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Boolean valued models, saturation, forcing axioms

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Introduction

This dissertation will focus on Boolean-valued models, giving some insight into the theory of Boolean ultrapowers, and developing the connection with forcing axioms and absoluteness results. This study will be divided into three chapters.

The first chapter provides the basic material to understand the subsequent work.

Boolean-valued models are well known in set theory for independence results and the development of forcing (on this vast subject, see [2]). In the second chapter of this dissertation, Boolean-valued models are studied from a general point of view.

In Section 2.1, we give the main definition of Boolean-valued model for an arbitrary first-order signature L. Suppose $\mathbb{B}$ be a complete Boolean algebra; a $\mathbb{B}$-*valued model* $\mathfrak{M}$ for L assigns to each $L_{\kappa,\omega}$-formula φ a *Boolean value* $[\![\varphi]\!]^{\mathfrak{M}} \in \mathbb{B}$, generalizing the usual two-valued Tarski semantics.

Given any full $\mathbb{B}$-valued model $\mathfrak{M}$ for L and an ultrafilter U on $\mathbb{B}$, one can define the L-structure $\mathfrak{M}/U$ as the quotient of $\mathfrak{M}$ by the relation of U-equivalence: that is, $\tau, \sigma \in M$ are U-equivalent if and only if $[\![\tau = \sigma]\!] \in U$. In Section 2.2 we study how some combinatorial properties of the ultrafilter U are related to the realization of types in the structure $\mathfrak{M}/U$. A first result is that if U is countably incomplete, then $\mathfrak{M}/U$ is countably saturated. More sophisticatedly, we prove that if U is a κ-*good* ultrafilter then $\mathfrak{M}/U$ is κ-saturated.

In Section 2.3 we develop the theory of *Boolean ultrapowers*, a generalization of usual (powerset) ultrapowers. Mansfield [17] presented this construction as a purely algebraic technique, and we follow his ideas expanding a number of aspects.

The saturation results of Section 2.2 and the constructions of Section 2.3 are then used in section 2.4 to produce saturated elementary extensions of a given structure. This includes, in particular, the construction of a κ-good ultrafilter on the *Lévy collapsing algebra* $\mathrm{Coll}(\aleph_0, {<}\kappa)$.

Finally, in Section 2.5 we introduce the $\mathbb{B}$-valued model $V^{\mathbb{B}}$, a classic topic in set theory, in order to present an different approach to Boolean ultrapowers, due to Hamkins and Seabold [7].

A more ambitious third chapter develops the connection with forcing axioms and absoluteness results. Let Γ be a class of partially ordered sets and κ a cardinal number. The *forcing axiom* $\mathsf{FA}_\kappa(\Gamma)$ is the following sentence: for all $\langle P, \leq \rangle \in \Gamma$, if $\mathcal{D} \subseteq \mathcal{P}(P)$ is a family of dense subsets of P with $|\mathcal{D}| \leq \kappa$, then there exists a filter G on P such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$. We give in Section 3.1 a formulation of bounded forcing axioms in terms of absoluteness.

From a philosophical point of view, forcing axioms are very appealing. Not only do they imply that the Continuum Hypothesis is false, but also they are particularly successful in deciding many independent statements in mathematics. However, the role of forcing axioms in the foundations of mathematics is quite debatable. This explains why it might be interesting to express commonly accepted principles in terms of forcing axioms. In fact, in Section 3.2 we prove that the Axiom of Choice is a "global" forcing axiom. In a final section, having the same idea in mind, we show that also large cardinal axioms are in fact natural generalizations of forcing axioms.

Notation

ZF	the Zermelo-Fraenkel theory.
ZFC	ZF plus the Axiom of Choice.
ω	the set of natural numbers.
$\mathcal{P}(X)$	the set of all subsets of X.
$\mathcal{P}_\omega(X)$	the set of finite subsets of X.
$\mathrm{dom}(R)$	the domain of a binary relation R.
$\mathrm{ran}(R)$	the range of a binary relation R.
$f\colon X \to Y$	f is a function, $\mathrm{dom}(f) = X$, and $\mathrm{ran}(f) \subseteq Y$.
Y^X	the set of all functions $f\colon X \to Y$.
$X^{<\alpha}$	$\bigcup_{\beta<\alpha} X^\beta$.
$f \restriction A$	the restriction of a function f to a set A.
$f \circ g$	the composition of f and g.
$f[A]$	$\{f(a) : a \in A\}$.
$f^{-1}[B]$	$\{a : f(a) \in B\}$.
$\mathrm{cf}(\delta)$	the cofinality of a limit ordinal δ.
$\mathrm{trcl}(x)$	the transitive closure of a set x.
V_α	the α-th stage of the cumulative hierarchy of sets.
$\mathrm{rank}(x)$	the ordinal number defined by recursion as $\mathrm{rank}(x) = \sup\{\mathrm{rank}(y) + 1 : y \in x\}$.
H_κ	$\{x : \lvert\mathrm{trcl}(x)\rvert < \kappa\}$.

Chapter 1

Basic Material

1.1 Model Theory

The purpose of this section is to fix some notations and to clarify some preliminary ideas. This is by no means a complete introduction to the subject: we refer the reader to [9] for further details.

Signatures and Structures

Definition 1.1.1. A *signature* is a set of symbols divided into three categories:

- Relation symbols: $\{P, Q, R, \dots\}$;
- Function symbols: $\{f, g, h, \dots\}$;
- Constant symbols: $\{a, b, c, \dots\}$.

To each symbol it is assigned a natural number, named *arity*. The arity is 0 for all constant symbols; otherwise it is a positive integer.

Definition 1.1.2. The signature of set theory is $\{\in\}$, where $\in$ is a 2-ary relation symbol.

Definition 1.1.3. Let L be a signature. An *L-structure* $\mathfrak{M}$ consists of:

1. A non-empty set M, called the *domain* of $\mathfrak{M}$.
2. The interpretations of symbols in L. That is:
   - for each n-ary relation symbol $R \in L$, a relation $R^{\mathfrak{M}} \subseteq M^n$;
   - for each n-ary function symbol $f \in L$, a function $f^{\mathfrak{M}} \colon M^n \to M$;
   - for each constant symbol $c \in L$, an element $c^{\mathfrak{M}} \in M$.

We shall often use the following notation for structures: $\mathfrak{M} = \langle M, R^{\mathfrak{M}}, \dots, f^{\mathfrak{M}}, \dots, c^{\mathfrak{M}}, \dots \rangle$.

Definition 1.1.4. Fix two signatures $L \subset L'$. Given an L'-structure $\mathfrak{M}'$, its *restriction* to L is the L-structure obtained from $\mathfrak{M}'$ by restricting the interpretation to L and leaving the domain unaltered. Given an L-structure $\mathfrak{M}$, an *expansion* of $\mathfrak{M}$ to L' is any L'-structure whose restriction to L is $\mathfrak{M}$.

Definition 1.1.5. Let $\mathfrak{M}$ be an L-structure and $A \subseteq M$. We define $L(A) = L \cup \{c_a : a \in A\}$ the signature obtained from L by adding a constant symbol c_a for every $a \in A$. We may expand $\mathfrak{M}$ to $L(A)$ in a natural way: the interpretation of the symbol c_a is simply a. This expansion is denoted by $\mathfrak{M}_A$.

Homomorphisms and Substructures

Definition 1.1.6. Let $\mathfrak{M}$ and $\mathfrak{N}$ be L-structures. A *homomorphism* from $\mathfrak{M}$ to $\mathfrak{N}$ is a function $h\colon M \to N$ satisfying:

- For every n-ary relation symbol $R \in L$ and $\langle a_1, \ldots, a_n\rangle \in M^n$, if $\langle a_1, \ldots, a_n\rangle \in R^{\mathfrak{M}}$ then $\langle h(a_1), \ldots, h(a_n)\rangle \in R^{\mathfrak{N}}$.
- For every n-ary relation symbol $f \in L$ and $\langle a_1, \ldots, a_n\rangle \in M^n$, $h\big(f^{\mathfrak{M}}(a_1, \ldots, a_n)\big) = f^{\mathfrak{N}}(h(a_1), \ldots, h(a_n))$.
- For every constant symbol $c \in L$, $h\big(c^{\mathfrak{M}}\big) = c^{\mathfrak{N}}$.

Definition 1.1.7. Let $\mathfrak{M}$ and $\mathfrak{N}$ be L-structures. An *embedding* of $\mathfrak{M}$ into $\mathfrak{N}$ is an injective function $e\colon M \to N$ satisfying:

- For every n-ary relation symbol $R \in L$ and $\langle a_1, \ldots, a_n\rangle \in M^n$, $\langle a_1, \ldots, a_n\rangle \in R^{\mathfrak{M}}$ if and only if $\langle e(a_1), \ldots, e(a_n)\rangle \in R^{\mathfrak{N}}$.
- For every n-ary relation symbol $f \in L$ and $\langle a_1, \ldots, a_n\rangle \in M^n$, $e\big(f^{\mathfrak{M}}(a_1, \ldots, a_n)\big) = f^{\mathfrak{N}}(e(a_1), \ldots, e(a_n))$.
- For every constant symbol $c \in L$, $e\big(c^{\mathfrak{M}}\big) = c^{\mathfrak{N}}$.

An *isomorphism* is a surjective embedding. $\mathfrak{M}$ and $\mathfrak{N}$ are *isomorphic*, in symbols $\mathfrak{M} \cong \mathfrak{N}$, if there is an isomorphism from $\mathfrak{M}$ to $\mathfrak{N}$.

Definition 1.1.8. Let $\mathfrak{M}$ and $\mathfrak{N}$ be L-structures. $\mathfrak{M}$ is a *substructure* of $\mathfrak{N}$ if $M \subseteq N$ and the inclusion $M \to N$ is an embedding.

The Language $L_{\kappa,\omega}$

We now define, for a signature L and an infinite cardinal κ, the language $L_{\kappa,\omega}$. The first ingredient is a set $\mathit{Var} = \{x, y, z, \ldots\}$ of variables; we require that $|\mathit{Var}| = \kappa$.

Definition 1.1.9. The *terms* of L are defined as follows:

- Every variable is a term of L.
- Every constant symbol $c \in L$ is a term of L.
- If $t_1, \ldots, t_n$ are terms of L and $f \in L$ is an n-ary function symbol, then $f(t_1, \ldots, t_n)$ is a term of L.

Definition 1.1.10. The *atomic formulas* of L are defined as follows:

- If t_1 and t_2 are terms of L, then $(t_1 = t_2)$ is an atomic formula of L.
- If $t_1, \ldots, t_n$ are terms of L and $R \in L$ is an n-ary relation symbol, then $R(t_1, \ldots, t_n)$ is an atomic formula of L.

Definition 1.1.11. The *$L_{\kappa,\omega}$-formulas* are defined as follows:

- Every atomic formula of L is an $L_{\kappa,\omega}$-formula.
- If φ is an $L_{\kappa,\omega}$-formula, then $\neg\varphi$ is an $L_{\kappa,\omega}$-formula.

- If Φ is a set of $L_{\kappa,\omega}$-formulas with $|\Phi| < \kappa$, then $\bigwedge \Phi$ is an $L_{\kappa,\omega}$-formula.
- If φ is an $L_{\kappa,\omega}$-formula and x is a variable, then $(\exists x\varphi)$ is an $L_{\kappa,\omega}$-formula.

We shall use standard abbreviations, such as $\bigwedge_{i\in I} \varphi_i$ instead of $\bigwedge \{\varphi_i : i \in I\}$, et cetera.

Definition 1.1.12. Given an $L_{\kappa,\omega}$-formula φ, we define the *free variables* of φ:

- If φ is atomic, then $\mathrm{FV}(\varphi)$ is the set of variables which appear in φ.
- $\mathrm{FV}(\neg\varphi) = \mathrm{FV}(\varphi)$.
- $\mathrm{FV}(\bigwedge \Phi) = \bigcup \{\mathrm{FV}(\varphi) : \varphi \in \Phi\}$.
- $\mathrm{FV}(\exists x\varphi) = \mathrm{FV}(\varphi) \setminus \{x\}$.

We say that φ is an $L_{\kappa,\omega}$*-sentence* if $\mathrm{FV}(\varphi) = \emptyset$. A *theory* in $L_{\kappa,\omega}$ is a set of $L_{\kappa,\omega}$-sentences.

Given an L-structure $\mathfrak{M}$, an $L_{\kappa,\omega}$-formula φ and an assignment $\nu\colon \mathit{Var} \to M$, we now define the relation $\mathfrak{M} \models \varphi[\nu]$. The definition is quite similar to the usual Tarski semantics, but we give it in full detail because we plan to generalize it in section 2.1.

First, if t is a term of L, we define $t^{\mathfrak{M}}[\nu] \in M$:

1. if x is a variable, then $x^{\mathfrak{M}}[\nu] = \nu(x)$;
2. if $c \in L$ is a constant symbol, then $c^{\mathfrak{M}}[\nu] = c^{\mathfrak{M}}$;
3. if $t_1, \dots, t_n$ are terms of L and $f \in L$ is an n-ary function symbol, then $f(t_1, \dots, t_n)^{\mathfrak{M}}[\nu] = f^{\mathfrak{M}}\big(t_1^{\mathfrak{M}}[\nu], \dots, t_n^{\mathfrak{M}}[\nu]\big)$.

Next, for $x \in \mathit{Var}$ and $a \in M$, define the assignment $\nu_{a/x}$ as follows: $\nu_{a/x}(x) = a$ and $\nu_{a/x}(y) = \nu(y)$ for all $x \in \mathit{Var} \setminus \{x\}$. Now we are ready for the definition, where $t_1, \dots, t_n$ are terms of L:

Definition 1.1.13. With the above notations, we define $\mathfrak{M} \models \varphi[\nu]$ by recursion:

1. $\mathfrak{M} \models (t_1 = t_2)[\nu]$ if and only if $t_1^{\mathfrak{M}}[\nu] = t_2^{\mathfrak{M}}[\nu]$.
2. $\mathfrak{M} \models R(t_1, \dots, t_n)[\nu]$ if and only if $\big(t_1^{\mathfrak{M}}[\nu], \dots, t_n^{\mathfrak{M}}[\nu]\big) \in R^{\mathfrak{M}}$.
3. $\mathfrak{M} \models \neg\varphi[s]$ if and only if it is not the case that $\mathfrak{M} \models \varphi[\nu]$.
4. $\mathfrak{M} \models \bigwedge \Phi[\nu]$ if and only if $\mathfrak{M} \models \varphi[\nu]$ for every $\varphi \in \Phi$.
5. $\mathfrak{M} \models (\exists x\varphi)[\nu]$ if and only if there exists $a \in M$ such that $\mathfrak{M} \models \varphi[\nu_{a/x}]$.

Given a formula $\varphi(x_1, \dots, x_n)$, if $a_1 = \nu(x_1), \dots, a_n = \nu(x_n)$ then whether $\mathfrak{M} \models \varphi[\nu]$ or not depends only on $a_1, \dots, a_n$; in this case we can use the notation $\mathfrak{M} \models \varphi(a_1, \dots, a_n)$. In particular, if φ is a sentence, the assignment ν is irrelevant, thus we can write $\mathfrak{M} \models \varphi$ ("$\mathfrak{M}$ satisfies φ").

Definition 1.1.14. Let T be a theory in $L_{\kappa,\omega}$. A *model* of T is an L-structure $\mathfrak{M}$ such that $\mathfrak{M} \models \varphi$ for every $\varphi \in \mathsf{T}$.

A sentence φ is a *consequence* of T, in symbols $\mathsf{T} \vdash \varphi$, if every model of T satisfies φ.

The next theorem, known as "compactness theorem", is a fundamental property of the language $L_{\omega,\omega}$. For a proof, see [9, Chapter 6].

Theorem 1.1.15. *Let* T *be a theory in* $L_{\omega,\omega}$*. If every finite subset of* T *has a model, then* T *has a model.*

Definition 1.1.16. The $L_{\kappa,\omega}$-theory of an L-structure $\mathfrak{M}$, denoted by $\mathrm{Th}_{L_{\kappa,\omega}}(\mathfrak{M})$, is the set of $L_{\kappa,\omega}$-sentences φ such that $\mathfrak{M} \models \varphi$. Two structures $\mathfrak{M}$ and $\mathfrak{N}$ are *elementarily equivalent*, in symbols $\mathfrak{M} \equiv \mathfrak{N}$, if $\mathrm{Th}_{L_{\omega,\omega}}(\mathfrak{M}) = \mathrm{Th}_{L_{\omega,\omega}}(\mathfrak{N})$.

Elementary Embeddings

Definition 1.1.17. Let $\mathfrak{M}$ and $\mathfrak{N}$ be L-structures. An *elementary embedding* from $\mathfrak{M}$ to $\mathfrak{N}$ is a function $j\colon M \to N$ such that for every $L_{\omega,\omega}$-formula $\varphi(x_1, \dots, x_n)$ and $\langle a_1, \dots, a_n\rangle \in M^n$,

$$\mathfrak{M} \models \varphi(a_1, \dots, a_n) \iff \mathfrak{N} \models \varphi(j(a_1), \dots, j(a_n)).$$

The reader can easily verify that every isomorphism is an elementary embedding, and that every elementary embedding is indeed an embedding in the sense of Definition 1.1.7.

Definition 1.1.18. Let $\mathfrak{M}$ and $\mathfrak{N}$ be L-structures. $\mathfrak{M}$ is an *elementary substructure* of $\mathfrak{N}$, in symbols $\mathfrak{M} \preceq \mathfrak{N}$, if $\mathfrak{M}$ is a substructure of $\mathfrak{N}$ and the inclusion $M \to N$ is an elementary embedding.

Here is another classic result in model theory (see [9, Chapter 3]):

Theorem 1.1.19 (Downward Löwenheim-Skolem)**.** *Let* $\mathfrak{M}$ *be an* L*-structure,* $A \subseteq M$*, and* κ *a cardinal number satisfying* $\aleph_0 + |L| + |A| \leq \kappa \leq |M|$*. Then there exists an elementary substructure* $\mathfrak{N} \preceq \mathfrak{M}$ *such that* $|N| = \kappa$ *and* $A \subseteq N$*.*

Elementary embeddings play a crucial role in set theory; we take this opportunity to present here a basic result.

Definition 1.1.20. Let $L = \{\in\}$ be the signature of set theory. An $L_{\omega,\omega}$-formula $\varphi(x_1, \dots, x_n)$ is *absolute* for $\langle M, \in\rangle$ if for every $a_1, \dots, a_n$,

$$\langle V, \in\rangle \models \varphi(a_1, \dots, a_n) \iff \langle M, \in\rangle \models \varphi(a_1, \dots, a_n).$$

Remark 1.1.21. The formulas "x is an ordinal" and "$\mathrm{rank}(x) = y$" are absolute for transitive models of ZF. This is a standard fact, and a proof can be found in [14, Chapter II].

Proposition 1.1.22. *Let* $j\colon V \to M$ *be an elementary embedding of the universe* V *into a transitive class* M*. Then:*

1. *For every ordinal number* α*,* $j(\alpha)$ *is an ordinal and* $\alpha \leq j(\alpha)$*.*
2. *If* j *is not the identity, there is an ordinal* δ *such that* $\delta < j(\delta)$*.*

Proof. First, note that $\langle M, \in\rangle$ is a model of ZFC, because j is an elementary embedding.

If α is an ordinal, then $\langle M, \in\rangle \models$ "$j(\alpha)$ is an ordinal" and, by Remark 1.1.21, we conclude that $j(\alpha)$ is an ordinal. Let α be the least ordinal such that $j(\alpha) < \alpha$. Then $j(j(\alpha)) < j(\alpha)$, contradicting the minimality of α. Hence $\alpha \leq j(\alpha)$ for all α.

For the second part, define

$$\delta = \min\{\mathrm{rank}(x) : j(x) \neq x\},$$

and take any x such that $\mathrm{rank}(x) = \delta$ and $j(x) \neq x$. We have $x \subset j(x)$, because $y \in x$ implies $y = j(y) \in j(x)$. Let $z \in j(x) \setminus x$. If $\mathrm{rank}(j(x)) \leq \delta$, then we would have $j(z) = z \in j(x)$, which implies $z \in x$, a contradiction. Therefore, it must be the case that $\delta < \operatorname{rank}(j(x))$. From this fact, using Remark 1.1.21 again, we conclude that

$$\delta < \operatorname{rank}(j(x)) = j(\operatorname{rank}(x)) = j(\delta). \qquad \square$$

Thanks to Proposition 1.1.22, we can define the *critical point* of a nontrivial elementary embedding $j \colon V \to M$ as the least ordinal δ such that $\delta < j(\delta)$. The critical point of j is denoted by $\operatorname{crit}(j)$.

