UNIVERSITÀ DEGLI STUDI DI PISA

Corso di Laurea Magistrale in Matematica

Tesi di Laurea

CHORD CATALOGS AND ESTRADA CLASSES: PARTIALLY ORDERED SET APPROACH.

Relatori:
FRANCESCA ACQUISTAPACE
MORENO ANDREATTA

Candidato:
GIULIO MASETTI

ANNO ACCADEMICO 2013-2014

Ringraziamenti

Il mio impegno nello scrivere sarebbe stato vano se non avessi incontrato l'aiuto di molte persone. Famiglia, amici, colleghi e relatori hanno sostenuto ed incoraggiato il mio lavoro. In loro ho trovato interlocutori sinceri e pronti allo scambio di idee. Un Grazie profondo voglio rivolgerlo ai miei genitori ed a Francesca, la mia ragazza, per l'immensa forza che mi hanno donato. Francesca Acquistapace e Moreno Andreatta mi hanno insegnato molto più di quanto avrei immaginato. Fabrizio Broglia per primo mi ha introdotto al mondo della matematica musicale, mi ha poi consigliato e sostenuto durante tutto il percorso. Giovanni Gaiffi mi ha aiutato a trovare il filo conduttore della tesi, aprendo la mia visuale in molte direzioni meravigliose. Cristiano Bocci, Mattia Bergomi, Jean-Louis Giavitto e Louis Bigo mi hanno donato interi giorni di lavoro, consentendomi di verificare ed approfondire i risultati esposti. Gabriele Caselli ha discusso con me la musica contenuta nella tesi. La tesi di Giulia Fidanza è stata il punto di partenza del mio lavoro. Tito Tonietti mi ha regalato questioni su cui riflettere. Giorgio Mossa ed Oscar Papini mi hanno consigliato riguardo questioni computazionali e di stile. L'istituto Ircam mi ha accolto ed un Grazie particolare voglio rivolgerlo a Gérard Assayag e Sylvie Benoit.

Introduction

The relationship between Mathematics and Music has an important role in the evolution of modern science [Ton14]. Math-music is involved in Physics, Physiology, Psychology and many other branches of human knowledge. Every attempt to define a mathematical framework that takes care of some physical, physiological, psychological and cultural aspects of Music has to deal with arbitrary philosophical trends. During the history we can see several *foundational* approaches, many languages and different points of view, *e.g.* discrete vs continuous, and the aim of this thesis is not to support one of them or to discuss their interplay in the analysis of some musical pieces.

We want to describe here one classical field of theoretical research and practical composition, the *Musical set theory*.

We will consider the finite set

$$X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

calling its elements *notes* and its subsets *chords*. The cardinality of a chord is called *length*.

The Musical set theory is the abstraction of musical choices of great composers as Babbitt, Xenakis and Vieru [And03] but we will describe this theory as the application of Algebra and Combinatorics to the study of chords, when the set of notes X is an arbitrary finite set.

Two chords are *equivalent* if they are equal up to some *symmetries*. We will formally describe this concept with *actions* of *groups* over notes and then with actions of groups over chords. We will define *catalog* the set of *equivalence classes* according to a specific group action over chords. The two most recent and preminent fundational works that we selected are: *Generalized Musical Intervals and Transformations* [Lew87] and *The Topos of Music: Geometric Logic of Concepts, Theory, and Performance* [Maz02]. The central role of group actions and *categories* is the backbone of this thesis, in particular of Chapter one, but we are interested as well in *partially ordered sets*, defined in a classical way and also with categories, *Pólya theory*, described in Chapter three and used to enumerate cardinalities of classes whose chords have fixed length, and *unimodal sequences*, described in Chapter four. Lewin's work does not include the property of a chord to be a sub-chord of another in the heart of the Musical set theory, so we will analyze it, wandering about the relationship between this theory and Logic.

Standard and non-standard topics of the theory will be treated: catalogs by *transposition* and *inverse-transposition*, the *intervallic structure* of a chord (related to the concept of *multiset* and developed by Allen Forte, see definition 40), the catalog of Julio Estrada (definition 39) and the *poset of bi-chord decompositions* (definition 42). We will apply results of Algebraic Combinatorics [Sta13] and Combinatorial Algebraic Topology [Koz08], [Wac07] to answer these questions:

1. Does exist a standard mathematical object that characterizes classical catalogs of chords?

2. Does exist a standard mathematical object that characterizes the non-standard catalog of Estrada?

3. How can we calculate the number of classes in a catalog with a fixed length?

4. Why the number of chords in a classical catalog with fixed length does increase and then decrease with a symmetric movement?

5. The number of chords in the Estrada catalog with a fixed length does increase and then decrease *but* without symmetric movement. How can we explain this phenomenon?

To answer 1) and 2) we will characterize standard and non-standard catalogs of chords as quotient of partially ordered sets by a group action with definitions 29 and 33. We will answer 3) with an application of theorem 11.

To answer 4) we will descibe theorems 15 and 21: for all n and for all subgroup G of the permutation group $\mathbb{S}_n$, the quotient poset of the Boolean lattice over a finite set of n elements by the group G is ranked and its ranks have cardinalities that increase and then decrease with a symmetric movement. This means that the action of any subgroup of $\mathbb{S}_n$ over a finite set of n notes always induces an action over the chords such that in the catalog the number of chords with a fixed length increase and then decrease with a symeetric movement.

We will present 5) as an open problem for the general case of Estrada catalog over n notes because the previous argument does not fit this non-standard framework. Nevertheless we have numerical results (see remark 13) that suggest an explaination for $n = 12$.

Most of these questions have been discussed during a stage at Ircam (Institut de Recherche et Coordination Acoustique/Musique) in Paris.

Contents

Chapter 1

Posets and their properties

In this chapter we will follow [Koz08] for the definitions in *category theory* and maintain standard definitions about posets (as in [Gro08]).

We will then define the poset of subset and the partitions of a set, our main objects.

1.1 Categories

Definition 1. A *category* $\mathcal{C}$ is a pair of classes $(\mathcal{O}, \mathcal{M})$ satisfying certain properties. The class $\mathcal{O}$ is called the class of *objects*, and the class $\mathcal{M}$ is called the class of *morphisms*. The class $\mathcal{M}$ is actually a disjoint union of sets $\mathcal{M}(a, b)$, for every pair $a, b \in \mathcal{O}$, with a given composition rule

$$\mathcal{M}(a, b) \times \mathcal{M}(b, c) \to \mathcal{M}(a, c) \text{ where } (m_1, m_2) \mapsto m_2 \circ m_1$$

This composition rule is required to satisfy the following axioms:

- composition is associative, when defined.
- for each $a \in \mathcal{O}$ there exists a (necessary unique) identity morphism $1_a \in \mathcal{M}(a, a)$ s.t. $1_a \circ f = f$ and $g \circ 1_a = g$, whenever the compositions are defined.

For a morphism $m : a \to b$ we call $a = dom(m)$ and $b = cod(m)$.

The first example is the category associated to a group.

Definition 2. Let G be a group. The category associated to G, called $\mathcal{C}G$, is defined as follows:

- The set $\mathcal{O}$ of objects consists of one element, the identity of the group, called o.

- The set $\mathcal{M}$ of morphisms $\mathcal{M}(o, o)$ is indexed by the elements of G, with the composition defined by the group multiplication.

Definition 3. Given two categories C_1, C_2, a *functor* $F : C_1 \mapsto C_2$ is a pair of maps $F_{\mathcal{O}} : \mathcal{O}(C_1) \mapsto \mathcal{O}(C_2), F_{\mathcal{M}} : \mathcal{M}(C_1) \mapsto \mathcal{M}(C_2)$ such that

- $F_{\mathcal{M}}(\mathcal{M}(x, y)) \subset \mathcal{M}(F_{\mathcal{O}}(x), F_{\mathcal{O}}(y))$, for all $x, y \in \mathcal{O}(C_1)$
- $F_{\mathcal{M}}(id_a) = id_{F_{\mathcal{O}}(a)}$, for all $a \in \mathcal{O}(C_1)$
- $F_{\mathcal{M}}(m_1 \circ m_2) = F_{\mathcal{M}}(m_1) \circ F_{\mathcal{M}}(m_2)$

Example 1. Let G, H be two groups. Given an homomorphism $f : G \to H$ we can consider the corresponding functor $F : \mathcal{C}G \to \mathcal{C}H$ where $F_{\mathcal{O}}(o_G) = o_H$ and $F_{\mathcal{M}}(g) = f(g)$.

A simple example is the following homomorphism between $G = \mathbb{Z}/2\mathbb{Z}$ and $H = \mathbb{Z}/2\mathbb{Z}$

$$\begin{aligned} f : \mathbb{Z}/2\mathbb{Z} &\to \mathbb{Z}/2\mathbb{Z} \\ \bar{0} &\mapsto \bar{0} \\ \bar{1} &\mapsto \bar{0} \end{aligned}$$

that corresponds to the functor

$$\begin{aligned} F : \mathcal{C}\mathbb{Z}/2\mathbb{Z} &\to \mathcal{C}\mathbb{Z}/2\mathbb{Z} \\ o_G &\mapsto o_H \end{aligned}$$

sending the morphism labelled as $\bar{0}$ in G to the morphism labelled as $\bar{0}$ in H and the morphism labelled $\bar{1}$ in G to the morphism labelled $\bar{0}$ in H.

$$\begin{tikzcd} o_H \arrow[loop left, "\bar{0}"] \arrow[loop right, dotted, "\bar{1}"] \end{tikzcd}$$

Here we dotted the morphism labelled $\bar{1}$ in H because it is not in the image of the functor F.

A powerful application of the categorial approach is the formalization of the class of all the categories.

Definition 4. We call $\mathcal{C}$ a *small category* if $\mathcal{O}(\mathcal{C})$ is a set. We define **Cat** the category whose objects are all the small categories and morphisms are all the functors between them.

We are interested only in *small categories*, so we will consider $\mathcal{O}$ that is a set. We will consider also particular kinds of $\mathcal{M}$, that are sets, so we don't need the most general context of classes.

Definition 5. Let $\mathcal{C}$ a category and $a, b \in \mathcal{O}(\mathcal{C})$. The *product* of a and b is an object $c = a \Pi b$ together with morphisms $\alpha : c \to a, \beta : c \to b$ satisfying the following universal property: for every object d and morphisms $\tilde{\alpha} : d \to a, \tilde{\beta} : d \to b$, there exists a unique morphism $\gamma : d \to c$ such that the diagram commutes

$$\begin{tikzcd} a & c \arrow[l, "\alpha"'] \arrow[r, "\beta"] & b \\ & d \arrow[ul, "\tilde{\alpha}"] \arrow[u, "\gamma"] \arrow[ur, "\tilde{\beta}"'] & \end{tikzcd}$$

Definition 6. Let $\mathcal{C}$ a category and $a, b \in \mathcal{O}(\mathcal{C})$. The *coproduct* of a and b is an object $c = a \amalg b$ together with morphisms $\alpha : a \to c, \beta : b \to c$ (called the structure morphisms) satisfying the following universal property: for every object d and morphisms $\tilde{\alpha} : a \to d, \tilde{\beta} : b \to d$, there exists a unique morphism $\gamma : c \to d$ such that the diagram commutes

$$\begin{tikzcd} a \arrow[r, "\alpha"] \arrow[dr, "\tilde{\alpha}"'] & c \arrow[d, "\gamma"] & b \arrow[l, "\beta"'] \arrow[dl, "\tilde{\beta}"] \\ & d & \end{tikzcd}$$

Remark 1. From now on, given a functor F, we will draw *only* objects and morphism that are in the image of F. We will draw with dotted lines morphisms that are out of the image of F.

Definition 7. We will call a *sink of the functor* $F : \mathcal{C}_1 \to \mathcal{C}_2$ an object L of $\mathcal{C}_2$, together with a collection of morphisms pointing from the object in the diagram to the object L, *i.e.* for all $a \in \mathcal{O}(\mathcal{C}_1)$ we consider morphisms $\lambda_a : F(a) \to L$ that commute with the other morphisms in the diagram (for all $m \in \mathcal{M}_{\mathcal{C}_1}(a_1, a_2)$ we have $\lambda_{a_2} \circ F(m) = \lambda_{a_1}$).

Example 2. Consider a category $\mathcal{C}_1$ with six objects and a functor $F : \mathcal{C}_1 \to \mathcal{C}_2$. In the following picture we label only the morphisms that commute well.

$$\begin{tikzcd} F(o_1) \arrow[r] \arrow[dr] \arrow[ddr, dotted] & F(o_2) \arrow[dd, dotted, bend left] & F(o_3) \arrow[dl] \arrow[d] \arrow[ddl, dotted] \\ F(o_4) \arrow[r] \arrow[dr, dotted, "\lambda_4"] & F(o_5) \arrow[d, dotted, "\lambda_5"] & F(o_6) \arrow[dl, dotted, "\lambda_6"'] \\ & L & \end{tikzcd}$$

Definition 8. Given a functor $F : \mathcal{C}_1 \to \mathcal{C}_2$, we call a sink of F, written as $(L, \{\lambda_a\}_{a \in \mathcal{O}(\mathcal{C}_1)})$, its *colimit* if it is *universal* in the following sense: for any other sink $(\tilde{L}, \{\tilde{\lambda}_a\}_{a \in \mathcal{O}(\mathcal{C}_1)})$ there exists a unique morphism $\phi : L \to \tilde{L}$ s.t. $\tilde{\lambda}_a = \phi \circ \lambda_a$, for all $a \in \mathcal{O}(\mathcal{C}_1)$. We will write $L = colim(X)$

$$
\begin{tikzcd}
F(o_1) \arrow[r] \arrow[d] \arrow[dr, dotted] \arrow[ddrr, dotted] & F(o_2) \arrow[d, dotted] \arrow[ddr, dotted] & \\
F(o_3) \arrow[r, dotted] \arrow[drr, dotted] & L \arrow[dr, squiggly, "\phi"] & \\
 & & \tilde{L}
\end{tikzcd}
$$

Definition 9. A small category $\mathcal{C}$ is called *acyclic* if only identity morphisms have inverses, and any morphism for an object to itself is an identity, *i.e.* for any pair of distinct objects $x, y \in \mathcal{O}(\mathcal{C})$ at most one of the sets $\mathcal{M}_{\mathcal{C}}(x, y)$ and $\mathcal{M}_{\mathcal{C}}(y, x)$ is non-empty.

In the next section we will study the main example of acyclic category.

1.2 Posets

From now on, we will consider only finite sets.

Definition 10 (Standard definition of a poset)**.** Let P be a set. A *partial order relation* is a binary relation $R \subset P \times P$ such that $\forall x, y, z \in P$ we have

- $(x, x) \in R$, that is *reflexivity*
- $(x, y) \in R$ and $(y, x) \in R$ iff $x = y$, that is *anti-symmetry*
- $(x, y) \in R$ and $(y, z) \in R$ iff $(x, z) \in R$, that is *transitivity*

The set P in which we consider the order relation is called *partially ordered set*, poset for short. In the following we will use a different notation: $x \leq y$ instead of $(x, y) \in R$. We will write P for *the set* P, and $(P, \leq)$ for *the partially ordered set* P. We will also write $x < y$ when $x \leq y$ and $x \neq y$.

Remark 2. A partially ordered set is not a totally ordered set. We lost that every two elements are *comparable*, *i.e.* $p, q \in P$ does not implies that $p \leq q$ or $q \leq p$, as in $\mathbb{Z}$ with the usual order.

Example 3. Consider the set of notes $X = \{C, D, E, G, A\}$ in anglo-saxon notation and call $Y = \mathscr{P}(X)$ the set of its subsets. Now $(Y, \subseteq)$ is a partially ordered set by inclusion, *i.e.* $y_1 \leq y_2$ iff $y_1 \subseteq y_2$ and inclusion is reflexive, anti-symmetric and transitive. We can observe that not every two elements are comparable, *e.g.* $\{C, E, G\}$ and $\{C, E, A\}$, so this is not a total order. We will define mathematically *notes* in definition 35, here we address to simple musical knowledge. Musicians can call X *major pentatonic scale* and subsets of X *chords*, considering y_1 a sub-chord of y_2 iff $y_1 \leq y_2$. The cardinality of Y is $2^5 = 32$, so there are 32 chords over 5 notes. We can also consider $\tilde{X} = \{D, E, G, A, B\}$ obtaining again 32 chords. The structure of $(Y, \leq)$ does not depend on the symbols in X or $\tilde{X}$, but only on the fact that X and $\tilde{X}$ are five-elements sets. We can then read in $(Y, \leq)$ properties that are common to all the pentatonic scales.

We can see a poset as a mathematical object with several totally ordered sub-sets.

Definition 11. A *chain* C in $(P, \leq)$ is a subset of P such that the partial order $\leq$ restricted to C is a *total order*, *i.e.* every two element $x, y \in C$ are comparable and for every $x \neq y \in C$ we have $x < y$ or $x > y$.

Example 4. For $X = \{C, D, E, G, A\}$ and Y the set of its subsets, we notice that

$$\{\{C\}, \{C, D\}, \{C, D, G\}\}$$
$$\{\{C\}, \{C, D\}, \{C, D, A\}\}$$

are chains and that they share the common *sub-chain* $\{\{C\}, \{C, D\}\}$. The partial order of $(Y, \subseteq)$ can be described listing its chains. We will see that this is a general property of posets.

Definition 12. A *sub-poset* of $(P, \leq)$ is a poset which elements are a subset of P with the order relation given by the restriction of $\leq$. An *embedding* of a poset $(P_1, \leq_1)$ into a poset $(P_2, \leq_2)$ is given by a bijection between P_1 and a subset of P_2 with $\leq_1 \subseteq \leq_2 \cap (P_1 \times P_1)$. In particular, an embedding is *weaker* then a sub-poset relation.

We want now to introduce some properties that are not common to every poset but peculiar of the posets of our interest.

Definition 13. Let $(P, \leq)$ be a poset, $V \subset P$, $x \in P$. We call x a *upper bound* of V if for all $v \in V$ we have $v \leq x$. We call x a *lower bound* of V if for all $v \in V$ we have $v \geq x$. An upper bound x of V is called the *least upper* *bound* or *join* if for all $\tilde{x}$ upper bound of V we have $x \leq \tilde{x}$. A lower bound x of V is called the *greatest lower bound* or *meet* if for all $\tilde{x}$ lower bound of V we have $x \geq \tilde{x}$. We will write $\vee V$ for the least upper bound and $\wedge V$ for the greatest lower bound.

Definition 14 (Categorial structure of a poset)**.** Let $(P, \leq)$ be a poset. The category $\mathcal{C}P$ associated to $(P, \leq)$ is defined as follows:

- The set $\mathcal{O}$ of objects consists of all elements of P.
- For two elements $x, y \in P$, the set $\mathcal{M}(x, y)$ is empty unless $x \geq y$; otherwise it consists of a unique morphism $x \to y$. The composition $\mathcal{M}(x, y) \circ \mathcal{M}(y, z)$ is defined only if $x \geq y$ and $y \geq z$, in which case we have
$$(x \to y) \circ (y \to z) = x \to z$$
- x, y are isomorphic in $(P, \leq)$ iff $x = y$.

Remark 3. This category is acyclic according to the anti-symmetry of the partial order relation.

Every partially ordered set $(P, \leq)$ can be structured as a category. So we are able to define many properties of interesting posets with the standard language of order relations but also with a categorial point of view. In particular an acyclic category $\mathcal{C}$ is a poset iff for every pair of objects $x, y \in \mathcal{O}(\mathcal{C})$ the cardinality of the set of morphisms $\mathcal{M}_{\mathcal{C}}(x, y)$ is at most 1. We can also define, starting from the acyclic category $\mathcal{C}$, the poset $(\mathcal{O}(C), \leq_{\mathcal{C}})$ where

$$x \leq_{\mathcal{C}} y \text{ iff } \mathcal{M}_{\mathcal{C}}(x, y) \neq \emptyset$$

A functor between acyclic categories $\mathcal{C}_1$ and $\mathcal{C}_2$ induces an order preserving map between $(\mathcal{O}(\mathcal{C}_1), \leq_{\mathcal{C}_1})$ and $(\mathcal{O}(\mathcal{C}_2), \leq_{\mathcal{C}_2})$.

Definition 15. We define **Ac** the sub-category of **Cat** in which the objects are all the acyclic categories and morphisms are all the functors between them.

The following definition is an example of the double interpretation of the same property in the context of order and categorial definition of a posets.

Definition 16. A poset $(P, \leq)$ is a *poset lattice* if for all $x, y \in P$ exists $\vee\{x, y\}$ and exists $\wedge\{x, y\}$. We will write $x \vee y$ instead of $\vee\{x, y\}$ and $x \wedge y$ instead of $\wedge\{x, y\}$. This notation is standard and reflects the idea that $\vee$ and $\wedge$ are "binary operations".

A poset $(P, \leq)$ is a lattice iff, when regarded as a category, has products and coproducts indeed for all $p, q \in P$ the product $p\Pi q$ is such that for every $r \in P$ with $p \leq r, q \leq r$ we have only one morphism from r to $p\Pi q$, *i.e.* $p\Pi q = p \vee q$, and similarly we have $p \amalg q = p \wedge q$.

Remark 4. Not all poset are poset lattice. If exists $\vee P$ we call it $\hat{1}$, if exists $\wedge P$ we call it $\hat{0}$; in this case the poset lattice is called *bounded poset lattice*.

We want now to define a strongher partial order relation between elements of $(P, \leq)$.

