p \in O$. Once p is fixed, T is a subset of X.*

6.1 Basic notions about toric varieties

In this chapter T will be a torus $T = (\mathbb{G}_m)^n$, M will be its group of characters and N its group of cocharacters. There is a natural perfect pairing $(\cdot,\cdot)$ between M and N, and thank to this pairing, given $A \subseteq M$, its orthogonal is well-defined: $A^{\perp} := \{n \in N : (n, a) = 0 \ \forall a \in A\}$.

A toric variety is a spherical variety, then we can associate to it a colored fan. But in this colored fan there are no colours because $T = B$, and moreover every cone is contained in $Hom(\Lambda, \mathbb{Z})_{\mathbb{Q}} = N_{\mathbb{Q}}$. So the combinatorial data associated to X now is a fan $\Sigma = \{\sigma_i\}_i$. Note that in our case T is the Borel subgroup of T, therefore by theorem 4.3 any simple toric variety is affine.

How to construct the cone associated to a simple toric variety: Given X an affine toric variety, we can decompose its ring of regular functions $O_X(X) = \bigoplus_{\chi \in M} V_{\chi}$ where V_{χ} is a representation on which T acts with character χ. Using observation 4.2, every V_{χ} has dimension less than or equal than 1, and if $S := \{m \in M : V_m \neq 0\}$, S is a finitely generated cone. The dual of this cone is a cone σ contained in N, and it is the cone associated to the simple spherical T-variety X. Notice that it is saturated:

Definition 6.2. *Given σ a cone in N, we say that σ is saturated if for every $n \in N$ and $c \in \mathbb{N}$, $cn \in L \Longrightarrow n \in L$.*

Notation 6.2. *Given X a toric variety with fan Σ and given $\sigma \in \Sigma$, it corresponds to a T orbit $O(\sigma)$. We will call $U_\sigma := X_0^{O(\sigma)}$ (see definition 4.6 for the definition of $X_0^{O(\sigma)}$). U_σ is an affine toric variety with associated cone σ.*

Closed T-stable subvarieties:
As for spherical varieties, given X a toric variety with fan Σ, every closed T-stable subvariety V of X is a toric variety as well. If τ is the cone associated to V, we will put $V(\tau) := V$. The dense torus in $V(\tau)$ is isomorphic to $\mathbb{G}_m \otimes_{\mathbb{Z}} Hom((\tau)^\perp, \mathbb{Z})$, where $(\tau)^\perp \subseteq M$ and thus $Hom((\tau)^\perp, \mathbb{Z}) \subseteq N$, (notice in particular that $dim(V(\tau)) = codim(\tau)$) and its corresponding lattice of cocharacters is N/N_τ where $N_\tau := < \tau \cap N >_{\mathbb{Z}}$.

Moreover, for every $\lambda \in \tau$, $s \in \mathbb{G}_m$ and $x \in V(\tau)$ $\lambda(s)x = x$, i.e. $< \tau >_{\mathbb{Z}} \otimes \mathbb{G}_m \subseteq N \otimes \mathbb{G}_m$ acts trivially on $V(\sigma)$.

We can characterize $V(\tau)$ in this ways: given $\lambda \in \mathring{\tau}$, $\lim_{s \to 0} \lambda(s)$ exists and $V(\tau) = \overline{T \lim_{s \to 0} \lambda(s)}$ (remember that for us a toric variety is an spherical embedding of T: T is canonically an open subset of X).

The toric variety $V(\tau)$ has a fan itself, the result is the following:

Theorem 6.1. *([4] proposition 3.2.7). With the same notations as before, the fan of the toric variety $V(\tau)$ is $\{\pi(\sigma) \subseteq N/N\tau : \tau$ is a face of $\sigma\}$, where $\pi : N \to N/N_\tau$ is the projection.*

The following definition and theorem generalizes what we have said about morphisms of spherical varieties:

Definition 6.3. *Given two toric varieties X_1, X_2 for two different tori T_1 and T_2 respectively, a morphism $\phi : X_1 \to X_2$ is a toric morphism if maps $T_1 \to T_2$ and restricted to T_1 is a group homomorphism.*

Theorem 6.2. *Let N_1 and N_2 be two lattices with Σ_i a fan in $(N_i)_{\mathbb{Q}}$. If $\overline{\phi} : N_1 \to N_2$ is a $\mathbb{Z}$-linear map such that for every cone $\sigma \in \Sigma_1$ there is $\tau \in \Sigma_2$ with $\overline{\phi}(\sigma) \subseteq \tau$, then there is a toric morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ compatible with the map $\overline{\phi} \otimes Id : N_1 \otimes \mathbb{G}_m \to N_2 \otimes \mathbb{G}_m$.*

On the other hand if $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is a toric morphism, then ϕ induces a linear map $\overline{\phi} : N_1 \to N_2$ such that for every cone $\sigma \in \Sigma_1$ there is $\tau \in \Sigma_2$ with $\overline{\phi}(\sigma) \subseteq \tau$.

Moreover, checking the proof of the previous theorem, we see that if $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is a toric morphism with associated map $\overline{\phi} : N_1 \to N_2$, then if $\overline{\phi}(\sigma) \subseteq \tau \Longrightarrow \phi(U_\sigma) \subseteq U_\tau$.

Theorem 6.3. *Let X be a smooth affine toric variety. Then $X \cong k^a \times (k - \{0\})^{n-a}$ for a suitable a where k is the ground field. The action is $(t_1, ..., t_n)(a_1, ..., a_n) = (t_1 a_1, ..., t_n a_n)$, the cone associated to this simple toric variety is simplicial (which means that the vectors generating the one-dimensional rays are linearly independent) and the open subset of X isomorphic to T is $(k - \{0\})^n$.*

Observation 6.1. *Assume that X is smooth. Then every closed T-stable subvariety of X is smooth as well: in fact we can use theorem 6.3: a smooth variety is covered by open T-stable affine sets isomorphic to $(k)^a \times (\mathbb{G}_m)^{n-a}$, and we know the action on these open subsets.*

Definition 6.4. *Given a fan Σ with $\sigma, \tau \in \Sigma$ of dimension $k+1$ and k respectively and τ a face of σ, let $n_{\sigma,\tau}$ be a lattice point in σ whose image generates the 1-dimensional lattice N_σ / N_τ.*

Observation 6.2. *Given a rational T-semiinvariant function on X, it is a rational T-semiinvariant function on T, i.e. a character of M. So, given $m \in M$, if X is a toric variety, one can show that the divisor of m is $\sum (m, n_\tau)[V(\tau)]$ where the sum is on all over the 1-dimensional cones τ of Σ, and n_τ is the generator of the face τ. (One can show it locally on every affine T-stable open subset).*