Types and Saturation

Definition 1.1.23. Let $\mathfrak{M}$ be an L-structure and $B \subseteq M$. Suppose that $\Sigma(x_1, \ldots, x_n)$ is a set of $L(B)_{\kappa,\omega}$-formulas. We say that a n-tuple $\langle a_1, \ldots, a_n \rangle \in M^n$ *realizes* $\Sigma(x_1, \ldots, x_n)$ in $\mathfrak{M}$ if $\mathfrak{M}_B \models \varphi(a_1, \ldots, a_n)$ for all $\varphi(x_1, \ldots, x_n) \in \Sigma(x_1, \ldots, x_n)$.

If $\Sigma(x_1, \ldots, x_n)$ is not realized by any n-tuple in $\mathfrak{M}$, we say that $\mathfrak{M}$ *omits* $\Sigma(x_1, \ldots, x_n)$.

Definition 1.1.24. Let T be a theory in $L_{\kappa,\omega}$. An *n-type* of T is any set $\Sigma(x_1, \ldots, x_n)$ of $L_{\kappa,\omega}$-formulas which is realized in some model of T.

An n-type $p(x_1, \ldots, x_n)$ is *complete* if for every formula $\varphi(x_1, \ldots, x_n)$ either $\varphi \in p(x_1, \ldots, x_n)$ or $\neg\varphi \in p(x_1, \ldots, x_n)$.

Remark 1.1.25. An n-type is complete if and only if it is maximal. In particular, every n-type can be extended to a complete n-type.

In most cases, T will be the $L_{\kappa,\omega}$-theory of a structure, possibly with parameters. Instead of "$\Sigma(x_1, \ldots, x_n)$ is an n-type of $\operatorname{Th}_{L_{\kappa,\omega}}(\mathfrak{M}_B)$" we shall say that "$\Sigma(x_1, \ldots, x_n)$ is an n-type over B in $L_{\kappa,\omega}$".

Proposition 1.1.26. *Let $\mathfrak{M}$ be an L-structure, $B \subseteq M$ and let $\Sigma(x_1, \ldots, x_n)$ be an n-type over B in $L_{\kappa,\omega}$. Then every subset $\Phi \subseteq \Sigma(x_1, \ldots, x_n)$ with $|\Phi| < \kappa$ is realized in $\mathfrak{M}$.*

Proof. Since $\Sigma(x_1, \ldots, x_n)$ is an n-type over B in $L_{\kappa,\omega}$, there is a model $\mathfrak{N}$ of $\operatorname{Th}_{L_{\kappa,\omega}}(\mathfrak{M}_B)$ such that $\Sigma(x_1, \ldots, x_n)$ is realized in $\mathfrak{N}$. In particular, Φ is realized in $\mathfrak{N}$. This means that $\mathfrak{N} \models \exists x_1 \ldots \exists x_n \bigwedge \Phi$, hence $\mathfrak{M}_B \models \exists x_1 \ldots \exists x_n \bigwedge \Phi$, which means that Φ is realized in $\mathfrak{M}$. $\square$

Definition 1.1.27. Let $\mathfrak{M}$ be an L-structure and λ a cardinal number. We say that $\mathfrak{M}$ is λ-*saturated* if, for every $A \subseteq M$ with $|A| < \lambda$, all complete 1-types over A in $L_{\omega,\omega}$ are realized in $\mathfrak{M}$. A structure $\mathfrak{M}$ is *saturated* if it is $|M|$-saturated.

Remark 1.1.28. If an infinite structure $\mathfrak{M}$ is κ-saturated, then $\kappa \leq |M|$.

We assume the reader is familiar with the following two theorems. Proofs can be found, for example, in [9, Chapter 10].

Theorem 1.1.29. *Let $\mathfrak{M}$ be an L-structure and λ a cardinal number satisfying $\aleph_0 + |L| \leq \lambda$. There exists a λ^+-saturated elementary extension $\mathfrak{N} \succeq \mathfrak{M}$ such that $|N| \leq |M|^\lambda$.*

Theorem 1.1.30. *Suppose that $\mathfrak{M}$ and $\mathfrak{N}$ are two elementarily equivalent saturated structures. If $|M| = |N|$, then $\mathfrak{M} \cong \mathfrak{N}$.*

1.2 Partially Ordered Sets

Definition 1.2.1. A *partially ordered set* $\langle P, \leq \rangle$ is a set P together with a transitive, reflexive and antisymmetric binary relation $\leq$.

For any partially ordered set, the notation $p < q$ stands for $p \leq q$ and $p \neq q$.

Definition 1.2.2. Let $\langle P, \leq \rangle$ be a partially ordered set, $X \subseteq P$ and $a \in P$.

1. a is an *upper bound* of X if $x \leq a$ for all $x \in X$.
2. a is a *lower bound* of X if $a \leq x$ for all $x \in X$.
3. a is the *greatest* element of X if $a \in X$ and a is an upper bound of X.
4. a is the *least* element of X if $a \in X$ and a is a lower bound of X.
5. a is the *supremum* of X if a is the least upper bound of X, denoted by $\sup(X)$
6. a is the *infimum* of X if a is the greatest lower bound of X, denoted by $\inf(X)$.

Definition 1.2.3. Two elements $p, q \in P$ are *compatible* if there is $r \in P$ such that $r \leq p$ and $r \leq q$ (otherwise, p and q are *incompatible*). A subset $A \subseteq P$ is an *antichain* if every $p, q \in A$ are incompatible. A subset $C \subseteq P$ is a *chain* if for every $p, q \in C$ we have $p \leq q$ or $q \leq p$.

Definition 1.2.4. Let κ be a cardinal number. P satisfies the *$<\kappa$-chain condition* if every antichain in P has cardinality $< \kappa$. P is *$<\kappa$-closed* if every chain $C \subseteq P$ with $|C| < \kappa$ has a lower bound.

Definition 1.2.5. A subset $O \subseteq P$ is *open* if $p \in O$, $q \in P$ and $q \leq p$ implies $q \in O$. A subset $D \subseteq P$ is *dense* if for every $p \in P$ there exists $d \in D$ such that $d \leq p$.

Definition 1.2.6. Let $\langle P, \leq \rangle$ be a partially ordered set. A *filter* on $\langle P, \leq \rangle$ is a subset $F \subseteq P$ such that:

- F is non-empty.
- If $p \in F$ and $q \in F$, then there is $r \in F$ such that $r \leq p$ and $r \leq q$.
- If $p \in F$, $q \in P$ and $p \leq q$, then $q \in F$.

Definition 1.2.7 (Martin, Solovay and many others)**.** Let Γ be a class of partially ordered sets and κ a cardinal number. The *forcing axiom* $\mathsf{FA}_\kappa(\Gamma)$ is the following sentence: for all $\langle P, \leq \rangle \in \Gamma$, if $\{D_\alpha : \alpha < \kappa\} \subseteq \mathcal{P}(P)$ is a family of dense subsets of P, then there exists a filter G on P such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$.

Lemma 1.2.8. *Let $\langle P, \leq \rangle$ be a partially ordered set and $p \in P$. If $\{D_n : n < \omega\} \subseteq \mathcal{P}(P)$ is a family of dense subsets of P, then there exists a filter G on P such that $p \in G$ and $G \cap D_n \neq \emptyset$ for all $n < \omega$. In particular, $\mathsf{FA}_{\aleph_0}(\Gamma)$ is true for every class Γ of partially ordered sets.*

Proof. We construct by recursion a sequence $\langle d_n : n < \omega \rangle$ in P. Let $d_0 = p$. Suppose d_n is already constructed; by density of D_n there is $d_{n+1} \in D_n$ such that $d_{n+1} \leq d_n$. Thus we have built a sequence $\langle d_n : n < \omega \rangle$ satisfying $d_n \in D_{n+1}$ and $d_{n+1} \leq d_n$, for all $n < \omega$.

Then
$$G = \{p \in P : \text{there exists } n < \omega \text{ such that } d_n \leq p\}$$
is a filter on P, such that $p \in G$ and $G \cap D_n \neq \emptyset$ for all $n < \omega$. □

1.3 Boolean Algebras

Definition 1.3.1. Let $L = \{\vee, \wedge, \neg, \mathbb{0}, \mathbb{1}\}$, where $\vee$, $\wedge$ are 2-ary fuction symbols, $\neg$ is a 1-ary function symbol and $\mathbb{0}, \mathbb{1}$ are constant symbols. A *Boolean algebra* is an L-structure that satisfies:

$$\begin{aligned} \forall x\forall y\forall z((x \vee y) \vee z = x \vee (y \vee z)), &\qquad \forall x\forall y\forall z((x \wedge y) \wedge z = x \wedge (y \wedge z)), \\ \forall x\forall y(x \vee y = y \vee x), &\qquad \forall x\forall y(x \wedge y = y \wedge x), \\ \forall x\forall y(x \vee (x \wedge y) = x), &\qquad \forall x\forall y(x \wedge (x \vee y) = x), \\ \forall x\forall y\forall z((x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)), &\qquad \forall x\forall y\forall z((x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)), \\ \forall x(x \vee (\neg x) = \mathbb{1}), &\qquad \forall x(x \wedge (\neg x) = \mathbb{0}). \end{aligned}$$

Example 1.3.2. Let X be any set. The structure

$$\langle \mathcal{P}(X), \cup, \cap, \neg, \emptyset, X \rangle,$$

where $\neg x = X \setminus x$, is a Boolean algebra.

Definition 1.3.3. A subset D of a Boolean algebra $\mathbb{B}$ is a *meet-semilattice* if $\mathbb{1} \in D$, $\mathbb{0} \notin D$, and $x, y \in D$ implies that $x \wedge y \in D$.

If $\mathbb{B}$ is any Boolean algebra, we define $a \leq b \overset{\text{def}}{\iff} a \wedge b = a$. Standard poset terminology applies to $\mathbb{B} \setminus \{\mathbb{0}\}$. For example, we may speak of antichains or dense subsets. Analogously, a filter on a Boolean algebra $\mathbb{B}$ is simply a filter on the partially ordered set $\mathbb{B} \setminus \{\mathbb{0}\}$. More explicitly:

Definition 1.3.4. Let $\mathbb{B}$ be a Boolean algebra. A *filter* on $\mathbb{B}$ is a subset $F \subset \mathbb{B}$ such that:

- $\mathbb{1} \in F$ and $\mathbb{0} \notin F$.
- If $a \in F$ and $b \in F$, then $a \wedge b \in F$.
- If $a \in F$, $b \in B$ and $a \leq b$, then $b \in F$.

A filter F on $\mathbb{B}$ is *principal* if $F = \{b \in \mathbb{B} : a \leq b\}$ for some $a \in \mathbb{B}$. An *ultrafilter* is a filter U that satisfies the following property: for all $b \in B$, either $b \in U$ or $\neg b \in U$.

Definition 1.3.5. Let $\mathbb{B}$ be a Boolean algebra. An *ideal* on $\mathbb{B}$ is a subset $I \subset \mathbb{B}$ such that:

- $\mathbb{0} \in I$ and $\mathbb{1} \notin I$.
- If $a \in I$ and $b \in I$, then $a \vee b \in I$.
- If $a \in I$, $b \in B$ and $b \leq a$, then $b \in I$.

Definition 1.3.6. Let $\mathbb{B}$ be a Boolean algebra. A subset $D \subset \mathbb{B}$ has the *finite intersection property* if for all $a_1, \ldots, a_n \in D$ we have $a_1 \wedge \cdots \wedge a_n > \mathbb{0}$.

Remark 1.3.7. Every $D \subset \mathbb{B}$ with the finite intersection property generates a filter F on $\mathbb{B}$: take

$$F = \{b \in \mathbb{B} : \text{there exist } d_1, \ldots, d_n \in D \text{ such that } d_1, \ldots, d_n \leq b\}.$$

Moreover, every filter on $\mathbb{B}$ can be extended to an ultrafilter on $\mathbb{B}$. This result, essentially due to Tarski, is well known; for a proof see [10, Chapter 7]. Of course, similar results apply to ideals.

Le $\mathbb{B}$ be a Boolean algebra and $I \subset \mathbb{B}$ an ideal. Consider this equivalence relation $\sim$ on $\mathbb{B}$:

$$a \sim b \overset{\text{def}}{\iff} (a \vee b) \wedge \neg(a \wedge b) \in I.$$

We define the quotient Boolean algebra $\mathbb{B}/I$ as the set of equivalence classes

$$[a]_\sim = \{b \in \mathbb{B} : b \sim a\}$$

equipped with the natural quotient of the operations on $\mathbb{B}$:

$$\begin{aligned} [a]_\sim \vee [b]_\sim &= [a \vee b]_\sim\,, \\ [a]_\sim \wedge [b]_\sim &= [a \wedge b]_\sim\,, \\ \neg[a]_\sim &= [\neg a]_\sim\,, \\ \mathbb{0} &= [\mathbb{0}]_\sim\,, \\ \mathbb{1} &= [\mathbb{1}]_\sim\,. \end{aligned}$$

Define $\mathrm{St}(\mathbb{B}) = \{U \subset \mathbb{B} : U \text{ is an ultrafilter on } \mathbb{B}\}$. By Stone's representation theorem (see [20]), $\mathbb{B}$ is isomorphic to a subalgebra of $\mathcal{P}(\mathrm{St}(\mathbb{B}))$ via the map

$$b \longmapsto \{U \in \mathrm{St}(\mathbb{B}) : b \in U\}\,.$$

Complete Boolean Algebras

Let $\mathbb{B}$ be a Boolean algebra and $X \subseteq \mathbb{B}$. We define $\bigvee X = \sup(X)$ and $\bigwedge X = \inf(X)$, whenever they actually exist.

Definition 1.3.8. A Boolean algebra $\mathbb{B}$ is *complete* if $\bigvee X$ and $\bigwedge X$ exist for all $X \subseteq \mathbb{B}$.

Theorem 1.3.9. *For every partially ordered set $\langle P, \leq\rangle$, there exist a complete Boolean algebra $\mathbb{B}$ and a function $e\colon P \to \mathbb{B} \setminus \{\mathbb{0}\}$ such that:*

1. *If $p \leq q$ then $e(p) \leq e(q)$.*

2. *p and q are incompatible in P if and only if $e(p) \wedge e(q) = \mathbb{0}$*

3. *$e[P]$ is dense in $\mathbb{B} \setminus \{\mathbb{0}\}$.*

Moreover, $\mathbb{B}$ is uniquely determined up to isomorphism, and is called $\mathrm{RO}(P)$*, the* regular open algebra *of P.*

Proof. This is a well known result: the reader can find a proof in [10, Corollary 14.12]. □

Here is an useful upper bound on the cardinality of $\mathrm{RO}(P)$.

Proposition 1.3.10. *Let $\langle P, \leq\rangle$ be a partially ordered set. If $\langle P, \leq\rangle$ satisfies the $<\kappa$-chain condition, then $|\mathrm{RO}(P)| \leq |P|^{<\kappa}$*

Proof. Let $\mathrm{A}(P) = \{A \subseteq P : A \text{ is an antichain}\}$. Note that, by hypothesis, $|\mathrm{A}(P)| \leq |P|^{<\kappa}$. To conclude the proof, it suffices to show that the map

$$\begin{aligned} \mathrm{A}(P) &\longrightarrow \mathrm{RO}(P) \setminus \{\mathbb{0}\} \\ A &\longmapsto \bigvee e[A] \end{aligned}$$

is surjective. Let $b \in \mathrm{RO}(P) \setminus \{\mathbb{0}\}$. Define

$$D = e[P] \cap \{a \in \mathrm{RO}(P) \setminus \{\mathbb{0}\} : a \leq b\}.$$

Since D is dense below b, by Zorn lemma we can construct a maximal antichain W in D. Note that W must satisfy $\bigvee W = b$, hence $A = e^{-1}[W] \in \mathrm{A}(P)$ has the property that $\bigvee e[A] = b$, concluding the proof. □

Definition 1.3.11. Let $\mathbb{B}$ be a complete Boolean algebra. A filter $F \subset \mathbb{B}$ is *κ-complete* if $X \subseteq F$ and $|X| < \kappa$ implies that $\bigwedge X \in F$. Similarly, an ideal $I \subset \mathbb{B}$ is *κ-complete* if $X \subseteq I$ and $|X| < \kappa$ implies that $\bigvee X \in I$.

Chapter 2

Boolean-Valued Models

2.1 Boolean-Valued Models

Definition 2.1.1. Let L be a signature and $\mathbb{B}$ a complete Boolean algebra. A $\mathbb{B}$-*valued model* $\mathfrak{M}$ for the signature L consists of:

1. A non-empty set M. The elements of M are called *names.*
2. The Boolean value of the equality symbol. That is, a function

$$\begin{aligned} M^2 &\longrightarrow \mathbb{B} \\ \langle \tau, \sigma \rangle &\longmapsto [\![\tau = \sigma]\!]^{\mathfrak{M}} \end{aligned} .$$

3. The interpretation of symbols in L. That is:
   - for each n-ary relation symbol $R \in L$, a function

$$\begin{aligned} M^n &\longrightarrow \mathbb{B} \\ \langle \tau_1, \dots, \tau_n \rangle &\longmapsto [\![R(\tau_1, \dots, \tau_n)]\!]^{\mathfrak{M}} \end{aligned} ;$$

   - for each n-ary function symbol $f \in L$, a function

$$\begin{aligned} M^{n+1} &\longrightarrow \mathbb{B} \\ \langle \tau_1, \dots, \tau_n, \sigma \rangle &\longmapsto [\![f(\tau_1, \dots, \tau_n) = \sigma]\!]^{\mathfrak{M}} \end{aligned} ; \tag{2.1}$$

   - for each constant symbol $c \in L$, a name $c^{\mathfrak{M}} \in M$.

We require that the following conditions hold:

1. For all $\tau, \sigma, \pi \in M$,

$$[\![\tau = \tau]\!]^{\mathfrak{M}} = \mathbb{1}, \tag{2.2}$$

$$[\![\tau = \sigma]\!]^{\mathfrak{M}} = [\![\sigma = \tau]\!]^{\mathfrak{M}}, \tag{2.3}$$

$$[\![\tau = \sigma]\!]^{\mathfrak{M}} \wedge [\![\sigma = \pi]\!]^{\mathfrak{M}} \leq [\![\tau = \pi]\!]^{\mathfrak{M}}. \tag{2.4}$$

2. If $R \in L$ is an n-ary function symbol, for all $\langle \tau_1, \dots, \tau_n \rangle, \langle \sigma_1, \dots, \sigma_n \rangle \in M^n$,

$$\left(\bigwedge_{i=1}^{n} \llbracket \tau_i = \sigma_i \rrbracket^{\mathfrak{M}} \right) \wedge \llbracket R(\tau_1, \dots, \tau_n) \rrbracket^{\mathfrak{M}} \leq \llbracket R(\sigma_1, \dots, \sigma_n) \rrbracket^{\mathfrak{M}} . \tag{2.5}$$

3. If $f \in L$ is an n-ary function symbol, for all $\sigma, \pi \in M$ and $\langle \tau_1, \dots, \tau_n \rangle, \langle \sigma_1, \dots, \sigma_n \rangle \in M^n$,

$$\left(\bigwedge_{i=1}^{n} \llbracket \tau_i = \sigma_i \rrbracket^{\mathfrak{M}} \right) \wedge \llbracket f(\tau_1, \dots, \tau_n) = \pi \rrbracket^{\mathfrak{M}} \leq \llbracket f(\sigma_1, \dots, \sigma_n) = \pi \rrbracket^{\mathfrak{M}} , \tag{2.6}$$

$$\bigvee_{\sigma \in M} \llbracket f(\tau_1, \dots, \tau_n) = \sigma \rrbracket^{\mathfrak{M}} = \mathbb{1}, \tag{2.7}$$

$$\llbracket f(\tau_1, \dots, \tau_n) = \sigma \rrbracket^{\mathfrak{M}} \wedge \llbracket f(\tau_1, \dots, \tau_n) = \pi \rrbracket^{\mathfrak{M}} \leq \llbracket \sigma = \pi \rrbracket^{\mathfrak{M}} . \tag{2.8}$$

This concludes the definition of $\mathbb{B}$-valued models.

Remark 2.1.2. Let $\mathbb{2} = \{\mathbb{0}, \mathbb{1}\}$. Then $\mathbb{2}$ is obviously a complete Boolean algebra, and every $\mathbb{2}$-valued model for a signature L is simply an L-structure.

Given a $\mathbb{B}$-valued model $\mathfrak{M}$ for L, an assignment $\nu\colon Var \to M$ and an $L_{\kappa,\omega}$-formula φ, we can define the *Boolean value* $\llbracket \varphi[\nu] \rrbracket^{\mathfrak{M}} \in \mathbb{B}$.

