IN SUPREMÆ DIGNITATIS
1343

Università di Pisa

Dipartimento di Matematica

Corso di Laurea Magistrale in Matematica

Heights of multiprojective cycles and small value estimates in dimension two

Tesi di Laurea Magistrale

17 luglio 2015

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Anno accademico 2014/2015

Abstract

In the first part we recall the theory of multiprojective elimination initiated by P.Philippon and developed by G.Rémond. In particular, we define the eliminant ideal, the resultant forms and the Hilbert-Samuel polynomial for multigraded modules. We then look at subvarieties and cycles of a product of projective spaces, over a number field, and we define their mixed degrees and mixed heights, which measure respectively their geometric and arithmetic complexity. Finally, we define the heights of multiprojective cycles relative to some sets of polynomials, generalizing a previous notion of height due to M.Laurent and D.Roy, and we give detailed proofs for their properties. In the second part we prove that if we have a sequence of polynomials with bounded degrees and bounded integer coefficients taking small values at a pair (a,b) together with their first derivatives, then both a and b need to be algebraic. The main ingredients of the proof include a translation of the problem in multihomogeneous setting, an interpolation result, the construction of a 0-dimensional variety with small height, a result for the multiplicity of resultant forms, and a final descent. This work is motivated by an arithmetic statement equivalent to Schanuel's conjecture, due to D.Roy.

To my family

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Introduction

The aim of this thesis is to present some arithmetic and geometric invariants for subvarieties and cycles of a product of projective spaces $\mathbb{P}^{\mathbf{n}} := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$, and to use them to prove estimates for the norm and the degrees of polynomials taking small values at a point together with some derivatives of them.
One of the main motivations for this work is Schanuel's conjecture, which is one of the outstanding open problems in Transcendental Number Theory, and ultimately asserts that there are no unexpected algebraic relations pertaining the exponential function $\exp\colon \mathbb{C} \to \mathbb{C}^*$ [Mac91]. In an article of 2001 [Roy01] D.Roy proves that Schanuel's conjecture is equivalent to an arithmetic statement which considers polynomials in only two variables and finitely generated subgroups of the linear algebraic group $\mathcal{G} := \mathbb{C} \times \mathbb{C}^*$. This formulation is interesting because it is similar to the currently known criteria of algebraic independence [Phi86a][Jad96][LR01], and so it suggests a reasonable approach towards Schanuel's Conjecture.
The first steps in the direction of the understanding and a proof of this statement were taken by D.Roy and N.A.Nguyen [Roy10][Roy08][Roy13][VR14], embedding $\mathcal{G}$ into the projective space $\mathbb{P}^2$, applying an homogenization of the polynomials involved, and then implementing arguments and techniques of Diophantine Approximation. In their approach, the interplay between two different concepts of *heights* for $\mathbb{Q}$-subvarieties of the projective space $\mathbb{P}^n$ proved to be very important.
The first of these heights was introduced by P.Philippon [Phi86a] and it consists in a measure of the absolute arithmetic complexity of a $\mathbb{Q}$-subvariety Z of $\mathbb{P}^n$. The definition takes into account the *Chow forms* $F_{Z,\mathtt{d}}$ of Z, which are irreducible polynomial forms in many variables with integer coefficients that describe the set of $(r+1)$-uples of homogeneous polynomials $(P_0, \ldots, P_r)$ with fixed degrees which have a common zero on Z.
The second of these concepts of heights was introduced and studied by D.Roy and M.Laurent in [LR01], and it measures the relative arithmetic complexity of a $\mathbb{Q}$-subvariety Z of $\mathbb{P}^n$ *with respect to* a given set (*convex body*) $\mathcal{C}$ of homogeneous polynomials with fixed degrees. The definition is again given in terms of the Chow forms $F_{Z,\mathtt{d}}$, but instead of looking at their coefficients, it considers their evaluation at polynomials in $\mathcal{C}$.
We see that the process of homogenization of polynomials in two variables doesn't permit to obtain estimates taking into account both the degrees in the first and second variable of the polynomials, but only in dependence of the total degree. To tackle this problem, D.Roy suggests in [Roy13] to embed $\mathcal{G}$ into a product of projective spaces $\mathbb{P}^1 \times \mathbb{P}^1$ and then to use the more recent theory of multiprojective elimination, introduced by P.Philippon [Phi93] and developed by G.Rémond [Rém01a]. In this master thesis we're going to carry on this

project, transposing the results and arguments of D.Roy in multihomogeneous setting, obtaining estimates in which the degrees in the variables x and y occur separately.
So, to evaluate the absolute arithmetic complexity of a $\mathbb{Q}$-subvariety Z of a product of projective spaces $\mathbb{P}^{\mathbf{n}} := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$, we use the heights introduced by G.Rémond in place of the height of P.Philippon. These invariants are defined in terms of the resultant forms $\mathrm{res}_{\mathbb{d}}(Z)$ of Z instead of Chow forms. The most evident complication in this context is that there is no more a unique canonical choice of height, but there are many, depending on a choice of components of the product of the spaces. In addition to this, resultant forms are no more irreducible polynomial forms, and so multiplicity arguments are to be taken into account.
One of the main contribution of this work is the definition of a multihomogeneous counterpart for the relative heights introduced by M.Laurent and D.Roy. Using again the theory of multiprojective elimination of G.Rémond we define a notion of height which measures the arithmetic complexity of a $\mathbb{Q}$-subvariety Z of a product of projective spaces $\mathbb{P}^{\mathbf{n}}$, with respect to a given set $\mathcal{C}$ (convex body) of multihomogeneous polynomials of given multidegrees. The idea is simply to consider the evaluation of the resultant form $\mathrm{res}_{\mathbb{d}}(Z)$ at polynomials in $\mathcal{C}$. We also give a proof of the basic properties of this height, following the exposition given in [LR01] for the homogeneous case.
The second main contribution of this thesis is the proof of a lower bound for the multiplicity of the resultant forms at certain $(r+1)$-uples of multihomogeneous polynomials, which generalizes a previous result of D.Roy ([Roy13], Theorem 5.2) for the Macaulay resultant. Here the proof combines a decomposition lemma due to D.Roy, an explicit description of the resultant forms of $\mathbb{P}^{\mathbf{n}}$ due to G.Rémond, and a lemma for calculating lenghts of modules over DVR rings.
Finally, after we've collected these results, valid on a generic product of projective spaces, we specialize them to the case $\mathbb{P}^{\mathbf{n}} = \mathbb{P}^1 \times \mathbb{P}^1$, in order to prove a small value estimate that generalizes that of [Roy13]. More precisely, if we define the $\mathcal{G}$-invariant derivation $\mathcal{D}_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y} : \mathbb{C}[x,y] \to \mathbb{C}[x,y]$, we prove that if we have a sequence of polynomials $P_N(x,y) \in \mathbb{Z}[x,y]$ for which we have bounds on their coefficients, their x-degree and their y-degree, such that they take small values when evaluated at a pair $(\xi,\nu) \in \mathbb{C} \times \mathbb{C}^{\times}$, together with their first invariant derivatives, then the pair (ξ,ν) must be in $\overline{\mathbb{Q}} \times \overline{\mathbb{Q}}^{\times}$ and actually the polynomials P_N vanish in (ξ,ν) with high multiplicity, for sufficiently big N. For the proof we essentially follow the arguments in [Roy13], but we replace the tools valid in homogeneous setting with those coming from multihomogeneous elimination and multiprojective geometry.
This thesis is naturally divided into two chapters, which are structured as follows. We refer to the specific sections for the notations and the definitions.

The first chapter: preliminaries. Heights of multiprojective cycles.

In the first chapter we collect algebraic, geometric and arithmetic tools useful to address problems of Diophantine Approximation in multiprojective setting. The main references here are [Rém01a], [Rém01b], [DKS11] and [LR01].
In the first section we present the theory of Multihomogeneous Elimination developed by G.Rémond. Given a multihomogeneous ideal I of a multigraded polynomial ring $K[\mathbf{X}]$, we define the eliminant ideals $\mathfrak{E}_{\mathbb{d}}(I)$ and we give an

algebraic interpretation of them. We then recall the basic properties of UFD rings and DVR rings. We make a choice of the representatives for the irreducible elements of a UFD ring, modulo multiplication by units, to define the principal part of an ideal and the annihilant of a module. We then state the principality theorem for the eliminant ideals and we define the eliminant forms $\mathrm{elim}_{\mathbb{d}}(I)$ and the resultant forms $\mathrm{res}_{\mathbb{d}}(I)$ of a multihomogeneous ideal I of $K[\mathbf{X}]$. We conclude by giving some relevant examples of resultant forms.
In the second section we study the geometry of multiprojective spaces, i.e. products of projective spaces $\mathbb{P}^{\mathbf{n}} := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$. We define a Hilbert-Samuel polynomial for multigraded modules of finite type and we give some definitions useful for the study of multivariate polynomials. We introduce the concepts of subvarieties, divisors and K-cycles of a multiprojective space, we recall their relation with multihomogeneous ideals and we define an intersection product between K-cycles and divisors. We then use the Hilbert-Samuel polynomial to give the definition of mixed degrees for multiprojective cycles, and we give a geometric interpretation for these invariants. We extend the definition of resultant forms to effective cycles. We study their behaviour under index permutations, field extensions and intersections by hypersurfaces. We then give explicit formulas for the resultant forms of 0-dimensional cycles and for the degrees of arbitrary resultant forms in terms of the mixed degrees of the cycle.
In the third section we measure the arithmetic complexity of cycles defined over a number field through the definition of their heights. We start by recalling the theory of absolute values over a number field. We then define the concept of convex bodies and of adelic convex bodies, and we give relevant examples of them. We define the absolute height of an algebraic number and of a polynomial form as a sum of local contributions and then, given a multiprojective cycle Z, we use its resultant forms $\mathrm{res}_{\mathbb{d}}(Z)$ to give the definition of its mixed heights $h_{\underline{\mu}}(Z)$ and of its heights relative to convex bodies $h_{\mathcal{C}}(Z)$.
In the fourth section we prove some of the basic properties of the heights of multiprojective cycles. We give estimates for the heights relative to convex bodies having a particular shape and we compare between the heights relative to different convex bodies. We study the relation between mixed heights and heights relative to specific convex bodies and then we give an arithmetic version of Bézout's inequality which estimates the heights of the intersection product of a cycle with an hypersurface.

The second chapter: results. Small value estimates in dimension two.

In the second chapter we specialize to a bihomogeneous setting and we apply the tools developed in the first chapter to prove a small value estimate for polynomials in two variables. Here we follow closely the arguments in [Roy13]. The main difference between that article and our work is that we replace $\mathbb{P}^2$ and homogeneous polynomials with $\mathbb{P}^1 \times \mathbb{P}^1$ and bihomogeneous polynomials. The outcome is a more finer statement in the end.
In the first section we state the main theorem, we discuss about the motivation of the statement, and we prove a corollary along the lines of the main conjecture.
In the second section we study in details the arithmetic and the geometry of $\mathbb{Q}$-cycles of $\mathbb{P}^1 \times \mathbb{P}^1$. We consider separately the subvarieties having dimension 0, 1 and 2 and we study the effect of cutting a subvariety by a hypersurface.

We then estimate the height of a 0-dimensional Z relative to a convex body $\mathcal{C}$ in terms of the mixed heights of Z and of the evaluations $P(\alpha)$ of polynomials $P \in \mathcal{C}$ at normalized representatives α of the complex points of Z.
In the third section we study the ring of polynomials $\mathbb{C}[\mathbf{X}, \mathbf{Y}]$. We define the translation automorphisms τ_γ and the invariant derivation $\mathcal{D} = X_0 \frac{\partial}{\partial X_1} + Y_2 \frac{\partial}{\partial Y_2}$ on $\mathbb{C}[\mathbf{X}, \mathbf{Y}]$, and we give basic estimates for them. We state an interpolation result due to D.Roy and V.Nguyen, which improves on a previous result of Mahler, and we use it to estimate the length of a polynomial for which we know the values of enough invariant derivatives at a point. We then consider the set $I_D^{(\gamma,T)}$ of multihomogeneous polynomials of multidegree D vanishing at a point γ with multiplicity at least T. We show that if $D \leq N$ and D is sufficiently big, then every $Q \in I_N^{(\gamma,T)}$ can be written as a combination of polynomials $P_\nu \in I_D^{(\gamma,T)}$, and we prove estimates for such a decomposition. The proof takes the form of a long division algorithm.
In the fourth section we define and estimate the multiprojective distance between two points of $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. We also define the distance from a point of $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ to some analytic curve A_γ. We introduce the quantity $|I_D^{(\gamma,T)}|_{(\alpha,\beta)}$ and we use it to estimate from below and from above the distances of a point (α, β) from γ and from A_γ.
In the fifth section we prove a decomposition lemma for a generic multigraded ring $K[\mathbf{X}]$ and a lower bound for the multiplicity of the resultant forms of an arbitrary multiprojective space $\mathbb{P}^{\mathbf{n}} := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$. We then specialize the result to prove that the resultant form $\mathrm{res}_{(D,D,D)}(\mathbb{P}^1 \times \mathbb{P}^1)$ vanishes to order at least T at each triple (P, Q, R) of polynomials in $I_D^{(\gamma,T)}$.
In the sixth section we perform the construction of a 0-dimensional subvariety Z with small height $h_{\mathcal{C}}(Z)$ relative to a specific convex body $\mathcal{C}$. We first provide an estimate for $h_{\mathcal{C}}(\mathbb{P}^1 \times \mathbb{P}^1)$ and we then cut $\mathbb{P}^1 \times \mathbb{P}^1$ with appropriate hypersurfaces to get the desired 0-dimensional subvariety Z. Finally, we estimate the mixed degrees and the mixed heights of Z.
The seventh section is devoted to the proof of the main theorem.
We end the chapter with some comments about the results of this thesis and with indications for possible developments.

Chapter 1

Heights of multiprojective cycles

1.1 Tools of Multiprojective Elimination

Throughout this text we denote by $\mathbb{N}$ the set of nonnegative integers and by $\mathbb{N}_+$ the set of strictly positive integers. Also, rings are intended to be commutative and with 1. For every $1 \leq i \leq q$ we define $\mathbf{e}_i \in \mathbb{N}^q$ to be the standard basis vector $(0, \ldots, 1, \ldots, 0)$, with 1 in position i. We also set $\mathbf{1} := \mathbf{e}_1 + \ldots + \mathbf{e}_q$. Given $\alpha \in \mathbb{N}^q$ we define

$$|\alpha| := \alpha_1 + \ldots + \alpha_q.$$

For $\mathbf{n} = (n_1, \ldots, n_q) \in \mathbb{N}^q$ we define $n := n_1 + \ldots + n_q$ and

$$\mathbb{N}^{\mathbf{n}} := \mathbb{N}^{n_1} \times \cdots \times \mathbb{N}^{n_q}.$$

Given two nonnegative integers $m, r \in \mathbb{N}$ we define $\mathbb{N}^m_r := \{\alpha \in \mathbb{N}^m : |\alpha| = r\}$, and if $\mathbf{d} = (d_1, \ldots, d_q) \in \mathbb{N}^q$, we set

$$\mathbb{N}^{\mathbf{n}+\mathbf{1}}_{\mathbf{d}} := \prod_{1 \leq i \leq q} \mathbb{N}^{n_i+1}_{d_i}.$$

A ring R is *multigraded* (or $\mathbb{N}^q$*-graded*) if it admits a decomposition

$$R = \bigoplus_{\alpha \in \mathbb{N}^q} R_\alpha$$

such that $R_\alpha R_\beta \subseteq R_{\alpha+\beta}$ for every $\alpha, \beta \in \mathbb{N}^q$. We say that the elements of R_α are *multihomogeneous of multidegree* α and we write $\deg(x) := \alpha$ for every $x \in R_\alpha$. A *multigraded* R-module M is a module over a multigraded ring R which admits a decomposition $M = \bigoplus_{\alpha \in \mathbb{Z}^q} M_\alpha$ such that $R_\alpha M_\beta \subseteq M_{\alpha+\beta}$ for every $\alpha \in \mathbb{N}^q$, $\beta \in \mathbb{Z}^q$.

1.1.1 Basic definitions

Let K be any field, $q \in \mathbb{N}_+$, $\mathbf{n} = (n_1, \ldots, n_q) \in \mathbb{N}^q$ and $n := n_1 + \ldots + n_q$. For $1 \leq i \leq q$, let $\mathbf{X}^{(i)} = \{X^{(i)}_0, \ldots, X^{(i)}_{n_i}\}$ be a group of $n_i + 1$ variables, and set

$$\mathbf{X} = \{\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(q)}\}.$$

We consider the ring $K[\mathbf{X}]$ multigraded by declaring $\deg(X_j^{(i)}) = \mathbf{e}_i$ for every $1 \le i \le q$, $0 \le j \le n_i$. For $\mathbf{d} = (d_1, \ldots, d_q) \in \mathbb{N}^q$ we denote by $K[\mathbf{X}]_{\mathbf{d}}$ its part of multidegree $\mathbf{d}$ and by $\mathfrak{M}_{\mathbf{d}}$ the set of monic monomials of multidegree $\mathbf{d}$. If $\alpha \in \mathbb{N}_{\mathbf{d}}^{\mathbf{n+1}}$ we define the monomial

$$\mathbf{X}^\alpha := X_0^{(1)\alpha_{1,0}} \cdots X_{n_1}^{(1)\alpha_{1,n_1}} \cdots X_0^{(q)\alpha_{q,0}} \cdots X_{n_q}^{(q)\alpha_{q,n_q}} \in K[\mathbf{X}]_{\mathbf{d}},$$

so that $\mathfrak{M}_{\mathbf{d}} = \{\mathbf{X}^\alpha : \alpha \in \mathbb{N}_{\mathbf{d}}^{\mathbf{n+1}}\}$.
Let $r \in \mathbb{N}$ and let $\mathbb{d} = (\mathbf{d}^{(0)}, \ldots, \mathbf{d}^{(r)}) \in (\mathbb{N}^q \backslash \{\mathbf{0}\})^{r+1}$ be a collection of multidegrees. For $0 \le i \le r$, we introduce the group of variables $\mathbf{u}^{(i)} = \{u_{\mathfrak{m}}^{(i)} : \mathfrak{m} \in \mathfrak{M}_{\mathbf{d}^{(i)}}\}$ and we consider, for $0 \le i \le r$, the general polynomial U_i of multidegree $\mathbf{d}^{(i)}$ in the variables $\mathbf{X}$:

$$U_i := \sum_{\alpha \in \mathbb{N}_{\mathbf{d}^{(i)}}^{\mathbf{n+1}}} u_{\mathbf{X}^\alpha}^{(i)} \mathbf{X}^\alpha \in K[\mathbf{u}^{(i)}][\mathbf{X}].$$

Set $\mathbf{u} = (\mathbf{u}^{(0)}, \ldots, \mathbf{u}^{(r)})$. We consider the ring $K[\mathbf{u}]$ multigraded by $\deg(u_{\mathfrak{m}}^{(i)}) = \mathbf{e}'_i$ for $i = 0, \ldots, r$, where $\mathbf{e}'_i$ is the $(i+1)$-th standard basis element of $\mathbb{N}^{r+1}$.
Given $\mathbb{d} = (\mathbf{d}^{(0)}, \ldots, \mathbf{d}^{(r)}) \in (\mathbb{N}^q \backslash \{\mathbf{0}\})^{r+1}$, it will be useful to think of a polynomial form $F \in K[\mathbf{u}]$ as an element of $\mathrm{Sym}(K[\mathbf{X}]^*_{\mathbf{d}^{(0)}}) \otimes \cdots \otimes \mathrm{Sym}(K[\mathbf{X}]^*_{\mathbf{d}^{(r)}})$ or, more concretely, as a polynomial map

$$F : \; K[\mathbf{X}]_{\mathbf{d}^{(0)}} \times \cdots \times K[\mathbf{X}]_{\mathbf{d}^{(r)}} \longrightarrow K.$$

This is possible, since the variables $\mathbf{u}^{(i)}$ are in bijection with coordinate functions of the affine space $K[\mathbf{X}]_{\mathbf{d}^{(i)}} \cong \mathbb{A}^N$, with

$$N = \binom{d_1^{(i)} + n_1}{n_1} \cdots \binom{d_q^{(i)} + n_q}{n_q}.$$

Therefore, with a slight abuse of notation, given $F \in K[\mathbf{u}]$ and $P_0, \ldots, P_r \in K[\mathbf{X}]_{\mathbf{d}^{(0)}} \times \cdots \times K[\mathbf{X}]_{\mathbf{d}^{(r)}}$, we will write

$$F(P_0, \ldots, P_r) := F(\mathrm{coeff}(P_0), \ldots, \mathrm{coeff}(P_r)),$$

where $\mathrm{coeff}(P_i)$ denotes the collection of the coefficients of P_i, so that

$$P = \sum_{\alpha \in \mathbb{N}_{\mathbf{d}^{(i)}}^{\mathbf{n+1}}} \mathrm{coeff}(P)_{\mathbf{X}^\alpha} \mathbf{X}^\alpha,$$

or, in other words, $P_i(\mathbf{X}) = U_i(\mathrm{coeff}(P))(\mathbf{X})$.
If I is an ideal of $K[\mathbf{X}]$, we denote by $I[\mathbb{d}]$ the ideal of $K[\mathbf{u}][\mathbf{X}]$ generated by I and by the general polynomials $U_0, \ldots, U_r$. We also consider the $K[\mathbf{u}]$-module

$$\mathcal{M}_{\mathbb{d}}(I) := K[\mathbf{u}][\mathbf{X}]/I[\mathbb{d}].$$

This module inherits a multigraded structure from $K[\mathbf{X}]$. For $\mathbf{k} \in \mathbb{N}^q$, we denote by $\mathcal{M}_{\mathbb{d}}(I)_{\mathbf{k}}$ its part of multidegree $\mathbf{k}$ in the variables $\mathbf{X}$. Every such multihomogeneous part is a finitely generated multigraded $K[\mathbf{u}]$-module.
We are now ready to introduce one of the most important objects studied by the theory of Multihomogeneous Elimination.

Definition 1.1.1. Let I be an ideal of $K[\mathbf{X}]$ and $\mathbb{d} = (\mathbf{d}^{(0)}, \dots, \mathbf{d}^{(r)}) \in (\mathbb{N}^q \backslash \{\mathbf{0}\})^{r+1}$. We define the *eliminant ideal of index* $\mathbb{d}$ by

$$\mathfrak{E}_{\mathbb{d}}(I) := \{f \in K[\mathbf{u}] : \exists \mathbf{k} \in \mathbb{N}^q \quad f\, \mathfrak{M}_{\mathbf{k}} \subseteq I[\mathbb{d}]\}.$$

The following theorem ([Rém01a], Théorème 2.2) gives an interpretation of the eliminant ideal in terms of polynomial equations.

Theorem 1.1.2. Let $\rho : K[\mathbf{u}] \to K$ a morphism of rings (i.e. a specialization for the general coefficients). Then, for all multihomogeneous ideals I of $K[\mathbf{X}]$, the following conditions are equivalent:

1. $\rho(\mathfrak{E}_{\mathbb{d}}(I)) = 0$;
2. there exists a (finite) field extension L/K and a nontrivial zero of $\rho(I[\mathbb{d}])$ in $L^{n_1+1} \times \cdots \times L^{n_q+1}$, that is, there exists $\mathbf{x} \in (L^{n_1+1} \backslash \{\mathbf{0}\}) \times \cdots \times (L^{n_q+1} \backslash \{\mathbf{0}\})$ such that $f(\mathbf{x}) = 0$ for all $f \in I$ and $\rho(U_i)(\mathbf{x}) = 0$ for all $0 \leq i \leq r$.

The following are easy properties of the eliminant ideal.

Proposition 1.1.3. If I is an ideal of $K[\mathbf{X}]$, then $\mathfrak{E}_{\mathbb{d}}(I) = K[\mathbf{u}]$ if and only if $\mathfrak{M}_{\mathbf{1}} \subset \sqrt{I}$. If I is a prime ideal of $K[\mathbf{X}]$ and $\mathfrak{M}_{\mathbf{1}} \not\subset I$, then $\mathfrak{E}_{\mathbb{d}}(I)$ is a prime ideal of $K[\mathbf{u}]$.

1.1.2 The annihilant and the principal part in UFD rings

A ring A is a *unique factorization domain*, or *UFD ring*, if it is an integral domain and every element $a \in A$ can be written as a finite product of irreducible elements of A, uniquely up to order and multiplication by units. We also say that A is *factorial*. A ring A is a *discrete valuation ring*, or *DVR ring*, if it is a unique factorization domain with a unique irreducible element (up to multiplication by units).

Facts 1.1.4. The following are well-known properties of UFD and DVR rings.

1. A polynomial ring $A[X_1, \dots, X_n]$ over a UFD ring A is again UFD;
2. if A is a UFD ring and $\pi \in A$ is an irreducible element, then (π) is a prime ideal of A and the localization $A_{(\pi)}$ is a DVR ring;
3. if A is a UFD ring and $\mathfrak{p}$ is a minimal nonzero prime ideal of A, then $\mathfrak{p}$ is principal, generated by an irreducible element of A;
4. if A is a DVR ring and M is a finitely generated torsion A-module, then M has finite length;
5. an integral domain A is a DVR ring if and only if there is some discrete valuation $v : K \to \mathbb{Z} \cup \{\infty\}$ on the field of fractions $K := \mathrm{Frac}(A)$ of A such that $A = \{x \in K : v(x) \geq 0\}$.

It will be convenient to fix once and for all a set $\mathrm{Irr}(A)$ of representatives for irreducible elements of A modulo multiplication by invertible elements. We also require that the representative for nonzero units is the element 1. In the

special case $A = \mathbb{Q}[X_1, \ldots, X_n]$ we take a natural choice for those representatives, modulo multiplication by ± 1, given by the condition that the coefficients of every polynomial in $\mathrm{Irr}(\mathbb{Q}[X_1, \ldots, X_n])$ lie in $\mathbb{Z}$ and are coprime.
Given an ideal J of A, we say that an element $a \in A$ is a *generator of the principal part of* J if it is a *gcd* for all elements of J or, in other words,

- $a \,|b$ for all $b \in J$;
- if $c \,|b$ for all $b \in J$ then $a \,|c$.

A generator for the principal part of an ideal J is unique up to multiplication by units, so we can give the following definition

Definition 1.1.5. Let J be a nonzero ideal of A. We define $\mathrm{ppr}(J)$ to be the unique generator for the principal part of J that can be written in the form

$$\mathrm{ppr}(J) = \prod_{\pi \in \mathrm{Irr}(A)} \pi^{n_\pi}.$$

If $\langle \mathcal{S} \rangle$ is the ideal generated by a nonempty subset of A we let $\gcd(\mathcal{S}) := \mathrm{ppr}(\langle \mathcal{S} \rangle)$. In the same vein we denote

$$\sqrt{a} := \prod_{\substack{\pi \in \mathrm{Irr}(A) \\ \pi \,| a}} \pi$$

for $a \in A \backslash \{0\}$. We complete these definitions by $\mathrm{ppr}(\{0\}) := 0$ and $\sqrt{0} := 0$.

Given a finitely generated module over a UFD ring, we are now going to define an element of the ring that, in some precise sense, encodes local multiplicities for the annihilator of the module. Since it does not seem to have a name in the literature, I will call it an annihilant for the module. It will be an essential tool in the definition of resultant forms in section 1.1.3.

Definition 1.1.6. Let A be a UFD ring and M a finitely generated A-module. If $\mathrm{Ann}_A(M) \neq 0$, we define the *annihilant* of M by

$$\chi_A(M) := \prod_{\pi \in \mathrm{Irr}(A)} \pi^{\ell(M_{(\pi)})} \quad \in A\,,$$

where $\ell(M_{(\pi)})$ is the length of the $A_{(\pi)}$-module $M_{(\pi)}$. If $\mathrm{Ann}_A(M) = 0$ we set $\chi_A(M) := 0$.

Remark 1.1.7. The annihilant of M is well-defined because for every $\pi \in \mathrm{Irr}(A)$ the ring $A_{(\pi)}$ is a DVR and, if $\mathrm{Ann}_A(M) \neq 0$, $M_{(\pi)}$ is a finitely generated torsion module over $A_{(\pi)}$, so that $\ell(M_{(\pi)})$ is finite; moreover, one has

$$\begin{aligned} M_{(\pi)} \neq 0 &\iff \mathrm{Ann}_{A_{(\pi)}}(M_{(\pi)}) \neq A_{(\pi)} \\ &\iff (\mathrm{Ann}_A(M))_{(\pi)} \neq A_{(\pi)} \\ &\iff \mathrm{Ann}_A(M) \subseteq (\pi) \end{aligned}$$

and therefore in the above product there are only a finite number of factors different by 1, precisely those which correspond to the π dividing $\mathrm{ppr}(\mathrm{Ann}(M))$.

We end this section with a result that proves to be useful for calculating length over DVR rings (see [Rém01a], Lemme 3.1).

Lemma 1.1.8. Let A a DVR ring, M a finitely generated A-module and

$$A^t \xrightarrow{\phi} A^s \to M \to 0$$

a finite presentation of M with $t \geq s$. Then $\ell(M)$ is the minimum of the valuations of the $s \times s$ minors of the matrix of ϕ.

1.1.3 Eliminant and resultant forms

We use the notations of the previous sections and we fix a collection of multidegrees $\mathbb{d} \in (\mathbb{N}^q \setminus \{\mathbf{0}\})^{r+1}$, where $q \in \mathbb{N}_+$ and $r \in \mathbb{N}$. We recall that $A = K[\mathbf{u}]$ is a factorial ring and $\mathcal{M}_{\mathbb{d}}(I)_{\mathbf{k}}$ is a finitely generated $K[\mathbf{u}]$-module for every $\mathbf{k} \in \mathbb{N}^q$ and every ideal I of $K[\mathbf{X}]$.
If $\mathfrak{p}$ is a prime ideal of $K[\mathbf{X}]$ we denote by $\mathrm{ht}(\mathfrak{p})$ its (algebraic) height[(1)], or rank, defined as the supremum of the lengths of all chains of prime ideals contained in $\mathfrak{p}$. More generally, if I is an ideal of $K[\mathbf{X}]$ we define $\mathrm{ht}(I)$ to be the infimum of the heights of all prime ideals containing I.
We have the following

Lemma 1.1.9. Let I be a multihomogeneous ideal of $K[\mathbf{X}]$ with $\mathrm{ht}(I) \geq n - r$. Then there exists $\mathbf{k}_0 \in \mathbb{N}^q$ such that

$$\mathrm{Ann}_{K[\mathbf{u}]}(\mathcal{M}_{\mathbb{d}}(I)_{\mathbf{k}}) = \mathrm{Ann}_{K[\mathbf{u}]}(\mathcal{M}_{\mathbb{d}}(I)_{\mathbf{k}_0}) \ , \quad \chi_{K[\mathbf{u}]}(\mathcal{M}_{\mathbb{d}}(I)_{\mathbf{k}}) = \chi_{K[\mathbf{u}]}(\mathcal{M}_{\mathbb{d}}(I)_{\mathbf{k}_0})$$

for all $\mathbf{k} \geq \mathbf{k}_0$. For such a $\mathbf{k} \in \mathbb{N}^q$, we have $\mathfrak{E}_{\mathbb{d}}(I) = \mathrm{Ann}_{K[\mathbf{u}]}(\mathcal{M}_{\mathbb{d}}(I)_{\mathbf{k}})$.

Proof. We begin with the following easy observation

$$\begin{aligned} \mathrm{Ann}_{K[\mathbf{u}]}(\mathcal{M}_{\mathbb{d}}(I)_{\mathbf{k}}) &= \{f \in K[\mathbf{u}] : \ f.K[\mathbf{u}][\mathbf{X}]_{\mathbf{k}} \subseteq I[\mathbb{d}]\} \\ &= \{f \in K[\mathbf{u}] : \ f\,\mathfrak{M}_{\mathbf{k}} \subseteq I[\mathbb{d}]\}. \end{aligned}$$

Since $\mathfrak{E}_{\mathbb{d}}(I) = \bigcup_{\mathbf{k} \in \mathbb{N}^q} \{f \in K[\mathbf{u}] : \ f\,\mathfrak{M}_{\mathbf{k}} \subseteq I[\mathbb{d}]\}$ and the ring $K[\mathbf{u}]$ is noetherian, we have $\mathfrak{E}_{\mathbb{d}}(I) = \{f \in K[d] : \ f\,\mathfrak{M}_{\mathbf{k}} \subseteq I[\mathbb{d}]\}$ for all $\mathbf{k} \geq \mathbf{k}_0$. This plainly gives the first and the last assertion. The second statement is [Rém01a], Lemme 3.2 and Théorème 3.3. □

Definition 1.1.10. Let I be a multihomogeneous ideal of $K[\mathbf{X}]$ with height at least $n - r$ and $\mathbb{d} \in (\mathbb{N}^q \setminus \{\mathbf{0}\})^{r+1}$. We define

$$\mathrm{elim}_{\mathbb{d}}(I) := \mathrm{ppr}(\mathfrak{E}_{\mathbb{d}}(I)),$$

and we call any nonzero scalar multiple of it an *eliminant form of index* $\mathbb{d}$ for I. We also define

$$\mathrm{res}_{\mathbb{d}}(I) := \chi_{K[\mathbf{u}]}(\mathcal{M}_{\mathbb{d}}(I)_{\mathbf{k}}),$$

for every sufficiently large $\mathbf{k}$. We call any nonzero scalar multiple of it a *resultant form of index* $\mathbb{d}$ for I.

(1) Not to be confused with the arithmetic height we define in section 1.3.2

We observe that the remark 1.1.7 implies that for every multihomogeneous ideal I with $\mathrm{ht}(I) \geq n-r$ one has the formula

$$\sqrt{\mathrm{elim}_{\mathbb{d}}(I)} = \sqrt{\mathrm{res}_{\mathbb{d}}(I)}. \qquad (1.1.1)$$

If I is a prime ideal, Proposition 1.1.3 implies that $\mathrm{elim}_{\mathbb{d}}(I) \in \mathrm{Irr}(K[\mathbf{u}])$ and that there exists $m \in \mathbb{N}_+$ (uniquely determined if $\mathrm{res}_{\mathbb{d}}(\mathfrak{p}) \neq 1$) such that $\mathrm{res}_{\mathbb{d}}(I) = \mathrm{elim}_{\mathbb{d}}(I)^m$.
The relevance of the definition of an eliminant form is essentially resumed in the following principality theorem ([Rém01a], Corollary 2.15, [DKS11], Lemma 1.34). Given a subset $J \subseteq \{1,\dots,q\}$ we denote by $\mathbf{X}^J$ the group of variables $\{\mathbf{X}^{(j)} : j \in J\}$ and we set $n_J := \sum_{j\in J} n_j$. If $\mathbb{d} \in (\mathbb{N}^q \backslash \{\mathbf{0}\})^{r+1}$ we denote $r_J := \#\{i: \ 0 \leq i \leq r, d_j^{(i)} = 0 \text{ for all } j \notin J\} - 1$. Equivalently, $r_J + 1$ counts the number of indices i for which the general form U_i of multidegree $\mathbf{d}^{(i)}$ lies in $K[\mathbf{u}][\mathbf{X}^J]$.

Theorem 1.1.11. Let K be any field, $r \in \mathbb{N}$, I a multihomogeneous ideal of $K[\mathbf{X}]$ and $\mathbb{d} \in (\mathbb{N}^q \backslash \{\mathbf{0}\})^{r+1}$.
Then $\mathfrak{E}_{\mathbb{d}}(I)$ is a principal ideal if and only if $n_J - \mathrm{ht}(I \cap K[\mathbf{X}^J]) \geq r_J$ for every $J \subseteq \{1,\dots,q\}$. In this case $\mathfrak{E}_{\mathbb{d}}(I) = (\mathrm{elim}_{\mathbb{d}}(I))$. Otherwise, $\mathfrak{E}_{\mathbb{d}}(I)$ is not principal and $\mathrm{elim}_{\mathbb{d}}(I) = 1$.

Nevertheless, we shall see that resultant forms are more suitable than eliminant forms for applications to Multiprojective Geometry and Diophantine Approximation.
We end this section providing an explicit formula for the resultant forms of the zero ideal $I = (0)$ of $K[\mathbf{X}]$.

Lemma 1.1.12. Let $\mathbb{d} = (\mathbf{d}^{(0)},\dots,\mathbf{d}^{(n)}) \in (\mathbb{N}^q \backslash \{\mathbf{0}\})^{n+1}$ be a collection of nontrivial multidegrees. For every multidegree $\mathbf{k} = (k_1,\dots,k_q)$ such that $\mathbf{k} \geq \mathbf{d}^{(i)}$ for every $i = 0,\dots,n$, we define the $K[\mathbf{u}]$-linear map

$$\begin{array}{cccc} \phi_{\mathbf{k}}: & K[\mathbf{u}][\mathbf{X}]_{\mathbf{k}-\mathbf{d}^{(0)}} \times \dots \times K[\mathbf{u}][\mathbf{X}]_{\mathbf{k}-\mathbf{d}^{(n)}} & \to & K[\mathbf{u}][\mathbf{X}]_{\mathbf{k}} \\ & (\ A_0 \quad , \ \dots \ , \quad A_n\ ) & \mapsto & \sum_{l=1}^{n} U_l A_l \end{array} . \qquad (1.1.2)$$

Let $M_{\mathbf{k}}$ be a matrix that represents the map $\phi_{\mathbf{k}}$ with respect to some basis of its domain and its codomain (they are free $K[\mathbf{u}]$-modules). Then, there exists a multidegree $\mathbf{k}_0 \in \mathbb{N}^q$ with $\mathbf{k}_0 \geq \mathbf{d}^{(i)}$ for $i = 0\dots,n$ and an element $\zeta \in K^\times$ such that, for all $\mathbf{k} \geq \mathbf{k}_0$, we have

$$\mathrm{res}_{\mathbb{d}}((0)) = \zeta \ \gcd(\{\det(\Delta) : \Delta \in \mathfrak{M}\}),$$

where $\mathfrak{M}$ is the set of minors of $M_{\mathbf{k}}$ having maximum rank and where the gcd is taken into $K[\mathbf{u}]$, which is an UFD ring.