Definition 17. We will write $x \prec y$, read y *covers* x, if $x \leq y$ and there are no z s.t. $x \leq z \leq y$. A chain is *saturated* if between two consecutive elements exists a cover relation. An *antichain* A is a subset of P no two elements of which are comparable in $(P, \leq)$.

We are now able to graphically represent posets.

Definition 18. An *Hasse diagram* of a poset $(P, \leq)$ is a directed graph $H = (V, E)$ in which every element of $(P, \leq)$ is a vertex in V and exists a directed edge in E between y and x iff $x \prec y$. A directed edge is a morphism arrow, so this definition reflect well the categorial interpretation of a poset.

Note that in litterature we can find different definitions of Hasse diagram, and in particular the original work of Hasse contains a different definition.

Definition 19. We call *distributive poset lattice* a lattice in which join and meet distribute over each other, *i.e.* $\forall p, q, r \in P, p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$.

Every subposet lattice of a distributive poset lattice is a distributive poset lattice.

Example 5. The Hasse diagram of $(Y, \subseteq)$, where $Y = \mathscr{P}(X)$ is the set of subsets of

$$X = \{C, D, E, G, A\}$$

is very complex (see fig. 1.1) but we can read in it a great amount of informations. We can gain a more readable diagram re-labeling of C, D, E, G, A as $1, 2, 3, 4, 5$.

We can observe a natural *layering structure*, given by the cardinalities of subsets of X, and we can also appreciate how different chains (total orders) interact to generate the partial order. This structure is generalized in the following definition.

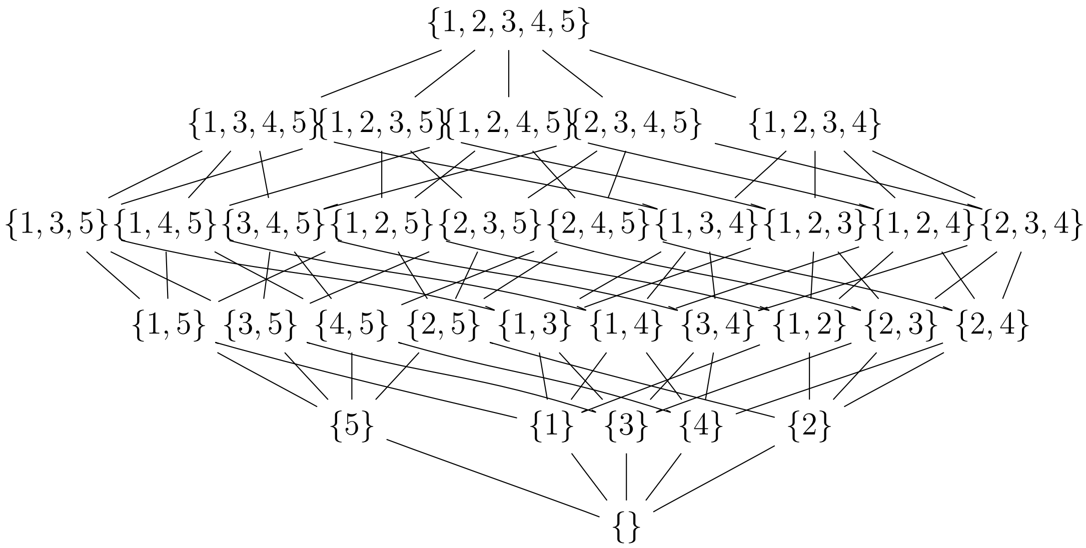

Figure 1.1: Hasse diagram of $(\mathscr{P}(\{1,2,3,4,5\}), \subseteq)$. This drawing is generated with Sage, see Appendix A.

Definition 20. A chain is *maximal* if it is not a proper sub-chain of another chain.

Definition 21. The finite poset $(P, \leq)$ is called *graded* if every maximal chains have the same cardinality. In this case there exists a map $\rho : P \mapsto \mathbb{N}$, called *rank*, such that

- ρ is *order-preserving*, *i.e.* for $x < y$ we have $\rho(x) < \rho(y)$.
- ρ is *cover-preserving*, *i.e.* for $x \prec y$ we have $\rho(y) = \rho(x) + 1$

If $(P, \leq)$ is graded then we have

$$P = P_0 \sqcup P_1 \sqcup \cdots \sqcup P_k$$

disjoint union of $P_i = \rho^{-1}(i)$, inverse images of ρ. We will call P_i the ith *rank* of $(P, \leq)$.

In fig. 1.1 we can verify that the cardinality of a subset of X is a rank function.

In a graded poset every rank is an antichain because two element p_1, p_1 of the rank P_i have the same image according to ρ, hence they are not comparable. In a general framework can exists other antichains in $(P, \leq)$, we will study interesting posets in which the only antichains are the ranks.

We had seen that chains describe completely the structure of a poset. We want now describe a different way to decompose a poset: the poset product.

Definition 22. Let $(P, \leq_P), (Q, \leq_Q)$ posets. We define the *poset product*

$$(P \times Q, \leq_{P \times Q})$$

with $(p_1, q_2) \leq_{P \times Q} (p_2, q_2)$ iff $p_1 \leq_P p_2$ and $q_1 \leq_Q q_2$.

Starting from $(P, \leq_P), (Q, \leq_Q)$ we can create a new poset, but sometimes it is useful to describe a given poset as the product of simpler posets. We will see in the context of Pólya theory an interesting application of this concept.

Definition 23. Let $(P, \leq_P), (Q, \leq_Q)$ posets. We call $\phi : P \mapsto Q$ an *homomorphism* if it is an *order-preserving* map, *i.e.* if $u \leq v$ for $u, v \in P$ then $\phi(u) \leq \phi(v)$ in Q. If ϕ is bijective we call it *isomorphism.* If ϕ is an isomorphism and $P = Q$, $R_P = R_Q$, we call ϕ an *automorphism.* We will call $Aut(P)$ the set of all automorphism over P. $(Aut(P), \circ)$ is a group with the standard composition of maps $\circ$.

If we are in a *bounded poset* we have $\phi(\hat{0}) = \hat{0}$.

1.3 Partitions of a set, the partition lattice

From now on we will write $[n]$ to say $\{1, \dots, n\}$.

Definition 24. Let X be a finite set. A *partition* of X is a finite decomposition

$$X = X_1 \cup \dots \cup X_k$$

with $X_i \cap X_j = \emptyset$ if $i \neq j$. We call $\Pi = \{\text{partitions of X}\}$.

Example 6. Consider $X = [3]$. We can write all its partitions

$$\Pi = \{\{\{1\}, \{2\}, \{3\}\}, \{\{1, 2\}, \{3\}\}, \{\{1, 3\}, \{2\}\}, \{\{2, 3\}, \{1\}\}, \{\{1, 2, 3\}\}\}$$

We will use the *more readable* notation

$$\Pi = \{1|2|3, 12|3, 13|2, 23|1, 123\}$$

Definition 25. We define *refinement* the following partial order on Π. For $\pi_1, \pi_2 \in \Pi$, we say $\pi_1 \leq \pi_2$ iff two or more parts in π_2 are assembled in π_1, *e.g.* $a|bc|\dots|z \leq a|b|c|\dots|z$.

Remark 5. $(\Pi, \leq)$ is a bounded poset, obviously $\hat{0} = [n]$ and $\hat{1} = 1|2|\dots|n$. With the rank function $\rho(\pi) =$ number of parts of π, our $(\Pi, \leq)$ is a graded poset. Meet $\pi_1 \wedge \pi_2$ and join $\pi_1 \vee \pi_2$ always exists, they are the greatest common assemblement and the smallest common division of parts of π_1 and π_2, so $(\Pi, \leq)$ is a lattice.

1.4 Subsets of a set, the Boolean lattice

Definition 26. Consider the finite set $X = [n]$ an call $\mathcal{B}_n = \mathscr{P}(X)$ the power set[1] of X. $\mathcal{B}_n$ has a natural partial order: $p \leq q$ iff $p \subset q$. The power set, ordered by *inclusion*, will be writed $(\mathcal{B}_n, \leq) = (\mathcal{B}_n, \subseteq)$.

The natural bijection between $\mathscr{P}(X)$ and $\{0,1\}^X = \{f : X \to \{0,1\}\}$ will be very useful in *Pólya theory.* To every subset of X we assign a function f that is equal to 1 over the elements of the subset and 0 elsewhere.

Remark 6. In our language $(\mathcal{B}_n, \subseteq)$ is isomorphic to

$$(\{0,1\}, \leq) \times \cdots \times (\{0,1\}, \leq)$$

where $(\{0,1\}, \leq)$ is the chain in which $0 \leq 1$. To prove this sentence it is sufficient to show that

$$\begin{aligned} f : \mathcal{B}_n &\to \{0,1\}^n \\ Y &\mapsto (c_1, \ldots, c_n) \end{aligned}$$

where c_i is equal to 1 if $i \in Y$ and 0 elsewhere, is order-preserving. Consider $f(Y) = (c_1, \ldots, c_n)$ and $f(T) = (b_1, \ldots, b_n)$, $Y \subseteq T$ means that $i \in Y$ implies $i \in T$ for $i = 1, \ldots, n$ and so $c_i \leq b_i$ in C_1. Then $(c_1, \ldots, c_n) \leq (b_1, \ldots, b_n)$ in $(C_1, \leq)^n$, *i.e.* $f(Y) \leq f(T)$. This isomorphism allows us to represent the elements of B_n as vertices of an n-cube.

We can also consider a richer structure over $\mathcal{B}$, the structure of *Boolean algebra*[2].

Definition 27. A 6-uple $(Y, \oplus, \otimes, \neg, 0, 1)$, where Y is a set, $\oplus, \otimes$ are binary operations over Y, $\neg$ is an unary operation, $0, 1$ two elements of Y, is called *Boolean algebra* if the following holds

- $y_1 \oplus (y_2 \oplus y_3) = (y_1 \oplus y_2) \oplus y_3$, that is *associativity*
- $y_1 \otimes (y_2 \otimes y_3) = (y_1 \otimes y_2) \otimes /y_3$, that is *associativity*
- $y_1 \oplus y_2 = y_2 \oplus y_1$, that is *commutativity*
- $y_1 \otimes y_2 = y_2 \otimes y_1$, that is *commutativity*

[1]The set X is finite, so we have no “logic troubles” working with $\mathscr{P}(X)$.

[2]The relationship between Logic and Algebra is an ispiring field of research, here we are interested only in some definitions.

- $y_1 \oplus 0 = y_1$ and $y_1 \otimes 0 = y_1$, that is *identity*
- $y_1 \oplus \neg y_1 = 1$ and $y_1 \otimes \neg y_1 = 0$, that is *complements*
- $y_1 \oplus (y_2 \otimes y_3) = (y_1 \oplus y_2) \otimes (y_1 \oplus y_3)$, that is *distributivity*
- $y_1 \otimes (y_2 \oplus y_3) = (y_1 \otimes y_2) \oplus (y_1 \otimes y_3)$, that is *distributivity*

We will write $(Y, \oplus, \otimes)$ instead of $(Y, \oplus, \otimes, \neg, 0, 1)$ when $\neg, 0, 1$ are clear.

Definition 28. We call $\mathcal{L} = (L, \oplus, \otimes)$ an *algebraic lattice* if it is a Boolean algebra with operations $\oplus, \otimes$ satisfying the two *absorption identities* $l_1 \otimes (l_1 \oplus l_2) = l_1$, $l_1 \oplus (l_1 \otimes l_2) = l_1$

Poset lattices and algebraic lattices came from different field of Mathematics but are strictly related.

Theorem 1 (Pierce)**.** *The following are equivalent.*

- *If $\mathcal{L} = (L, \leq)$ is poset lattice, then we can build the algebraic lattice $\mathcal{L}^a = (L, \oplus, \otimes)$.*
- *If $\mathcal{L} = (L, \oplus, \otimes)$ is algebraic lattice, then we can build the poset lattice $\mathcal{L}^p = (L, \leq)$.*

Proof. We can simply build what we need

- Set $l_1 \otimes l_2 = \wedge\{l_1, l_2\}$ and $l_1 \oplus l_2 = \vee\{l_1, l_2\}$.
- Set $l_1 \leq l_2$ iff $l_1 \otimes l_2 = l_1$.

□

So we have to pay attention on additional properties over Boolean algebras if we want to build a poset lattice starting from them. When it is possible, we will write $\oplus$ instead of $\vee$ ad $\otimes$ istead of $\wedge$ and *vice versa.*

$\mathcal{B}$ with $\cup$ as $\oplus$ and $\cap$ as $\otimes$, is a Boolean algebra and we will call it $(\mathscr{P}(X), \cup, \cap)$. We can easilly see that $\cup, \cap$ satisfy the two *absorption identities.* This is also an algebraic lattice and it is *bounded* by X as 1 and $\emptyset$ as 0.

From theorem 1 we obtain that $(\mathcal{B}, \subseteq)$ is a poset lattice.

We can easily see that $(\mathcal{B}, \subseteq)$ is graded with the rank function $\rho(p) = |p|$.

Remark 7. There exists an injective omomorphism between $(\mathcal{B}, \subseteq)$ and $(\Pi, \leq)$ given by

$$(\mathcal{B}, \subset) \to (\Pi, \leq)$$
$$\{x_1, \ldots, x_k\} \mapsto \{\{x_1\}, \ldots, \{x_k\}, \{x_1, \ldots, x_k\}^C\}$$

1.5 Quotient of a poset by a group

We will define the quotient of a poset by an equivalence relation, following [Hal14], and then the quotient of a poset by a group action, following [Koz08] and his categorical point of view. A group action induces an equivalence relation so the second construction can be seen as a particular case of the first, but we want also point out that the quotient of a poset by an equivalence relation (or by a group action) is not, in general, a poset and the second construction works with more general objects: a quotient of a poset by a group action is an acyclic category.

Definition 29. Let $(P, \leq)$ a finite poset and let $\sim$ be an equivalence relation on P. We define $P/\sim$ to be the *set* of the equivalence classes with the *binary relation* $\blacktriangleleft$ defined by $X \blacktriangleleft Y$ in $P/\sim$ iff there are $x \in P$ and $y \in P$ such that $x \leq y$ in $(P, \leq)$.

Remark 8. The binary relation $\blacktriangleleft$ on $P/\sim$ is reflexive and transitive, but -in general- it s not antisymmetric. For example, let P be the chain $0 \leq 1 \leq 2$ and take $X = \{0, 2\}$ and $Y = \{1\}$. As equivalence classes $X \blacktriangleleft Y$ and $Y \blacktriangleleft X$ but $X \neq Y$ in $P/\sim$.

Definition 30. Let $(P, \leq)$ a finite poset and let $\sim$ be an equivalence relation on P. Ordering the equivalence classes as in the previous definition, $(P/\sim, \blacktriangleleft)$ is an *homogeneous quotient* if

- $\hat{0}$ is in an equivalence class by itself
- if $X \blacktriangleleft Y$ in $P/\sim$ and $x \in X$ then exists a $y \in Y$ such that $x \leq y$

Theorem 2. *An homogeneous quotient is a poset.*

Proof. We have a finite set P and an equivalence relation $\sim$. Denote by $[x]$ the equivalence class of $x \in P$. We have to show that the binary relation is anti-symmetric.

Assume $[x] \blacktriangleleft [y] \blacktriangleleft [x]$, hence for any $p_1 \in [x]$ there is $q_1 \in [y]$ such that $p_1 \leq q_1$ and there is $p_2 \in [x]$ such that $q_1 \leq p_2$. This way we construct the sequence

$$p_1 \leq q_1 \leq p_2 \leq q_2 \leq \dots$$

Since P is finite for some k we get $p_r = p_j$ with $j < k$, hence

$$p_j \leq q_j \leq \dots \leq q_{k-1} \leq p_j = p_k$$

In particular $p_j \leq q_j \leq p_j$ that implies $q_j = p_j$. Since $q_j \in [y]$ and $p_j \in [x]$ we obtain $[x] = [y]$.

□

Definition 31. An *action of a group G over a finite set P* is a map $\cdot$ with

$$\begin{aligned} \cdot : G \times P &\to P \\ (g, p) &\mapsto g \cdot p \end{aligned}$$

that satifies

- The *compatibility* law $(gh) \cdot p = g \cdot (h \cdot p)$.
- The *identity* law $e \cdot p = p$ for all $g, h \in G$ and $p \in P$.

Denote the *permutation group* of P as

$$\mathcal{S}_P = \{f : P \to P \text{ s.t. } f \text{ is bijective}\}$$

The group action $\cdot$ defines a homomorphism between G and $\mathcal{S}_P$ namely

$$\begin{aligned} \phi : G &\to \mathcal{S}_P \\ g &\mapsto \phi_g \end{aligned}$$

where $\phi_g(p)$ is the permutation on P induced by g,*i.e.* $\phi_g(p) = g \cdot p$. Its inverse is $\phi_{g^{-1}}$.

We are now able to define the action of a group over a poset.

Definition 32. An *action of a group G over a finite poset* $(P, \leq)$ is an action of the group G over the finite set P that is *order-preserving*, *i.e.* $p \leq q$ implies $g \cdot p \leq g \cdot q$ for all $g \in G$. An action of a group over a finite poset induces the equivalence relation $\sim_G$ given by $p \sim_G q$ iff exists $g \in G$ such that $p = g \cdot q$. Then we can consider the set of equivalence classes $P/\sim_G$ with the binary relation $\blacktriangleleft$ defined by $X \blacktriangleleft Y$ iff there are $x \in X, y \in Y$ and $g, h \in G$ such that $g \cdot x \leq h \cdot y$ in $(P, \leq)$. We call $(P/\sim_G, \blacktriangleleft)$ the *quotient*.

We can re-define the action of a group over a poset, and the quotient, also according to the categorial framework.

Definition 33. Let G a finite group and $\mathcal{C}$ an acyclic category. Remember that **Ac** is the category of all the acyclic categories. Consider the category associated to G, called $\mathcal{C}G$, and define an *action of the group G over the acyclic category* $\mathcal{C}$ as the functor

$$\begin{aligned} F : \mathcal{C}G &\to \mathbf{Ac} \\ o &\mapsto \mathcal{C} \end{aligned}$$

This functor maps the marphisms of $\mathcal{C}G$ to some auto-functors of $\mathcal{C}$ because definition 3 implies

$$F_{\mathcal{M}}(o, o) \subseteq \mathcal{M}(F_{\mathcal{O}}(o), F_{\mathcal{O}}(o)) = \mathcal{M}(\mathcal{C}, \mathcal{C})$$

Remark 9. In our case $\mathcal{C}$ is $(P, \leq)$ and $\mathcal{M}(\mathcal{C}, \mathcal{C})$ is $Aut(P, \leq)$, so the action of a group G over the poset $(P, \leq)$ selects the automorphisms of $(P, \leq)$ that comutes well with the morphisms $F_{\mathcal{M}}(o, o)$ (labelled with the elements of G).

Definition 34. The colimit of the action always exists and we will call it $\mathcal{C}(P/G)$, the *quotient* of the action. We can list its objects and morphism. For x an object or morphism of C we will denote $G(x)$ the orbit of x under G.

- The objects of $\mathcal{C}(P/G)$ are $\{G(a) \text{ s.t. } a \in \mathcal{O}(C)\}$.

The construction of morphisms is more difficult because we have to deal with the "rule of composition". We can define the relation $x \leftrightarrow y$ between morphisms if $G(y_i) = G(x_i)$ for $i \in [t]$ and $x = x_1 \circ \cdots \circ x_t, y = y_1 \circ \cdots \circ y_t$. The *transitive closure* of $\leftrightarrow$ is an equivalent relation, the class of x in this equivalence is called $[x]$.

- The morphisms of $\mathcal{C}(P/G)$ are $\{[x] \text{ s.t. } x \in \mathcal{M}(C)\}$ with

$$\begin{aligned} dom([x]) &= [dom(x)] \\ cod([x]) &= [cod(x)] \\ [x] \circ [y] &= [x \circ y], \text{ whenever } \circ \text{ is defined} \end{aligned}$$

Consider now $(P, \leq)$ a finite poset. It is an acyclic category so, for $a, b \in P$, $\mathcal{M}(\mathcal{C}(P/G))$ contains $[\emptyset]$ if a and b are not comparable and $[\mathcal{M}(a, b)]$ if $a \geq b$; in the latter case, $[\mathcal{M}(a, b)] = [\text{unique element}] = \mathcal{M}(\alpha, \beta)$ if exists $\alpha \in G(a)$ and exists $\beta \in G(b)$ such that $\alpha \geq \beta$. Hence we found that $\mathcal{C}(P/G)$ is a quotient by the equivalence relation $a \sim_G b$ iff exists $g \in G$ such that $b = g(a)$ and then the two definition of quotient are equivalent. In general $\mathcal{C}(P/G)$ is not a poset, but when it is a poset we will use the notation $\mathcal{C}(P/G) = (P/G, \leq)$.

Theorem 3 (Pendergrass [Pen14], Theorem 1)**.** *Given an action of the group G over the poset $(P, \leq)$ If the action is* transitive, i.e. *for all $p \in P$ and for all $g \in G$, $g \cdot p \leq p$ implies $g \cdot p = p$, then the quotient $(P/\sim, \blacktriangleleft)$ is homogeneous and hence a poset $(P/G, \leq)$.*

Proof. If $[p] \blacktriangleleft [q]$ then exists $p_0 \in [p], q_0 \in [q]$ s.t. $p_0 \leq q_0$. For all $\tilde{p} \in [p]$ exists $g \in G$ such that $\tilde{p} = g \cdot p_0$ and then, remember that $g \cdot p_0 \leq g \cdot q_0$, we have $\tilde{p} \leq g \cdot q_0$. $g \cdot q_0$ is in $[q]$, hence for all $\tilde{p}$ in $[p]$ exists $\tilde{q} = g \cdot q_0$ in $[q]$ such that $\tilde{p} \leq \tilde{q}$.