Theorem 6.4. *The Chow group $A_k(X)$ of a complete toric variety X with fan Σ is generated by the classes $[V(\sigma)]$ where $\sigma \in \Sigma$ has codimension k, and the relations are $< \{\sum_\sigma (u, n_{\sigma,\tau})[V(\sigma)]\} >_{\mathbb{Z}}$ with $u \in M$; $\sigma, \tau \in \Sigma$ with σ of codimension k and τ of codimension $k+1$ is a face of σ. This follows from the previous observation and theorem 2.1.*

We will mainly deal with complete toric varieties:

Observation 6.3. *Restricting our study to complete toric varieties is not a great restriction: in fact given a toric variety which is not complete we can always find $\overline{X}$ another complete toric variety with an open embedding $X \hookrightarrow \overline{X}$, and we can use the following sequence (lemma 1.3)*

$$A_k(\overline{X} - X) \to A_k(\overline{X}) \to A_k(X) \to 0$$

Now we state a result, by which it is clear why switching from the naive definitions of $A^k(X) := A_{dim(X)-k}(X)$ to the bivariant definition is important:

Proposition 6.1. *([7] corollary 2.4). Let X be a complete toric variety. Then $Pic(X) \cong A^1(X)$.*

This is not true if we define $A^1(X) := A_{n-1}(X)$: here it is important to define it using the bivariant definition of Chow rings. In fact it is not true that for a complete toric variety $Cl(X) = Pic(X)$, this is true if and only if X is smooth.

There is a descriprtion of the Chow ring of a smooth complete variety due to Danilov-Jurkiewicz, [4] theorem 12.4.1. Here it is proved that for a smooth complete toric variety the Chow ring is isomorphic to the ring of Stanley-Reisner. This is a ring described by generators and relations in this way: suppose that our toric variety is associated to the fan Σ, then $A^*(X)$ is isomorphic to $\mathbb{Z}[X_\rho$ where ρ is a 1-dimensional ray of $\Sigma]/I$. I is the ideal generated by $x_{\rho_1} \cdot ... \cdot x_{\rho_r}$ if $\rho_1, ..., \rho_r$ do not generate a cone in Σ, and by $\sum < m, u_\rho > x_\rho$ for every $m \in M$, where u_ρ is the generator of the ray ρ.

6.2 Computing $m^\gamma_{\sigma,\tau}$

Looking at theorem 2.4 and observation 2.2, the main combinatorial data to describe how the intersection product works on a complete spherical variety are the integers $m^\gamma_{\sigma,\tau}$. The first part of this chapter is aimed at describing these integers: we have to describe how $\delta([V(\gamma)])$ decomposes in $A_*(X \times X) \cong A_*(X) \otimes A_*(X)$. A reference for this section is [12].

Definition 6.5. *Given L a saturated sublattice of N of dimension d, it corresponds to a subtorus $T_L \subseteq T$. Given $v \in N$, we define*

$$\Delta_{L,N}(v) := \{\sigma \in \Sigma \text{ such that } |(L_\mathbb{R} + v) \cap \mathring{\sigma}| = 1\}$$

If both L and N are clear, we will omit the subscript L, N.

Definition 6.6. *Given a toric variety X with fan Σ, and L a saturated sublattice of N, we say that $v \in N$ is generic if it does not belong to L and $\sigma \in \Delta(v) \implies dim(\sigma) = n - d$. Note that if v is generic, $|(L_\mathbb{R} + v) \cap \mathring{\sigma}| = 1 \implies |(L_\mathbb{R} + v) \cap \sigma| = 1$.*

Lemma 6.1. *Let v be a generic lattice point in N for the lattice L, with corresponding subtorus T_L, we have that $[(\overline{T_L})_X] = \sum_{\sigma \in \Delta(v)} [N : L + N_\sigma][V(\sigma)]$.*

We can use this lemma to prove the following theorem, which solves the problem of finding $m^\gamma_{\sigma,\tau}$:

Theorem 6.5. *Let (v_1, v_2) be a generic element for the diagonal sublattice of $N/N_\gamma \times N/N_\gamma$, $< \{(n, n) \in N/N_\gamma \times N/N_\gamma\} >$. Given a toric variety with fan Σ and given $\gamma \in \Sigma$,*

$$[\delta(V(\gamma))] = \sum_{\sigma,\tau} [N : N_\sigma + N_\tau][V(\sigma) \times V(\tau)]$$

The sum is over all pairs (σ, τ) of cones in Σ such that γ is a face of σ and τ such that if $\pi : N \to N/N_\gamma$ is the projection, $\pi(\sigma)$ meets $\pi(\tau) + (v_1 - v_2)$, $codim(\sigma) + codim(\tau) = codim(\gamma)$.

Before proving the theorem, we make some considerations about the property of being generic. Let L be a saturated sublattice of N of dimension d, and let $v \in N$. For every cone $\tau \in \Sigma$ with $dim(\tau) < n-d$, we can consider $H_\tau := L_\mathbb{R} + < \tau >_\mathbb{R}$.

Observation 6.4. *If* $v \in N - \bigcup_{\tau \in \Sigma,\ dim(\tau) < n-d} H_\tau$ *then it is generic.*

Proof. In fact, if we pick $v \in N - \bigcup_{\tau \in \Sigma,\ dim(\tau) < n-d} H_\tau$ then $\sigma \in \Delta(v) \implies dim(\sigma) \geq n-d$. On the other hand if $\sigma \in \Delta(v)$ then for dimensional reasons, $dim(\sigma) \leq n-d$, therefore $dim(\sigma) = n-d$: v is generic. □

With the notations of the theorem above, given a generic vector $(v_1, v_2) \in N/N_\gamma \times N/N_\gamma$, if $\pi(v) = v_1$ and $\pi(w) = v_2$ then there exists an element $y_{\sigma,\tau} \in < \gamma >$ and two elements x_τ, x_σ in τ, σ respectively such that $x_\tau + v - w - x_\sigma = y_{\sigma,\tau}$. But γ is a face of both σ and τ then up to changing x_τ and x_σ we can assume $0 = y_{\sigma,\tau}$. This means that up to choosing v in N which is not contained in a certain finite family of hyperplanes, the sum of the theorem is made over all pairs (σ, τ) such that σ meets $\tau + v$, $codim(\sigma) + codim(\tau) = codim(\gamma)$ and γ is a face of both σ and τ.

We now prove the theorem:

Proof. It is enough to prove the theorem with the whole variety X. In fact, assume that the thesis is true for X, then we can replace X with $V(\gamma)$, and the multiplicity which we obtain is $[N/N_\gamma : (N_\sigma + N_\tau)/N_\gamma] = [N : N_\sigma + N_\tau]$. Note that $codim(\pi(\sigma)) + codim(\pi(\tau)) = dim(N/N_\gamma)$ if and only if $codim(\sigma) + codim(\tau) = codim(\gamma)$.

So let us prove the theorem when $X = V(\gamma)$. We apply the previous lemma to the diagonal embedding $X \hookrightarrow X \times X$.