Proof. We see that $\times_{i=0}^{n} K[\mathbf{u}][\mathbf{X}]_{\mathbf{k}-\mathbf{d}^{(i)}} \cong K[\mathbf{u}]^{\alpha(\mathbf{k})}$ and $K[\mathbf{u}][\mathbf{X}]_{\mathbf{k}} \cong K[\mathbf{u}]^{\beta(\mathbf{k})}$, where $\alpha(\mathbf{k}), \beta(\mathbf{k})$ are polynomial expressions in the q numbers $k_1,\dots,k_q$, with coefficients depending on $\mathbb{d}$. If we denote by $T = (T_1,\dots,T_q)$ a group of q variables and by $\alpha(T)$ and $\beta(T)$ the polynomials corresponding to $\alpha(\mathbf{k})$ and $\beta(\mathbf{k})$, we see that $\beta(T)$ has degree n_i in the variable T_i, for $i = 1,\dots,q$, and that it has a unique monomial with maximum total degree, namely $LP(\beta(T)) = \frac{T_1^{n_1}\cdots T_q^{n_q}}{n_1!\cdots n_q!}$

(see also formula (1.2.2)). Moreover $\alpha(T)$ has degree $(n+1)n_i$ in the variable T_i, for $i = 1, \dots, q$, and has a leading monomial $LP(\alpha(T)) = LP(\beta(T))^{n+1}$. This implies that we can assume $\alpha(\mathbf{k}) \geq \beta(\mathbf{k})$, for $\mathbf{k}$ sufficiently big, say $\mathbf{k} \geq \mathbf{k}_1$. We also notice that the map $\phi_{\mathbf{k}}$ fits into a presentation for the $K[\mathbf{u}]$-module $N_{\mathbf{k}} := \mathcal{M}_{\mathbb{d}}((0))_{\mathbf{k}} = (K[\mathbf{u}][\mathbf{X}]/(0)[\mathbb{d}])_{\mathbf{k}}$:

$$K[\mathbf{u}]^{\alpha(\mathbf{k})} \xrightarrow{\phi_{\mathbf{k}}} K[\mathbf{u}]^{\beta(\mathbf{k})} \longrightarrow N_{\mathbf{k}} \to 0.$$

Given $\pi \in \mathrm{Irr}(K[\mathbf{u}])$, tensoring by $K[\mathbf{u}]_{(\pi)}$ preserves the exactness of the presentation

$$(K[\mathbf{u}])_{(\pi)}^{\alpha(\mathbf{k})} \to (K[\mathbf{u}])_{(\pi)}^{\beta(\mathbf{k})} \to (N_{\mathbf{k}})_{(\pi)} \to 0\,,$$

and $\mathrm{res}_{\mathbb{d}}((0))$ is calculated, for $\mathbf{k}$ sufficiently big, say $\mathbf{k} \geq \mathbf{k}_2$, from the lengths of the modules $(N_{\mathbf{k}})_{(\pi)}$ through the formula

$$\mathrm{res}_{\mathbb{d}}((0)) = \prod_{\pi \in \mathrm{Irr}(K[\mathbf{u}])} \pi^{\ell((N_{\mathbf{k}})_{(\pi)})}.$$

Since $K[\mathbf{u}]_{(\pi)}$ is a DVR ring, Lemma 1.1.8 applies:

$$\ell((N_{\mathbf{k}})_{(\pi)}) = \max\{e \in \mathbb{N} :\ \det(\Delta) \in (\pi)^e \quad \text{for all } \Delta \in \mathfrak{M}\}.$$

This proves the assertion with $\mathbf{k}_0 = \max\{\mathbf{k}_1, \mathbf{k}_2\}$. □

1.2 Multiprojective Geometry

1.2.1 The multihomogeneous Hilbert-Samuel polynomial

In [Rém01a] G.Rémond develops a theory of Hilbert-Samuel polynomials for multigraded modules, which extends the usual theory for graded modules. He use this tool to prove his theorems on the eliminant and resultant forms and to give the definition of mixed degrees, which are measures of the geometric complexity of multihomogeneous ideals. In what follows it will be convenient to denote by I_J the intersection $I \cap K[\mathbf{X}^{(J)}]$, for I an ideal of $K[\mathbf{X}]$ and $J \subseteq \{1, \dots, q\}$. We also consider a family of variables $T_1, \dots, T_q$, and $T^{(J)}$ will denote the sub-family $\{T_i :\ i \in J\}$.
If M is a multigraded $K[\mathbf{X}]$-module, we give the following definitions.

$$\begin{aligned} e(M) &:= \max_{Ann(M) \subset \mathfrak{p}, \mathfrak{M}_1 \not\subset \mathfrak{p}} (n - \mathrm{ht}(\mathfrak{p})), \\ e_J(M) &:= \max_{Ann(M) \subset \mathfrak{p}, \mathfrak{M}_1 \not\subset \mathfrak{p}} (n_J - \mathrm{ht}(\mathfrak{p}_J)). \end{aligned} \tag{1.2.1}$$

Definition 1.2.1. Let K be a field and M a multigraded finitely generated $K[\mathbf{X}]$-module. We define the *Hilbert-Samuel function* of M:

$$\begin{array}{cccc} \Psi_M : & \mathbb{Z}^q & \to & \mathbb{N} \\ & \mathbf{k} & \mapsto & \dim_K M_{\mathbf{k}}. \end{array}$$

The following theorem asserts that the Hilbert-Samuel function is eventually a polynomial function ([Rém01a], Théorème 2.10).

Theorem 1.2.2. Let M be a multigraded finitely generated $K[\mathbf{X}]$-module. Then there exists an unique polynomial $H_M \in \mathbb{Q}[T_1, \ldots, T_q]$ (called *the Hilbert-Samuel polynomial of* M) such that for some $\mathbf{k}_0 \in \mathbb{N}^q$ and for all $\mathbf{k} \geq \mathbf{k}_0$ we have $\Psi_M(\mathbf{k}) = H_M(\mathbf{k})$.

Moreover the polynomial H_M has the following properties.

1. H_M has total degree $e(M)$ and, more generally, has partial degree $e_J(M)$ in the variables $T^{(J)}$ for every $J \subseteq \{1, \ldots, q\}$;
2. The coefficients of the monomial of H_M having maximal total degree are positive.

1.2.2 The $*$-operator and operations on polynomials

Given a multihomogeneous ideal I of $K[\mathbf{X}]$ we are going to study the multihomogeneous Hilbert-Samuel polynomial for the multigraded $K[\mathbf{X}]$-module $M = K[\mathbf{X}]/I$. To this extent, we first give some general definitions concerning polynomials in many variables. We denote the collection of coefficients of a polynomial $P \in \mathbb{Q}[T_1, \ldots, T_q]$ by $\mathrm{coeff}(P) \in \mathbb{Q}^{\mathbb{N}^q}$, so that we can write

$$P(T_1, \ldots, T_q) = \sum_{\alpha \in \mathbb{N}^q} \mathrm{coeff}(P)_\alpha T^\alpha.$$

We define the *total degree* of P by $\mathrm{totdeg}(P) := \max\{|\alpha| : \mathrm{coeff}(P)_\alpha \neq 0\} \in \mathbb{N}$. Given this, we introduce the leading part of a polynomial, defined as the sum of the terms of higher degree

$$LP(P) := \sum_{|\alpha| = \mathrm{totdeg}(P)} \mathrm{coeff}(P)_\alpha T^\alpha. \tag{1.2.2}$$

We also define the collection $\widetilde{\mathrm{coeff}}(P) \in \mathbb{Q}^{\mathbb{N}^q}$ of *normalized coefficients* of P by

$$\widetilde{\mathrm{coeff}}(P)_\alpha := \frac{1}{\alpha!} \mathrm{coeff}(P)_\alpha,$$

where $\alpha! := \alpha_1! \cdots \alpha_q!$. We then denote the collection of normalized leading coefficients of P by

$$LC(P) := \widetilde{\mathrm{coeff}}(LP(P)).$$

For a polynomial $P \in \mathbb{Q}[T_1, \ldots, T_q]$ and a multi-rational $\mathbf{k} \in \mathbb{Q}^q$ we define another polynomial by

$$\Delta_{\mathbf{k}}(P)(T_1, \ldots, T_q) := P(T_1, \ldots, T_q) - P(T_1 - k_1, \ldots, T_q - k_q)$$

To study the coefficients of $\Delta_{\mathbf{k}}(P)$, we introduce the $*$-operator, following [Rém01a].

Definition 1.2.3. We define the $*$*-operator*, $* : \mathbb{Q}^{\mathbb{N}^q} \times \mathbb{Q}^q \to \mathbb{Q}^{\mathbb{N}^q}$, by the formula

$$(c * \mathbf{k})_\alpha := \sum_{i=1}^{q} k_i \cdot c_{\alpha + \mathbf{e}_i}.$$

Given $c \in \mathbb{Q}^{\mathbb{N}^q}$ and a collection of multi-rationals $(\mathbf{k}_1, \ldots, \mathbf{k}_r) \in (\mathbb{Q}^q)^r$ we will write $c * \mathbf{k}_1 * \cdots * \mathbf{k}_r$ for $(\ldots(c * \mathbf{k}_1) * \cdots) * \mathbf{k}_r$. We also notice that this operation is *commutative*, meaning that for every permutation σ of $\{1, \ldots, r\}$ we have $c * \mathbf{k}_1 * \cdots * \mathbf{k}_r = c * \mathbf{k}_{\sigma(1)} * \cdots * \mathbf{k}_{\sigma}(r)$. This is easy to prove as, for $\mathbf{a}, \mathbf{b} \in \mathbb{Q}^q$, we have

$$(c * \mathbf{a} * \mathbf{b})_\alpha = \sum_{i=1}^{q} b_i \cdot \sum_{j=1}^{q} a_j c_{\alpha+\mathbf{e}_i+\mathbf{e}_j} = \sum_{i,j=1}^{q} a_i b_j c_{\alpha+\mathbf{e}_i+\mathbf{e}_j} = (c * \mathbf{b} * \mathbf{a})_\alpha.$$

For this reason we allow us to abbreviate $c * \mathbf{k}_1 * \cdots * \mathbf{k}_r$ with $c *_{i=1}^{r} \mathbf{k}_i$. With a slight abuse of notation, in case we have $c \in \mathbb{Q}^{\mathbb{N}^q}$ for which $c_\alpha = 0$ for all $\alpha \neq (0, \ldots, 0)$, we consider it simply as a number $c \in \mathbb{Q}$. In other words, we identify $\mathbb{Q} \cong \mathbb{Q}^{\mathbb{N}_0^q} \hookrightarrow \mathbb{Q}^{\mathbb{N}^q}$. We thus also use the following notation.

Definition 1.2.4. Let $c \in \mathbb{Q}^{\mathbb{N}_r^q}$, and $\mathbb{d} = (\mathbf{d}^{(1)}, \ldots, \mathbf{d}^{(r)}) \in (\mathbb{Q}^q)^r$ a collection of multi-rationals. We denote

$$\langle c; \mathbb{d} \rangle := c * \mathbf{d}^{(1)} * \cdots * \mathbf{d}^{(r)} \quad \in \mathbb{Q}.$$

We will also consider a degenerate case of this definition: if $c \in \mathbb{Q} = \mathbb{Q}^{\mathbb{N}_0^q}$ we define $\langle c; \emptyset \rangle := c$.

We have the following easy proposition about the coefficients of $\Delta_{\mathbf{d}}(P)$.

Proposition 1.2.5. If $P \in \mathbb{Q}[T_1, \ldots, T_q]$ is a polynomial without negative coefficients in its leading part $LP(P)$ and if $\mathbf{d} = (d_1, \ldots, d_q) \in \mathbb{Q}^q$ is a multi-rational with $d_i > 0$ for all $i = 1, \ldots, q$, then $\operatorname{totdeg}(\Delta_{\mathbf{d}}(P)) = \operatorname{totdeg}(P) - 1$ and

$$LC(\Delta_{\mathbf{d}}(P)) = LC(P) * \mathbf{d}.$$

Proof. Let $t := \operatorname{totdeg}(P)$ and $c := \operatorname{coeff}(P)$, and write

$$P(T) = \sum_{|\alpha|=t} c_\alpha T^\alpha + \sum_{|\alpha|=t-1} c_\alpha T^\alpha + \cdots$$

so that

$$P(T - \mathbf{d}) = \sum_{|\alpha|=t} c_\alpha T^\alpha - \sum_{|\alpha|=t} \sum_{i=1}^{q} d_i \alpha_i c_\alpha T^{\alpha-\mathbf{e}_i} + \sum_{|\alpha|=t-1} c_\alpha T^\alpha + \cdots$$

and therefore

$$\Delta_{\mathbf{d}}(P) = \sum_{|\alpha|=t-1} \sum_{i=1}^{q} d_i \alpha_i c_{\alpha+\mathbf{e}_i} T^\alpha + \cdots,$$

where the monomials that are not shown in the above formulas all have total degree $\leq t-2$. The proposition is then proved because the positivity assumptions avoid cancellation in total degree $t - 1$. $\square$

Actually, one can give more general formulas for the $*$-operator taking into consideration generating functions (see for example the Chow class of a cycle and the corresponding Bézout formula in [DKS11]).

1.2.3 Multiprojective cycles

Up to now we have considered ideals of $K[\mathbf{X}]$. We recall now the link with closed subschemes of $\mathbb{P}^{\mathbf{n}}_K := \mathbb{P}^{n_1}_K \times \cdots \times \mathbb{P}^{n_q}_K$.

Definition 1.2.6. If I is an ideal of $K[\mathbf{X}]$ we define its *multisaturation* to be the ideal

$$\bar{I} := \{f \in K[\mathbf{X}] : \exists k \in \mathbb{N}^q \quad f\mathfrak{M}_k \subseteq I\}.$$

An ideal I is said *multisatured* if $I = \bar{I}$.

We introduce, for every $j \in \{0, \dots, n_1\} \times \{0, \dots, n_q\}$, an indeterminate Z_j, and we denote by $\mathbf{Z}$ the collection of all the Z_j. We consider the map of K-algebras

$$\begin{array}{rccc} \theta : & K[\mathbf{Z}] & \to & K[\mathbf{X}] \\ & Z_j & \mapsto & X^{(1)}_{j_1} \cdots X^{(q)}_{j_q}. \end{array}$$

The kernel of θ is generated(2) by the polynomials of the form $Z_j Z_k - Z_l Z_m$ such that $\{j_i, k_i\} = \{l_i, m_i\}$ for all $i = 1, \dots, q$, and we know (see [Har77], II, Ex.5.12 for the case $q = 2$) that there is a canonical isomorphism

$$\operatorname{Proj}(K[Z]/\operatorname{Ker}(\theta)) \cong \mathbb{P}^{n_1}_K \times \cdots \times \mathbb{P}^{n_q}_K.$$

If I is a multihomogeneous ideal of $K[\mathbf{X}]$, it is clear that $\theta^{-1}(I)$ is a homogeneous ideal of $K[Z]$ (graded by $\deg(Z_i) = 1$). The latter gives rise to a multihomogeneous ideal of $K[Z]/\operatorname{Ker}(\theta)$ and so to a closed subscheme of $\operatorname{Proj}(K[Z]/\operatorname{Ker}(\theta))$. Finally, the above isomorphism permits to attach to a multihomogeneous ideal I a closed subscheme $\mathcal{Z}(I)$ of $\mathbb{P}^{\mathbf{n}}_K$. The relevance of this procedure is given by the following ([Rém01a], 2.17)

Proposition 1.2.7. The application $I \mapsto \mathcal{Z}(I)$ described above induces a decreasing bijection between multisatured multihomogeneous ideals of $K[\mathbf{X}]$ and closed subschemes of $\mathbb{P}^{\mathbf{n}}_K$. The adjective *decreasing* means that if $I \subseteq J$ the closed immersion $\mathcal{Z}(J) \hookrightarrow \mathbb{P}^{\mathbf{n}}_K$ factorizes as $\mathcal{Z}(J) \hookrightarrow \mathcal{Z}(I) \hookrightarrow \mathbb{P}^{\mathbf{n}}_K$. We denote the inverse of $I \mapsto \mathcal{Z}(I)$ by $Z \mapsto \mathcal{I}(Z)$.

The only multisatured ideal containing $\mathfrak{M}_{\mathbf{1}}$ is $K[\mathbf{X}]$ itself. A multihomogeneous prime ideal of $K[\mathbf{X}]$ that does not contain $\mathfrak{M}_{\mathbf{1}}$ is multisatured. The reduced closed subschemes of $\mathbb{P}^{\mathbf{n}}_K$, alternatively called *subvarieties*, correspond to radical multihomogeneous ideals of $K[\mathbf{X}]$ not containing $\mathfrak{M}_{\mathbf{1}}$. The integral closed subschemes of $\mathbb{P}^{\mathbf{n}}_K$ correspond to multihomogeneous prime ideals not containing $\mathfrak{M}_{\mathbf{1}}$ and will be alternatively called *irreducible subvarieties.* A closed subscheme Z of $\mathbb{P}^{\mathbf{n}}_K$ have an underlying noetherian (Zariski) topological space and so, as it is customary, one can define its dimension $\dim(Z)$ using chains of irreducible subvarieties. The dimension of a nonempty closed subscheme of $\mathbb{P}^{\mathbf{n}}_K$ is a nonnegative integer bounded by $n = \dim(\mathbb{P}^{\mathbf{n}}_K)$. We observe that for a multisatured ideal we have $\dim(\mathcal{Z}(I)) = e(K[\mathbf{X}]/I)$; in fact the first number is obtained, by proposition 1.2.7, considering the length of chains of multihomogeneous prime ideals; the second is calculated with chains of any prime ideals; the equality is obtained combining [Rém01a], Lemme 2.5 and the fact that $K[\mathbf{X}]$ is a catenary ring. Analogously, $e_J(K[\mathbf{X}]/I)$ is interpreted as the dimension of the projection of $\mathcal{Z}(I)$ over the product of the factors $\mathbb{P}^{n_i}_K, i \in J$.
We now introduce the concept of *cycle* following [DKS11].

(2) About this fact, there is a mistake in [Rém01a].

Definition 1.2.8. A *K-cycle* of $\mathbb{P}_K^{\mathbf{n}}$ is a finite $\mathbb{Z}$-linear combination

$$Z = \sum_V m_V V \tag{1.2.3}$$

of irreducible subvarieties of $\mathbb{P}_K^{\mathbf{n}}$. The subvarieties V such that $m_V \neq 0$ are the *irreducible components of* X. The dimension of a cycle is the maximum dimension of its irreducible components. A K-cycle is *pure dimensional* or *equidimensional* if its components have all the same dimension. It is *effective* (respectively, *reduced*) if it can be written as in (1.2.3) with $m_V \geq 0$ (respectively, $m_V \in \{0, 1\}$).

For $0 \leq r \leq n$, we denote by $Z_r(\mathbb{P}_K^{\mathbf{n}})$ the group of K-cycles of $\mathbb{P}_K^{\mathbf{n}}$ of pure dimension r and by $Z_r^+(\mathbb{P}_K^{\mathbf{n}})$ the semigroup of those which are effective. We also set $Z_{\leq s}(\mathbb{P}_K^{\mathbf{n}}) := \bigoplus_{r=0}^{s} Z_r(\mathbb{P}_K^{\mathbf{n}})$, $Z_{\leq s}^+(\mathbb{P}_K^{\mathbf{n}}) := \bigoplus_{r=0}^{s} Z_r^+(\mathbb{P}_K^{\mathbf{n}})$, $Z(\mathbb{P}_K^{\mathbf{n}}) := Z_{\leq n}(\mathbb{P}_K^{\mathbf{n}})$ and $Z^+(\mathbb{P}_K^{\mathbf{n}}) := Z_{\leq n}^+(\mathbb{P}_K^{\mathbf{n}})$. We define the group of divisors[(3)] $\mathrm{Div}(\mathbb{P}_K^{\mathbf{n}}) := Z_{n-1}(\mathbb{P}_K^{\mathbf{n}})$ and the semigroup of effective divisors $\mathrm{Div}^+(\mathbb{P}_K^{\mathbf{n}}) := Z_{n-1}^+(\mathbb{P}_K^{\mathbf{n}})$.
For every multihomogeneous ideal I of $K[\mathbf{X}]$ we associate the effective K-cycle

$$Z(I) := \sum_{\mathfrak{p}} m_{\mathfrak{p}} \mathcal{Z}(\mathfrak{p}),$$

where the sum ranges over all minimal primes $\mathfrak{p} \supseteq I$ and $m_{\mathfrak{p}} = \ell((K[\mathbf{X}]/I)_{\mathfrak{p}})$.
If $f \in K[\mathbf{X}]$ is a nonzero multihomogeneous polynomial, we get from the well-known Krull's hauptidealsatz that every minimal prime $\mathfrak{p}$ containing f has $\mathrm{ht}(\mathfrak{p}) = 1$. Since $K[\mathbf{X}]$ is factorial, we also observe that every such $\mathfrak{p}$ is principal, generated by a multihomogeneous element of $\mathrm{Irr}(K[\mathbf{X}])$. Then, for every multihomogeneous $f \in K[\mathbf{X}]$ we define

$$\mathrm{div}(f) := Z((f)) \in \mathrm{Div}^+(\mathbb{P}_K^{\mathbf{n}}).$$

We say that a hypersurface $\mathrm{div}(f)$ *intersects properly* an irreducible subvariety V of $\mathbb{P}_K^{\mathbf{n}}$ if f is not contained in the prime ideal $\mathcal{I}(V)$. We say that a divisor D intersects properly a K-cycle Z if every irreducible component of D (which is a hypersurface) intersects properly all the irreducible components of Z.

Definition 1.2.9. Let $f \in K[\mathbf{X}]$ be a multihomogeneous polynomial and V an irreducible subvariety of $\mathbb{P}_K^{\mathbf{n}}$. Assume that $\mathrm{div}(f)$ intersects properly V. Then we define the *intersection product* of $\mathrm{div}(f)$ and V by

$$V \cdot \mathrm{div}(f) := Z((\mathcal{I}(V), f)).$$

It is an effective cycle of pure dimension $\dim(V) - 1$. By linearity, the intersection product extends to a pairing

$$Z_r(\mathbb{P}_K^{\mathbf{n}}) \times \mathrm{Div}(\mathbb{P}_K^{\mathbf{n}}) \dashrightarrow Z_{r-1}(\mathbb{P}_K^{\mathbf{n}}) \quad , \quad (Z, D) \mapsto Z \cdot D,$$

well-defined whenever Z and D intersect properly.

(3) In this setting there is no distinction between Weil and Cartier divisors.

1.2.4 Mixed degrees

If I is a multihomogeneous ideal of $K[\mathbf{X}]$, we see that $K[\mathbf{X}]/I$ is a finitely generated multigraded $K[\mathbf{X}]$-module. Therefore, thanks to theorem 1.2.2, we can consider the polynomial $H_{K[\mathbf{X}]/I} \in \mathbb{Q}[T_1, \ldots, T_q]$, with total degree $e(K[\mathbf{X}]/I)$. One can prove (see [Rém01a]) that its normalized leading coefficients are non-negative integers. Moreover, if I is a multihomogeneous multisatured ideal not containing $\mathfrak{M}_{\mathbf{1}}$, we have that $e(K[\mathbf{X}]/I) = \mathrm{ht}(I) = \dim(\mathcal{Z}(I))$. We can therefore give the following definitions.

Definition 1.2.10. Given a multihomogeneous multisatured ideal I of $K[\mathbf{X}]$ we define the collection of *mixed degrees* of I by

$$\deg(I) := \widetilde{\mathrm{coeff}}(LP(H_{K[\mathbf{X}]/I})) \quad \in \mathbb{N}^{\mathbb{N}^q_{e(K[\mathbf{X}]/I)}}.$$

If V is an irreducible subvariety of $\mathbb{P}^{\mathbf{n}}_K$, we define

$$\deg(V) := \deg(\mathcal{I}(V)) \quad \in \mathbb{N}^{\mathbb{N}^q_{\dim(V)}}$$

If $Z = \sum_V m_V V$ is a K-cycle of $\mathbb{P}^{\mathbf{n}}_K$ we define

$$\deg(Z) := \sum_{\dim(V)=\dim(Z)} m_V \deg(V) \quad \in \mathbb{N}^{\mathbb{N}^q_{\dim(Z)}}.$$

These definitions agree on the geometric and algebraic sides, meaning that $\deg(I) = \deg(Z(I))$ for every multihomogeneous multisatured ideal I of $K[\mathbf{X}]$. This is easily seen using the following decomposition formula (see [Rém01a])

$$\deg(I) = \sum_{\substack{\mathfrak{p} \supseteq I \\ \mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(I)}} \ell(K[\mathbf{X}]_{\mathfrak{p}}/I_{\mathfrak{p}}) \deg(\mathfrak{p}).$$

Mixed degrees behave well with respect to intersection product, because they satisfy the following version of *Bézout*'s theorem ([DKS11], Theorem 1.11, [Rém01a], Lemme 2.11, [Rém01b], Théorème 3.4).

Theorem 1.2.11. Let Z a K-cycle, $\mathbf{d} \in \mathbb{N}^q \backslash \{\mathbf{0}\}$ and $f \in K[\mathbf{X}]_{\mathbf{d}}$ a multihomogeneous polynomial of multidegree $\mathbf{d}$ such that Z and $\mathrm{div}(f)$ intersect properly. Then

$$\deg(Z \cdot \mathrm{div}(f)) = \deg(Z) * \mathbf{d}.$$

In case $Z \in Z_0(\mathbb{P}^{\mathbf{n}}_K)$ is a 0-dimensional cycle, we can give a nice interpretation of its degree by passing to an algebraic closure $\bar{K}$ of K.

Definition 1.2.12. If V is an irreducible subvariety of $\mathbb{P}^{\mathbf{n}}_K$ and L/K is an extension of fields, we define the *scalar extension* of V by L as the L-cycle

$$V_L := Z(\mathcal{I}(V) \otimes_K L).$$

This notion extends to K-cycles by linearity and induces an inclusion $Z_r(\mathbb{P}^n_K) \hookrightarrow Z_r(\mathbb{P}^{\mathbf{n}}_L)$. If Z is an effective K-cycle, then Z_L is also effective.

It is easy to see that 0-dimensional irreducible subvarieties of $\mathbb{P}^{\mathbf{n}}_{\bar{K}}$ correspond to points in $\mathbb{P}^{\mathbf{n}}(\bar{K}) = \mathbb{P}^{n_1}(\bar{K}) \times \cdots \times \mathbb{P}^{n_q}(\bar{K})$ (i.e. points in $(\bar{K}^{n_1+1}\backslash\{\mathbf{0}\}) \times \cdots \times (\bar{K}^{n_q+1}\backslash\{\mathbf{0}\})$ modulo multiplication by scalars on each factor). Therefore, for a general 0-dimensional K-cycle Z, we write $Z_{\bar{K}} = \sum_{\boldsymbol{\xi}} m_{\boldsymbol{\xi}} \boldsymbol{\xi}$ for some points $\boldsymbol{\xi} \in \mathbb{P}^{\mathbf{n}}(\bar{K})$ and $m_{\boldsymbol{\xi}} \in \mathbb{Z}$. Then $\deg(Z) \in \mathbb{N}^{\mathbb{N}_0^q}$ and if we identify $\mathbb{N}^{\mathbb{N}_0^q} \cong \mathbb{N}$ and write $\deg(Z) = \deg(Z)_{\mathbf{0}}$, we have ([DKS11], Proposition 1.10 (3))

$$\deg(Z) = Z_{\bar{K}} = \sum_{\boldsymbol{\xi}} m_{\boldsymbol{\xi}} \boldsymbol{\xi} = \sum_{\boldsymbol{\xi}} m_{\boldsymbol{\xi}}. \tag{1.2.4}$$

In general, we see that mixed degrees can be also defined geometrically ([DKS11], Corollary 1.14).

Proposition 1.2.13. Let $Z \in Z_r(\mathbb{P}^{\mathbf{n}}_K)$ and $\beta \in \mathbb{N}^q_r$. For $1 \leq i \leq q$ and $0 \leq j \leq \beta_i$ we denote by $H_{i,j}$ the inverse image with respect to the projection $\mathbb{P}^{\mathbf{n}}_K \to \mathbb{P}^{n_i}_K$ of a generic hyperplane of $\mathbb{P}^{n_i}_K$ (i.e. $H_{i,j} = \mathrm{div}(\ell_{i,j})$ for a generic $\ell_{i,j} \in K[\mathbf{X}]_{\mathbf{e}_i}$). Then

$$\deg(Z)_\beta = \deg\left(Z \cdot \prod_{i=1}^{q} \prod_{j=1}^{\beta_i} H_{i,j} \right).$$

1.2.5 Properties of the resultant forms

Definition 1.2.14. If $\mathbb{d} = (\mathbf{d}^{(0)}, \ldots, \mathbf{d}^{(r)}) \in (\mathbb{N}^q\backslash\{\mathbf{0}\})^{r+1}$ and $Z = \sum_V m_V V \in Z^+_{\leq r}(\mathbb{P}^{\mathbf{n}}_K)$ is an effective K-cycle of dimension at most r, we define

$$\mathrm{res}_{\mathbb{d}}(Z) := \prod_V \mathrm{res}_{\mathbb{d}}(\mathcal{I}(V))^{m_V},$$

and we call any nonzero scalar multiple of $\mathrm{res}_{\mathbb{d}}(Z)$ a *resultant form* for Z of index $\mathbb{d}$.

We remark that $\mathrm{res}_{\mathbb{d}}(\mathcal{I}(V)) = 1$ if $\dim(V) < r$. This follows from Theorem 1.1.11, formula (1.1.1) and the equality $\mathrm{ht}(\mathcal{I}(V)) = n - \dim(V)$. We now collect some of the basic and more relevant properties of resultant forms. For details and proofs, we address the reader to [Rém01a] and [DKS11].
The definitions of resultant forms agree on the geometric and algebraic sides ([Rém01a], Théorème 3.3).

Proposition 1.2.15. If I is a multihomogeneous ideal of $K[\mathbf{X}]$, then we have the following decomposition formula

$$\mathrm{res}_{\mathbb{d}}(I) = \prod_{\mathfrak{p} \supseteq I} \mathrm{res}_{\mathbb{d}}(\mathfrak{p})^{\ell(K[\mathbf{X}]_{\mathfrak{p}}/I_{\mathfrak{p}})}.$$

In particular, we have $\mathrm{res}_{\mathbb{d}}(I) = \mathrm{res}_{\mathbb{d}}(Z(I))$ for every multihomogeneous multisatured ideal I of $K[\mathbf{X}]$.

Resultant forms are invariant under index permutations ([DKS11], Proposition 1.27).

Proposition 1.2.16. Let $Z \in Z_r^+(\mathbb{P}_K^{\mathbf{n}})$, $\mathbb{d} = (\mathbf{d}^{(0)}, \ldots, \mathbf{d}^{(r)}) \in (\mathbb{N}^q \backslash \{\mathbf{0}\})^{r+1}$ and $\mathbf{u} = (\mathbf{u}^{(0)}, \ldots, \mathbf{u}^{(r)})$ the group of variables corresponding to $\mathbb{d}$. Let σ be a permutation of the set $\{0, \ldots, r\}$ and write $\sigma\mathbb{d} = (\mathbf{d}^{(\sigma(0))}, \ldots, \mathbf{d}^{(\sigma(r))})$ and $\sigma\mathbf{u} = (\mathbf{u}^{(\sigma(0))}, \ldots, \mathbf{u}^{(\sigma(r))})$. Then

$$\mathrm{res}_{\sigma\mathbb{d}}(Z)(\sigma\mathbf{u}) = \mathrm{res}_{\mathbb{d}}(Z)(\mathbf{u}).$$

Resultant forms are invariant under field extensions ([DKS11], Proposition 1.28).

Proposition 1.2.17. Let $Z \in Z_r^+(\mathbb{P}_K^{\mathbf{n}})$, $\mathbb{d} = (\mathbf{d}^{(0)}, \ldots, \mathbf{d}^{(r)}) \in (\mathbb{N}^q \backslash \{\mathbf{0}\})^{r+1}$ and L/K a field extension. Then there exists $\lambda \in L^\times$ such that

$$\mathrm{res}_{\mathbb{d}}(Z_L) = \lambda\, \mathrm{res}_{\mathbb{d}}(Z),$$

where Z_L is the scalar extension of Z by L (see Definition 1.2.12).

The resultant forms of effective pure dimensional K-cycles are multihomogeneous polynomial forms and their partial degrees can be expressed in terms of mixed degrees ([Rém01a], Théorème 3.4).

Proposition 1.2.18. Let $Z \in Z_r^+(\mathbb{P}_K^{\mathbf{n}})$ and $\mathbb{d} = (\mathbf{d}^{(0)}, \ldots, \mathbf{d}^{(r)}) \in (\mathbb{N}^q \backslash \{\mathbf{0}\})^{r+1}$. Then $\mathrm{res}_{\mathbb{d}}(Z)$ is a multihomogeneous element of $K[\mathbf{u}]$ and its degree in the group of variables $\mathbf{u}^{(i)}$ is

$$\deg(Z) *_{\substack{0 \le j \le r \\ j \neq i}} \mathbf{d}^{(j)}.$$

Resultant forms behave well with respect to the intersection product of a cycle with an hypersurface. In particular, it transforms it in an evaluation ([Rém01a], Proposition 3.6).

Proposition 1.2.19. Let $Z \in Z_r^+(\mathbb{P}_K^{\mathbf{n}})$, $\mathbb{d} = (\mathbf{d}^{(0)}, \ldots, \mathbf{d}^{(r)}) \in (\mathbb{N}^q \backslash \{\mathbf{0}\})^{r+1}$, $\mathbb{d}' = (\mathbf{d}^{(0)}, \ldots, \mathbf{d}^{(r-1)})$, $\mathbf{u}' = (\mathbf{u}^{(0)}, \ldots, \mathbf{u}^{(r-1)})$ the group of variables corresponding to $\mathbb{d}'$ and $f \in K[\mathbf{X}]_{\mathbf{d}^{(r)}}$ such that Z and $\mathrm{div}(f)$ intersect properly. Then there exists $\lambda \in K^\times$ such that

$$\mathrm{res}_{\mathbb{d}'}(Z \cdot \mathrm{div}(f))(\mathbf{u}') = \lambda\, \mathrm{res}_{\mathbb{d}}(Z)(\mathbf{u}', \mathrm{coeff}(f)).$$

Remark 1.2.20. If Z and $\mathrm{div}(f)$ doesn't intersect properly, we see that there is an irreducible component V of Z such that $f \in \mathcal{I}(V)$. Since $Z \in Z_r^+(\mathbb{P}_K^{\mathbf{n}})$ is a pure-dimensional cycle, we have $\dim(V) = r$. Therefore, for every choice of multihomogeneous polynomials $p_i \in K[\mathbf{X}]_{\mathbf{d}^{(i)}}$ for $i = 0, \ldots, r-1$, it is easy to see that there is a nontrivial common zero of $\mathcal{I}(V)$ and $p_0, \ldots, p_{r-1}$ in $\bar{K}^{n_1+1} \times \cdots \times \bar{K}^{n_q+1}$, where $\bar{K}$ is an algebraic closure of K. Hence, we see by the definition of an eliminant form and by Theorem 1.1.2 that for every specialization $\rho : K[\mathbf{u}] \to K$ extending $\mathbf{u}^{(r)} \mapsto \mathrm{coeff}(f)$ we have $\rho(\mathrm{elim}_{\mathbb{d}}(V)) = 0$. This implies (if the field K is infinite, but we can address the case of K finite passing to an algebraic closure $\bar{K}$) that the polynomial form $\mathrm{elim}_{\mathbb{d}}(V)(\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r-1)}, \mathrm{coeff}(f))$ is identically zero. Since $\mathrm{res}_{\mathbb{d}}(Z)$ is a multiple of $\mathrm{elim}_{\mathbb{d}}(V)$, we conclude that $\mathrm{res}_{\mathbb{d}}(Z)(\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r-1)}, \mathrm{coeff}(f)) \equiv 0$ in case Z and $\mathrm{div}(f)$ doesn't intersect properly.

Resultant forms have the following specialization property.

Proposition 1.2.21. Let $Z \in Z_r^+(\mathbb{P}_K^{\mathbf{n}})$, $\mathbb{d} = (\mathbf{d}^{(0)}, \dots, \mathbf{d}^{(r)}) \in (\mathbb{N}^q \backslash \{\mathbf{0}\})^{r+1}$, and let I be a finite set of indices. For every $i \in I$ let $\boldsymbol{\delta}_i^{(r)} \in \mathbb{N}^q \backslash \{\mathbf{0}\}$ and denote by $\mathbb{d}_i$ the collection of multidegrees $(\mathbf{d}^{(0)}, \dots, \mathbf{d}^{(r-1)}, \boldsymbol{\delta}_i^{(r)})$. Assume that $\mathbf{d}^{(r)} = \sum_{i \in I} \boldsymbol{\delta}_i^{(r)}$ and let $\mathbf{u}' = (\mathbf{u}^{(0)}, \dots, \mathbf{u}^{(r-1)})$. Then there exists $\lambda \in K^\times$ such that

$$\mathrm{res}_{\mathbb{d}}(Z)(\mathbf{u}', \mathrm{coeff}(\prod_{i \in I} f_i)) = \lambda \prod_{i \in I} \mathrm{res}_{\mathbb{d}_i}(Z)(\mathbf{u}', \mathrm{coeff}(f_i)) \tag{1.2.5}$$

for every field L containing K and every choice of polynomials $f_i \in L[\mathbf{X}]_{\boldsymbol{\delta}_i^{(r)}}$.

Proof. Since by Proposition 1.2.17 a change of coefficients affects the resultant forms only by multiplication by a scalar, we can assume $L = K$. It is easy to see that the divisor $\mathrm{div}(\prod_{i \in I} f_i)$ intersects properly the cycle Z if and only if for every $i \in I$ the divisor $\mathrm{div}(f_i)$ intersects properly Z. Thus, if there is some $i \in I$ such that $\mathrm{div}(f_i)$ doesn't intersect properly Z, Remark 1.2.20 implies that both sides of (1.2.5) are zero, and we are done in this case. Otherwise, we observe that $X \cdot \mathrm{div}(\prod_{i \in I} f_i) = \sum_{i \in I}(X \cdot \mathrm{div}(f_i))$ and we conclude with Proposition 1.2.19. □

We conclude this section by providing an explicit formula for the resultant forms of 0-dimensional cycles ([DKS11], Corollary 1.38).