Now we want to prove the antisymmetric property of $\blacktriangleleft$, so let $[p], [q]$ such that $[p] \blacktriangleleft [q]$ and $[q] \blacktriangleleft [p]$, then

$$\begin{aligned} \forall p \in [p], \exists q \in [q] \text{ s.t. } p \leq q \\ \exists r \in [p] \text{ s.t. } q \leq r \end{aligned}$$

and exists $g \in G$ s.t. $p = g \cdot r$. Transitive action implies that $g \cdot r \leq r$ and then $p = r$, so for all $p \in [p]$ exists $q \in [q]$ such that $p = q$.

□

Remark 10. If G acts transitively on P then p and $g \cdot p$ are always either incomparable or identical. Then all equivalence classes in the quotient are antichains in $(P, \leq)$.

We will consider in the following only the case in which $G < Aut(P, \leq)$, *i.e.* $g \in G$ is a bijective order-preserving map.

Theorem 4. *Let $(P, \leq)$ be a finite poset and $G < Aut(P, \leq)$. Then $(P/\sim, \blacktriangleleft)$ is a poset $(P/G, \leq)$.*

Proof. Remember that P finite implies $Aut(P, \leq)$ finite, and then $g^{-1}h$ has finite order. As for the previous remark, $[p]$ is an antichain because the action by automorphism has the following property

$$g \cdot x \leq h \cdot x \implies x \leq g^{-1}h \cdot x \leq (g^{-1}h)^2 \cdot x \leq \cdots \leq x$$

Then reflexivity of $\leq$ implies $x = g^{-1}h \cdot x$, *i.e.* $g = h$ and then $g \cdot x = h \cdot x$.

We can then prove the anti-symmetry of $\blacktriangleleft$ showing that $[p] \blacktriangleleft [q]$ and $[q] \blacktriangleleft [p]$ implies $[p] = [q]$. From the first we obtain $\exists \tilde{g}, \tilde{h} \in G$ such that $\tilde{g} \cdot p \leq \tilde{h} \cdot q$, *i.e.* $\tilde{h}^{-1}\tilde{g} \cdot p \leq q$. From the latter we obtain $\exists \bar{g}, \bar{h} \in G$ such that $\bar{g} \cdot p \geq \bar{h} \cdot q$. Thus

$$\begin{aligned} \bar{h}\tilde{h}^{-1}\tilde{g} \cdot p \leq \bar{h} \cdot q \leq \bar{g} \cdot p \\ \bar{g}^{-1}\bar{h}\tilde{h}^{-1}\tilde{g} \cdot p \leq p \end{aligned}$$

$[p]$ is an antichain, then $\bar{h}\tilde{h}^1 = \bar{g}\tilde{g}^{-1}$. We can now apply this element of G to $\tilde{g} \cdot p \leq \tilde{h}q$ and obtain $\bar{g} \cdot p \leq \bar{h} \cdot q$. We have $\bar{g} \cdot p \leq \bar{h} \cdot q$ and $\bar{g} \cdot p \geq \bar{h} \cdot q$ thus $\bar{g} \cdot p = \bar{h} \cdot q$, in particular $p \in [q]$ and $q \in [p]$, *i.e.* $[p] = [q]$.

$\square$

Chapter 2

Catalogs of chords

In this section we will define classical and new catalogs of chords, important field of *math-musicology*, in a non-standard perspective: the partially ordered set point of view.

Definition 35. We will call *notes* $\mathcal{N}$ the infinite set of symbols

$$\{\dots, A_{-1}\sharp, B_{-1}, C_0, C_0\sharp, D_0, D_0\sharp, E_0, F_0, F_0\sharp, G_0, G_0\sharp, A_0, A_0\sharp, B_0, C_1, C_1\sharp \dots\}$$

Musicians can recognize in $\mathcal{N}$ the anglo-saxon musical notation and equal temperament. Every *note* W_k in $\mathcal{N}$ has a literal part W lying in $\mathcal{O}$ and an integer subscript $k \in \mathbb{Z}$ where

$$\mathcal{O} = \{A, A\sharp, B, C, C\sharp, D, D\sharp, E, F, F\sharp, G, G\sharp\}$$

Two notes are equivalent when they have the same literal part W, hence we can label every representative of this equivalence relation with an element of $\mathcal{O}$.

We can also label representatives according to the total order $R \subset \mathcal{O} \times \mathcal{O}$ where $R = L \cup S$ with L the lexicographic total order over $\{A, B, C, D, E, F, G\}$ and S which elements are the relations

$$A < A\sharp, C < C\sharp, D < D\sharp, F < F\sharp, G < G\sharp$$

Each of our 12 representatives has an unique position in the order hence we label them with numbers from 1 to 12.

Definition 36. We define *notes* the set $X = [12]$ of representatives, labelled as above. The sets X and $\mathcal{N}$ share the same name for historical reasons but in this thesis we will work only with X and then there are no misunderstandings.

In the following we will define actions of permutation groups $G < \mathcal{S}_X$ over X and partially ordered sets starting from X, hence we will set up an abstract context in which mathematicians, composers and musicologists can work.

2.1 All-chords catalog

Consider the set of notes $X = [12]$ and $\mathcal{B}_{12} = \mathscr{P}(X)$ the set of all subsets of X, *i.e.* the set of all the chords. Chords are partially ordered naturally by *inclusion*, *e.g.* $\{C, E, G\} \subseteq \{C, E, G, A\sharp\}$, so the first and most important catalog of chords is $(\mathcal{B}_{12}, \subseteq)$. This is a relevant enrichment of the Musical set theory that focuses the attention on the relation between chords instead of on the relation between notes.

Unfortunally this poset has $2^n = 2^{12} = 4096$ elements (chords) and then from the point of view of composers and musicologists it is not a simple object to work with.

2.2 (Inverse-)Transposition catalog

Musicologists are interested in *data mining*, *i.e.* extracting relevant informations from a great source of data (like our 2^{12}-elements poset) and composers want a small set of musical objects with well defined rules to work with. So we will compress informations through symmetry, *i.e.* we will consider an equivalence relation between chords that is defined by the action of a group and then construct a quotient. The quotient is a *catalog* of useful musical objects and the group action rules the game. This is the heart of Musical set theory. We have seen in Chapter 1 how to take care of the structure of poset according to a group action and we want to organize the catalog of chords following this perspective. This is our non-standard analysis in the context of Musical set theory.

We need to select the symmetry group among the subgroups of $Aut(\mathcal{B}, \subseteq)$, then we need a complete description of the automorphisms of the Boolean lattice.

Theorem 5. *Let $X = [n]$ and $\mathcal{B} = \mathscr{P}(X)$. Then $Aut(\mathcal{B}, \subseteq) \cong \mathbb{S}_n$*

Proof. If $\phi \in Aut(\mathscr{P}(X))$, it preserves the partial order, and in particular the cardinality, so ϕ sends singletons in singletons

$$
\begin{aligned}
f : Aut(2^X, \subseteq) &\to \mathcal{S}_X \\
\phi &\mapsto \phi|_{\text{singletons}}
\end{aligned}
$$

where we consider $\mathscr{P}(X)$ in bijection with 2^X and $\phi|_{\text{singletons}}$ make sense only over subsets of X with cardinality 1. Hence we have the mutual homomorphism

$$
\begin{aligned}
g : \mathcal{S}_X &\to Aut(2^X, \subseteq) \\
\sigma &\mapsto g(\sigma)
\end{aligned}
$$

where $g(\sigma)(\{x_1, \dots, x_k\}) = \{\sigma(x_1), \dots, \sigma(x_k)\}$. The homomorphism $g(\sigma)$ preserves the cardinality and the inclusion. We can observe that $f \circ g = id_{Aut(\mathscr{P}(X))}$ and $g \circ f = id_{\mathbb{S}_n}$.

□

For every subgroup $G < Aut(\mathcal{B}, \subseteq)$ we can consider the acyclic category $G//\mathcal{B}$ with the relation $\blacktriangleleft$. Is this a poset?

The action of G over $\mathcal{B}$ is increasing because $p \leq q$ means

$$p = \{x_1, \dots, x_k\} \subset q = \{x_1, \dots, x_k, x_{k+1}, \dots, x_h\}$$

and then for all $g \in G$ we have the corresponding element $\sigma \in \mathbb{S}_n$ and

$$g \cdot p = \{x_{\sigma(1)}, \dots, x_{\sigma(k)}\} \subset g \cdot q = \{x_{\sigma(1)}, \dots, x_{\sigma(k)}, x_{\sigma(k+1)}, \dots, x_{\sigma(h)}\}$$

The action of G over $\mathcal{B}$ is transitive because $\mathcal{B}$ is graded (the rank funtion is the cardinality of an element of the poset) and $\rho(g \cdot p) = \rho(p)$ for all $p \in \mathcal{B}$, then $g \cdot p \leq p$ means $g \cdot p = p$.

Theorem 3 implies that $\mathcal{B}_n/G$ is a poset. We can now select the symmetries that we want to use in the data mining, *i.e.* the subgroup G of $Aut(\mathcal{B}, \subseteq)$ to be used in the equivalence relation. We are interested in $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{D}_n$ so we need to recognize them as subgroup of $\mathbb{S}_n$.

Definition 37. Let $\tau = (1, 2, \dots, n)$ be an n-cycle of $\mathbb{S}_n$ and consider the cyclic group $\mathbb{Z}/n\mathbb{Z} =< \tau >$. We define *transposition* the action

$$
\begin{aligned}
\alpha : G \times X &\to X \\
(\tau^k, x) &\mapsto \tau^k \cdot x
\end{aligned}
$$

of $< \tau >$ over X. Consider now the dihedral group presented as

$$\mathbb{D}_n = \{\tau, \sigma \,|\, \tau^n = e, \sigma^2 = e, \tau\sigma\tau\sigma = e\}$$

We can see it as a subgroup of $\mathbb{S}_n$ choosing τ as before and σ the cyclic decomposition

$$(1,n)(2,n-1)\cdots(\frac{n-1}{2},\frac{n+3}{2})(\frac{n+1}{2}) \text{ if } n \text{ is odd}$$
$$(1,n)(2,n-1)\cdots(\frac{n}{2},\frac{n}{2}+1) \text{ if } n \text{ is even}$$

It easy to see that $\tau^n = \sigma^2 = e$ and $\tau\sigma = \sigma\tau^{-1}$. We define *inverse-transposition* the action

$$\beta : G \times X \to X$$
$$(\tau^i\sigma^j, x) \mapsto \tau^i\sigma^j \cdot x$$

of $\mathbb{D}_n$ over X.

The names transposition and inverse-transposition came from Geometry and refer to the action of $< \tau >$ and $\mathbb{D}_n$ on a regular polygon inscribed in a circle. The elements of $< \tau >$ *rotate* the vertices of the polygon and in $\mathbb{D}_n$ we can find elements that *reflect* the vertices according to an axis. As we can appreciate in the following example, in the musical context a rotation corresponds to a transposition and a reflection corresponds to an invertion.

Example 7. Consider $X = \{1, \ldots, 12\}$. Here we label X starting from 0. The action of τ^3 over X sends x in $x + 3 \pmod{12}$. If we use the musical notation $\tau^3 \cdot C = D\sharp$, so we transpose the note C to the note $D\sharp$. Consider now the action of σ over the note C. We have $\sigma \cdot C = \sigma \cdot 4 = 9 = F$, so we invert C in F.

We can also consider a different musical notation that involves [12]. Untill now we have seen only the anglo-saxon notation, in which A is 1 and so on. If we consider $F\sharp$ as 1 we have B as 6 and then the inversion with σ is a reflection of the notes in one chord respect to the middle line of the staff, in violin key, when we use the usual *pentagram*. This is, for istance, the mechanism involved in the *canon cancrizans* of Bach's *Musikalisches Opfer*.

We have seen the action of G over X and that the action of G over $(\mathcal{B}, \subseteq)$ gives us the quotient poset $(\mathcal{B}_n/G, \leq)$. We will call transposition the action of $< \tau >$ over $(\mathcal{B}, \subseteq)$ and inverse-transposition the action of $\mathbb{D}_n$. We can represent geometrically a chord as a (not necessary regular) polygon

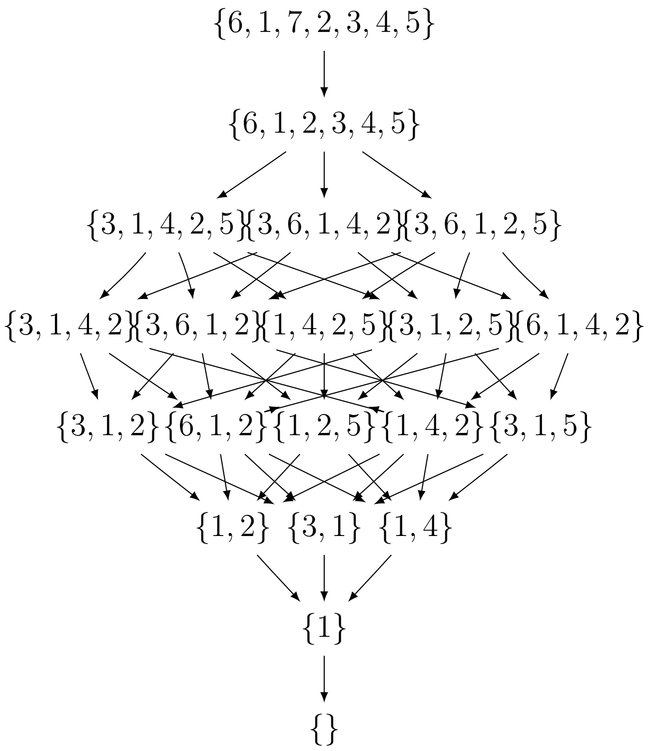

Figure 2.1: Hasse diagram of $(\mathcal{B}_7/(\mathbb{Z}/7\mathbb{Z}), \leq)$. We choose one representative element for any orbit, so for istance we write here $\{1,3\} \leq \{1,2,6\}$ because $\tau^6 \cdot \{1,3\} = \{1,6\} \subseteq \{1,2,6\}$.

inscribed in a circle the vertices of which are the notes of the chord. So a transposition of a chord is a rotation of the polygon and the invertion is a reflection.

We will study the cardinality of ranks $(\mathcal{B}/G)_{i=0,\ldots,n}$ using Pólya theory in Chapter 3 and we will investigate how these numbers increase and then decrease using the unimodal theory.

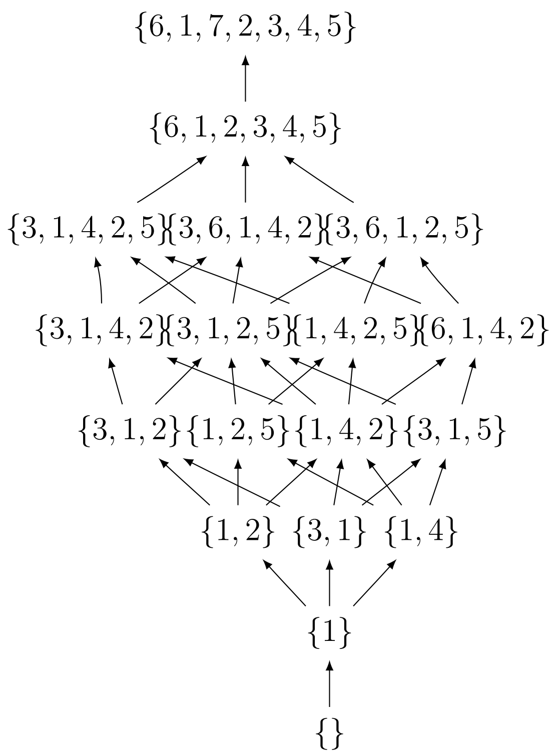

Figure 2.2: Hasse diagram of $(\mathcal{B}_7/\mathbb{D}_7, \leq)$.

2.3 Partitions of integers

We can also classify musical chords with a different criterion: *intervals* between notes.

Definition 38. Consider two distinct notes $x, y \in [n]$ and define *interval* between them the number $x - y$. The interval between x and y is the opposit of the interval between y and x.

Consider a chord c with k notes. As a subset of $[n]$, the notes of c are totally ordered, say $c_1, \ldots, c_k$. Between them we can consider $k - 1$ positive intervals $c_2 - c_1$, $c_3 - c_2$, ..., $c_k - c_{k-1}$. Observe that

$$\begin{aligned}(c_2 - c_1) + (c_3 - c_2) + \cdots + (c_k - c_{k-1}) = \\ c_2 + \cdots + c_k - (c_1 + \cdots + c_{k-1}) = \\ c_k - c_1 < n\end{aligned}$$

hence we build a sequence of elements in $[n]$ that sums to n, precisely

$$(c_2 - c_1) + \dots (c_k - c_{k-1}) + (n - c_k)$$

Consider now the generic positive sequence $a_1, \dots, a_k$ whose elements are in $[n]$ and

$$a_1 + \cdots + a_k = n$$

We can simply build the following chord starting from this sequence.

$$\{a_1, a_1 + a_2, \dots, a_1 + \cdots + a_{k-1}\}$$

So each chord generates a sequence that sums to n and any positive sequence that sums to n generates a chord. We will define a poset that describes the set of sequence in $[n]$ whose elements sum to n and take advantage of its structure to define a non-standard catalog of chords, the *Estrada catalog*.

Consider the poset $(\Pi, \leq_{ref})$ of partitions of the set $[n]$. What can we say about the cardinalities of the parts of its elements?

Example 8. If $n = 12$, we have for istance $123|45|67|89|10|11|12 \in \Pi_{12}$ and the cardinalities of its parts are $3, 2, 2, 2, 1, 1, 1$. They sums to 12.

Theorem 6. *The cardinalities of the parts of $\pi \in \Pi_n$ sum to n.*

Proof. Consider $\pi = \{p_1, \dots, p_k\}$. By definition $[n]$ is the disjoint union of its parts, then $|p_1| + \cdots + |p_k| = n$.

□

Can we associate to every $\pi \in \Pi_n$ an unique chord? No, because we want a bijection that takes care only of the cardinalities of the parts of π and then two partitions with different parts, but with the same cardinalities, are associated to the same chord. We can easily solve this problem considering two partitions equivalent if there exists a permutation that sends the first in the latter. The following theorems and definitions characterize the poset we are looking for.

Theorem 7. *Let* $(\Pi_n, \leq_{ref})$ *be the poset of partitions of the set* $[n]$. *Then* $Aut(\Pi, \leq_{ref}) \cong \mathbb{S}_n$.

Proof. We can simply refer to the commutative law of addition in the case of a finite set of integers. □

Definition 39 (Estrada classes)**.** Define the catalog $\mathcal{P}_n = \mathbb{S}_n // \Pi_n$ *Estrada catalog* and call its elements *Estrada classes*.

Theorem 8. *The acyclic category* $(\mathcal{P}_n, \blacktriangleleft)$ *is a poset. We will call it* integer partition poset $(\mathcal{P}, \leq_{ref})$.

Proof. If $\sigma \cdot \pi \leq_{ref} \pi$ then $\sigma \cdot \pi = \pi$ because the action of σ doesn't change the cardinalities of the parts of π. So the action is transitive and then theorem 3 implies the thesis.

□

A partition of the positive integer n is a way to write n as the sum of integers, not considering the order in which we choose the summands. The partial order is again the refinement, this time between summands and not between parts. We define the rank function $\rho(p)$ from $(\mathcal{P}, \leq_{ref})$ to $\mathbb{N}$ equal to the number of summands of p. So $(\mathcal{P}, \leq_{ref})$ is a graded poset.

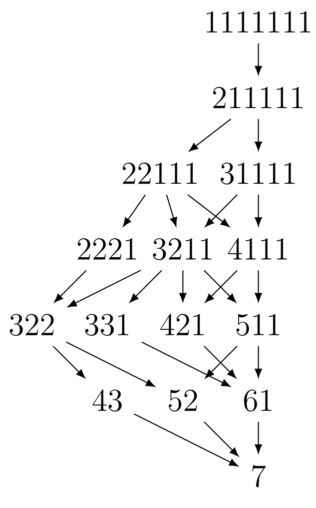

Figure 2.3: Hasse diagram, as directed graph, of $(\mathcal{P}_7, \leq_{ref})$.

From the musical point of view we have a refinement of a chord if we add one or more notes, *i.e.* if we split one or more intervals.

Example 9. Consider $n = 12$ and $c = \{1, 5, 8\}$. The partition associated to its Estrada class is 3+4+5. We can refine the chord in many ways, for istance $c' = \{1, 5, 8, 11\}$. The Estrada class of c' has partition 2+3+3+4 and we can observe the splitting of 5 in $2 + 3$. Consider also that in the Estrada class of c we find the chord $\{4, 9, 12\}$ and in the Estrada class of c' we find the chord $\{1, 3, 6, 10\}$. Between these two chords there is no inclusion relation. Hence the Estrada catalog heaps together chords that are considered different in every catalog of the form $(\mathcal{B}/G, \leq)$.

Remark 11. In Combinatorics it is well studied also the *Young lattice*, another poset built over the same set $\mathbb{S}_n//\Pi_n$ but with a different partial order. We are not interested here in the relationship between them. For a preminent article about this topic refers to [Zie86].