This corresponds to the diagonal inclusion of lattices $N \hookrightarrow N \times N$: if we take the dual of this inclusion we get $M \times M \to M$, $(a, b) \to a + b$ which is in terms of regular functions $O_T(T) \otimes O_T(T) \to O_T(T)$, $p \otimes q \to pq$: we get the diagonal embedding.

Note now that using a generic vector of the form (v_1, v_2) gives the same formula in the previous lemma as using $(v_1 - v_2, 0)$, because $diag(N) + (v_1, v_2) = \{(n + v_1, n + v_2) \in N \times N\} = \{(n + v_1 - v_2, n) \in N \times N\}$.

Points of intersection of interior part of the cone $\sigma \times \tau$ with $diag(N_\mathbb{R}) + (v, 0)$ corresponds to points of intersection of σ with $\tau + v$, and for such a v there is at most one intersection point when $dim(\sigma) + dim(\tau) = n$.

Lemma 6.1 implies that $diag_*([X]) = \sum m_{\sigma,\tau}[V(\sigma) \times V(\tau)]$, where the sum is over all pairs σ, τ such that $|\mathring{\sigma} \cap (\mathring{\tau} + v)| = 1$ with $dim(\sigma) + dim(\tau) = n$, and the coefficient is $[N \times N : \{(n, n)\}_{n \in N} + N_\sigma \times N_\tau] = [N : N_\sigma + N_\tau]$. □

Actually using the previous lemma it is possible to prove even more: assume that $c \in A^k(X)$ and let $f : X' \to X$. Then it is possible to describe

$$A_p(X') \to A_{p-k}(X') \text{ sending } z \mapsto f^*(c) \cap z.$$

This actually is a generalization of the previous theorem: take $f = Id$. We will not prove this result because it goes beyond the purpose of this thesis, a reference is [12] theorem 3.5.

Now we come back to the proof of the lemma:

Proof. First notice that we can assume $L + \mathbb{Z}v$ to be a saturated sublattice, in fact for every $l \in L$ and every $k \in \mathbb{N}$, $\Delta(v) = \Delta(v+l)$ and $\Delta(kv) = \Delta(v)$.

Case 1: $N = L \oplus \mathbb{Z}v$

Let $u \in M$ be an element which vanishes on L such that $m(v) = -1$, consider the function $f = x^u - 1$.Given a 1-dimensional cone $\sigma \in \Sigma$, $[N : L + N_\sigma] < \infty \iff \sigma \nsubseteq L$, and if this is the case we define $m_\sigma = min\{m \in \mathbb{N} :$ there is an $l \in L$ with $mv + l \in \sigma\}$, if $\sigma \subseteq L$ we put $m_\sigma = 0$.

We want to show the following equality:

$$[div(f)] = [Y] - \sum_{\sigma \in \Delta(v)} m_\sigma [V(\sigma)] \tag{6.1}$$

The two sides of equation (6.1) have the same restriction to the open set T because $([Y] - \sum_{\sigma\in\Delta(v)} m_\sigma[V(\sigma)])|_T = ([Y])|_T$ is the torus associated to the lattice L, which is the divisor of f by definition of u. Then it is enough to show that for every ray $\sigma \in \Sigma$ the irreducible divisor $[V(\sigma)]$ has coefficient $-m_\sigma$ in the divisor of f if $\sigma \in \Delta(v)$ and 0 otherwise.

Note that $ord_{V(\sigma)}(f)$ is a nonpositive integer. In fact this is a local computation: assume that f has not a pole along $V(\sigma)$. Therefore x^u has not a pole, thus if $O_{X,V(\sigma)}$ is the local ring along the generic point of $V(\sigma)$, $x^u \in O_{X,V(\sigma)}$. Let m be the maximal ideal of $O_{X,V(\sigma)}$, then being m T-stable we have a T-equivariant isomorphism of vector spaces $O_{X,V(\sigma)} \cong_T k(V(\sigma)) \oplus_T m$, $k(V(\sigma)) \cong_T < 1 > \oplus_T \bigoplus_{\lambda\in\Lambda'} k\lambda$ for a certain subset $\Lambda \in M - \{0\}$. Therefore $O_{X,V(\sigma)} \cong_T < 1 > \oplus_T \bigoplus_{\lambda\in\Lambda'} k\lambda \oplus_T m$. The eigenspace associated to the trivial character is 1 and x^u is not constant, therefore if $\pi : O_{X,V(\sigma)} \to k(V(\sigma))$ is the projection, x^u is either 0 or a T-eigenfunction with nontrivial character: $x^u|_{V(\sigma)}$ can not be constantly 1.

Then f can have just poles along $V(\sigma)$, so $ord_{V(\sigma)}(f) = min\{0, ord_{V(\sigma)}(x^u)\} = min\{0, (u, r)\}$ where r is the generator of the 1-dimensional ray σ (the last equality follows from observation 6.2). So the point is to describe this generator r.

If $N_\sigma \subseteq L$ then x^u is a rational function on $V(\sigma)$: its order is 0.

Let σ be a 1-dimensional cone in Σ such that $[N : L + N_\sigma]$ is finite. $[N : L + N_\sigma] = [L \oplus \mathbb{Z}v : L + N_\sigma] = m_\sigma$, therefore a generator of the 1-dimensional lattice N_σ is $m_\sigma n + l$ for a certain l.

The 1-dimensional lattice N_σ contains the 1-dimensional cone σ, which is a saturated cone: it is generated by either $m_\sigma n + l$ or $-m_\sigma n - l$. In the first case, f has a pole along $V(\sigma)$ of order m_σ, in the other case x^u vanishes on $V(\sigma)$, so $f|_{V(\sigma)} \equiv 1$.

Finally, the cone is generated by the former if and only if $\sigma \subseteq \{x \in N : (n,u) < 0\}$, if and only if σ belongs to $\Delta(v)$. Summarizing:

$$f \text{ has a pole along } V(\sigma) \Longleftrightarrow x^u \text{ has a pole along } V(\sigma) \Longleftrightarrow \sigma \in \Delta(v).$$

and

$$x^u \text{ has a pole along } V(\sigma) \Longrightarrow ord_{V(\sigma)}(f) = ord_{V(\sigma)}(x^u) = -m_\sigma$$

General case:
Let $L' := L \oplus \mathbb{Z}v < N$. Let Σ' be the fan in L' whose cones are the intersections $\sigma \cap L'_{\mathbb{R}}$, $\sigma \in \Sigma$. This gives a toric variety X' and a proper morphism $p : X' \to X$ (it is proper because X' is complete and $X \to spec(k)$ is separated).