Proposition 1.2.22. Let $Z \in Z_0^+(\mathbb{P}_K^{\mathbf{n}})$ a 0-dimensional effective K-cycle and $\mathbf{d} \in \mathbb{N}^q \backslash \{\mathbf{0}\}$ a nonzero multidegree. Write $Z_{\bar{K}} = \sum_{i=1}^s m_i \boldsymbol{\xi}_i$ with $\boldsymbol{\xi}_i \in \mathbb{P}^{\mathbf{n}}(\bar{K})$ and $m_i \in \mathbb{N}$, and let ξ_i a choice of representatives for the points $\boldsymbol{\xi}_i$ in $(\bar{K}^{n_1+1} \backslash \{\mathbf{0}\}) \times \cdots \times (\bar{K}^{n_q+1} \backslash \{\mathbf{0}\})$. Then there exists $\lambda \in \bar{K}^\times$ such that

$$\mathrm{res}_{(\mathbf{d})}(Z) = \lambda \prod_{i=1}^s U_0(\xi)^{m_i},$$

where $U_0 \in K[\mathbf{u}][\mathbf{X}]$ is the general polynomial of multidegree $\mathbf{d}$.

1.3 Heights

A *number field* is a field which is a finite extension of $\mathbb{Q}$.

Definition 1.3.1. An *absolute value* on a field K is a function $|\cdot| : K \to \mathbb{R}_{\geq 0}$ which satisfies

1. $|x| = 0 \iff x = 0$;
2. $|x + y| \leq |x| + |y|$ for all $x, y \in K$ (triangle inequality);
3. $|xy| = |x|\ |y|$ for all $x, y \in K$.

An absolute value $|\cdot|$ is called *ultrametric* if it satisfies the ultrametric inequality $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$. It is called *archimedean* if it is not ultrametric.

An absolute value defines a metric on K and thus a topology on K. We say that two absolute values on K are equivalent if they define the same topology, and a *place* is a class of absolute values on K modulo equivalence. Given a number field K, we denote by $\mathcal{M}_K$ the set of nontrivial places of K and by $\mathcal{M}_{K,0}$ the subset of the archimedean places of K. Ostrowski's theorem says that the elements of $\mathcal{M}_{K,0}$ correspond to the embeddings $\sigma : K \hookrightarrow \mathbb{C}$ up to conjugation, whereas elements of $\mathcal{M}_K \backslash \mathcal{M}_{K,0}$ are in bijection with the prime ideals of the ring of integers $\mathcal{O}_K$.
For each $v \in \mathcal{M}_K$ we select a corresponding absolute value $|\cdot|_v$ on K and a real number $\lambda_v > 0$ such that the following *product formula* holds

$$\prod_{v \in \mathcal{M}_K} |x|_v^{\lambda_v} = 1 \tag{1.3.1}$$

for all $x \in K^\times$. Since there are only a finite number of places such that $|x|_v \neq 1$, this product is well-defined. We can also choose the numbers λ_v such that

$$\sum_{v \in \mathcal{M}_{K,0}} \lambda_v = 1. \tag{1.3.2}$$

For each place $v \in \mathcal{M}_K$ we choose an absolute value of $\overline{\mathbb{Q}}$ which extends $|\cdot|_v$ on K. We denote by $\mathbb{C}_v$ the completion of $\overline{\mathbb{Q}}$ with respect to this absolute value and we again denote by $|\cdot|_v$ the absolute value of $\mathbb{C}_v$ which coincides with the chosen absolute value on $\overline{\mathbb{Q}}$.

Definition 1.3.2. Let $v \in \mathcal{M}_K$ a place of K and $P = \sum_{\alpha \in \mathbb{N}^{\mathbf{n+1}}} c_\alpha \mathbf{X}^\alpha \in \mathbb{C}_v[\mathbf{X}]$ a polynomial in some set of variables $\mathbf{X}$ with coefficients from $\mathbb{C}_v$. We define the *local v-norm* and the *local v-length* of P respectively by

$$\|P\|_v := \max\{|c_\alpha|_v\} \quad , \quad \mathcal{L}_v(P) := \sum_{\alpha \in \mathbb{N}^{\mathbf{n+1}}} |c_\alpha|_v .$$

We will use the fact that for every $P, Q \in \mathbb{C}_v[\mathbf{X}]$ one has $\|P\|_v \leq \mathcal{L}_v(P)$ and $\mathcal{L}_v(PQ) \leq \mathcal{L}_v(P)\mathcal{L}_v(Q)$ (see Lemma 2.3.2). We give analogous definitions for polynomial forms $f \in K[\mathbf{u}] \subseteq \mathbb{C}_v[\mathbf{u}]$.

1.3.1 Convex bodies

Let L be a local field, that is a field which is complete with respect to an absolute value $|\cdot|$. It is well known that if V is a finite dimensional L-vector space, all norms on V are equivalent, so they define the same topology and we endow V with this topology. The notion of a *bounded* subset of V is also independent of the choice of a norm. We may therefore give the following definition.

Definition 1.3.3. We say that a subset $\mathcal{C}$ of V is a *convex body* if it contains a neighbourhood of zero, it is bounded and it satisfies the condition

$$\lambda x + \mu y \in \mathcal{C}$$

for any choice of points $x, y \in \mathcal{C}$ and any $\lambda, \mu \in L$ with

$$\begin{cases} |\lambda| + |\mu| \leq 1 & \text{if } |\cdot| \text{ is archimedean} \\ \max\{|\lambda|, |\mu|\} \leq 1 & \text{if } |\cdot| \text{ is ultrametric.} \end{cases}$$

To each convex body $\mathcal{C}$ of V, we attach a norm on the vector space of all polynomial maps $F:\ V \to L$.

Definition 1.3.4. If $\mathcal{C}$ is a convex body of a finite dimensional L-vector space V and $F:\ V \to L$ is a polynomial map, we define

$$\|F\|_{\mathcal{C}} := \sup\{|F(x)|:\ x \in \mathcal{C}\}.$$

We observe that the norm attached to a convex body $\mathcal{C}$ is well-defined because $\mathcal{C}$ is a bounded set. Moreover, since $\mathcal{C}$ is a neighbourhood of zero, we have $\|F\|_{\mathcal{C}} > 0$ for all nonzero polynomial maps $F:\ V \to L$.
Let now K be a number field and let V be a finite dimensional K-vector space. For every $v \in \mathcal{M}_K$ we view V as a K-subspace of $\mathbb{C}_v \otimes_K V$ under the map $x \mapsto 1 \otimes x$. Given a polynomial map $F:\ V \to K$, we also denote by the same letter F the unique polynomial map from $\mathbb{C}_v \otimes_K V$ to $\mathbb{C}_v$ which coincides with F on V.

Definition 1.3.5. Let V be a vector space over K of finite dimension $n > 0$. An *adelic convex body* for V is a product

$$\mathcal{C} = \prod_{v \in \mathcal{M}_K} \mathcal{C}_v \subseteq \prod_{v \in \mathcal{M}_K} \mathbb{C}_v \otimes_K V$$

which satisfies the following conditions:

1. for each $v \in \mathcal{M}_K$, the set $\mathcal{C}_v$ is a convex body of $\mathbb{C}_v \otimes_K V$;
2. there exist an invertible linear map $\phi:\ K^n \to V$ such that, except for finitely many places $v \in \mathcal{M}_K$, the set $\mathcal{C}_v$ is the image of the unit ball of $\mathbb{C}_v^n$, $\mathcal{B}_v := \{\mathbf{x} \in \mathbb{C}_v^n:\ \max |x_i|_v = 1\}$, under the linear map $\phi_v:\ \mathbb{C}_v^n \to \mathbb{C}_v \otimes_K V$ which extends ϕ.

In the sequel we will be interested in a particular type of adelic convex bodies, suitable for the study of Diophantine Approximation in multihomogeneous setting.

Definition 1.3.6. Given a collection of multidegrees $\mathbb{d} = (\mathbf{d}^{(0)}, \ldots, \mathbf{d}^{(r)}) \in (\mathbb{N}^q \backslash \{\mathbf{0}\})^{r+1}$, we define an *adelic convex body of index* $\mathbb{d}$ to be an adelic convex body for the K-vector space $K[\mathbf{X}]_{\mathbf{d}^{(0)}} \times \cdots \times K[\mathbf{X}]_{\mathbf{d}^{(r)}}$ which has the form of a cartesian product $\mathcal{C} = \mathcal{C}_0 \times \cdots \times \mathcal{C}_r$ where $\mathcal{C}_j$ is an adelic convex body for $K[\mathbf{X}]_{\mathbf{d}^{(j)}}$, $0 \le j \le r$.

For any nonempty bounded subset S of a finite dimensional vector space W over a local field L, which contains a basis for W, there is a smallest convex body of W containing S. We call it the *symmetric convex hull* of S. In consists of all points of W of the form $\lambda_1 x_1 + \ldots + \lambda_s x_s$ with $x_1, \ldots, x_s \in S$ and $\lambda_1, \ldots, \lambda_s \in L$ satisfying $\sum |\lambda_i| = 1$ if L is archimedean and $\max\{|\lambda_i|\} \le 1$ otherwise. Given this, we define the following operation on adelic convex bodies.

Definition 1.3.7. Let $\mathbf{d}', \mathbf{d}'' \in \mathbb{N}^q \backslash \{\mathbf{0}\}$ be nonzero multidegrees and let $\mathbf{d} = \mathbf{d}' + \mathbf{d}''$. Moreover, let $\mathcal{C}' = \prod_v \mathcal{C}'_v$ and $\mathcal{C}'' = \prod_v \mathcal{C}''_v$ be adelic convex bodies for $K[\mathbf{X}]_{\mathbf{d}'}$ and $K[\mathbf{X}]_{\mathbf{d}''}$ respectively (hence of indices $(\mathbf{d}')$ and $(\mathbf{d}'')$). We define their *product* $\mathcal{C} = \mathcal{C}'\mathcal{C}''$ as the adelic convex body $\mathcal{C} = \prod_v \mathcal{C}_v$ for $K[\mathbf{X}]_{\mathbf{d}}$ whose v-component $\mathcal{C}_v$ is the symmetric convex hull of the set of products $Q'Q''$ with $Q' \in \mathcal{C}'_v$ and $Q'' \in \mathcal{C}''_v$, for each place $v \in \mathcal{M}_K$.

It is easy to see that this construction gives indeed an adelic convex body of index $(\mathbf{d})$. It is also easy to prove that the product of convex bodies is commutative and associative.
We are now going to introduce examples of relevant convex bodies and to establish comparisons between them.

Definition 1.3.8. For every multidegree $\mathbf{d} = (d_1, \dots, d_q) \in \mathbb{N}^q \backslash \{\mathbf{0}\}$ and every place $v \in \mathcal{M}_K$ we define the following convex bodies

$$\begin{aligned}
\mathcal{B}_v^{(\mathbf{d})} &:= \{P \in \mathbb{C}_v[\mathbf{X}]_{\mathbf{d}} \colon \|P\|_v \leq 1\}, \\
\mathcal{D}_v^{[i]} &:= \{L \in \mathbb{C}_v[\mathbf{X}]_{\mathbf{e}_i} \colon \|L\|_v \leq 1\}, \qquad \text{for all } i = 1, \dots, q, \\
\mathcal{D}_v^{\mathbf{d}} &:= (\mathcal{D}_v^{[1]})^{d_1} \cdots (\mathcal{D}_v^{[q]})^{d_q} \subseteq \mathbb{C}_v[\mathbf{X}]_{\mathbf{d}},
\end{aligned}$$

and the adelic convex bodies $\mathcal{B}^{(\mathbf{d})} = \prod_v \mathcal{B}_v^{(\mathbf{d})}$, $\mathcal{D}^{[i]} = \prod_v \mathcal{D}_v^{[i]}$, $\mathcal{D}^{\mathbf{d}} = \prod_v \mathcal{D}_v^{\mathbf{d}}$.

For an index $\underline{\mu} \in \{1, \dots, q\}^{r+1}$ we define by $\mathbf{e}_{\underline{\mu}}$ the collection of multidegrees

$$\mathbf{e}_{\underline{\mu}} := (\mathbf{e}_{\mu_0}, \dots, \mathbf{e}_{\mu_r}) \quad \in (\mathbb{N}^q \backslash \{\mathbf{0}\})^{r+1}. \tag{1.3.3}$$

Definition 1.3.9. For a collection of multidegrees $\mathbb{d} = (\mathbf{d}^{(0)}, \dots, \mathbf{d}^{(r)}) \in (\mathbb{N}^q \backslash \{\mathbf{0}\})^{r+1}$ and $\underline{\mu} \in \{1, \dots, q\}^{r+1}$ we define the following (cartesian adelic) convex bodies

$$\begin{aligned}
\mathcal{B}^{(\mathbb{d})} &:= \mathcal{B}^{(\mathbf{d}^{(0)})} \times \cdots \times \mathcal{B}^{(\mathbf{d}^{(r)})}, \\
\mathcal{D}^{\underline{\mu}} &:= \mathcal{D}^{[\mu_0]} \times \cdots \times \mathcal{D}^{[\mu_r]}, \\
\mathcal{D}^{\mathbb{d}} &:= \mathcal{D}^{\mathbf{d}^{(0)}} \times \cdots \times \mathcal{D}^{\mathbf{d}^{(r)}},
\end{aligned}$$

of indices $\mathbb{d}$, $\mathbf{e}_{\underline{\mu}}$ and $\mathbb{d}$ respectively.

We remark that $\mathcal{D}^{[i]} = \mathcal{D}^{\mathbf{e}_i} = \mathcal{B}^{(\mathbf{e}_i)}$ for all $i = 1, \dots, q$ and $\mathcal{D}^{\underline{\mu}} = \mathcal{D}^{\mathbf{e}_{\underline{\mu}}} = \mathcal{B}^{(\mathbf{e}_{\underline{\mu}})}$ for all $\underline{\mu} \in \{1, \dots, q\}^{r+1}$.

Lemma 1.3.10. Let v be any place of K and $\mathbf{d} = (d_1, \dots, d_q) \in \mathbb{N}^q \backslash \{\mathbf{0}\}$ a nonzero multidegree. We have inclusions

$$\prod_{i=1}^q (n_i + 1)^{-\epsilon_v d_i} \mathcal{D}_v^{\mathbf{d}} \subseteq \mathcal{B}_v^{(\mathbf{d})} \subseteq \dim(\mathbf{d})^{\epsilon_v} \mathcal{D}_v^{\mathbf{d}}, \tag{1.3.4}$$

where we put $\epsilon_v := 1$ if v is archimedean and $\epsilon_v := 0$ otherwise, and where $\dim(\mathbf{d}) := \#\mathfrak{M}_{\mathbf{d}} = \dim_{\mathbb{C}_v}(\mathbb{C}_v[\mathbf{X}]_{\mathbf{d}})$ denotes the number of monic monomials of multidegree $\mathbf{d}$.

Proof. Given $|\mathbf{d}|$ linear forms $L_{i,j} \in \mathcal{D}_v^{\mathbf{e}_i}$ for $i = 1, \dots, q$ and $j = 1, \dots, d_i$ it is easy to see that their product $P = \prod_{i=1}^q \prod_{j=1}^{d_i} L_{i,j}$ is a polynomial of multidegree $\mathbf{d}$ and norm $\|P\|_v \leq \mathcal{L}_v(P) \leq \left| \prod_{i=1}^q (n_i+1)^{d_i} \right| = \prod_{i=1}^q (n_i+1)^{\epsilon_v d_i}$ (see Definition 1.3.2). Conversely, let $P \in \mathbb{C}_v[\mathbf{X}]_{\mathbf{d}}$ with $\|P\|_v \leq 1$. Then $\frac{1}{\dim(\mathbf{d})} P$ is clearly in the symmetric convex hull of all the monic monomials in $\mathbb{C}_v[\mathbf{X}]_{\mathbf{d}}$, and since monomials are trivially products of linear polynomials, we have $\frac{1}{\dim(\mathbf{d})} P \in \mathcal{D}_v^{\mathbf{d}}$. $\square$

1.3.2 Heights of cycles

Let K be a number field. Following G.Rémond [Rém01b], we introduce the concept of mixed heights for a multiprojective cycle $Z \in Z(\mathbb{P}_K^{\mathbf{n}})$. In fact, our definition is slightly different from that of G.Rémond, because we use a different notion of height for a polynomial. The effect is that our definition is easier to use, but behaves worse with respect to geometric operations.
Then, following the work of M.Laurent and D.Roy [LR01], we introduce a concept of height attached to a convex body. We then give the definition of the height of a multiprojective cycle Z with respect to a convex body $\mathcal{C}$. This last notion of height has not been defined previously in the literature, but it is a natural generalization to multiprojective setting of the ideas in [LR01], and it will be an essential tool in the proof of the main theorem of the next chapter. For this reason we devote the next section to give some useful estimates related to this new concept of height.
One of the most basilar notions of height that one can define in the theory of Diophantine Approximation is the Weil absolute height of an algebraic number.

Definition 1.3.11. Let K be a number field, and $\alpha \in K$ an algebraic number. We define the *Weil absolute (logarithmic) height* of α by

$$h(\alpha) = h_K(\alpha) := \sum_{v \in \mathcal{M}_K} \lambda_v \log \max\{1, |\alpha|_v\}.$$

The Weil absolute height of an algebraic number α is a nonnegative real number that measure the arithmetic complexity of α. It does not depend on the field K, meaning that if $\alpha \in K \subseteq L$, then $h_K(\alpha) = h_L(\alpha)$. It is an important tool in many problems of Diophantine Approximation, and there are plenty of references on this subject. For example one can see [Lan02] for a detailed presentation of the basic results. We will use the following basic properties of Weil absolute logarithmic height.

Proposition 1.3.12. Let α, β algebraic numbers. Then

1. $h(\alpha + \beta) \leq h(\alpha) + h(\beta) + \log(2)$;
2. $h(\alpha\beta) \leq h(\alpha) + h(\beta)$.

We stress the fact that the Weil absolute height is defined as a sum of local contributions. We are going to give definitions of more general concepts of height based on this feature.
We recall that we defined in Definition 1.3.2 the local norm of a polynomial to be the maximum of the absolute values of its coefficients, with respect to an absolute value $|\cdot|_v$. Summing up the logarithms of the local norms we define the absolute height of a polynomial.

Definition 1.3.13. Let $P \in K[\mathbf{X}]$ a polynomial with coefficients in K over some set of variables $\mathbf{X}$. We define the *absolute height* of P by

$$h(P) := \sum_{v \in \mathcal{M}_K} \lambda_v \log \|P\|_v.$$

We remark that the Weil absolute height of a number $\alpha \in K$ is equal to the absolute height of the polynomial $x-\alpha \in K[\mathbf{X}]$. It is worth saying that if $K=\mathbb{Q}$ and P is a polynomial with integer coprime coefficients, then its height is simply the logarithm of its norm $h(P)=\log\|P\|$.
We notice that a resultant form $\mathrm{res}_{\mathbb{d}}$ is a polynomial form in $K[\mathbf{u}]$, and so we give the following

Definition 1.3.14. Let $\underline{\mu} \in \{1,\dots,q\}^{r+1}$ and let $\mathbf{e}_{\underline{\mu}}$ be as in (1.3.3). Let V be an irreducible subvariety of $\mathbb{P}_K^{\mathbf{n}}$ of dimension r. We define the *mixed height of V of index $\underline{\mu}$* by

$$h_{\underline{\mu}}(V) := h(\mathrm{res}_{\mathbf{e}_{\underline{\mu}}}(V)).$$

We extend this definition to all irreducible subvarieties putting $h_{\mathcal{C}}(V):=0$ if $\dim(V)\neq r$, and then by linearity to an arbitrary cycle $Z=\sum_V m_V V \in Z(\mathbb{P}_K^{\mathbf{n}})$:

$$h_{\underline{\mu}}(Z) := \sum_{\dim(V)=r} m_V h_{\underline{\mu}}(V).$$

We use the product formula on K and the norm attached to the convex bodies $\mathcal{C}_v$ (see Definition 1.3.4) to define the height of a nonzero polynomial map $F:\ V\to K$.

Definition 1.3.15. If $\mathcal{C}$ is an adelic convex body for the finite dimensional K-vector space V and $F:\ V\to\mathcal{C}$ is a nonzero polynomial map, we define

$$h_{\mathcal{C}}(F) := \sum_{v\in\mathcal{M}_K} \lambda_v \log\|F\|_{\mathcal{C}_v}.$$

We extend this definition with $h_{\mathcal{C}}(0):=0$.

We recall that for every $Z\in Z_r(\mathbb{P}_K^{\mathbf{n}})$ and every $\mathbb{d}=(\mathbf{d}^{(0)},\dots,\mathbf{d}^{(r)})\in(\mathbb{N}^q\backslash\{\mathbf{0}\})^{r+1}$ the resultant form $\mathrm{res}_{\mathbb{d}}(Z)$ can be seen as a polynomial map $K[\mathbf{X}]_{\mathbf{d}^{(0)}}\times\cdots\times K[\mathbf{X}]_{\mathbf{d}^{(r)}}\to K$. We can then give the following

Definition 1.3.16. If V is an irreducible subvariety of $\mathbb{P}_K$ of dimension r, $\mathbb{d}\in(\mathbb{N}^q\backslash\{\mathbf{0}\})^{r+1}$ is a collection of multidegrees and $\mathcal{C}$ is an adelic convex body of index $\mathbb{d}$, we define the *height of Z relative to* $\mathcal{C}$ by the formula

$$h_{\mathcal{C}}(V) := h_{\mathcal{C}}(\mathrm{res}_{\mathbb{d}}(V)).$$

We extend this definition to all irreducible subvarieties putting $h_{\mathcal{C}}(V):=0$ if $\dim(V)\neq r$, and then by linearity to an arbitrary cycle $Z=\sum_V m_V V\in Z(\mathbb{P}_K^{\mathbf{n}})$:

$$h_{\mathcal{C}}(Z) := \sum_{\dim(V)=r} m_V h_{\mathcal{C}}(V).$$

We remark that this definition does not depend on the choice of $\mathrm{res}_{\mathbb{d}}(V)$ because of the product formula on K.

1.4 Estimates for the heights

We set some notation that will help us to get more readable formulas in the following propositions.

Definition 1.4.1. If $Z \in Z_r(\mathbb{P}^{\mathbf{n}}_K)$ is an r-dimensional cycle of $\mathbb{P}^{\mathbf{n}}_K$ and $\mathbb{d} = (\mathbf{d}^{(0)}, \dots, \mathbf{d}^{(r)}) \in (\mathbb{N}^q \backslash \{\mathbf{0}\})^{r+1}$ is a collection of nonzero multidegrees, we let

$$N_j := \langle \deg(Z); \widehat{\mathbb{d}}^{(j)} \rangle,$$

where $\widehat{\mathbb{d}}^{(i)} := (\mathbf{d}^{(0)}, \dots, \widehat{\mathbf{d}^{(i)}}, \dots, \mathbf{d}^{(r)}) \in (\mathbb{N}^q \backslash \{\mathbf{0}\})^r$ is obtained by deleting the i-th entry from $\mathbb{d}$. In terms of the $*$-operator (see Definition 1.2.3) we have

$$N_j = \deg(Z) * \mathbf{d}^{(0)} * \cdots * \widehat{\mathbf{d}^{(i)}} * \cdots * \mathbf{d}^{(r)}.$$

More explicitly, we have

$$N_j = \sum_{\substack{1 \le \mu_i \le q \\ 0 \le i \le r \\ i \neq j}} \Big(\prod_{\substack{k=0 \\ k \neq j}}^{r} d^{(k)}_{\mu_k} \Big) (\deg(Z))_{\sum_{i \neq j} \mathbf{e}_{\mu_i}},$$

where the first sum is taken over all the $\underline{\mu} = \mu_0, \dots, \widehat{\mu_j}, \dots, \mu_r$ in $\{1, \dots, q\}^r$. We also let

$$\dim(\mathbf{d}^{(i)}) := \#\mathfrak{M}_{\mathbf{d}^{(i)}} = \dim_K K[\mathbf{X}]_{\mathbf{d}^{(i)}} = \binom{d^{(i)}_1 + n_1}{n_1} \cdots \binom{d^{(i)}_q + n_q}{n_q},$$

and

$$\langle h(Z); \mathbb{d} \rangle := \sum_{\substack{1 \le \mu_i \le q \\ 0 \le i \le r}}^{q} \Big(\prod_{k=0}^{r} d^{(k)}_{\mu_k} \Big) h_{\underline{\mu}}(Z),$$

where the sum is taken over all the $\underline{\mu} = \mu_0, \dots, \mu_r$ in $\{1, \dots, q\}^{r+1}$.

We also put $\|n\| := \max\{n_1, \dots, n_q\}$ and we will often use the estimate

$$\log \dim(\mathbf{d}^{(i)}) \le \log(\|n\| + 1) \left| \mathbf{d}^{(i)} \right|$$

coming from the general inequality $\binom{a+b}{b} \le (b+1)^a$.
For the statements and the proofs of our estimates, we also need the following definitions.

Definition 1.4.2. Let $v \in \mathcal{M}_K$ and $\mathbf{d} \in \mathbb{N}^q \backslash \{\mathbf{0}\}$. For any point $\alpha \in \mathbb{C}_v^{n_1+1} \times \cdots \times \mathbb{C}_v^{n_q+1}$, we consider the linear map of evauation at α:

$$\begin{array}{cccc} \mathcal{L}_\alpha : & \mathbb{C}_v[\mathbf{X}]_{\mathbf{d}} & \longrightarrow & \mathbb{C}_v \\ & Q & \longmapsto & Q(\alpha) \end{array}.$$

We observe that, according to Definition 1.3.4, we have, for a convex body $\mathcal{C}_v$ of $\mathbb{C}_v[\mathbf{X}]_{\mathbf{d}}$:

$$\|\mathcal{L}_\alpha\|_{\mathcal{C}_v} = \sup_{Q \in \mathcal{C}_v} |\mathcal{L}_\alpha(Q)|_v = \sup_{Q \in \mathcal{C}_v} |Q(\alpha)|_v .$$

Lemma 1.4.3. Let $v \in \mathcal{M}_K$ any place and let $\alpha \in \mathbb{C}_v^{n_1+1} \times \cdots \times \mathbb{C}_v^{n_q+1}$ any point. Let $\mathbf{d}, \mathbf{d}', \mathbf{d}'', \mathcal{C}_v, \mathcal{C}'_v, \mathcal{C}''_v$ as in Definition 1.3.7. We have

$$\|\mathcal{L}_\alpha\|_{\mathcal{C}_v} = \|\mathcal{L}_\alpha\|_{\mathcal{C}'_v} \|\mathcal{L}_\alpha\|_{\mathcal{C}''_v}.$$

Proof. Every element $Q \in \mathcal{C}_v$ can be written as a finite sum $Q = \sum_{i=1}^s \lambda_i Q'_i Q''_i$ where $Q'_i \in \mathcal{C}'_v$ and $Q''_i \in \mathcal{C}''_v$ for $i = 1, \ldots, s$, and where the coefficients $\lambda_1, \ldots, \lambda_s \in \mathbb{C}_v$ satisfy $\sum_{i=1}^s |\lambda_i|_v \leq 1$ if v is archimedean and $\max_i |\lambda_i|_v \leq 1$ otherwise. In both cases, we get $|Q(\alpha)|_v \leq \|\mathcal{L}_\alpha\|_{\mathcal{C}'_v} \|\mathcal{L}_\alpha\|_{\mathcal{C}''_v}$ and thus $\|\mathcal{L}_\alpha\|_{\mathcal{C}_v} \leq \|\mathcal{L}_\alpha\|_{\mathcal{C}'_v} \|\mathcal{L}_\alpha\|_{\mathcal{C}''_v}$. To establish the reverse inequality, we choose $Q = Q'Q''$ with $Q' \in \mathcal{C}'_v$ and $Q'' \in \mathcal{C}''_v$, and we take the supremum of both sides of the equality $|Q'(\alpha)|_v |Q''(\alpha)|_v = |Q(\alpha)|_v$ over the set of all such polynomials Q. □

Definition 1.4.4. If $P \in \mathbb{C}_v[\mathbf{X}]_{\mathbf{d}}$ is a polynomial of multidegree $\mathbf{d} \in \mathbb{N}^q$ and $\mathcal{C}_v$ is a convex body of $\mathbb{C}_v[\mathbf{X}]_{\mathbf{d}}$, we define *the norm of P relative to $\mathcal{C}_v$* by

$$\|P\|_{\mathcal{C}_v} := \inf\{|\rho|_v : \rho \in \mathbb{C}_v \text{ and } P \in \rho\,\mathcal{C}_v\}.$$

If $P \in K[\mathbf{X}]_{\mathbf{d}}$ is a nonzero polynomial of multidegree $\mathbf{d} \in \mathbb{N}^q$ and $\mathcal{C} = \prod_v \mathcal{C}_v$ is an adelic convex body of index $(\mathbf{d})$, we define *the height of P relative to $\mathcal{C}$* by

$$h_{\mathcal{C}}(P) := \sum_{v \in \mathcal{M}_K} \lambda_v \log\|P\|_{\mathcal{C}_v}.$$

1.4.1 Comparison between convex bodies

We prove two propositions that compare the relative heights with respect to different adelic convex bodies.
For the proof of the first proposition we need the following result ([LR01], Proposition 3.7).

Proposition 1.4.5. Let L be a local field and assume that $V = V_1 \times \cdots \times V_k$ is a product of L-vector spaces of dimension $\dim(V_j) = n_j$ for $j = 1, \ldots, k$. Moreover, let $\mathcal{C}$ be a convex body of V in the form of a cartesian product $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_k$ where $\mathcal{C}_j$ is a convex body of V_j for $j = 1, \ldots, k$. Then, if $F_1, \ldots, F_s$ are multihomogeneous polynomial maps from $V_1 \times \cdots \times V_k$ to L and if their product $F = F_1 \cdots F_s$ has multidegree $(d_1, \ldots, d_k)$, we have

$$\|F\|_{\mathcal{C}} \leq \prod_{i=1}^s \|F_i\|_{\mathcal{C}} \leq \left(\prod_{j=1}^k n_j^{2d_j\epsilon} \right) \|F\|_{\mathcal{C}},$$

where $\epsilon := 1$ if $(L, |\cdot|)$ is archimedean and $\epsilon := 0$ otherwise.

Proposition 1.4.6. Let $\mathcal{C} = \mathcal{C}_0 \times \cdots \times \mathcal{C}_r$ be an adelic convex body of index $\mathbb{d} = (\mathbf{d}^{(0)}, \ldots, \mathbf{d}^{(r)})$ and let s be an integer with $0 \leq s \leq r$. Suppose that $\mathbf{d}^{(s)}$ is written as a finite sum of nonzero multi-integers $\mathbf{d}^{(s)} = \sum_{i \in I} \boldsymbol{\delta}_i^{(s)}$ and that we have a corresponding decomposition of $\mathcal{C}_s$ into a product

$$\mathcal{C}_s = \prod_{i \in I} \mathcal{C}_{s,i},$$

where $\mathcal{C}_{s,i}$ is an adelic convex body for $K[\mathbf{X}]_{\boldsymbol{\delta}_i^{(s)}}$ for each $i \in I$. For each $i \in I$, denote by $\mathcal{C}^{[i]}$ the adelic convex body which, as a cartesian product, has the same factors as $\mathcal{C}$ except that the s-th factor $\mathcal{C}_s$ is replaced by $\mathcal{C}_{s,i}$. Then, for any effective cycle Z of dimension r, we have

$$-2 \sum_{\substack{0 \le j \le r \\ j \ne s}} N_j \log \dim(\mathbf{d}^{(j)}) \le h_{\mathcal{C}}(Z) - \sum_{i \in I} h_{\mathcal{C}^{[i]}}(Z) \le 2N_s \sum_{i \in I} \log \dim(\boldsymbol{\delta}_i^{(s)}).$$

Proof. The height of Z relative to a cartesian product of adelic convex bodies does not change under a permutation of its factors, by Proposition 1.2.16, so we may assume, without loss of generality, that $s = r$. By the effectivity assumption and by linearity, we may assume that $Z = V$ is an irreducible subvariety of $\mathbb{P}_K^{\mathbf{n}}$. Let F be a resultant form of V of index $\mathbb{d}$ and, for $i \in I$, let F_i be a resultant form of V of index $\mathbb{d}_i = (\mathbf{d}^{(0)}, \dots, \mathbf{d}^{(r-1)}, \boldsymbol{\delta}_i^{(r)})$. Lemma 1.2.21 shows that there exists $\zeta \in K^\times$ such that

$$F\left(P_1, \dots, P_{r-1}, \prod_{i \in I} Q_i\right) = \zeta \prod_{i \in I} F_i(P_0, \dots, P_{r-1}, Q_i) \tag{1.4.1}$$

for any field L containing K and any choice of polynomials $Q_i \in L[\mathbf{X}]_{\boldsymbol{\delta}_i^{(r)}}$ for $i \in I$ and $P_j \in L[\mathbf{X}]_{\mathbf{d}^{(j)}}$ for $j = 0, \dots, r-1$. Define polynomial maps $G : \mathbb{C}_v[\mathbf{X}]_{\mathbf{d}^{(r)}} \to \mathbb{C}_v$ and $G_i : \mathbb{C}_v[\mathbf{X}]_{\boldsymbol{\delta}_i^{(r)}}$ for $i \in I$ by putting

$$G(Q) = F(P_0, \dots, P_{r-1}, Q) \quad \text{and} \quad G_i(Q_i) = F_i(P_0, \dots, P_{r-1}, Q_i),$$

for any $Q \in \mathbb{C}_v[\mathbf{X}]_{\mathbf{d}^{(r)}}$ and any $Q_i \in \mathbb{C}_v[\mathbf{X}]_{\boldsymbol{\delta}_i^{(r)}}$ with $i \in I$. If $P_0, \dots, P_{r-1}$ is not a regular sequence in the ring $K[\mathbf{X}]/\mathcal{I}(V)$, then Remark 1.2.20 implies that G and G_i are identically zero, for all $i \in I$. Otherwise, $Z' := Z \cdot \operatorname{div}(P_0) \cdots \operatorname{div}(P_{r-1})$ is a well-defined 0-dimensional cycle, and, by Proposition 1.2.19, G and G_i are resultant forms for Z', respectively of indices $\mathbb{d}$ and $\mathbb{d}_i$, for $i \in I$. In any case, thanks to Proposition 1.2.22 there exist elements $\alpha_1, \dots, \alpha_{N_r} \in \mathbb{C}_v^{n_1+1} \backslash \{\mathbf{0}\} \times \cdots \times \mathbb{C}_v^{n_q+1} \backslash \{\mathbf{0}\}$, and (eventually zero) constants $\xi \in \mathbb{C}_v$ and $\xi_i \in \mathbb{C}_v$ for $i \in I$ such that

$$G(Q) = \xi \prod_{k=1}^{N_r} Q(\alpha_k) \quad \text{and} \quad G_i(Q_i) = \xi_i \prod_{k=1}^{N_r} Q_i(\alpha_k),$$

for every choice of polynomials Q and Q_i with $i \in I$ as above, and where $N_r = \deg(Z) * \mathbf{d}^{(0)} * \cdots * \mathbf{d}^{(r-1)}$ (see Proposition 1.2.11). By virtue of (1.4.1), we have $G(Q) = \zeta \prod_{i \in I} G_i(Q_i)$ whenever $Q = \prod_{i \in I} Q_i$ and therefore $\xi = \zeta \prod_{i \in I} \xi_i$. Applying Proposition 1.4.5 to the above factorization of the maps G and G_i into products of linear forms, we find

$$\|G\|_{\mathcal{C}_{r,v}} \le |\xi|_v \prod_{k=1}^{N_r} \|\mathcal{L}_{\alpha_k}\|_{\mathcal{C}_{r,v}}$$

and

$$|\xi_i|_v \prod_{k=1}^{N_r} \|\mathcal{L}_{\alpha_k}\|_{\mathcal{C}_{r,i,v}} \le \dim(\boldsymbol{\delta}_i^{(r)})^{2N_r \epsilon_v} \|G_i\|_{\mathcal{C}_{r,i,v}}.$$

On the other hand, Lemma 1.4.3 shows that, for any $\alpha \in \mathbb{C}_v^{n_1+1}\backslash\{\mathbf{0}\} \times \cdots \times \mathbb{C}_v^{n_q+1}\backslash\{\mathbf{0}\}$, we have $\|\mathcal{L}_\alpha\|_{\mathcal{C}_{r,v}} = \prod_{i\in I}\|\mathcal{L}_\alpha\|_{\mathcal{C}_{r,i,v}}$. Combining this with the previous inequalities and using the relation $\xi = \zeta \prod_{i\in I} \xi_i$, we get

$$\|G_{\mathcal{C}_{r,v}}\| \leq |\zeta|_v \prod_{i\in I} \left(\dim(\boldsymbol{\delta}_i^{(r)})^{2N_r\epsilon_v}\|G_i\|_{\mathcal{C}_{r,i,v}}\right).$$

Taking the supremum of both sides of this inequality over all choices of $P_0, \ldots, P_{r-1}$, we deduce

$$\|F\|_{\mathcal{C}_v} \leq |\zeta|_v \prod_{i\in I} \left(\dim(\boldsymbol{\delta}_i^{(r)})^{2N_r\epsilon_v}\|F_i\|_{\mathcal{C}_v^{[i]}}\right).$$

Finally, taking the logarithms of both sides, multiplying them by λ_v, summing over $v \in \mathcal{M}_K$ and taking into account the product formula for ζ, we get

$$h_{\mathcal{C}}(V) \leq 2N_r \sum_{i\in I} \log\dim \boldsymbol{\delta}_i^{(r)} + \sum_{i\in I} h_{\mathcal{C}^{[i]}}(V).$$

For the lower bound, fix a place $v \in \mathcal{M}_K$ and a choice of polynomials $Q_i \in \mathcal{C}_{r,i,v}$ for $i \in I$. Put $Q = \prod_{i\in I} Q_i$ and define $\mathbb{C}_v$-valued polynomial maps E and E_i for $i \in I$ on the product space $\prod_{j=0}^{r-1} \mathbb{C}_v[\mathbf{X}]_{\mathbf{d}^{(j)}}$ by putting

$$E(P_0, \ldots, P_{r-1}) := F(P_0, \ldots, P_{r-1}, Q)$$

and

$$E_i(P_0, \ldots, P_{r-1}) := F_i(P_0, \ldots, P_{r-1}, Q_i)$$

for any choice of $P_j \in \mathbb{C}_v[\mathbf{X}]_{\mathbf{d}^{(j)}}$ for $j = 0, \ldots, r-1$. By virtue of (1.4.1), we have $E = \zeta \prod_{i\in I} E_i$. Since E_i is multihomogeneous of multidegree $(N_0, \ldots, N_{r-1})$ if not identically zero, Proposition 1.4.5 gives

$$|\zeta|_v \prod_{i\in I}\|E_i\|_{\mathcal{C}'_v} \leq \left(\prod_{j=0}^{r-1} \dim(\mathbf{d}^{(j)})^{2N_j\epsilon_v}\right) \|E\|_{\mathcal{C}'_v},$$

where $\mathcal{C}'_v := \mathcal{C}_{0,v} \times \cdots \times \mathcal{C}_{r-1,v}$. Taking the supremum of both sides over all choices of polynomials $Q_i \in \mathcal{C}_{r,i,v}$ with $i \in I$ and using the fact that $Q = \prod_{i\in I} Q_i$ belongs to $\mathcal{C}_{r,v}$, we get

$$|\zeta|_v \prod_{i\in I}\|F_i\|_{\mathcal{C}_v^{[i]}} \leq \left(\prod_{j=0}^{r-1} \dim(\mathbf{d}^{(j)})^{2N_j\epsilon_v}\right) \|F\|_{\mathcal{C}_v}.$$

Arguing as in the previous situation we deduce

$$\sum_{i\in I} h_{\mathcal{C}^{[i]}}(V) \leq h_{\mathcal{C}}(V) + 2\sum_{j=0}^{r-1} N_j \log\dim(\mathbf{d}^{(j)}).$$

The thesis follows. □

The second proposition is simpler.