2.4 Subsets of multiset

A *multiset* is a set in which repetitions are allowed. For instance $\{0, 1, 1, 2, 2, 3\}$ is a multiset. The set associated to the previous is $\{0, 1, 2, 3\}$. To avoid confusion we will write the previous multiset as $\{0^1, 1^2, 2^2, 3^1\}$; so we will write $\{0^1, 1^1, 2^1, 3^1\} = \{0, 1, 2, 3\}$ for the associated set.

Definition 40 (Forte intervallic content)**.** Let $c = \{c_1, \ldots, c_k\}$ be an element of $\mathcal{B}_n$, *i.e.* a chord. We can compute all the intervals between c_i and $c \setminus \{c_i\}$ for $i = 1, \ldots, k$ and store this information in a multiset

$$\{0^{\#\text{ 0-intervals}}, 1^{\#\text{ 1-intervals}}, \ldots, n^{\#\text{ n-intervals}}\}$$

where between c_h and c_k we consider only the non-negative interval. We define this multiset the *Forte intervallic content*.

Definition 41. The poset of sub-multiset of $\{1^{\lambda_1}, \ldots, k^{\lambda_k}\}$ is composed by multisets of the form $\{1^{\mu_1}, \ldots, k^{\mu_k}\}$ where $\mu_i \leq \lambda_i$ and $\mu_i = 0$ means that the element i is not in the multiset. $\{1^{\tau_1}, \ldots, k^{\tau_k}\} \subseteq \{1^{\xi_1}, \ldots, k^{\xi_k}\}$ iff for all $i = 1, \ldots, k$ we have $\tau_i \leq \xi_i$. We call $\mathscr{P}(X)$ the power multiset where X is a multiset and $(\mathscr{P}(X), \subseteq)$ the sub-multiset poset.

2.5 Bi-chords decompositions poset and its quotients

Consider $[n] = \{1, \ldots, n\}$ the set of notes and let X be the set of two-elements subsets of $[n]$. Call $m = |X| = \binom{n}{2}$. X is the set of bi-chords. We can describe an element of $\mathcal{B}_n$, *i.e.* a chord, listing *all* its sub-chords of lenght 2, *i.e.* its bi-chords, and then associate to it a subset of X. This point of view reflects the aim of *tertian harmony* and then *quartal harmony* (from the *Tristan chord* to Bill Evans and further). Here we want to characterize the *color of a chord* according to the bi-chords we find in it.

Definition 42. Let $[n]$ the set of notes. We call *decomposed in bi-chords* an element of $\mathcal{B}_n$ seen as the superimposition of *some* of its bi-chords. We call *completely decomposed in bi-chords* an element of $\mathcal{B}_n$ seen as the superimposition of *all* its bi-chords. We call $(\mathcal{B}_X, \subseteq)$ the poset of *bi-chords decompositions*, complete and not.

Example 10. Consider the chord $c = \{1, 5, 6, 10\}$ of $\mathcal{B}_{n=12}$. Its completely bi-chords decomposition is

$$\{\{1,5\},\{1,6\},\{1,10\},\{5,6\},\{5,10\},\{6,10\}\}$$

We can also see $[n]$ as a set of vertices and X the set of all possible edges between these vertices. A subset of X is then a simple graph. The Boolean lattice $(\mathcal{B}_X, \subseteq)$ describes how two simple graphs are related by the *sub-graph relation.*

A chord of k notes that is completely bi-chords decomposed corresponds to a complete sub-graph over k vertices in $\mathcal{B}_X$. Obviously in $\mathcal{B}_X$ we have many elements that are not a completely bi-chords decomposition, *e.g.*

$$\{\{1,2\},\{2,3\},\dots,\{k-1,k\}\}$$

Thus we can associate different decompositions to the same chord.

We can associate to any subgroup G of $\mathbb{S}_n$, acting on $i \in [n]$, a subgroup $G^{(2)}$ of $\mathbb{S}_m$, acting on $\{i, j\} \in \mathcal{B}_X$ by

$$\pi^{(2)} \cdot \{i,j\} = \{\pi \cdot i, \pi \cdot j\} \text{ for all } \pi \in G$$

We will call $G^{(2)}$ a *binary permutation group.* Two graphs of $\mathcal{B}_X$ are in the same orbit of the action of $\mathbb{S}_n^{(2)}$ iff they are *isomorphic*, *i.e.* exists a relabelling of the vertices (and edges as pairs of vertices) that sends one in the other.

There is a bijection between bi-chords and non-negative intervals of notes hence listing all the bi-chords that compose a given chord is equivalent to compute the Forte intervallic content. Unfortunately the action of $\mathbb{S}_n^{(2)}$ doesn't preserve intervals. Nevertheless we are able to follow the construction of Forte, without the multiset framework, according to the action of $\mathbb{Z}_n^{(2)}$ over $(\mathcal{B}_X, \leq)$. This action preserves the intervals.

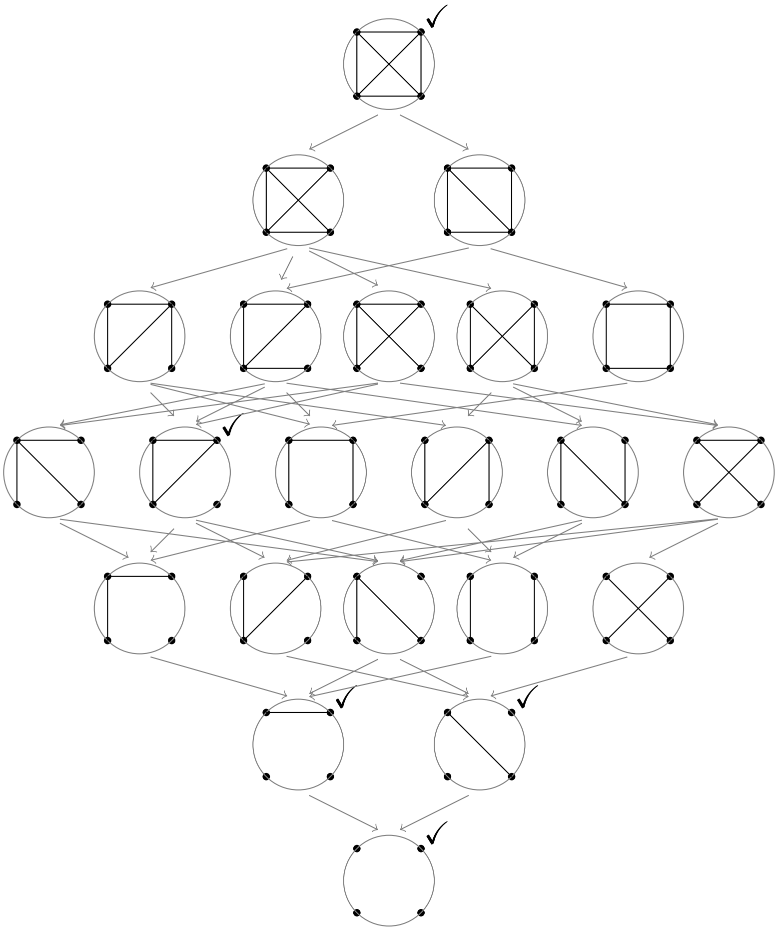

Figure 2.4: Hasse diagram of $(\mathcal{B}_X/(\mathbb{Z}/4\mathbb{Z})^{(2)}, \leq)$ for $n = 4$. Completely bichords decompositions of chords are marked.

Example 11. Consider $n = 12$ and $[n]$ labelled as

$$A, A\sharp, B, C, C\sharp, D, D\sharp, E, F, F\sharp, G, G\sharp$$

In the catalog $\mathcal{B}_X/(\mathbb{Z}/n\mathbb{Z})^{(2)}$ we have that the decomposition $\{\{C, F\}, \{D, G\}\}$ and the decomposition $\{\{A, D\}, \{D\sharp, G\sharp\}\}$ are the same element because they are two *perfect fourth chords* with empty intersection.

We are also able to preserve the Estrada classes according to the action of $\mathbb{D}_n^{(2)}$, where $\mathbb{D}_n$ is the group of symmetries of the regular n-agon, as we can see in the following example.

Example 12. Consider $n = 5$. As we already know, the intervallic content doesn't change under the action of $(\mathbb{Z}/n\mathbb{Z})^{(2)}$, in particular $\{1, 3, 5\}$ and $\{1, 2, 4\}$ give us the same positive sequence $(2, 2, 1)$ of consecutive intervals. Consider now the action of $(12)(35)(4) \in \mathbb{D}_n^{(2)}$. This sends $\{1, 3, 5\}$ in $\{2, 3, 5\}$, that has $(1, 2, 2)$ as consecutive intervals sequence. Hence $\{1, 3, 5\}$ and $\{2, 3, 5\}$ are in the same Estrada class.

The action of $\mathbb{D}_n^{(2)}$ preserves angles of the the n-agon associated to a chord, *i.e.* intervals of the chord, but not the order in which they appear.

We can then change our point of view and associate to every Estrada class, *i.e.* to every element $a_1 + \cdots + a_k \in (\mathcal{P}, \leq_{ref})$, the non-crossing and single-path-connected graph

$$\{\{1, 1+a_1\}, \{1+a_1, 1+a_1+a_2\}, \ldots, \{1+\cdots+a_{k-1}, 1+\cdots+a_k\}, \{1+\cdots+a_k, 1\}\}$$

Its orbit in $(\mathcal{B}_X/\mathbb{D}_n^{(2)}, \leq)$ is in bijection with the Estrada class, then we can embed $(\mathcal{P}, \leq_{ref})$ in $(\mathcal{B}_X/\mathbb{D}_n^{(2)}, \leq)$. Hence we have an upper boundary for the cardinality of the ranks of $(\mathcal{P}, \leq_{ref})$ given by the cardinality of the ranks of $(\mathcal{B}_X/\mathbb{D}_n^{(2)}, \leq)$.

We can also associate to any non crossing and single path connected graph its *completion* to a complete sub-graph of $(\mathcal{B}_X/\mathbb{D}_n^{(2)}, \leq)$, and this association is a bijection. Hence we can say that, for a fixed n, exists $\tilde{k}$ such that the number of Estrada classes of cardinality $k \geq \tilde{k}$ *decreases.*

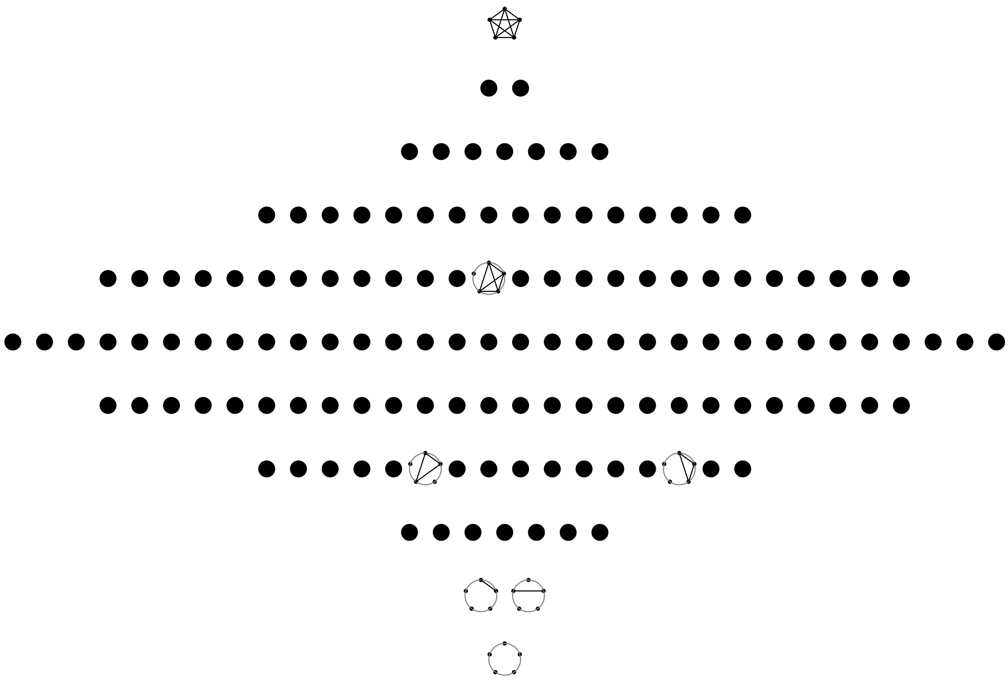

Figure 2.5: $(\mathcal{B}_X/\mathbb{D}_n^{(2)}, \leq)$ for $n = 5$. In evidence are the completed non-crossing and single-path-connected graphs. The upper bound is far from being optimal.

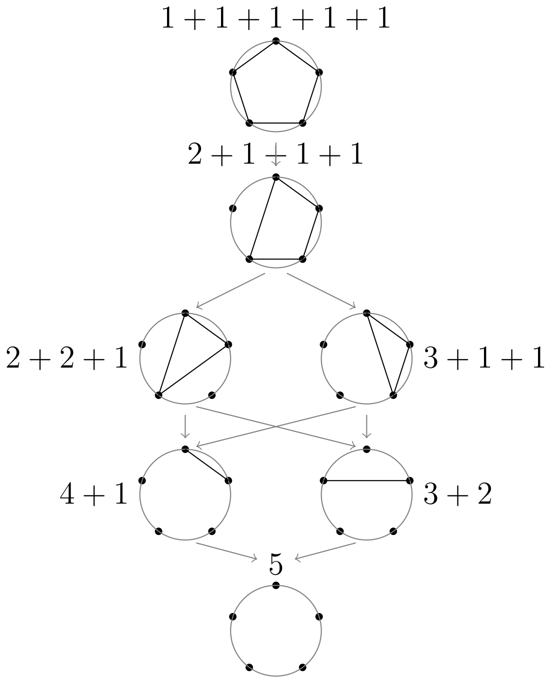

Figure 2.6: Representatives of elements of $(\mathcal{P}, \leq_{ref})$ seen in $(\mathcal{B}_X/\mathbb{D}_n^{(2)}, \leq)$ for $n = 5$.

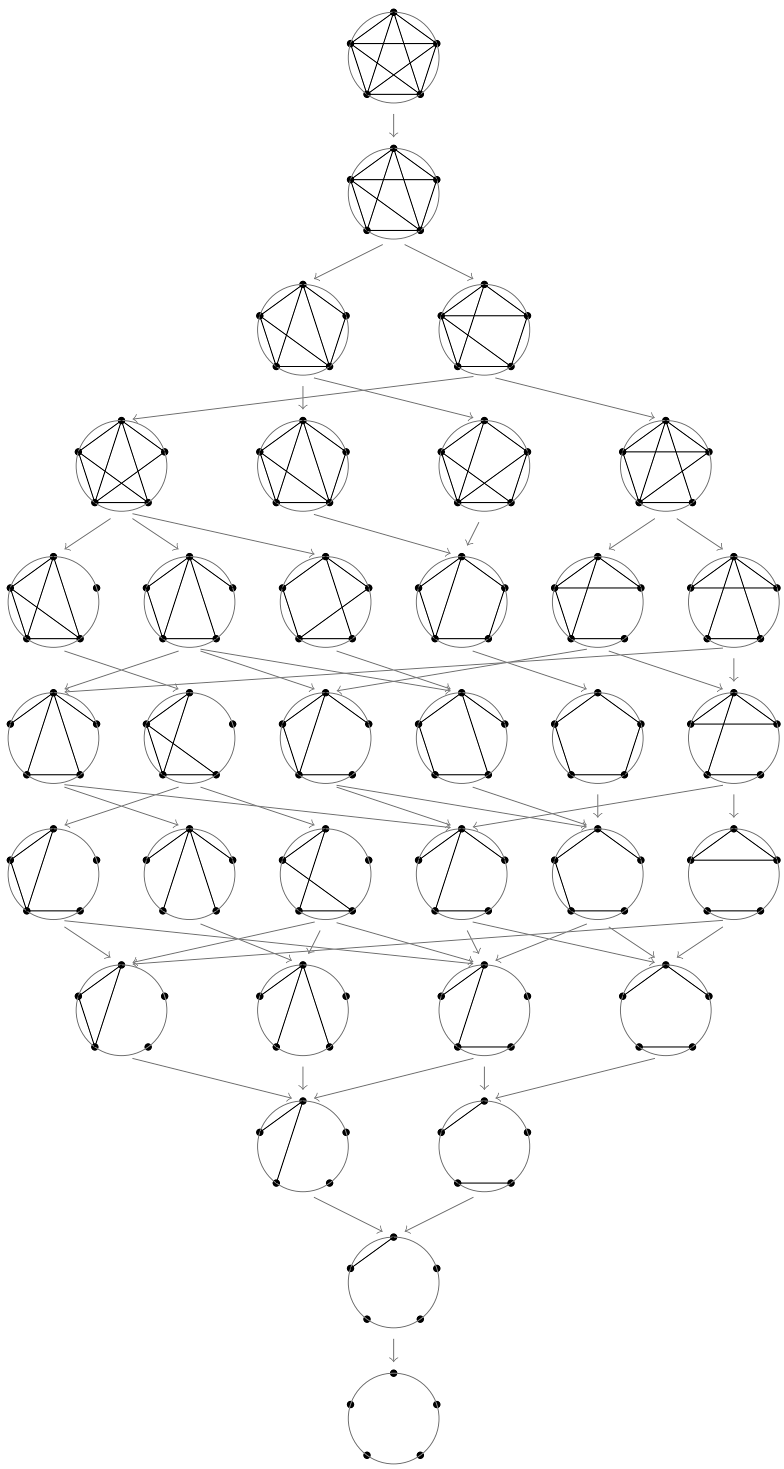

Figure 2.7: Hasse diagram of $(\mathcal{B}_X/\mathbb{S}_n^{(2)})$. Here we list all the isomorphisms classes of graph over $n = 5$ vertices.

Chapter 3

Pólya theory

Remember that an action of G over a set X is given by an homomorphism $\alpha : G \to \mathcal{S}_X$. We consider only finite sets X, so the permutation group $\mathcal{S}_X$ over the set X is a finite. If $|X| = n$ we will use the standard notation $\mathbb{S}_n$, *symmetric group*, for the permutation group $\mathcal{S}_X$.

The action is *faithful* if the corresponding homomorphism is injective, *i.e.*

$$ker(\alpha) = \{g \in G \text{ s.t. } \alpha(g) = e_{\mathbb{S}_n}\} = \{e_G\}$$

or equivalently there are no $g \in G$ such that $g \cdot p = p$ for all $p \in P$. In this perspective we observe that G is isomorphic to $Im_\alpha(G) < \mathbb{S}_n$ and in the following we will identify G with $Im_\alpha(G)$. So the notation $G < \mathbb{S}_n$ makes sence.

In this chapter we will follow standard notations of Permutation group theory and Combinatorics [Sta13]. From now on X will be a finite set of cardinality n. We can consider a total order over X, because it is finite, then we can label it as $[n] = \{1, 2, \ldots, n\}$.

Definition 43. Given an action of G over X, we will write

$$\begin{aligned}
Gx &= \{g \cdot x \text{ s.t. } g \in G\} \text{ orbit of } x \in X \\
G_x &= \{g \in G \text{ s.t. } g \cdot x = x\} \text{ stabilizer of } x \in X \\
X_g &= \{x \in X \text{ s.t. } g \cdot x = x\} \text{ fixed points of } g \in G
\end{aligned}$$

We will consider the (already introduced) equivalence relation

$$x \sim_G y \text{ iff exists } g \in G \text{ s.t. } y = g \cdot x$$

and we will call $G//X = \{Gx \text{ s.t. } x \in X\}$.

Clearly $G_x < G$ so we get the injective (and surjective) map from the left laterals of G_x to the orbit Gx given by

$$\begin{aligned} G/G_x &\to Gx \\ g \cdot G_x &\mapsto g \cdot x \end{aligned}$$

This map is well defined because $h \cdot G_x = g \cdot G_X$ implies $\exists \tau \in G_x$ such that $h = g\tau$ and then $h \cdot x = g\tau \cdot x = g \cdot x$. It is injective because $g \cdot x = h \cdot x$ implies $h^{-1}g \in G_x$ and then $h \cdot G_X = h(h^{-1}g) \cdot G_x = g \cdot G_x$. Hence G finite implies

$$|Gx| = |G/G_x| = \frac{|G|}{|G_x|}$$

The previous equation gives us an useful formula that involves the cardinality of the orbit Gx, the cardinality of G and the cardinality of the stabilizer G_x.

We are interested here in explicit computation of cardinalities of equivalence classes of chords *with a fixed length*, so a finer computation. We will focus on this particular aspect of Combinatorics because it will produce a qualitative difference between $(\mathcal{B}/G, \subseteq)$ and $(\mathcal{P}, \leq_{ref})$. In this chapter we will compute

- Cardinalities of orbits according to the action of a group over a set (just done for a general orbit, we will introduce a finer approach for particular orbits)

- Cardinalities of orbits according to the action of a group over the Boolean lattice $(\mathcal{B}, \subseteq)$

- Cardinalities of orbits according to the action of $\mathbb{S}_n$ over $(\Pi, \leq_{ref})$

3.1 Burnside theorem

A *cycle* is a permutation $\sigma \in \mathbb{S}_n$ of the form $(a_1, \ldots, a_k)$ with the usual meaning $\sigma(a_i) = a_{i+1}$ for $1 \leq i \leq k-1$ and $\sigma(a_k) = a_1$. The action of $G =< \sigma >$ over X induces at most only one orbit whose cardinality is greater then one.