By the case we have proved above we have that if Y' is the closure of the torus associated with L in X',

$$[Y'] = \sum_{\sigma'} m_{\sigma'}[V(\sigma')] \tag{6.2}$$

where σ' runs over all rays of $\Delta_{L'}$ that have the form $\sigma \cap L'_{\mathbb{R}}$ for some $\sigma \in \Delta(v)$: in fact v is generic also for the lattice $L \subseteq L'$ and if $N \supseteq \sigma \in \Delta_{L,N}(v) \Longrightarrow \sigma \cap L' \in \Delta_{L,L'}(v)$, and $m_{\sigma'} = [L' : L + N_{\sigma'}] = [L' + N_\sigma : L + N_\sigma]$.

p is proper, then the push forward on the chow group is defined: $p_*([Y']) = \sum_{\sigma'} m_{\sigma'} p_*([V(\sigma')])$. But $p_*([Y']) = [Y]$, we have to compute $p_*([V(\sigma')])$.

If $\sigma \in \Delta_{L,N}(v)$ then $\sigma \cap L' \in \Delta_{L,L'}(v)$ and $U_{\sigma \cap L'}$ maps to U_σ. Given $\lambda \in \sigma \mathring{\cap} L'$, it belongs to $\mathring{\sigma}$. This is because if it belongs to a face τ then $\tau \cap L' = \sigma \cap L'$ which implies that v is not general.

Now: the orbit corresponding to $\sigma \cap L'$ is $O(\sigma \cap L') := (L' \otimes \mathbb{G}_m) \cdot \lim_{s \to 0} \lambda(s)$ where as regards the limit we think (as usual) at $\mathbb{G}_m \otimes L'$ as imbedded in $X_{\Sigma'}$. Therefore $p(O(\sigma \cap L')) = p((L' \otimes \mathbb{G}_m) \cdot \lim_{s \to 0} \lambda(s)) = (L' \otimes \mathbb{G}_m) p(\lim_{s \to 0} \lambda(s)) = (L' \otimes \mathbb{G}_m) \lim_{s \to 0} \lambda(s)$ where the last λ is thought as a cocharacter of T composing it with the inclusion $L' \otimes \mathbb{G}_m \subseteq N \otimes \mathbb{G}_m = T$.

But $(L' \otimes \mathbb{G}_m) \lim_{s \to 0} \lambda(s) \subseteq T \lim_{s \to 0} \lambda(s) = O(\sigma)$ (remember what we have said before theorem 6.1) because $\lambda \in \mathring{\sigma}$.

Moreover, for every $s \in \mathbb{G}_m$, for every $\mu \in < \sigma >_{\mathbb{Z}}$ and for every $x \in O(\sigma)$ $\mu(s)x = x$. Thus if we write $N = < \sigma >_{\mathbb{Z}} \oplus N'$ (which is possible being σ saturated) we get a decomposition of T: $T = (\mathbb{G}_m \otimes < \sigma >_{\mathbb{Z}}) \times (\mathbb{G}_m \otimes N')$ and $O(\sigma) = (\mathbb{G}_m \otimes N') \lim_{s \to 0} \lambda(s)$.

But now notice that we can choose N' to be L: in fact it is saturated and $L \cap < \sigma >_{\mathbb{Z}} = \{0\}$ being $\sigma \in \Delta_{L,N}(v)$. So $(\mathbb{G}_m \otimes N') \lim_{s \to 0} \lambda(s) = (\mathbb{G}_m \otimes L) \lim_{s \to 0} \lambda(s) : p(O(\sigma \cap L)) = O(\sigma)$. Thus $p(V(\sigma')) = V(\sigma)$ because p is closed.

The degree is $a := [k(V(\sigma')) : k(V(\sigma))]$. $k(V(\sigma))$ is a T-stable subfield of $k(V(\sigma'))$ then to compute a we can decompose $k(V(\sigma')) = k(V(\sigma')) \oplus_T \bigoplus_{i=1}^{a} k\chi_i$ where χ_i are characters and k is the ground field. This inclusion corresponds to the inclusion of lattices $L'/N_{\sigma'} = L'/(L' \cap N_\sigma) = (L' + N_\sigma)/N_\sigma \hookrightarrow N/N_\sigma$, which gives us $a = [N : L' + N_\sigma]$.

Then $[N : L' + N_\sigma]m_{\sigma'} = [N : L' + N_\sigma][L' + N_\sigma : L + N_\sigma] = [N : L + N_\sigma] = m_\sigma$: the coefficient of $[V(\sigma)]$ is m_σ. □

We now present an example in which we use the theory developed: the Chow ring of $\mathbb{P}^n$. Notice that when the ground field is $\mathbb{C}$ we obtain the cohomology of $\mathbb{P}^n$ (corollary 2.1).

Example 6.1. The Chow ring of the projective space $\mathbb{P}^n$:

Choose a basis $x_0, ..., x_n$ for $O_{\mathbb{P}^n}(1)(\mathbb{P}^n)$. Then consider the following action of the torus on the projective space: $(t_1, ..., t_n)[a_0, ..., a_n] := [t_1 a_0, ..., t_n a_{n-1}, a_n]$. With this action $X := \mathbb{P}^n$ is a toric variety, and let Σ be its fan. Let $m_i : T \to k^*$, $(t_1, ..., t_n) \mapsto t_i$ and $n_i : k^* \to T$, $s \mapsto (1, ..., 1, s, 1, ..., 1)$ be the element dual to m_i with respect to the basis of M $m_1, ..., m_n$, and let $n_0 := -n_1, ..., -n_n$.

What is Σ?

For every $J \subseteq \{0, ..., n\}$ the set of closed points are $\{\bigcap_{i\in J}\{x_i = 0\}\}_{|J|=n}$: for every ($T$-stable) open set of the form $D(x_i)$ the unique closed point is $[0, ..., 0, 1, 0, ..., 0]$ where the non-zero component is the i-th. Let σ_i^n be the (n-dimensional) cone corresponding to the closed point in $D(x_i)$. The ring of regular functions on $D(x_i)$ is $k[y_1, ..., y_m] = \bigoplus_{\lambda \in S_{\sigma_i^1}} k\lambda$. Looking at the action on the regular functions we get that $S_{\sigma_i^1}$ is the cone generated by $m_1 - m_i, ..., m_n - m_i, -m_i$ when $i \neq 0$, with dual cone $\sigma_i^1 =< n_1, ..., \hat{n}_i, .., n_n, n_0 >_{\mathbb{N}}$ (the hat means that n_i is missing); and when $i = 0$ it is $m_1, ..., m_n$ with dual cone $\sigma_0^1 =< n_1, ..., n_n >_{\mathbb{N}}$ w. These are the maximal cones of Σ.

What is $A_*(X)$?