Proposition 1.4.7. Let $\mathcal{C} = \mathcal{C}_0 \times \cdots \times \mathcal{C}_r$ and $\mathcal{C}' = \mathcal{C}'_0 \times \cdots \times \mathcal{C}'_r$ be adelic convex bodies of index $\mathbb{d} \in (\mathbb{N}^q \backslash \{\mathbf{0}\})^{r+1}$, and let Z be an effective cycle of dimension r. Then, there exist a finite subset $\mathcal{S}$ of $\{0, \ldots, r\} \times \mathcal{M}_K$ such that $\mathcal{C}'_{j,v} = \mathcal{C}_{j,v}$ for any $(j,v) \notin \mathcal{S}$. Moreover, for each $(j,v) \in \mathcal{S}$, there exists $\rho_{j,v} \in \mathbb{C}_v^{\times}$ such that $\mathcal{C}'_v \subseteq \rho_{j,v}\mathcal{C}_{j,v}$. For such a choice of set $\mathcal{S}$ and numbers $\rho_{j,v}$ with $(j,v) \in \mathcal{S}$, we have

$$h_{\mathcal{C}'}(Z) \leq h_{\mathcal{C}}(Z) + \sum_{(j,v)\in\mathcal{S}} \lambda_v N_j \log|\rho_{j,v}|_v.$$

Proof. The existence of $\mathcal{S}$ and $(\rho_{j,v})_{(j,v)\in\mathcal{S}}$ follows from the definition of adelic convex bodies (see Definition 1.3.5). To prove the inequality, we may assume, by induction on the cardinality of $\mathcal{S}$, that $\mathcal{S}$ consists of only one point (i,w). Moreover, by permuting the factors of $\mathcal{C}$ and $\mathcal{C}'$ if necessary, we may assume that $i = 0$. We may also assume that $Z = V$ is an irreducible subvariety of $\mathbb{P}^{\mathbf{n}}_K$, because of the linearity of the heights and of the numbers N_j. This being so, let F be a resultant form of V of index $\mathbb{d}$. Since F is homogeneous of degree N_0 on the factor $K[\mathbf{X}]_{\mathbf{d}^{(0)}}$ by Proposition 1.2.18, we find

$$\|F\|_{\mathcal{C}'_w} \leq \|F\|_{\mathcal{C}_w} \, |\rho_{0,w}|_w^{N_0}.$$

For the other places $v \neq w$, we have $\|F\|_{\mathcal{C}'_v} = \|F\|_{\mathcal{C}_v}$, since $\mathcal{C}'_v = \mathcal{C}_v$. Hence, we get the desired inequality summing up the logarithms of these local data. $\square$

1.4.2 Comparison between mixed and relative heights

The following results compare the mixed heights of G.Rémond with the heights relative to the convex bodies $\mathcal{D}^{\underline{\mu}}$ and $\mathcal{B}^{(\mathbb{d})}$.

Lemma 1.4.8. Let $Z \in Z_r^+(\mathbb{P}^{\mathbf{n}}_K)$ be an effective cycle of dimension r, let $\underline{\mu} \in \{1, \ldots, q\}^{r+1}$ and for $j = 0, \ldots, r$ let $\beta_j := \sum_{i\neq j} \mathbf{e}_{\mu_i} \in \mathbb{N}^q_r$. Then we have

$$h_{\underline{\mu}}(Z) \leq h_{\mathcal{D}^{\underline{\mu}}}(Z) \leq h_{\underline{\mu}}(Z) + \log(\|\mathbf{n}\| + 1) \sum_{j=0}^{r} \deg(Z)_{\beta_j}, \tag{1.4.2}$$

where $\|\mathbf{n}\| := \max\{n_1, \ldots, n_q\}$.

Proof. By the effectivity assumption and by the linearity of mixed heights, relative heights and mixed degrees, it suffices to prove the assertion for an irreducible subvariety $Z = V$. Let F be a resultant form of V of index $\mathbf{e}_{\underline{\mu}}$ and let $v \in \mathcal{M}_K$. It is easily seen that if v is ultrametric, we have $\|F\|_v = \|F\|_{\mathcal{D}^{\underline{\mu}}_v}$, whereas for v archimedean we have $\|F\|_v \leq \|F\|_{\mathcal{D}^{\underline{\mu}}_v} \leq \mathcal{L}_v(F) \leq N\|F\|_v$, where N is the number of nonzero coefficients of F. Since the form F is multihomogeneous of multidegree $\langle \deg(Z); \widehat{\mathbf{e}_{\underline{\mu}}}^{(j)} \rangle = \deg(Z)_{\beta_j}$ on the factor $K[\mathbf{X}]_{\mathbf{e}_{\mu_j}}$, and since the dimension of $K[\mathbf{X}]_{\mathbf{e}_{\mu_j}}$ is $n_{\mu_j} + 1$, we have

$$N \leq \binom{\deg(Z)_{\beta_0} + \|\mathbf{n}\|}{\|\mathbf{n}\|} \cdots \binom{\deg(Z)_{\beta_r} + \|\mathbf{n}\|}{\|\mathbf{n}\|},$$

thus, using the general crude estimate $\binom{a+b}{b} \leq (b+1)^a$ we get

$$\log N \leq \log(\|\mathbf{n}\|+1)\sum_{j=0}^{r} \deg(Z)_{\beta_j}.$$

We conclude adding up the logarithms of the local norms and applying formula (1.3.2). □

Proposition 1.4.9. Let $Z \in Z_r^+(\mathbb{P}_K^{\mathbf{n}})$ be an effective cycle of dimension r, $\mathbb{d} \in (\mathbb{N}^q \backslash \{\mathbf{0}\})^{r+1}$ a collection of multidegrees, and $\mathcal{B} := \mathcal{B}^{(\mathbb{d})}$ as in Definition 1.3.9. We have the following estimate

$$\Big| h_{\mathcal{B}}(Z) - \langle h(Z); \mathbb{d}\rangle \Big| \leq (2r+3)\log(\|\mathbf{n}\|+1)\sum_{j=0}^{r} N_j |\mathbf{d}^{(j)}|,$$

where $\|\mathbf{n}\| = \max\{n_1, \dots, n_q\}$ and $|\mathbf{d}^{(j)}| = d_0^{(j)} + \dots + d_{n_j}^{(j)}$.

Proof. We first compare between the heights relative to the convex bodies $\mathcal{B}^{(\mathbb{d})}$ and $\mathcal{D}^{\mathbb{d}}$. In order to get more readable formulas, we set

$$\Omega := \log(\|n\|+1)\sum_{j=0}^{r} N_j \left| \mathbf{d}^{(j)} \right|.$$

By Proposition 1.3.10 we have that $\mathcal{B}_v^{(\mathbb{d})} = \mathcal{D}_v^{\mathbb{d}}$ for every $v \in \mathcal{M}_K \backslash \mathcal{M}_{K,0}$. Moreover, for every $v \in \mathcal{M}_{K,0}$ and every $j \in \{0, \dots, r\}$ we have $\rho_{j,v}^{-1}\mathcal{D}_v^{\mathbf{d}^{(j)}} \subseteq \mathcal{B}_v^{(\mathbf{d}^{(j)})} \subseteq \mathcal{D}_v^{\mathbf{d}^{(j)}}$ for the choice $\rho_{j,v} = (\|n\|+1)^{d_1^{(j)}+\dots+d_q^{(j)}}$. We can then apply Proposition 1.4.7 with the choices $\mathcal{S} := \{0, \dots, r\} \times \mathcal{M}_{K,0}$, $\mathcal{C} = \mathcal{B}^{(\mathbb{d})}$, $\mathcal{C}' = \mathcal{D}^{\mathbb{d}}$ and with $\mathcal{S} := \{0, \dots, r\} \times \mathcal{M}_{K,0}$, $\mathcal{C}' = \mathcal{B}^{(\mathbb{d})}$, $\mathcal{C} = \mathcal{D}^{\mathbb{d}}$ to get the estimates

$$h_{\mathcal{D}^{\mathbb{d}}}(Z) - \Omega \leq h_{\mathcal{B}^{(\mathbb{d})}}(Z) \leq h_{\mathcal{D}^{\mathbb{d}}}(Z) + \Omega. \tag{1.4.3}$$

We now write $\mathcal{D}^{\mathbb{d}} = \widetilde{\mathcal{D}} \times \mathcal{D}^{\mathbf{d}^{(r)}}$ and we consider $\mathcal{D}^{\mathbf{d}^{(r)}} = (\mathcal{D}^{[1]})^{d_1^{(r)}} \dots (\mathcal{D}^{[}q])^{d_q^{(r)}}$ as a product of $\left|\mathbf{d}^{(r)}\right|$ convex bodies. By Proposition 1.4.6 we deduce that

$$\left| h_{\widetilde{\mathcal{D}} \times \mathcal{D}^{\mathbf{d}^{(r)}}}(Z) - \sum_{j=1}^{q} d_j^{(r)} h_{\widetilde{\mathcal{D}} \times \mathcal{D}^{[j]}}(Z) \right| \leq 2\Omega.$$

We can therefore prove by induction that

$$\Big| h_{\widetilde{\mathcal{D}} \times \mathcal{D}^{\mathbf{d}^{(r)}}}(Z) - \langle h(Z); \mathbb{d}\rangle \Big| \leq 2(r+1)\Omega. \tag{1.4.4}$$

The thesis is proved by combining (1.4.3) and (1.4.4). □

1.4.3 An arithmetic Bézout inequality

We report the following result ([LR01], Proposition 3.10).

Proposition 1.4.10. Let $r \in \mathbb{N}$ and $s \in \mathbb{N}_+$. For $j = 0, \dots, r$, let V_j be a vector space over K of finite dimension $n_j > 0$ and let $\mathcal{C}_j$ be an adelic convex body for V_j. Put $\mathcal{C} = \mathcal{C}_0 \times \cdots \times \mathcal{C}_r$ and $V = V_0 \times \cdots \times V_r$. Moreover, for $i = 1, \dots, s$, let F_i be a multihomogeneous polynomial map from V to K. Denote by $F = F_1 \cdots F_s$ the product of these maps and by $(d_1, \dots, d_r)$ the multidegree of F. Then $\mathcal{C}$ is an adelic convex body for V and we have

$$h_{\mathcal{C}}(F) \leq \sum_{i=1}^{s} h_{\mathcal{C}}(F_i) \leq h_{\mathcal{C}}(F) + 2\sum_{j=0}^{r} d_j \log n_j.$$

We remark that with the notation set in Definition 1.4.1 we have that a resultant form of an effective cycle $Z \in Z_r^+(\mathbb{P}_K^{\mathbf{n}})$ of index $\mathbb{d} \in (\mathbb{N}^q \backslash \{\mathbf{0}\})^{r+1}$ is a multihomogeneous polynomial map from $K[\mathbf{X}]_{\mathbf{d}^{(0)}} \times \cdots \times K[\mathbf{X}]_{\mathbf{d}^{(r)}}$ with multidegree $\underline{N} = (N_0, \dots, N_r)$. We have the following

Proposition 1.4.11. Let $\mathbb{d} = (\mathbf{d}^{(0)}, \dots, \mathbf{d}^{(r)}) \in (\mathbb{N}^q \backslash \{\mathbf{0}\})^{r+1}$ and $Z \in Z_r^+(\mathbb{P}_K^{\mathbf{n}})$ an effective cycle. If F is a resultant form of Z of index $\mathbb{d}$ then

$$0 \leq h_{\mathcal{C}}(Z) - h_{\mathcal{C}}(F) \leq 2\sum_{j=0}^{r} N_j \log \dim(\mathbf{d}^{(j)}). \tag{1.4.5}$$

Proof. Write $Z = \sum_{i=1}^{s} m_i V_i$, where $V_1, \dots, V_s$ are the distinct components of Z and $m_i > 0$ for $i = 1, \dots, s$. By definition we have $h_{\mathcal{C}}(Z) = \sum_{i=1}^{s} m_i h_{\mathcal{C}}(V_i)$ and $F = \lambda F_1^{m_1} \cdots F_s^{m_s}$, where F_i is a resultant form of V_i of index $\mathbb{d}$ and $\lambda \in K^\times$. Then Proposition 1.4.10 together with the equality $h_{\mathcal{C}}(F_i) = h_{\mathcal{C}}(V_i)$ and the previous remark plainly give (1.4.5). ☐

The following proposition is an arithmetic version of Bézout's inequality. It estimates the relative height of the intersection of an effective cycle with an hypersurface.

Proposition 1.4.12. Let $r \in \mathbb{N}_+$, $\mathbb{d} = (\mathbf{d}^{(0)}, \dots, \mathbf{d}^{(r)}) \in (\mathbb{N}^q \backslash \{\mathbf{0}\})^{r+1}$, $Z \in Z_r^+(\mathbb{P}_K^{\mathbf{n}})$, and $P \in K[\mathbf{X}]_{\mathbf{d}^{(r)}}$. Let also $\mathcal{C} = \mathcal{C}_0 \times \cdots \times \mathcal{C}_r$ be an adelic convex body of index $\mathbb{d}$ and $\mathcal{C}' = \mathcal{C}_0 \times \cdots \times \mathcal{C}_{r-1}$. Assume that P intersects properly the effective cycle Z, and let Z' be the intersection product $Z \cdot \mathrm{div}(P)$ defined in Definition 1.2.9. Then, we have

$$h_{\mathcal{C}'}(Z') \leq h_{\mathcal{C}}(Z) + N_r h_{\mathcal{C}_r}(P) + 2\sum_{j=0}^{r-1} N_j \log \dim(\mathbf{d}^{(j)}).$$

Proof. Let F be a resultant form of Z of index $\mathbb{d}$. Let F' the polynomial map from $K[\mathbf{X}]_{\mathbf{d}^{(0)}} \times \cdots \times K[\mathbf{X}]_{\mathbf{d}^{(r-1)}}$ to K given by

$$F'(P_0, \dots, P_{r-1}) := F(P_0, \dots, P_{r-1}, P).$$

Proposition 1.2.19 shows that F' is a resultant form of Z' of index $\widehat{\mathbb{d}}^{(r)}$. Since F is homogeneous of degree N_r on the factor $K[\mathbf{X}]_{\mathbf{d}^{(r)}}$, we have the upper bound

$$\|F'\|_{\mathcal{C}'_v} \leq \|P\|_{\mathcal{C}_{r,v}}^{N_r} \|F\|_{\mathcal{C}_v}$$

for any place $v \in \mathcal{M}_K$. Summing up the logarithms over all the places, we get

$$h_{\mathcal{C}'}(F') \leq h_{\mathcal{C}}(F) + N_r h_{\mathcal{C}_r}(P).$$

Proposition 1.4.11 gives the inequalities

$$h_{\mathcal{C}'}(Z') \leq h_{\mathcal{C}'}(F') + 2\sum_{j=0}^{r-1} N_j \log \dim(\mathbf{d}^{(j)}) \quad \text{and} \quad h_{\mathcal{C}}(F) \leq h_{\mathcal{C}}(Z),$$

and this completes the proof. $\square$

Chapter 2

A small value estimate

2.1 Statement of the main theorem

We define $\mathcal{D}_1 : \mathbb{C}[x,y] \to \mathbb{C}[x,y]$ to be the differential operator on the ring $\mathbb{C}[x,y]$ defined by the formula $\mathcal{D}_1 := \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$. Given a polynomial $P \in \mathbb{Z}[x,y]$, we define its norm $\|P\|$ to be the largest absolute value of its coefficients (see Definition 2.3.1). For a real number $\alpha \in \mathbb{R}$, the expression $\lfloor \alpha \rfloor$ denotes the largest integer less than or equal α.
The aim of this chapter is to prove the following

Theorem 2.1.1. Let $\gamma = (\xi, \eta) \in \mathbb{C} \times \mathbb{C}^\times$ and let $\beta, \tau, \nu, \delta, t_0, t_1, t$ be positive real numbers satisfying

$$\max\{t_0, t_1\} = 1, \quad \min\{t_0, t_1\} = t, \quad 1 < \tau < 1 + t,$$

$$\tau < \beta, \quad \nu = 1 + t + \beta - \tau + \delta$$

and $\delta > (\tau - t)(1 + t - \tau)/(\beta + 1 - \tau)$. Suppose that, for each sufficiently large positive integer N, there exists a nonzero polynomial $P_N \in \mathbb{Z}[x,y]$ with partial degrees $\deg_x(P_N) \le \lfloor N^{t_0} \rfloor$, $\deg_y(P_N) \le \lfloor N^{t_1} \rfloor$ and norm $\|P_N\| \le \exp(N^\beta)$, such that

$$\max_{0 \le i < 3\lfloor N^\tau \rfloor} |\mathcal{D}_1^i P_N(\xi, \eta)| \le \exp(-N^\nu). \tag{2.1.1}$$

Then, we have $\xi, \eta \in \overline{\mathbb{Q}}$ and moreover for each sufficiently large integer N we have $\mathcal{D}_1^i P_N(\xi, \eta) = 0$ for every $0 \le i < 3\lfloor N^\tau \rfloor$.

This is a generalization of an analogous statement proved in [Roy13], Main Theorem. The proof will follow the line of that article and relies on the theory developed in the previous chapter.
As it was mentioned in the introduction, theorem 2.1.1 is motivated by Schanuel's conjecture, whose statement reads as follows

Conjecture 2.1.2. Let ℓ be a positive integer and let $\alpha_1, \dots, \alpha_\ell \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$. Then

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(\alpha_1, \dots, \alpha_\ell, e^{\alpha_1}, \dots, e^{\alpha_\ell}) \ge \ell.$$

This far-reaching conjecture contains nearly all known results and all generally accepted conjectures on the transcendence of the values of the exponential function and it is currently widely open. However in [Roy01] D.Roy proposes an approach to tackle Schanuel's Conjecture, and he proves that it is equivalent to the following arithmetic statement.

Conjecture 2.1.3. Let ℓ be a positive integer, let $y_1, \dots, y_\ell \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$ and $\alpha_1, \dots, \alpha_\ell \in \mathbb{C}^\times$. Moreover, let s_0, s_1, t_0, t_1, ν be positive numbers satisfying

$$\max\{1, t_0, 2t_1\} < \min\{s_0, 2s_1\} < \nu < \frac{1}{2}(1 + t_0 + t_1). \tag{2.1.2}$$

Assume that, for any sufficiently large positive integer N, there exists a nonzero polynomial $P_N \in \mathbb{Z}[x, y]$ with partial degrees $\deg_x(P_N) \le \lfloor N^{t_0} \rfloor$, $\deg_y(P_N) \le \lfloor N^{t_1} \rfloor$ and norm $\|P_N\| \le e^N$, such that

$$\left| \mathcal{D}_1^k P_N\Big(\sum_{j=1}^{\ell} m_j y_j, \prod_{j=1}^{\ell} \alpha_j^{m_j}\Big) \right| \le \exp(-N^\nu), \tag{2.1.3}$$

for any integers $k, m_1, \dots, m_\ell \in \mathbb{N}$ with $k \le N^{s_0}$ and $\max\{m_1, \dots, m_\ell\} \le N^{s_1}$. Then

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(y_1, \dots, y_\ell, \alpha_1, \dots, \alpha_\ell) \ge \ell.$$

This conjecture was given in the form reported here in [Van]. It can be shown that if Conjecture 2.1.3 is true for some positive integer ℓ and some choice of parameters s_0, s_1, t_0, t_1, ν satisfying (2.1.2), then Schanuel's Conjecture 2.1.2 is true for this value of ℓ. Conversely, if Conjecture 2.1.2 is true for some positive integer ℓ, then Conjecture 2.1.3 is also true for the same value of ℓ and for any choice of parameters satisfying (2.1.2).
Actually, there are both analogies and differences between the statements of our Theorem 2.1.1 and of Conjecture 2.1.3, but we postpone a discussion on this topic to the final section of this thesis § 2.8.

2.1.1 A corollary

We are going to deduce from Theorem 2.1.1 a statement that considers polynomials in two variables evaluated at many points of a finitely generated subgroup of $\mathcal{G}$, together with their first invariant derivatives. To do this, we need to state some results.
The following is a version of the well-known Liouville's inequality ([Wal00], Proposition 3.14).

Theorem 2.1.4. Let K be a number field with $d = [K : \mathbb{Q}]$, let $v \in \mathcal{M}_K$ an archimedean place of K, and let $q \in \mathbb{N}_+$ be a positive integer. For $1 \le i \le q$, let $\gamma_i \in K$ and let $P \in \mathbb{Z}[x_1 \dots, x_q]$ be a polynomial in q variables, with coefficients in $\mathbb{Z}$, which does not vanish at the point $\underline{\gamma} = (\gamma_1, \dots, \gamma_q)$. Assume that P has partial degree at most N_i with respect to the variable x_i. Then

$$\log \left| P(\underline{\gamma}) \right|_v \ge -(d-1) \log \mathcal{L}(P) - d \sum_{i=1}^{q} N_i h(\gamma_i).$$

where $\mathcal{L}(P) = \mathcal{L}_v(P)$ denotes the length of P, i.e. the sum of the absolute values of the coefficients of P (see Definitions 2.3.1 and 1.3.2).

The following is essentially a special case of Philippon's multiplicity estimate on commutative linear algebraic groups [Phi86b]. We refer to [Ber85], [roy] and [Wal00] for more on this result.

Theorem 2.1.5. Let $\mathcal{G} := \mathbb{C} \times \mathbb{C}^\times$ and let Σ be a finite subset of $\mathcal{G}$ containing $e = (0, 1)$. Assume that, for $D \in N_+^2$ and $S_0 \in \mathbb{N}_+$, there exists a nonzero polynomial $P \in \mathbb{C}[x, y]$ with partial degrees $\deg_x(P) \leq D_x$ and $\deg_y(P) \leq D_y$ such that $\mathcal{D}_1^k P$ vanishes on the sumset $\Sigma + \Sigma$ (where the sum is with respect to the group law of $\mathcal{G}$), for every $k = 0, \ldots, 2S_0$. Let $|\Sigma|$, $|\pi_1(\Sigma)|$ and $|\pi_2(\Sigma)|$ be respectively the cardinalities of Σ and of its images through the projections $\pi_1 : \mathcal{G} \to \mathbb{C}$ and $\pi_2 : \mathcal{G} \to \mathbb{C}^\times$. Then

$$\min\{|\Sigma|, D_x |\pi_2(\Sigma)|, 2D_y |\pi_1(\Sigma)|\} \leq \frac{4D_x D_y}{S_0 + 1}.$$

Proof. We use the notation of [Wal00], Chapters 5 and 8. Since $\mathbb{C}[x, y] \subseteq \mathbb{C}[x, y, y^{-1}]$, we can apply [Wal00], Theorem 8.1 for the choice $K := \mathbb{C}$, $G = G^+ := \mathcal{G}$, $G^- := \{e\}$, $d_0 = d_1 := 1$, $d = d^+ := 2$, $D_0 := D_x$, $D_1 := D_y$, $S_0 := S_0$ and $\mathcal{W}$ the subspace of $T_e(G) = \mathbb{C}^2$ generated by the vector $w = (1, 1)$ to get a connected algebraic subgroup G^* of G^+ of dimension < 2 such that, if we set

$$\ell_0' = \dim_{\mathbb{C}} \left(\frac{\mathcal{W} + T_e(G^*)}{T_e(G^*)} \right),$$

then

$$\binom{S_0 + \ell_0'}{\ell_0'} \operatorname{Card}\left(\frac{\Sigma + G^*}{G^*}\right) \mathcal{H}(G^*; D) \leq \mathcal{H}(G^+; D).$$

There are only three cases for the choice of G^*.

- If $G^* = \{e\}$, then $\ell_0' = 1$, $\mathcal{H}(G^*; D) = 1$ and $\operatorname{Card}\left(\frac{\Sigma + G^*}{G^*}\right) = |\Sigma|$;
- if $G^* = \mathbb{C} \times \{1\}$, then $\ell_0' = 1$, $\mathcal{H}(G^*; D) = D_x$ and $\operatorname{Card}\left(\frac{\Sigma + G^*}{G^*}\right) = |\pi_2(\Sigma)|$;
- if $G^* = \{0\} \times \mathbb{C}^\times$, then $\ell_0' = 1$, $\mathcal{H}(G^*; D) = 2D_y$ and $\operatorname{Card}\left(\frac{\Sigma + G^*}{G^*}\right) = |\pi_1(\Sigma)|$.

Since in each of these cases he have $\binom{S_0 + \ell_0'}{\ell_0'} = S_0 + 1$ and $\mathcal{H}(G^+; D) = 4D_x D_y$, the thesis is proved. $\square$

We are now ready to prove the following consequence of Theorem 2.1.1.

Corollary 2.1.6. Let ℓ be a positive integer, let $(\xi_j, \eta_j) \in \mathbb{C} \times \mathbb{C}^\times$ for $j = 1, \ldots, \ell$, and let $\beta, \tau, \nu, \delta, t_0, t_1, t$ be as in Theorem 2.1.1. Suppose that for any sufficiently large positive integer N, there exists a nonzero polynomial $P_N \in \mathbb{Z}[x, y]$ of partial degrees $\deg_x(P_N) \leq \lfloor N^{t_0} \rfloor$, $\deg_y(P_N) \leq \lfloor N^{t_1} \rfloor$ and norm $\|P_N\| \leq \exp(N^\beta)$, such that

$$\left| \mathcal{D}_1^k P_N \left(\sum_{j=1}^{\ell} m_j \xi_j, \prod_{j=1}^{\ell} \eta_j^{m_j} \right) \right| \leq \exp(-N^\nu), \tag{2.1.4}$$

for any choice of integers $k, m_1, \ldots, m_\ell$ with $0 \leq k \leq 3N^\tau$ and $0 \leq m_1, \ldots, m_\ell \leq 8N^{(1+t-\tau)/\ell}$. Then, $\xi_1, \ldots, \xi_\ell$ are linearly dependent over $\mathbb{Q}$.

Proof. A direct application of Theorem 2.1.1 shows that ξ_j and η_j belong to the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$ for $j = 0, \ldots \ell$. We are now going to see that Liouville's inequality 2.1.4 implies that, for sufficiently large N, the left hand side of (2.1.4) vanishes for all admissible choices of $k, m_1, \ldots, m_\ell$. Indeed, let K be a number field containing all the numbers $\xi_1, \ldots, \xi_\ell, \eta_1, \ldots, \eta_\ell$, let $d := [K : \mathbb{Q}]$ and let $H := \max\{h(\xi_1), \ldots, h(\xi_\ell), h(\eta_1), \ldots, h(\eta_\ell)\}$. We consider on K the archimedean absolute value determined by the given inclusion $K \subseteq \mathbb{C}$. Let $N, k, m_1, \ldots, m_\ell$ such that the left hand side of (2.1.4) does not vanish. Adapting the arguments of Lemma 2.3.6 one can show, since $\tau < \beta$, that

$$\mathcal{L}(\mathcal{D}_1^k P_N) \leq \exp(2N^\beta)$$

for sufficiently big N. Moreover, by the basic properties of Weil absolute height stated in Proposition 1.3.12, we have

$$\begin{aligned} h(m_1\xi_j + \ldots + m_\ell\xi_\ell) &\leq \sum_{j=1}^{\ell}(\log(m_j) + h(\xi_j)) + (\ell - 1)\log(2) \\ &\leq (1 + t - \tau)\log(N) + \ell H + (\ell - 1)\log(2) \leq N^{\frac{1+t-\tau}{\ell}}, \end{aligned}$$

for sufficiently big N, and

$$h(\eta_1^{m_1} \cdots \eta_\ell^{m_\ell}) \leq \sum_{j=1}^{\ell} m_j h(\eta_j) \leq 2\ell H N^{\frac{1+t-\tau}{\ell}}.$$

Therefore, using $t_0, t_1 \leq 1$, Liouville's inequality 2.1.4 gives

$$\left|\mathcal{D}_1^k P_N(\sum_{j=1}^{\ell} m_j\xi_j, \prod_{j=1}^{\ell} \eta_j^{m_j})\right| \geq \exp(-2(d-1)N^\beta - d(2\ell H + 1)N^{\frac{1+t-\tau}{\ell}+1}).$$

Since $1 < \beta < \nu$ and $\frac{1+t-\tau}{\ell} + 1 < 1 + t + \beta - \tau < \nu$, this is contradicts (2.1.4) for sufficiently big N.
Thus, for such N and for $0 \leq k < 3\lfloor N^\tau \rfloor$, $\mathcal{D}_1^k P_N$ vanishes on the sumset $\Sigma_N + \Sigma_N$ where Σ_N consists of all points $(m_1\xi_j + \ldots + m_\ell\xi_\ell, \eta_1^{m_1} \cdots \eta_\ell^{m_\ell})$ with $0 \leq m_1, \ldots, m_\ell \leq 4N^{(1+t-\tau)/\ell}$. Since the projections of Σ_N on both factors of $\mathbb{C} \times \mathbb{C}^\times$ have cardinality at least 1, and since $2\lfloor N^\tau \rfloor < 3\lfloor N^\tau \rfloor$, it follows from Proposition 2.1.5 that

$$\min\{|\Sigma|, \lfloor N^{t_0} \rfloor, 2\lfloor N^{t_1} \rfloor\} \leq \frac{4\lfloor N^{t_0} \rfloor \lfloor N^{t_1} \rfloor}{\lfloor N^\tau \rfloor + 1}.$$

We have $t_0, t_1 \leq 1 < \tau$, so $\lfloor N^{t_0} \rfloor$ and $2\lfloor N^{t_1} \rfloor$ are greater than $\frac{4\lfloor N^{t_0} \rfloor \lfloor N^{t_1} \rfloor}{\lfloor N^\tau \rfloor + 1}$ for sufficiently large N. This implies

$$|\Sigma| \leq \frac{4\lfloor N^{t_0} \rfloor \lfloor N^{t_1} \rfloor}{\lfloor N^\tau \rfloor + 1} \leq 4N^{1+t-\tau}$$

On the other hand, if the numbers $\xi_1, \ldots, \xi_\ell$ were linearly independent over $\mathbb{Q}$ we would get the estimate

$$|\Sigma| \geq (4\lfloor N^{\frac{1+t-\tau}{\ell}} \rfloor + 1)^\ell > 4N^{1+t-\tau},$$

and this is a contradiction. $\square$

2.2 Subvarieties of $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$

In this chapter we study in detail the geometric and arithmetic invariants of $\mathbb{Q}$-subvarieties of the multiprojective space $\mathbb{P} := \mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$. We establish a few basic estimates for the heights of cycles cut by an hypersurface and of 0-dimensional cycles. We derive these estimates as corollaries of the general results of Sections § 1.3 and § 1.4. In this low-dimensional setting, cycles can only have dimension $0, 1$ or 2, and we have $n = q = 2$ and $n_1 = n_2 = 1$.
We identify the set of indices $\{1, 2\}$ with $\{x, y\}$. With this notation, we let $\mathbf{e}_x = \mathbf{e}_1 = (1, 0)$, $\mathbf{e}_y = \mathbf{e}_2 = (0, 1)$ be the standard basis elements of $\mathbb{N}^2$ and we denote the mixed heights of index $\underline{\mu} \in \{1, 2\}^{r+1}$ of an r-dimensional cycle by

$$h_{\underline{\sigma}}(Z),$$

where $\underline{\sigma} \in \{x, y\}^{r+1}$ is a string composed by letters of type x and y.
An element $D \in \mathbb{N}^2$ will be called *nonnegative bi-integer*, or *bidegree*. For $D \in \mathbb{N}^2$ we denote by D_x and D_y its two components and we define

$$|D| := D_x + D_y \quad \text{and} \quad \|D\| := \max\{D_x, D_y\}.$$

For $L, D \in \mathbb{N}^2$ we write $L \leq D$ if $L_x \leq D_x$ and $L_y \leq D_y$. We also write $L < D$ to mean $L_x < D_x$ and $L_y < D_y$.
An element $c \in \mathbb{R}^2_+$ will be called a *bi-constant* and we denote by c_x, c_y its two components. Given two bi-constants $c_1, c_2 \in \mathbb{R}^2_+$ we perform componentwise multiplication and addition, so that we have

$$\begin{aligned} c_1 + c_2 &:= (c_{1,x} + c_{2,x}, c_{1,y} + c_{2,y}), \\ c_1 c_2 &:= (c_{1,x} c_{2,x}, c_{1,y} c_{2,y}), \end{aligned}$$

and we define

$$c_1^{c_2} := c_{1,x}^{c_{2,x}} \cdot c_{1,y}^{c_{2,y}}. \tag{2.2.1}$$

2.2.1 Two-dimensional cycles

The only subvariety of $\mathbb{P} = \mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ having dimension $r = 2$ is $\mathbb{P}$ itself. It is easy to see that

$$H_{\mathbb{Q}[\mathbf{X},\mathbf{Y}]} = (T_1 + 1)(T_2 + 1).$$

This implies that the only nonzero term of $\deg(\mathbb{P}) \in \mathbb{N}^{\mathbb{N}^2}$ is

$$\deg(\mathbb{P})_{(1,1)} = 1.$$

An arbitrary collection of $r + 1 = 3$ multidegrees takes the form

$$\mathbb{d} = (D_0, D_1, D_2)$$

and in this situation we have (see Definition 1.4.1)

$$N_0 = \langle \deg(\mathbb{P}); (D_1, D_2) \rangle = D_{1,x} D_{2,y} + D_{1,y} D_{2,x} \tag{2.2.2}$$

$$N_1 = \langle \deg(\mathbb{P}); (D_0, D_2) \rangle = D_{0,x} D_{2,y} + D_{0,y} D_{2,x} \tag{2.2.3}$$

$$N_2 = \langle \deg(\mathbb{P}); (D_0, D_1) \rangle = D_{0,x} D_{1,y} + D_{0,y} D_{1,x} \tag{2.2.4}$$

so that

$$\sum_{j=0}^{r} N_j|D_j| = |D_0|\,|D_1|\,|D_2| - D_{0,x}D_{1,x}D_{2,x} - D_{0,y}D_{1,y}D_{2,y}. \tag{2.2.5}$$

We are now going to see that the mixed heights $h_{xxx}(\mathbb{P})$, $h_{xxy}(\mathbb{P})$, $h_{xyy}(\mathbb{P})$, $h_{yyy}(\mathbb{P})$ of $\mathbb{P}$ are pretty easy to describe. We then use this calculation to estimate the height of $\mathbb{P}$ with respect to the convex bodies $\mathcal{B}^{(\mathbb{d})}$.

Lemma 2.2.1. All the primitive multiprojective heights of $\mathbb{P}^1 \times \mathbb{P}^1$ are 0.

Proof. In [Rém01a], Lemme 3.7 there is an explicit description of the resultant form of $\mathbb{P}^1 \times \mathbb{P}^1$ of index $\mathbb{d} = (\mathbf{e}_x, \mathbf{e}_y, D)$, with $D \in \mathbb{N}^2 \backslash \{\mathbf{0}\}$. We recall the construction in the case $D = \mathbf{e}_x$. We consider the set of variables

$$\mathbf{u}^{(0)} = \{\mathbf{u}^{(0)}_{X_0}, \mathbf{u}^{(0)}_{X_1}\}, \quad \mathbf{u}^{(0)} = \{\mathbf{u}^{(1)}_{Y_0}, \mathbf{u}^{(1)}_{Y_1}\}, \quad \mathbf{u}^{(0)} = \{\mathbf{u}^{(2)}_{X_0}, \mathbf{u}^{(2)}_{X_1}\},$$

and $\mathbf{u} = \{\mathbf{u}^{(0)}, \mathbf{u}^{(1)}, \mathbf{u}^{(2)}\}$. We consider first the 1×2 matrices

$$M_x = \begin{bmatrix} u^{(0)}_{X_0} & u^{(0)}_{X_1} \end{bmatrix} \quad \text{and} \quad M_y = \begin{bmatrix} u^{(1)}_{Y_0} & u^{(1)}_{Y_1} \end{bmatrix}.$$

We also introduce the vector of their minors

$$\Delta = (u^{(0)}_{X_1}, -u^{(0)}_{X_0}, u^{(1)}_{Y_1}, -u^{(1)}_{Y_0}),$$

and then, for the generic linear forms

$$L_0 = u^{(0)}_{X_0}X_0 + u^{(0)}_{X_1}X_1, \quad L_1 = u^{(1)}_{Y_0}Y_0 + u^{(1)}_{Y_1}Y_1, \quad L_2 = u^{(2)}_{X_0}X_0 + u^{(2)}_{X_1}X_1,$$

we have

$$\mathrm{elim}_{\mathbb{d}}(\mathbb{P}) = \mathrm{res}_{\mathbb{d}}(\mathbb{P}) = L_2(\Delta) = u^{(2)}_{X_0}u^{(0)}_{X_1} - u^{(2)}_{X_1}u^{(0)}_{X_0}. \tag{2.2.6}$$

With analogous notation and procedure, we check that for $\mathbb{d}' = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_x)$ we have

$$\mathrm{elim}_{\mathbb{d}'}(\mathbb{P}) = \mathrm{res}_{\mathbb{d}'}(\mathbb{P}) = u^{(2)}_{Y_0}u^{(1)}_{Y_1} - u^{(2)}_{Y_1}u^{(1)}_{Y_0}.$$

From these explicit formulas we deduce $h_{xyx}(\mathbb{P}) = h_{xyy}(\mathbb{P}) = \log(1) = 0$. Since by Proposition 1.2.16 the mixed heights are invariant by index permutation, we get also $h_{yxx}(\mathbb{P}) = h_{xxy}(\mathbb{P}) = h_{yxy}(\mathbb{P}) = h_{yyx}(\mathbb{P}) = 0$. For indices $(\mathbf{e}_x, \mathbf{e}_x, \mathbf{e}_x)$ and $(\mathbf{e}_y, \mathbf{e}_y, \mathbf{e}_y)$ we get by the formulas (2.2.2)-(2.2.4) that $N_0 = N_1 = N_2 = 0$. Proposition 1.2.18 thus states that the resultant forms $\mathrm{res}_{(\mathbf{e}_x,\mathbf{e}_x,\mathbf{e}_x)}(\mathbb{P})$ and $\mathrm{res}_{(\mathbf{e}_y,\mathbf{e}_y,\mathbf{e}_y)}(\mathbb{P})$ are homogeneous of degree 0 in all their variables. So, in this case they are equal to the constant polynomial form 1 and again we deduce $h_{xxx}(\mathbb{P}) = h_{yyy}(\mathbb{P}) = 0$. $\square$

Remark 2.2.2. We notice that formula (2.2.6) makes sense in virtue of the specialization property of the eliminant forms (Theorem 1.1.2): indeed it is easy to see that when the six variables in $\mathbf{u}$ are specialized to elements of a field $\mathbb{L}$, then the resulting linear forms $\widetilde{L_0}$, $\widetilde{L_1}$, $\widetilde{L_1}$ have a common zero in $\mathbb{P}^1_{\mathbb{L}} \times \mathbb{P}^1_{\mathbb{L}}$ iff $\det\begin{pmatrix} u^{(0)}_{X_0} & u^{(0)}_{X_1} \\ u^{(2)}_{X_0} & u^{(2)}_{X_1} \end{pmatrix} = 0$. We also notice that (2.2.6) is also consistent with the formula for the degrees of resultant forms from Proposition 1.2.18: we have $\mathbb{d} = ((1,0),(0,1),(1,0))$, so the resultant form is homogeneous of degree $N_0 = 1$, $N_1 = 0$, $N_2 = 1$ respectively in the coordinates $\mathbf{u}^{(0)}, \mathbf{u}^{(1)}, \mathbf{u}^{(2)}$.