We know that every permutation $\pi \in \mathbb{S}_n$ is the product of disjoint cycles. These cycles are univocally determined and commute. As a consequence, given a permutation $\pi \in \mathbb{S}_n$ and calling $c(\pi)$ the number of cycles of π, we can order the cycles according to their smallest element. So we get

$$1 \leq a_1 \leq a_2 \leq \cdots \leq a_{c(\pi)}$$

Calling k_i the length of the ith cycle, we obtain the standard decomposition of π into disjoint cycles

$$\pi = \circ_{i=1}^{c(\pi)} (a_i, \pi(a_i), \ldots, \pi^{k_i - 1}(a_i))$$

Definition 44. We will call *shape* of the permutation $\pi \in \mathbb{S}_n$ the $n$uple

$$\lambda(\pi) = (\lambda_1(\pi), \ldots, \lambda_n(\pi))$$

where $\lambda_i(\pi)$ is the number of cycles of length i in the standard decomposition of π.

We now want to introduce a function that characterizes different elements of X according to a fixed property.

Definition 45. Let X be a finite set and $\mathfrak{R}$ a ring that contains $\mathbb{Q}$. A *weight function* is a map $w : X \to \mathfrak{R}$. The *inventory* of X is the sum

$$I_X = \sum_{x \in X} w(x)$$

where $w(x)$ is the *weight* of x.

Remark 12. We will consider weight functions w that are costant over orbits of the action of G. We can then test whether an element $x \in X$ belongs to an orbit Ω comparing $w(x)$ and $w(\Omega)$.

Theorem 9. *Let X be a finite set, $G < \mathbb{S}_n$ and $w : X \to \mathfrak{R}$ weight function costant over the orbits of the action of G. Then*

$$I_{G//X} = \sum_{\Omega \in G//X} w(\Omega) = \frac{1}{|G|} \sum_{g \in G} \sum_{x \in X_g} w(x)$$

Proof. The following sets are equals

$$\{(g, x) \text{ s.t. } g \in G, x \in X_g\}$$
$$\{(g, x) \text{ s.t. } g \in G, x \in X, g \cdot x = x\}$$
$$\{(g, x) \text{ s.t. } x \in X, g \in G_x\}$$

Now $w(x) = w(y)$ if $x, y \in \Omega$ gives us that $w(\Omega) = w(x)$ with $x \in \Omega$ is well defined. Then

$$
\begin{aligned}
\sum_{g\in G}\sum_{x\in X_g} w(x) &= \sum_{x\in X}\sum_{g\in G_x} w(x) \\
&= \sum_{x\in X} |G_x| w(x) \\
&= \sum_{x\in X} \frac{|G|}{|Gx|} w(x) \\
&= |G| \sum_{\Omega\in G//X}\sum_{x\in\Omega} \frac{w(\Omega)}{|\Omega|} \\
&= |G| \sum_{\Omega\in G//X} w(\Omega)
\end{aligned}
$$

where we used $|Gx| = \frac{|G|}{|G_x|}$. The thesis follows.

□

Theorem 10 (Cauchy-Frobenius, also known as Burnside)**.** *Let X a finite set with $|X| = n$ and $G < \mathbb{S}_n$. Then*

$$|G//X| = \frac{1}{|G|}\sum_{g\in G} |X_g|$$

Proof. Consider the costant weight function $w(x) = 1$. Then theorem 9 implies

$$\sum_{g\in G}\sum_{x\in X_g} 1 = \sum_{g\in G} |X_g| = |G| \sum_{\Omega\in G//X} 1 = |G| \cdot |G//X|$$

□

We have now the freedom to select a weight function w that focus our attention on particular orbits. For istance, consider $w(x)$ equal to 1 if x is in the orbit $\tilde{\Omega}$ and 0 elsewhere (w is then costant over all the orbits of the action) we notice that the inventory is equal to $|\tilde{\Omega}|$. If $w(x)$ is non-zero only for x that lies in orbits with odd cardinality, and costant over all the orbits, we then ignore in the inventory the orbits with even cardinality. In general the weight function reflects the role that we want to assign to our orbits, according to what we are looking for.

3.2 Pólya theory

We have a combinatorial object, the inventory, that gathers orbits according to the rule given by the weight function w that is costant over the orbits of the action of G over X. We want now an algebraic tool, useful to characterize groups of permutations.

Definition 46. Let $\alpha : G \to \mathbb{S}_n$ an action of G over X, finite set of cardinality n. Consider $\mathfrak{R}[z_1, \dots, z_n]$ the polynomial ring over n variables, where $\mathfrak{R}$ is a ring that contains $\mathbb{Q}$. We will define *cycle-index* the multivariate polynomial

$$P_G(\mathbf{z}) = \frac{1}{|G|} \sum_{g \in G} \mathbf{z}^{\lambda(g)}$$

where $\lambda(g)$ is the shape of g and $\mathbf{z}^{\lambda(g)}$ is the monomial $z_1^{\lambda_1(g)} \cdots z_n^{\lambda_n(g)}$.

We have *encoded* the structure of G in the polynomial P_G. From a computational point of view, we have now a *data structure* that contains informations about the group G and we are able to manipolate these informations simply working with polynomials. The inventory contains informations about what we want, *i.e.* select particular orbits with the weight function, and the cycle-index is a powerful tool to performe the computation.

Until now we have introduced the inventory and the cycle-index in the context of the action of a group G over a set X. How can we shift to the study of the action of a group G over a poset?

Consider again the Boolean lattice $(\mathcal{B}, \subseteq)$. We know that $\mathcal{B}$ is equal to the set of subsets of X, and we know that there is a bijection between $\mathscr{P}(X)$ and $\{0,1\}^X$. More generally we can consider any finite set Y and define the following action of G over Y^X. Given $f \in Y^X$ and $g \in G, x \in X$ define

$$(g \cdot f)(x) = f(g^{-1}(x))$$

If $W : Y \to \mathfrak{R}$ is a weight function then

$$\begin{aligned} w : Y^X &\to \mathfrak{R} \\ f &\mapsto \prod_{x \in X} W(f(x)) \end{aligned}$$

is a weight function over Y^X costant over the orbits of the action f G because

$$w(g \cdot f) = \prod_{x \in X} W(f(g^{-1}x)) = \prod_{x \in X} W(f(x)) = w(f)$$

In the context of the Boolean lattice a weight function $W : 2 \to \mathfrak{R}$ gives us the weight function

$$w(f) = \Big(\prod_{x \in f^{-1}(1)} W(1)\Big) \cdot \Big(\prod_{x \in f^{-1}(0)} W(0)\Big)$$

that characterizes the cardinality of $f^{-1}(1)$. So we are able (with the well known bijection) to select orbits of the action of G over $\mathcal{B}$ that contains only subsets of X with a fixed chardinality.

This is what we were looking for. The following theorem illustrates the power of cycle-index to compute the inventory.

Theorem 11 (Pólya-de Bruijn). *Let X be a finite set of cardinality n, $G < \mathbb{S}_n$ a group acting over X, Y a finite set and the weight function $W : Y \to \mathfrak{R}$. Then*

$$I_{G//Y^X} = \sum_{\Omega \in G//Y^X} w(\Omega) = P_G\Big(\sum_{y \in Y} W(y), \sum_{y \in Y} W(y)^2, \dots, \sum_{y \in Y} W(y)^n\Big)$$

Where P_G is the cycle-index polynomial.

Proof. If we prove that

$$\sum_{f \in Y_g^X} w(f) = \prod_{i=1}^{n} \Big(\sum_{y \in Y} W(y)^i\Big)^{\lambda_i(g)}$$

We can apply theorem 9 and complete the proof. We observe that $f \in Y_g^X$ iff f is costant over the cycles of g, *i.e.* f is costant over the orbits of the action of the permutation group $< g >$ over the set X, in fact $f \in Y_g^X$ means that $g \cdot f = f$. So we can identify Y_g^X with $Y^{<g>//X}$. From now on call $C =< g > //X$. The weight function w is costant over orbits $\Omega \in C$ so

$$\sum_{f \in Y_g^X} w(f) = \sum_{f \in Y_g^X} \prod_{x \in X} W(f(x)) = \sum_{f \in Y^C} \prod_{\Omega \in C} W(f(\Omega))^{|\Omega|}$$

Now $\prod \sum W(y)^{|\Omega|}$ is the product of $\Big(W(y_1)^{|\Omega_1|} + \dots + W(y_m)^{|\Omega_1|}\Big)$ by $\Big(W(y_1)^{|\Omega_2|} + \dots + W(y_m)^{|\Omega_2|}\Big)$ by $\dots$ by $\Big(W(y_1)^{|\Omega_k|} + \dots + W(y_m)^{|\Omega_k|}\Big)$. Rearranging these finite terms, and using the bijective map

$$\begin{aligned} \{1, \dots, m\}^k &\to Y^{<g>//X} \\ (i_1, \dots, i_k) &\mapsto f \end{aligned}$$

such that $f(\Omega_j) = y_{i_j}$, we have

$$\prod_{\Omega\in C}\sum_{y\in Y} W(y)^{|\Omega|} =$$

$$= \sum_{(i_1,\ldots,i_k)\in\{1,\ldots,m\}^k} \prod_{j=1}^{k} W(y_{i_j})^{|\Omega_j|}$$
$$= \sum_{f\in Y^C}\prod_{\Omega\in C} W(f(\Omega))^{|\Omega|}$$

So we obtain a new writing for our sum

$$\sum_{f\in Y_g^X} w(f) = \prod_{\Omega\in C}\sum_{y\in Y} W(y)^{|\Omega|}$$

Since $< g >$ is cyclic, if x belongs to an orbit Ω then Ω is composed by the images of x under the iterates of g. But g has its cyclic decomposition and x is moved by one and only one of these cycles. So $|\Omega|$ is the lenght of the only cycle that moves x. Hence there are exactly $\lambda_i(g)$ orbits of cardinality i, for all $i = 1, \ldots, n$. Then we obtain our thesis with the following re-writing

$$\sum_{f\in Y_g^X} w(f) = \prod_{\Omega\in C}\sum_{y\in Y} W(y)^{|\Omega|} = \prod_{i=1}^{n}\Big(\sum_{y\in Y} W(y)^i\Big)^{\lambda_i(g)}$$

$\square$

We have seen that the bijection between $\mathcal{B}$ and $\{0,1\}^X$ has an analogue in the context of posets: the isomorphism between $(\mathcal{B},\subseteq)$ and $(\{0,1\},\leq)^n$. A critical point in the previous proof is the identification of Y_g^X with $Y^{<g>//X}$, so we wonder: does this fact reflect some useful property of the poset $(\mathcal{B},\subseteq)$?

Example 13. For $n = 5$ and $g = (13)$ we have

$$< (13) > //\{1,2,3,4,5\} =$$
$$\{\{e,(13)\}\cdot 1, \{e,(13)\}\cdot 2, \{e,(13)\}\cdot 3,$$
$$\{e,(13)\}\cdot 4, \{e,(13)\}\cdot 5\}$$
$$\{\{1,3\},\{2\},\{4\},\{5\}\}$$

So there exists an isomorphism between $(\mathcal{B},\subseteq)_{(13)}$ and $(\{0,1\},\leq)^4$. The Hasse diagram of $(\mathcal{B},\subseteq)_{(13)}$ can be easily embedded in $\{0,1\}^4$ with the map that sends an element p of $\mathcal{B}_5$ in (i_1,i_2,i_3,i_4) where i_1 is equal to 1 iff both 1

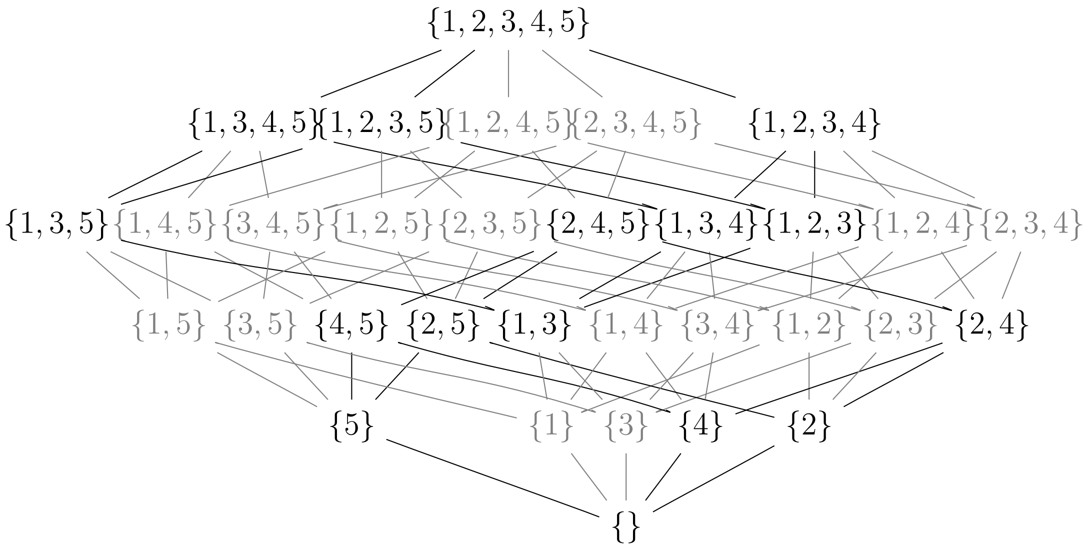

Figure 3.1: $(\mathcal{B}, \subseteq)_{(13)}$ inside the poset $(\mathcal{B}, \subseteq)$.

and 3 are in p (and 0 elsewhere), i_2 is equal to 1 iff 2 is in p (and 0 elsewhere), i_3 is equal to 1 iff 4 is in p (and 0 elsewhere) and i_4 is equal to 1 iff 5 is in p (and 0 elsewhere). From the musical point of view, this means that we have an algebraic tool that emphasize the presence or absence of a particular group of notes in the All-notes catalog and a standard embedding in an hypercube.

Consider the chain $Y = (\{0, \dots, m-1\}, \leq)$ and the antichain $A = ([n], \leq)$, *i.e.* any two elements of $[n]$ are not comparable. We will use only $m = 2$. Call $Hom((P, \leq), Y)$ the set of order-preserving map from a finite poset $(P, \leq)$ to the chain Y. An order-preserving action of $G < Aut(P, \leq)$ on $(P, \leq)$ induces an action of G over $Hom((P, \leq), Y)$ as subset of $[n]^P$, then, by theorem 4, the acyclic category $(Hom((P, \leq_P), Y)/G, \leq)$ is a poset. We have

Theorem 12 ([Joc14] Theorem 2.5)**.** *Let G be a finite group acting by automorphisms on a finite poset $(P, \leq)$. Thus*

$$|Hom((P, \leq), Y)/G| = \frac{1}{|G|} \sum_{g \in G} |Hom((P/ < g >, \leq), Y)|$$

Proof. Theorem 10 implies that

$$|Hom((P, \leq), Y)/G| = \frac{1}{|G|} \sum_{g \in G} |Hom((P, \leq), Y)_g|$$

By definition $\phi \in Hom((P, \leq), Y)_g$ iff $\phi(g^{-1} \cdot p) = \phi(p)$ for all $p \in P$, *i.e.* ϕ is constant over the orbit $< g > p$, then

$$Hom((P, \leq_P), Y)_g \to Hom((P/ < g >, \leq), Y)$$
$$f \mapsto \psi_f$$

with $\psi_f(< g > p) = f(p)$, is a one-to-one correspondence. So

$$|Hom((P, \leq), Y)/G| = \frac{1}{|G|} \sum_{g \in G} |Hom((P/ < g >, \leq), Y)|$$

□

If we consider now $(P, \leq) = A$ and $m = 2$ we obtain

$$Hom(A, Y) = (Y^A, \subseteq) = (\mathcal{B}, \subseteq)$$

Thus we are able to compute the cardinality of $(\mathcal{B}_n/G, \subseteq)$ in a simple way

Corollary 1. *Let $G < Aut(\mathcal{B}, \subseteq) = \mathbb{S}_n$. Then*

$$|\mathcal{B}_n/G| = \frac{1}{G} \sum_{g \in G} m^{c(g)}$$

where $c(g)$ is the number of cycles of g.

Proof. We can consider $c(g) = |[n]/ < g > |$ for all $g \in G$ and observe that $A/ < g >$ is an antichain with $c(g)$ elements. Then

$$|Hom(A/ < g >, Y)| = m^{|A/<g>|} = m^{c(g)}$$

Thus theorem 12 implies

$$|\mathcal{B}_n/G| = |Hom(A, Y)/G| = \frac{1}{|G|} \sum_{g \in G} m^{c(g)}$$

□

3.3 Application: $(\mathcal{B}_n/G, \subseteq)$

We have seen that if $Y = \{0, 1\}$ then

$$w(f) = \Big(\prod_{x\in f^{-1}(1)} W(1)\Big) \cdot \Big(\prod_{x\in f^{-1}(0)} W(0)\Big)$$

Now consider $\mathfrak{R} = \mathbb{Q}[t]$ and the particular weight function

$$\begin{aligned} W : \{0,1\} &\to \mathbb{Q}[t] \\ 0 &\mapsto 1 \\ 1 &\mapsto t \end{aligned}$$

Then

$$w(f) = t^{|f^{-1}(1)|}$$

So $\sum_{y\in\{0,1\}} w(f)^i = 1 + t^i$ and the Pólya-de Bruijn theorem gives us the formula

$$I_{G//\mathcal{B}} = P_G(1 + t, 1 + t^2, \ldots, 1 + t^n)$$

We will consider two particular group actions over $(\mathcal{B}, \subseteq)$ and we will compute the cardinality of their orbits with a fixed length.

3.3.1 Transposition $G = \mathbb{Z}/n\mathbb{Z}$

We label the elements of $G = \mathbb{Z}/n\mathbb{Z}$ as $0, 1, \ldots, n-1$. The elements of G are equivalence classes but we don't use here a specific symbol to emphasize this fact. Call $\tau = (1, 2, \ldots, n) \in \mathbb{S}_n$ and consider the action of G over X given by

$$\begin{aligned} \alpha : \mathbb{Z}/n\mathbb{Z} &\to \mathbb{S}_n \\ k &\mapsto \tau^k \end{aligned}$$

As we have seen at the beginning of this chapter, we will write $< \tau >= G < \mathbb{S}_n$.

The cycle-index of G is then

$$P_{\mathbb{Z}/n\mathbb{Z}}(\mathbf{z}) = \frac{1}{n}\sum_{k=0}^{n-1} \mathbf{z}^{\lambda(\tau^k)}$$

where $\mathbf{z}^{\lambda(\tau^k)} = z_1^{\lambda_1(\tau^k)} z_2^{\lambda_2(\tau^k)} \cdots z_n^{\lambda_n(\tau^k)}$. So we need to compute $\lambda_i(\tau^k)$ for every $i = 1, \ldots, n$.

For all $a \in \mathbb{Z}/n\mathbb{Z}$ we have a in a cycle of the decomposition of τ^k of length l iff $(\tau^k)^l(a) = a$ and $(\tau^k)^h(a) \neq a$ for all $h < l$, *i.e.* l is the smallest non-negative solution of the equation $a + kx = a$ (mod n). So l is the solution of $kx = 0$ (mod n), *i.e.* $l = \frac{n}{(k,n)}$, where (u, v) is the greatest common divisor between u and v.

So $\lambda_i(\tau^k) = (k, n)$ if $i = \frac{n}{(k,n)}$ otherwise $\lambda_i(\tau^k) = 0$. We can also observe that for all $k \in \{0, \ldots, n-1\}$ we have $(k, n)|n$ and for all $d|n$ we have $|\{k \text{ s.t. } (k, n) = d\}| = \phi(\frac{n}{d})$ where ϕ is the *Euler function*. Hence

$$P_{\mathbb{Z}/n\mathbb{Z}}(\mathbf{z}) = \frac{1}{n} \sum_{k=0}^{n-1} z_{\frac{n}{(k,n)}}^{(k,n)} = \frac{1}{n} \sum_{d|n} \phi(\frac{n}{d}) z_{\frac{n}{d}}^{d}$$

that allows us to consider only a small number of variables z_i in the cycle-index. For istance if n is prime we have $P_{\mathbb{Z}/n\mathbb{Z}} = \frac{1}{n} z_1^n$.