First, we need to generalize (2.1). Specifically, if t is any term of L and $\sigma \in M$, we define by recursion $\llbracket (t = \sigma)[\nu] \rrbracket^{\mathfrak{M}}$:

1. if x is a variable, then $\llbracket (x = \sigma)[\nu] \rrbracket^{\mathfrak{M}} = \llbracket \nu(x) = \sigma \rrbracket^{\mathfrak{M}}$;
2. if $c \in L$ is a constant symbol, then $\llbracket (c = \sigma)[\nu] \rrbracket^{\mathfrak{M}} = \left\llbracket c^{\mathfrak{M}} = \sigma \right\rrbracket^{\mathfrak{M}}$.
3. If $t_1, \dots, t_n$ are terms of L and $f \in L$ is an n-ary function symbol, then

$$\llbracket (f(t_1, \dots, t_n) = \sigma)[\nu] \rrbracket^{\mathfrak{M}} = \bigvee_{\tau_1, \dots, \tau_n \in M} \left(\bigwedge_{i=1}^{n} \llbracket (t_i = \tau_i)[\nu] \rrbracket^{\mathfrak{M}} \right) \wedge \llbracket f(\tau_1, \dots, \tau_n) = \sigma \rrbracket^{\mathfrak{M}} .$$

Now we are ready for the definition, where $t_1, \dots, t_n$ are terms of L:

Definition 2.1.3. With the above notations, we define $\llbracket \varphi[\nu] \rrbracket^{\mathfrak{M}}$ by recursion:

1. $\llbracket (t_1 = t_2)[\nu] \rrbracket^{\mathfrak{M}} = \bigvee_{\tau \in M} \llbracket (t_1 = \tau)[\nu] \rrbracket^{\mathfrak{M}} \wedge \llbracket (t_2 = \tau)[\nu] \rrbracket^{\mathfrak{M}}$.
2. If $R \in L$ is an n-ary relation symbol, then

$$\llbracket R(t_1, \dots, t_n)[\nu] \rrbracket^{\mathfrak{M}} = \bigvee_{\tau_1, \dots, \tau_n \in M} \left(\bigwedge_{i=1}^{n} \llbracket (t_i = \tau_i)[\nu] \rrbracket^{\mathfrak{M}} \right) \wedge \llbracket R(\tau_1, \dots, \tau_n) \rrbracket^{\mathfrak{M}} .$$

3. $\llbracket \neg\varphi[\nu] \rrbracket^{\mathfrak{M}} = \neg \llbracket \varphi[\nu] \rrbracket^{\mathfrak{M}}$.
4. $\llbracket \bigwedge \Phi[\nu] \rrbracket^{\mathfrak{M}} = \bigwedge_{\varphi \in \Phi} \llbracket \varphi[\nu] \rrbracket^{\mathfrak{M}}$.
5. $\llbracket (\exists x \varphi)[\nu] \rrbracket^{\mathfrak{M}} = \bigvee_{\tau \in M} \left\llbracket \varphi[\nu_{\tau/x}] \right\rrbracket^{\mathfrak{M}}$.

For $L_{\kappa,\omega}$-sentences φ the assignment ν is irrelevant, thus we can write $[\![\varphi]\!]^{\mathfrak{M}}$ instead of $[\![\varphi[\nu]]\!]^{\mathfrak{M}}$. Finally, we say that a sentence φ is *valid* in $\mathfrak{M}$ if and only if $[\![\varphi]\!]^{\mathfrak{M}} = \mathbb{1}$.

From now on we may often drop the superscript $\mathfrak{M}$ and denote Boolean values simply by $[\![\varphi]\!]$.

Remark 2.1.4. For every $L_{\kappa,\omega}$ formula $\varphi(x)$ (possibly with parameters) and $\tau, \sigma \in M$, it is true that

$$[\![\tau = \sigma]\!] \wedge [\![\varphi(\tau)]\!] \leq [\![\varphi(\sigma)]\!]\,;$$

this can be proved by induction on the complexity of φ.

Full Boolean-Valued Models

Definition 2.1.5. Let $\mathfrak{M}$ be a $\mathbb{B}$-valued model for L. We say that $\mathfrak{M}$ is *full* when it satisfies the following property: if $A \subset \mathbb{B}$ is an antichain and $\{\tau_a : a \in A\} \subseteq M$, then there is $\tau \in M$ such that $a \leq [\![\tau = \tau_a]\!]$ for all $a \in A$.

The next proposition states a fundamental property of full Boolean-valued models.

Proposition 2.1.6. *Let $\mathfrak{M}$ be a full $\mathbb{B}$-valued model for L and κ an infinite cardinal. For every $L_{\kappa,\omega}$-formula $\varphi(x, y_1, \ldots, y_n)$ and $\langle \sigma_1, \ldots, \sigma_n \rangle \in M^n$ there exists $\tau \in M$ such that*

$$[\![\exists x \varphi(x, \sigma_1, \ldots, \sigma_n)]\!] = [\![\varphi(\tau, \sigma_1, \ldots, \sigma_n)]\!]\,.$$

Proof. Define

$$D = \{a \in \mathbb{B} : \text{there exists } \tau_a \in M \text{ such that } a \leq [\![\varphi(\tau_a, \sigma_1, \ldots, \sigma_n)]\!]\}\,.$$

It is easy to show that D is open and dense below $[\![\exists x \varphi(x, \sigma_1, \ldots, \sigma_n)]\!]$. By Zorn lemma, we can construct a maximal antichain A in D. Note that A must satisfy $[\![\exists x \varphi(x, \sigma_1, \ldots, \sigma_n)]\!] \leq \bigvee A$. By hypothesis, there exists $\tau \in M$ such that $a \leq [\![\tau = \tau_a]\!]$ for all $a \in A$. By Remark 2.1.4 we have $a \leq [\![\varphi(\tau, \sigma_1, \ldots, \sigma_n)]\!]$ for all $a \in A$, hence

$$[\![\exists x \varphi(x, \sigma_1, \ldots, \sigma_n)]\!] \leq [\![\varphi(\tau, \sigma_1, \ldots, \sigma_n)]\!]\,.$$

Since the inequality $\geq$ is trivial, this concludes the proof. □

Remark 2.1.7. Let $\mathfrak{M}$ be a full $\mathbb{B}$-valued model for L. For every function symbol $f \in L$ and $\langle \tau_1, \ldots, \tau_n \rangle \in M^n$, there exists $\sigma \in M$ such that $[\![f(\tau_1, \ldots, \tau_n) = \sigma]\!] = \mathbb{1}$. In fact, it suffices to choose any σ such that $[\![\exists x (f(\tau_1, \ldots, \tau_n) = x)]\!] = [\![f(\tau_1, \ldots, \tau_n) = \sigma]\!]$. Moreover, if σ_1 and σ_2 are two possible choices, then $[\![\sigma_1 = \sigma_2]\!] = \mathbb{1}$ thanks to (2.8). This remark will be useful in the next definition.

Definition 2.1.8. Let $\mathbb{B}$ be a complete Boolean algebra, $F \subset \mathbb{B}$ a filter and $\mathfrak{M}$ a full $\mathbb{B}$-valued model for L. The *quotient* of $\mathfrak{M}$ by F is the L-structure $\mathfrak{M}/F$ defined as follows:

1. Its domain, called M/F, is the quotient of M by the equivalence relation $\equiv_F$ defined as

$$\tau \equiv_F \sigma \overset{\text{def}}{\iff} [\![\tau = \sigma]\!] \in F.$$

2. These are the interpretations of symbols in L:

   - If $R \in L$ is an n-ary relation symbol, then

$$R^{\mathfrak{M}/F} = \{\langle [\tau_1]_F, \ldots, [\tau_n]_F \rangle \in (M/F)^n : [\![R(\tau_1, \ldots, t_n)]\!] \in F\}\,.$$

- If $f \in L$ is an n-ary function symbol, then

$$\begin{aligned} f^{\mathfrak{M}/F} \colon (M/F)^n &\longrightarrow M/F \\ \langle [\tau_1]_F, \ldots, [\tau_n]_F \rangle &\longmapsto [\sigma]_F \end{aligned},$$

  where σ is any element of M such that $[\![f(\tau_1, \ldots, \tau_n) = \sigma]\!] = \mathbb{1}$.
- If $c \in L$ is a constant symbol, then $c^{\mathfrak{M}/F} = \left[c^{\mathfrak{M}}\right]_F \in M/F$.

It is an easy exercise to show, using the fact that F is a filter, that the relation $\equiv_F$ is indeed an equivalence relation, and that the interpretations of symbols are well defined.

Remark 2.1.9. When M is a proper class, the equivalence class $[\tau]_F$ may be a proper class. This makes Definition 2.1.8 problematic. However, we can get round this problem by defining $[\tau]_F$ as the set of all $\sigma \in M$ *of minimal rank* such that $\tau \equiv_F \sigma$.

Theorem 2.1.10 (Łoś)**.** *Let κ be an infinite cardinal, $\mathfrak{M}$ a full $\mathbb{B}$-valued model for L and $U \subset \mathbb{B}$ a κ-complete ultrafilter. Then, for every $L_{\kappa,\omega}$-formula $\varphi(x_1, \ldots, x_n)$ and $\langle \tau_1, \ldots, \tau_n \rangle \in M^n$ we have*

$$\mathfrak{M}/U \models \varphi([\tau_1]_U, \ldots, [\tau_n]_U) \iff [\![\varphi(\tau_1, \ldots, \tau_n)]\!] \in U. \tag{2.9}$$

Proof. We say that φ is an unnested atomic formula if it is of the form

$$\begin{gathered} (x = y), \\ (c = y), \\ f(x_1, \ldots, x_n) = y, \\ R(x_1, \ldots, x_n). \end{gathered}$$

φ is an unnested formula if all its atomic subformulas are unnested.

It suffices to prove the theorem by induction on the complexity of unnested formulas φ, for the case for φ any atomic formula can be reduced to the case of unnested formulas by removing compositions between function symbols: for example, $f(g(x)) = y$ is logically equivalent to $\exists z(g(x) = z \wedge f(z) = y)$.

So, if φ is an unnested atomic formula, then (2.9) holds by definition of $\mathfrak{M}/U$. Suppose (2.9) holds for $\varphi(x_1, \ldots, x_n)$. Then, using the fact that U is an ultrafilter, we have

$$\begin{aligned} \mathfrak{M}/U \models \neg\varphi([\tau_1]_U, \ldots, [\tau_n]_U) &\iff \text{it is not the case that } \mathfrak{M}/U \models \varphi([\tau_1]_U, \ldots, [\tau_n]_U) \\ &\iff [\![\varphi(\tau_1, \ldots, \tau_n)]\!] \notin U \\ &\iff [\![\neg\varphi(\tau_1, \ldots, \tau_n)]\!] \in U. \end{aligned}$$

Now suppose (2.9) holds for every $\varphi(x_1, \ldots, x_n) \in \Phi$. Then, by κ-completeness of U, we have

$$\begin{aligned} \mathfrak{M}/U \models \bigwedge_{\varphi\in\Phi} \varphi([\tau_1]_U, \ldots, [\tau_n]_U) &\iff \mathfrak{M}/U \models \varphi([\tau_1]_U, \ldots, [\tau_n]_U) \text{ for every } \varphi \in \Phi \\ &\iff [\![\varphi(\tau_1, \ldots, \tau_n)]\!] \in U \text{ for every } \varphi \in \Phi \\ &\iff \left[\!\!\left[\bigwedge_{\varphi\in\Phi} \varphi(\tau_1, \ldots, \tau_n) \right]\!\!\right] \in U. \end{aligned}$$

Finally, suppose (2.9) holds for $\varphi(x_1, \ldots, x_n, y)$. Then, by Proposition 2.1.6, we have

$$\begin{aligned} \mathfrak{M}/U \models \exists y \varphi([\tau_1]_U, \ldots, [\tau_n]_U, y) &\iff \mathfrak{M}/U \models \varphi([\tau_1]_U, \ldots, [\tau_n]_U, [\sigma]_U) \text{ for some } \sigma \in M \\ &\iff [\![\varphi(\tau_1, \ldots, \tau_n, \sigma)]\!] \in U \text{ for some } \sigma \in M \\ &\iff [\![\exists y \varphi(\tau_1, \ldots, \tau_n, y)]\!] \in U. \end{aligned}$$

□

2.2 Realization of Types

The purpose of this section is to show how some combinatorial properties of U are related to the realization of types in $\mathfrak{M}/U$.

Theorem 2.2.1. *Let κ be an infinite cardinal, $\mathfrak{M}$ a full $\mathbb{B}$-valued model for L and $U \subset \mathbb{B}$ an ultrafilter that is κ-complete and κ^+ incomplete. Then every 1-type $\Sigma(x)$ over M/U in $L_{\kappa,\omega}$ such that $|\Sigma(x)| \leq \kappa$ is realized in $\mathfrak{M}/U$.*

Proof. Fix an enumeration $\Sigma(x) = \{\varphi_\alpha(x) : \alpha < \kappa\}$. First, note that for all $\alpha < \kappa$ we have

$$\left[\!\left[\exists x \bigwedge_{\beta \leq \alpha} \varphi_\beta(x) \right]\!\right] \in U. \tag{2.10}$$

Indeed, by Proposition 1.1.26 the set $\{\varphi_\beta(x) : \beta \leq \alpha\}$ is realized in $\mathfrak{M}/U$, which means that $\mathfrak{M}/U \models \exists x \bigwedge_{\beta \leq \alpha} \varphi_\beta(x)$. We can now apply Theorem 2.1.10 to get (2.10). By κ^+-incompleteness of U, there exists $\{a_\alpha : \alpha < \kappa\} \subseteq U$ such that $\bigwedge_{\alpha<\kappa} a_\alpha \notin U$. For $\alpha < \kappa$, define

$$b_\alpha = \bigwedge_{\beta \leq \alpha} a_\beta \wedge \left[\!\left[\exists x \bigwedge_{\beta \leq \alpha} \varphi_\beta(x) \right]\!\right],$$

and notice that $\{b_\alpha : \alpha < \kappa\} \subseteq U$ (by κ-completeness of U and (2.10)) but $\bigwedge_{\alpha<\kappa} b_\alpha \notin U$. Moreover, $\{b_\alpha \wedge \neg b_{\alpha+1} : \alpha < \kappa\}$ is clearly an antichain. By Proposition 2.1.6, we can find $\{\sigma_\alpha : \alpha < \kappa\} \subseteq M$ such that for all $\alpha < \kappa$

$$\left[\!\left[\exists x \bigwedge_{\beta \leq \alpha} \varphi_\beta(x) \right]\!\right] = \left[\!\left[\bigwedge_{\beta \leq \alpha} \varphi_\beta(\sigma_\alpha) \right]\!\right],$$

and by fullness of $\mathfrak{M}$ there exists $\tau \in M$ such that for all $\alpha < \kappa$

$$b_\alpha \wedge \neg b_{\alpha+1} \leq [\![\tau = \sigma_\alpha]\!].$$

To conclude the proof, we fix $\alpha < \kappa$ and show that $[\![\varphi_\alpha(\tau)]\!] \in U$. For every $\beta \geq \alpha$ we have

$$[\![\varphi_\alpha(\tau)]\!] \geq \left[\!\left[\bigwedge_{\gamma \leq \beta} \varphi_\gamma(\tau) \right]\!\right] \geq [\![\tau = \sigma_\beta]\!] \wedge \left[\!\left[\bigwedge_{\gamma \leq \beta} \varphi_\gamma(\sigma_\beta) \right]\!\right] \geq b_\beta \wedge \neg b_{\beta+1},$$

and therefore

$$[\![\varphi_\alpha(\tau)]\!] \geq \bigvee_{\beta \geq \alpha} (b_\beta \wedge \neg b_{\beta+1}) = b_\alpha \wedge \neg \bigwedge_{\beta<\kappa} b_\beta \in U.$$ □

As a special instance of the previous theorem, we obtain the following result:

Corollary 2.2.2. *Let $\mathfrak{M}$ be a full $\mathbb{B}$-valued model for L and $U \subset \mathbb{B}$ an $\aleph_1$-incomplete ultrafilter. Suppose that $|L| \leq \aleph_0$. Then $\mathfrak{M}/U$ is $\aleph_1$-saturated.*

At this point, the case $\kappa > \aleph_1$ is left open. What could be an adequate condition on U for the quotient $\mathfrak{M}/U$ to be κ-saturated? The next subsection will provide an answer to this question.

Good Ultrafilters

If X is any set, let $\mathcal{P}_\omega(X)$ denote the set of finite subsets of X.

Definition 2.2.3. Let $\mathbb{B}$ be a Boolean algebra and $f\colon \mathcal{P}_\omega(X) \to \mathbb{B}$.

- f is *multiplicative* if for all $S, T \in \mathcal{P}_\omega(X)$, $f(S \cup T) = f(S) \wedge f(T)$.
- f is *monotonically decreasing* if for all $S, T \in \mathcal{P}_\omega(X)$, $S \subseteq T$ implies $f(T) \leq f(S)$.

Definition 2.2.4. Let $\mathbb{B}$ be a Boolean algebra, $U \subset \mathbb{B}$ an ultrafilter and κ a cardinal number. U is *κ-good* if for every $\lambda < \kappa$ and for every monotonically decreasing function $f\colon \mathcal{P}_\omega(\lambda) \to U$, there exists a multiplicative function $g\colon \mathcal{P}_\omega(\lambda) \to U$ such that $g(S) \leq f(S)$ for all $S \in \mathcal{P}_\omega(\lambda)$.

The following theorem is a generalization of [4, Theorem 6.1.8]; see also [17, Theorem 4.1].

Theorem 2.2.5. *Let $\mathfrak{M}$ be a full $\mathbb{B}$-valued model for L and $U \subset \mathbb{B}$ an $\aleph_1$-incomplete ultrafilter. Let κ a cardinal such that $\aleph_0 + |L| < \kappa$. If U is κ-good, then $\mathfrak{M}/U$ is κ-saturated.*

Proof. Let $A \subseteq M/U$ be any subset such that $|A| < \kappa$ and let $p(x)$ be a complete 1-type over A in $L_{\omega,\omega}$. Define $\lambda = |p(x)|$, observe that $\lambda < \kappa$, and fix an enumeration $p(x) = \{\varphi_\alpha(x) : \alpha < \lambda\}$.

First, note that for all $S \in \mathcal{P}_\omega(\lambda)$ we have

$$\left[\!\!\left[\exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x) \right]\!\!\right] \in U.$$

Indeed, by Proposition 1.1.26 the set $\{\varphi_\alpha(x) : \alpha \in S\}$ is realized in $\mathfrak{M}/U$, which means that $\mathfrak{M}/U \models \exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x)$. We can now apply Theorem 2.1.10 to get (2.10). By $\aleph_1$-incompleteness of U, there exists $\{a_n : n < \omega\} \subseteq U$ such that $\bigwedge_{n<\omega} a_n \notin U$. Define a monotonically decreasing function $f\colon \mathcal{P}_\omega(\lambda) \to U$ as follows: for every $S \in \mathcal{P}_\omega(\lambda)$

$$f(S) = \bigwedge_{n \leq |S|} a_n \wedge \left[\!\!\left[\exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x) \right]\!\!\right].$$

Using the fact that U is κ-good, we can find a multiplicative function $g\colon \mathcal{P}_\omega(\lambda) \to U$ such that $g(S) \leq f(S)$ for all $S \in \mathcal{P}_\omega(\lambda)$. Define another function $h\colon \mathcal{P}_\omega(\lambda) \to \mathbb{B}$ as follows: for every $S \in \mathcal{P}_\omega(\lambda)$

$$h(S) = g(S) \wedge \bigwedge \{\neg g(T) : |T| > |S|\}.$$

We prove two claims about the function h.

Claim 1. $\operatorname{ran}(h) \setminus \{\mathbb{0}\}$ *is an antichain.*

Proof of Claim 1. The first thing to prove is that for all $S, T \in \mathcal{P}_\omega(\lambda)$

$$g(S) \wedge h(T) > \mathbb{0} \implies S \subseteq T. \tag{2.11}$$

Suppose not; then $|T| < |S \cup T|$ and, using the fact that g is multiplicative,

$$g(S) \wedge h(T) \leq g(S) \wedge g(T) \wedge \neg g(S \cup T) = g(S) \wedge g(T) \wedge \neg(g(S) \wedge g(T)) = \mathbb{0},$$

a contradiction. To conclude, just observe that $h(S) \wedge h(T) > \mathbb{0}$ implies both $g(S) \wedge h(T) > \mathbb{0}$ and $g(T) \wedge h(S) > 0$, which together imply $S = T$. □

Claim 2. $\bigvee \{h(T) : T \supseteq S\} \in U$.