Lemma 2.2.3. Let $\mathbb{d} = (D_0, D_1, D_2) \in (\mathbb{N}^q \backslash \{\mathbf{0}\})^3$ be a collection of nonzero bidegrees and let $\mathcal{B} = \mathcal{B}^{(\mathbb{d})}$ be as in Definition 1.3.9. Then, for $\mathbb{P} = \mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$, we have

$$h_{\mathcal{B}}(\mathbb{P}) \leq 7\log(2)\big(|D_0|\,|D_1|\,|D_2| - D_{0,x}D_{1,x}D_{2,x} - D_{0,y}D_{1,y}D_{2,y}\big).$$

Proof. The result follows directly from (2.2.5) and from Proposition 1.4.9, where $Z = \mathbb{P}$, $r = 2$, $\|n\| = 1$ and $h_{\underline{\sigma}} = 0$ for all $\underline{\sigma} \in \{x, y\}^3$ thanks to Lemma 2.2.1. ☐

We notice that in the case $D_0 = D_1 = D_2 = D$ Lemma 2.2.3 reduces to

$$h_{\mathcal{B}}(\mathbb{P}) \leq 21\log(2)D_xD_y\,|D|\,.$$

2.2.2 One-dimensional cycles

The Hilbert-Samuel polynomial of an irreducible subvariety Z of $\mathbb{P}$ of dimension $r = 1$ is a polynomial in two variables of total degree 1, so it takes the form

$$H = aT_1 + bT_2 + c.$$

This implies that there are only two nonzero components of $\deg(Z)$:

$$\deg_x(Z) := \deg(Z)_{(1,0)} \qquad \deg_y(Z) := \deg(Z)_{(0,1)}.$$

An arbitrary collection of $r + 1 = 2$ multidegrees takes the form

$$\mathbb{d} = (D_0, D_1) = ((D_{0,x}, D_{0,y}), (D_{1,x}, D_{1,y}))$$

and in this situation we have

$$\begin{aligned} N_0 &= \langle \deg(Z); (D_1)\rangle \deg_x(Z)D_{1,x} + \deg_y(Z)D_{1,y} \\ N_1 &= \langle \deg(Z); (D_0)\rangle = \deg_x(Z)D_{0,x} + \deg_y(Z)D_{0,y} \end{aligned}$$

so that

$$\begin{aligned} \sum_{j=0}^{r} N_j|D_j| = {} & \deg_x(Z)\,(2D_{0,x}D_{1,x} + D_{0,x}D_{1,y} + D_{0,y}D_{1,x}) + \\ & + \deg_y(Z)\,(2D_{0,y}D_{1,y} + D_{0,x}D_{1,y} + D_{0,y}D_{1,x}). \end{aligned}$$

Finally, we have three different mixed heights in this situation, namely

$$h_{xx}(Z), h_{xy}(Z) = h_{yx}(Z), h_{yy}(Z).$$

The easiest way to produce a 1-dimensional cycle of $\mathbb{P}$ is to consider hypersurfaces. Actually, since $\mathbb{Q}[\mathbf{X}, \mathbf{Y}]$ is a factorial ring, all its primes of height 1 are principal, generated by an irreducible polynomial. Thus all integral closed subschemes of dimension 1 in $\mathbb{P}$ are in fact hypersurfaces.

Proposition 2.2.4. Let $D \in \mathbb{N}^2 \backslash \{\mathbf{0}\}$ be a nonzero bidegree, let $\mathcal{C}$ be a convex body of $\mathbb{C}[\mathbf{X}, \mathbf{Y}]_D$, $\mathcal{C}' = \mathcal{C} \times \mathcal{C}$ and $\mathcal{C}'' = \mathcal{C}' \times \mathcal{C}$. Suppose that there exists a nonzero polynomial $P \in \mathbb{Z}[\mathbf{X}, \mathbf{Y}]_D \cap \mathcal{C}$. Then $Z' = \mathrm{div}(P)$ is an effective cycle of $\mathbb{P} = \mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ of pure dimension 1 which satisfies

(i) $\deg_x(Z') = D_y$ and $\deg_y(Z') = D_x$,

(ii) $h_{xx}(Z') \leq 11\log(2)D_y$,
$h_{xy}(Z') \leq \log\|P\| + 9\log(2)|D|$,
$h_{yy}(Z') \leq 11\log(2)D_x$,

(iii) $h_{\mathcal{C}'}(Z') \leq h_{\mathcal{C}''}(\mathbb{P}) + 8\log(2)D_xD_y|D|$.

Proof. We consider Z' as the intersection product $Z' = \mathbb{P}\cdot\operatorname{div}(P)$ as in Definition 1.2.9. Then, (i) follows from Theorem 1.2.11 while (iii) derives from Lemma 1.4.12, because $h_{\mathcal{C}}(P) \leq 0$, $N_0 = N_1 = N_2 = 2D_xD_y$ and $\log\dim(D) \leq \log(2)\,|D|$. To prove (ii), we note that we have $P \in \|P\|\,\mathcal{B}$ for the convex body $\mathcal{B} = \mathcal{B}^{(D)}$ of Definition 1.3.8 and so, by Lemma 1.4.12 applied to the convex body $\mathcal{E} = \mathcal{B}^{(\mathbf{e}_\mu,\mathbf{e}_\nu,D)} = \mathcal{D}^{\mathbf{e}_\mu} \times \mathcal{D}^{\mathbf{e}_\nu} \times \mathcal{B}$ for $\mu,\nu \in \{x,y\}$, we get

$$h_{\mathcal{D}^{\mathbf{e}_\mu}\times\mathcal{D}^{\mathbf{e}_\nu}}(Z') \leq h_{\mathcal{E}}(\mathbb{P}) + N_2'\log\|P\| + 2\log(2)(N_0' + N_1'),$$

with

$$N_0' = D_{\nu^c}, \quad N_1' = D_{\mu^c}, \quad \text{and} \quad N_2' = \delta_{\mu,\nu^c} = \begin{cases} 1 & \text{if } \mu \neq \nu \\ 0 & \text{if } \mu = \nu \end{cases},$$

thanks to the explicit calculations in equations (2.2.2)-(2.2.4), where μ^c (resp. ν^c) is defined to be the only element in $\{x,y\}$ different from μ (resp. ν). Then, since $\mathcal{D}^{\mathbf{e}_\mu} \times \mathcal{D}^{\mathbf{e}_\nu} = \mathcal{D}^{(\mu,\nu)}$, we see that (ii) follows by combining this upper bound with the estimates

$$h_{\mu\nu}(Z') \leq h_{\mathcal{D}^{(\mu,\nu)}}(Z')$$

coming from Lemma 1.4.8 and

$$h_{\mathcal{E}}(\mathbb{P}) \leq 7\log(2)(|D| - D_\mu\delta_{\mu,\nu}).$$

coming from Lemma 2.2.3. □

2.2.3 Zero-dimensional cycles

The Hilbert-Samuel polynomial of an irreducible subvariety Z of $\mathbb{P}$ of dimension $r = 0$ is a polynomial of total degree 0, so it must be constant. This implies that there is only one nonzero component of $\deg(Z)$, namely

$$\deg(Z) := \deg(Z)_{(0,0)} \in \mathbb{N}.$$

With a bit of abuse of notation we omit parentheses from collections of multidegrees with only one element, writing

$$\mathbb{d} = D = (D_x, D_y).$$

Here we have a degenerate calculation for N_0

$$N_0 = \langle \deg(Z); \emptyset \rangle = \deg(Z),$$

so that

$$\sum_{j=0}^{r} N_j\,|D_j| = N_0\,|D| = \deg(Z)(D_x + D_y).$$

In this case we have to deal with only two mixed heights, which we collect into

$$h(Z) = (h_x(Z), h_y(Z)).$$

As we have seen in Section § 1.2.4, we have a simple description for 0-dimensional subvarieties of $\mathbb{P}$. Indeed, let Z be a 0-dimensional irreducible subvariety of $\mathbb{P}$. Then $Z(\mathbb{C})$ consists of exactly $\deg(Z)$ points, which actually lie in $Z(\overline{\mathbb{Q}})$ and are conjugate to each other through the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.
We can obtain a 0-dimensional cycle by cutting a 1-dimensional subvariety with an hypersurface.

Proposition 2.2.5. Let $\mathbb{d} = (D_0, D) \in (\mathbb{N}^2 \backslash \{\mathbf{0}\})^2$ be a pair of bidegrees, let $\widetilde{\mathcal{C}} = \mathcal{C}_0 \times \mathcal{C}$ be a convex body of $\mathbb{C}[\mathbf{X},\mathbf{Y}]_{D_0} \times \mathbb{C}[\mathbf{X},\mathbf{Y}]_D$, and let Z be an irreducible subvariety of $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ of dimension $r = 1$. Suppose that there exists a polynomial $P \in \mathbb{Z}[\mathbf{X},\mathbf{Y}]_D \cap \mathcal{C}$ which does not belong to the ideal of Z. Put $N_0 = D_x \deg_x(Z) + D_y \deg_y(Z)$ and let $\mu \in \{x, y\}$. Then there exists an effective cycle Z' of $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ of pure dimension 0 which satisfies

(i) $\deg(Z') = N_0$,

(ii) $$\begin{aligned} h_\mu(Z') \le \quad & D_x h_{\mu x}(Z) + D_y h_{\mu y}(Z) + \\ & + \deg_\mu(Z)(5\log(2)\,|D| + \log\|P\|) + 7\log(2) N_0, \end{aligned}$$

(iii) $h_{\mathcal{C}_0}(Z') \le h_{\widetilde{\mathcal{C}}}(Z) + 2\log(2)|D_0| N_0$.

Proof. Define Z' to be the intersection product $Z' = Z \cdot \mathrm{div}(P)$ as in Definition 1.2.9. Then, (i) follows from Theorem 1.2.11 while (iii) derives from Lemma 1.4.12, because $h_{\mathcal{C}}(P) \le 0$. To prove (ii), we note that we have $P \in \|P\|\,\mathcal{B}$ for the convex body $\mathcal{B} = \mathcal{B}^{(D)}$ of Definition 1.3.8 and so, by Lemma 1.4.12 applied to the convex body $\mathcal{E} = \mathcal{B}^{(\mathbf{e}_\mu, D)} = \mathcal{D}^{\mathbf{e}_\mu} \times \mathcal{B}$ for $\mu = x, y$, we get

$$h_{\mathcal{D}^{\mathbf{e}_\mu}}(Z') \le h_{\mathcal{E}}(Z) + N_1' \log\|P\| + 2\log(2) N_0,$$

where $N_1' = \deg_\mu(Z)$. Then, since $\mathcal{D}^{\mathbf{e}_\mu} = \mathcal{D}^\mu$, we see that (ii) follows by combining this upper bound with the estimates

$$h_\mu(Z') \le h_{\mathcal{D}^\mu}(Z')$$

coming from Lemma 1.4.8 and

$$h_{\mathcal{E}}(Z) - D_x h_{\mu x}(Z) - D_y h_{\mu y}(Z) \le 5\log(2)(N_0 + |D| \deg_\mu(Z)).$$

coming from Lemma 1.4.9. □

Definition 2.2.6. Given $\alpha \in \mathbb{P}^{n_1}(\mathbb{C}) \times \cdots \times \mathbb{P}^{n_q}(\mathbb{C})$, we say that a representative $\underline{\alpha} = (\alpha^{(1)}, \ldots, \alpha^{(q)}) \in \mathbb{C}^{n_1+1} \times \cdots \times \mathbb{C}^{n_q+1} = \mathbb{C}^{n+q}$ of α is *normalized* if it satisfies $\|\alpha^{(i)}\| = 1$ for $i = 1, \ldots, q$, where $\|\cdot\|$ denotes the sup norm of $\mathbb{C}^{n_i+1}$.

Proposition 2.2.7. Let $D = (D_x, D_y)$ be a nonzero bidegree, let $\mathcal{C}$ be a convex body of $\mathbb{C}[\mathbf{X},\mathbf{Y}]_D$, let Z be a subvariety of $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ of dimension 0, and let $\underline{Z}$ be a set of normalized representatives of the points of $Z(\mathbb{C})$ by elements of $\mathbb{C}^4$. Then, we have

$$\left| h_{\mathcal{C}}(Z) - D_x h_x(Z) - D_y h_y(Z) - \sum_{\underline{\alpha} \in \underline{Z}} \log \sup_{P \in \mathcal{C}} |P(\underline{\alpha})| \right| \le 6\log(2)|D| \deg(Z). \tag{2.2.7}$$

Moreover, if there exists a polynomial $P \in \mathbb{Z}[\mathbf{X},\mathbf{Y}]_D \cap \mathcal{C}$ which does not belong to the ideal of Z, then we have $h_{\mathcal{C}}(Z) \geq 0$ and

$$0 \leq 3\log(2)\,|D|\deg(Z) + D_x h_x(Z) + D_y h_y(Z) + \sum_{\underline{\alpha}\in\underline{Z}} \log|P(\underline{\alpha})|. \qquad (2.2.8)$$

Proof. Let F be a resultant form of Z in degree D with integer coefficients. There is a constant $a \in \mathbb{C}^\times$ depending only on F and Z such that, for any $P \in \mathbb{C}[\mathbf{X},\mathbf{Y}]_D$, we have

$$F(P) = a \prod_{\underline{\alpha}\in\underline{Z}} P(\underline{\alpha})\,. \qquad (2.2.9)$$

As this is a factorization of F into a product of $\deg(Z)$ linear forms on $\mathbb{C}[\mathbf{X},\mathbf{Y}]_D$, Proposition 1.4.5 gives

$$0 \leq h_{\mathcal{C}}(Z) - \log|a| - \sum_{\underline{\alpha}\in\underline{Z}} \log\sup\{|P(\underline{\alpha})| \,:\, P\in\mathcal{C}\} \leq 2\deg(Z)\log\dim(D). \qquad (2.2.10)$$

Applying this estimate to the convex body $\mathcal{B} = \mathcal{B}^{(D)}$ instead of $\mathcal{C}$, we get

$$0 \leq h_{\mathcal{B}}(Z) - \log|a| \leq 3\deg(Z)\log\dim(D) \qquad (2.2.11)$$

because for each of the $\deg(Z)$ points $\underline{\alpha}$ of $\underline{Z}$, we have

$$0 \leq \log\sup\{|P(\underline{\alpha})| \,:\, P\in\mathcal{B}\} \leq \log\dim(D).$$

Combining Proposition 1.4.9 with the explicit calculations given at the beginning of Section § 2.2.3 we get

$$\Big|h_{\mathcal{B}}(Z) - D_x h_x(Z) - D_y h_y(Z)\Big| \leq 3\log(2)\,|D|\deg(Z). \qquad (2.2.12)$$

Putting together the inequalities (2.2.10),(2.2.11) and (2.2.12) and using that $\log\dim(D) \leq \log(2)\,|D|$ we deduce (2.2.7). Finally, if a polynomial $P \in \mathbb{Z}[\mathbf{X},\mathbf{Y}]_D \cap \mathcal{C}$ does not belong to the ideal of Z, then we have $F(P) \in \mathbb{Z}\setminus\{0\}$ and so $\log|F(P)| \geq 0$. The estimate (2.2.8) then follows from (2.2.9), (2.2.11) and (2.2.12), because we have

$$\log|a| \leq h_{\mathcal{B}}(Z) \leq D_x h_x(Z) + D_y h_y(Z) + 3\log(2)\,|D|\deg(Z)$$

and

$$-\log|a| \leq \log|F(P)| - \log|a| = \sum_{\underline{\alpha}\in\underline{Z}} \log|P(\underline{\alpha})|\,.$$

□

2.3 Estimates on $\mathbb{C}[\mathbf{X},\mathbf{Y}]$

2.3.1 Definitions and basic estimates

Here we shall work with the ring $\mathbb{C}[\mathbf{X},\mathbf{Y}]$ in the two groups of variables $\mathbf{X} = (X_0, X_1)$ and $\mathbf{Y} = (Y_0, Y_1)$. Given $D = (D_x, D_y) \in \mathbb{N}^2$ we denote by $\mathbb{C}[\mathbf{X},\mathbf{Y}]_D$ the set of bihomogeneous polynomial of bidegree D, i.e. homogeneous of degree

D_x in the variables $\mathbf{X}$ and homogeneous of degree D_y in the variables $\mathbf{Y}$. We recall the definitions of $|D| := D_x + D_y$ and $\|D\| := \max\{D_x, D_y\}$, and we define $\dim D := \dim_{\mathbb{C}} \mathbb{C}[\mathbf{X},\mathbf{Y}]_D = (D_x+1)(D_y+1)$.
When the bidegree D is clear from the context we shall for brevity denote by $\mathbf{X}^a\mathbf{Y}^b$ the monomial $X_0^{D_x-a}X_1^aY_0^{D_y-b}Y_1^b$, with the convention that this expression is 0 when one of the exponents involved is negative.

Definition 2.3.1. Given a (nonzero) polynomial $Q \in \mathbb{C}[\mathbf{X},\mathbf{Y}]$, we define its norm $\|Q\|$ as the largest absolute value of its coefficients and its length $\mathcal{L}(Q)$ as the sum of the absolute values of its coefficients. We define somewhat consistently $\|0\| = 0$ and $\mathcal{L}(0) = 0$.

We will often use the following properties of the length

Lemma 2.3.2. For every $P, Q \in \mathbb{C}[\mathbf{X},\mathbf{Y}]$ we have

$$\mathcal{L}(P+Q) \le \mathcal{L}(P) + \mathcal{L}(Q) \qquad \mathcal{L}(PQ) \le \mathcal{L}(P)\mathcal{L}(Q). \tag{2.3.1}$$

Proof. The idea for the proof of both inequalities is to replace the coefficients by their absolute values and to evaluate the resulting polynomial in $(1,1,1,1)$. If write the polynomials P, Q in the standard form

$$P = \sum_{\nu\in\mathbb{N}^2} p_\nu \mathbf{X}^{\nu_x}\mathbf{Y}^{\nu_y}, \quad Q = \sum_{\nu\in\mathbb{N}^2} q_\nu \mathbf{X}^{\nu_x}\mathbf{Y}^{\nu_y}.$$

we have

$$\mathcal{L}(P+Q) = \sum_{\nu\in\mathbb{N}^2} |p_\nu + q_\nu| \le \sum_{\nu\in\mathbb{N}^2} (|p_\nu| + |q_\nu|) = \mathcal{L}(P) + \mathcal{L}(Q),$$

thus the first inequality is proven. Similarly, we have

$$\mathcal{L}(PQ) = \sum_{\nu\in\mathbb{N}^2} |\sum_{\substack{\alpha,\beta\in\mathbb{N}^2\\ \alpha+\beta=\nu}} p_\alpha q_\beta| \le \sum_{\nu\in\mathbb{N}^2} \sum_{\substack{\alpha,\beta\in\mathbb{N}^2\\ \alpha+\beta=\nu}} |p_\alpha|\,|q_\beta| = \mathcal{L}(P)\mathcal{L}(Q).$$

□

Let $\mathcal{G}$ denote the commutative group $(\mathbb{G}_a \times \mathbb{G}_m)(\mathbb{C}) = \mathbb{C} \times \mathbb{C}^\times$ with its group law written additively. For each $\gamma = (\xi, \eta) \in \mathcal{G}$ we define $\vartheta_\gamma := (1, \xi, 1, \eta)$.

Definition 2.3.3. We denote by τ_γ the $\mathbb{C}$-algebra automorphism of $\mathbb{C}[\mathbf{X},\mathbf{Y}]$ given by

$$\tau_\gamma(P(\mathbf{X},\mathbf{Y})) := P(X_0, \xi X_0 + X_1, Y_0, \eta Y_1),$$

so that, for every $\gamma, \gamma' \in \mathcal{G}$ and any $P \in \mathbb{C}[\mathbf{X},\mathbf{Y}]$, we have

$$(\tau_\gamma P)(\vartheta_{\gamma'}) = P(\vartheta_{\gamma+\gamma'}) \quad \text{and} \quad \tau_\gamma \circ \tau_{\gamma'} = \tau_{\gamma+\gamma'}.$$

We also define on $\mathbb{C}[\mathbf{X},\mathbf{Y}]$ the following derivation

Definition 2.3.4.

$$\mathcal{D} := X_0 \frac{\partial}{\partial X_1} + Y_2 \frac{\partial}{\partial Y_2}.$$

This operator is $\mathcal{G}$-invariant, meaning that that $\tau_\gamma \circ \mathcal{D} = \mathcal{D} \circ \tau_\gamma$ for any $\gamma \in \mathcal{G}$. For every bidegree $D \in \mathbb{N}^2$ we have that

$$\mathcal{D}(\mathbf{X}^a\mathbf{Y}^b) = a\mathbf{X}^{a-1}\mathbf{Y}^b + b\mathbf{X}^a\mathbf{Y}^b.$$

Finally, we introduce, for every $T \in \mathbb{N}_+$, an ideal of $\mathbb{C}[\mathbf{X},\mathbf{Y}]$ consisting of polynomials vanishing in γ to order at least T.

Definition 2.3.5. Let $T \in \mathbb{N}_+$. We denote by $I^{(\gamma,T)}$ the ideal of $\mathbb{C}[\mathbf{X},\mathbf{Y}]$ generated by all bihomogeneous polynomials P satisfying $\mathcal{D}^i P(\vartheta_\gamma) = 0$ for $i = 0,\dots,T-1$. For $D \in \mathbb{N}^2$, the symbol $I_D^{(\gamma,T)}$ represents its homogeneous part of degree D.

We will see in Section § 2.4 that this ideal can be used to estimate the distance of a point α from γ.
We first establish a lemma providing estimates for $\tau_\gamma Q$ and $\mathcal{D}^i Q$.

Lemma 2.3.6. Let $D \in \mathbb{N}^2$ and $Q \in \mathbb{C}[\mathbf{X},\mathbf{Y}]_D$. For any $\gamma = (\xi,\eta) \in \mathcal{G}$ and $i \in \mathbb{N}$, we have

$$\mathcal{L}(\tau_\gamma Q) \le c_1(\gamma)^D \|Q\|, \qquad \mathcal{L}(\mathcal{D}^i Q) \le |D|^i \, \mathcal{L}(Q)$$

and

$$\left|\mathcal{D}^i Q(\vartheta_\gamma)\right| \le c_2(\gamma)^D \, |D|^i \, \mathcal{L}(Q),$$

where $c_1(\gamma)$ is a pair $(2 + |\xi|, 1 + |\eta|)$ and similarly $c_2(\gamma)$ is a bi-constant $(\max\{1,|\xi|\}, \max\{1,|\eta|\})$. We recall (see the beginning of Section § 2.2) that we use the notation $(A,B)^D$ to shorten the expression $A^{D_x} \cdot B^{D_y}$.

Proof. We write the bihomogeneous polynomial Q in the standard form

$$Q = \sum_{(a,b)\le D} q_{a,b}\mathbf{X}^a\mathbf{Y}^b$$

so that

$$\begin{aligned}
\mathcal{L}(\tau_\gamma Q) &\le \sum_{(a,b)\le D} \mathcal{L}\left(q_{a,b}\, X_0^{D_x-a}(\xi X_0 + X_1)^a Y_0^{D_y-b}(\eta Y_1)^b\right) \\
&\le \sum_{(a,b)\le D} |q_{a,b}| \cdot 1 \cdot (|\xi|+1)^a \cdot 1 \cdot (\eta)^b \\
&\le \|Q\| \sum_{a=0}^{D_x} (|\xi|+1)^a \sum_{b=0}^{D_y} |\eta|^b \le c_1(\gamma)^D \|Q\|.
\end{aligned}$$

For the second inequality we inductively apply the following

$$\begin{aligned}
\mathcal{L}(\mathcal{D}Q) &= \mathcal{L}\Big(\sum_{(a,b)\le D} q_{a,b}\, a\mathbf{X}^{a-1}\mathbf{Y}^b + q_{a,b}\, b\mathbf{X}^a\mathbf{Y}^b\Big) \\
&\le D_x\mathcal{L}(Q) + D_y\mathcal{L}(Q) = |D|\,\mathcal{L}(Q).
\end{aligned}$$

As for the third, it is a direct consequence of the second inequality because

$$Q(1,\xi,1,\eta) \le \mathcal{L}(Q) \max_{(a,b)\le D} |\xi|^a\,|\eta|^b$$

for every $Q \in \mathbb{C}[\mathbf{X},\mathbf{Y}]_D$. ∎

2.3.2 An interpolation result

We report the following proposition due to D.Roy and V.Nguyen (Lemma 3.1 in [VR14]), which is an improvement of Mahler's formula (7) from page 88 of [Mah67], also stated as Lemma 2 in [Tij73]. We use the notation

$$i^{(0)} := 1, \quad i^{(\mu)} := i(i-1)\dots(i-\mu+1) \quad , \text{ for } i \in \mathbb{N},\ \mu \in \mathbb{N}_+$$

and the convention that the empty product is 1, in particular $z^0 = 1$ for any $z \in \mathbb{C}$.

Lemma 2.3.7. Consider the linear recurrence sequence $u = (u_i)_{i\in\mathbb{N}}$ given by

$$u_i = \sum_{\nu=0}^{n-1} \sum_{\mu=0}^{m_\nu - 1} A_{\mu,\nu} i^{(\mu)} \alpha_\nu^{i-\mu} \qquad (i \in \mathbb{N}),$$

for fixed $n \in \mathbb{N}_+$, $\alpha_\nu \in \mathbb{C}$, $m_\nu \in \mathbb{N}_+$ and $A_{\mu,\nu} \in \mathbb{C}$, with $\alpha_0, \dots, \alpha_{n-1}$ distinct. Set

$$M = \sum_{\nu=0}^{n-1} m_\nu \ , \qquad a_0 = \left(\max_{0\le\nu<n} \binom{M}{m_\nu} \right) \prod_{\nu=0}^{n-1} (1 + |\alpha_\nu|)^{m_\nu}$$

$$a_1 = \min_{0\le\nu<n} \prod_{\substack{0\le\nu'<n \\ \nu'\neq\nu}} |\alpha_{\nu'} - \alpha_\nu|^{m_{\nu'}} \ , \quad a_2 = \min_{\substack{0\le\nu,\nu'<n \\ \nu\neq\nu'}} \min\{1, |\alpha_{\nu'} - \alpha_\nu|^{m_\nu}\}$$

with the understanding that $a_1, a_2 = 1$ if $n = 1$. Then, we have

$$\max |A_{\mu,\nu}| \le \frac{a_0}{a_1 a_2} \max_{0\le i<M} |u_i| \, .$$

With the help of the above result we can prove the following interpolation estimate.

Proposition 2.3.8. Let $\gamma \in \mathcal{G}$, $D \in \mathbb{N}^2$ and put $M = (D_x + 1)(D_y + 1)$. Then the map

$$\begin{array}{ccc} \mathbb{C}[\mathbf{X}, \mathbf{Y}]_D & \to & \mathbb{C}^M \\ Q & \mapsto & (\mathcal{D}^i Q(\vartheta_\gamma))_{0\le i<M} \end{array} \tag{2.3.2}$$

is an isomorphism of $\mathbb{C}$-vector spaces. Moreover, for each $Q \in \mathbb{C}[\mathbf{X}, \mathbf{Y}]_D$, we have

$$\mathcal{L}(Q) \le c_1(-\gamma)^D 8^M \max_{0\le i<M} \left| \mathcal{D}^i Q(\vartheta_\gamma) \right| \tag{2.3.3}$$

Proof. The second assertion is a quantitative version of the first because it implies that the linear map (2.3.2) is injective and so is an isomorphism, its domain and codomain having the same dimension M. Therefore it suffices to prove the second assertion. To this end, we fix a polynomial $Q \in \mathbb{C}[\mathbf{X}, \mathbf{Y}]_D$.
We first consider the case when $\gamma = e = (0, 1)$ is the neutral element of $\mathcal{G} = \mathbb{C}\times\mathbb{C}^\times$, so that $\vartheta_e = (1, 0, 1, 1)$.
Writing the polynomial $Q \in \mathbb{C}[\mathbf{X}, \mathbf{Y}]_D$ in the form

$$Q = \sum_{(\mu,\nu)\le D} q_{\mu,\nu} \, \mathbf{X}^\mu \mathbf{Y}^\nu$$

it is easy to see that for every $i \geq 0$

$$\mathcal{D}^i Q(\vartheta_e) = \sum_{\mu=0}^{D_x} q_{\mu,\nu}\, i^{(\mu)} \nu^{i-\mu}.$$

Therefore, by Lemma 2.3.7 with $n = D_y+1$, $A_{\mu,\nu} = q_{\mu,\nu}$, $\alpha_\nu = \nu$ and $m_\nu = D_x+1$ we get

$$|q_{\mu,\nu}| \leq \frac{a_0}{a_1 a_2} \left|\mathcal{D}^i Q(\vartheta_e)\right|$$

where

$$a_2 = 1$$

and

$$\begin{aligned}
\frac{a_0}{a_1} &= \max_{0\leq\nu\leq D_y} \binom{M}{D_x+1}(1+\nu)^{D_x+1} \prod_{\substack{0\leq\nu'\leq D_y\\ \nu'\neq\nu}} \left(\frac{1+\nu'}{|\nu'-\nu|}\right)^{D_x+1}\\
&= \max_{0\leq\nu\leq D_y} \binom{M}{D_x+1}\left(\frac{(D_y+1)!}{\nu!(D_y-\nu)!}\right)^{D_x+1}\\
&= \max_{0\leq\nu\leq D_y} \left(\binom{M}{D_x+1}\binom{D_y}{\nu}(D_y+1)\right)^{D_x+1}\\
&\leq 2^M\cdot 2^M\cdot 2^M = 8^M
\end{aligned}$$

using the general estimates $\binom{n}{k} \leq 2^n$ and $n \leq 2^n$. We conclude that

$$\|Q\| \leq 8^M \max_{0\leq i<M} \left|\mathcal{D}^i Q(\vartheta_e)\right|$$

For the general case, we apply the previous result to $\tau_\gamma Q$ instead of Q. Since

$$\mathcal{D}^i(\tau_\gamma Q)(\vartheta_e) = \tau_\gamma(\mathcal{D}^i Q)(\vartheta_e) = \mathcal{D}^i Q(\vartheta_\gamma) \quad \text{for each } i\in\mathbb{N},$$

this gives

$$\|\tau_\gamma Q\| \leq 8^M \max_{0\leq i<M} \left|\mathcal{D}^i Q(\vartheta_\gamma)\right|.$$

The conclusion follows as Lemma 2.3.6 gives $\mathcal{L}(Q) \leq c_1(-\gamma)^D\|\tau_\gamma Q\|$. □

Corollary 2.3.9. Let $\gamma\in\mathcal{G}$ and $T\in\mathbb{N}_+$. Define $I_\gamma := I^{(\gamma,1)}$. Then I_γ is a prime ideal of rank 2 and $I^{(\gamma,T)}$ is I_γ-primary of degree T.

Proof. The ideal I_γ is generated by the homogeneous polynomials vanishing at the point ϑ_γ. Therefore it is prime of rank 2. As $(I_\gamma)^T \subseteq I^{(\gamma,T)} \subseteq I_\gamma$, the radical of $I^{(\gamma,T)}$ is I_γ. Since I_γ is a prime of maximal rank, this is sufficient to conclude that $I^{(\gamma,T)}$ is I_γ-primary. Moreover, for any choice of homogeneous polynomials $P,Q\in\mathbb{C}[\mathbf{X},\mathbf{Y}]$ with $P\notin I_\gamma$ and $Q\notin I^{(\gamma,T)}$, we find that $PQ\notin I^{(\gamma,T)}$. Thus, $I^{(\gamma,T)}$ is I_γ-primary. Finally, consider the linear map $\varphi\colon \mathbb{C}[\mathbf{X},\mathbf{Y}]\to\mathbb{C}^T$ given by $\varphi(Q) = (\mathcal{D}^i Q(1,\gamma))_{0\leq i<T}$ for each $Q\in\mathbb{C}[\mathbf{X},\mathbf{Y}]$. Then, $I_D^{(\gamma,T)}$ is the kernel of the restriction of φ to $\mathbb{C}[\mathbf{X},\mathbf{Y}]_D$, for each $D\in\mathbb{N}^2$. Thus, the Hilbert function of $I^{(\gamma,T)}$ is given by $H(I^{(\gamma,T)};D) = \dim_{\mathbb{C}}\varphi(\mathbb{C}[\mathbf{X},\mathbf{Y}]_D)$. However, Proposition 2.3.8 shows that $\varphi(\mathbb{C}[\mathbf{X},\mathbf{Y}]_D) = \mathbb{C}^T$ when $(D_x+1)(D_y+1)\geq T$. Thus, for each large enough bi-integer D, the value $H(I^{(\gamma,T)};D)$ is constant equal to T and so $I^{(\gamma,T)}$ has degree T. □

2.3.3 A division algorithm

The following technical lemma provides the inductive step needed in the proof of the next two propositions.

Lemma 2.3.10. Let $\gamma \in \mathcal{G}$, let $L, K, N \in \mathbb{N}^2$ and $T \in \mathbb{N}_+$ with

$$\begin{aligned} L_xL_y + \|L\| < T &\le (L_x+1)(L_y+1) \\ L \le K \le N \le 2K - L, &\quad L \ne K. \end{aligned} \tag{2.3.4}$$

and let $Q \in I_N^{(\gamma,T)}$. Then, we can write

$$Q = \sum_{i,j=0}^{1} X_i^{N_x-K_x} Y_j^{N_y-K_y} Q_{i,j}$$

for a choice of polynomials $Q_{i,j} \in I_K^{(\gamma,T)}$ $(0 \le i,j \le 1)$ satisfying

$$\sum_{i,j=0}^{1} \mathcal{L}(Q_{i,j}) \le c_1(\gamma)^L c_2(\gamma)^K (64|K|)^T \mathcal{L}(Q). \tag{2.3.5}$$

Proof. Let us denote $\mathbf{M}_{i,j}^D := X_i^{D_x} Y_j^{D_y}$ and $\mathcal{A} := \{(1,0),(0,1),(1,1)\}$. Since (2.3.4) implies $N > 2(N - K - (1,1))$, any monomial in $\mathbf{X},\mathbf{Y}$ of bidegree N is divisible by at least one of the four monomials $\mathbf{M}_{i,j}^{N-K}$ $(0 \le i,j \le 1)$. So, we can write

$$Q = \sum_{i,j=0}^{1} \mathbf{M}_{i,j}^{N-K} P_{i,j}$$

for some homogeneous polynomials $P_{i,j}$ of bidegree K with

$$\sum_{i,j=0}^{1} \mathcal{L}(P_{i,j}) = \mathcal{L}(Q). \tag{2.3.6}$$

Put $M = (L_x+1)(L_y+1)$. Then, for each $(i,j) \in \mathcal{A}$, Proposition 2.3.8 ensures the existence of a unique polynomial $R_{i,j} \in \mathbb{C}[\mathbf{X},\mathbf{Y}]_L$ satisfying

$$\mathcal{D}^k R_{i,j}(\vartheta_\gamma) = \begin{cases} \mathcal{D}^k P_{i,j}(\vartheta_\gamma) & \text{for } 0 \le k < T, \\ 0 & \text{for } T \le k < M, \end{cases}$$

and shows, with the help of Lemma 2.3.6, that it has length

$$\mathcal{L}(R_{i,j}) \le c_1(-\gamma)^L 8^M \max_{0 \le k < T} |\mathcal{D}^k P_{i,j}(\vartheta_\gamma)| \le c_1(-\gamma)^L 8^M c_2(\gamma)^K |K|^T \mathcal{L}(P_{i,j}).$$

As $M \le 2(L_xL_y + \|L\| + 1) - 1 \le 2T - 1$, the above estimate simplifies to

$$\mathcal{L}(R_{i,j}) \le \frac{1}{8} c_1(-\gamma)^L c_2(\gamma)^K (64|K|)^T \mathcal{L}(P_{i,j}) \quad ((i,j) \in \mathcal{A}). \tag{2.3.7}$$

Furthermore, since $2K - L \ge N$, $N \ge K$ and $K \ge L$, the expressions

$$Q_{0,0} := P_{0,0} + \mathbf{M}_{0,0}^{2K-L-N} \left(\sum_{(i,j)\in\mathcal{A}} \mathbf{M}_{i,j}^{N-K} R_{i,j} \right)$$

$$Q_{i,j} := P_{i,j} - \mathbf{M}_{0,0}^{K-L} R_{i,j} \qquad ((i,j) \in \mathcal{A})$$

are bihomogeneous polynomials of bidegree K which satisfy

$$\sum_{i,j=0}^{1} \mathbf{M}_{i,j}^{N-K} Q_{i,j} = Q. \tag{2.3.8}$$

By construction, we have $Q_{i,j} \in I^{(\gamma,T)}$ for $(i,j) \in \mathcal{A}$. Since Q as well belongs to $I^{(\gamma,T)}$, we deduce that $\mathbf{M}_{0,0}^{N-K} Q_{0,0} \in I^{(\gamma,T)}$ and so $Q_{0,0} \in I^{(\gamma,T)}$ because $X_0, Y_0 \notin I_\gamma$ (see Corollary 2.3.9). Thus, (2.3.8) provides a decomposition of Q with polynomials $Q_{i,j} \in I_K^{(\gamma,T)}$, $0 \le i,j \le 1$. Using (2.3.6) and (2.3.7), we find as announced

$$\sum_{i,j=0}^{1} \mathcal{L}(Q_{i,j}) \le 2 \sum_{(i,j)\in\mathcal{A}} \mathcal{L}(R_{i,j}) + \sum_{i,j=0}^{1} \mathcal{L}(P_{i,j}) \le c_1(-\gamma)^L c_2(\gamma)^K (64|K|)^T \mathcal{L}(Q),$$

because $6\,\frac{64^T}{8} + 1 \le 64^T$ for $T \ge 1$. □

On the qualitative side, this lemma has the following useful consequence.