We are now able to compute the inventory

$$\begin{aligned}
I_{(\mathbb{Z}/n\mathbb{Z})//\mathcal{B}} &= \\
&= P_{\mathbb{Z}/n\mathbb{Z}}(1 + t, 1 + t^2, \ldots, 1 + t^n) \\
&= \frac{1}{n} \sum_{d|n} \phi\Big(\frac{n}{d}\Big) \Big(1 + t^{\frac{n}{d}}\Big)^d \\
&= \frac{1}{n} \sum_{d|n} \phi\Big(\frac{n}{d}\Big) \sum_{j=0}^{d} \binom{d}{j} t^{\frac{nj}{d}}
\end{aligned}$$

Two powers of t have the same exponent k iff $\frac{nj}{d} = k$, *i.e.* $\frac{n}{k} = \frac{d}{j}$, so

$$\begin{aligned}
I_{(\mathbb{Z}/n\mathbb{Z})//\mathcal{B}} &= \\
&= \sum_{k=0}^{n} \Big(\frac{1}{n} \sum_{d,j \text{ s.t. } \frac{d}{j} = \frac{n}{k}} \phi\Big(\frac{n}{d}\Big) \binom{d}{j} \Big) t^k \\
&= \sum_{k=0}^{n} \Big(\frac{1}{n} \sum_{d,j \text{ s.t. } \frac{d}{j} = \frac{n}{k}} \phi\Big(\frac{k}{j}\Big) \binom{\frac{jn}{k}}{j} \Big) t^k \\
&= \sum_{k=0}^{n} \Big(\frac{1}{n} \sum_{m|(n,k)} \phi(m) \binom{\frac{n}{m}}{\frac{k}{m}} \Big) t^k
\end{aligned}$$

where $m = \frac{k}{j} = \frac{n}{d}$. So we have an explicit formula for the number of orbits with cardinality equal to k, it is

$$\frac{1}{n} \sum_{m|(n,k)} \phi(m) \binom{\frac{n}{m}}{\frac{k}{m}}$$

The following table show the computation for $n = 12$

k	0	1	2	3	4	5	6	7	8	9	10	11	12
orbits with card. k	1	1	6	19	43	66	80	66	43	19	6	1	1

We have 352 orbits, the number of orbits with cardinality k increases and the decreases *with a symmetric movement.*

Example 14. We want to apply corollary 1 to $n = 7$ and verify our results with fig. 2.1

$$\begin{aligned}
g^1 &= (1234567) \text{ has 1 cycle} \\
g^2 &= (1357246) \text{ has 1 cycle} \\
g^3 &= (1473625) \text{ has 1 cycle} \\
g^4 &= (1526374) \text{ has 1 cycle} \\
g^5 &= (1642753) \text{ has 1 cycle} \\
g^6 &= (1765432) \text{ has 1 cycle} \\
e = g^7 &= (1)(2)(3)(4)(5)(6)(7) \text{ has 7 cycles}
\end{aligned}$$

So

$$|\mathcal{B}_7/(\mathbb{Z}/7\mathbb{Z})| = \frac{1}{7}\left(2^7 + 6 \cdot 2^1\right) = 20$$

3.3.2 Inverse-transposition $G = \mathbb{D}_n$

We have to follow the same pattern of the previous computation, with a different cycle-index. The Pólya theory is a flexible approach to Combinatorics because we have only to select the weight function (here fixed) and describe the cycle-index of a given group G, now equal to $\mathbb{D}_n$.

As usual in the context of permutation groups, we will consider the dihedral group $\mathbb{D}_n < \mathbb{S}_n$ as generated by $\tau = (1, 2, \ldots, n)$ and σ with the cyclic decomposition

$$(1, n)(2, n-1)\cdots(\frac{n-1}{2}, \frac{n+3}{2})(\frac{n+1}{2}) \text{ if } n \text{ is odd}$$
$$(1, n)(2, n-1)\cdots(\frac{n}{2}, \frac{n}{2}+1) \text{ if } n \text{ is even}$$

It easy to see that $\tau^n = \sigma^2 = e$ and $\tau\sigma = \sigma\tau^{-1}$, so

$$\mathbb{D}_n = \{\tau^i\sigma^j \text{ s.t. } i = 0, \ldots, n-1 \text{ and } j = 0, 1\}$$

Then the cycle-index is

$$\begin{aligned} P_{\mathbb{D}_n}(\mathbf{z}) &= \\ &= \frac{1}{2n} \sum_{j\in\{0,1\}} \sum_{i=0}^{n-1} \mathbf{z}^{\lambda(\tau^i\sigma^j)} \\ &= \frac{1}{2} P_{\mathbb{Z}/n\mathbb{Z}}(\mathbf{z}) + \frac{1}{2n} \sum_{i=0}^{n-1} \mathbf{z}^{\lambda(\tau^i\sigma)} \end{aligned}$$

We already know $P_{\mathbb{Z}/n\mathbb{Z}}(\mathbf{z})$, so we need to compute $\lambda(\tau^i\sigma)$. The expression $(\tau^i\sigma)^2 = \tau^i(\sigma\tau^i\sigma) = \tau^i\tau^{-i} = e$ implies that in the cyclic decomposition of $\tau^i\sigma$ there are only cycles of length 1 or 2, so it is sufficient to compute the number of cycle of lenght 1, *i.e.* fixed points of the action, to have a complete description of $\lambda(\tau^i\sigma)$.

$$(\tau^i\sigma)(x) = x \text{ iff } n+1-x+i = m \pmod{\mathrm{n}} \text{ iff } 2x = i+1 \pmod{\mathrm{n}}$$

has solutions iff $(2, n)|(i+1)$ and in this case there exist exactly $(2, n)$ solutions.

- If n is odd then exists only one fixed point and $\frac{n-1}{2}$ transpositions
- If n is even then
  - If $i+1$ is even then exist 2 fixed points and $\frac{n-2}{2}$ transpositions
  - If $i+1$ is odd then there is no fixed point and there are $\frac{n}{2}$ transpositions

This means that

$$P_{\mathbb{D}_n}(\mathbf{z}) = \frac{1}{2}P_{\mathbb{Z}/n\mathbb{Z}}(\mathbf{z}) + \frac{1}{2}z_1 z_2^{\frac{n-1}{2}} \text{ if } n \text{ is odd}$$
$$P_{\mathbb{D}_n}(\mathbf{z}) = \frac{1}{2}P_{\mathbb{Z}/n\mathbb{Z}}(\mathbf{z}) + \frac{1}{4}\left(z_1^2 z_2^{\frac{n-2}{2}} + x_2^{\frac{n}{2}}\right) \text{ if } n \text{ is even}$$

It is possible to find explicit formulas for the coefficient of t^k in the inventory $I_{\mathbb{D}_n//\mathcal{B}}$ but we can also use a software like Sage (see Appendix A) to performe the computation.

For $n = 12$ we obtain the following table

k	0	1	2	3	4	5	6	7	8	9	10	11	12
orbits with card. k	1	1	6	12	29	38	50	38	29	12	6	1	1

Again the number of orbits with length k increases and the decreases with a symmetric movement.

3.4 Application: $(\Pi_n/\mathbb{S}_n, \leq_{ref})$

We already know that $(\mathcal{P}, \leq_{ref}) = (\Pi_n/\mathbb{S}_n, \leq_{ref})$. Here we want to compute the cardinality of a fixed rank of $(\mathcal{P}, \leq_{ref})$, *i.e.* the number of Estrada classes with a fixed length. We will follow [WC05].

We can performe our computation in a more general combinatorial framework. Fix a non-empty subset S of $\mathbb{Z}$ and allow S to be infinite. We will consider $S = \mathbb{Z}$ but we can think also S be a different set like the odd numbers, the even numbers, *etc.* We want to compute the number of partitions of n with parts in S, called $p(n; S)$.

For any positive integer k let $S^{[k]}$ the set of function between $[k] = \{1, \ldots, k\}$ and S. $\mathbb{S}_k$ acts on $[k]$ ($f \sim g$ iff there exists $\sigma \in \mathbb{S}_k$ such that $f(\sigma d) = g(d)$ for every $d \in [k]$) so we are able to use Pólya theory. Consider $\mathfrak{R} = \mathbb{Q}[x]$ and the weight function

$$W : S \to \mathbb{Q}[x]$$
$$i \mapsto x^i$$

We have to show that this weight functions allows to compute the number of Estrada classes.

Theorem 13. *Each orbit of the action of* $\mathbb{S}_k$ *over* $S^{[k]}$ *with weight* n *determines a partition of* n *with parts into* S *that has* k *parts, and* viceversa. *In other words, there is a bijection between* $\mathbb{S}_k//S^{[k]}$ *and* $(\Pi_n/\mathbb{S}_n)_k$ *the* k*th rank of the poset* $(\Pi_n/\mathbb{S}_n, \leq_{ref})$.

Proof. Let E be an equivalence class with weight n in $S^{[k]}$ and let f be a function in E. We know that S is a subset of $\mathbb{Z}$ and then we can arrange $Im(f)$ in a non-increasing order $j_1 \leq \cdots \leq j_k$, repetition allowed, and obtain

$$x^n = w(f) = \prod_{i+1}^{k} W(f(i)) = x^{j_1+\cdots+j_k}$$

Then $n = j_1 + \ldots j_k$, *i.e.* f corresponds to a partition of n with k parts into S.

The equivalence relation given by the condition $f(\sigma d) = g(d)$ implies that every g has image $Im(g) = \{j_1, \ldots, j_k\}$, thus each equivalence class with weight n corresponds to a partition of n with k parts into S.

Conversely, let $t_1 \leq \cdots \leq t_k$ a partition of n with k parts into S. The function $h(i) = t_i$ for $i = 1, \ldots, k$ is an element of $S^{[k]}$ and

$$w(h) = \prod_{i=1}^{k} x^{t_i} = x^{t_1+\cdots+t_k} = x^n$$

Then any partition of n with k parts into S determines the equivalence class containing h in $S^{[k]}$.

□

The Pólya-de Bruijn result (theorem 11) implies that the number of partitions of a positive integer n in k parts in S, called $p(n,k;S)$, is the coefficient of X^n in the counting series

$$P_{\mathbb{S}_k}\Big(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \ldots, \sum_{i\in S} x^{ki}\Big)$$

The number $p(n;S)$ of partitions of n with parts in S is the coefficient of X^n in the counting series

$$\sum_{k=1}^{\infty} P_{\mathbb{S}_k}\Big(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \ldots, \sum_{i\in S} x^{ki}\Big)$$

Example 15. Consider $S = \mathbb{Z}$ and $[3] = \{1, 2, 3\}$, we have

$$\mathbb{S}_3 = \{e, (123), (132), (12), (13), (23)\}$$

Then we obtain

$$P_{\mathbb{S}_3}(z_1, z_2, z_3) = \frac{1}{6}(x_1^3 + 2x_3 + 3x_1x_2)$$

From $w : S \to \mathbb{Q}[x]$ with $W(i) = x^i$ we have the formal power series $\sum_{i\in S} W(i) = \sum_{i=1}^{\infty} x^i, \sum_{i\in S} W(i)^2 = \sum_{i=1}^{\infty} x^{2i}$ and $\sum_{i\in S} W(i)^3 = \sum_{i=1}^{\infty} x^{3i}$. Hence we have

$$\begin{aligned}
P_{\mathbb{S}_3}\Big(\sum_{i=1}^{\infty} x^i, \sum_{i=1}^{\infty} x^{2i}, \sum_{i=1}^{\infty} x^{3i}\Big) = \\
&= \frac{1}{6}\Big((x^1 + x^2 + x^3 + \dots)^3 + 2(x^3 + x^6 + x^9 + \dots) \\
&+ 3(x^1 + x^2 + x^3)(x^2 + x^4 + x^6 + \dots)\Big) \\
&= \frac{1}{6}\Big((x^3 + 3x^4 + 6x^5 + 10x^6 + 15x^7 + 21x^8 + \dots) \\
&+ (2x^3 + 2x^6 + \dots) \\
&+ (3x^3 + 3x^4 + 6x^5 + 6x^6 + 9x^7 + 9x^8)\Big) \\
&= x^3 + x^4 + 2x^5 + 3x^6 + 4x^7 + 5x^8 + \dots
\end{aligned}$$

There exists no partition of $n = 1, 2$ with 3 parts, there is 1 partition of $n = 3$ with 3 parts, there is 1 partition of $n = 4$ with 3 parts, there are 2 partitions of $n = 5$ with 3 parts, there are 3 partitions of $n = 6$ with 3 parts, and so on.

How can we describe the cycle-index $P_{\mathbb{S}_n}$ for a general n?

Consider the poset $(\Pi, \leq_{ref})$ and in particular one partition π of $[n]$ with j_k parts of length k. Every part of π give us $\frac{k!}{k}$ cycles of length k because we have k choices for its first element, $k-1$ for the second, ..., and we have to divide by k because $(a_1, \dots, a_k)$ is equal to $(a_k, a_1, \dots, a_{k-1})$, $(a_{k-1}, a_k, a_1, \dots, a_{k-2})$, ..., $(a_2, \dots, a_k, a_1)$. We want to ignore the order in which our j_k parts appear in π, so we have to consider the action of the group $\mathbb{S}_{j_k}$ over them. So for every partition π we have to count

$$\frac{n!}{\prod_{k=1}^{n}(k!)^{j_k}} \prod_{k=1}^{n} \Big(\frac{k!}{k}\Big)^{j_k} \prod_{k=1}^{n} \frac{1}{j_k} = \frac{n!}{\prod_{k=1}^{n} k^{j_k} j_k!}$$

The cardinalities of the parts of π are related by $1 \cdot j_1 + 2 \cdot j_2 + \dots n \cdot j_n = n$, then

$$P_{\mathbb{S}_n}(\mathbf{z}) = \frac{n!}{n!} \sum_{1\cdot j_1+2\cdot j_2+\ldots n\cdot j_n=n} \frac{1}{\prod_{k=1}^{n} k^{j_k} j_k!} \prod_{k=1}^{n} z_k^{j_k}$$

We wonder: there exists a recursive formula for $p(n;S)$? We will describe some Combinatorics tools. Observe that

$$\begin{aligned} exp\Big(z_1 x + z_2\frac{x^2}{2} + z_3\frac{x^3}{3} + \ldots\Big) &= \\ &= e^{z_1 x} \cdot e^{z_2\frac{x^2}{2}} \cdot e^{z_3\frac{x^3}{3}} \cdots \\ &= \Big(\sum_{n\geq 0} \frac{z_1 x^n}{n!}\Big) \cdot \Big(\sum_{n\geq 0} \frac{z_2 x^{2n}}{2^n n!}\Big) \cdot \Big(\sum_{n\geq 0} \frac{z_3 x^{3n}}{3^n n!}\Big) \cdots \end{aligned}$$

and in the previous product the coefficient of $z_1^{c_1} z_2^{c_2} \cdots x^l$, with $l = c_1 + 2c_2 + \ldots$, is equal to

$$\frac{1}{1^{c_1} c_1! 2^{c_2} c_2! \cdots} = \frac{1}{l!}\Big(\frac{l!}{1^{c_1} c_1! 2^{c_2} c_2! \cdots}\Big)$$

Then

$$\sum_{l\geq 0} P_{\mathbb{S}_l}(z_1, z_2, \ldots) x^l = exp\Big(z_1 x + z_2\frac{x^2}{2} + z_3\frac{x^3}{3} + \ldots\Big)$$

In [HP73] we can find a more general equation

$$1 + \sum_{k=1}^{\infty} P_{\mathbb{S}_k}(f(x), f(x^2), \ldots, f(x^k)) x^k = exp\Big(\sum_{k=1}^{\infty} \frac{f(x^k)}{k}\Big)$$

where $f(x)$ is a function of x or a series of x, and also (for all $m \geq 1$) we have

$$a_m = A_m - m^{-1}\Big(\sum_{k=1}^{m-1} k a_k A_{m-k}\Big)$$

starting from

$$\sum_{m=0}^{\infty} A_m x^m = exp\Big(\sum_{m=1}^{\infty} a_m x^m\Big)$$

Then we can re-arrenge the previous result and obtain a recursive formula.

Theorem 14. *The number of partition of $n > 1$ with parts in S is equal to*

$$p(n;S) = \frac{1}{n}\Big(\sum_{i|n, i\in S} i + \sum_{k=1}^{n-1}\Big(\sum_{i|k, i\in S} i\Big)p(n-k;S)\Big)$$

Proof. We know that

$$1 + \sum_{n=1}^{\infty} p(n;S)x^n = 1 + \sum_{k=1}^{\infty} P_{\mathbb{S}_k}\Big(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \ldots, \sum_{i\in S} x^{ki}\Big)$$

Now we can re-write this equation as

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} p(n;S)x^n = & \\ &= exp\Big(\sum_{k=1}^{\infty}\Big(\sum_{i\in S}\frac{x^{ki}}{k}\Big)\Big) \\ &= exp\Big(\sum_{k=1}^{\infty}\Big(\sum_{i\in S}\frac{ix^{ki}}{ik}\Big)\Big) \\ &= exp\Big(\sum_{k=1}^{\infty}\frac{1}{n}\Big(\sum_{i|n, i\in S} i\Big)x^n\Big) \end{aligned}$$

So

$$\frac{1}{n}\sum_{i|n, i\in S} i = p(n;S) - \frac{1}{n}\Big(\sum_{k=1}^{n-1} k \cdot \frac{1}{k}\Big(\sum_{i|k, i\in S} i\Big)p(n-k;S)\Big)$$

Then

$$p(n;S) = \frac{1}{n}\Big(\sum_{i|n, i\in S} i + \sum_{k=1}^{n-1}\Big(\sum_{i|k, i\in S} i\Big)p(n-k;S)\Big)$$

□

We can compile the following table, in which we list the number of partition of 12 with k parts.

parts	1	2	3	4	5	6	7	8	9	10	11	12
cardinality	1	6	12	15	13	11	7	5	3	2	1	1

These are the cardinalities of Estrada classes with a fixed length. We observe that this distribution increases and then decreases, *but not with a symmetric movement*. For $(\mathcal{B}/G, \leq)$, when G is $\mathbb{Z}/n\mathbb{Z}$ or $\mathbb{D}_n$, we have seen that the distribution is symmetric, so we wonder if a more careful analysis of our posets can explain this fenomenon. We need more theory to answer this question and we will devote the following chapter to this purpose.

3.5 Application: $(\mathcal{B}_n/\mathbb{D}_n^{(2)}, \leq)$

We already know how to list the elements of $\mathbb{D}_n$, here we want to compute the cardinality of the ranks of $(\mathcal{B}_n/\mathbb{D}_n^{(2)}, \leq)$ for $n = 5$. We can then draw fig. 2.5 and have an upper bound for the cardinality of the ranks of $(\mathcal{P}, \leq_{ref})$.

We can label all the 2-subsets of [5] as in the following table and perform the computation of $(\tau^i\sigma^j)^{(2)}$.

$\mathbb{D}_5$	labels	$\mathbb{D}_5^{(2)}$
$\tau = (12345)$	{1,2}=a	$\tau^{(2)} = (aehld)(bficg)$
$\tau^2 = (13524)$	{1,3}=b	$(\tau^2)^{(2)} = (ahdel)(bigfc)$
$\tau^3 = (14253)$	{1,4}=c	$(\tau^3)^{(2)} = (aledh)(bcfgi)$
$\tau^4 = (15432)$	{1,5}=d	$(\tau^4)^{(2)} = (adlhe)(bgcif)$
$\sigma = (15)(24)(3)$	{2,3}=e	$\sigma^{(2)} = (al)(bi)(cg)(d)(eh)(f)$
$\sigma\tau = (1)(25)(34)$	{2,4}=f	$(\sigma\tau)^{(2)} = (ad)(bc)(el)(fi)(g)(h)$
$\sigma\tau^2 = (12)(35)(4)$	{2,5}=g	$(\sigma\tau^2)^{(2)} = (a)(bg)(cf)(de)(hl)(i)$
$\sigma\tau^3 = (13)(2)(45)$	{3,4}=h	$(\sigma\tau^3)^{(2)} = (ae)(b)(ci)(dh)(fg)(l)$
$\sigma\tau^4 = (14)(23)(5)$	{3,5}=i	$(\sigma\tau^4)^{(2)} = (ah)(bf)(c)(dl)(e)(gi)$
$e = (1)(2)(3)(4)(5)$	{4,5}=l	$e^{(2)} = (a)(b)(c)(d)(e)(f)(g)(h)(i)(l)$

Hence

$$P_{\mathbb{D}_5^{(2)}}(\mathbf{z}) = \frac{1}{10}\Big(x_1^{10} + 4x_5^2 + 4x_1^2x_2^4\Big)$$

Thus with $x_i = 1 + t^i$ for $i = 1, \ldots, 10$ we have

$$I_{\mathbb{D}_5^{(2)}//\mathcal{B}_{[5]}} = t^{10} + 2t^9 + 7t^8 + 16t^7 + 26t^6 + 32t^5 + 26t^4 + 16t^3 + 7t^2 + 2t + 1$$

3.6 Summary table

We can summarize the computational results of this chapter in the following table

$n = 12$	0	1	2	3	4	5	6	7	8	9	10	11	12	tot.
$(\mathcal{B}_n/(\mathbb{Z}/n\mathbb{Z}), \subseteq)$	1	1	6	19	43	66	80	66	43	19	6	1	1	352
$(\mathcal{B}_n/\mathbb{D}_n, \subseteq)$	1	1	6	12	29	38	50	38	29	12	6	1	1	224
$(\mathcal{P}, \leq_{ref})$	0	1	6	12	15	13	11	7	5	3	2	1	1	77

The catalog by transpositions $(\mathcal{B}_n/(\mathbb{Z}/n\mathbb{Z}), \leq)$ contains more chords than the catalog by transpositions and invertions $(\mathcal{B}_n/\mathbb{D}_n, \subseteq)$, that contains more chords than the catalog $(\mathcal{P}, \leq_{ref})$. We consider equivalent two chords according to some symmetry group, more this group is elaborated less chords we obtain.

Chapter 4

Unimodality, rank-symmetry and Sperner

We have seen that the number of classes of cardinality k in $\mathcal{B}/G$, with $G = \mathbb{Z}/n\mathbb{Z}, \mathbb{D}_n$, increases and then decreases when k runs from 0 to n. This movement is symmetric. We have also seen that the number of partitions of n in k parts increases and then decreases, but the movement is not symmetric. This is an interesting property and we want to describe it with the language of partially ordered set.

We have seen that $(\mathcal{B}, \subseteq)$ is a finite graded poset and the ith rank contains exactly the subsets of $X = [n]$ with cardinality equal to i. We wonder

- For which group G the quotient poset $(\mathcal{B}/G, \leq)$ is graded? (the answer will be: for every $G < Aut(\mathcal{B}, \subseteq)$)
- When it is well defined, how can we relate the cardinality of its ranks, *e.g.* $|(\mathcal{B}/G)_i|$, with the cardinality of the ranks of $(\mathcal{B}, \subseteq)$?
- What can we say about the cardinality of ranks when i increases? (we will introduce the concept of unimodality)

Definition 47. We define *rank-unimodal* the finite sequence $a_0, \ldots, a_n$ of real numbers iff exists j such that

$$a_0 \leq \cdots \leq a_j \geq \cdots \geq a_n$$

So we call rank-unimodal a finite sequence that increases and then decreases.