Proof of Claim 2. Let

$$b = g(S) \wedge \bigwedge \{\neg h(T) : T \supseteq S\}.$$

Observe that

$$b \vee \bigvee \{h(T) : T \supseteq S\} = \Big(g(S) \wedge \bigwedge \{\neg h(T) : T \supseteq S\}\Big) \vee \bigvee \{h(T) : T \supseteq S\} =$$
$$= \Big(g(S) \vee \bigvee \{h(T) : T \supseteq S\}\Big) \wedge \Big(\bigwedge \{\neg h(T) : T \supseteq S\} \vee \bigvee \{h(T) : T \supseteq S\}\Big) \geq$$
$$\geq g(S) \wedge \mathbb{1} = g(S) \in U.$$

Thus it suffices to show that $b \notin U$. Assume, by contradiction, that $b \in U$. Define, for $n < \omega$,

$$c_n = \bigvee \{g(T) : |T| = n\}.$$

It is easy to verify that $c_{n+1} \leq c_n$ for all $n < \omega$ and that $b \leq c_{|S|}$. Moreover, there is $i < \omega$ such that $b \wedge c_i \wedge \neg c_{i+1} > \mathbb{0}$ (otherwise, we would have $b \leq \bigwedge_{n<\omega} c_n \leq \bigwedge_{n<\omega} a_n \notin U$, hence $b \notin U$). Therefore, by definition of c_i, there exists $I \in \mathcal{P}_\omega(\lambda)$ such that $|I| = i$ and

$$\mathbb{0} < b \wedge g(I) \wedge \neg c_{i+1},$$

but $g(I) \wedge \neg c_{i+1} = h(I)$ and so we have $\mathbb{0} < b \wedge h(I)$. Finally, by (2.11) we have $S \subseteq I$, thus

$$\mathbb{0} < b \wedge h(I) = g(S) \wedge \bigwedge \{\neg h(T) : T \supseteq S\} \wedge h(I) \leq \neg h(I) \wedge h(I) = \mathbb{0}$$

a contradiction. □

By Proposition 2.1.6, we can find $\{\sigma_S : S \in \mathcal{P}_\omega(\lambda)\} \subseteq M$ such that for all $S \in \mathcal{P}_\omega(\lambda)$

$$\left[\!\left[\exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x) \right]\!\right] = \left[\!\left[\bigwedge_{\alpha \in S} \varphi_\alpha(\sigma_S) \right]\!\right],$$

and by fullness of $\mathfrak{M}$ there exists $\tau \in M$ such that for all $S \in \mathcal{P}_\omega(\lambda)$

$$h(S) \leq [\![\tau = \sigma_S]\!].$$

Now fix $S \in \mathcal{P}_\omega(\lambda)$. For every $T \supseteq S$ we have

$$\left[\!\left[\bigwedge_{\alpha \in S} \varphi_\alpha(\tau) \right]\!\right] \geq \left[\!\left[\bigwedge_{\alpha \in T} \varphi_\alpha(\tau) \right]\!\right] \geq [\![\tau = \sigma_T]\!] \wedge \left[\!\left[\bigwedge_{\alpha \in T} \varphi_\alpha(\sigma_T) \right]\!\right] \geq h(T). \tag{2.12}$$

Then, by (2.12) and Claim 2, we have

$$\left[\!\left[\bigwedge_{\alpha \in S} \varphi_\alpha(\tau) \right]\!\right] \geq \bigvee \{h(T) : T \supseteq S\} \in U;$$

this means that $[\tau]_U$ realizes $p(x)$ in $\mathfrak{M}/U$. □

2.3 Boolean Powers

In this section we take an L-structure $\mathfrak{M}$, a complete Boolean algebra $\mathbb{B}$ and we construct a $\mathbb{B}$-valued model $\mathfrak{M}^{\downarrow\mathbb{B}}$.

First, we need some notation. Let A and W be maximal antichains of $\mathbb{B}$. We say that W is a *refinement* of A if for every $w \in W$ there is $a \in A$ such that $w \leq a$. This element $a \in A$ is unique. Note that maximal antichains $A_1, \dots, A_n$ always admit a common refinement W.

Let X be any set, A a maximal antichain of $\mathbb{B}$ and $f\colon A \to X$. If W is a refinement of A, the *reduction* of f to W is the function

$$\begin{aligned} (f \downarrow W)\colon W &\longrightarrow X \\ w &\longmapsto f(a) \end{aligned},$$

where a is the element of A such that $w \leq a$.

Definition 2.3.1. Let $\mathfrak{M}$ be an L-structure and $\mathbb{B}$ a complete Boolean algebra. The $\mathbb{B}$-*power* of $\mathfrak{M}$ is the $\mathbb{B}$-valued model $\mathfrak{M}^{\downarrow\mathbb{B}}$ defined as follows:

1. Its domain is the set $M^{\downarrow\mathbb{B}} = \{\tau\colon A \to M : A \subseteq \mathbb{B} \text{ is a maximal antichain}\}$.

2. Let $\tau, \sigma \in M^{\downarrow\mathbb{B}}$. Choose a common refinement W of $\mathrm{dom}(\tau)$ and $\mathrm{dom}(\sigma)$, and define

$$[\![\tau = \sigma]\!] = \bigvee \{w \in W : (\tau \downarrow W)(w) = (\sigma \downarrow W)(w)\}.$$

3. Now define the interpretations of symbols in L:

   - If $R \in L$ is an n-ary function symbol and $\tau_1, \dots, \tau_n \in M^{\downarrow\mathbb{B}}$, choose a common refinement W of $\mathrm{dom}(\tau_1), \dots, \mathrm{dom}(\tau_n)$, and define

$$[\![R(\tau_1, \dots, \tau_n)]\!] = \bigvee \{w \in W : \mathfrak{M} \models R((\tau_1 \downarrow W)(w), \dots, (\tau_n \downarrow W)(w))\}; \qquad (2.13)$$

   - If $f \in L$ is an n-ary function symbol and $\tau_1, \dots, \tau_n, \sigma \in M^{\downarrow\mathbb{B}}$, choose a common refinement W of $\mathrm{dom}(\tau_1), \dots, \mathrm{dom}(\tau_n), \mathrm{dom}(\sigma)$, and define

$$[\![f(\tau_1, \dots, \tau_n) = \sigma]\!] = \bigvee \{w \in W : \mathfrak{M} \models f((\tau_1 \downarrow W)(w), \dots, (\tau_n \downarrow W)(w)) = (\sigma \downarrow W)(w)\};$$

   - if $c \in L$ is a constant symbol, its interpretation is the function

$$\begin{aligned} c^{\mathfrak{M}^{\downarrow\mathbb{B}}}\colon \{\mathbb{1}\} &\longrightarrow M \\ \mathbb{1} &\longmapsto c^{\mathfrak{M}} \end{aligned}.$$

Is is easy to prove that, in the previous definition, the Boolean value of the equality symbol and the interpretation of symbols in L are well defined and do not depend on the choice of common refinements.

Lemma 2.3.2. *$\mathfrak{M}^{\downarrow\mathbb{B}}$ satisfies conditions* (2.2) *to* (2.8). *That is, $\mathfrak{M}^{\downarrow\mathbb{B}}$ is a $\mathbb{B}$-valued model.*

Proof. In each point of this proof, to avoid cumbersome notation, we shall implicitly choose a suitable common refinement W and we shall assume that names $\tau \in M^{\downarrow\mathbb{B}}$ have already been reduced to W.

That said, (2.2) holds because

$$[\![\tau = \tau]\!] = \bigvee \{w \in W : \tau(w) = \tau(w)\} = \bigvee W = \mathbb{1}.$$

It is obvious that (2.3) holds, because the right-hand side of (2.13) is symmetric in τ and σ. We now prove (2.4):

$$[\![\tau = \sigma]\!] \wedge [\![\sigma = \pi]\!] = \bigvee \{w_1 \in W : \tau(w_1) = \sigma(w_1)\} \wedge \bigvee \{w_2 \in W : \sigma(w_2) = \pi(w_2)\} =$$
$$= \bigvee \{w_1 \wedge w_2 : \tau(w_1) = \sigma(w_1),\ \sigma(w_2) = \pi(w_2)\} = \bigvee \{w \in W : \tau(w) = \sigma(w),\ \sigma(w) = \pi(w)\} \leq$$
$$\leq \bigvee \{w \in W : \tau(w) = \pi(w)\} = [\![\tau = \pi]\!] .$$

The same idea can be used to prove (2.5), (2.6) and (2.8).

As for (2.7), we prove the stronger assertion that, given $\tau_1, \dots, \tau_n \in M^{\downarrow\mathbb{B}}$, there exists $\sigma \in M^{\downarrow\mathbb{B}}$ such that $[\![f(\tau_1, \dots, \tau_n) = \sigma]\!] = \mathbb{1}$. Indeed, it suffices to define $\sigma(w) = f^{\mathfrak{M}}(\tau_1(w), \dots, \tau_n(w))$ for every $w \in W$. □

According to Definition 2.1.3, $\mathfrak{M}^{\downarrow\mathbb{B}}$ assigns a Boolean value $[\![\varphi]\!]$ to every formula φ. The next proposition clarifies this point.

Proposition 2.3.3. *Let $\mathfrak{M}^{\downarrow\mathbb{B}}$ be the $\mathbb{B}$-power of an L-structure $\mathfrak{M}$. Let $\varphi(x_1, \dots, x_n)$ be an $L_{\omega,\omega}$-formula and $\tau_1, \dots, \tau_n \in M^{\downarrow\mathbb{B}}$. If W is any common refinement of* $\mathrm{dom}(\tau_1), \dots, \mathrm{dom}(\tau_n)$, *then*

$$[\![\varphi(\tau_1, \dots, \tau_n)]\!] = \bigvee \{w \in W : \mathfrak{M} \models \varphi((\tau_1 \downarrow W)(w), \dots, (\tau_n \downarrow W)(w))\} . \tag{2.14}$$

Proof. We carry out the proof by induction on unnested atomic formulas φ. We shall assume that $\tau_1, \dots, \tau_n$ have already been reduced to W.

For unnested atomic formulas, (2.14) is simply the definition. Suppose (2.14) holds for $\varphi(x_1, \dots, x_n)$. Then, keeping in mind that W is a maximal antichain,

$$[\![\neg\varphi(\tau_1, \dots, \tau_n)]\!] = \neg\, [\![\varphi(\tau_1, \dots, \tau_n)]\!] = \neg \bigvee \{w \in W : \mathfrak{M} \models \varphi(\tau_1(w), \dots, \tau_n(w))\} =$$
$$= \bigvee \{w \in W : \mathfrak{M} \models \neg\varphi(\tau_1(w), \dots, \tau_n(w))\}$$

Suppose that (2.14) holds for $\varphi(x_1, \dots, x_n)$ and $\psi(x_1, \dots, x_n)$. Then

$$[\![\varphi(\tau_1, \dots, \tau_n) \wedge \psi(\tau_1, \dots, \tau_n)]\!] = [\![\varphi(\tau_1, \dots, \tau_n)]\!] \wedge [\![\psi(\tau_1, \dots, \tau_n)]\!] =$$
$$= \bigvee \{w_1 \in W : \mathfrak{M} \models \varphi(\tau_1(w_1), \dots, \tau_n(w_1))\} \wedge \bigvee \{w_2 \in W : \mathfrak{M} \models \psi(\tau_1(w_2), \dots, \tau_n(w_2))\} =$$
$$= \bigvee \{w_1 \wedge w_2 : \mathfrak{M} \models \varphi(\tau_1(w_1), \dots, \tau_n(w_1)),\ \mathfrak{M} \models \psi(\tau_1(w_2), \dots, \tau_n(w_2))\} =$$
$$= \bigvee \{w \in W : \mathfrak{M} \models (\varphi(\tau_1(w), \dots, \tau_n(w)) \wedge \psi(\tau_1(w), \dots, \tau_n(w)))\} .$$

Finally, suppose (2.14) holds for $\varphi(x_1,\dots,x_n,y)$. Then

$$\begin{aligned}
[\![\exists y\varphi(\tau_1,\dots,\tau_n,y)]\!] &= \bigvee_{\sigma\in M^{\downarrow\mathbb{B}}} [\![\varphi(\tau_1,\dots,\tau_n,\sigma)]\!] = \\
&= \bigvee_{\sigma\in M^{\downarrow\mathbb{B}}} \bigvee \{w\in W : \mathfrak{M}\models\varphi(\tau_1(w),\dots,\tau_n(w),\sigma(w))\} = \\
&= \bigvee \bigcup_{\sigma\in M^{\downarrow\mathbb{B}}} \{w\in W : \mathfrak{M}\models\varphi(\tau_1(w),\dots,\tau_n(w),\sigma(w))\} = \\
&= \bigvee \left\{w\in W : \text{there exists } \sigma\in M^{\downarrow\mathbb{B}} \text{ such that } \mathfrak{M}\models\varphi(\tau_1(w),\dots,\tau_n(w),\sigma(w))\right\} = \\
&= \bigvee \{w\in W : \mathfrak{M}\models\exists y\varphi(\tau_1(w),\dots,\tau_n(w),y)\}. \quad \square
\end{aligned}$$

Theorem 2.3.4. *The* $\mathbb{B}$*-valued model* $\mathfrak{M}^{\downarrow\mathbb{B}}$ *is full.*

Proof. Let $A\subset\mathbb{B}$ be an antichain and $\{\tau_a : a\in A\}\subseteq M^{\downarrow\mathbb{B}}$. Without loss of generality, we may assume that A is maximal. For every $a\in A$ define

$$D_a = \{b\wedge a : b\in\operatorname{dom}(\tau_a)\},$$

and note that $D_{a_1}\cap D_{a_2}=\emptyset$ whenever $a_1\neq a_2$. We define $\tau\in M^{\downarrow\mathbb{B}}$ as follows: first, its domain is $\operatorname{dom}(\tau)=\bigcup_{a\in A}D_a$. If $d\in D_a$, then $\tau(d)=\tau_a(b)$, where b is the unique element of $\operatorname{dom}(\tau_a)$ such that $d\leq b$.

Let $a\in A$; we have to prove that $a\leq[\![\tau=\tau_a]\!]$. For simplicity of notation, we assume that $\operatorname{dom}(\tau)$ is a refinement of $\operatorname{dom}(\tau_a)$ and that τ_a has already been reduced. We have

$$\begin{aligned}
a\wedge[\![\tau=\tau_a]\!] = a\wedge\bigvee\{d\in\operatorname{dom}(\tau):\tau(d)=\tau_a(d)\} = \bigvee\{a\wedge d : d\in\operatorname{dom}(\tau),\ \tau(d)=\tau_a(d)\} \geq \\
\geq \bigvee\{a\wedge d : d\in D_a,\ \tau(d)=\tau_a(d)\} = \bigvee\{a\wedge d : d\in D_a\} = a,
\end{aligned}$$

which completes the proof. $\square$

Since $\mathfrak{M}^{\downarrow\mathbb{B}}$ is full, it is legitimate to consider its quotients by ultrafilters. This is what we do in the next subsection.

Boolean Ultrapowers

Definition 2.3.5. Let $\mathfrak{M}$ be an L-structure, $\mathbb{B}$ a complete Boolean algebra and $U\subset\mathbb{B}$ an ultrafilter. The $\mathbb{B}$-*ultrapower* of $\mathfrak{M}$ by U is the quotient $\mathfrak{M}^{\downarrow\mathbb{B}}/U$.

What is important here is that we are able to elementarily embed $\mathfrak{M}$ into $\mathfrak{M}^{\downarrow\mathbb{B}}/U$. For every $x\in M$, define $c_x\in M^{\downarrow\mathbb{B}}$ as the function

$$\begin{aligned} c_x\colon \{\mathbb{1}\} &\longrightarrow M \\ \mathbb{1} &\longmapsto x \end{aligned}.$$

Theorem 2.3.6. *The map* $j\colon x\mapsto[c_x]_U$ *is an elementary embedding of* $\mathfrak{M}$ *into* $\mathfrak{M}^{\downarrow\mathbb{B}}/U$*.*

Proof. Let $\varphi(x_1,\dots,x_n)$ be an $L_{\omega,\omega}$-formula and $\langle a_1,\dots,a_n\rangle\in M^n$. Using Theorem 2.1.10 and Proposition 2.3.3, we have

$$\begin{aligned}
\mathfrak{M}^{\downarrow\mathbb{B}}/U\models\varphi(j(a_1),\dots,j(a_n)) &\iff [\![\varphi(c_{a_1},\dots,c_{a_n})]\!]\in U \\
&\iff \bigvee\{w\in\{\mathbb{1}\} : \mathfrak{M}\models\varphi(c_{a_1}(w),\dots,c_{a_n}(w))\}\in U \\
&\iff \mathfrak{M}\models\varphi(a_1,\dots,a_n). \qquad \square
\end{aligned}$$

Consider the special case in which $\mathbb{B}$ is a power set algebra (Example 1.3.2). If $\mathfrak{M}$ is an L-structure and $U \subset \mathbb{B}$ is an ultrafilter, the corrisponding Boolean ultrapower is denoted by $\mathrm{Ult}(\mathfrak{M}, U)$ and will be referred to as the *ultrapower* of $\mathfrak{M}$ by U.

Canjar [3] has shown that not every Boolean ultrapower is isomorphic to a power set ultrapower.

2.4 Construction of Saturated Structures

The Lévy Collapse

Definition 2.4.1. Let $\kappa > \aleph_0$ be a cardinal number. We say that κ is *inaccessible* if it is regular and for all $\lambda < \kappa$ we have $2^\lambda < \kappa$.

For the rest of this section, fix an inaccessible cardinal κ. For every $\alpha < \kappa$, define

$$P_\alpha = \{p \subset \aleph_0 \times \alpha : p \text{ is a finite partial function}\};$$

let $p \leq q$ if and only if $q \subseteq p$.

Remark 2.4.2. P_α has the following property: for all $p \in P_\alpha$, we can find $q \leq p$ and $r \leq p$ such that q and r are incompatible.

If $\langle p_\alpha : \alpha < \kappa\rangle \in \prod_{\alpha<\kappa} P_\alpha$, define $\mathrm{supp}(\langle p_\alpha : \alpha < \kappa\rangle) = \{\alpha < \kappa : p_\alpha \neq \emptyset\}$. Now define

$$P = \left\{ \langle p_\alpha : \alpha < \kappa\rangle \in \prod_{\alpha<\kappa} P_\alpha : \mathrm{supp}(\langle p_\alpha : \alpha < \kappa\rangle) \text{ is finite} \right\}; \tag{2.15}$$

let $\langle p_\alpha : \alpha < \kappa\rangle \leq \langle q_\alpha : \alpha < \kappa\rangle$ if and only if $p_\alpha \leq q_\alpha$ for all $\alpha < \kappa$.

Lemma 2.4.3. *In the above notations, let $C \subset P$ any subset such that*

1. *If $p, q \in C$, then p and q are compatible.*
2. *If $p \in C$, $q \in P$ and $p \leq q$, then $q \in C$.*

Then, for every infinite cardinal $\lambda < \kappa$, there is a maximal antichain $A \subset P$ such that $|A| = \lambda$ and $A \cap C = \emptyset$.

Proof. Consider the projection

$$C_\lambda = \{p_\lambda : \langle p_\alpha : \alpha < \kappa\rangle \in C\} \subset P_\lambda.$$

It obvious that C_λ satisfies conditions (1) and (2). We claim that $P_\lambda \setminus C_\lambda$ is a dense subset of P_λ. Let $p \in P_\lambda$; by Remark 2.4.2, there are $q \leq p$ and $r \leq p$ such that q and r are incompatible, hence at least one of them does not belong to C_λ.

It is now easy to construct a maximal antichain $A_\lambda \subseteq P_\lambda \setminus C_\lambda$ such that $|A_\lambda| = \lambda$: take $p \in P_\lambda \setminus C_\lambda$, take $n \in \aleph_0 \setminus \mathrm{dom}(p)$ and extend the antichain $\{p \cup \{\langle n, \alpha\rangle\} : \alpha < \lambda\}$ to a maximal one.

Now, it suffices to define

$$A = \{\langle p_\alpha : \alpha < \kappa\rangle : p_\lambda \in A_\lambda \text{ and } p_\beta = \emptyset \text{ if } \beta \neq \lambda\}$$

and note that A is a maximal antichain with the desired properties. □

By Theorem 1.3.9, there exist a unique complete Boolean algebra $\mathrm{Coll}(\aleph_0, {<}\kappa)$ (usually called *Lévy collapse*), and a function $e\colon P \to \mathrm{Coll}(\aleph_0, {<}\kappa) \setminus \{\mathbb{0}\}$ such that:

1. If $p \leq q$ then $e(p) \leq e(q)$.
2. p and q are incompatible in P if and only if $e(p) \wedge e(q) = \mathbb{0}$
3. $e[P]$ is dense in $\mathrm{Coll}(\aleph_0, {<}\kappa) \setminus \{\mathbb{0}\}$.

Filters on $\mathrm{Coll}(\aleph_0, {<}\kappa)$ are very far from being complete: this is made precise by the next proposition. We have already seen that incompleteness properties of U result in saturation properties of the quotient $\mathfrak{M}/U$. In fact, we will soon be able to construct saturated structures of the form $\mathfrak{M}^{\downarrow \mathrm{Coll}(\aleph_0, <\kappa)}/U$.