Proposition 2.3.11. Let $\gamma = (\xi,\eta) \in \mathcal{G}$, let $D \in \mathbb{N}_+^2$ and $T \in \mathbb{N}_+$ with $T \le D_x D_y + \min\{D_x, D_y\}$. Then, any homogeneous element of $I^{(\gamma,T)}$ of degree $\ge D$ belongs to the ideal J of $\mathbb{C}[\mathbf{X},\mathbf{Y}]$ generated by $I_D^{(\gamma,T)}$. Moreover, for any finite set of points S of $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ not containing $([1:\xi],[1:\eta])$, there exists an element of $I_D^{(\gamma,T)}$ which does not vanish at any point of S.

Proof. The hypotheses on D and T imply that there exists a bi-integer $L \neq D$ such that $L \le D - (1,0)$ and $L_x L_y + \|L\| < T \le (L_x+1)(L_y+1)$. Then, for any $N \in \mathbb{N}$ with $N \ge D + (1,0)$, the conditions (2.3.4) of Lemma 2.3.10 are fulfilled with $K = N - (1,0)$ and the lemma shows that $I_N^{(\gamma,T)}$ is contained in the ideal of $\mathbb{C}[\mathbf{X},\mathbf{Y}]$ generated by $I_{N-(1,0)}^{(\gamma,T)}$. The same argument works for another choice of $L \le D - (0,1)$, $N \ge D + (0,1)$ and $K = N - (0,1)$. By induction, we conclude that $I_N^{(\gamma,T)} \subseteq J$ for each $N \ge D$. This proves the first assertion of the proposition. It also implies that $I^{(\gamma,T)}$ and J have the same zero set in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$, namely $\{([1:\xi],[1:\eta])\}$, which leads to the second assertion. □

For the next proposition, in comparison with the homogeneous case (Proposition 3.7 of [Roy13]), the simplification of the hypothesis in 2.3.10 allows one to nearly halve the coordinates of N when they are very big, and we get a little improvement for the choice of T. The main new problem that occurs in the proof is that when only one coordinate of N is very different from that of D, you can halve only one coordinate and so it becomes more intricate to deal with this case.
In addition to this, we remark that we have $N \log |N|$ in the exponent of $c_3(\gamma)$, while in the homogeneous version one has simply N. It is possible to get an estimate with exponent N also in this setting, at the cost of strengthening the hypothesis on T (to a condition of the form $T \le c D_x D_y$, with $0 < c < 1$), but this minor improvement is ineffective, for the application he have in mind.

Proposition 2.3.12. Let $\gamma \in \mathcal{G}$, $T \in \mathbb{N}_+$ and $D \in \mathbb{N}_+^2$ with $|D| \geq 3$ and $1 \leq T \leq D_x D_y$. For any $N \in \mathbb{N}^2$ with $N \geq D$ and any polynomial $Q \in I_N^{(\gamma,T)}$, we can write $Q = \sum_{\nu \leq N-D} P_\nu \mathbf{X}^{\nu_x} \mathbf{Y}^{\nu_y}$ for a choice of polynomials $P_\nu \in I_D^{(\gamma,T)} (\nu \in \mathbb{N}^2, \nu \leq N - D)$ satisfying

$$\sum_{\nu \leq N-D} \mathcal{L}(P_\nu) \leq c_3(\gamma)^{15N \log(|N|)} |N|^{28T \log(|N|)} \mathcal{L}(Q), \tag{2.3.9}$$

where $c_3(\gamma) = c_1(-\gamma) c_2(\gamma)$.

Proof. We proceed by induction on N. For $N = D$, the result is clear. Suppose that $N \geq D$, $N \neq D$ and let $Q \in I_N^{(\gamma,T)}$. The hypothesis imply that we can choose a non-negative bi-integer $L \in \mathbb{N}^2$ satisfying $L < D$ and $L_x L_y + \|L\| < T \leq (L_x + 1)(L_y + 1) = \dim(L)$. We shall prove the thesis with the finer inequality

$$\sum_{\nu \leq N-D} \mathcal{L}(P_\nu) \leq c_2(\gamma)^N c_3(\gamma)^{\beta D \log(\dim(N-L))} |N|^{2T\beta \log(\dim(N-L))} \mathcal{L}(Q), \tag{2.3.10}$$

where $\beta = 2 \log_{\frac{4}{3}} e = 6.95\ldots$.
To this extent, we also define

$$K = \left(\max\{D_x, \left\lceil \frac{N_x + L_x}{2} \right\rceil\}, \max\{D_y, \left\lceil \frac{N_y + L_y}{2} \right\rceil\} \right).$$

For this choice of K, we have $N \geq K \geq D > L$ and $2K \geq N + L$, so that the conditions (2.3.4) of Lemma 2.3.10 are fulfilled. Moreover, since $|N| \geq |D| + 1 \geq 4$, we have $64|K| \leq 64|N| \leq |N|^4$ and so this lemma provides polynomials $Q_{i,j} \in I_K^{(\gamma,T)}$ satisfying

$$Q = \sum_{i,j=0}^{1} \mathbf{M}_{i,j}^{N-K} Q_{i,j} \quad \text{and} \quad \sum_{i,j=0}^{1} \mathcal{L}(Q_{i,j}) \leq c_2(\gamma)^{K-L} c_3(\gamma)^L |N|^{4T} \mathcal{L}(Q). \tag{2.3.11}$$

Since $c_1(\gamma) \geq 1$, $c_2(\gamma) \geq 1$, $\beta \geq 2$, $\dim(N - L) \geq 2 \cdot 3 \geq e$, if $K = D$, this decomposition of Q has all the requested properties. Otherwise, assume that $K_x > D_x$ and $K_y \geq D_y$ (the other case is symmetric), so that $K_x = \lceil \frac{N_x + L_x}{2} \rceil$. We first notice that

$$2K - N \leq 2D. \tag{2.3.12}$$

Indeed $2K_x - N_x \leq L_x + 1 \leq 2D_x$, and if $K_y > D_y$ then again $2K_y - N_y \leq 2D_y$, else $K_y = D_y$ and $2K_y - N_y \leq D_y$ becomes obvious. We also observe that

$$\beta \log(\dim(N - L)) \geq \beta \log(\dim(K - L)) + 2. \tag{2.3.13}$$

To see this, we notice that $N_x > K_x > D_x > L_x$, so $N_x \geq L_x + 3$, hence

$$K_x \leq \frac{N_x + L_x}{2} + \frac{1}{2}$$

$$K_x - L_x + 1 \leq \frac{N_x - L_x + 1}{2} + 1$$

$$\frac{K_x - L_x + 1}{N_x - L_x + 1} \leq \frac{1}{2} + \frac{1}{N_x - L_x + 1} \leq \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

Since in addition $N_y - L_y + 1 \geq K_y - L_y + 1$, we have $\beta \log(\dim(N-L)) - \beta\log(\dim(K-L)) \geq \beta \log\left(\frac{N_x - L_x + 1}{K_x - L_x + 1}\right) \geq \beta \log\left(\frac{4}{3}\right) \geq 2$.
Now, by induction, we may assume that each $Q_{i,j}$ in (2.3.11) admits a decomposition $Q_{i,j} = \sum_{\nu \leq K-D} \mathbf{X}^{\nu_x}\mathbf{Y}^{\nu_y} P_{i,j,\nu}$ with polynomials $P_{i,j,\nu} \in I_D^{(\gamma,T)}$ satisfying

$$\sum_{\nu \leq K-D} \mathcal{L}(P_{i,j,\nu}) \leq c_2(\gamma)^K c_3(\gamma)^{\beta D \log(\dim(K-L))} |K|^{2T\beta \log(\dim(K-L))} \mathcal{L}(Q_{i,j}).$$

If we substitute these expressions in the decomposition (2.3.11) of Q and we collect terms, we obtain a new decomposition $Q = \sum_{\nu \leq N-D} \mathbf{X}^{\nu_x}\mathbf{Y}^{\nu_y} P_\nu$ with polynomials $P_\nu \in I_D^{(\gamma,T)}$ satisfying

$$\begin{aligned}\sum_{\nu \leq N-D} \mathcal{L}(P_\nu) &\leq c_2(\gamma)^K c_3(\gamma)^{\beta D \log(\dim(K-L))} |K|^{2T\beta\log(\dim(K-L))} \sum_{i,j=0}^{1} \mathcal{L}(Q_{i,j}) \\ &\leq c_2(\gamma)^{2K-L} c_3(\gamma)^{\beta D\log(\dim(K-L))+L} |N|^{4T} |K|^{2T\beta \log(\dim(K-L))} \mathcal{L}(Q).\end{aligned}$$

As $2K - L \geq N$ and $c_3(\gamma) \geq c_2(\gamma)$ we have $c_2(\gamma)^{2K-L} c_3(\gamma)^{\beta D \log(\dim(K-L))+L} \leq c_2(\gamma)^N c_3(\gamma)^{\beta D \log(\dim(K-L))+2K-N}$. Using (2.3.12) and (2.3.13) we deduce $\beta D \log(\dim(K-L)) + 2K - N \leq \beta D \log(\dim(N-L))$ and $4T + 2T\beta\log(\dim(K-L) \leq 2T\beta \log(\dim(N-L)))$, so that (2.3.10) holds, hence the thesis follows easily from $\dim(N) \leq |N|^2$, valid for N positive bi-integer. $\square$

2.4 Distance

Throughout this section, we fix a point $\gamma = (\xi, \eta) \in \mathcal{G} = \mathbb{C} \times \mathbb{C}^\times$ and denote by $\boldsymbol{\vartheta}_\gamma = ([1:\xi],[1:\eta])$ the class of $\vartheta_\gamma = (1, \xi, 1, \eta)$ in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. To alleviate the notation, we simply write c_1 and c_2 to denote respectively the bi-constants $c_1(\gamma)$ and $c_2(\gamma)$ of Lemma 2.3.6, and c_3 to denote the bi-constant $c_3(\gamma)$ from Proposition 2.3.12. In particular, we have

$$c_2 = (c_{2,x}, c_{2,y}) = (\max\{1, |\xi|\}, \max\{1, |\eta|\}) = (\|(1,\xi)\|, \|(1,\eta)\|).$$

For each bi-integer $D \geq 0$, each positive integer $T \geq 1$, and each point $(\alpha, \beta) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ with representative $(\underline{\alpha}, \underline{\beta}) = (\alpha_0, \alpha_1, \beta_0, \beta_1) \in \mathbb{C}^4$ of norm $\|\underline{\alpha}\| = \|\underline{\beta}\| = 1$, we also define

$$|I_D^{(\gamma,T)}|_{(\alpha,\beta)} = \sup\{|P(\underline{\alpha}, \underline{\beta})| \,:\, P \in I_D^{(\gamma,T)},\, \|P\| \leq 1\},$$

where $I_D^{(\gamma,T)}$ stands for the bihomogeneous part of degree D of the ideal $I^{(\gamma,T)}$ introduced in the preceding section. The goal of this section is to estimate this quantity in terms of the projective distances between (α, β) and $\boldsymbol{\vartheta}_\gamma$, defined by

$$\begin{aligned}\mathrm{dist}_x(\alpha, (1:\xi)) &= \frac{\|\alpha_1 - \alpha_0\xi\|}{\|\underline{\alpha}\|\, \|(1,\xi)\|} = c_{2,x}^{-1}|\alpha_1 - \alpha_0 \xi|, \\ \mathrm{dist}_y(\beta, (1:\eta)) &= \frac{\|\beta_1 - \beta_0\eta\|}{\|\underline{\beta}\|\,\|(1,\eta)\|} = c_{2,y}^{-1}|\beta_1 - \beta_0\xi|, \\ \mathrm{bidist}((\alpha,\beta), \boldsymbol{\vartheta}_\gamma) &= (\mathrm{dist}_x(\alpha,(1:\xi)), \mathrm{dist}_x(\beta,(1:\eta))), \\ \mathrm{dist}((\alpha,\beta),\boldsymbol{\vartheta}_\gamma) &= \|\mathrm{bidist}((\alpha,\beta),\boldsymbol{\vartheta}_\gamma)\|,\end{aligned} \tag{2.4.1}$$

and in terms of the distance from (α, β) to the analytic curve $A_\gamma = \{([1 : \xi + z], [1 : \eta e^z]) : z \in \mathbb{C}\}$ defined by

$$\operatorname{dist}(\alpha, A_\gamma) = \left| \frac{\beta_1}{\beta_0} - \eta \exp\Big(\frac{\alpha_1}{\alpha_0} - \xi\Big) \right|$$

when $\alpha_0, \beta_0 \neq 0$.

2.4.1 Lower bound for the distance

For our first estimate, we use the following lemma.

Lemma 2.4.1. Let $((\alpha, \beta) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ with $\operatorname{bidist}((\alpha, \beta), \boldsymbol{\vartheta}_\gamma)) \leq (2c_2)^{-1} := ((2c_{2,x})^{-1}, (2c_{2,y})^{-1})$, and let $(\underline{\alpha}, \underline{\beta}) \in \mathbb{C}^4$ be a normalized representative of (α, β). Then we have $(|\alpha_0|, |\beta_0|) \geq (2c_2)^{-1}$.

Proof. We have $\|\alpha_1 - \alpha_0 \xi\| = c_{2,x} \operatorname{dist}_x(\alpha, (1 : \xi)) \leq 1/2$, so $|\alpha_0 \xi| \geq |\alpha_1| - 1/2 = 1/2$ and therefore $|\alpha_0| \geq (2c_{2,x})^{-1}$. Analogously, we have also $|\beta_0| \geq (2c_{2,y})^{-1}$. ☐

Proposition 2.4.2. Let $D \in \mathbb{N}^2 \backslash \{\mathbf{0}\}$, $T \in \mathbb{N}_+$, $P \in \mathbb{C}[\mathbf{X}, \mathbf{Y}]_D$ with $P \neq 0$, and let (α, β), $\underline{\alpha}, \underline{\beta}$ be as in Lemma 2.4.1. Then we have

$$\frac{|P(\underline{\alpha}, \underline{\beta})|}{\|P\|} \leq c_4 \max_{0 \leq i < T} \frac{|\mathcal{D}^i P(\vartheta_\gamma)|}{\|P\|} + c_4^{|D|} \big(\operatorname{dist}(\alpha, (1 : \xi))^T + \operatorname{dist}((\alpha, \beta), A_\gamma) \big) \tag{2.4.2}$$

where $c_4 = 12 \|c_2\| \exp(2c_{2,x}^2)$.

Proof. Since by Lemma 2.4.1 we have $\alpha_0, \beta_0 \neq 0$, we put

$$\delta_1 = \frac{\alpha_1}{\alpha_0} - \xi \quad \text{and} \quad \delta_2 = \frac{\beta_1}{\beta_0} - \eta e^{\delta_1},$$

and consider the entire function $f \colon \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = P(1, \xi + z, 1, \eta e^z) \quad (z \in \mathbb{C}).$$

Since $\alpha_0^{D_x} \beta_0^{D_y} f(\delta_1) = P(\alpha_0, \alpha_1, \beta_0, \beta_1 - \delta_2 \beta_0)$, we have

$$|P(\underline{\alpha}, \underline{\beta})| \leq |\alpha_0|^{D_x} |\beta_0|^{D_y} |f(\delta_1)| + |P(\alpha_0, \alpha_1, \beta_0, \beta_1) - P(\alpha_0, \alpha_1, \beta_0, \beta_1 - \delta_2 \beta_0)|, \tag{2.4.3}$$

and since f is an entire function and $f^{(i)}(0) = \mathcal{D}^i P(\vartheta_\gamma)$ for each $i \in \mathbb{N}$, we get

$$\begin{aligned} f(\delta_1) &\leq \sum_{i=0}^{\infty} \frac{1}{i!} |\mathcal{D}^i P(\vartheta_\gamma)| \, |\delta_1|^i \\ &= \sum_{i=0}^{T-1} \frac{1}{i!} |\mathcal{D}^i P(1, \vartheta_\gamma)| \, |\delta_1|^i + \sum_{i=T}^{\infty} \frac{1}{i!} |\mathcal{D}^i P(1, \xi, \eta)| \, |\delta_1|^i \\ &\leq e^{|\delta_1|} \max_{0 \leq i < T} |\mathcal{D}^i P(\vartheta_\gamma)| + \sum_{i=T}^{\infty} \frac{1}{i!} \|P\| \dim(D) c_2^D |D|^i |\delta_1|^i, \end{aligned}$$

where the last estimate uses the upper bound $|\mathcal{D}^i P(\vartheta_\gamma)| \le c_2^D |D|^i \mathcal{L}(P) \le c_2^D |D|^i \cdot \|P\|(D_x+1)(D_y+1)$ coming from Lemma 2.3.6. To provide an upper bound for the last series, we note that, since $|\alpha_0| \ge (2c_{2,x})^{-1}$, we have

$$|\delta_1| = |\alpha_0|^{-1}|\alpha_1 - \alpha_0\xi| \le 2c_{2,x}^2 \operatorname{dist}_x(\alpha, (1:\xi)).$$

As $\operatorname{dist}_x(\alpha,(1:\xi)) \le (2c_{2,x})^{-1} \le 1$, this gives $|\delta_1| \le c_{2,x}$ and, for each integer $i \ge T$, we can write $|\delta_1|^i \le (2c_{2,x}^2)^i \operatorname{dist}_x(\alpha,(1:\xi))^T$. Therefore,

$$\begin{aligned}
&\sum_{i=T}^{\infty} \frac{1}{i!}\|P\| \dim(D) c_2^D |D|^i |\delta_1|^i \\
&\le \|P\| \dim(D) c_2^D \operatorname{dist}_x(\alpha,(1:\xi))^T \sum_{i=T}^{\infty} \frac{1}{i!}(2c_{2,x}^2|D|)^i \\
&\le \|P\| \dim(D) c_2^D \exp(2c_{2,x}^2)^{|D|} \operatorname{dist}(\alpha,(1:\xi))^T \\
&\le c_4^D \|P\| \operatorname{dist}(\alpha,(1:\xi))^T.
\end{aligned}$$

The second term in (2.4.3) is easily estimated by writing explicitly the polynomial $P = \sum_{(a,b)\le D} p_{a,b}\,\mathbf{X}^a\mathbf{Y}^b$:

$$\begin{aligned}
&|P(\underline{\alpha},\underline{\beta}) - P(\underline{\alpha},\beta_0,\beta_1-\delta_2\beta_0)| \\
&\le \|P\| \sum_{a=0}^{D_x} \big(|\alpha_0|^{D_x-a}|\alpha_1|^a\big)\Big(\sum_{b=0}^{D_y} |\beta_0|^{D_y-b}|\beta_1^b - (\beta_1-\delta_2\beta_0)^b|\Big) \\
&\le \|P\|(D_x+1)\sum_{b=0}^{D_y} b(|\beta_1| + |\delta_2\beta_0|)^{b-1}|\delta_2\beta_0| \\
&\le \|P\|(D_x+1)\frac{D_y(D_y+1)}{2}(2 + c_{2,y}\exp(c_{2,x}))^{D_y-1}\operatorname{dist}((\alpha,\beta),A_\gamma),
\end{aligned}$$

where the last inequality comes from $|\delta_2\beta_0| \le |\delta_2| = \operatorname{dist}((\alpha,\beta),A_\gamma)$ and $|\beta_1| + |\delta_2\beta_0| \le 2 + |\eta|e^{|\delta_1|}$. The conclusion follows putting together all the preceding inequalities, since $e^{|\delta_1|} \le \exp(c_{2,x}) \le c_4$, and $\dim(D)\frac{D_y}{2}(2+c_{2,y}\exp(c_{2,x}))^{D_y-1} \le 4^{|D|}(2+|c_2|\exp(c_{2,x}))^{|D|-1} \le c_4^{|D|}$. □

As an immediate consequence, we get

Corollary 2.4.3. With $(\alpha,\beta), \underline{\alpha}, \underline{\beta}$ as in Lemma 2.4.1 and $D \in \mathbb{N}_+^2$, $T \in \mathbb{N}_+$, we have

$$|I_D^{(\gamma,T)}|_{(\alpha,\beta)} \le c_4^{|D|}\big(\operatorname{dist}(\alpha,(1:\xi))^T + \operatorname{dist}((\alpha,\beta),A_\gamma)\big).$$

2.4.2 Upper bound for the distance

We now turn to the problem of finding a lower bound for $|I_D^{(\gamma,T)}|_{(\alpha,\beta)}$. To this end, we first note the following consequence of Proposition 2.3.12.

Lemma 2.4.4. Let $D \in \mathbb{N}^2\backslash\{0\}$, $T \in \mathbb{N}^*$ with $|D| \ge 3$ and $\|D\| \le T \le D_xD_y$, let $(\alpha,\beta) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ and let $\underline{\alpha},\underline{\beta} \in \mathbb{C}^4$ be a normalized representative of it. Then, for any $Q \in I_{(T,T)}^{(\gamma,T)}$, we have

$$|Q(\underline{\alpha},\underline{\beta})| \le c_3^{15(T,T)\log(2T)}(2T)^{28T\log(2T)}\mathcal{L}(Q)\,|I_D^{(\gamma,T)}|_{(\alpha,\beta)}.$$

Proof. Fix a polynomial $Q \in I_N^{(\gamma,T)}$ and consider a decomposition of Q as given by Proposition 2.3.12 for the choice of $N = (T,T)$. Since $|P(\underline{\alpha},\underline{\beta})| \leq \mathcal{L}(P)\,|I_D^{(\gamma,T)}|_{(\alpha,\beta)}$ for any $P \in I_D^{(\gamma,T)}$, we obtain

$$|Q(\underline{\alpha},\underline{\beta})| \leq \sum_{|\nu|\leq N-D} |P_\nu(\underline{\alpha},\underline{\beta})| \leq c_3^{15(T,T)\log(2T)}(2T)^{28T\log(2T)}\mathcal{L}(Q)\,|I_D^{(\gamma,T)}|_{(\alpha,\beta)}.$$

□

Proposition 2.4.5. With the notation and hypotheses of Lemma 2.4.4, we have

$$\operatorname{dist}(\alpha,(1:\xi))^T \leq (c_5 T)^{28T\log(2T)}|I_D^{(\gamma,T)}|_{(\alpha,\beta)}\,.$$

Moreover, if $\operatorname{bidist}((\alpha,\beta),\boldsymbol{\vartheta}_\gamma) \leq (2c_2)^{-1}$, we also have

$$\operatorname{dist}((\alpha,\beta),A_\gamma) \leq c_4(c_5T)^{28T\log(2T)}|I_D^{(\gamma,T)}|_{(\alpha,\beta)}$$

where $c_5 = (2c_2^{(1,1)}+2)2c_3^{(1,1)}$, and c_4 is as in Proposition 2.4.2.

Proof. The formula (2.4.1) shows that $\operatorname{dist}(\alpha,(1:\xi)) = |M'(\underline{\alpha})|$ for the linear form $M' = c_{2,x}^{-1}(X_1 - \xi X_0) \in \mathbb{C}[\mathbf{X},\mathbf{Y}]_{(1,0)}$. If $|\beta_0| = 1$ we put $M = Y_0 M'$ and if $|\beta_1| = 1$ we put $M = Y_1 M'$. For such a choice of $M \in \mathbb{C}[\mathbf{X},\mathbf{Y}]_{(1,1)}$ we have $\|M\| \leq 1$, $M(\vartheta_\gamma) = 0$ and $M(\underline{\alpha},\underline{\beta}) = \operatorname{dist}(\alpha,(1:\xi))$. Then, as $M^T \in I_{(T,T)}^{(\gamma,T)}$ and $\mathcal{L}(M) \leq 2$, Lemma 2.4.4 gives

$$\operatorname{dist}(\alpha,(1:\xi))^T = |M(\underline{\alpha},\underline{\beta})|^T \leq 2^T c_3^{15(T,T)\log(2T)}(2T)^{28T\log(2T)}|I_D^{(\gamma,T)}|_{(\alpha,\beta)}. \tag{2.4.4}$$

So the first thesis is proved, as $c_2^{(1,1)} := c_{2,x}c_{2,y} \geq 1 \geq 0$. Now, assume that $\operatorname{bidist}((\alpha,\beta),\vartheta_\gamma) \leq (2c_2)^{-1}$, and write $\underline{\alpha},\underline{\beta} = (\alpha_0,\alpha_1,\beta_0,\beta_1)$. As the polynomial

$$Q(\mathbf{X},\mathbf{Y}) = X_0^T Y_0^{T-1} Y_1 - \eta\sum_{i=0}^{T-1}\frac{1}{i!}(X_1 - \xi X_0)^i X_0^{T-i} Y_0^T$$

also belongs to $I_{(T,T)}^{(\gamma,T)}$, the same result combined with Lemma 2.4.1 leads to the estimate

$$\begin{aligned}\Big|\frac{\beta_1}{\beta_0} - \eta\sum_{i=0}^{T-1}\frac{1}{i!}\Big(\frac{\alpha_1}{\alpha_0}-\xi\Big)^i\Big| &= |\alpha_0|^{-T}|\beta_0|^{-T}|Q(\underline{\alpha},\underline{\beta})| \\ &\leq c_4(2c_2)^{(T,T)}c_3^{15(T,T)\log(2T)}(2T)^{28T\log(2T)}|I_D^{(\gamma,T)}|_{(\alpha,\beta)}\end{aligned} \tag{2.4.5}$$

using $\mathcal{L}(Q) \leq 1 + |\eta|\exp(1+|\xi|) \leq c_4$. Arguing as in the proof of Proposition 2.4.2, we also note that, for each integer $i \geq T$, we have $|\alpha_1/\alpha_0 - \xi|^i \leq \operatorname{dist}(\alpha,(1:\xi))^T(2c_{2,x}^2)^i$ and therefore

$$\Big|\eta\sum_{i=T}^{\infty}\frac{1}{i!}\Big(\frac{\alpha_1}{\alpha_0}-\xi\Big)^i\Big| \leq \operatorname{dist}(\alpha,(1:\xi))^T c_{2,y}\sum_{i=T}^{\infty}\frac{(2c_{2,x}^2)^i}{i!} \leq c_4\operatorname{dist}(\alpha,(1:\xi))^T. \tag{2.4.6}$$

Combining (2.4.4), (2.4.5) and (2.4.6), we get

$$\begin{aligned}\operatorname{dist}((\alpha,\beta),A_\gamma) &\leq |\alpha_0|^{-T}|\beta_0|^{-1}|Q(\underline{\alpha},\underline{\beta})| + c_4\operatorname{dist}(\alpha,(1:\xi))^T \\ &\leq c_4((2c_2)^{(T,T)}+2^T)c_3^{15(T,T)\log(2T)}(2T)^{28T\log(2T)}|I_D^{(\gamma,T)}|_{(\alpha,\beta)},\end{aligned}$$

hence we obtain the desired estimate from $(2c_2)^{(T,T)}+2^T \leq ((2c_2)^{(1,1)}+2)^T$ □

We will also need the following

Proposition 2.4.6. With the notation and hypotheses of Lemma 2.4.4, we have

$$\operatorname{dist}(\beta,(1:\eta))^T \leq (c_5 T)^{28T\log(2T)}|I_D^{(\gamma,T)}|_{(\alpha,\beta)}\,.$$

where $c_5 = (2c_2^{(1,1)}+2)2c_3^{(1,1)}$, as in Proposition 2.4.5.

Proof. The formula (2.4.1) shows that $\operatorname{dist}(\beta,(1:\eta)) = |M'(\underline{\beta})|$ for the linear form $M' = c_{2,y}^{-1}(Y_1-\eta Y_0) \in \mathbb{C}[\mathbf{X},\mathbf{Y}]_{(0,1)}$. If $|\alpha_0|=1$ we put $M = X_0M'$ and if $|\alpha_1|=1$ we put $M = X_1M'$. For such a choice of $M \in \mathbb{C}[\mathbf{X},\mathbf{Y}]_{(1,1)}$ we have $\|M\| \leq 1$, $M(\vartheta_\gamma)=0$ and $M(\underline{\alpha},\underline{\beta}) = \operatorname{dist}(\beta,(1:\eta))$. Then, as $M^T \in I_{(T,T)}^{(\gamma,T)}$ and $\mathcal{L}(M) \leq 2$, Lemma 2.4.4 gives

$$\operatorname{dist}(\beta,(1:\eta))^T = |M(\underline{\alpha},\underline{\beta})|^T \leq 2^T c_3^{15(T,T)\log(2T)}(2T)^{28T\log(2T)}|I_D^{(\gamma,T)}|_{(\alpha,\beta)}. \tag{2.4.7}$$

The proposition follows from $c_2^{(1,1)} := c_{2,x}c_{2,y} \geq 0$. □

We remark that for the first two lemmas of this section we can adapt the arguments to achieve estimates involving smaller exponents in case D_y is a bit smaller than D_x, but not too much. Since we don't need such improvements, we didn't include them and we opted for simpler statements and proofs.

2.5 Multiplicity of the resultant form

In this section, we introduce the last crucial tool that we need for the proof of our main theorem. In consists in a lower bound for the multiplicity of the resultant form of multihomogeneous polynomials in $n+q = (n_1+1)+(n_q+1)$ variables at certain $n+1$-tuples of such polynomials. In the applications, we will restrict to $n=q=2$ and $n_1=n_2=1$.
Let $\mathbb{P} := \mathbb{P}^{n_1}\times\ldots\times\mathbb{P}^{n_q}$. We set its multidegree in the trivial embedding: $\deg(\mathbb{P}) := \deg_{K[\mathbf{X}]}((0))$.

2.5.1 The main result

In this section we use many of the definitions of the first chapter. We start with a decomposition lemma.

Lemma 2.5.1. Let $P_0,\ldots,P_n$ be a regular sequence of $K[\mathbf{X}]$ made of multihomogeneous polynomials, with $P_i \in K[\mathbf{X}]_{D_i}$ respectively, for some $D_i \in \mathbb{N}_+^q$, and let $\nu \in \mathbb{N}^q$ with $\nu \geq (\sum_{h=0}^n D_h) - (n_1,\ldots,n_q)$. Then there exist subspaces $E_0,\ldots,E_n$ respectively of $K[\mathbf{X}]_{\nu-D_i}$ with $\dim_{\mathbb{C}}(E_n) = \deg(\mathbb{P}) * D_0 * \cdots * D_{n-1}$ such that

$$K[\mathbf{X}]_\nu = E_0P_0 \oplus\cdots\oplus E_nP_n.$$

Proof. For $j = 0, \ldots, n$, define the multihomogeneous ideal $I_j = (P_0, \ldots, P_j)$ and, for each multi-integer $\nu \in \mathbb{N}^q$, choose a subspace $E_{j+1}(\nu)$ of $K[\mathbf{X}]_\nu$ such that

$$K[\mathbf{X}]_\nu = (I_j)_\nu \oplus E_{j+1}(\nu).$$

Put also $I_{-1} = (0)$ and $E_0(\nu) = K[\mathbf{X}]_\nu$ so that the above holds for $j = -1$, and extend the definition to multi-integers $\nu \in \mathbb{Z}^q$ with some negative coordinate by putting $K[\mathbf{X}]_\nu = (I_j)_\nu = E_{j+1}(\nu) = \{0\}$ for $j = -1, 0, \ldots, n$. Then, for each $\nu \in \mathbb{Z}^q$ and each $j = 0, \ldots, n$, we have an exact sequence

$$0 \longrightarrow (K[\mathbf{X}]/I_{j-1})_{\nu - D_j} \xrightarrow{\times P_j} (K[\mathbf{X}]/I_{j-1})_\nu \longrightarrow (K[\mathbf{X}]/I_j)_\nu \longrightarrow 0, \quad (2.5.1)$$

where the first non-trivial map comes from multiplication by P_j in $K[\mathbf{X}]$ while the second is induced by the identity map in $K[\mathbf{X}]$. As the inclusion of $E_{j+1}(\nu)$ in $K[\mathbf{X}]_\nu$ induces an isomorphism between $E_{j+1}(\nu)$ and $(K[\mathbf{X}]/I_j)_\nu$ for each $\nu \in \mathbb{Z}^q$ and $j = -1, 0, \ldots, n$, and as all these maps are clearly $\mathbb{C}$-linear, it follows that

$$(I_j)_\nu = E_j(\nu - D_j)P_j \oplus (I_{j-1})_\nu \qquad (\nu \in \mathbb{Z}^q,\ 0 \le j \le n).$$

Since $(I_{-1})_\nu = \{0\}$, combining these decompositions leads to

$$(I_n)_\nu = \bigoplus_{j=0}^{n} E_j(\nu - D_j)P_j \quad (2.5.2)$$

for each $\nu \in \mathbb{Z}^q$. On the other hand, at the level of dimensions, the exactness of the sequence (2.5.1) gives

$$\dim_{\mathbb{C}} E_{j+1}(\nu) = \dim_{\mathbb{C}} E_j(\nu) - \dim_{\mathbb{C}} E_j(\nu - D_j) \qquad (\nu \in \mathbb{Z}^q,\ 0 \le j \le n). \quad (2.5.3)$$

Since

$$\dim_{\mathbb{C}} E_0(\nu) = \binom{\nu_1 + n_1}{n_1} \cdots \binom{\nu_q + n_q}{n_q} \quad \text{for each } \nu \ge (-n_1, \ldots, -n_q)$$

we notice that for $\nu \ge (-n_1, \ldots, -n_q)$ it is a polynomial in q variables evaluated at ν and it coincides with the Hilbert-Samuel polynomial of the multihomogeneous ideal (0) of $K[\mathbf{X}]$:

$$\dim_{\mathbb{C}} E_0(\nu) = H(K[\mathbf{X}], \nu) \quad \text{for } \nu \ge -(n_1, \ldots, n_q).$$

We also have $\operatorname{totdeg}(H(K[\mathbf{X}], \nu)) = n$, $LP(H(K[\mathbf{X}], \nu)) = T_1^{n_1} \cdots T_q^{n_q}$ and $LC(H(K[\mathbf{X}], \nu)) = \deg(\mathbb{P})$. From (2.5.3) we also have that

$$\dim_{\mathbb{C}} E_j(\nu) = \Delta_{D_{j-1}} \cdots \Delta_{D_0}(H(K[\mathbf{X}], \nu) \quad \text{for } \nu \ge \sum_{h=0}^{j-1} D_h - (n_1, \ldots, n_q).$$

We deduce by induction, using proposition (1.2.5), that these are polynomials of total degree $n - j$. In particular, this gives $\dim_{\mathbb{C}}(E_n) = \deg(\mathbb{P}) * D_0 * \cdots * D_{n-1}$ for all $\nu \ge \sum_{h=0}^{n-1} D_h - (n_1, \ldots, n_q)$. Then (2.5.3) with $j = n$ implies that $E_{n+1}(\nu) = \{0\}$ for each $\nu \ge \sum_{h=0}^{n} D_h - (n_1, \ldots, n_q)$ and so $(I_n)_\nu = K[\mathbf{X}]_\nu$ for these values of ν. The conclusion of the lemma then follows from (2.5.2). $\square$

Theorem 2.5.2. Let I be a multihomogeneous ideal of $K[\mathbf{X}]$. Suppose that, for some positive multidegree $D \in \mathbb{N}_+^q$, the set of common zeros of the elements of I_D in $\mathbb{P}(\mathbb{C}) = \mathbb{P}^{n_1}(\mathbb{C}) \times \ldots \times \mathbb{P}^{n_q}(\mathbb{C})$ is finite and non-empty. Then, the resultant form for $n+1$ multihomogeneous polynomials of multidegree D vanishes up to order $\deg(I)$ at each point of $(I_D)^{n+1}$.