Let $0 \leq k \leq n-1$ be an integer, we want to understand what is $A_k(X)$ ($A_n(X) = \mathbb{Z}$ is clear). First we start with $A_{n-1}(X)$, where the setting is easier but not so far from the general case. A set of generators is $\{V_{n-1}(i)\}_{i=0}^n$ where $V_{n-1}(i) := V(< n_i >_{\mathbb{N}})$ (theorem 2.1). In order to determine the relations we use theorem 6.4, and it is enough to look at the relations given by the m_i: they are $[V_{n-1}(i)] - [V_{n-1}(0)]$. Therefore

$$A_{n-1}(X) \cong (\mathbb{Z}e_0 \oplus ... \oplus \mathbb{Z}e_n)/\{(a_0, ..., a_n) : \sum a_i = 0\}$$

Now, let $1 \leq k \leq n$. For every subset $J \subseteq \{0, ..., n\}$ let $V_{n-|J|}(J) := V(< \{n_j\}_{j\in J} >_{\mathbb{N}})$: $A_{n-k}(X) =< \{[V_{n-k}(J)]\}_{|J|=k} >_{\mathbb{Z}}$. Now we want to understand which are the relations, so let $1 \leq i_1 < ... < i_{k-1} \leq n$: we want to understand which are the relations given by the T-semiinvariant rational functions on the $n-k+1$-dimensional subvariety $V_{n-k+1}(\{i_1,...,i_{k-1}\})$. The lattice of cocharacters of this subvariety is $N' := N/ < n_{i_1},...,n_{i_{k-1}} >_{\mathbb{Z}}$ with projection $\pi : N \to N'$, and if $\tau =< n_{i_1},...,n_{i_{k-1}} >_{\mathbb{N}}$, for every $\sigma =< n_{i_1},...,n_{i_{k-1}}, n_j >_{\mathbb{N}}$, $n_{\sigma,\tau} = \pi(n_j)$. Thus the relation given by m_j with j not in $\{i_1,...,i_{k-1}\}$ is $[V_{n-k}(\{i_1,...,i_{k-1},j\})] - [V_{n-k}(\{i_1,...,i_{k-1},0\})]$: this means that when we start with $V_{n-k}(J)$ with 0 not belonging to J, we can replace an element of J with 0. This implies that when we have J, J' two sets of cardinality k, $[V_{n-k}(J)]-[V_{n-k}(J')] = 0$ (we can prove it by induction on $|J \cap J'| = r$).

Now we have to add the relations given by any divisor on $V_{n-k+1}(J)$ with $0 \in J$. But these relations are already included because if we consider the change of coordinates sending $n_1,...,n_n \mapsto n_0, n_2,...,n_n$, and $m_1,...,m_n \to -m_1, m_2-m_1..., m_n-m_1$ we can repeat the same argument as before, and get that any divisor of a T-eigenfunction has the shape of $[V_{n-k}(J)]-[V_{n-k}(J')]$ for two suitable subsets J, J' of cardinality k. Therefore we have again

$$A_k(X) \cong (\mathbb{Z}e_1 \oplus ... \oplus \mathbb{Z}e_{\binom{n+1}{k}})/\{(a_1,...,a_{\binom{n+1}{k}}) : \sum a_i = 0\}$$

So summarizing, if V_k is a T-stable subvariety of timension k,

$$\forall 0 \leq k \leq n,\ A_k(X) = \mathbb{Z}[V_k]$$

(Notice that all these subvarieties intersect transversally, theorem 6.3).

Now with this description is easy to see how intersection product works. Instead, we will try to understand how this product works using what we have done before.

The intersection product and Poincaré duality:

Finally we want to understand how to intersect two such cycles using what we have done in this section. Using Poincaré duality $A^k(X) \cong A_{n-k}(X)$, let $f : A^k(X) \to A_{n-k}(X)$ be this isomorphism. By theorem 2.3 $A^k(X) = Hom(A_k(X), \mathbb{Z}) \cong \mathbb{Z}$, let c be the map which has value 1 on $[V_k]$. Then c generates $A^k(X)$, thus its dual could be either $[V_k]$ or $-[V_k]$. Now, looking at theorem 2.4 we see that $c \cap [X] = \sum_{|J|=k} n_J[V(J)]$, each of the coefficients n_J is greater or equal than 0, thus $c \cap [X] = [V_k]$.

Now we want to compute $c \cap [V_m]$. If $[V_m] = [V(\lambda)]$, we have to understand what are the coefficients $m^{\lambda}_{\sigma,\tau}$. We take $\lambda =< n_1,...,n_{n-m} >_{\mathbb{N}}$, then $N/N_\lambda \cong < n_{n-m+1},...,n_n >$ and a generic vector for the diagonal sublattice in $N/N_\lambda \times N/N_\lambda$ could be $(0; n_{n-m+1} + 2n_{n-m+2} + ... + mn_n)$. In fact it does not belong to any subspace $H_{\sigma\times\tau}$ with $dim(\sigma \times \tau) < m$ (observation 6.4). Thus the vector we need to consider is

$$v = n_{n-m+1} + 2n_{n-m+2} + ... + mn_n$$

Now, this is the question: if we consider the fan Σ' in $N' := N/N_\lambda$ given by the maximal cones $< e_0,...,\hat{e_i},...,e_m >_{\mathbb{N}}$ where $e_i = n_{n-m+i}$ and $e_0 = -e_1 - ... - e_m$, we want to find all the couples of cones σ, τ such that $dim(\sigma) + dim(\tau) = m$ and $|(\sigma \times \tau) \cap (\{w, w - v\}_{w \in N'})| = 1$ where $-v = -e_1 - 2e_2 - ... - me_m$.

Remark 6.1. *Notice in particular the following fact: if we complete e_0 as a basis for N' with elements belonging to $\{e_1, ..., e_m\}$, and we write $-v$ in the coordinates relative to such a basis, we get that if the basis is $\{e_0, e_1, ..., \hat{e_i}, ..., e_m\}$ then the coordinate relative to e_0 is i.*

First notice that σ and τ can not share a face (but the face $\{0\}$): if $0 \neq w \in \sigma \cap \tau$ and we have $z \in \tau$ and $s \in \sigma$ such that $z + v = s \Longrightarrow z + v' + v = s + v'$: $1 < |\sigma \times \tau \cap (\{w, w - v\}_{w \in N'})|$. So, given $I \subseteq \{0, ..., n\}$ we want to see whether there is a $J \subseteq \{0, ..., n\} - I$ with $< \{e_j\} >_{j \in J} \times < \{e_i\}_{i \in I} > \in \Delta((0, -v))$.

Case 1: assume first that 0 does not belong to I.