We want to prove that the ranks of our quotient posets have cardinality that increases and then decreases showing that they are rank-unimodal, according to the following definition.

Definition 48. Let $(P, \leq)$ a finite graded poset with $P = P_0 \sqcup P_1 \sqcup \cdots \sqcup P_k$ the decomposition of P according to the rank function, *i.e.* $P_i = \rho^{-1}(i)$.

- If exists j such that $|P_0| \leq |P_1| \leq \cdots \leq |P_j| \geq |P_{j+1}| \geq \cdots \geq |P_k|$ then we call $(P, \leq)$ *rank-unimodal.*
- If for all $k = 0, \ldots, k$ we have $|P_i| = |P_{k-i}|$ then we call $(P, \leq)$ *rank-symmetric.*

Example 16. $(\mathcal{B}_n, \subseteq)$ is rank-symmetric and rank-unimodal because $|P_i| = \binom{n}{i} = \binom{n}{n-i} = |P_{n-i}|$ and $\binom{n}{0} \leq \cdots \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \geq \cdots \geq \binom{n}{n}$. In a rank unimodal and rank symmetric poset we can have $|P_0| \leq |P_1| \leq \cdots < |P_j| = |P_{j+1}| = \cdots = |P_{h-1}| = |P_h| > |P_{h+1}| \geq \cdots \geq |P_k|$. In $(\mathcal{B}_n, \subseteq)$ if n is odd then $\binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lfloor \frac{n}{2} \rfloor + 1}$ so it has two *middle elements.*

We can easily answer our question about rank-symmetry of $(\mathcal{B}/G, \leq)$. From now on we will write $\mathcal{B}_n$ instead $\mathcal{B}$ to point out the cardinality of $X = [n]$.

Theorem 15. *For all $G < Aut(\mathcal{B}, \subseteq)$, $(\mathcal{B}/G, \leq)$ is rank-symmetric.*

Proof. The rank of $o \in \mathcal{B}/G$ is simply the rank of y for $y \in o$, where $y \in \mathcal{B}$, then $\mathcal{B}/G$ is graded and

$$|(\mathcal{B}_n/G)_i| = |(\mathcal{B}_n)_i/G|$$

For $o = \{y_1, \ldots, y_j\}$ where $|y_h| = i$ for every $h = 1, \ldots, j$ we can consider $\{y_1^C, \ldots, y_j^C\}$, an orbit of subsets of X with cardinality $n - i$; so

$$|(\mathcal{B}_n)_i/G| = |(\mathcal{B}_n)_{n-i}/G|$$

Then we have

$$|(\mathcal{B}_n/G)_i| = |(\mathcal{B}_n)_i/G| = |(\mathcal{B}_n)_{n-i}/G| = |(\mathcal{B}_n/G)_{n-i}|$$

□

We have seen, in the context of Pólya theory, that all our quotient posets are rank-unimodal but in general not rank-symmetric, *e.g.* $(\Pi, \leq_{ref})$ is not rank-symmetric. We know that $(\mathcal{P}, \leq_{ref})$ is a quotient of $(\Pi, \leq_{ref})$ and we wonder if the latter is rank-unimodal. In the following section we will answer this question and we will introduce new theoric matherial.

4.1 Unimodal sequences and log-concave sequences

Consider a stronger definition about finite real number sequences.

Definition 49. A finite sequence $a_0, a_1, \ldots, a_n$ of real numbers is called

- *logarithmically concave* (log-concave for short) if for all $1 \leq i \leq n-1$ the inequality $a_i^2 \geq a_{i-1}a_{i+1}$ holds.
- *strongly log-concave* if the sequence $\left\{b_i = \frac{a_i}{\binom{n}{i}}\right\}_i$ is log-concave, *i.e.* $b_i^2 \geq b_{i-1}b_{i+1}$ for $1 \leq i \leq n-1$. Strong log-concavity is equivalent to

$$a_i^2 \geq \left(1+\frac{1}{i}\right)\left(1+\frac{1}{n-i}\right)a_{i-1}a_{i+1}$$

for all $1 \leq i \leq n-1$ because $\binom{n}{m+1} = \frac{n-m}{m+1}\binom{n}{m}$ and then

$$\begin{aligned}
a_i^2 \geq & \\
&\geq \frac{\binom{n}{i}^2}{\binom{n}{i-1}\binom{n}{i+1}}a_{i-1}a_{i+1} \\
&= \frac{\binom{n}{i}^2}{\binom{n}{i}\frac{n-i}{i+1}\binom{n}{i}}\frac{n-i+1}{i}a_{i-1}a_{i+1} \\
&= \frac{i+1}{i}\frac{n-i+1}{n-i}a_{i-1}a_{i+1} \\
&\geq a_{i-1}a_{i+1}
\end{aligned}$$

So strongly log-concavity implies log-concavity. Does log-concavity implies unimodality? In general no, but

Theorem 16. *Let $a_0, \ldots, a_n$ a finite sequence of real numbers. If*

- *it is non-negative*
- *has no internal zeros,* i.e. *whenever $i < j < k$ the condition $a_i \neq 0$ and $a_k \neq 0$ implies $a_j \neq 0$*

then log-concave implies unimodality.

Proof. If there are only two elements of the sequence that are different from zero then they are consecutive, *i.e.* exists j such that $a_j, a_{j+1} \neq 0$, so for all i we have $a_{i-1}a_{i+1} = 0$ hence the sequence is unimodal.

$$0 \leq 0 \leq \cdots \leq 0 < a_j \gtreqless a_{j+1} > 0 \geq 0 \cdots \geq 0$$

If there are at least three elements of the sequence that are different from zero then they are consecutive by hypothesis. By contraddiction, if exists $1 \leq j \leq n-1$ such that $a_{j-1} > a_j \leq a_{j+1}$ and $a_{j+1} > 0$ then $a_j^2 < a_{j-1}a_{j+1}$, that is in contraddiction with the log-concavity. So we have

$$a_1 \leq \cdots \leq a_{l-2} \leq a_{j-1} \leq a_j \geq a_{j+1} \geq a_{j+2} \cdots \geq a_n$$

Hence the sequence is unimodal.

□

We now show that $(\Pi_n, \leq_{ref})$ is rank-unimodal with a proof *by induction* (for more information [Eng97]).

Definition 50. Define $S_{n,k}$ *Stirling number of second kind* the number of different ways to partition the set $[n]$ into k parts.

We can characterize $S_{n,k}$ with the following recursive formula.

Theorem 17. *The number of different ways to partition the set* $[n]$ *into* k *parts is equal to*

$$S_{n,k} = kS_{n-1,k} + S_{n-1,k-1}$$

Proof. We can explain this equation observing that a partition of $[n]$ can contain the singleton $\{n\}$, and we have $S_{n-1,k-1}$ choises, or not, and we have $kS_{n-1,k}$ choises because we have $S_{n-1,k}$ partition of $[n] \setminus \{n\}$ in k parts and we have to select which join with $\{n\}$.

□

For a graded poset $(P, \leq)$ we call $W_i(P) = |P_i|$ *Whitney numbers* for $i = 0, \ldots, n$. The definition of rank function implies $W_i(\Pi_n, \leq_{ref}) = S_{n,n-i}$.

Theorem 18. *The Stirling sequence* $S_{n,k}$ *is rank-unimodal.*

Proof. By induction we will prove that $S_{n,k}$ for $k = 1, \ldots, n$ is strictly log-concave. It is non-negative and has no internal zeros, hence theorem 16 implies unimodality. For $n = 1, 2$ it is trivial, and

$$
\begin{aligned}
S_{n,k}^2 - S_{n,k-1}S_{n,k+1} = & \\
& \Big(S_{n-1,k-1}^2 - S_{n-1,k-2}S_{n-1,k}\Big) + \\
& \Big(k^2 S_{n-1,k}^2 - (k^2-1)S_{n-1,k-1}S_{n-1,k+1}\Big) + \\
& (k+1)\Big(S_{n-1,k-1}S_{n-1,k} - S_{n-1,k-2}S_{n-1,k+1}\Big)
\end{aligned}
$$

Direct application of induction implies that the first two summands are positive. The third is positive because

$$S_{n-1,k-1}^2 S_{n-1,k}^2 \geq (S_{n-1,k-2}S_{n-1,k})(S_{n-1,k+1}S_{n-1,k-1})$$

□

Another important result is the log-concavity of the partitions of integers n. The proof requires Hardy-Ramanujan asymptotic formula [HR00] and a carefull error analysis. We will only sketch the main steps of the proof, refer to [Pak14] for details.

Theorem 19. *The number of partitions of n is a log-concave sequence for all $n > 2600$.*

Proof. The number of partitions of n, according to the Rademacher's truncated series, is equal to

$$p(n) = \frac{\sqrt{12}}{24n-1}\sum_{k=1}^{N} A_k^*(n)\Big[\Big(1-\frac{k}{\mu}\Big)e^{\frac{\mu}{k}} + \Big(1+\frac{k}{\mu}\Big)e^{-\frac{\mu}{k}}\Big] + R_2(n,N)$$

where $\mu(n) = \frac{\pi}{6}\sqrt{24n-1}$, N is the truncation term, $A_k^*(n)$ is a complicated arithmetic function and $R_2(n,N)$ is the error.

We can isolate th largest individual contributions to the sum and write $p(n) = T(n) + R(n)$ where

$$
\begin{aligned}
T(n) &= \frac{\sqrt{12}}{24n-1}\Big[\Big(1-\frac{1}{\mu}e^{\mu}\Big) + \frac{(-1)^n}{\sqrt{12}}e^{\frac{\mu}{2}}\Big] \\
R(n) &= \frac{\pi^2}{6\sqrt{3}\mu^2}\Big[\Big(1+\frac{1}{\mu}\Big)e^{-\mu} - \frac{2(-1)^n}{\sqrt{2}\mu}e^{\frac{\mu}{2}} + \frac{(-1)^n}{\sqrt{2}}\Big(1+\frac{2}{\mu}\Big)e^{-\frac{\mu}{2}}\Big] + R_2(n,2)
\end{aligned}
$$

We want to prove that $p(n)$ is log-concave by showing

$$p_2(n) = \log(p(n-1)) - 2\log(p(n)) + \log(p(n+1)) \leq 0$$

We can estimate with complex computations the values of $2\log(T(n)) - \log(T(n-1)) - \log(T(n+1))$ and $\frac{|R(n)|}{T(n)}$, so we are able to bound $p(n)$ with

$$T(n)\Big(1 - \frac{|R(n)|}{T(n)}\Big) < p(n) < T(n)\Big(1 + \frac{|R(n)|}{T(n)}\Big)$$

and then $p_2(n)$ as

$$\frac{1}{(24n)^{\frac{3}{2}}} < p_2(n) < \frac{2}{n^{\frac{3}{2}}}$$

for all $n > 2600$, that implies the thesis.

□

We can verify by *brute force* that $p(n)$ is log-concave for $n < 2600$ and then we obtain log-concavity for all n.

Establishing rank-unimodality of $(\mathcal{P}, \leq_{ref})$,*i.e.* unimodality of the sequence $p(n,k)$ for a fixed n, is an *open problem*. We shortly report here some partial results.

Theorem 20. *Let $n > 4$ a fixed integer. Then $p(n,k)$ is not log-concave when $0 < k \leq n$.*

Proof. Observe that $p(n,n-1) = p(n,n) = 1$ and $p(n,n-2) \geq 2$. Then

$$p(n,n-1)^2 = 1 < 2 \leq p(n,n-2) = p(n,n-2)p(n,n)$$

Hence $p(n,k)$ is not log-concave.

□

Remark 13. By brute force it is possible to check log-concavity of $p(n,k)$ for $51 \leq n \leq 10000$ and $1 < k < n-25$, and there exists a bijection between $p(\tilde{n},k)$ and $p(n)$ for $\lceil \frac{\tilde{n}}{2} \rceil \leq k \leq \tilde{n}$ and $n \leq \lfloor \frac{\tilde{n}}{2} \rfloor$. So we have partial results about log-concavity of $p(n,k)$, and then rank-unimodality of $(\mathcal{P}, \leq_{ref})$, but not a full result. Figure 2.5 gives us some bounds but not a complete proof of the rank-unimodality of $(\mathcal{P}, \leq_{ref})$.

4.2 Sperner property

Remember that an antichain is a subset $A \subset P$ no two elements of which are comparable in $(P, \leq)$ and that in a graded poset every P_i is an antichain, hence

$$\max_{A \text{ antichain of } P} |A| \geq \max_i |P_i|$$

Definition 51. When the equality holds in the previous equation we say that $(P, \leq)$ is *Sperner*.

In a Sperner poset the largest rank provides an antichain of maximum cardinality, but there may exists other antichains with the same maximum cardinality. If a poset is rank-unimodal, rank-symmetric and Sperner, then only the middle ranks (remember it's possible the existance of multiple middle ranks) are the antichains with maximum cardinality.

The study of maximum or minimum cardinality of a collection of finite objects in combinatorics is called *extremal theory*.

4.2.1 Linear algebraic approach

We will follow Stanley's work (original paper [Sta91], and more recently [Sta13]) discussing a combinatorial condition which guarantees rank unimodality and Sperner, and then an algebraic machinary to prove this condition in particular cases.

Definition 52. Let $(P, \leq)$ a graded poset in which $\max \rho(p) = n$. In this case we say that $(P, \leq)$ has *rank* n. We define an *order-matching* from P_i to P_{i+1} to be an injective function $\mu : P_i \to P_{i+1}$ such that , for all $p \in P_i$, we have $p < \mu(p)$. Similarly we define an order-matching from P_i to P_{i-1} to be a injective function $\mu : P_i \to P_{i-1}$ such that , for all $x \in P_i$, we have $\mu(x) < x$.

Theorem 21. *$(P, \leq)$ a graded poset of rank n. If there exists an integer $0 \leq j \leq n$ and order-machings such that*

$$P_0 \to P_1 \to P_2 \to \cdots \to P_j \leftarrow P_{j+1} \leftarrow P_{j+2} \leftarrow \cdots \leftarrow P_n$$

then $(P, \leq)$ is rank-unimodal and Sperner.

Proof. The injective property of order-matchings implies rank-unimodality because $P_i \to P_{i+1}$ injective implies $|P_i| \leq |P_{i+1}|$. To prove that $(P, \leq)$ is Sperner we consider G the sub-graph of the Hasse diagram of $(P, \leq)$ the vertices of which are the elements of P and p, q are connected by an edge iff one of the order-matchings μ satisfies $\mu(p) = q$. Injectivity implies that

G is a disjoint union of paths, including single-vertex paths. Every path is a chain, so we have partitioned $(P, \leq)$ in disjoint chains. We have just seen that $(P, \leq)$ is rank-unimodal with biggest rank P_j so all of our chains contain a vertex in P_j (again we use injectivity) and then the number of our chains is exacly $|P_j|$. Any antichain A can intersect each of our chains at most once (otherwise in A we can find two elements that are comparable) so $|A|$ cannot exceed the number of our chains: $|A| \leq |P_j|$, so $(P, \leq)$ is Sperner. □

Definition 53. S a finite set, $\mathfrak{K}$ a field that contains $\mathbb{Q}$,*i.e.* with characteristic 0. Define $\mathfrak{K}S$ the vector space consisting of all formal linear combinations (with coefficients in $\mathfrak{K}$) of elements of S. S is a basis for $\mathfrak{K}S$. If $(P, \leq)$ is a graded poset then $\mathfrak{K}P = \mathfrak{K}P_0 \oplus \cdots \oplus \mathfrak{K}P_n$.

Theorem 22. *If exists a linear transformation $U : \mathfrak{K}P_i \to \mathfrak{K}P_{i+1}$ (U stands for* up*) such that*

- *U is injective*
- *For all $p \in P_i$, $U(p)$ is a linear combination of elements $q \in P_{i+1}$ such that $p < q$, in which case U is said* order-raising *operator,*

then there exists an order-matching $\mu : P_i \to P_{i+1}$. Similarly, if exists a linear transformation $U : \mathfrak{K}P_i \to \mathfrak{K}P_{i+1}$ such that

- *U is onto*
- *U is order-raising operator*

then there exist an order-matching $\mu : P_{i+1} \to P_i$

Proof. $U : \mathfrak{K}P_i \to \mathfrak{K}P_{i+1}$ injective order-rising operator. We will denote $[U]$ the matrix associated to U with respect to the bases P_i of $\mathfrak{K}P_i$ and P_{i+1} of $\mathfrak{K}P_{i+1}$, so the rows of $[U]$ are indexed by the elements $q_1, \ldots, q_{|P_{i+1}|}$ of P_{i+1}, in some order, and the columns by the elements $p_1, \ldots, p_{|P_i|}$. U injective implies that the rank of $[U]$ is equal to $|P_i|$, and then $[U]$ has $|P_i|$ linearly independent rows. We can consider an order between elements of P_{i+1} such that the first $|P_i|$ rows of $[U]$ are linearly independent. Call A the square $|P_i| \times |P_i|$ top-left sub-matrix of $[U]$, we have

$$det(A) = \sum_{\sigma \in \mathbb{S}_{|P_i|}} \pm a_{1\sigma(1)} \ldots a_{|P_i|\sigma(|P_i|)} \neq 0$$

where the sum is over all permutations σ of $\{1, \ldots, |P_i|\}$. Then exists at least one permutation τ such that $a_{1\tau(1)} \ldots a_{|P_i|\tau(|P_i|)} \neq 0$, so for all $1 \leq h \leq$

$|P_i|$ we have $a_{h\tau(h)} \neq 0$. This means that $U(p_{\tau(h)})$ is a linear combination of the elements q_k, for $k \leq h$, and in particular (we use the order-raising property) $p_{\tau(h)} < q_h$. We can now easily construct the order-matching map $\mu : P_i \to P_{i+1}$ as $\mu(p_h) = q_{\tau^{-1}(h)}$. Considering the transpose of $[U]$ we can prove the same statement for U onto instead of injective. □

We can now produce a proof of the Sperner property of $(\mathcal{B}, \subseteq)$. We consider $(\mathcal{B}_n)_i$ a basis for $\mathfrak{K}(\mathcal{B}_n)_i$, so it is well defined the linear transformation

$$\begin{aligned} \mathfrak{K}(\mathcal{B}_n)_i &\to \mathfrak{K}(\mathcal{B}_n)_{i+1} \\ p \mapsto U_i(p) &= \sum_{(\mathcal{B}_n)_{i+1} \ni q > p} q \end{aligned}$$

and observe that U_i is an order-raising by definition.
Define the linear transformation (D stands for *down*)

$$\begin{aligned} \mathfrak{K}(\mathcal{B}_n)_i &\to \mathfrak{K}(\mathcal{B}_n)_{i-1} \\ q \mapsto D_i(q) &= \sum_{(\mathcal{B}_n)_{i-1} \ni p < q} p \end{aligned}$$

Denote $[U_i]$ the matrix associated with U_i with respect to the basis $(\mathcal{B}_n)_i$ and $(\mathcal{B}_n)_{i+1}$, and $[D_i]$ the matrix associated with D_i with respect to the basis $(\mathcal{B}_n)_i$ and $(\mathcal{B}_n)_{i-1}$.

Example 17. Consider the Boolean lattice $(\mathcal{B}_3, \subseteq)$. We want to compute $[U_1]$ and $[D_2]$ according to the previous definitions.

$$\begin{tikzcd}
 & \{1,2,3\} \arrow[dl] \arrow[d] \arrow[dr] & \\
\{1,2\} \arrow[d] \arrow[dr] & \{1,3\} \arrow[dl] \arrow[dr] & \{2,3\} \arrow[dl] \arrow[d] \\
\{1\} \arrow[dr] & \{2\} \arrow[d] & \{3\} \arrow[dl] \\
 & \{\} &
\end{tikzcd}$$

Figure 4.1: $(\mathcal{B}_3, \subseteq)$

We can choose an order, *e.g.* $\{1\}$,$\{2\}$,$\{3\}$ for $(\mathcal{B}_3)_1$ and $\{1,2\}$,$\{1,3\}$,$\{2,3\}$ for $(\mathcal{B}_3)_2$. Hence we have

$$\begin{aligned}
U_1 : \mathfrak{K}(\mathcal{B}_3)_1 &\to \mathfrak{K}(\mathcal{B}_3)_2 \\
\{1\} &\mapsto \{1,2\} + \{1,3\} \\
\{2\} &\mapsto \{1,2\} + \{2,3\} \\
\{3\} &\mapsto \{1,3\} + \{2,3\} \\
D_2 : \mathfrak{K}(\mathcal{B}_3)_2 &\to \mathfrak{K}(\mathcal{B}_3)_1 \\
\{1,2\} &\mapsto \{1\} + \{2\} \\
\{1,3\} &\mapsto \{1\} + \{3\} \\
\{2,3\} &\mapsto \{2\} + \{3\}
\end{aligned}$$

and then

$$[U_1] = [U_1]^T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = [D_2]$$

We can observe that

$$det[U_1] = -1^{(12)} - 0^{(13)} - 1^{(23)} + 0^e + 0^{(123)} + 0^{(132)} = -2 \neq 0$$

where we mark each $sgn(\sigma) \prod_{j=1}^{|(\mathcal{B}_3)_1|} a_{j,\sigma(j)}$ with the apex σ. So we can build two order-matchings $\mu_{(12)}, \mu_{(23)}$.

$$\begin{tikzcd}
 & \{1,2,3\} & \\
\{1,2\} & \{1,3\} & \{2,3\} \\
\{1\} \arrow[ur] & \{2\} \arrow[ul] & \{3\} \arrow[u] \\
 & \{\} &
\end{tikzcd}$$

Figure 4.2: $(\mathcal{B}_3, \subseteq)$ with $\mu_{(12)}$

We are able to generalize something of the previous example, as we can read in the following theorem.