Proposition 2.4.4. *Let $F \subset \mathrm{Coll}(\aleph_0, {<}\kappa)$ be any filter. For every infinite cardinal $\lambda < \kappa$, there exists a maximal antichain $A \subset \mathrm{Coll}(\aleph_0, {<}\kappa)$ such that $|A| = \lambda$ and $A \cap F = \emptyset$.*

Proof. Note that $e^{-1}[F] \subset P$ satisfies conditions (1) and (2) of Lemma 2.4.3. Therefore, we can find a maximal antichain $W \subset P$ such that $|W| = \lambda$ and $W \cap e^{-1}[F] = \emptyset$. Then $A = e[W]$ has the desired properties. □

Theorem 2.4.5. $\mathrm{Coll}(\aleph_0, {<}\kappa)$ *satisfies the ${<}\kappa$-chain condition.*

Proof. Clearly, it suffices to show that the partially ordered set P defined in (2.15) satisfies the ${<}\kappa$-chain condition.

Let $W \subseteq P$ be an antichain. We construct by recursion a sequence $\langle A_n : n < \omega \rangle$ such that $A_n \subseteq A_{n+1} \subseteq \kappa$ for all $n < \omega$, and another sequence $\langle W_n : n < \omega \rangle$ such that $W_n \subseteq W_{n+1} \subseteq W$ for all $n < \omega$. Let $A_0 = W_0 = \emptyset$. Suppose A_n and W_n are constructed. For every $p \in P$ such that $\mathrm{supp}(p) \subseteq A_n$, choose $q_p \in W$ such that $q_p \restriction A_n = p$, whenever it exists.[1] Then define

$$W_{n+1} = W_n \cup \{q_p : p \in P,\ \mathrm{supp}(p) \subseteq A_n\}, \quad A_{n+1} = \bigcup \{\mathrm{supp}(q) : q \in W_{n+1}\}, \quad A = \bigcup_{n<\omega} A_n.$$

We now prove that $W = \bigcup_{n<\omega} W_n$. Let $q \in W$; since $\mathrm{supp}(q)$ is finite, we can choose $n < \omega$ such that $\mathrm{supp}(q) \cap A = \mathrm{supp}(q) \cap A_n$. By construction, there is $q' \in W_{n+1}$ such that $q' \restriction A_n = q \restriction A_n$. But $\mathrm{supp}(q') \subseteq A$, hence

$$\mathrm{supp}(q) \cap \mathrm{supp}(q') = \mathrm{supp}(q) \cap \mathrm{supp}(q') \cap A = \mathrm{supp}(q) \cap \mathrm{supp}(q') \cap A_n \subseteq A_n,$$

therefore we have proved that q and q' are compatible. Since W is an antichain, we conclude that $q = q'$ and $q \in W_{n+1}$.

We finish the proof by showing that $|W_n| < \kappa$ by induction on $n < \omega$. Suppose that $|W_n| < \kappa$. First, note that

$$|A_n| = \left| \bigcup \{\mathrm{supp}(q) : q \in W_n\} \right| \leq \aleph_0 \cdot |W_n| < \kappa.$$

It follows easily that $|\{p \in P : \mathrm{supp}(p) \subseteq A_n\}| < \kappa$ and so $|W_{n+1}| < \kappa$. □

Corollary 2.4.6. $|\mathrm{Coll}(\aleph_0, {<}\kappa)| = \kappa$

Proof. By Proposition 1.3.10 and Theorem 2.4.5, we have $|\mathrm{Coll}(\aleph_0, {<}\kappa)| \leq \kappa^{<\kappa} = \kappa$; the other inequality is obvious. □

[1] Recall that every $p \in P$ is a function $\langle p_\alpha : \alpha < \kappa \rangle$; it makes sense to consider the restriction of p to a subset of κ.

Good Ultrafilters Exist

We now construct a κ-good ultrafilter on $\mathrm{Coll}(\aleph_0, {<}\kappa)$. Let us start the construction with two technical lemmas.

Lemma 2.4.7. *Let X be a set with $|X| < \kappa$. For every function $f\colon X \to \mathrm{Coll}(\aleph_0, {<}\kappa)\setminus\{\mathbb{0}\}$, there exists a function $g\colon X \to \mathrm{Coll}(\aleph_0, {<}\kappa) \setminus \{\mathbb{0}\}$ such that* $\mathrm{ran}(g)$ *is an antichain and $g(x) \leq f(x)$ for all $x \in X$.*

Proof. Without loss of generality, suppose X is a cardinal λ; by hypothesis, we have $\lambda < \kappa$. Let P be the partially ordered set defined in (2.15). Since $e[P]$ is dense in $\mathrm{Coll}(\aleph_0, {<}\kappa) \setminus \{\mathbb{0}\}$, it is sufficient to prove this sentence: for every function $f\colon \lambda \to P$, there is a function $g\colon \lambda \to P$ such that $\mathrm{ran}(g)$ is an antichain and $g(\alpha) \leq f(\alpha)$ for all $\alpha < \lambda$.

By definition, $\mathrm{supp}(f(\alpha))$ is finite for every $\alpha < \lambda$, hence

$$\left|\bigcup_{\alpha<\lambda} \mathrm{supp}(f(\alpha))\right| \leq \lambda < \kappa.$$

Thus we can find a cardinal μ, with $\lambda < \mu < \kappa$, such that $f(\alpha)_\mu = \emptyset$ for all $\alpha < \lambda$. Now define a function $p\colon \lambda \to P$ as follows: for every $\alpha < \kappa$,

$$p(\alpha)_\mu = \{\langle 0, \alpha\rangle\}, \quad \text{and} \quad p(\alpha)_\beta = \emptyset \text{ if } \beta \neq \mu.$$

Observe that $\mathrm{ran}(p)$ is an antichain. By construction, for all $\alpha < \lambda$ $f(\alpha)$ and $p(\alpha)$ are compatible, hence we can find $g(\alpha) \in P$ such that $g(\alpha) \leq f(\alpha)$ and $g(\alpha) \leq p(\alpha)$. The function $g\colon \lambda \to P$ has the desired properties. □

Lemma 2.4.8. *Let $D \subset \mathrm{Coll}(\aleph_0, {<}\kappa)$ be a meet-semilattice such that $|D| < \kappa$. Let $f\colon \mathcal{P}_\omega(\lambda) \to D$ be a monotonically decreasing function, for some $\lambda < \kappa$. There exist a meet-semilattice $D' \supseteq D$ such that $|D'| < \kappa$, and a multiplicative function $g\colon \mathcal{P}_\omega(\lambda) \to D'$ such that $g(S) \leq f(S)$ for all $S \in \mathcal{P}_\omega(\lambda)$.*

Proof. By applying Lemma 2.4.7 to the function

$$\begin{aligned} \mathcal{P}_\omega(\lambda) \times D &\longrightarrow \mathrm{Coll}(\aleph_0, {<}\kappa) \setminus \{\mathbb{0}\} \\ \langle S, d\rangle &\longmapsto f(S) \wedge d \end{aligned},$$

we can find a function

$$h\colon \mathcal{P}_\omega(\lambda) \times D \longrightarrow \mathrm{Coll}(\aleph_0, {<}\kappa) \setminus \{\mathbb{0}\}$$

such that:

- $\mathrm{ran}(h)$ is an antichain.
- For all $S \in \mathcal{P}_\omega(\lambda)$ and $d \in D$, we have $h(S,d) \leq f(S) \wedge d$.

Then, define $g\colon \mathcal{P}_\omega(\lambda) \to \mathrm{Coll}(\aleph_0, {<}\kappa) \setminus \{\mathbb{0}\}$ as follows: for every $S \in \mathcal{P}_\omega(\lambda)$,

$$g(S) = \bigvee \{h(T, d) : T \supseteq S,\ d \in D\}.$$

We prove that g has the required properties.

The first thing to prove is that $g(S) \leq f(S)$ for all $S \in \mathcal{P}_\omega(\lambda)$. But this is immediate, for if $T \supseteq S$ and $d \in D$ then

$$h(T, d) \leq f(T) \wedge d \leq f(T) \leq f(S).$$

Secondly, we show that g is multiplicative. For all $S_1, S_2 \in \mathcal{P}_\omega(\lambda)$, using the fact that $\operatorname{ran}(h)$ is an antichain we obtain

$$g(S_1) \wedge g(S_2) = \bigvee \{h(T_1, d_1) : T_1 \supseteq S_1,\ d_1 \in D\} \wedge \bigvee \{h(T_2, d_2) : T_2 \supseteq S_2,\ d_2 \in D\} =$$
$$= \bigvee \{h(T_1, d_1) \wedge h(T_2, d_2) : T_1 \supseteq S_1,\ T_2 \supseteq S_2,\ d_1 \in D,\ d_2 \in D\} =$$
$$= \bigvee \{h(T, d) : T \supseteq S_1,\ T \supseteq S_2,\ d \in D\} = \bigvee \{h(T, d) : T \supseteq S_1 \cup S_2,\ d \in D\} = g(S_1 \cup S_2).$$

To conclude, define

$$D' = \{g(S) \wedge d : S \in \mathcal{P}_\omega(\lambda),\ d \in D\}.$$

Note that $D \subseteq D'$, because $g(\emptyset) = \mathbb{1}$. Moreover, $\mathbb{0} \notin D'$ because $g(S) \wedge d \geq h(S, d) > \mathbb{0}$. The thesis now follows. □

Theorem 2.4.9. *There exists a κ-good ultrafilter on* $\operatorname{Coll}(\aleph_0, {<}\kappa)$.

Proof. By Corollary 2.4.6, we can fix an enumeration

$$\operatorname{Coll}(\aleph_0, {<}\kappa) = \{b_\alpha : \alpha < \kappa\}.$$

Let $\{f_\alpha : \alpha < \kappa\}$ be an enumeration of all monotonically decreasing functions $f : \mathcal{P}_\omega(\lambda) \to \operatorname{Coll}(\aleph_0, {<}\kappa)$, for $\lambda < \kappa$. This enumeration has order-type $\kappa \cdot \kappa$, which means that every such function is listed κ times.

Now, we construct by recursion a sequence $\langle D_\alpha : \alpha < \kappa\rangle$ of meet-semilattices in $\operatorname{Coll}(\aleph_0, {<}\kappa)$. These three conditions will be satisfied at each stage $\alpha < \kappa$:

1. $|D_\alpha| < \kappa$
2. If $f_\alpha : \mathcal{P}_\omega(\lambda) \to D_\alpha$, then there is a multiplicative function $g : \mathcal{P}_\omega(\lambda) \to D_{\alpha+1}$ such that $g(S) \leq f(S)$ for all $S \in \mathcal{P}_\omega(\lambda)$.
3. Either $b_\alpha \in D_{\alpha+1}$ or $\neg b_\alpha \in D_{\alpha+1}$.

Let $D_0 = \{\mathbb{1}\}$. Suppose D_α is already defined, and construct $D_{\alpha+1}$ as follows. First, check whether $\operatorname{ran}(f_\alpha) \subseteq D_\alpha$ or not. If not, define ${D_\alpha}' = D_\alpha$. Otherwise, if $f_\alpha : \mathcal{P}_\omega(\lambda) \to D_\alpha$, then apply Lemma 2.4.8 to construct a meet-semilattice ${D_\alpha}' \supseteq D_\alpha$ such that $\left|{D_\alpha}'\right| < \kappa$, and a multiplicative function $g : \mathcal{P}_\omega(\lambda) \to {D_\alpha}'$ such that $g(S) \leq f(S)$ for all $S \in \mathcal{P}_\omega(\lambda)$. By Remark 1.3.7, ${D_\alpha}'$ is contained in some ultrafilter. It follows that either ${D_\alpha}' \cup \{b_\alpha\}$ has the finite intersection property, or ${D_\alpha}' \cup \{\neg b_\alpha\}$ has the finite intersection property. Then define

$$D_{\alpha+1} = {D_\alpha}' \cup \left\{d \wedge b_\alpha : d \in {D_\alpha}'\right\}$$

in the former case, or

$$D_{\alpha+1} = {D_\alpha}' \cup \left\{d \wedge \neg b_\alpha : d \in {D_\alpha}'\right\}$$

in the latter case. Finally, if δ is a limit ordinal, define $D_\delta = \bigcup_{\alpha<\delta} D_\alpha$. This concludes the recursive definition.

Let $U = \bigcup_{\alpha<\kappa} D_\alpha$. By (3), U is an ultrafilter on $\operatorname{Coll}(\aleph_0, {<}\kappa)$. To terminate the proof, we show that U is κ-good. Let $\lambda < \kappa$ and let $f : \mathcal{P}_\omega(\lambda) \to U$ be a monotonically decreasing function. There exists $\beta < \kappa$ such that $\operatorname{ran}(f) \subseteq D_\beta$, because κ is regular. Choose some $\alpha > \beta$ such that $f = f_\alpha$; by (2) there exists a multiplicative function $g : \mathcal{P}_\omega(\lambda) \to D_{\alpha+1}$ such that $g(S) \leq f(S)$ for all $S \in \mathcal{P}_\omega(\lambda)$. □

Finally, we are able to use Boolean ultrapowers to construct saturated elementary extensions of a given structure. Compare this result with Theorem 1.1.29.

Corollary 2.4.10. *Let $\mathfrak{M}$ be an L-structure and κ an inaccessible cardinal satisfying $|L|^+ + |M| \leq \kappa$. There exist a complete Boolean algebra $\mathbb{B}$ and an ultrafilter $U \subset \mathbb{B}$ such that $\mathfrak{M}^{\downarrow\mathbb{B}}/U$ is a saturated structure of cardinality κ.*

Proof. Take $\mathbb{B} = \mathrm{Coll}(\aleph_0, {<}\kappa)$ and let U be any κ-good ultrafilter on $\mathbb{B}$. Notice that U is automatically $\aleph_1$-incomplete, by Proposition 2.4.4. Thus we may apply Theorem 2.2.5 to obtain that $\mathfrak{M}^{\downarrow\mathbb{B}}/U$ is κ-saturated. To finish the proof, we show that $\left|M^{\downarrow\mathbb{B}}\right| \leq \kappa$. By Theorem 2.4.5, we have

$$\left|M^{\downarrow\mathbb{B}}\right| = |\{\tau \colon A \to M : A \subseteq \mathbb{B} \text{ is a maximal antichain}\}| \leq \left|\bigcup \left\{M^A : A \subseteq \mathbb{B},\ |A| < \kappa\right\}\right|.$$

Since κ is inaccessible, we have $\left|M^A\right| \leq \kappa$ whenever $|A| < \kappa$. Hence, we can conclude that $\left|M^{\downarrow\mathbb{B}}\right| \leq \kappa \cdot \kappa^{<\kappa} = \kappa \cdot \kappa = \kappa$. □

2.5 The Boolean-Valued Model $V^{\mathbb{B}}$

We conclude this chapter with some remarks on the $\mathbb{B}$-valued model $V^{\mathbb{B}}$. We are not concerned with giving a complete treatment of the subject. Rather, we focus on a set-theoretic approach to Boolean ultrapowers, due to Hamkins and Seabold [7].

In this section, fix the signature $L = \{\in\}$. Unless otherwise specified, $\mathbb{B}$ will be a complete Boolean algebra.

Definition 2.5.1. τ is a $\mathbb{B}$-*name* if and only if τ is a set of pairs of the form $\langle\sigma, b\rangle$, where σ is a $\mathbb{B}$-name and $b \in \mathbb{B}$. The class of $\mathbb{B}$-names is denoted by $V^{\mathbb{B}}$.

Theorem 2.5.2. *$V^{\mathbb{B}}$ is a full $\mathbb{B}$-valued model for L, if we define by recursion*

$$\llbracket \tau \in \sigma \rrbracket = \bigvee_{\langle\pi,b\rangle\in\sigma} \Big(\llbracket \tau = \pi \rrbracket \wedge b\Big),$$

$$\llbracket \tau \subseteq \sigma \rrbracket = \bigwedge_{\pi\in\mathrm{dom}(\tau)} \Big(\neg \llbracket \pi \in \tau \rrbracket \vee \llbracket \pi \in \sigma \rrbracket\Big),$$

$$\llbracket \tau = \sigma \rrbracket = \llbracket \tau \subseteq \sigma \rrbracket \wedge \llbracket \sigma \subseteq \tau \rrbracket .$$

Moreover, if φ is an axiom of ZFC*, then $\llbracket \varphi \rrbracket^{V^{\mathbb{B}}} = \mathbb{1}$.*

Proof. See, for example, [10, Chapter 14]. □

Every $x \in V$ has its canonical $\mathbb{B}$-name $\check{x} \in V^{\mathbb{B}}$, defined recursively as $\check{x} = \{\langle\check{y}, \mathbb{1}\rangle : y \in x\}$. Now, for any class $M \subseteq V$, define

$$\check{M}^{\mathbb{B}} = \left\{\tau \in V^{\mathbb{B}} : \bigvee_{x\in M} \llbracket \tau = \check{x} \rrbracket = \mathbb{1}\right\}. \tag{2.16}$$

We regard $\check{M}^{\mathbb{B}}$ as a $\mathbb{B}$-valued model for L, with $\llbracket \tau = \sigma \rrbracket$ and $\llbracket \tau \in \sigma \rrbracket$ inherited from $V^{\mathbb{B}}$.

Theorem 2.5.3. *The $\mathbb{B}$-valued model $\check{M}^{\mathbb{B}}$ is full.*

Proof. Let $A \subset \mathbb{B}$ be an antichain and $\{\tau_a : a \in A\} \subseteq \check{M}^{\mathbb{B}}$. Without loss of generality, we may assume that A is maximal. By fullness of $V^{\mathbb{B}}$, there is $\tau \in V^{\mathbb{B}}$ such that $a \leq [\![\tau = \tau_a]\!]$ for all $a \in A$. Observe that $\tau \in \check{M}^{\mathbb{B}}$, because for all $a \in A$

$$\bigvee_{x \in M} [\![\tau = \check{x}]\!] \geq \bigvee_{x \in M} ([\![\tau = \tau_a]\!] \wedge [\![\tau_a = \check{x}]\!]) = [\![\tau = \tau_a]\!] \wedge \bigvee_{x \in M} [\![\tau_a = \check{x}]\!] = [\![\tau = \tau_a]\!] \wedge \mathbb{1} = [\![\tau = \tau_a]\!] \geq a,$$

hence

$$\bigvee_{x \in M} [\![\tau = \check{x}]\!] \geq \bigvee A = \mathbb{1}.$$

The proof is complete. ☐

We shall see that every $L_{\omega,\omega}$-formula is preserved between $\langle V, \in \rangle$ and $\check{V}^{\mathbb{B}}$. In general, this will not happen between $\langle V, \in \rangle$ and $V^{\mathbb{B}}$.

Proposition 2.5.4. *Let $\varphi(x_1, \dots, x_n)$ be an $L_{\omega,\omega}$-formula and $a_1, \dots, a_n \in M$. Then*

$$\langle M, \in \rangle \models \varphi(a_1, \dots, a_n) \iff [\![\varphi(\check{a_1}, \dots, \check{a_n})]\!]^{\check{M}^{\mathbb{B}}} = \mathbb{1}. \tag{2.17}$$

Proof. We show that (2.17) holds by induction on the complexity of φ.

If φ is an atomic formula, then

$$(x = y) \iff [\![\check{x} = \check{y}]\!] = \mathbb{1}, \quad \text{and} \quad (x \in y) \iff [\![\check{x} \in \check{y}]\!] = \mathbb{1}$$

are clear from the definition of canonical names. Moreover, the inductive step for $\neg$ and $\wedge$ is straightforward.