$I = \{i_1, ..., i_r\}$ with $a < b \Longrightarrow i_a < i_b$, then we have to look at the r-codimensional cones in the fan Σ which intersect in just one point the set $-v+ < \{e_i\}_{i \in I} >_{\mathbb{N}}$: there is such a cone if and only if $I = \{m, m - 1, ..., m - r + 1\}$. In fact if $I = \{m, m - 1, ..., m - r + 1\}$ we can pick the cone $< e_1, ..., e_{r-1}, e_0 >_{\mathbb{N}}$, pick $e_{m-r+1} + 2e_{m-r+2} + ... + re_m \in < \{e_i\}_{i \in I} >_{\mathbb{N}}$ and $-v + e_{m-r+1} + 2e_{m-r+2} + ... + re_m \in < e_1, ..., e_{r-1}, e_0 >_{\mathbb{N}}$. If I is not in that way, assume that there is such a J, then we note that:
i) $0 \in J$: for every point $p \in -v+ < \{e_i\}_{i \in I} >_{\mathbb{N}}$ and for every j not in I, $(m_j, p) < 0$, thus $0 \in J$.
ii) Let p be the unique point in $< \{e_j\}_{j \in J} >_{\mathbb{N}} \cap - v+ < \{e_i\}_{i \in I} >_{\mathbb{N}}$ and $j_0 := max\{j \in \{1, ..., m\} - I\}$, then if we complete the set $\{e_j\}_{j \in J}$ as a basis of N' adding elements of $\{e_i\}_{i \in I}$, the coefficient of e_0 that we get writing p in this basis is j_0. This follows from $p \in -v+ < \{e_i\}_{i \in I} >_{\mathbb{N}}$ and from what are the coordinates of p in the basis $\{e_1, ..., e_m\}$.
iii) Let $j_0 := max\{j \in \{1, ..., m\} - I\}$, then every $i < j_0$ should belong to J because $\forall p \in -v+ < \{e_i\}_{i \in I} >_{\mathbb{N}}$, $(m_i, p) > -j_0$.

Case 2: assume now that $0 \in I$.

Then 0 does not belong to J because the two cones can't share a face, and this is absurd because writing the points of $-v+ < \{e_i\}_{i \in I} >_{\mathbb{N}}$ in the basis $\{e_1, ..., e_m\}$ we see that there must be some coordinate with a negative coefficient.

Therefore we get that the coefficient $m^{\lambda}_{\sigma,\tau} = [N : N_\sigma + N_\tau]$ is either 0 or 1, it is 1 if and only if $\sigma = \lambda+ < \{n_j\}_{j > j_0} >_{\mathbb{N}}$ for a certain $n - m \leq j_0$ and $\tau = \lambda+ < n_{n-m+1}, ..., n_{j_0} >_{\mathbb{N}}$. Therefore for a certain $a \geq dim(\lambda)$ there is just a σ face of λ with dimension a and just one τ such that $\sigma \times \tau \in \Delta(v)$, and the coefficient $m^{\lambda}_{\sigma,\tau} = 1$. Therefore we get that, if $A := \{I \subseteq \{n-m+1, ..., n\}$ with $i \in I \Longrightarrow i + 1 \in I\}$, then

$$c \cap [V(\lambda)] =$$
$$\sum_{I \in A} c([V(\lambda + (< \{n_{n-m+i}\}_{i \in I^c} >_{\mathbb{N}}))])[V(\lambda+ < \{n_{n-m+j}\}_{j \in I} >_{\mathbb{N}})] =$$
$$[V(\lambda+ < \{n_i\}_{i \in I_k} >_{\mathbb{N}})]$$

where I_k is the unique $I \in A$ with $|I| = k$ and it is 0 if there are no $I \in A$ with $|I| = k$.

6.3 Another description of the Halphen ring of a torus

In this section we will give another description for $A^*(X)_{\mathbb{Q}}$, in the case in which X is a toric projective variety.

Remark 6.2. *In this section we will work just with rational coefficients, therefore in order to simplify the notations we will drop the subscript* $\mathbb{Q}$ *in* $A_*(X)_{\mathbb{Q}}$.

Our aim is to relate this ring with the polytope algebra. So let me recall some facts about this algebra, a reference is [18].

Definition 6.7. *The polytope algebra* Π *is a* $\mathbb{Q}$*-algebra with a generator* $[P]$ *for every polytope* $P \subseteq \mathbb{Q}^n$*, and* $[\emptyset] = 0$*. These generators satisfy the following relations:*
$[P \cup Q] + [P \cap Q] = [P] + [Q]$ *when* $P \cup Q$ *is a convex polytope;*
$[P + t] = [P]$ *for all* $t \in \mathbb{Q}^n$*.*
The multiplication in Π *is given by the Minkowski sum:* $[P] \cdot [Q] = [P + Q]$ *where the last* $+$ *is the Minkowski sum.*

Observation 6.5. *The identity is the class of a point* $[p]$*.*

The fundamental relation that should be kept in mind is the following one:

Theorem 6.6. *([18] lemma 13). For every polytope* P*,* $([P] - 1)^{n+1} = 0$*.*

This is a very important relation, in fact it permits us to compute $log([P]) = \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i}([P] - 1)^i$, and now $exp(log([P])) = [P]$, where the map exp is defined as the usual infinite sum, which in this case is finite again because of the previous theorem. Therefore using the expansion of exp it is clear, at least as vector spaces, the following result:

Theorem 6.7. $\Pi = \bigoplus_{r=0}^{n} \Xi_r$ *where* $\Xi_i =< \{(log([P]))^i : [P] \in \Pi\} >_{\mathbb{Q}}$ *has degree* i*.*

Definition 6.8. *Let* $[P]$ *be a polytope in* Π*. We define* $\Pi[P]$ *to be the subalgebra of* Π *generated by all classes* $[Q]$ *such that there exists a positive* c *and a polytope* $[R]$ *with* $P = cQ + R$*.*

Given a polytope P, we can consider its normal fan Σ_P, defined in this way:

Definition 6.9. *If $P = Conv(x_1, ..., x_r)$ and $P(\{1, ..., r\})$ is the set of subsets of $\{1, ..., r\}$ then $\Sigma_P = \{(< \{x_j - x_i\}_{j\in J} >_{\mathbb{R}^+})^\vee\}_{J\in P(\{1,...,r\})-\emptyset; i\in\{1,...,r\}}$.*

Therefore they determine a toric variety. Notice that the resulting toric variety is complete when the polytope has maximal dimension (theorem 4.9, again because in this case Σ_P is supported on the whole space).

Proposition 6.2. *Let $i : X \hookrightarrow \mathbb{P}^N$ be a projective toric variety with fan Σ, consider $i^*O_{\mathbb{P}^N}(1)(X)$. This is finite dimensional ([13] chapter 5) and if we write $i^*O_{\mathbb{P}^N}(1)(X) = \bigoplus_{\lambda\in P} k\lambda$ a decomposition of the global functions in T-eigenfunctions with $P \subseteq M$, then $conv(P) \subseteq N_\mathbb{R}$ is a polytope and $\Sigma_P = \Sigma$.*

P_D will be the polytope associated to the very ample line bundle $D := i^*O_{\mathbb{P}^N}(1)$.

If moreover we have a Cartier divisor which is globally generated, we can again consider its associated polytope Q, then Σ refines the fan Σ_Q. ([4] proposition 6.2.5)

In the example of the projective space in this chapter, the polytope associated to that action of T on $\mathbb{P}^n$ is the standard n-dimensional symplex.