Proof. Since the elements of I_D have finitely many common zeros in $\mathbb{P}(\mathbb{C})$, the subspace I_D of $K[\mathbf{X}]_D$ contains a regular sequence $P_0, \ldots, P_{n-1}$ of length n. Moreover, as the elements of $K[\mathbf{X}]_D$ have no common zeros in $\mathbb{P}(\mathbb{C})$, this sequence can be extended to a regular sequence $P_0, \ldots, P_{n-1}, P_n$ for some $P_n \in K[\mathbf{X}]_D$. Fix a multi-integer $\nu \geq (n+1)D - (n_1, \ldots, n_q)$ large enough so that the Hilbert function of I at ν is $H(I;\nu) = \deg(I)$, and choose subspaces $E_0, \ldots, E_n$ of $K[\mathbf{X}]_{\nu-D}$ as in Lemma 2.5.1. For each $(n+1)$-tuple $\mathbf{Q} = (Q_0, \ldots, Q_n) \in K[\mathbf{X}]_D^{n+1}$, we define a linear map

$$\begin{aligned} \varphi_{\mathbf{Q}} \,:\, E_0 \times \cdots \times E_n &\longrightarrow K[\mathbf{X}]_\nu \\ (A_0, \ldots, A_n) &\longmapsto A_0 Q_0 + \cdots + A_n Q_n \,. \end{aligned}$$

Then, by construction, for the choice of $\mathbf{P} = (P_0, \ldots, P_n)$, the map $\varphi_{\mathbf{P}}$ is an isomorphism. Put $\delta := \dim_{\mathbb{C}} E_n$.
Form a basis $\mathcal{A}$ of $E_0 \times \cdots \times E_n$ by concatenating bases $\mathcal{A}^{(i)} = (A_j^{(i)})_{1 \leq j \leq \dim_{\mathbb{C}} E_i}$ of $0 \times \cdots \times E_i \times \cdots \times 0$ for $i = 0, \ldots, n$, so that the last δ elements of $\mathcal{A}$ form a basis of $0 \times \cdots \times 0 \times E_n$. Since $H(I;\nu) = \deg(I)$, the set I_ν is a subspace of $K[\mathbf{X}]_\nu$ of codimension $\deg(I)$ and so there is also a basis $\mathcal{B}$ of $K[\mathbf{X}]_\nu$ whose last elements past the first $\deg(I)$ form a basis of I_ν. For each $\mathbf{Q} \in E_0 \times \cdots \times E_n$, we denote by $M_{\mathbf{Q}}$ the matrix of the linear map $\varphi_{\mathbf{Q}}$ with respect to the bases $\mathcal{A}$ and $\mathcal{B}$. Define $N_i := \sum_{k=0}^{i-1} \dim_{\mathbb{C}} E_k$. Then the $(N_i + j)$-th column of $M_{\mathbf{Q}}$ represents $A_j^{(i)} Q_i$ properly written in terms of the basis $\mathcal{B}$. Then, if we write explicitly $Q_i = \sum_{\mathfrak{m} \in \mathfrak{M}_D} q_{\mathfrak{m}}^{(i)} \mathfrak{m}$ we see that the entries of the $(N_i + j)$-th column of $M_{\mathbf{Q}}$ are fixed $\mathbb{C}$-linear combinations (depending only on $\mathcal{A}$ and $\mathcal{B}$) of the coefficients $q_{\mathfrak{m}}^{(i)}$ of Q_i. Then, using Laplace's expansion for the determinant, we see that the map

$$\begin{aligned} \Phi \,:\, \quad K[\mathbf{X}]_D^{n+1} \quad &\longrightarrow \mathbb{C} \\ \mathbf{Q} = (Q_0, \ldots, Q_n) &\longmapsto \det(M_{\mathbf{Q}}) \end{aligned}$$

is a multihomogenous polynomial map in the coefficients of $Q_0, \ldots, Q_n$, which is homogeneous of degree $\dim_{\mathbb{C}} E_i$ in the coefficients of Q_i, for $i = 0, \ldots, n$. For each $\mathbf{Q} \in I_D^{n+1}$, the first $\deg(I)$ rows of $M_{\mathbf{Q}}$ vanish because the image of $\varphi_{\mathbf{Q}}$ is contained in I_ν. It follows from this that all partial derivatives of Φ of order less than $\deg(I)$ vanish at each point of I_D^{n+1}.

Claim 2.5.3. If ν is sufficiently large, Φ is divisible by the resultant form in multidegree $d := (D, \ldots, D) \in \mathbb{N}^{n+1}$: we have

$$\Phi(\mathbf{Q}) = \Psi(\mathbf{Q}) \operatorname{res}_d(\mathbf{Q}) \tag{2.5.4}$$

where $\Psi \colon K[\mathbf{X}]_D^{n+1} \to \mathbb{C}$ is also a polynomial map.

Proof. To see this, we include each E_i into $K[\mathbf{X}]_{\nu-D}$ and extend the basis $\mathcal{A}$ of $E_0 \times \ldots \times E_n$ to a basis $\widetilde{\mathcal{A}}$ of $K[\mathbf{X}]_{\nu-D}^{n+1}$. We also consider the $\mathbb{C}$-linear map

$\widetilde{\varphi}_{\mathbf{Q}} \colon K[\mathbf{X}]^{n+1}_{\nu-D} \to K[\mathbf{X}]_\nu$ extending $\varphi_{\mathbf{Q}}$, given by $\widetilde{\varphi}_{\mathbf{Q}}(A_0, \ldots, A_n) = \sum_{i=0}^n A_i Q_i$. We notice that $M_{\mathbf{Q}}$ is now a minor of maximum rank extracted by the matrix $\widetilde{M_{\mathbf{Q}}}$ representing $\widetilde{\varphi}_{\mathbf{Q}}$ in the basis $\widetilde{\mathcal{A}}$ and $\mathcal{B}$. We consider now the map ϕ_ν considered in Lemma 1.1.12 with d as above. After tensorizing by the flat (free) $\mathbb{C}$-module $\mathbb{C}[d]$, we obtain from $\widetilde{A}$ and $\mathcal{B}$ two basis $\widetilde{\mathcal{A}'}, \mathcal{B}'$, respectively for $K[\mathbf{u}][\mathbf{X}]^{n+1}_{\nu_D}$ and $K[\mathbf{u}][\mathbf{X}]_\nu$, and then we consider the matrix M' representing ϕ_ν in these two basis. It is then clear that for every choice of $\mathbf{Q} \in K[\mathbf{X}]^{n+1}_D$ the matrix $\widetilde{M_{\mathbf{Q}}}$ is obtained from M' by specialization of the variables $u^{(i)}_{\mathfrak{m}}$ to the coefficients $q^{(i)}_{\mathfrak{m}}$ of $Q_0, \ldots, Q_n$. Thus, the multihomogeneous polynomial representing Φ exactly coincides with $\det(\Delta)$, where Δ is the minor of M' corresponding to the basis $\mathcal{A}' = \mathcal{A} \otimes 1 \subseteq \widetilde{\mathcal{A}'}$. Lemma 1.1.12 says precisely that res_d divides Φ as a polynomial map. $\square$

Since by Proposition 1.2.18 the resultant form is homogeneous of degree $\deg \mathbb{P} * D * \cdots * D$ on each factor of $K[\mathbf{X}]^{n+1}_D$ and since by Lemma 2.5.1 Φ is of the same degree on the last factor, the map Ψ has degree 0 on that factor. This means that $\Psi(Q_0, \ldots, Q_n)$ is independent of Q_n. Since $\Phi(\mathbf{P}) \neq 0$ and since $P_0, \ldots, P_{n-1} \in I_D$, we deduce that the restriction of Ψ to I^{n+1}_D is not the zero map and so the condition $\Psi(\mathbf{Q}) \neq 0$ defines a non-empty Zariski open subset $\mathcal{U}$ of I^{n+1}_D. As the map Φ vanishes to order at least $\deg(I)$ at each point of $\mathcal{U}$, the factorization (2.5.4) implies that the resultant vanishes up to order $\deg(I)$ at the same points and therefore, by continuity, vanishes up to order $\deg(I)$ at each point of the closure of $\mathcal{U}$ in I^{n+1}_D. Since I^{n+1}_D is a $\mathbb{C}$-vector subspace of $K[\mathbf{X}]^{n+1}$, it is irreducible as algebraic set and so the closure of $\mathcal{U}$ in I^{n+1}_D is all of I^{n+1}_D. $\square$

2.5.2 Corollaries in dimension two

Lemma 2.5.4. Let R be an irreducible bihomogeneous polynomial of $\mathbb{Q}[\mathbf{X}, \mathbf{Y}]$. Then R divides $\mathcal{D}R$ if and only if R is a constant multiple of either X_0, Y_0 or Y_1.

Proof. Suppose first that $R \,|\, \mathcal{D}R$ and let $D = (D_x, D_y) := (\deg_x R, \deg_y R)$. Since $\mathcal{D}R$ is also bihomogeneous of bidegree D, this hypothesis means that R is an eigenvector of the differential operator $\mathcal{D}$ acting on $\mathbb{Q}[\mathbf{X}, \mathbf{Y}]_D$. We observe that, for each $k = 0, \ldots, D_y$ the subspace $\mathbf{Y}^k \mathbb{Q}[\mathbf{X}]_{Dx}$ is in the kernel of $(\mathcal{D} - k)^{D_x+1}$ (indeed, it is also easy to see that it coincides with it) and so the product $\prod_{k=0}^{D_y} (\mathcal{D} - k)^{D_x+1}$ induces the zero operator on $\mathbb{Q}[\mathbf{X}, \mathbf{Y}]_D$. Thus the eigenvalues of $\mathcal{D}$ are the integers $1, \ldots, D_y$ and we find that, for each $k = 0, \ldots, D_y$, the eigenspace for k is generated by the monomial $Y_0^{D_y - k} Y_1^k X_0^{D_x}$. So R is a multiple of such a monomial and, as it is irreducible, we conclude that it has total degree $D_x + D_y = 1$ and is a multiple of X_0, Y_0 or Y_1. The converse is clear. $\square$

Lemma 2.5.5. Let $D \in \mathbb{N}^2_+$ be a positive bidegree and let $P \in \mathbb{Q}[\mathbf{X}, \mathbf{Y}]_D$ with $X_0 \nmid P$, $Y_0 \nmid P$ and $Y_1 \nmid P$. If an irreducible bihomogeneous polynomial $R \in \mathbb{Q}[\mathbf{X}, \mathbf{Y}]$ divides $P, \mathcal{D}P, \ldots \mathcal{D}^k P$ for some integer $k \geq 0$, then R^{k+1} divides P. In particular, the polynomials $P, \mathcal{D}P, \ldots, \mathcal{D}^{\|D\|} P$ have no common irreducible factor in $\mathbb{Q}[\mathbf{X}, \mathbf{Y}]$ and $P, \mathcal{D}P, \ldots, \mathcal{D}^{D_y} P$ have no common irreducible factor in $\mathbb{Q}[\mathbf{X}, \mathbf{Y}]$ with positive y-degree.

Proof. Let R be an irreducible factor of P in $\mathbb{Q}[\mathbf{X}, \mathbf{Y}]$, and write $P = R^e Q$ for some positive integer $e \leq \|D\|$ and some bihomogeneous polynomial $Q \in \mathbb{Q}[\mathbf{X}, \mathbf{Y}]$

not divisible by R. Then, for $i = 0, \dots, e-1$, the polynomial $\mathcal{D}^i P$ is divisible by R while $\mathcal{D}^e P$ is congruent to $(\mathcal{D}R)^e Q$ modulo R. However, by Lemma 2.5.4 the hypothesis on P implies that $R \nmid \mathcal{D}R$. So e is the largest integer for which R divides $P, \mathcal{D}P, \dots, \mathcal{D}^{e-1}P$, and the result follows. $\square$

For the next results, we denote respectively by $\pi_1 : \mathcal{G} \to \mathbb{C}$ and by $\pi_2 : \mathcal{G} \to \mathbb{C}^\times$ the projections from $\mathcal{G} = \mathbb{C} \times \mathbb{C}^\times$ to its first and second factors.

Lemma 2.5.6. Let R be an irreducible bihomogeneous polynomial of $\mathbb{Q}[\mathbf{X}, \mathbf{Y}]$. Then $\tau_\gamma R$ is irreducible for any $\gamma \in \mathcal{G}$. Moreover, assume that R is not a multiple of either X_0, Y_0 or Y_1, and denote by Γ_R the set of all $\gamma \in \mathcal{G}$ such that R divides $\tau_\gamma R$. Then if $\deg_y(R) = 0$, then $\pi_1(\Gamma_R)$ is reduced to $\{0\}$, otherwise $\pi_2(\Gamma_R)$ is a cyclic subgroup of $\mathbb{C}^\times$ of order at most $\deg_y(R)$.

Proof. The first assertion follows simply from the fact that each τ_γ is an automorphism of $\mathbb{Q}[\mathbf{X}, \mathbf{Y}]$. To prove the second one, we first note that Γ_R is a subgroup of $\mathcal{G}$. Let $\gamma = (\xi, \eta) \in \mathcal{G}$ be an arbitrary element of Γ_R. Since $\tau_\gamma R$ has the same degree as R, we have $\tau_\gamma R = \lambda R$ for some $\lambda \in \mathbb{C}^\times$. Writing $R = \sum_{k=0}^{\deg_y(R)} \mathbf{Y}^k A_k(\mathbf{X})$, this condition translates into $\eta^k A_k(X_0, \xi X_0 + X_1) = \lambda A_k(X_0, X_1)$ for each $k = 0, \dots, \deg_y(R)$. When $A_k \neq 0$, this relation implies that $\eta^k = \lambda$. So, if $\deg_y(R) > 0$, being R irreducible and not divisible by Y_0 and Y_1, there are at least two indices k with $A_k \neq 0$. Then η is a root of unity of order at most $\deg_y(R)$ and, the choice of $(\xi, \eta) \in \Gamma_R$ being arbitrary, we conclude that $\pi_2(\Gamma_R)$ is a finite thus cyclic subgroup of $\mathbb{C}^\times$ of order at most $\deg_y(R)$. Otherwise, assuming that X_0 does not divide R, we obtain that $R = A_0(\mathbf{X})$ is of positive degree in X_1, and the equality $A_0(X_0, \xi X_0 + X_1) = \lambda A_0(X_0, X_1)$ implies that $\lambda = 1$ and $\xi = 0$. Thus, in that case, we have $\pi_1(\Gamma_R) = \{0\}$. $\square$

Theorem 2.5.7. Let Σ be a non-empty finite subset of $\mathcal{G}$ and let T be a positive integer. Denote by I the ideal of $\mathbb{C}[\mathbf{X}, \mathbf{Y}]$ generated by the bihomogeneous polynomials P satisfying

$$(\mathcal{D}^i P)(\vartheta_\gamma) = 0 \quad \text{for each } \gamma \in \Sigma \text{ and each } i = 0, \dots, T-1.$$

Suppose that there exist a finite subset Σ_1 of $\mathcal{G}$, an integer $T_1 \geq 0$ and a positive bi-integer $D = (D_x, D_y) \in \mathbb{N}_+^2$ such that

$$D < (T_1 + 1)(|\pi_1(\Sigma_1)|, |\pi_2(\Sigma_1)|) \quad \text{and} \quad (T + T_1)\,|\Sigma + \Sigma_1| < \dim(D) \quad (2.5.5)$$

where $\Sigma + \Sigma_1 = \{\gamma + \gamma_1 : \ \gamma \in \Sigma,\ \gamma_1 \in \Sigma_1\}$ denotes the sumset of Σ and Σ_1 in $\mathcal{G}$ and $\dim(D) = (D_x + 1)(D_y + 1)$. Then, the resultant form in bidegrees (D, D, D) vanishes up to order $T\,|\Sigma|$ at each point of $(I_D)^3$.

Proof. We have an irredundant primary decomposition $I = \bigcap_{\gamma \in \Sigma} I^{(\gamma, T)}$ where, according to Corollary 2.3.9, the ideals $I^{(\gamma, T)}$ are primary for distinct prime ideals of rank 2. Furthermore they all have the same degree T, and so $\deg(I) = T\,|\Sigma| \in \mathbb{N}$. The second condition in (2.5.5) implies the existence of a non-zero polynomial $P \in \mathbb{C}[\mathbf{X}, \mathbf{Y}]_D$ satisfying

$$(\mathcal{D}^i P)(\vartheta_\gamma) = 0 \text{ for each } \gamma \in \Sigma + \Sigma_1 \text{ and each } i = 0, \dots, T + T_1 - 1.$$

Fix such a polynomial P. If it is divisible by X_0, Y_0 or Y_1, then its quotient by any of these variables possesses the same vanishing property. Thus, upon dividing P by a suitable monomial of the form $X_0^h Y_0^k Y_1^l$ and multiplying the result by $X_1^h(Y_0+Y_1)^{k+l}$ to restore the bidegree, we may assume that P is not divisible by X_0, Y_0, nor Y_1. By construction, the polynomials $\tau_\gamma(\mathcal{D}^i P)$ belong to I for each $\gamma \in \Sigma_1$ and each $i = 0, \dots, T_1$. We claim that the latter have no non-constant common factor. For, suppose they have such a common factor R. Choose it to be bihomogeneous and irreducible. As P is not divisible by X_0, Y_0, nor Y_1, the same holds for R. Define Γ_R as in Lemma 2.5.6, and denote by Σ_2 a minimal subset of Σ_1 such that $\Sigma_2 + \Gamma_R = \Sigma_1 + \Gamma_R$. For any pair of distinct elements γ, γ' of Σ_2, we have $\gamma - \gamma' \notin \Gamma_R$, thus R does not divide $\tau_{\gamma-\gamma'}(R)$, and so, applying $\tau_{-\gamma}$, we see that the irreducible polynomials $\tau_{-\gamma}(R)$ and $\tau_{-\gamma'}(R)$ are not associated. Moreover, the choice of R implies that $\tau_{-\gamma}(R)$ divides $\mathcal{D}^i P$ for $i = 0, \dots, T_1$. By Lemma 2.5.5, this means that P is divisible by $\tau_{-\gamma}(R)^{T_1+1}$. Thus P is divisible by $\prod_{\gamma\in\Sigma_2} \tau_{-\gamma}(R)^{T_1+1}$ and so we have this inequality in $\mathbb{N}^2$

$$D = \deg(P) \geq (T_1+1)\,|\Sigma_2|\deg(R). \tag{2.5.6}$$

According to Lemma 2.5.6, either we have $\deg_y(R) = 0$ and $\pi_1(\Gamma_R) = \{0\}$ or the group $\pi_2(\Gamma_R)$ is cyclic of order at most $\deg_y(R)$. In the first case the equality $\Sigma_2 + \Gamma_R = \Sigma_1 + \Gamma_R$ implies that $\pi_1(\Sigma_2) = \pi(\Sigma_1)$ and from (2.5.6) we deduce that $D_x \geq (T_1+1)\,|\pi_1(\Sigma_1)|$ against the hypothesis (2.5.5). In the second case, it implies that $|\pi_2(\Sigma_2)| \geq |\pi_1(\Sigma_1)| \,/\deg_y(R)$ and (2.5.6) leads to $D_y \geq (T_1+1)\,|\pi_2(\Sigma_1)|$ once again in contradiction with (2.5.5).
Since the polynomials $\tau_\gamma(\mathcal{D}^i P)$ with $\gamma \in \Sigma_1$ and $i = 0, \dots, T_1$ all belong to I_D and share no common factor, the set of zeros of I_D in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ is finite (as $\mathbb{C}[\mathbf{X},\mathbf{Y}]$ is an UFD and so its prime ideals of rank 1 are principal...). As this set contains Σ, it is also non-empty. Therefore, by Theorem 2.5.2, the resultant form $\mathrm{res}_{(D,D,D)}$ vanishes up to order $\deg(I) = T\,|\Sigma|$ at each point of $(I_D)^3$. $\square$

In the case where Σ consists of just one point γ, the ideal I of the theorem is simply $I^{(\gamma,T)}$, and for the choice of $\Sigma_1 = \{e\}$ and $T_1 = \|D\|$, the condition 2.5.5 reduces to $T \leq D_x D_y + \min\{D_x, D_y\}$.

Corollary 2.5.8. Let $\gamma \in \mathcal{G}$, $D \in \mathbb{N}_+^2$ and $T \in \mathbb{N}_+$ with $T \leq D_x D_y + \min\{D_x, D_y\}$. Then, the resultant form $\mathrm{res}_{(D,D,D)}$ vanishes up to order T at each triple (P,Q,R) of elements of $I_D^{(\gamma,T)}$.

2.6 Construction of a subvariety of dimension 0

The first part of the proof of our main theorem consists in constructing, for each sufficiently large bi-integer D, a zero-dimensional subvariety Z of $\mathbb{P}^1_\mathbb{Q} \times \mathbb{P}^1_\mathbb{Q}$ with small height relative to a certain convex body. In this section, we define a convex body $\mathcal{C}$ of $\mathbb{C}[\mathbf{X},\mathbf{Y}]_D = \mathbb{C}[X_0,X_1,Y_0,Y_1]_D$ of the appropriate form and provide an estimate for the height of $\mathbb{P}^1 \times \mathbb{P}^1$ relative to $\mathcal{C}$. Then, we use this to construct a zero-dimensional subvariety Z with small height $h_\mathcal{C}(Z)$ assuming the existence of a non-zero homogeneous polynomial $P \in \mathbb{Z}[\mathbf{X},\mathbf{Y}]_D$ whose first derivatives with respect to $\mathcal{D}$ belong to $\mathcal{C}$. The rest of the section is devoted to a posteriori estimates for the mixed degree and the mixed heights of Z. Since we don't need sharp constants, we shall often the estimate $\log(3) \geq \log(2)$.

2.6.1 The relevant convex body

Proposition 2.6.1. Let $T \in \mathbb{N}_+$, let $D \in \mathbb{N}^2 \backslash \{\mathbf{0}\}$ and let $Y, U > 0$ with

$$T \leq D_x D_y + \min\{D_x, D_y\} \quad \text{and} \quad 2T \log(c_6) \leq Y \tag{2.6.1}$$

where $c_6 = 8 \max\{2 + |\xi|, 1 + |\eta|^{-1}\}$. Then, for the choice of convex body

$$\mathcal{C} = \{P \in \mathbb{C}[\mathbf{X}, \mathbf{Y}]_D \;:\; \|P\| \leq e^Y, \max_{0 \leq i < T} |\mathcal{D}^i P(\vartheta_\gamma)| \leq e^{-U}\},$$

we have $h_{\mathcal{C}}(\mathbb{P}^1 \times \mathbb{P}^1) \leq -TU + 6D_x D_y Y + 24\log(3) D_x D_y |D|$.

Proof. Let $\mathrm{Res}_D := \mathrm{res}_{(D,D,D)} \colon \mathbb{C}[\mathbf{X}, \mathbf{Y}]_D^3 \to \mathbb{C}$ denote the resultant form for $\mathbb{P}^1 \times \mathbb{P}^1$ of index $\mathbb{d} = (D, D, D)$. Using the notation of Lemma 2.2.3, we have, by that lemma,

$$h_{\mathcal{B}}(\mathrm{Res}_D) = h_{\mathcal{B}}(\mathbb{P}^1 \times \mathbb{P}^1) \leq 21 \log(2)(D_x^2 D_y + D_x D_y^2). \tag{2.6.2}$$

Since Res_D is a polynomial with integer coprime coefficients (see the discussion in Section § 1.1.2 and the comments immediately after Definition 1.3.13) we also have,

$$h_{\mathcal{C}}(\mathbb{P}^1 \times \mathbb{P}^1) = h_{\mathcal{C}}(\mathrm{Res}_D) = \log \sup\{|\mathrm{Res}_D(P_0, P_1, P_2)| \;:\; P_0, P_1, P_2 \in \mathcal{C}\}.$$

As $\mathcal{C}$ is compact, there exist $P_0, P_1, P_2 \in \mathcal{C}$ for which

$$h_{\mathcal{C}}(\mathbb{P}^1 \times \mathbb{P}^1) = \log |\mathrm{Res}_D(P_0, P_1, P_2)|.$$

The first hypothesis in (2.6.1) implies the existence of a non-negative bi-integer $L \in \mathbb{N}^2$ such that $L \leq D$, $L \neq D$ and $L_x L_y + \|L\| < T \leq \dim(L) =: M$. Then, we have $|L| < M \leq 2T$. For this choice of L and for each $j = 0, 1, 2$, Proposition 2.3.8 ensures the existence of a unique polynomial $Q_j \in \mathbb{C}[\mathbf{X}, \mathbf{Y}]_L$ such that

$$\mathcal{D}^i Q_j(\vartheta_\gamma) = \begin{cases} \mathcal{D}^i P_j(\vartheta_\gamma) & \text{for } i = 0, \ldots, T-1, \\ 0 & \text{for } i = T, \ldots, M-1, \end{cases}$$

and shows that it has norm

$$\|Q_j\| \leq c_1(-\gamma)^L 8^M \max_{0 \leq i \leq T-1} |\mathcal{D}^i P_j(\vartheta_\gamma)| \leq (8\|c_1(-\gamma)\|)^{2T} e^{-U} \leq e^{Y-U}$$

since $8\|c_1(-\gamma)\| = c_6$ and $\|Q_j\| \leq \mathcal{L}(Q_j)$. Put $\mathbf{M}_{(0,0)}^{D-L} = X_0^{D_x - L_x} Y_0^{D_y - L_y}$. By construction, the differences $P_j - \mathbf{M}_{(0,0)}^{D-L} Q_j$ are elements of $I_D^{(\gamma,T)}$ (notice that multiplication by X_0 or Y_0 commutes with differentiation by $\mathcal{D}$) and so, according to Corollary 2.5.8, the polynomial

$$f(z) = \mathrm{Res}_D(P_0 - (1-z)\mathbf{M}_{(0,0)}^{D-L} Q_0, \ldots, P_2 - (1-z)\mathbf{M}_{(0,0)}^{D-L} Q_2) \in \mathbb{C}[z]$$

vanishes to order at least T at $z = 0$. Applying the standard Schwarz lemma, this leads to

$$\begin{aligned} \exp(h_{\mathcal{C}}(\mathrm{Res}_D)) &= |f(1)| \\ &\leq e^{-TU} \sup\{|f(z)|\,;\, |z| = e^U\} \\ &\leq e^{-TU} \sup\{|\mathrm{Res}_D(P_0', P_1', P_2')| \;:\; P_j' \in \mathbb{C}[\mathbf{X}, \mathbf{Y}]_D, \|P_j'\| \leq 3e^Y\} \\ &\leq e^{-TU} (3e^Y)^{6D_x D_y} \exp(h_{\mathcal{B}}(\mathrm{Res}_D)), \end{aligned}$$

where the second estimate follows from $\|P_j + (1-e^U)Q_j\| \le e^Y + (1+e^U)e^{Y-U} \le 3e^Y$ and the last estimate follows from the fact that Res_D is homogeneous of degree $2D_xD_y$ on each of its three arguments. Using (2.6.2) we deduce

$$h_{\mathcal{C}}(\mathbb{P}^1 \times \mathbb{P}^1) \le -TU + 6D_xD_yY + 6D_xD_y\log(3) + 21\log(2)D_xD_y|D|.$$

The thesis follows from $6\log(3)+21\log(2) = 21,14\ldots \le 24\log(3) = 26,36\ldots$ $\square$

2.6.2 Existence of a subvariety with small relative height

Proposition 2.6.2. Let D, T, Y, U and $\mathcal{C}$ be as in Proposition 2.6.1. Define a real number $C > 0$ by the condition $TU = CD_xD_yY$ and suppose moreover that

$$8 < C, \quad \|D\| \le 2T - 1 \quad \text{and} \quad 32\log(3)|D| \le Y.$$

Finally, suppose that there exists a non-zero homogeneous polynomial $P \in \mathbb{Z}[\mathbf{X},\mathbf{Y}]_D$ not divisible by X_0, Y_0 or Y_1 such that $\mathcal{D}^iP \in \mathcal{C}$ for $i = 0,\ldots,2T-1$. Then, there exists a subvariety Z of $\mathcal{Z}(\mathcal{D}^iP : 0 \le i < 2T)$ of dimension 0 with

$$h_{\mathcal{C}}(Z) \le -C''(Y\deg(Z) + D_xh_x(Z) + D_yh_y(Z)), \tag{2.6.3}$$

where $C'' = (C-8)/14$.

Proof. Since P is a non-zero element of $\mathbb{Z}[\mathbf{X},\mathbf{Y}]_D \cap \mathcal{C}$, Proposition 2.2.4 ensures the existence of a non-zero cycle Z' of $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ of dimension 1 which satisfies

$$\begin{aligned}
\deg_x(Z') &\le (=)D_y,\\
\deg_y(Z') &\le (=)D_x,\\
h_{xx}(Z') &\le 11\log(2)D_y\\
h_{yy}(Z') &\le 11\log(2)D_x\\
h_{xy}(Z') &\le \log\|P\| + 9\log(2)|D| \le 2Y
\end{aligned} \tag{2.6.4}$$

and also, thanks to Proposition 2.6.1 and the definition of C:

$$\begin{aligned}
h_{\mathcal{C}}(Z') &\le h_{\mathcal{C}}(\mathbb{P}) + 8\log(2)D_xD_y|D|\\
&\le -TU + 6D_xD_yY + 32\log(3)D_xD_y|D|\\
&\le -(C-7)D_xD_yY.
\end{aligned}$$

We derive from (2.6.4) the following inequalities

$$\begin{aligned}
4D_xY\deg_x(Z') &\le 4D_xD_yY,\\
4D_yY\deg_y(Z') &\le 4D_xD_yY,\\
D_x^2h_{xx}(Z') &\le D_xD_yY,\\
D_y^2h_{yy}(Z') &\le D_xD_yY,\\
2D_xD_yh_{xy}(Z') &\le 4D_xD_yY,
\end{aligned}$$

and then

$$\begin{aligned}
h_{\mathcal{C}}(Z') \le &-C'(4D_xY\deg_x(Z') + 4D_yY\deg_y(Z') +\\
&+ D_x^2h_{xx}(Z') + D_y^2h_{yy}(Z') + 2D_xD_yh_{xy}(Z')),
\end{aligned}$$

where $C' = (C-7)/14$. From the last estimate and the additivity of the degree and heights on one-dimensional cycles, we deduce the existence of a component Z_1 of Z' with

$$h_{\mathcal{C}}(Z_1) \leq -C'(4YN_0 + D_x^2 h_{xx}(Z_1) + D_y^2 h_{yy}(Z_1) + 2D_x D_y h_{xy}(Z_1)), \quad (2.6.5)$$

where we put for convenience $N_0 := D_x \deg_x(Z_1) + D_y \deg_y(Z_1)$. We also observe that all the inequalities in (2.6.4) are valid for Z_1 replacing Z'. By Lemma 2.5.5, the polynomials $P, \mathcal{D}P, \ldots, \mathcal{D}^{\|D\|}P$ have no common irreducible factor in $\mathbb{Q}[\mathbf{X},\mathbf{Y}]$. Therefore, at least one of them does not belong to the ideal of Z_1. Since it has integral coefficients and since, by hypothesis, it belongs to $\mathcal{C}$, Proposition 2.2.5 ensures the existence of a non-zero cycle Z'' of $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ of dimension 0 with

$$\begin{aligned}
\deg(Z'') &= N_0,\\
h_x(Z'') &\leq D_x h_{xx}(Z_1) + D_y h_{xy}(Z_1) + 2Y\deg_x(Z_1) + 7\log(2)N_0\\
h_y(Z'') &\leq D_x h_{xy}(Z_1) + D_y h_{yy}(Z_1) + 2Y\deg_y(Z_1) + 7\log(2)N_0\\
h_{\mathcal{C}}(Z'') &\leq h_{\mathcal{C}}(Z_1) + 2\log(2)|D|N_0 \leq h_{\mathcal{C}}(Z_1) + (1/14)4YN_0\\
&\leq -C''(4YN_0 + D_x^2 h_{xx}(Z_1) + D_y^2 h_{yy}(Z_1) + 2D_xD_y h_{xy}(Z_1))\\
&\leq -C''(Y\deg(Z'') + D_x h_x(Z'') + D_y h_y(Z'')),
\end{aligned}$$

with $C'' = (C-8)/14$. Thus, by linearity, at least one component Z of Z'' satisfies (2.6.3). Since $C > 8$, we have $h_{\mathcal{C}}(Z) < 0$. So, by Proposition 2.2.7, the ideal of Z contains $\mathbb{Z}[\mathbf{X},\mathbf{Y}]_D \cap \mathcal{C}$ and so contains $\mathcal{D}^i P$ for $i = 0, \ldots, 2T-1$. $\square$

2.6.3 A posteriori estimates for degree and heights

Lemma 2.6.3. Let $D \in \mathbb{N}^2 \setminus \{\mathbf{0}\}$ a nonzero bidegree[(1)]. Let $(\alpha,\beta) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ and take a representative of it $(\underline{\alpha},\underline{\beta}) = (\alpha_0,\alpha_1,\beta_0,\beta_1) \in \mathbb{C}^4$. Suppose that there exists a nonzero polynomial $P \in \mathbb{C}[\mathbf{X},\mathbf{Y}]_D$ not divisible by X_0 such that $\mathcal{D}^i P(\underline{\alpha},\underline{\beta}) = 0$ for $i = 0, \ldots, D_y$. Then, either we have $\alpha_0 \neq 0$ or α is one of the points $(\infty,0) = ([0:1],[1:0])$ or $(\infty,\infty) = ([0:1],[0:1])$.

Proof. Write

$$P = \sum_{(a,b)\leq D} p_{a,b}\mathbf{X}^a\mathbf{Y}^b.$$

If $\alpha_0 = 0$, we have, for $i = 0, 1, \ldots, D_y$,

$$0 = \mathcal{D}^i P(\underline{\alpha},\underline{\beta}) = \Big(Y_1\frac{\partial}{\partial Y_1}\Big)^i P(\underline{\alpha},\underline{\beta}) = \sum_{b=0}^{D_y} p_{D_x,b} b^i \alpha_1^{D_x}\beta_0^{D_y-b}\beta_1^b,$$

with the usual convention that $b^i = \delta_{i,0}$ for $b = 0$. As the matrix $\big(b^i\big)_{\substack{0\leq i\leq D_y\\ 0\leq b\leq D_y}}$ is invertible, this yields $p_{D_x,b}\alpha_1^{D_x}\beta_0^{D_y-b}\beta_1^b = 0$ for $b = 0, \ldots, D_y$. However, as $X_0 \nmid P$, we also have $p_{D_x,b} \neq 0$ for at least one of these values of b, and thus we conclude that $\alpha_1\beta_0\beta_1 = 0$. Since α_0 implies $\alpha_1 \neq 0$, we have $\beta_0\beta_1 = 0$, and so $\alpha = (\infty,0)$ or $\alpha = (\infty,\infty)$. $\square$

(1)Indeed, the proof still works if $D_x = 0$ or $D_y = 0$.

Lemma 2.6.4. Let $D \in \mathbb{N}^2 \backslash \{\mathbf{0}\}$ a nonzero bidegree. Let $(\alpha, \beta) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ and take a representative of it $(\underline{\alpha}, \underline{\beta}) = (\alpha_0, \alpha_1, \beta_0, \beta_1) \in \mathbb{C}^4$. Suppose that there exists a non-zero polynomial $P \in \mathbb{C}[\mathbf{X}, \mathbf{Y}]_D$ not divisible by Y_0 or Y_1 such that $\mathcal{D}^i P(\underline{\alpha}, \underline{\beta}) = 0$ for $i = 0, \dots, D_x$. Then, either we have $\beta_0 \beta_1 \neq 0$ or α is one of the points $(\infty, 0) = ([0:1],[1:0])$ or $(\infty, \infty) = ([0:1],[0:1])$.

Proof. Write

$$P = \sum_{(a,b) \leq D} p_{a,b} \mathbf{X}^a \mathbf{Y}^b.$$

If $\beta_1 = 0$, we find, for $i = 0, 1, \dots, D_x$,

$$0 = \mathcal{D}^i P(\underline{\alpha}, \underline{\beta}) = \Big(X_0 \frac{\partial}{\partial X_1}\Big)^i P(\underline{\alpha}, \underline{\beta}) = \sum_{a=i}^{D_x} p_{a,0} \frac{a!}{(a-i)!} \alpha_0^{D_x - a + i} \alpha_1^{a-i} \beta_0^{D_y}.$$

As $Y_1 \nmid P$, we also note that $p_{a,0} \neq 0$ for some a with $0 \leq a \leq D_x$. If a_0 is the largest such index then, for $i = a_0$, this yields $0 = p_{a_0,0} a_0! \alpha_0^{D_x} \beta_0^{D_y}$. Since now $\beta_0 \neq 0$, we have $\alpha_0 = 0$, and so $\alpha = (\infty, 0)$.
Similarly, if $\beta_0 = 0$, we find, for $i = 0, 1, \dots, D_x$,

$$0 = \mathcal{D}^i P(\underline{\alpha}, \underline{\beta}) = \sum_{k=0}^{i} \binom{i}{k} (D_y)^{i-k} \Big(X_0 \frac{\partial}{\partial X_1}\Big)^k P(\underline{\alpha}, \underline{\beta}),$$

where we define $D_y^0 := 1$ if $D_y = 0$. We therefore see by induction on i that, for $i = 0, 1, \dots, D_x$,

$$0 = \Big(X_0 \frac{\partial}{\partial X_1}\Big)^i P(\underline{\alpha}, \underline{\beta}) = \sum_{a=i}^{D_x} p_{a,D_y} \frac{a!}{(a-i)!} \alpha_0^{D_x - a + i} \alpha_1^{a-i} \beta_1^{D_y}.$$

As $Y_0 \nmid P$, we also note that $p_{a,D_y} \neq 0$ for some a with $0 \leq a \leq D_x$. If a_0 is the largest such index then, for $i = a_0$, this yields $0 = p_{a_0,D_y} a_0! \alpha_0^{D_x} \beta_1^{D_y}$. Since now $\beta_1 \neq 0$, we have $\alpha_0 = 0$, and so $\alpha = (\infty, \infty)$. $\square$

Corollary 2.6.5. Let $D \in \mathbb{N}^2 \backslash \{\mathbf{0}\}$ a nonzero bidegree. Let $(\alpha, \beta) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ and take a representative of it $(\underline{\alpha}, \underline{\beta}) = (\alpha_0, \alpha_1, \beta_0, \beta_1) \in \mathbb{C}^4$. Suppose that there exists a non-zero polynomial $P \in \mathbb{C}[\mathbf{X}, \mathbf{Y}]_D$ not divisible by X_0, Y_0 or Y_1 such that $\mathcal{D}^i P(\underline{\alpha}, \underline{\beta}) = 0$ for $i = 0, \dots, \|D\|$. Then, either we have $\alpha_0 \beta_0 \beta_1 \neq 0$ or α is one of the points $(\infty, 0) = ([0:1],[1:0])$ or $(\infty, \infty) = ([0:1],[0:1])$.

Remark 2.6.6. Conversely, if $\alpha = (\infty, 0)$ (resp. $\alpha = (\infty, \infty)$), then, for any positive bi-integer $D \in \mathbb{N}_+^2$, the point α is a common zero of all the derivatives $\mathcal{D}^i P$, $i \in \mathbb{N}$, where $P = X_0^{D_x} Y_0^{D_y} + X_1^{D_x} Y_1^{D_y}$ (resp. $P = X_0^{D_x} Y_1^{D_y} + X_1^{D_x} Y_0^{D_y}$) is not divisible by X_0, Y_0 or Y_1.