Suppose (2.17) holds for $\varphi(x_1, \dots, x_n, y)$. If

$$\langle M, \in \rangle \models \exists y \varphi(a_1, \dots, a_n, y),$$

then there exists $b \in M$ such that $\langle M, \in \rangle \models \varphi(a_1, \dots, a_n, b)$. By inductive hypothesis, this implies $[\![\varphi(\check{a_1}, \dots, \check{a_n}, \check{b})]\!]^{\check{M}^{\mathbb{B}}} = \mathbb{1}$. Since $\check{b} \in \check{M}^{\mathbb{B}}$, we can conclude that $[\![\exists y \varphi(\check{a_1}, \dots, \check{a_n}, y)]\!]^{\check{M}^{\mathbb{B}}} = \mathbb{1}$, as desired. Conversely, assume

$$[\![\exists y \varphi(\check{a_1}, \dots, \check{a_n}, y)]\!]^{\check{M}^{\mathbb{B}}} = \mathbb{1}.$$

For every $\tau \in \check{M}^{\mathbb{B}}$, we have

$$[\![\varphi(\check{a_1}, \dots, \check{a_n}, \tau)]\!]^{\check{M}^{\mathbb{B}}} = [\![\varphi(\check{a_1}, \dots, \check{a_n}, \tau)]\!]^{\check{M}^{\mathbb{B}}} \wedge \bigvee_{b \in M} [\![\tau = \check{b}]\!] \leq \bigvee_{b \in M} [\![\varphi(\check{a_1}, \dots, \check{a_n}, \check{b})]\!]^{\check{M}^{\mathbb{B}}},$$

hence

$$\mathbb{1} = [\![\exists y \varphi(\check{a_1}, \dots, \check{a_n}, y)]\!]^{\check{M}^{\mathbb{B}}} = \bigvee_{\tau \in \check{M}^{\mathbb{B}}} [\![\varphi(\check{a_1}, \dots, \check{a_n}, \tau)]\!]^{\check{M}^{\mathbb{B}}} \leq \bigvee_{b \in M} [\![\varphi(\check{a_1}, \dots, \check{a_n}, \check{b})]\!]^{\check{M}^{\mathbb{B}}}.$$

As a consequence, there is $b \in M$ such that $[\![\varphi(\check{a_1}, \dots, \check{a_n}, \check{b})]\!]^{\check{M}^{\mathbb{B}}} > \mathbb{0}$. By inductive hypothesis, it cannot happen that $\langle M, \in \rangle \models \neg\varphi(a_1, \dots, a_n, b)$, otherwise we would have $[\![\varphi(\check{a_1}, \dots, \check{a_n}, \check{b})]\!]^{\check{M}^{\mathbb{B}}} = \mathbb{0}$. Therefore, $\langle M, \in \rangle \models \varphi(a_1, \dots, a_n, b)$, which means that $\langle M, \in \rangle \models \exists y \varphi(a_1, \dots, a_n, y)$, as desired. ☐

Let $U \subset \mathbb{B}$ be an ultrafilter; consider the quotient $\check{M}^{\mathbb{B}}/U$. As the next theorem shows, the canonical $\mathbb{B}$-names provide us with a natural way to elementarily embed M into $\check{M}^{\mathbb{B}}/U$.

Theorem 2.5.5. *The map $i\colon x \mapsto [\check{x}]_U$ is an elementary embedding of M into $\check{M}^{\mathbb{B}}/U$.*

Proof. Let $\varphi(x_1, \ldots, x_n)$ be an $L_{\omega,\omega}$-formula and $a_1, \ldots, a_n \in M$; suppose that

$$\langle M, \in \rangle \models \varphi(a_1, \ldots, a_n).$$

By Proposition 2.5.4, we have $[\![\varphi(\check{a_1}, \ldots, \check{a_n})]\!]^{\check{M}^{\mathbb{B}}} = \mathbb{1}$, hence $[\![\varphi(\check{a_1}, \ldots, \check{a_n})]\!]^{\check{M}^{\mathbb{B}}} \in U$. Now apply Theorem 2.1.10 to get

$$\check{M}^{\mathbb{B}}/U \models \varphi([\check{a_1}]_U, \ldots, [\check{a_n}]_U),$$

which is what we wanted. □

We come to the main theorem, which establishes that the two approaches to the Boolean ultrapower are equivalent.

Theorem 2.5.6. *Let $U \subset \mathbb{B}$ be an ultrafilter. Let $j\colon x \mapsto [c_x]_U$ be the elementary embedding of M into $M^{\downarrow\mathbb{B}}/U$, and let $i\colon x \mapsto [\check{x}]_U$ be the elementary embedding of M into $\check{M}^{\mathbb{B}}/U$. There is an isomorphism $\pi\colon M^{\downarrow\mathbb{B}}/U \to \check{M}^{\mathbb{B}}/U$ making this diagram commute:*

$$\begin{array}{ccc} M^{\downarrow\mathbb{B}}/U & \xrightarrow{\quad\pi\quad} & \check{M}^{\mathbb{B}}/U \\ & {\scriptstyle j}\nwarrow \quad \nearrow{\scriptstyle i} & \\ & M & \end{array}$$

Proof. Let $A \subset \mathbb{B}$ be a maximal antichain and $f\colon A \to M$ a function in $M^{\downarrow\mathbb{B}}$. By fullness of $\check{M}^{\mathbb{B}}$, there is $\tau_f \in \check{M}^{\mathbb{B}}$ such that $a \leq \left[\!\!\left[\tau_f = \check{f(a)}\right]\!\!\right]^{\check{M}^{\mathbb{B}}}$ for all $a \in A$.

Suppose W is a refinement of A. Then

$$[\![\tau_f = \tau_{(f\downarrow W)}]\!]^{\check{M}^{\mathbb{B}}} = \mathbb{1}. \tag{2.18}$$

Indeed, let $w \in W$ and let $a \in A$ be the unique element such that $w \leq a$. We have

$$w \leq a \leq \left[\!\!\left[\tau_f = \check{f(a)}\right]\!\!\right]^{\check{M}^{\mathbb{B}}} = \left[\!\!\left[\tau_f = (f \downarrow \check{W})(w)\right]\!\!\right]^{\check{M}^{\mathbb{B}}}.$$

On the other hand, we have by definition $w \leq \left[\!\!\left[\tau_{(f\downarrow W)} = (f \downarrow \check{W})(w)\right]\!\!\right]^{\check{M}^{\mathbb{B}}}$. It follows that

$$w \leq \left[\!\!\left[\tau_f = (f \downarrow \check{W})(w)\right]\!\!\right]^{\check{M}^{\mathbb{B}}} \wedge \left[\!\!\left[\tau_{(f\downarrow W)} = (f \downarrow \check{W})(w)\right]\!\!\right]^{\check{M}^{\mathbb{B}}} \leq [\![\tau_f = \tau_{(f\downarrow W)}]\!]^{\check{M}^{\mathbb{B}}},$$

hence $\mathbb{1} = \bigvee W \leq [\![\tau_f = \tau_{(f\downarrow W)}]\!]^{\check{M}^{\mathbb{B}}}$. This proves (2.18).

Now, we prove that

$$[\![f = g]\!]^{M^{\downarrow\mathbb{B}}} \in U \iff [\![\tau_f = \tau_g]\!]^{\check{M}^{\mathbb{B}}} \in U, \tag{2.19}$$

$$[\![f \in g]\!]^{M^{\downarrow\mathbb{B}}} \in U \iff [\![\tau_f \in \tau_g]\!]^{\check{M}^{\mathbb{B}}} \in U. \tag{2.20}$$

Actually, we shall prove (2.19) only, because the proof of (2.20) is quite similar. Choose a common refinement W of $\mathrm{dom}(f)$ and $\mathrm{dom}(g)$, and suppose that

$$[\![f = g]\!]^{M^{\downarrow\mathbb{B}}} = \bigvee \{w \in W : (f \downarrow W)(w) = (g \downarrow W)(w)\} \in U.$$

If $w \in W$ is such that $(f \downarrow W)(w) = (g \downarrow W)(w)$, then, using (2.18),

$$w \leq \left[\!\left[\tau_{(f\downarrow W)} = (f \downarrow \check{W})(w)\right]\!\right]^{\check{M}^{\mathbb{B}}} \wedge \left[\!\left[\tau_{(g\downarrow W)} = (g \downarrow \check{W})(w)\right]\!\right]^{\check{M}^{\mathbb{B}}} \leq [\![\tau_{(f\downarrow W)} = \tau_{(g\downarrow W)}]\!]^{\check{M}^{\mathbb{B}}} = [\![\tau_f = \tau_g]\!]^{\check{M}^{\mathbb{B}}},$$

and we obtain $[\![\tau_f = \tau_g]\!]^{\check{M}^{\mathbb{B}}} \in U$. Conversely, assume $[\![\tau_f = \tau_g]\!]^{\check{M}^{\mathbb{B}}} \in U$. If $w \in W$ is such that $(f \downarrow W)(w) \neq (g \downarrow W)(w)$ then, by a similar argument,

$$w \leq [\![\tau_f \neq \tau_g]\!]^{\check{M}^{\mathbb{B}}} \notin U,$$

and it is easy to conclude.

We can now define the map

$$\begin{aligned} \pi\colon M^{\downarrow\mathbb{B}}/U &\longrightarrow \check{M}^{\mathbb{B}}/U \\ [f]_U &\longmapsto [\tau_f]_U \end{aligned}.$$

By (2.19) and (2.20), π is an embedding. We show that π is surjective, hence an isomorphism. Let $\tau \in \check{M}^{\mathbb{B}}$. The set

$$A = \left\{[\![\tau = \check{x}]\!]^{\check{M}^{\mathbb{B}}} : x \in M\right\} \setminus \{\mathbb{0}\}$$

is easily seen to be a maximal antichain in $\mathbb{B}$. Then, the function

$$\begin{aligned} f\colon A &\longrightarrow M \\ [\![\tau = \check{x}]\!]^{\check{M}^{\mathbb{B}}} &\longmapsto x \end{aligned}.$$

satisfies $\pi([f]_U) = [\tau]_U$.

Finally, we show that $\pi \circ j = i$. For each $x \in M$ we have $[\![\tau_{c_x} = \check{x}]\!]^{\check{M}^{\mathbb{B}}} = \mathbb{1}$, therefore

$$\pi(j(x)) = \pi([c_x]_U) = [\tau_{c_x}]_U = [\check{x}]_U = i(x).$$

This concludes the proof. □

Forcing

Forcing is one of the central themes in modern set theory. It would not be convenient to give here a full treatment of this subject. However, we state the main results just to have a reference for the next sections. The reader can find a good introduction in Kunen [14], and an all-purpose reference is Jech [10].

Definition 2.5.7. Let M be a transitive model of ZFC, and $\langle P, \leq\rangle \in M$ a partially ordered set. A filter $G \subset P$ is *M-generic* if for all $D \subseteq P$, if D is dense in P and $D \in M$, then $G \cap D \neq \emptyset$.

Let M be a transitive model of ZFC, and $\langle P, \leq\rangle \in M$ a partially ordered set. Theorem 1.3.9 is true in M, consequently M believes that there is a complete Boolean algebra $\mathrm{RO}(P)^M$ which is the regular open algebra of P, together with the map

$$e\colon P \longrightarrow \mathrm{RO}(P)^M \setminus \{\mathbb{0}\}.$$

Note that $\mathrm{RO}(P)^M$ need not be a complete Boolean algebra. We only know that

$$\langle M, \in\rangle \models \text{“}\,\mathrm{RO}(P)^M \text{ is a complete Boolean algebra”}$$

We may form the class of $\mathrm{RO}(P)^M$-names for M, denoted by $M^{\mathrm{RO}(P)}$. More explicitly,

$$M^{\mathrm{RO}(P)} = \left\{\tau \in M : \langle M, \in\rangle \models \text{“}\tau \text{ is a } \mathrm{RO}(P)^M\text{-name”}\right\}.$$

Then, if $G \subset P$ is an M-generic filter and $\tau \in M^{\mathrm{RO}(P)}$, we define

$$\mathrm{val}(\tau, G) = \{\mathrm{val}(\sigma, G) : \text{there exist } p \in G \text{ and } b \in B \text{ such that } e(p) \leq b \text{ and } \langle \sigma, b\rangle \in \tau\},$$

and

$$M[G] = \left\{\mathrm{val}(\tau, G) : \tau \in M^{\mathrm{RO}(P)}\right\}.$$

Theorem 2.5.8. *Let M be a transitive model of* ZFC*, $\langle P, \leq\rangle \in M$ a partially ordered set and $G \subset P$ an M-generic filter. The* generic extension *$M[G]$ has the following properties:*

1. *$M[G]$ is a transitive model of* ZFC*.*
2. *$M \subseteq M[G]$ and $G \in M[G]$.*
3. *M and $M[G]$ have the same ordinals.*
4. *If N is a transitive model of* ZF *such that $M \subseteq N$ and $G \in N$, then $M[G] \subseteq N$.*

Definition 2.5.9. The *P-forcing language* is the signature of set theory with elements of $M^{\mathrm{RO}(P)}$ added as constant symbols. Let $\varphi(\tau_1, \dots, \tau_n)$ be a sentence in the P-forcing language and $p \in P$; we define the *forcing relation*:

$$p \Vdash \varphi(\tau_1, \dots, \tau_n) \iff e(p) \leq [\![\varphi(\tau_1, \dots, \tau_n)]\!]^{M^{\mathrm{RO}(P)}}.$$

Theorem 2.5.10. *Let M be a transitive model of* ZFC*, $\langle P, \leq\rangle \in M$ a partially ordered set and $\varphi(\tau_1, \dots, \tau_n)$ a sentence in the P-forcing language. For every M-generic filter $G \subset P$, we have*

$$\langle M[G], \in\rangle \models \varphi(\mathrm{val}(\tau_1, G), \dots, \mathrm{val}(\tau_n, G)) \iff \textit{there is } p \in G \textit{ such that } p \Vdash \varphi(\tau_1, \dots, \tau_n).$$

The basic properties of the forcing relation can be summarized in the following theorem.

Theorem 2.5.11. *Let M be a transitive model of* ZFC*, and $\langle P, \leq\rangle \in M$ a partially ordered set. Then:*

- *$p \Vdash \neg\varphi$ if and only if there is no $q \leq p$ such that $q \Vdash \varphi$.*
- *$p \Vdash (\varphi \wedge \psi)$ if and only if $p \Vdash \varphi$ and $p \Vdash \psi$.*
- *$p \Vdash \exists x \varphi$ if and only if for all $q \leq p$ there exist $r \leq q$ and $\tau \in M^{\mathrm{RO}(P)}$ such that $r \Vdash \varphi(\tau)$.*

We conclude this brief treatment of forcing with a practical result; we shall use it soon.

Theorem 2.5.12. *Let κ be a regular cardinal. If $\langle P, \leq\rangle$ satisfies the $<\kappa$-chain condition, then forcing with $\langle P, \leq\rangle$ preserves the regularity of κ.*

Proof. Let $\lambda < \kappa$. We have to prove that for every name $\dot{f} \in V^{\mathrm{RO}(P)}$ and for every $p \in P$,

$$\text{if} \quad p \Vdash \dot{f} \text{ is a function from } \check{\lambda} \text{ to } \check{\kappa}, \quad \text{then} \quad p \Vdash \dot{f} \text{ is bounded.} \tag{2.21}$$

For all $\alpha < \lambda$ define

$$B_\alpha = \left\{\beta < \kappa : \text{there is } q_\beta \leq p \text{ such that } q_\beta \Vdash \dot{f}(\check{\alpha}) = \check{\beta}\right\}.$$

Clearly, the set $A_\alpha = \{q_\beta : \beta \in B_\alpha\}$ is an antichain, hence $|B_\alpha| = |A_\alpha| < \kappa$ for all $\alpha < \lambda$. It follows from the regularity of κ that $\bigcup_{\alpha<\lambda} B_\alpha$ is bounded by a $\gamma < \kappa$. Then for all $\alpha < \lambda$, $p \Vdash \dot{f}(\check{\alpha}) < \check{\gamma}$, establishing (2.21). □

The Boolean Ultrapower of H_κ and Absoluteness

We finish this chapter with an application of what we have done up to this time. This subsection also provides a bridge towards the next Chapter 3.

First of all, we fix some standard notation. Let $\mathrm{trcl}(x)$ denote the transitive closure of x; it is the smallest transitive set containing x. For every infinite cardinal κ, let

$$H_\kappa = \{x : |\mathrm{trcl}(x)| < \kappa\} .$$

We start with a basic result.

Proposition 2.5.13. *For every infinite cardinal κ, H_κ is a transitive set of cardinality $2^{<\kappa}$.*

Proof. We prove that $H_\kappa \subseteq V_\kappa$; in particular, this will establish that H_κ is a set. Let $x \in H_\kappa$. For every $\alpha < \mathrm{rank}(x)$, there is some $y \in \mathrm{trcl}(x)$ such that $\mathrm{rank}(y) = \alpha$ (this is easily proved by induction). Conversely, if $y \in \mathrm{trcl}(x)$ then $\mathrm{rank}(y) < \mathrm{rank}(\mathrm{trcl}(x)) = \mathrm{rank}(x)$. Thus we have proved that

$$\mathrm{rank}(x) = \{\mathrm{rank}(y) : y \in \mathrm{trcl}(x)\} ,$$

whence $\mathrm{rank}(x) \leq |\mathrm{trcl}(x)| < \kappa$ and so $x \in V_\kappa$. The transitivity of H_κ is immediate from the definitions.

We now prove that $|H_\kappa| = 2^{<\kappa}$. The inequality $|H_\kappa| \geq 2^{<\kappa}$ is easy, because for every cardinal $\lambda < \kappa$ we have $\mathcal{P}(\lambda) \subseteq H_\kappa$. To prove that $|H_\kappa| \leq 2^{<\kappa}$, define a function

$$f \colon H_\kappa \longrightarrow \bigcup_{\lambda<\kappa} \mathcal{P}(\lambda \times \lambda)$$

as follows: if $x \in H_\kappa$, let $\lambda = |\mathrm{trcl}(x) \cup \{x\}| < \kappa$, and choose a binary relation $f(x) \subseteq \lambda \times \lambda$ such that $\langle \lambda, f(x)\rangle \cong \langle \mathrm{trcl}(x) \cup \{x\}, \in\rangle$. A well known result is that for all sets x, y

$$\langle \mathrm{trcl}(x) \cup \{x\}, \in\rangle \cong \langle \mathrm{trcl}(y) \cup \{y\}, \in\rangle \implies x = y; \tag{2.22}$$

we just sketch how to prove it (see [14, Chapter I] for further details). If π is an isomorphism between the two structures in (2.22), then π necessarily coincides with the Mostowski collapsing function, whose restriction to transitive sets is the identity. It follows that $\mathrm{trcl}(x) \cup \{x\} = \mathrm{trcl}(y) \cup \{y\}$, hence $x = y$. From (2.22), we obtain that the function f is injective, and the proof is complete. □

Definition 2.5.14. Let κ be an infinite cardinal and $\mathbb{B}$ a complete Boolean algebra. We define

$$H_\kappa^{V^\mathbb{B}} = \left\{\tau \in V^\mathbb{B} : [\![|\mathrm{trcl}(\tau)| < \check{\kappa}]\!] = \mathbb{1}\right\} .$$

To be specific, $H_\kappa^{V^\mathbb{B}}$ is the relativization of H_κ to $V^\mathbb{B}$.

For the next remark, remember that an $L_{\omega,\omega}$-formula is Δ_0 if all its quantifiers are bounded, that is of the form $\exists x((x \in y) \wedge \varphi)$ and $\forall x((x \in y) \rightarrow \varphi)$. Let us recall in passing that Δ_0 formulas are absolute for transitive models of ZF. We recall also that a formula is Σ_1 if it is of the form $\exists x \varphi$, where φ is Δ_0.

Remark 2.5.15. Let $\varphi(x_1, \ldots, x_n)$ be a Δ_0 $L_{\omega,\omega}$-formula and $a_1, \ldots, a_n \in H_\kappa$. Then

$$\langle H_\kappa, \in\rangle \models \varphi(a_1, \ldots, a_n) \iff [\![\varphi(\check{a}_1, \ldots, \check{a}_n)]\!]^{H_\kappa^{V^\mathbb{B}}} = \mathbb{1}.$$

This is just the Boolean-valued version of absoluteness of Δ_0 formulas: the proof is an easy exercise.

Proposition 2.5.16. *Let κ be inaccessible and $\mathbb{B} = \mathrm{Coll}(\aleph_0, {<}\kappa)$. Then* $\left[\!\left[H_\kappa^{V^{\mathbb{B}}} = H_\kappa \cap V^{\mathbb{B}} \right]\!\right] = \mathbb{1}$.

Proof. It follows directly from the definitions that

$$\left[\!\left[H_\kappa^{V^{\mathbb{B}}} \supseteq H_\kappa \cap V^{\mathbb{B}} \right]\!\right] = \mathbb{1};$$

actually, this is true for any complete Boolean algebra $\mathbb{B}$ and any cardinal κ.