A consequence of this theorem is that $H^0(i^*O_{\mathbb{P}^N}(1), X) = |P \cap M|$, and for every integer $m > 0$, $H^0(i^*O_{\mathbb{P}^N}(m), X) = |mP \cap M|$. Using Serre's vanishing theorem and the basic properties of the Euler polynomial $\mathbb{Z} \ni n \mapsto \chi i^*O_{\mathbb{P}^N}(n)$ ([13]) we get that $m^n deg(X)/n! = |mP \cap M| = |P \cap \frac{1}{m}M|$, thus $deg(X)/n! = \frac{1}{m^n}|P \cap \frac{1}{m}M| \xrightarrow{n\to\infty} vol(P)$ where vol is the volume form on $M_\mathbb{R}$ invariant under translation which has value 1 on the standard polytope of the lattice of M.

Form now on we will identify N with the standard sublattice $\mathbb{Z}^n \subseteq \mathbb{Q}^n$. Let me remind the following result:

Lemma 6.2. *([4] proposition 6.2.13). Let P, Q be two lattice polytopes in $\mathbb{Q}^n$. Then $[P] \in \Pi[Q] \iff$ the fan Σ_Q refines Σ_P.*

Then we are ready to state the main theorem of this chapter:

Theorem 6.8. *Let X be a projective toric variety, with a very ample line bundle with polytope P. There exists an injective homomorphism of graded $\mathbb{Q}$-algebras $\theta : \Pi[P] \to A^*(X)$ such that $\theta([P_D]) = \sum_{i=0}^{n} \frac{[D]^i}{i!}$ for every very ample line bundle $O_X(D)$. The image of θ equals the subalgebra of $A^*(X)$ generated by $A^1(X)$.*

Notice in particular that when X is smooth, the whole Chow ring is generated in degree 1 because it is generated by the classes of the orbit closures $[V(\tau)]$ and $V(\tau) = \bigcup_{\rho \text{ ray of } \tau} V(\rho)$ and the intersection is transversal because it is a local computations and we have theorem 6.3. Before starting this proof, we need to recall some results about weights, a reference is [19].

Definition 6.10. *Given a polytope $P \subseteq \mathbb{R}^n$, let $F_k(P)$ the set of k-dimensional faces of the polytope. A k-weight on P is a function $\omega : F_k(P) \to \mathbb{R}$ which satisfies the following equation:*

$$\sum_{F \subseteq G} \omega(F) v_{F,G} = 0 \text{ for all } G \in F_{k+1}(P)$$

where the sum is over all k-faces F of G, and $v_{F,G}$ is the unit outer normal vector (in the vector subspace parallel to G) at the face F. The real vector space of k-weights on P is denoted by $\Omega_k(P)$.

Example 6.2.

If we put an euclidean norm (and therefore a volume form on every subspace) on $\mathbb{R}^n$ and P has dimension n, the map which sends a face $F \mapsto vol_{n-1}(F)$ is a $n-1$-weight (divergence theorem).

Assume that P is the Minkowski sum of Q and R. Then we can associate to every face of P a face of Q (we will call it the corresponding face of Q). In fact the following result holds:

Proposition 6.3. *Let $P_1, ..., P_r$ be polytopes in $\mathbb{R}^d$ and let F be a face of the Minkowski sum $P_1 + ... + P_r$. Then there are faces F_i of P_i such that $F = F_1 + ... + F_r$. The faces F_i are uniquely determined by F.*

Definition 6.11. *With the same notations as before, we define a map $F_k(P) \to \mathbb{R}$, $F \mapsto vol_k(F')$ where F' is the face of Q corresponding to F. We call it $vol_k(Q)$.*

The main theorem which we will use is the following, a reference is [19]:

Theorem 6.9. *With the same notations as before, $vol_k(Q) \in \Omega_k(P)$; there is a well-defined map*

$$\phi : \Pi[P] \to \Omega(P) := \bigoplus_{k=0}^{n} \Omega_k(P),\ [Q] \to \oplus_k vol_k(Q)$$

which is injective.

Now we are ready to prove theorem 6.8:

Proof. **Part 1:**

Using theorem 2.3 and 2.1, we can think at an element $a \in A^k(X)$ as a function $c : \Sigma^{(k)} \to \mathbb{Z}$, where $\Sigma^{(k)}$ is the set of cones in Σ which have codimension k, such that it satisfies the following relation: for every cone $\tau \in \Sigma^{(k+1)}$ and every element $u \in M(\tau)$,

$$\sum_{\sigma \in \Sigma^{(k)} : \tau \subseteq \sigma} (u, n_{\sigma,\tau}) c(\sigma) = 0 \tag{6.3}$$

Using this identification we show that $\Omega_k(P)$ is canonically isomorphic to $A^k(X)_{\mathbb{R}}$. Let $\widetilde{M} := M \otimes \mathbb{R}$ and $\widetilde{N} := N \otimes \mathbb{R}$, and we identify both these spaces with $\mathbb{R}^n$ with the euclidean norm. When we have an euclidean norm, we can associate to it a volume form, which has volume 1 on the cube with sides of length 1. Therefore for every linear subspace we have a canonical volume form on it, the one associated to the restriction of the norm to this subspace. We will call such a volume form vol_k if it is on a k-dimesnional subspace.

For every cone $\sigma \in \Sigma$, let $\widetilde{M}(\sigma) := M(\sigma) \otimes \mathbb{R}$ with an orthonormal basis $\{m_1, ..., m_k\}$, and $\widetilde{N_\sigma} := N_\sigma \otimes \mathbb{R}$. We define

$$proj_\sigma : \widetilde{N}/\widetilde{N_\sigma} \to \widetilde{M}(\sigma),\ v \mapsto \sum_{i=1}^{k} (v, m_i) m_i$$

Notice that with this definition, if $< \cdot, \cdot >$ is the scalar product on $\widetilde{M}(\sigma)$, then $(n, m) =< proj_\sigma(n), m >$.

Part 2:

Let Vol_σ be a multiple of the standard volume form vol_k on $\widetilde{M}(\sigma)$ such that it has value 1 on every primitive simplex T in $M(\sigma)$, and let ν_σ be a constant such that $Vol_\sigma(\cdot) = \nu_\sigma k! vol_k(\cdot)$.

For every $F \in F_k(P)$, let σ_F be the normal cone to P at F. We claim now that the map

$$\psi_k : A^k_{\mathbb{R}} \to \Omega_k(P),\ \psi_k(c)(F) := c(\sigma_F)/\nu_{\sigma_F}$$

is a well-defined isomorphism.

We have to show that the function $\psi_k(c)$ is a weight function. The point is that $||proj_\tau(n_{\sigma,\tau})|| = \frac{\nu_\tau}{\nu_\sigma}$, where τ is a facet of σ. In fact, first notice that $\widetilde{M}(\sigma) \subseteq \widetilde{M}(\tau)$. Then we can decompose $\widetilde{M}(\tau)$ as $\widetilde{M}(\sigma) \oplus < v >_{\mathbb{R}}$ with v orthogonal to $\widetilde{M}(\sigma)$ and $||v|| = 1$. Then $proj_\tau(n_{\sigma,\tau}) = cv$ for a certain $c \in \mathbb{R}$ because for every $m \in M(\sigma)$, $< proj_\tau(n_{\sigma,\tau}), m >= (n_{\sigma,\tau}, m) = 0$. The absolute value of c will be the norm of $proj_\tau(n_{\sigma,\tau})$: we are interested in c.