Proposition 2.6.7. Let $D \in \mathbb{N}_+^2$, $T \in \mathbb{N}_+$, let $P \in \mathbb{C}[\mathbf{X}, \mathbf{Y}]_D$ with $X_0 \nmid P, Y_0 \nmid P$ and $Y_1 \nmid P$, and let $Y \in \mathbb{R}$. Suppose that

$$T \leq D_x D_y + \min\{D_x, D_y\},$$

$$\max\{34 \log(2)|D|, \log\|P\|, \log\|\mathcal{D}P\|, \dots, \log\|\mathcal{D}^{\|D\|} P\|\} \leq Y,$$

and that $W = \mathcal{Z}(\mathcal{D}^i P\,;\, 0 \le i < \|D\| + T)$ is not empty. Then any irreducible component Z of W in $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ has dimension 0 with

$$\deg(Z) \le \frac{2D_xD_y}{T} \quad \text{and} \quad \langle h(Z); \mathcal{D}\rangle \le \frac{5D_xD_yY}{T}. \tag{2.6.6}$$

Proof. By Lemma 2.5.5, the polynomials $P, \mathcal{D}P, \dots, \mathcal{D}^{\|D\|}P$ are relatively prime as a set. Since they are all bihomogeneous of degree D, we conclude that there exist integers $a_1, \dots, a_{\|D\|}$ of absolute values at most $|D|$ such that $Q = \sum_{i=1}^{\|D\|} a_i \mathcal{D}^i P$ is relatively prime to P[(2)]. Then $\mathcal{Z}(P, Q)$ has dimension 0 and since W is a closed subset of $\mathcal{Z}(P, Q)$, it also has dimension 0.
Let Z be an irreducible component of W. Since $Z \subseteq \mathcal{Z}(\mathcal{D}^i P\,;\, 0 \le i \le \|D\|)$, Lemma 2.6.5 shows that either $Z(\mathbb{C})$ is contained in the open set $\mathcal{G}$ of $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ or it consists of one of the points $(\infty, 0)$ or (∞, ∞) (the points of $Z(\mathbb{C})$ are conjugate over $\mathbb{Q}$). In the latter case, we have $\deg(Z) = 1$ and $h(Z) = 0$, and the estimates in (2.6.6) follow. Thus, in order to prove these estimates, we may assume, without loss of generality that $Z(\mathbb{C}) \subseteq \mathcal{G}$.
Let $G = \mathrm{res}_{(D)}(Z) \colon \mathbb{C}[\mathbf{X}, \mathbf{Y}]_D \to \mathbb{C}$ be a resultant form for Z of index (D), and let $F \colon \mathbb{C}[\mathbf{X}, Y]_D \to \mathbb{C}$ denote the map given by $F(R) = \mathrm{res}_{(D,D,D)}(\mathbb{P})(P, Q, R)$ for each $R \in \mathbb{C}[\mathbf{X}, \mathbf{Y}]_D$. We claim that G^T divides F.
To prove this claim, choose a system of representatives $\underline{\alpha}_1, \dots, \underline{\alpha}_s \in \mathbb{C}^4$ of the points of $Z(\mathbb{C})$ and complete it to a system of representatives $\underline{\alpha}_1, \dots, \underline{\alpha}_t$ of those of $\mathcal{Z}(P, Q)(\mathbb{C})$. Then, there exist $a, b \in \mathbb{C}^\times$ and $e_1, \dots, e_t \in \mathbb{N}_+$ such that

$$F(R) = aR(\underline{\alpha}_1)^{e_1} \cdots R(\underline{\alpha}_t)^{e_t} \quad \text{and} \quad G(R) = bR(\underline{\alpha}_1) \cdots R(\underline{\alpha}_s) \tag{2.6.7}$$

for each $R \in \mathbb{C}[\mathbf{X}, \mathbf{Y}]_D$. Moreover, $e_1 = \cdots = e_s$ represents the multiplicity of G as a factor of F over $\mathbb{Q}$. So, our claim reduces to showing that $e_1 \ge T$. Denote by α the point of $Z(\mathbb{C})$ corresponding to $\underline{\alpha}_1$. According to Proposition 2.3.11, there exists a polynomial R in $I_D^{(\alpha,T)}$ such that $R(\underline{\alpha}_i) \ne 0$ for $i = 2, \dots, t$. Since P and Q also belong to $I_D^{(\alpha,T)}$, Corollary 2.5.8 shows that $\mathrm{res}_{(D,D,D)}(Z)$ vanishes to order at least T at the point (P, Q, R). Therefore, for any fixed $S \in \mathbb{C}[\mathbf{X}, Y]_D$, the polynomial $F(R + zS) \in \mathbb{C}[z]$ is divisible by z^T. Choosing S so that $S(\underline{\alpha}_1) \ne 0$, formula (2.6.7) for F provides

$$F(R + zS) = aS(\underline{\alpha}_1)^{e_1} R(\underline{\alpha}_2)^{e_2} \cdots R(\underline{\alpha}_t)^{e_t} z^{e_1} + O(z^{e_1+1}),$$

and then $e_1 \ge T$. Therefore, G^T divides F, and we obtain

$$T \deg(Z) = T \deg(G) \le \deg(F) = 2D_xD_y, \tag{2.6.8}$$

which proves the first half of (2.6.6). In terms of the convex body $\mathcal{B} = \mathcal{B}^{(D,D,D)}$ of Definition 1.3.9, we also find, thanks to 1.4.5,

$$Th_{\mathcal{B}}(Z) = Th_{\mathcal{B}}(G) \le h_{\mathcal{B}}(F) + 4 \log \dim(D) D_xD_y \le h_{\mathcal{B}}(F) + 4 \log(2) D_xD_y|D|.$$

To translate this inequality in terms of the standard height $h(Z)$, we first observe that Lemma 1.4.9 and the degree estimate (2.6.8) lead to

$$T\langle h(Z); D\rangle \le Th_{\mathcal{B}}(Z) + 3T \log(2)|D| \deg(Z) \le Th_{\mathcal{B}}(Z) + 6 \log(2) D_xD_y|D|.$$

(2) Proof by induction, based on the fact that P has at most $|D|$ irreducible factors.

Moreover, since F is obtained by specializing the first two arguments of a resultant form of $\mathbb{P}$ into P and Q with $\|P\| \leq e^Y$ and $\|Q\| \leq |D|^2 e^Y \leq e^{|D|+Y}$, and since that resultant form is homogeneous of degree $2D_xD_y$ in each of its three arguments, we also find, using $2 < 3\log(2)$:

$$h_{\mathcal{B}}(F) \leq 2D_xD_y(Y + |D| + Y) + h_{\mathcal{B}}(\mathbb{P}) \leq 4D_xD_yY + 24\log(2)D_xD_y|D|,$$

using the upper bound for $h_{\mathcal{B}}(\mathbb{P})$ provided by Lemma 2.2.3. Combining the last three estimates, we conclude that

$$T\langle h(Z); D\rangle \leq 4D_xD_yY + 34\log(2)D_xD_y|D| \leq 5D_xD_yY,$$

which proves the second half of (2.6.6). □

2.7 Proof of the main Theorem

We are now going to prove Theorem 2.1.1. We divide the proof into several steps and we use the results obtained in the previous sections of this chapter.
We shall put ourself in a multiprojective setting. To this extent, we think of $\mathcal{G} = \mathbb{C} \times \mathbb{C}^\times$ as embedded into $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ and we use the notation of Section 2.4. We shall argue by contradiction, assuming on the contrary that $\boldsymbol{\vartheta}_\gamma = ([1 : \xi], [1 : \eta])$ is not a point of $\mathbb{P}^1(\overline{\mathbb{Q}}) \times \mathbb{P}^1(\overline{\mathbb{Q}})$. From there we proceed in several steps.

Step 1. We define relevant convex bodies $\mathcal{C}_N$ and we define bihomogeneous polynomials $\widetilde{P}_N$ that belong to $\mathcal{C}_N$ together with their first $2\lfloor N^\tau \rfloor$ derivatives.

For each positive integer $N \in \mathbb{N}_+$, we put

$$D := (\lfloor N^{t_0} \rfloor, \lfloor N^{t_1} \rfloor), \quad T := \lfloor N^\tau \rfloor, \quad Y := 2N^\beta, \quad U := N^\nu/2,$$

and we define a convex body $\mathcal{C}_N$ of $\mathbb{C}[\mathbf{X}, \mathbf{Y}]_D$ by

$$\mathcal{C}_N = \Big\{ P \in \mathbb{C}[\mathbf{X}, \mathbf{Y}]_D \ : \ \|P\| \leq \exp(Y), \ \max_{0 \leq i < T} |\mathcal{D}^i P(\vartheta_\gamma)| \leq \exp(-U) \Big\}.$$

Given a polynomial $P = \sum_{(a,b) \leq D} p_{a,b} x^a y^b \in \mathbb{C}[x, y]_{\leq D}$ we define its bihomogenization in bidegree D to be $H_D P := \sum_{(a,b) \leq D} p_{a,b} \mathbf{X}^a \mathbf{Y}^b \in \mathbb{C}[\mathbf{X}, \mathbf{Y}]_D$. We then denote by $\widetilde{P}_N$ the bihomogeneous polynomial of $\mathbb{Z}[\mathbf{X}, \mathbf{Y}]_D$ defined by

$$\widetilde{P}_N = X_0^{-h_0} X_1^{h_0} Y_0^{-k_0} Y_1^{-k_1} (Y_0 + Y_1)^{k_0 + k_1} H_D P_N,$$

where h_0, k_0, k_1 stand for the largest integers such that $X_0^{h_0}, Y_0^{k_0}, Y_1^{k_1}$ divide $H_D P_N$. Then, by construction, $\widetilde{P}_N$ is not divisible by X_0, Y_0 or Y_1. We claim that, for any sufficiently large N, the polynomials $\mathcal{D}^j \widetilde{P}_N$ with $0 \leq j < 2\lfloor N^\tau \rfloor$ all belong to $\mathcal{C}_N$.

To prove this, fix a choice of integer j with $0 \leq j < 2\lfloor N^\tau \rfloor$, and put $Q = \mathcal{D}^j \widetilde{P}_N$. Using Lemma 2.3.6, and the fact that $\tau < \beta$ and $t_1 < \beta$ we find

$$\|Q\| \leq |D|^j \mathcal{L}(\widetilde{P}_N) \leq |D|^{2N^\tau} 2^{2D_y} \dim(D) \|P_N\| = \exp((1 + o(1))N^\beta).$$

Moreover, for any $i = 0, \ldots, \lfloor N^\tau \rfloor - 1$, Leibniz' rule of differentiation for a product leads to the following estimate, where we also use that multiplication by X_0 or Y_0 commutes with $\mathcal{D}$, that $\mathcal{D}H_D = H_D\mathcal{D}_1$, that X_0, Y_0 are evaluated to 1, and that $1, t, \tau < \nu$:

$$\begin{aligned}
&|\mathcal{D}^i Q(\vartheta_\gamma)| = \\
&= |\mathcal{D}_1^{i+j}(x^{h_0} y^{-k_1}(1+y)^{k_0+k_1} P_N(x,y))|_{x=\xi,\, y=\eta} \\
&\leq \sum_{r+s+t+u=i+j} \frac{(i+j)!}{r!\, s!\, t!\, u!} |\mathcal{D}_1^r x^{h_0}|_{x=\xi} \, |\mathcal{D}_1^s y^{-k_1}|_{y=\eta} \, |\mathcal{D}_1^t (1+y)^{k_0+k_1}|_{y=\eta} \, |\mathcal{D}_1^u P_N|_{\substack{x=\xi \\ y=\eta}} \\
&\leq \sum_{r+s+t+u=i+j} \frac{(i+j)!}{r!\, s!\, t!\, u!} \, h_0^r \max\{1, |\xi|\}^{h_0} \cdot k_1^s \, |\eta|^{-k_1} \cdot 2^{k_0+k_1}(k_0+k_1)^t \cdot \\
&\qquad\qquad \cdot \max\{1, |\eta|\}^{k_0+k_1} \cdot \max_{0 \leq k < 3\lfloor N^\tau \rfloor} |\mathcal{D}_1^k P_N(\xi, \eta)| \cdot 1^u \\
&\leq \max\{2, |\xi|, |\eta|^{-1}, |2\eta|\}^{h_0+k_0+k_1} (h_0 + k_1 + k_0 + k_1 + 1)^{3\lfloor N^\tau \rfloor} \exp(-N^\nu) \\
&= \exp(-(1 - o(1)) N^\nu).
\end{aligned}$$

Step 2. For N sufficiently large, we construct a 0-dimensional subvariety $Z = Z_N$ of $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ contained in $\mathcal{Z}(\mathcal{D}^i \widetilde{P_N} \,:\, 0 \leq i < 2T)$ such that

$$h_{\mathcal{C}_N}(Z) \leq -\frac{N^\delta}{57}(2N^\beta \deg(Z) + \langle h(Z); D \rangle).$$

We fix $N \in \mathbb{N}_+$ and observe that the convex set $\mathcal{C}$ defined in Proposition 2.6.1 coincide with $\mathcal{C}_N$. We now check that if N is sufficiently large the hypothesis of 2.6.2 are all fulfilled. This will imply the existence of Z_N. Recall that we defined $\delta = \nu + \tau - 1 - t - \beta$, and fix an arbitrarily large integer N.

- $T \leq D_x D_y + \min\{D_x, D_y\}$ because $\tau < 1 + t$,
- $2T \log(c_6) \leq Y$ because $\tau < \beta$,
- $8 < C$ because $C \asymp N^\delta$ and $\delta = \nu + \tau - 1 - t - \beta > 0$,
- $\|D\| \leq 2T - 1$ because $\tau \geq 1$.
- $32 \log(3)|D| \leq Y$ because $\beta > \tau \geq 1$.

Moreover, we have $C'' = (C - 8)/14 \geq N^\delta/57$ because $C \approx N^\delta/4$.

Step 3. We let $\underline{Z}$ be a set of normalized representatives of the points of $Z(\mathbb{C})$ by elements of $\mathbb{C}^4$ and we define $\mathcal{U}$ to be the set of points (α, β) of $Z(\mathbb{C})$ with $\operatorname{bidist}((\alpha, \beta), \boldsymbol{\vartheta}_\gamma) \leq (2c_2)^{-1}$. We show that the quantity

$$\sum_{(\alpha,\beta) \in \mathcal{U}} \max\{T \log \operatorname{dist}(\alpha, (1 : \xi)), \log \operatorname{dist}((\alpha, \beta), A_\gamma)\}$$

is bounded above by $-\frac{N^\delta}{57}(2N^\beta \deg(Z) + \langle h(Z); D \rangle)$.

Here the starting point is Proposition 2.2.7, from which we get, since $|D| \leq 2N$:

$$\sum_{(\underline{\alpha}, \underline{\beta}) \in \underline{Z}} \log \sup\{|P(\underline{\alpha}, \underline{\beta})| \,;\, P \in \mathcal{C}_N\} \leq h_{\mathcal{C}_N}(Z) - \langle h(Z); D \rangle + 12 \log(2) N \deg(Z).$$

For each $(\alpha,\beta) \in Z(\mathbb{C})$ with corresponding point $(\underline{\alpha},\underline{\beta}) \in \underline{Z}$, we also have, according to the definitions,

$$\sup\{|P(\underline{\alpha},\underline{\beta})|\,;\, P \in \mathcal{C}_N\} \geq \sup\{|P(\underline{\alpha},\underline{\beta})|\,;\, P \in I_D^{(\gamma,T)}, \|P\| \leq 1\,\} =^{(3)} |I_D^{(\gamma,T)}|_{(\underline{\alpha},\underline{\beta})}.$$

For each $(\alpha,\beta) \in Z(\mathbb{C}) \setminus \mathcal{U}$, Proposition 2.4.5 and Proposition 2.4.6 give

$$\begin{aligned}|I_D^{(\gamma,T)}|_{(\alpha,\beta)} &\geq (c_5T)^{-28T\log(2T)} \max\{\mathrm{dist}(\alpha,(1:\xi))^T, \mathrm{dist}(\beta,(1:\eta))^T\} \\ &\geq (2\|c_2\|c_5T)^{-28T\log(2T)} \\ &\geq T^{-30T\log(T)},\end{aligned}$$

assuming N large enough so that $(2\|c_2\|c_5)^{28T\log(2T)}T^{28\log(2)} \leq T^{2T\log(T)}$. For the more interesting points $(\alpha,\beta) \in \mathcal{U}$, Proposition 2.4.5 gives

$$\begin{aligned}|I_D^{(\gamma,T)}|_{(\alpha,\beta)} &\geq c_4^{-1}(c_5T)^{-28T\log(2T)} \max\{\mathrm{dist}(\alpha,(1:\xi))^T, \mathrm{dist}((\alpha,\beta),A_\gamma)\} \\ &\geq T^{-30T\log(T)} \max\{\mathrm{dist}(\alpha,(1:\xi))^T, \mathrm{dist}((\alpha,\beta),A_\gamma)\},\end{aligned}$$

provided that N is large enough. Putting all these estimates together, taking into account that $Z(\mathbb{C})$ consists of $\deg(Z)$ points, we conclude that

$$\begin{aligned}&\sum_{(\alpha,\beta)\in\mathcal{U}} \max\{T\log\mathrm{dist}(\alpha,(1:\xi)), \log\mathrm{dist}((\alpha,\beta),A_\gamma)\} \\ &\qquad\leq \sum_{(\alpha,\beta)\in\mathcal{U}} \Big(\log|I_D^{(\gamma,T)}|_{(\alpha,\beta)} + 30T\log(T)^2\Big) \\ &\qquad\leq \sum_{(\alpha,\beta)\in Z(\mathbb{C})} \Big(\log|I_D^{(\gamma,T)}|_{(\alpha,\beta)} + 30T\log(T)^2\Big) \\ &\qquad\leq h_{\mathcal{C}_N}(Z) - \langle h(Z);D\rangle + 12\log(2)N\deg(Z) + 30T\log(T)^2\deg(Z) \\ &\qquad\leq -\frac{N^\delta}{57}(N^\beta\deg(Z) + \langle h(Z);D\rangle)\end{aligned}$$

if N is large enough, because $\beta > \tau \geq 1$ and $h_x(Z), h_y(Z) \geq 0$.
In particular, the set $\mathcal{U}$ is not empty and contains at least one point (α,β) for which $\log\mathrm{dist}(\alpha,(1:\xi)) \leq -\frac{N^{\delta+\beta}}{57T}$ and $\log\mathrm{dist}((\alpha,\beta),A_\gamma) \leq -\frac{N^{\delta+\beta}}{57} \leq -\frac{N^{\delta+\beta}}{57T}$. Moreover, if $\alpha = (\alpha_0 : \alpha_1)$, $\beta = (\beta_0 : \beta_1)$, $\delta_1 = \frac{\alpha_1}{\alpha_0} - \xi$, and if we have the normalization $\|(\alpha_0,\alpha_1)\| = \|(\beta_0,\beta_1)\| = 1$, we have

$$\begin{aligned}\mathrm{dist}(\beta,(1:\eta)) &= c_{2,y}^{-1}\,|\beta_1 - \eta\beta_0| \\ &\leq |\beta_1 - \eta\beta_0\,e^{\delta_1}| + c_{2,y}^{-1}\,|\eta||\beta_0||e^{\delta_1} - 1| \\ &\leq |\beta_0|\,\mathrm{dist}((\alpha,\beta),A_\gamma) + e^{|\delta_1|}|\delta_1| \\ &\leq \mathrm{dist}((\alpha,\beta),A_\gamma) + c_{2,x}e^{c_{2,x}}|\alpha_0|^{-1}\,\mathrm{dist}(\alpha,(1:\xi)), \\ &\leq 2c_{2,x}e^{c_{2,x}}\exp\Big(-\frac{N^{\delta+\beta}}{57T}\Big)\end{aligned}$$

In the first inequality we used $c_{2,y} \geq 1$, in the second we used $c_{2,y} \geq |\eta|$ and the general inequality $e^x - 1 \geq xe^x$, valid for $x \geq 0$, and in the third inequality we

(3) True also with a constant e^Y, which changes nothing.

used $e^{|\delta_1|} \le e^{c_{2,x}}$ (see the proof of Proposition 2.4.2). Thus, as N goes to infinity, the point (α_0, β_0) runs through an infinite sequence of points of $\mathbb{P}^1(\overline{\mathbb{Q}}) \times \mathbb{P}^1(\overline{\mathbb{Q}})$ converging to $\boldsymbol{\vartheta}_\gamma$ but distinct from $\boldsymbol{\vartheta}_\gamma$, because $\boldsymbol{\vartheta}_\gamma \notin \mathbb{P}^1(\overline{\mathbb{Q}}) \times \mathbb{P}^1(\overline{\mathbb{Q}})$.

Step 4. We derive upper bounds for $\deg(Z)$, $h_x(Z)$ and $h_y(Z)$ in terms of the smallest positive integer N_* for which

$$Z \subseteq \mathcal{Z}\big(\mathcal{D}^i \widetilde{P}_{N_*+1} \,:\, 0 \le i < 2\lfloor (N_*+1)^\tau \rfloor\big).$$

If $N \ge 2$, such an integer exists and is at most equal to $N-1$. Moreover, N_* goes to infinity with N because, as $\widetilde{P}_{N_*+1}$ is not divisible by X_0, Y_0 or Y_1, it follows from Lemma 2.5.5 that $\mathcal{Z}(\mathcal{D}^i \widetilde{P}_{N_*+1} \,:\, 0 \le i \le N_*+1)(\mathbb{C})$ is a finite subset of $\mathbb{P}^1(\overline{\mathbb{Q}}) \times \mathbb{P}^1(\overline{\mathbb{Q}})$ and so, for fixed $N_* \ge 1$, this set does not contain the point (α_0, β_0) of $Z(\mathbb{C})$ when N is large enough. Thus, assuming N large enough, it follows from Step 1 that $\mathcal{D}^i \widetilde{P}_{N_*+1}$ belongs to $\mathcal{C}_{N_*+1}$ for $i = 0, \dots, N_*+1$ and

$$\max\{34\log(2)(N_*+1),\, \log\|\widetilde{P}_{N_*+1}\|, \dots, \log\|\mathcal{D}^{N_*+1}\widetilde{P}_{N_*+1}\|\} \le 2(N_*+1)^\beta.$$

For N large enough, putting $D_+ := (\lfloor (N_*+1)^{t_0} \rfloor, \lfloor (N_*+1)^{t_1} \rfloor)$, we also have $\lfloor (N_*+1)^\tau \rfloor \le D_{+,x} D_{+,y}$, because $\tau < 1+t$. By Proposition 2.6.7, we conclude that

$$\deg(Z) \le \frac{2(N_*+1)^{1+t}}{\lfloor (N_*+1)^\tau \rfloor} \le 3N_*^{1+t-\tau}$$

and

$$\langle h(Z); D_+ \rangle \le \frac{5(N_*+1)^{1+t+\beta}}{\lfloor (N_*+1)^\tau \rfloor} \le 6N_*^{1+t+\beta-\tau}.$$

Thus expanding $\langle h(Z); D_+ \rangle$ we get

$$h_x(Z) \le 6N_*^{t_1+\beta-\tau} \quad \text{and} \quad h_y(Z) \le 6N_*^{t_0+\beta-\tau}.$$

Step 5. Put $T^* := \lfloor N_*^\tau \rfloor$ and $D_* := (\lfloor N_*^{t_0} \rfloor, \lfloor N_*^{t_1} \rfloor) = D_{N_*}$. We show that for every subset $\mathcal{S}$ of $\mathcal{U}$, the quantity

$$\sum_{(\alpha,\beta)\in\mathcal{S}} \max\big\{T^* \log \operatorname{dist}(\alpha, (1:\xi)),\, \log \operatorname{dist}((\alpha,\beta), A_\gamma)\big\}$$

is bounded below by $-8N_*^\beta \deg(Z) - \langle h(Z); D_* \rangle$.

For N large enough, we have $N_* \ge 2$ and so, by the very choice of N_*, there exists an integer i_0 with $0 \le i_0 < 2\lfloor N_*^\tau \rfloor$ such that Z is not contained in the curve of $\mathbb{P}^1 \times \mathbb{P}^1$ defined by the polynomial $P^* := \mathcal{D}^{i_0} \widetilde{P}_{N_*}$. For N large enough, we also have $P^* \in \mathcal{C}_{N_*} \cap \mathbb{Z}[\mathbf{X}, \mathbf{Y}]$. Then, Proposition 2.2.7, together with $|D_*| \le 2N_*$, gives

$$0 \le 6\log(2)N_* \deg(Z) + \langle h(Z); D_* \rangle + \sum_{(\underline{\alpha},\underline{\beta})\in\underline{Z}} \log|P^*(\underline{\alpha},\underline{\beta})|.$$

Moreover, the fact that $P^* \in \mathcal{C}_{N_*}$ leads to the crude estimate

$$\max_{(\underline{\alpha},\underline{\beta})\in\underline{Z}} \log|P^*(\underline{\alpha},\underline{\beta})| \le \log(2)|D_*| + \log\|P^*\| \le (2\log(2)+2)N_*^\beta.$$

Combining the last two results and using $-6\log(2) + 2\log(2) + 2 \geq -2$, we deduce that, for N large enough:

$$\sum_{(\underline{\alpha},\underline{\beta})\in \underline{Z}} \min\{0, \log|P^*(\underline{\alpha},\underline{\beta})|\} \geq -2N_*^\beta \deg(Z) - \langle h(Z); D_*\rangle. \qquad (2.7.1)$$

For a point $(\alpha,\beta) \in \mathcal{U}$ with representative $(\underline{\alpha},\underline{\beta}) \in \underline{Z}$, Proposition 2.4.2 provides the more precise estimate

$$\begin{aligned}
&|P^*(\underline{\alpha},\underline{\beta})| \leq \\
&\quad \leq c_4 \max_{0\leq i<T^*} |\mathcal{D}^i P^*(\vartheta_\gamma)| + c_4^{2N_*} \|P^*\| \big(\operatorname{dist}(\alpha,(1:\xi))^{T^*} + \operatorname{dist}((\alpha,\beta),A_\gamma)\big)\\
&\quad \leq c_4 e^{-N_*^\nu/2} + c_4^{2N_*} e^{2N_*^\beta} \big(\operatorname{dist}(\alpha,(1:\xi))^{T^*} + \operatorname{dist}((\alpha,\beta),A_\gamma)\big).
\end{aligned}$$

However, if N_* is large enough, the inequality (2.7.1) combined with the estimates for $\deg(Z)$ and $h_x(Z), h_y(Z)$ obtained in Step 4 leads to

$$\log|P^*(\underline{\alpha},\underline{\beta})| \geq -2N_*^\beta \deg(Z) - \langle h(Z); D_*\rangle \geq -(6+12)N_*^{1+t+\beta-\tau},$$

thus $|P^*(\underline{\alpha},\underline{\beta})| \geq 2c_4 e^{-N_*^\nu/2}$, and so

$$\log|P^*(\underline{\alpha},\underline{\beta})| \leq 3N_*^\beta + \max\big\{T^* \log\operatorname{dist}(\alpha,(1:\xi)),\, \log\operatorname{dist}((\alpha,\beta),A_\gamma)\big\}.$$

Note that this holds for any $(\alpha,\beta) \in \mathcal{U}$ with a lower bound on N_* not depending on (α,β). Therefore, if N is large enough, we conclude using (2.7.1) that, for any subset $\mathcal{S}$ of $\mathcal{U}$, we have

$$\begin{aligned}
&\sum_{(\alpha,\beta)\in\mathcal{S}} \max\big\{T^* \log\operatorname{dist}(\alpha,(1:\xi)),\, \log\operatorname{dist}((\alpha,\beta),A_\gamma)\big\} \geq \\
&\qquad\qquad\qquad\qquad \geq -5N_*^\beta \deg(Z) - \langle h(Z); D_*\rangle.
\end{aligned}$$

Step 6. We derive a contradiction by combining the results of Step 3 and Step 5 with the estimates found in Step 4. We then prove that $(\xi,\eta) \in \overline{\mathbb{Q}} \times \overline{\mathbb{Q}}^\times$.

We partition $\mathcal{U}$ into the following two disjoint subsets $\mathcal{U}'$ and $\mathcal{U}''$

$$\mathcal{U}' := \{(\alpha,\beta) \in \mathcal{U}\,;\, T^* \log\operatorname{dist}(\alpha,(1:\xi)) \geq \log\operatorname{dist}((\alpha,\beta),A_\gamma)\} \quad , \quad \mathcal{U}'' := \mathcal{U}\backslash\mathcal{U}'.$$

We also put

$$S_1 := \sum_{(\alpha,\beta)\in\mathcal{U}'} \log\operatorname{dist}(\alpha,(1:\xi)) \quad , \quad S_2 := \sum_{(\alpha,\beta)\in\mathcal{U}''} \log\operatorname{dist}((\alpha,\beta),A_\gamma).$$

According to Step 3, we have

$$TS_1 + S_2 \leq -\frac{N^\delta}{57}(N^\beta \deg(Z) + \langle h(Z); D\rangle),$$

whereas the result of Step 5 applied to the sets $\mathcal{S} = \mathcal{U}'$ and $\mathcal{S} = \mathcal{U}''$ gives respectively

$$\begin{aligned}
T^* S_1 &\geq -5N_*^\beta \deg(Z) - \langle h(Z); D_*\rangle,\\
S_2 &\geq -5N_*^\beta \deg(Z) - \langle h(Z); D_*\rangle.
\end{aligned}$$

Combining these three inequalities, we obtain

$$-\frac{N^{\delta}}{57}(N^{\beta}\deg(Z)+\langle h(Z);D\rangle)\geq-\Big(\frac{T}{T^{*}}+1\Big)\big(5N_{*}^{\beta}\deg(Z)+\langle h(Z);D_{*}\rangle\big)$$

and so expanding we get that the quantity

$$N^{\delta+\beta}\deg(Z)+N^{\delta+t_0}h_x(Z)+N^{\delta+t_1}h_y(Z)$$

is asymptotically dominated by

$$N^{\tau}N_{*}^{t_0-\tau}h_x(Z)+N^{\tau}N_{*}^{t_1-\tau}h_y(Z)$$

(we may omit the term $N^{\tau}N_{*}^{\beta-\tau}\deg(Z)$ in the right hand side as it is negligible with respect to $N^{\delta+\beta}\deg(Z)$). Suppose that

$$N^{\tau}N_{*}^{t_0-\tau}h_x(Z)\geq N^{\tau}N_{*}^{t_1-\tau}h_y(Z). \tag{2.7.2}$$

This implies that

$$N^{\delta+\beta-\tau}\deg(Z)\ll N_{*}^{t_0-\tau}h_x(Z)\quad\text{and}\quad N_{*}^{\tau-t_0}\ll N^{\tau-\delta-t_0}. \tag{2.7.3}$$

Since $\deg(Z)\geq 1$ and $h_x(Z)\ll N_{*}^{t_1+\beta-\tau}$ (see Step 4), from the first estimate in (2.7.3) we get

$$N^{\delta+\beta-\tau}\ll N_{*}^{\beta+1+t-2\tau}.$$

As $\tau\geq 1$, combining this with the second estimate from (2.7.3) yields

$$(\tau-t_0)(\delta+\beta-\tau)\leq(\tau-\delta-t_0)(\beta+1+t-2\tau)$$

which after simplifications is equivalent to $\delta\leq(\tau-t_0)(1+t-\tau)/(\beta+t_1-\tau)$. In case we have the opposite inequality in (2.7.2) we obtain with the same arguments $\delta\leq(\tau-t_1)(1+t-\tau)/(\beta+t_0-\tau)$. We see that the inequality

$$\frac{\tau-t_0}{\beta+t_1-\tau}\geq\frac{\tau-t_1}{\beta+t_0-\tau}$$

is equivalent to $(\tau-t_0)(\beta+t_0-\tau)\geq(\tau-t_1)(\beta+t_1-\tau)$, which in turn is equivalent to $(t_1-t_0)((\beta-\tau)+(t_0+t_1-\tau))\geq 0$ and so to $t_1\geq t_0$. This means that in any case we have proven

$$\delta\leq\max\left\{\frac{\tau-t_0}{\beta+t_1-\tau},\frac{\tau-t_1}{\beta+t_0-\tau}\right\}=\frac{(\tau-t)(1+t-\tau)}{\beta+1-\tau}$$

This contradicts the hypothesis on δ in the statement of Theorem 2.1.1, and therefore proves that $\xi,\eta\in\overline{\mathbb{Q}}$.

Step 7. We use Liouville's inequality to prove that for sufficiently large N the polynomials $\mathcal{D}_1^k P_N$ vanish at (ξ,η) for all $k=0,\dots,d\lfloor N^{\tau}\rfloor$.

The argument is similar to the one presented in the proof of Corollary 2.1.6, so we only give a sketch of the calculations. Let $K=\mathbb{Q}(\xi,\eta)$, let $d=[K:\mathbb{Q}]$ and let H be an upper bound for the Weil absolute height of ξ and η (see Definition 1.3.11). If $\mathcal{D}_1^k P_N$ does not vanish at (ξ,η), Proposition 2.1.4 gives

$$\left|\mathcal{D}_1^k P_N(\xi,\nu)\right|\geq\exp(-2(d-1)N^{\beta}-dH(N^{t_0}+N^{t_1})),$$

for sufficiently big N. Since $t_0,t_1<\beta<\nu$, this is contradicts (2.1.1) for sufficiently big N.

2.8 Further research

It should be mentioned that the main results of this thesis, Theorem 2.1.1 and Corollary 2.1.6, should be considered only as first steps towards the full comprehension and a proof of Conjecture 2.1.3. Our work treats the case of a single point taken with high multiplicity, although Corollary 2.1.6 shows that we can deduce a result valid for a finitely generated subgroup of $\mathcal{G} = \mathbb{C} \times \mathbb{C}^{\times}$. A dual problem was considered by V.Nguyen in [VR14], where she considers polynomials taking small values at translates of a point $\gamma \in \mathcal{G}$ by multiples of a rational point $(r, s) \in \mathbb{Q} \times \mathbb{Q}^{\times}$. Using the theory developed in the first chapter of this thesis and adapting the techniques used in the second chapter, we are able to translate into a multihomogeneous setting also the arguments of [VR14] and to prove the following result.

Theorem 2.8.1. Let $\gamma = (\xi, \eta) \in \mathbb{C} \times \mathbb{C}^{\times}$ and $(r, s) \in \mathbb{Q} \times \mathbb{Q}^{\times}$ with $r \neq 0$ and $s \neq \pm 1$. Let $\beta, \sigma, \nu, t_0, t_1, t$ be positive real numbers satisfying

$$\max\{t_0, t_1\} = 1, \quad \min\{t_0, t_1\} = t, \quad 1 < \sigma < 1 + t,$$

$$\sigma + 1 < \beta, \quad \nu > \begin{cases} 1 + t + \beta - \sigma & \text{if } \sigma \geq 1 + \frac{t}{2} \\ 1 + t + \beta - \sigma + \delta & \text{if } \sigma < 1 + \frac{t}{2} \end{cases}$$

where $\delta = (\sigma - t)(2 + t - 2\sigma)/(\beta + 2 - 2\tau)$. Suppose that, for each sufficiently large positive integer N, there exists a nonzero polynomial $P_N \in \mathbb{Z}[x, y]$ with partial degrees $\deg_x(P_N) \leq \lfloor N^{t_0} \rfloor$, $\deg_y(P_N) \leq \lfloor N^{t_1} \rfloor$ and norm $\|P\| \leq \exp(N^{\beta})$, such that

$$\max_{0 \leq i < 4\lfloor N^{\sigma} \rfloor} |P_N(\xi + ir, \eta s^i)| \leq \exp(-N^{\nu}), \tag{2.8.1}$$

Then, we have $\xi, \eta \in \overline{\mathbb{Q}}$ and moreover for each sufficiently large integer N we have $P_N(\xi + ir, \eta s^i) = 0$ for every $0 \leq i < 4\lfloor N^{\sigma} \rfloor$.

The striking point of this statement is that in the range $1 + \frac{t}{2} \leq \sigma < 1 + t$ the estimate for ν proves to be best-possible. This can be showed with an easy application of Dirichlet's Box Principle.

Therefore, the first achievement one could hope to get is a result that deals with both multiplicities and translations. This should be done with appropriate generalizations of our lemmas. For example, it is possible to prove an interpolation result that bounds the length of a polynomial for which we know the values of sufficiently many invariant derivatives of it, at sufficiently many points of a finitely generated subgroup of $\mathcal{G}$.

However, if one compares the estimates one gets from these methods to the conjectural ones from Conjecture 2.1.3, one can see that the main obstruction is represented by the fact that, for each N, the number of conditions imposed on P_N need to be less than the dimension of the space of bihomogeneous polynomials of bi-degree D. This limitation is required by the interpolation result of Section § 2.3.2 and for an efficient application of the multiplicity estimate of Section § 2.5. Thus, the understanding of the phenomena that come out when the number of conditions slightly exceeds the dimension of the space of polynomials should be one of the future direction of research.

Moreover, one of the main differences between our theorem 2.1.1 and conjecture 2.1.3 is that the second is a question about high-transcendence degree, while the first is not.
To overcome this problem one My try to work within higher-dimensional linear algebraic groups and with higher-dimensional subvarieties, and to induct on the dimension. An example of this technique can be found in the proof of the Main Theorem of [Phi86a].
Another approach that looks tantalizing is to extend the results of sections 1.3 and 1.4 of this thesis to the case when K is not a number field, but a generic M-field with a product formula. An M-field is a field that comes together with a set of nontrivial places, parametrized by a measure space, so that a notion of height can be defined by integrating the local contributions. We refer to [Gub94][Gub98] for more on this topic. The main fact we need here is that, by a result of Moriwaki [Mor00], any field finitely generated over $\mathbb{Q}$ can be given a canonical structure of M-field with a product formula. One could hope that such a generalization would permit to work within $\mathbb{P}^1_K \times \mathbb{P}^1_K$ and to prove the desired statements by induction on the transcendence degree of K.
As for the lower bound for the multiplicity of resultant forms, it should be remarked that an adaptation of the arguments of section § 2.5 may show that Theorem 2.5.2 can be extended, with minor modifications, to arbitrary resultant forms of subvarieties of $\mathbb{P}^{\mathbf{n}}$. This should be one of the themes of a forthcoming article on the results of this thesis.
Finally, it is worth saying that there is a very active area of research that studies different concepts of heights in the field of Arithmetic Geometry. For this reason, it seems compelling to deepen the study of the heights introduced in the first chapter, and to compare them with analogous concepts coming, for example, from Arakelov Theory [Rém01b] or from the study of toric varieties [GPS14].

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