We prove that the converse inclusion has Boolean value $\mathbb{1}$ as well. Let $\tau \in H_\kappa^{V^{\mathbb{B}}}$. We have seen in the proof of Proposition 2.5.13 that τ can be "coded" as a name for an element of $2^{<\kappa}$. Combining Theorem 2.4.5 with Theorem 2.5.12, we see that $\mathrm{Coll}(\aleph_0, {<}\kappa)$ preserves the regularity of κ. As a consequence, τ can ultimately be coded as $\dot{f} \colon \kappa \to 2$, a name for the characteristic function of a bounded subset of κ. More explicitly, $\dot{f}$ satisfies

$$\left[\!\left[\exists \alpha \big(\alpha < \kappa \wedge \dot{f}^{-1}[\{1\}] \subseteq \alpha \big) \right]\!\right] = \mathbb{1}.$$

We show that for every $p \in \mathbb{B}$ there are $q \leq p$ and $\dot{g} \in H_\kappa \cap V^{\mathbb{B}}$ such that $q \Vdash \dot{f} = \dot{g}$. This will prove the proposition (applying Theorem 2.5.11). We define

$$D = \left\{ p \in \mathbb{B} : \text{there is } \alpha < \kappa \text{ such that } p \Vdash \dot{f}^{-1}[\{1\}] \subseteq \alpha \right\}$$

and, for all $\xi < \kappa$,

$$E_\xi = \left\{ p \in \mathbb{B} : \text{there is } i \in 2 \text{ such that } p \Vdash \dot{f}(\xi) = i \right\}.$$

Note that, if $p \Vdash \dot{f}^{-1}[\{1\}] \subseteq \alpha$, then $p \in \bigcap_{\xi \geq \alpha} E_\xi$. It is plain that D and the sets E_ξ are open and dense. Therefore, for every $\xi < \kappa$, we can construct a maximal antichain $A_\xi \subseteq E_\xi \cap D$. Now fix $p \in \mathbb{B}$; choose $q \leq p$ such that $q \in D$, and let $\alpha < \kappa$ such that $q \Vdash \dot{f}^{-1}[\{1\}] \subseteq \alpha$. Define for all $\xi < \alpha$

$$B_\xi = \{ r \in A_\xi : r \text{ and } q \text{ are compatible} \}.$$

By ${<}\kappa$-chain condition, we have $|B_\xi| < \kappa$ for all $\xi < \alpha$. Finally, let

$$\dot{g} = \left\{ \langle \langle \xi, i \rangle, r \rangle : r \in B_\xi \text{ and } r \Vdash \dot{f}(\xi) = i \right\}.$$

Since $\mathrm{Coll}(\aleph_0, {<}\kappa) \subset H_\kappa$ and $|B_\xi| < \kappa$ for all $\xi < \alpha$, we have $\dot{g} \in H_\kappa$. Moreover, by construction we have $q \Vdash \dot{f} = \dot{g}$, as desired. □

Theorem 2.5.17. *Let κ be inaccessible, $\mathbb{B} = \mathrm{Coll}(\aleph_0, {<}\kappa)$, and $U \subset \mathbb{B}$ a κ-good ultrafilter. Then $\check{H}_{\aleph_1}^{\mathbb{B}}/U$ and $H_{\aleph_1}^{V^{\mathbb{B}}}/U$ are both saturated structures of cardinality κ.*

Proof. From the results of Section 2.4, we already know that the Boolean ultrapower $\check{H}_{\aleph_1}^{\mathbb{B}}/U$ is a saturated structure of cardinality κ.

On the other hand, Theorem 2.2.5 implies that $H_{\aleph_1}^{V^{\mathbb{B}}}/U$ is κ-saturated. Moreover, Proposition 2.5.16 and the inaccessibility of κ imply that

$$\left| H_{\aleph_1}^{V^{\mathbb{B}}}/U \right| \leq \left| H_\kappa^{V^{\mathbb{B}}}/U \right| \leq |H_\kappa| = 2^{<\kappa} = \kappa,$$

concluding the proof. □

If $\check{H}^{\mathbb{B}}_{\aleph_1}/U$ and $H^{V^{\mathbb{B}}}_{\aleph_1}/U$ were elementarily equivalent, then by means of Theorem 1.1.30 we could prove that they are isomorphic. The next theorem will provide a first, partial answer.

Note that Theorem 2.5.18 is a direct consequence of Cohen's forcing theorem.

Theorem 2.5.18 (Cohen's Absoluteness)**.** *Let $\varphi(x, y)$ be a Δ_0 $L_{\omega,\omega}$-formula, and let $r \subseteq \omega$. Then the following are equivalent:*

- $\langle H_{\aleph_1}, \in\rangle \models \exists x\varphi(x, r)$.
- *There is a complete Boolean algebra $\mathbb{B}$ such that* $[\![\exists x\varphi(x, \check{r})]\!]^{V^{\mathbb{B}}} > \mathbb{0}$.

Proof. Suppose that

$$\langle H_{\aleph_1}, \in\rangle \models \exists x\varphi(x, r).$$

By absoluteness of Δ_0 formulas, this implies $\langle V, \in\rangle \models \exists x\varphi(x, r)$. Now it suffices to take the complete Boolean algebra $\mathbb{2} = \{\mathbb{0}, \mathbb{1}\}$ to have

$$[\![\exists x\varphi(x, \check{r})]\!]^{V^{\mathbb{2}}} = \mathbb{1} > \mathbb{0}.$$

Conversely, suppose there is a complete Boolean algebra $\mathbb{B}$ such that

$$[\![\exists x\varphi(x, \check{r})]\!]^{V^{\mathbb{B}}} > \mathbb{0}.$$

Let λ be an inaccessible cardinal such that $\mathbb{B} \in H_\lambda$ and $[\![\exists x\varphi(x, \check{r})]\!]^{H_\lambda^{V^{\mathbb{B}}}} > \mathbb{0}$. Recall that, in this case, $\langle H_\lambda, \in\rangle$ is a model of ZFC. Use Theorem 1.1.19 to produce a countable elementary substructure

$$\langle M, \in\rangle \preceq \langle H_\lambda, \in\rangle$$

such that $\mathbb{B} \in M$ and $r \in M$. Let

$$\pi \colon M \longrightarrow N$$

be the transitive collapse of $\langle M, \in\rangle$ (Mostowski [18]). Notice that $\pi(r) = r$, because $r \subseteq \omega$ and the restriction of π to any transitive subset of M is the identity. If we define $\mathbb{C} = \pi(\mathbb{B})$, then

$$\langle N, \in\rangle \models \text{“}\mathbb{C} \text{ is a complete Boolean algebra”}, \quad \text{and} \quad [\![\exists x\varphi(x, \check{r})]\!]^{N^{\mathbb{C}}} > \mathbb{0}.$$

Take an N-generic filter $G \subset \mathbb{C}$ such that $[\![\exists x\varphi(x, \check{r})]\!]^{N^{\mathbb{C}}} \in G$ (use Lemma 1.2.8). Then

$$\langle N[G], \in\rangle \models \exists x\varphi(x, r),$$

which means that there is $a \in N[G]$ such that $\langle N[G], \in\rangle \models \varphi(a, r)$. Since $N[G]$ is countable and transitive, $N[G] \in H_{\aleph_1}$, hence $a \in H_{\aleph_1}$. Since $\varphi(x, y)$ is Δ_0, we obtain $\langle H_{\aleph_1}, \in\rangle \models \varphi(a, r)$ and finally

$$\langle H_{\aleph_1}, \in\rangle \models \exists x\varphi(x, r).$$

□

Corollary 2.5.19. *Let κ be inaccessible, $\mathbb{B} = \mathrm{Coll}(\aleph_0, {<}\kappa)$, and $U \subset \mathbb{B}$ a κ-good ultrafilter. Let $\varphi(x)$ be a Δ_0 $L_{\omega,\omega}$-formula. Then*

$$\left\langle \check{H}^{\mathbb{B}}_{\aleph_1}/U, \in\right\rangle \models \exists x\varphi(x) \iff \left\langle H^{V^{\mathbb{B}}}_{\aleph_1}/U, \in\right\rangle \models \exists x\varphi(x).$$

Proof. Suppose that

$$\langle \check{H}^{\mathbb{B}}_{\aleph_1}/U, \in \rangle \models \exists x \varphi(x);$$

since $H_{\aleph_1}$ and $\check{H}^{\mathbb{B}}_{\aleph_1}/U$ are elementarily equivalent (see Theorem 2.5.5), also $\langle H_{\aleph_1}, \in \rangle \models \exists x \varphi(x)$. This means that there is $a \in H_{\aleph_1}$ such that $\langle H_{\aleph_1}, \in \rangle \models \varphi(a)$. By the Boolean-valued version of absoluteness of Δ_0 formulas, Remark 2.5.15, we have $[\![\varphi(\check{a})]\!]^{H^{V^{\mathbb{B}}}_{\aleph_1}} = \mathbb{1}$. Hence

$$[\![\exists x \varphi(x)]\!]^{H^{V^{\mathbb{B}}}_{\aleph_1}} \geq [\![\varphi(\check{a})]\!]^{H^{V^{\mathbb{B}}}_{\aleph_1}} = \mathbb{1} \in U$$

and the thesis follows from Theorem 2.1.10.

Conversely, suppose that

$$\langle H^{V^{\mathbb{B}}}_{\aleph_1}/U, \in \rangle \models \exists x \varphi(x).$$

From Theorem 2.1.10 again we have $[\![\exists x \varphi(x)]\!]^{H^{V^{\mathbb{B}}}_{\aleph_1}} \in U$. In particular, this Boolean value is not $\mathbb{0}$, and a fortiori

$$[\![\exists x \varphi(x)]\!]^{V^{\mathbb{B}}} \geq [\![\exists x \varphi(x)]\!]^{H^{V^{\mathbb{B}}}_{\aleph_1}} > \mathbb{0}.$$

We apply Theorem 2.5.18 to get $\langle H_{\aleph_1}, \in \rangle \models \exists x \varphi(x)$, hence

$$\langle \check{H}^{\mathbb{B}}_{\aleph_1}/U, \in \rangle \models \exists x \varphi(x). \qquad \square$$

Corollary 2.5.19 means that $\check{H}^{\mathbb{B}}_{\aleph_1}/U$ and $H^{V^{\mathbb{B}}}_{\aleph_1}/U$ satisfy the same Σ_1 sentences. If we are willing to assume large cardinals, a deep result of Woodin entails that not only $\check{H}^{\mathbb{B}}_{\aleph_1}/U$ and $H^{V^{\mathbb{B}}}_{\aleph_1}/U$ satisfy the same Σ_1 sentences, but also they are elementarily equivalent.

Definition 2.5.20. κ is a *Woodin cardinal* if for every function $f \colon \kappa \to \kappa$ there exists $\delta < \kappa$ such that $f[\delta] \subseteq \delta$ and an elementary embedding $j \colon V \to M$ with $\operatorname{crit}(j) = \delta$ and $V_{j(f)(\delta)} \subseteq M$.

Theorem 2.5.21 (Woodin's Absoluteness [15])**.** *Suppose there is a proper class of Woodin cardinals. Let $\varphi(x)$ be an $L_{\omega,\omega}$-formula, and let $r \subseteq \omega$. Then the following are equivalent:*

- $\langle H_{\aleph_1}, \in \rangle \models \varphi(r)$.
- *There is a complete Boolean algebra $\mathbb{B}$ such that* $[\![\varphi(\check{r})]\!]^{H^{V^{\mathbb{B}}}_{\aleph_1}} > \mathbb{0}$.

Corollary 2.5.22. *Let κ be inaccessible, $\mathbb{B} = \operatorname{Coll}(\aleph_0, {<}\kappa)$, and $U \subset \mathbb{B}$ a κ-good ultrafilter. Suppose there is a proper class of Woodin cardinals. Then $\check{H}^{\mathbb{B}}_{\aleph_1}/U$ and $H^{V^{\mathbb{B}}}_{\aleph_1}/U$ are isomorphic.*

Proof. Combining Theorem 1.1.30 and Theorem 2.5.17, it suffices to prove that $\check{H}^{\mathbb{B}}_{\aleph_1}/U$ and $H^{V^{\mathbb{B}}}_{\aleph_1}/U$ are elementarily equivalent.

Let φ be an $L_{\omega,\omega}$-sentence. If $\langle H^{V^{\mathbb{B}}}_{\aleph_1}/U, \in \rangle \models \varphi$, then $[\![\varphi]\!]^{H^{V^{\mathbb{B}}}_{\aleph_1}} \in U$. In particular, this Boolean value is not $\mathbb{0}$. Then we apply Theorem 2.5.21 to get $\langle H_{\aleph_1}, \in \rangle \models \varphi$, hence $\langle \check{H}^{\mathbb{B}}_{\aleph_1}/U, \in \rangle \models \varphi$.

Conversely, if $\langle H^{V^{\mathbb{B}}}_{\aleph_1}/U, \in \rangle \models \neg\varphi$, then the same argument establishes that $\langle \check{H}^{\mathbb{B}}_{\aleph_1}/U, \in \rangle \models \neg\varphi$. Thus, the two structures are elementarily equivalent $\square$

Chapter 3

Forcing Axioms

3.1 Bounded Forcing Axioms

The last section of the previous chapter has given us a first glimpse of absoluteness results. In this section we shall elaborate on this idea. The connection with forcing axioms is given by the crucial Theorem 3.1.3.

From now on, L will always denote the signature of set theory.

Definition 3.1.1 (Goldstern and Shelah [6]). Let Γ be a class of partially ordered sets and κ a cardinal number. The *bounded forcing axiom* $\mathsf{BFA}_\kappa(\Gamma)$ is the following sentence: for all $\langle P, \leq\rangle \in \Gamma$, if $\{A_\alpha : \alpha < \kappa\} \subseteq \mathcal{P}(\mathrm{RO}(P))$ is a family of maximal antichains in $\mathrm{RO}(P)$ such that $|A_\alpha| \leq \kappa$ for all $\alpha < \kappa$, then there exists a filter G on $\mathrm{RO}(P)$ such that $G \cap A_\alpha \neq \emptyset$ for all $\alpha < \kappa$.

Proposition 3.1.2. *Let Γ be a class of partially ordered sets and κ a cardinal number. Then $\mathsf{FA}_\kappa(\Gamma)$ implies $\mathsf{BFA}_\kappa(\Gamma)$.*

Proof. Let $\langle P, \leq\rangle \in \Gamma$, and $\{A_\alpha : \alpha < \kappa\} \subseteq \mathcal{P}(\mathrm{RO}(P))$ a family of maximal antichains in $\mathrm{RO}(P)$ such that $|A_\alpha| \leq \kappa$ for all $\alpha < \kappa$. For every $\alpha < \kappa$, define

$$D_\alpha = \{d \in P : \text{there exists } a \in A_\alpha \text{ such that } e(d) \leq a\}.$$

We show that every D_α is dense in P. Let $\alpha < \kappa$ and $p \in P$; since A_α is a maximal antichain, we can find some $a \in A$ such that $e(p) \wedge a > \mathbb{0}$. Since $e[P]$ is dense in $\mathrm{RO}(P) \setminus \{\mathbb{0}\}$, there is $d \in P$ such that $e(d) \leq e(p) \wedge a$. Hence, $d \leq p$ and $d \in D_\alpha$, as desired. By $\mathsf{FA}_\kappa(\Gamma)$, there is a filter F on P such that $F \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$. Then

$$G = \{b \in \mathrm{RO}(P) : \text{there exists } p \in F \text{ such that } e(p) \leq b\}$$

is clearly a filter on $\mathrm{RO}(P)$ such that $G \cap A_\alpha \neq \emptyset$ for all $\alpha < \kappa$. □

Theorem 3.1.3 (Bagaria [1]; Stavi and Väänänen [19]). *Let Γ be a class of partially ordered sets and κ a cardinal number such that $\mathrm{cf}(\kappa) > \omega$. Then $\mathsf{BFA}_\kappa(\Gamma)$ is equivalent to the following sentence: for every $\langle P, \leq\rangle \in \Gamma$, every Δ_0 $L_{\omega,\omega}$-formula $\varphi(x, y)$ and every $r \subseteq \kappa$, we have*

$$\langle H_{\kappa^+}, \in\rangle \models \exists x \varphi(x, r) \iff [\![\exists x \varphi(x, \check{r})]\!]^{V^{\mathrm{RO}(P)}} = \mathbb{1}. \tag{3.1}$$

Proof. Assume that $\mathsf{BFA}_\kappa(\Gamma)$ holds. Let $\langle P, \leq \rangle \in \Gamma$; define $\mathbb{B} = \mathrm{RO}(P)$. Let $\varphi(x, y)$ be a Δ_0 formula and $r \subseteq \kappa$; we prove that (3.1) is true. The "$\Longrightarrow$" implication is immediate and does not need $\mathsf{BFA}_\kappa(\Gamma)$: it suffices to adapt the first part of the proof of Corollary 2.5.19.

To prove the "$\Longleftarrow$" implication, let us assume that

$$[\![\exists x \varphi(x, \check{r})]\!]^{V^{\mathbb{B}}} = \mathbb{1}.$$

Let $G \subset P$ be a generic filter, and let $\lambda > \kappa$ be an inaccessible cardinal such that $P \in H_\lambda$ and $H_\lambda[G] \models \exists x \varphi(x, r)$. Use Theorem 1.1.19 to produce an elementary substructure

$$\langle M', \in \rangle \preceq \langle H_\lambda[G], \in \rangle$$

such that $|M'| = \kappa$ and $\kappa \cup \{r\} \subseteq M'$. Let

$$\pi \colon M' \longrightarrow M$$

be the transitive collapse of $\langle M', \in \rangle$ (Mostowski [18]). Notice that $\pi(r) = r$, because $r \subseteq \kappa$ and the restriction of π to any transitive subset of M' is the identity. Since

$$\langle M, \in \rangle \models \exists x \varphi(x, r),$$

we can find $a \in M$ such that $\langle M, \in \rangle \models \varphi(a, r)$.

Define $N = (\kappa + \kappa) \cup \{r\}$ and choose a binary relation $E \subseteq N \times N$, such that $\langle M, \in \rangle$ and $\langle N, E \rangle$ are isomorphic via an isomorphism $f \colon M \to N$ whose restriction to $\kappa \cup \{r\}$ is the identity. The relation E is well founded on N, and we may consider its rank function. More specifically, for all $\alpha \in (\kappa + \kappa) \setminus \kappa$ we define recursively the restricted rank function ρ_α as follows: for every $\kappa \leq x \leq \alpha$,

$$\rho_\alpha(x) = \sup \{\rho_\alpha(y) + 1 : \kappa \leq y \leq \alpha \text{ and } \langle y, x \rangle \in E\}.$$

Let $\dot{\mathfrak{N}}$ be a name for $\langle N, E \rangle$ and σ a name for $f(a)$, so that $\left[\!\!\left[\dot{\mathfrak{N}} \models \varphi(\sigma, \check{r})\right]\!\!\right] = \mathbb{1}$. Furthermore, let $\dot{E}$ be a name for E and, for all $\alpha \in (\kappa + \kappa) \setminus \kappa$, let $\dot{\rho_\alpha}$ be a name for ρ_α.

Now, let

$$T = \{\check{x} : x \in N\} \cup \{\sigma\};$$

for every Δ_0 formula $\theta(x_1, \ldots, x_n)$ and every $\tau_1, \ldots, \tau_n \in T$ we construct a maximal antichain $A_{\theta(\tau_1, \ldots, \tau_n)} \subset \mathbb{B}$ such that $|A_{\theta(\tau_1, \ldots, \tau_n)}| \leq \kappa$. The purpose of $A_{\theta(\tau_1, \ldots, \tau_n)}$ is to *decide* $\dot{\mathfrak{N}} \models \theta(\tau_1, \ldots, \tau_n)$. Here is the definition: if $\theta(x_1, \ldots, x_n)$ is quantifier-free, let

$$A_{\theta(\tau_1, \ldots, \tau_n)} = \left\{ \left[\!\!\left[\dot{\mathfrak{N}} \models \theta(\tau_1, \ldots, \tau_n)\right]\!\!\right], \left[\!\!\left[\dot{\mathfrak{N}} \models \neg\theta(\tau_1, \ldots, \tau_n)\right]\!\!\right] \right\}.$$

If $\theta(x_1, \ldots, x_n)$ is the formula $\exists x_0((x_0 \in x_1) \wedge \psi(x_0, x_1, \ldots, x_n))$, we define $A_{\theta(\tau_1, \ldots, \tau_n)}$ to be any maximal antichain $A \subset \mathbb{B}$ such that

- $|A| \leq \kappa$.
- For all $a \in A$, either $a \leq \left[\!\!\left[\dot{\mathfrak{N}} \models \theta(\tau_1, \ldots, \tau_n)\right]\!\!\right]$ or $a \leq \left[\!\!\left[\dot{\mathfrak{N}} \models \neg\theta(\tau_1, \ldots, \tau_n)\right]\!\!\right]$.
- For all $a \in A$, if $a \leq \left[\!\!\left[\dot{\mathfrak{N}} \models \theta(\tau_1, \ldots, \tau_n)\right]\!\!\right]$ then there is $\tau_0 \in T$ such that $a \leq [\![\psi(\tau_0, \tau_1, \ldots, \tau_n)]\!]$.

Similarly, for all $\alpha \in (\kappa+\kappa)\setminus\kappa$ and every $\kappa \le \beta \le \alpha$, let $A_{\alpha,\beta}$ be a maximal antichain, with $|A_{\alpha,\beta}| \le \kappa$, deciding the value of $\dot{\rho_\alpha}(\beta)$.

Finally, let $A_\sigma \subseteq \{[\![\sigma = \check{x}]\!] : x \in N\}$ be a maximal antichain.