Let v' be such that $M(\tau) = M(\sigma) \oplus \mathbb{Z}v'$, $v' = \alpha v + w$ with $w \in \widetilde{M}(\sigma)$, and if T_σ is the primitive simplex of $M(\sigma)$ and T_τ the one of $M(\tau)$, $vol_{k+1}(T_\tau) = \alpha \frac{vol_k(V_\sigma)}{k+1}$. Therefore we get:

$$1 = Vol_{k+1}(T_\tau) = (k+1)! \nu_\tau vol_{k+1}(T_\tau) = \alpha \frac{vol_k(V_\sigma)}{k+1}(k+1)! \nu_\tau =$$
$$\alpha k! vol_k(T_\sigma)\nu_\tau = \alpha \frac{\nu_\tau}{\nu_\sigma} \longrightarrow \alpha = \frac{\nu_\sigma}{\nu_\tau}$$

Now, $1 = (n_{\sigma,\tau}, v')$ because $n_{\sigma,\tau}$ generates the lattice N_σ/N_τ (thus we have either $1 = (n_{\sigma,\tau}, v')$ or $-1 = (n_{\sigma,\tau}, v')$) and $n_{\sigma,\tau}$ belongs to the image of σ (thus $1 = (n_{\sigma,\tau}, v')$). So finally: $1 = (n_{\sigma,\tau}, v') =< proj_\tau(n_{\sigma,\tau}), v' >= c\alpha$ thus $c = \frac{\nu_\tau}{\nu_\sigma}$. Then equation (6.3) translates into

$$0 = \sum_{\tau \subseteq \sigma} proj_\tau(n_{\sigma,\tau}) c(\sigma) = \frac{1}{\nu_\tau} \sum_{\tau \subseteq \sigma} proj_\tau(n_{\sigma,\tau}) c(\sigma) = \sum_{\tau \subseteq \sigma} \psi_k(c) \frac{proj_\tau(n_{\sigma,\tau})}{||proj_\tau(n_{\sigma,\tau})||}$$

But the latter condition is exactly the condition of being a k-weight: ψ_k is an isomorphism.

Part 3:

If we consider the composition $\theta := \psi^{-1} \circ \phi$ we get a graded injective homomorphism of vector spaces $\theta : \Pi[P] \to A^*(X)_{\mathbb{R}}$.

For every ample divisor D, the corresponding polytope P_D belongs to $\Pi[P_D]$ (prop 6.2), and the element $D^k \in A^k(X)$ is represented as the homomorphism $\sigma \mapsto Vol_\sigma(F)$ where $V(\sigma)$ has dimension k and F is the k-face of P_D polar to σ. In fact, if $X \hookrightarrow \mathbb{P}^m$ is the embedding associated to the ample sheaf $O_X(D)$, any face F of P of codimension k corresponds to a T-stable subvariety $V(\sigma)$ of X of dimension k. For such a subvariety we can consider the projective embedding $j : V(\sigma) \hookrightarrow X \hookrightarrow \mathbb{P}^m$, the result is that the polytope associated to this embedding is the face F. Thus $deg(D^k \cap [V(\sigma)]) = deg(V(\sigma)) = vol'(F)/k!$ where vol' is the volume which has value 1 on the polytope of the lattice $M(\sigma)$. Noting that $vol' = k!Vol_\sigma$ we are done.

Part 4:

Finally, we get

$$\psi_k(D^k) = k!vol_k(P_D) = \phi(log([P_D])^k) \Longrightarrow \theta([P_D]) = exp(D) \text{ and } \theta(log([P_D])^k) = D^k$$

Now we use prop 6.2: the Picard group of X is generated by ample line bundles, and $A^1(X)$ is the Picard group. Therefore θ is surjective in degree 1.

Moreover θ is a ring homomorphism:

$$\theta([P_D][P'_D]) = \theta([P_{D+D'}]) = exp(D + D') = exp(D)exp(D') \qquad (6.4)$$

where the first equality is true because if we consider the ebmedding given by $O_X(D) \otimes O_X(D')$, this is the composition of the maps $X \to \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ and the Segre embedding $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \to \mathbb{P}^{(n_1+1)(n_2+1)-1}$. Thus, recalling how the Segre embedding is defined, we get $O_X(D_1) \otimes O_X(D_2)(X) = \{fg : f \in O_X(D_1)$ and $g \in O_X(D_2)\}$: $P_{D_1+D_2} = P_{D_1} + P_{D_2}$.

Equation (6.4) implies the thesis. □

Thank to this description, we can compute the Halphen ring with rational coefficients of a torus:

Theorem 6.10. *The algebra* $\varinjlim A^*(X)$*, where the limit is over all the compactifications of* T*, is isomorphic to the polytope algebra.*

Proof. First notice that we can reduce ourselves to projective smooth toric varieties, because these are a cofinal family. Let $f : X_1 \to X_2$ be a morphism of projective smooth toric varieties, and let P_i be the polytope associated to X_i. Then we have the isomorphisms of the theorem $\theta^{(i)} : \Pi[P_i] \to A^*(X)$.

Using Lemma 6.2, we get that the polytope P_2 is a Minkowski summand of P_1, thus we have an inclusion of the polytope algebras $i : \Pi[P_2] \hookrightarrow \Pi[P_1]$. The point is that the following diagram commutes:

$$\begin{array}{ccc} \Pi[P_2] & \xrightarrow{\theta^{(2)}} & A^*(X_2) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f^*} \\ \Pi[P_1] & \xrightarrow[\theta^{(1)}]{} & A^*(X_1) \end{array}$$

In fact, let D be an ample divisor D with its associated sheaf $O_{X_2}(D)$ on X_2 and polytope $[P_D]$. Then the divisor $f^*O_{X_2}(D)$ is a globally generated sheaf on X_1 with the same associated polytope, thus $\theta^{(1)}(i(P_D)) = exp(f^*(D))$ by definition of θ, $exp(f^*D) = f^*exp(D)$ being f^* a ring homomorphism, and $f^*(exp(D)) = f^*(\theta^{(2)}([P_D]))$ which is the thesis because ample line bundles generate $A^1(X)$, and $A^1(X)$ generates $A^*(X)$ for X smooth.

Therefore the Halphen ring of a torus is isomorphic to the limit of the algebras $\Pi[P]$, where the fan of P gives us a smooth variety and P has maximal dimension. Using resolution of singularities and using proposition 6.2, we see that for every politope Q there is another polytope P whose fan gives us a smooth toric variety such that $Q \in \Pi[P]$, thus this limit is the whole polytope algebra, which is the thesis.

□

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