As a consequence of $\mathsf{BFA}_\kappa(\Gamma)$, there is a filter F on $\mathbb{B}$ such that F has non-empty intersection with every $A_{\theta(\tau_1,\dots,\tau_n)}$, every $A_{\alpha,\beta}$, and with A_σ. For every $x \in N$, define $\check{x}^F = x$. Let σ^F be the unique $x \in N$ such that $[\![\sigma = \check{x}]\!] \in F$. Thus, for every $\tau \in T$ we have defined $\tau^F \in N$. Now we define a binary relation $\dot{E}^F \subseteq N \times N$ in this way:

$$\dot{E}^F = \left\{\left\langle \tau_1^F, \tau_2^F\right\rangle : \tau_1, \tau_2 \in T \text{ and } \left[\!\!\left[\langle \tau_1, \tau_2\rangle \in \dot{E}\right]\!\!\right] \in F\right\}.$$

This means that, for all $\tau_1, \tau_2 \in T$,

$$\left\langle N, \dot{E}^F\right\rangle \models \left(\tau_1^F \in \tau_2^F\right) \iff \left[\!\!\left[\dot{\mathfrak{N}} \models (\tau_1 \in \tau_2)\right]\!\!\right] \in F. \tag{3.2}$$

Actually, an easy induction on the complexity of formulas shows that (3.2) is true not only for atomic formulas, but also for arbitrary Δ_0 formulas. In other words, for every Δ_0 formula $\theta(x_1,\dots,x_n)$ and every $\tau_1,\dots,\tau_n \in T$ we have

$$\left\langle N, \dot{E}^F\right\rangle \models \theta\left(\tau_1^F,\dots,\tau_n^F\right) \iff \left[\!\!\left[\dot{\mathfrak{N}} \models \theta(\tau_1,\dots,\tau_n)\right]\!\!\right] \in F;$$

in particular, we conclude that $\left\langle N, \dot{E}^F\right\rangle \models \varphi\left(\sigma^F, r\right)$.

Now, we would like to apply Mostowski's collapsing function to the structure $\left\langle N, \dot{E}^F\right\rangle$, but first we have to show that the relation $\dot{E}^F$ is extensional and well founded on N. $\dot{E}^F$ is extensional simply because $\forall z(z \in x \to z \in y) \wedge \forall w(w \in y \to w \in x)$ is Δ_0. Moreover, suppose by contradiction that $\dot{E}^F$ is not well founded on N. Clearly, $\dot{E}^F$ cannot be ill-founded on $\kappa \cup \{A\}$, because on this subset $\dot{E}^F$ coincides with the well founded relation $\in$. Then, also using the fact that $\mathrm{cf}(\kappa) > \omega$, there must be some $\alpha \in (\kappa+\kappa)\setminus\kappa$ such that $\dot{E}^F$ is ill-founded on the interval $[\kappa,\alpha]$. But, using the fact that $F \cap A_{\alpha,\beta}$ for all $\kappa \le \beta \le \alpha$, we have a rank function

$$\dot{\rho}_\alpha^F = \{\langle \beta, \gamma\rangle : \kappa \le \beta \le \alpha \text{ and } [\![\dot{\rho_\alpha}(\beta) = \gamma]\!] \in F\}$$

witnessing the well-foundedness of $\dot{E}^F$ on $[\kappa,\alpha]$, a contradiction.

That said, let

$$\pi' \colon N \longrightarrow N'$$

be the transitive collapse of $\left\langle N, \dot{E}^F\right\rangle$. Hence, $\langle N', \in\rangle \models \varphi\left(\pi'\left(\sigma^F\right), r\right)$. Since $\varphi(x,y)$ is Δ_0 and $\pi'\left(\sigma^F\right) \in H_{\kappa^+}$, we obtain that $\langle H_{\kappa^+}, \in\rangle \models \varphi\left(\pi'\left(\sigma^F\right), r\right)$ and, finally,

$$\langle H_{\kappa^+}, \in\rangle \models \exists x \varphi(x, r).$$

This concludes the first part of the proof.

Conversely, assume that (3.1) holds; we show that $\mathsf{BFA}_\kappa(\Gamma)$ is true. Let $\langle P, \le\rangle \in \Gamma$; define $\mathbb{B} = \mathrm{RO}(P)$. Let $\{A_\alpha : \alpha < \kappa\} \subseteq \mathcal{P}(\mathbb{B})$ be a family of maximal antichains in $\mathbb{B}$ such that $|A_\alpha| \le \kappa$ for all $\alpha < \kappa$.

Consider the structure $\left\langle \mathbb{B}, \le^{\mathbb{B}}\right\rangle$, where $\le^{\mathbb{B}}$ is the usual partial order relation on $\mathbb{B}$. We use Theorem 1.1.19 again to produce an elementary substructure

$$\left\langle \mathbb{C}, \le^{\mathbb{C}}\right\rangle \preceq \left\langle \mathbb{B}, \le^{\mathbb{B}}\right\rangle$$

such that $|\mathbb{C}| = \kappa$ and $\bigcup_{\alpha<\kappa} A_\alpha \subseteq \mathbb{C}$. Note that the relation of incompatibility is preserved; in particular, every A_α is still a maximal antichain in $\mathbb{C}$. Now, choose a binary relation $\le^\kappa \subseteq \kappa \times \kappa$ such that $\left\langle \mathbb{C}, \le^{\mathbb{C}}\right\rangle$ and $\left\langle \kappa, \le^\kappa\right\rangle$ are isomorphic, via an isomorphism $f \colon \mathbb{C} \to \kappa$.

If $G \subset P$ is a V-generic filter, then we have

$$V[G] \models \text{“there exists a filter } F \text{ on } \kappa \text{ such that } F \cap f[A_\alpha] \neq \emptyset \text{ for all } \alpha < \kappa\text{”}. \tag{3.3}$$

It is easy to see that we can translate “there exists a filter F on κ” as

$$\exists F \varphi(F, \kappa, \leq^\kappa),$$

where φ is an appropriate Δ_0 formula with parameters κ and $\leq^\kappa$. Also, if we define

$$r = \{\langle \alpha, \beta \rangle \in \kappa \times \kappa : \beta \in f[A_\alpha]\},$$

then we can translate “$F \cap f[A_\alpha] \neq \emptyset$ for all $\alpha < \kappa$” as

$$\forall \alpha \in \kappa (\exists \beta \in F(\langle \alpha, \beta \rangle \in r)).$$

Again, this is a Δ_0 formula with parameter r. Both $\leq^\kappa$ and r belong to H_{κ^+}; as already observed, they can be coded as elements of $\mathcal{P}(\kappa)$. Note that the coding function is Σ_1-definable.

In conclusion, the right-hand side of (3.3) is equivalent to a Σ_1 formula with parameters in $\mathcal{P}(\kappa)$. Thus, by (3.1), it is true in V. Hence, $f^{-1}[F]$ is a filter on $\mathbb{C}$ such that $f^{-1}[F] \cap A_\alpha \neq \emptyset$ for all $\alpha < \kappa$. Finally,

$$F' = \left\{ b \in \mathbb{B} : \text{there exists } c \in f^{-1}[F] \text{ such that } c \leq b \right\}$$

is a filter on $\mathbb{B}$ such that $F' \cap A_\alpha \neq \emptyset$ for all $\alpha < \kappa$. □

3.2 Some Remarks on the Axiom of Choice

In this section, we shall present the axiom of choice as a global forcing axiom; this idea appears in Todorčević [21]. A fundamental reference for anything related to the axiom of choice is Jech [11].

Definition 3.2.1. The *Axiom of Choice* AC is the following sentence: for every set S such that $\emptyset \notin S$, there is a function f on S such that $f(Y) \in Y$ for all $Y \in S$.

Definition 3.2.2. Let κ be an infinite cardinal. The *principle of Dependent Choices* DC_κ is the following sentence: for every non-empty set X and every function $F\colon X^{<\kappa} \to \mathcal{P}(X) \setminus \{\emptyset\}$, there exists $g\colon \kappa \to X$ such that $g(\alpha) \in F(g \restriction \alpha)$ for all $\alpha < \kappa$.

Theorem 3.2.3 (Lévy [16])**.** AC *is equivalent to* $\forall \kappa\, \mathsf{DC}_\kappa$ *modulo* ZF.

Proof. Assume AC. Let κ be an infinite cardinal, X a non-empty set and $F\colon X^{<\kappa} \to \mathcal{P}(X) \setminus \{\emptyset\}$. By AC, there is a function f on $\mathcal{P}(X) \setminus \{\emptyset\}$ such that $F(Y) \in Y$ for all $Y \in \mathcal{P}(X) \setminus \{\emptyset\}$. We can define $g\colon \kappa \to X$ by recursion: for all $\alpha < \kappa$, let $g(\alpha) = f(F(g \restriction \alpha))$.

We turn to the converse implication. The first step is to prove that DC_κ implies that, for all X, either $|X| \leq \kappa$ or $|X| \geq \kappa$. Suppose $|X| \nleq \kappa$; we can define

$$\begin{aligned} F\colon X^{<\kappa} &\longrightarrow \mathcal{P}(X) \setminus \{\emptyset\} \\ s &\longmapsto X \setminus \operatorname{ran}(f) \end{aligned}.$$

Apply DC_κ to obtain a function $g\colon \kappa \to X$ such that $g(\alpha) \in X \setminus \operatorname{ran}(g \restriction \alpha)$. This means that g is injective, and $\kappa \leq |X|$, as desired.

Now assume $\forall\kappa\,\mathsf{DC}_\kappa$. Then the following must be true:

$$\text{for every } \kappa \text{ and for every } X, \text{ either } |X| \leq \kappa \text{ or } |X| \geq \kappa. \tag{3.4}$$

It easy to see that (3.4) implies that every set can be well ordered, hence AC: for every set X, consider its *Hartogs number* $h(X)$. A classic result (Hartogs [8]) is that $h(X)$ is a cardinal number and $h(X) \not\leq |X|$. Thus $|X| \leq h(X)$, and X can be well ordered. □

Let Γ_κ be the class of all $<\!\kappa$-closed partially ordered sets.

Theorem 3.2.4. *Let κ be an infinite cardinal. Then DC_κ is equivalent to $\mathsf{FA}_\kappa(\Gamma_\kappa)$ modulo the theory $\mathsf{ZF} + \forall\lambda(\lambda < \kappa \to \mathsf{DC}_\lambda)$.*

Proof. Assume DC_κ; we prove (in ZF) that $\mathsf{FA}_\kappa(\Gamma_\kappa)$ holds. Let $\langle P, \leq\rangle$ be a $<\!\kappa$-closed partially ordered set, and $\{D_\alpha : \alpha < \kappa\} \subseteq \mathcal{P}(P)$ a family of dense subsets. We want to define a function $F\colon P^{<\kappa} \to \mathcal{P}(P) \setminus \{\emptyset\}$. Let $\alpha < \kappa$ and $\langle p_\beta : \beta < \alpha\rangle \in P^{<\kappa}$. If $\{p_\beta : \beta < \alpha\}$ is not a chain, for completeness define $F(\langle p_\beta : \beta < \alpha\rangle)$ arbitrarily (e.g. $= P$). Otherwise, if $\{p_\beta : \beta < \alpha\}$ is a chain, let

$$F(\langle p_\beta : \beta < \alpha\rangle) = \{d \in D_\alpha : d \leq p_\beta \text{ for all } \beta < \alpha\},$$

which is non-empty because $\langle P, \leq\rangle$ is $<\!\kappa$-closed and D_α is dense. By DC_κ, we find $g\colon \kappa \to P$ such that $g(\alpha) \in F(g \restriction \alpha)$ for all $\alpha < \kappa$. This means that $g(\alpha) \in D_\alpha$ and $g(\alpha) \leq g(\beta)$ for all $\beta < \alpha < \kappa$, as can be easily proved by induction. Then

$$G = \{p \in P : \text{there exists } \alpha < \kappa \text{ such that } g(\alpha) \leq p\}$$

is a filter on P, such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$.

Conversely, assume $\mathsf{FA}_\kappa(\Gamma_\kappa)$ and $\forall\lambda(\lambda < \kappa \to \mathsf{DC}_\lambda)$; we prove (in ZF) that DC_κ holds. We distinguish two cases, according to whether κ is regular or singular.

Suppose κ is regular. Let X be a non-empty set and $F\colon X^{<\kappa} \to \mathcal{P}(X) \setminus \{\emptyset\}$. Define the partially ordered set

$$P = \left\{s \in X^{<\kappa} : \text{for all } \alpha \in \mathrm{dom}(s),\ s(\alpha) \in F(s \restriction \alpha)\right\},$$

with $s \leq t$ if and only if $t \subseteq s$. Let $\lambda < \kappa$ and let $s_0 \geq s_1 \geq \cdots \geq s_\alpha \geq \ldots$, for $\alpha < \lambda$, be a chain in P. Then $\bigcup_{\alpha<\lambda} s_\alpha$ is clearly a lower bound for the chain. Since κ is regular, we have $\bigcup_{\alpha<\lambda} s_\alpha \in P$ and so P is $<\!\kappa$-closed. For every $\alpha < \kappa$, define

$$D_\alpha = \{s \in P : \alpha \in \mathrm{dom}(s)\},$$

and note that D_α is dense in P. Using $\mathsf{FA}_\kappa(\Gamma_\kappa)$, there exists a filter $G \subset P$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$. Then $g = \bigcup G$ is a function $g\colon \kappa \to X$ such that $g(\alpha) \in F(g \restriction \alpha)$ for all $\alpha < \kappa$. Note that, in this case, we did not use the assumption $\forall\lambda(\lambda < \kappa \to \mathsf{DC}_\lambda)$.

Suppose κ is singular. This means that there is an increasing sequence $\langle \alpha_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ such that $\kappa = \sup\{\alpha_\xi : \xi < \mathrm{cf}(\kappa)\}$. Let X be a non-empty set and $F\colon X^{<\kappa} \to \mathcal{P}(X) \setminus \{\emptyset\}$. Define

$$T = \bigcup_{\xi < \mathrm{cf}(\kappa)} X^{\alpha_\xi}.$$

For every $t \in T^{<\mathrm{cf}(\kappa)}$, let $s_t \in X^{<\kappa}$ denote the sequence obtained by concatenation of the values of t, that is $s_t = t(0)^\frown t(1)^\frown \ldots$. Define a function $G\colon T^{<\mathrm{cf}(\kappa)} \to \mathcal{P}(T) \setminus \{\emptyset\}$ as follows: if $t \in T^\xi$, then

$$G(t) = \{z \in X^{\alpha_\xi} : z_\eta \in F(s_t{}^\frown(z \restriction \eta)) \text{ for all } \eta < \alpha_\xi\}.$$

Observe that $G(t)$ is always non-empty, because DC_λ holds for all $\lambda < \kappa$. Now, by $\mathsf{DC}_{\mathrm{cf}(\kappa)}$ applied to G, there is a function $f\colon \mathrm{cf}(\kappa) \to T$ such that $f(\xi) \in G(f \restriction \xi)$ for all $\xi < \mathrm{cf}(\kappa)$. More explicitly, $f(\xi)$ is a sequence with the property that

$$f(\xi)_\eta \in F(s_{f\restriction\xi}{}^\frown(f(\xi) \restriction \eta)) \text{ for all } \eta < \alpha_\xi. \tag{3.5}$$

Finally, let g be the concatenation of the sequences $f(\xi)$, for $\xi < \mathrm{cf}(\kappa)$. Then $g\colon \kappa \to X$ and, by (3.5), $g(\alpha) \in F(g \restriction \alpha)$ for all $\alpha < \kappa$. Note that, in this case, we did not use the assumption $\mathsf{FA}_\kappa(\Gamma_\kappa)$. □

Combining Theorem 3.2.3 and Theorem 3.2.4, we obtain this interesting result.

Corollary 3.2.5 (Todorčević). AC *is equivalent to* $\forall\kappa\, \mathsf{FA}_\kappa(\Gamma_\kappa)$ *modulo* ZF.

3.3 Measurability and Large Cardinals

We conclude with some brief remarks on large cardinals. A comprehensive reference on this vast subject is [12].

Definition 3.3.1. Let κ be a cardinal number. We say that κ is *measurable* if $\kappa > \aleph_0$ and there is a κ-complete nonprincipal ultrafilter on $\mathcal{P}(\kappa)$.

In the definition of $\mathsf{BFA}_\kappa(\Gamma)$, if we drop the condition that every maximal antichain must be of size $\leq \kappa$, we obtain $\mathsf{FA}_\kappa(\Gamma)$. On the other hand, if we drop the condition that the family of maximal antichains must be of size $\leq \kappa$, we obtain essentially the notion of measurability:

Proposition 3.3.2. *Let* $\kappa > \aleph_0$ *be a cardinal number. The following are equivalent:*

1. κ *is measurable.*
2. *If* $\{A_i : i \in I\} \subseteq \mathcal{P}(\mathcal{P}(\kappa))$ *is a family of maximal antichains in* $\mathcal{P}(\kappa)$ *such that* $|A_i| < \kappa$ *for all* $i \in I$*, then there exists a nonprincipal ultrafilter* G *on* $\mathcal{P}(\kappa)$ *such that* $G \cap A_i \neq \emptyset$ *for every* $i \in I$.

Proof. Before proving the proposition, we remark that a maximal antichain in $\mathcal{P}(\kappa)$ is simply a partition of κ.

Suppose κ is measurable; let $\{A_i : i \in I\} \subseteq \mathcal{P}(\mathcal{P}(\kappa))$ be a family of maximal antichains such that $|A_i| < \kappa$ for all $i \in I$. Let $U \subset \mathcal{P}(\kappa)$ be a κ-complete nonprincipal ultrafilter on $\mathcal{P}(\kappa)$; we show that $U \cap A_i \neq \emptyset$ for all $i \in I$. Assume by contradiction that there is $i \in I$ such that $U \cap A_i = \emptyset$. Since U is an ultrafilter, we have $\kappa \setminus a \in U$ for all $a \in A_i$. By κ-completeness of U, we obtain

$$\bigcap_{a \in A_i} (\kappa \setminus a) \in U;$$

but this is impossible, for

$$\bigcap_{a \in A_i} (\kappa \setminus a) = \kappa \setminus \bigcup A_i = \kappa \setminus \kappa = \emptyset.$$

Conversely, suppose (2) is true. We apply (2) to the family of all maximal antichains A in $\mathcal{P}(\kappa)$ such that $|A| < \kappa$, and we find a nonprincipal ultrafilter U on $\mathcal{P}(\kappa)$ having nonempty intersection with every such antichain. We show that U is κ-complete. Assume by contradiction that there exist $\lambda < \kappa$ and $\{b_\alpha : \alpha < \lambda\} \subseteq U$ such that $\bigcap_{\alpha<\lambda} b_\alpha \notin U$. Define, for every $\alpha < \lambda$,

$$a_\alpha = (\kappa \setminus a_\alpha) \cap \bigcap_{\beta<\alpha} a_\beta.$$

Observe that κ can be partitioned as

$$\kappa = \bigcup_{\alpha<\lambda} a_\alpha \cup \bigcap_{\alpha<\lambda} b_\alpha,$$

therefore some set in this partition must belong to U. Since $\bigcap_{\alpha<\lambda} b_\alpha \notin U$ we conclude that there exists $\alpha < \lambda$ such that $a_\alpha \in U$. This implies $\emptyset = a_\alpha \cap b_\alpha \in U$, a contradiction. □

The concept of measurability can be further generalized in at least two natural directions. Perhaps the first that comes to mind is the following:

Definition 3.3.3. Let κ be a cardinal number. We say that κ is *strongly compact* if $\kappa > \aleph_0$ and, for all sets X, every κ-complete filter on $\mathcal{P}(X)$ can be extended to a κ-complete ultrafilter on $\mathcal{P}(X)$.

Remark 3.3.4. Every strongly compact cardinal is measurable, for any ultrafilter U on $\mathcal{P}(\kappa)$ extending the filter

$$\{X \subseteq \kappa : |\kappa \setminus X| < \kappa\}$$

is necessarily nonprincipal.

On the other hand, let us consider this theorem:

Theorem 3.3.5 (Keisler and Tarski [13])**.** *Let κ be a cardinal number. The following are equivalent:*

1. *κ is measurable.*
2. *There is an elementary embedding $j\colon V \to M$ such that* $\mathrm{crit}(j) = \kappa$ *and $M^\kappa \subseteq M$.*

Theorem 3.3.5 suggests another natural generalization of measurability:

Definition 3.3.6. Let $\kappa \leq \lambda$ be cardinal numbers. We say that κ is λ-*supercompact* if there is an elementary embedding $j\colon V \to M$ such that:

- $\mathrm{crit}(j) = \kappa$,
- $j(\kappa) > \lambda$,
- $M^\lambda \subseteq M$.

Furthermore, we say that κ is *supercompact* if κ is λ-supercompact for every $\lambda \geq \kappa$.

The relation between forcing, elementary embeddings and large cardinals is one of the central themes in set theory; for example, see [5]. The considerations presented in this section are to be intended as a brief survey, because a deeper analysis would constitute an entire new work.

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