Theorem 23. *We have* $[D_{i+1}] = [U_i]^T$

$\{1,2,3\}$

$\{1,2\}\ \{1,3\}\ \{2,3\}$

$\{1\}\ \ \{2\}\ \ \{3\}$

$\{\}$

Figure 4.3: $(\mathcal{B}_3, \subseteq)$ with $\mu_{(23)}$

Proof. For $x \in \mathfrak{K}(\mathcal{B}_n)_i, y \in \mathfrak{K}(\mathcal{B}_n)_{i+1}$ we have

$$U_i(x) = \sum_h a_h U_i(p_h) = \sum_h a_h \sum_{q > p_h, q \in (\mathcal{B}_n)_{i+1}} q = \sum_{h,k} \alpha_{h,k} q_k$$
$$D_{i+1}(y) = \sum_k a_k D_{i+1}(q_k) = \sum a_k \sum_{p < q_k, p \in (\mathcal{B}_n)_i} p = \sum_{h,k} \beta_{k,h} p_h$$

where $\alpha_{h,k} = a_h \chi_{C_h^+}(q_k), \beta_{k,h} = a_k \chi_{C_k^-}(p_h)$ with

$$C_h^+ = \{q \in (\mathcal{B}_n)_{i+1} \text{ s.t. } q > p_h\}$$
$$C_k^- = \{p \in (\mathcal{B}_n)_i \text{ s.t. } p < q_k\}$$

and $\chi_{C_h^+}(q_k) = 1 \Leftrightarrow \chi_{C_k^-}(p_h) = 1$. Then $[U_i]_{h,k} = [D_{i+1}]_{k,h}$. □

Remark 14. We can define U_i, D_i for a general graded poset $(P, \leq)$ in the same way obtaining $[U_i]^T = [D_{i+1}]$.

Theorem 24 (Only for $(\mathcal{B}, \subseteq)$)**.** *Define* $U_n = D_0 = 0$*. Then for all* $0 \leq i \leq n$ *we have*

$$D_{i+1}U_i - U_{i-1}D_i = (n - 2i)I_i$$

Proof. We will prove that $(D_{i+1}U_i - U_{i-1}D_i)(p) = (n-2i)p$ for all $p \in (\mathcal{B}_n)_i$.

It's clear that

$$D_{i+1}U_i(p) = D_{i+1}\big(\sum_{|q|=i+1, p \subset q} q\big)$$
$$= \sum_{|q|=i+1, p\subset q} \sum_{|z|=i, z \subset q} r$$

So

- if $p, r \in (\mathcal{B}_n)_i$ such that $|p \cap r| < i - 1$ then there is no $q \in (\mathcal{B}_n)_{i+1}$ such that $r \subset q, p \subset q$, that implies: the coefficient of r in $D_{i+1}U_i(p)$ is 0.
- if $p, r \in (\mathcal{B}_n)_i$ such that $|p \cap r| = i - 1$ then there is only one q such that $r \subset q, p \subset q$, namely $q = p \cup r$.
- if $r = p$ then q can be any element of $(\mathcal{B}_n)_{i+1}$ that contains p, so we have $n - i$ elements that give us a non-empty summation.

It follows that

$$D_{i+1}U_i(p) = (n-i)p + \sum_{|r|=i, |r \cap p| = i-1} r$$

Similarly we have

$$U_{i-1}D_i(p) = ip + \sum_{|r|=i, |r \cap p| = i-1} r$$

Then $(D_{i+1}U_i - U_{i-1}D_i)(p) = (n - 2i)p$.

□

With the characterization of D and U we can now prove what we want.

Theorem 25. *U_i is injective if $i < \frac{n}{2}$ and onto if $i \geq \frac{n}{2}$. Then $(\mathcal{B}, \subseteq)$ is (rank-unimodal and) Sperner.*

Proof. From linear algebra we know that a (rectangular) matrix times its transpose is *symmetric* and *positive semidefinite*, *i.e.* has non-negative (real) eigenvalues. We know that $[D_i] = [U_{i-1}]^T$ and $D_{i+1}U_i = U_{i-1}D_i + (n - 2i)I_i$, so *if* $i < \frac{n}{2}$ then $n - 2i > 0$ and the eigenvalues of $D_{i+1}U_i$ are strictly positive. $U_{i-1}D_i$ and $(n - 2i)I_i$ commutes. Then $D_{i+1}U_i$ is invertible bacause

$$Ker(D_{i+1}U_i) = \{x \in \mathfrak{K}(\mathcal{B}_n)_i \text{ s.t. } x \neq 0 \text{ and } D_{i+1}U_i x = 0 \cdot x = 0\}$$

and we know that there is no zero eigenvalue because by *simultaneous diagonalization* we can say that all its eigenvalues are strictly positive. Now U_i is *left-invertible* because $(D_{i+1}U_i)^{-1}D_{i+1} \circ U_i = I$, then U_i is injective. *If* $i \geq \frac{n}{2}$ then we can consider $U_iD_{i+1} = D_{i+2}U_{i+1} + (2i + 2 - n)I_{i+1}$, so U_iD_{i+1} is invertible, then U_i is onto becouse U_i is *right-invertible.*

□

The previous proof works only because we have a general property about U_i, D_i and a particular property of $(\mathcal{B}, \subseteq)$.

Corollary 2. *$(\mathcal{B}, \subseteq)$ is rank-symmetric, rank-unimodal and Sperner.*

Proof. We have found that $(\mathcal{B}, \subseteq)$ is rank-unimodal and Sperner. We already know that $(\mathcal{B}, \subseteq)$ is rank-symmetric.

□

In the next section we will apply our linear algebra approach to the more complex case of $(\mathcal{B}/G, \leq)$.

4.3 $(\mathcal{B}/G, \leq)$ is rank-symmetric, rank-unimodal and Sperner

To every $\pi \in \mathbb{S}_n$ we can associate a linear transformation (still denoted π)

$$\pi : \mathfrak{K}(\mathcal{B}_n)_i \to \mathfrak{K}(\mathcal{B}_n)_i$$
$$\sum_{p \in (\mathcal{B}_n)_i} c_p p \mapsto \sum_{p \in (\mathcal{B}_n)_i} c_p \pi(p)$$

and then an action of any subgroup G of $\mathbb{S}_n$ on the vector space $\mathfrak{K}(\mathcal{B})_i$. The matrix associated to π is a *permutation matrix*, *i.e.* a matrix with one 1 in every row and column and 0's elsewhere. We will denote the fixed elements $\mathfrak{K}(\mathcal{B}_n)_i^G$.

Theorem 26. *A basis for $\mathfrak{K}(\mathcal{B}_n)_i^G$ consists of the elements*

$$\left\{ v_o = \sum_{p \in o} p \right\}$$

where $o \in (\mathcal{B}_n)_i/G$, the set of G-orbit for the action of G over $(\mathcal{B}_n)_i$.

Proof. It's clear that if $\pi \in G$ then $\pi(v_o) = v_o$, because π permutes the elements of o, so $v_o \in \mathfrak{K}(\mathcal{B}_n)_i^G$. It's also clear that v_o's are linearly independent because any $p \in (\mathcal{B}_n)_i$ appears with nonzero coefficient in exactly one v_o.

We want to show that every $v = \sum_{p \in (\mathcal{B})_i} c_p p \in \mathfrak{K}(\mathcal{B}_n)_i^G$ can be written as a linear combination of v_o's.

We know that $\pi(p) = \sigma(p)$ iff $\pi, \sigma \in G_p$, where G_p is the left coset of p. It follows that

$$\sum_{\pi \in G} \pi(p) = |G_p| \cdot v_{o_x}$$

where $x \in v_{o_x}$. We can then apply π to $v = \sum_{p \in (\mathcal{B})_i} c_p p$ and sum on all $\pi \in G$

$$\begin{aligned} |G| \cdot v &= \sum_{\pi \in G} \pi(v) \\ &= \sum_{\pi \in G} \Big(\sum_{p \in (\mathcal{B}_n)_i} c_p \pi(p) \Big) \\ &= \sum_{p \in (\mathcal{B}_n)_i} c_p \Big(\sum_{\pi \in G} \pi(p) \Big) \\ &= \sum_{p \in (\mathcal{B}_n)_i} c_p \cdot (\# o_x) \cdot v_{o_x} \end{aligned}$$

We are in a field $\mathfrak{K}$ so we can divide by $|G|$ and obtain v as a linear combination of elements v_{o_x}.

□

Theorem 27. *If $v \in \mathfrak{K}(\mathcal{B}_n)_i^G$ then $U_i(v) \in \mathfrak{K}(\mathcal{B}_n)_{i+1}^G$*

Proof. For $p \in (\mathcal{B}_n)_i$, the order-preserving property of $\pi \in G$ implies

$$U_i(\pi(p)) = \sum_{(\mathcal{B}_i)_{i+1} \ni q > \pi(p)} q = \sum_{(\mathcal{B}_n)_{i+1} \ni \pi^{-1}(q) > p} q = \pi(U_i(p))$$

For $v \in \mathfrak{K}(\mathcal{B}_n)_i^G$, the linearity of U_i implies $\pi(U_i(v)) = U_i(\pi(v)) = U_i(v)$, so $U_i(v) \in \mathfrak{K}(\mathcal{B}_n)_{i+1}^G$.

□

Theorem 28. *Let G a subgroup of $\mathbb{S}_n$. Then the quotient poset $(\mathcal{B}_n/G, \leq)$ is Sperner.*

Proof. We will denote $P = \mathcal{B}_n/G$ and define order-raising operators $\hat{U}_i : \mathfrak{K}P_i \to P_{i+1}$ and order-lowering operators $\hat{D}_i : \mathfrak{K}P_i \to P_{i-1}$. We now know how the usual order-raising operator U_i works:

$$U_i(v_o) = \sum_{o' \in (\mathcal{B}_n)_{i+1}/G} c_{o,o'} v_{o'}$$

where $o \in (\mathcal{B}_n)_i/G$. Then we can define the linear operator

$$\begin{aligned} \hat{U}_i : \mathfrak{K}((\mathcal{B}_n)_i/G) &\to \mathfrak{K}((\mathcal{B}_n)_{i+1}/G) \\ o &\mapsto \sum_{o' \in (\mathcal{B}_n)_{i+1}/G} c_{o,o'} o' \end{aligned}$$

Now we have the *commutative diagram*

$$\begin{array}{ccc} (\mathfrak{K}\mathcal{B}_n)_i^G & \xrightarrow{U_i} & (\mathfrak{K}\mathcal{B}_n)_{i+1}^G \\ \downarrow & & \downarrow \\ \mathfrak{K}(\mathcal{B}_n/G)_i & \xrightarrow{\hat{U}_i} & \mathfrak{K}(\mathcal{B}_n/G)_{i+1} \end{array}$$

where the vertical arrows refer to the linear transformation $v_o \mapsto o$. We claim that $\hat{U}_i$ is order-raising. We need to show that if $c_{o,o'} \neq 0$ in the definition of $\hat{U}_i$ then $o' > o$ in P. The only way $c_{o,o'} \neq 0$ is for some $p' \in o'$ to satisfy $p' > p$ for some $p \in o$ but this condition is exactly the definition of quotient of poset, so $\hat{U}_i$ is order-raising by contruction.

Now, if $i < \frac{n}{2}$ then U_i is injective, but U_i and $\hat{U}_i$ are exactly the same transformation, except for the names of the basis elements, so $\hat{U}_i$ is also injective.

An exactly analogous argument can be applied to D_i instead of U_i, so if $i > \frac{n}{2}$ then $\hat{D}_i$ is order-lowering and onto. Then $\mathcal{B}_n/G$ is Sperner.

□

We can resume our result about rank-unimodality, rank-symmetry and Sperner property in the following corollary.

Corollary 3. *G a subgroup of $\mathbb{S}_n$. Then the quotient poset $(\mathcal{B}_n/G, \leq)$ is graded of rank n, rank-symmetric, rank-unimodal and Sperner.*

4.4 $(\Pi, \leq_{ref})$ and $(\mathcal{P}, \leq_{ref})$ are Sperner?

A theorem of Canfield [Can78] implies that, for large n, $(\Pi_n, \leq_{ref})$ is not Sperner. Other intersting computations can be found in [Can03].

Remark 15. $(\mathcal{P}_n, \leq_{ref})$ Sperner is an open problem for large n. Nevertheless $(\mathcal{P}, \leq_{ref})$ is Sperner for $n < 45$ and then with $n = 12$ we have that the greatest antichain is the rank $(\mathcal{P}_{12})_3$.

4.5 Extremal theory over the Bi-chord decompositions poset

As we had seen, the Bi-chord decompositions poset is a Boolean lattice, then graded, rank-symmetric, rank-unimodal and Sperner. We had also seen that $(\mathcal{B}_X/G, \leq)$ is the quotient of the Boolean lattice $(\mathcal{B}_X, \subset)$ by the re-labelling group G, so it's graded, rank-symmetric, rank-unimodal and Sperner.

We can find in this posets informations about Estrada catalog and Forte catalog. The relation with graphs theory motivate future studies.

Chapter 5

Conclusions

We gave the following answers to our initial questions:

- The object we chosen to represent standard catalogs of chords is

$$(\mathcal{B}/G, \leq)$$

- The object we chosen to represent the Estrada catalog is

$$(\mathcal{P}, \leq_{ref})$$

- For any n and any symmetry, *i.e.* for any $G < Aut(\mathcal{B}, \subseteq)$, the standard catalog $(\mathcal{B}_n/G, \leq)$ has ranks that increase and then decrease with a symmetric movement.

- For $n = 5, 7$ and 12 the Estrada catalog has ranks that increase and then decrease but we have no general result.

We have seen how simple questions that arise in Musical set theory can lead us in wide fields of Mathematics as Pólya theory, posets quotients, unimodality theory and rank-simmetry theory. We focused on enumeration and representation of chords catalogs not only to answer these questions but also because composers and musicologists are looking for math-musical objects that are both general and concrete. Thus we have exposed accessible technologies such as categorial objects, for which a computer science framework already exists, tangible "data structure" like the Cycle-index or the order-raising operator, and pictures as the Hasse diagram of a poset. Further work can involves

- the beautiful proof of the Pólya theorem given by Gian Carlo Rota [RS77] because *Galois connections* and *Möbius inversion theorem*, involved in the proof, can enlarge our knowledge.

- the *order complex* associated to a poset as defined in [Koz08]. Do exist n and $G < Aut(\mathcal{B}, \subseteq)$ such that the order complex associated to $(\mathcal{B}/G, \leq)$ has non-trivial (co)homology groups?

- the research of a proof of the rank-unimodality or $(\mathcal{P}, \leq_{ref})$.

Beside other properties, Music contains also formal structures and Mathematics can provide contexts, science and technology to work with. Following [Ala10], from the composition to the performance, composers have to overcame many difficulties and often they need a different language to express their ideas.

Appendix A

Sage

We will describe here how to represent in Sage[1] our posets and poset quotients, and to perform computations related to Pólya theory.

A.1 Posets

In Sage there are many tools devoted to the construction and study of posets, in particular we can work with

```
sage: P = Posets.IntegerPartitions(7); P
Finite poset containing 15 elements
sage: len(P.cover_relations())
28
```

Or we can manage

```
sage: Posets.BooleanLattice(5)
Finite lattice containing 32 elements
```

but we can encounter some troubles working with the labels of elements. Why? Sage is built on top of the Python[2] programming language but has its own objects, so for istance "0.6**2" in Python returns something like "0.35999999999999999" but Sage pre-parses the expression and transforms the input like this

[1]Sage (http://www.sagemath.org) is a GPL licensed software. We are interested in its Combinatorics framework.

[2]In particular it is developed with Cython (http://cython.org) and many libraries caming from different open source projects. The power of Object-Oriented paradigm, that works well thanks to the *hash table* data structure, is then used to deal with numerical analysis tools (like the LAPACK library), symbolic computation tools and computational discrete Algebra (like GAP).

```
sage: preparse(``0.6**2'')
``RealNumber0(``0.6'')**Integer(2)''
```

The result is "0.360000000000000", and the computation is performed with the help of RealNumber and Integer, that are not in Python. We are interested in the Combinat library of Sage, that contains the general framework objects and the particular posets objects we are looking for. Reading the source code (sage-6.4.1/ src/ sage/ combinat/ posets/ poset_examples.py at line 89) we find that the BooleanLattice object is defined as

```
LatticePoset([[Integer(x|(1<<y)) for \
y in range(0,n) if x&(1<<y)==0] for\
x in range(0,2**n)])
```

and the LatticePoset has the following definition (sage-6.4.1/ src/ sage/ combinat/ posets/ lattices.py at line 332)

```
def LatticePoset(data, *args, **options):
#we omit the comments
    if isinstance(data,FiniteLatticePoset) and\
len(args) == 0 and len(options) == 0:
        return data
    P = Poset(data, *args, **options)
    if not P.is_lattice():
        raise ValueError("Not a lattice.")
    return FiniteLatticePoset(P, \
category = FiniteLatticePosets(), facade = P._is_facade)
```

From which we understand that in the Combinat library every ojbects is built with a categorial construction, for istance

```
sage: L = LatticePoset([[1,2],[3],[3]])
sage: type(L)
<class ``sage.combinat.posets.lattices.
FiniteLatticePoset_with_category''>
```

It is clear that the BooleanLattice function labels its elements not according to the subset relation, so we prefere to build our $(\mathcal{B}, \subseteq)$, this time with the right labels, as follows

```
sage: elements=[Set(b) for b in\
Set([1,2,3,4,5]).subsets().list()]
sage: relation = [[i,j] for i in elements for j in\
elements if i.issubset(j) ]
```

```
sage: P=Poset([elements,relation]); P
Finite poset containing 32 elements
sage: hasse=P.hasse_diagram(); hasse
Digraph on 32 vertices
#for istance we can now print the Hasse diagram in
#a convinient way, useful to work with
#the tikz package of latex
sage: hasse.set_latex_options(tkz_style = ``Classic'')
sage: latex(hasse)
```

The categorial approach of Sage in the context of Combinatorics fits our aim to build $(\mathcal{B}/G, \leq)$ as colimit of the functor between $\mathcal{C}G$ and **Ac**, but in Sage 6.4.1 the colimit object is not implemented yet. So we are forced to perform an explicit construction of the relation $R \subseteq \mathcal{B}/G \times \mathcal{B}/G$ that defines the partial order $\leq$.

```
sage: gap.eval(\
``G:=Group((1,2,3,4,5,6,7,8,9,10,11,12))'')
sage: null = gap.eval(\
``s:=Combinations([1,2,3,4,5,6,7,8,9,10,11,12])'')
sage: G = gap(\
``Group((1,2,3,4,5,6,7,8,9,10,11,12))'')
sage: s = gap(\
``Combinations([1,2,3,4,5,6,7,8,9,10,11,12])'')
#list is a list of orbist, that are lists of lists
sage: list = gap(``Orbits(G,s,OnSets)'')
#slist is a list of obits,
#but now seen as lists of sets
sage: slist = [ map(lambda x: Set(set(x)),l) for l \
in list]
#R is the relation, seen as a subset of rep\times rep
sage: R = []
sage: RepSet = set([])
sage: for p in slist:
    for q in slist:
        #if len(p) <= len(q) and len(q) > 0:
        if p[0].cardinality() <= q[0].cardinality()\
and len(q) > 0:
            y = q[0]
            for z in p:
                if z.issubset(y):
                    RepSet.add(p[0])
```

```
                    RepSet.add(y)
                    R.append((p[0],y))
                    break
#rep contains one representative element for each orbit
sage: rep=Set(RepSet).list()
sage: len(R)
23831
#means that we have 23831 lesEqual relations
sage: len(rep)
352
#means that we have 352 elements in our quotient poset
sage: QuotPoset = Poset([rep,R])
#QuotPoset is the data structure that embed our poset
```

With the QuotPoset object we can list all the orbits, or all the orbits with a fixed length, we can check some property, *e.g.* gradness, and we can plot the Hasse diagrams as directed graph.

A.2 Pólya theory

We will define the cycle-index P_G for a permutation group G with an explicit coputation. For $G = \mathbb{Z}/n\mathbb{Z}$ we have

```
sage: n=12
sage: T, t = QQ[``t''].objgen()
sage: R = PolynomialRing(QQ,n,``z'')
sage: z=[R.gen(i) for i in range(n)]
sage: P = 0
for d in divisors(n):
    P=P+euler_phi(n/d)*z[n/d-1]^d
P = 1/n * P
sage: P([1+t^(i+1) for i in range(n)]).coefficients()
[1, 1, 6, 19, 43, 66, 80, 66, 43, 19, 6, 1, 1]
```

For $G = \mathbb{S}_k$, useful to compute the number of partition of n with k parts, we have

```
sage: n=12
sage: k=4
sage: T.<t> = PowerSeriesRing(QQ)
sage: R = PolynomialRing(QQ,k,``z'')
sage: z=[R.gen(i) for i in range(k)]
```

```
sage: def select(m,l):
    if m == l:
        return 1
    else:
        return 0
sage: P = 0
for p in Partitions(k):
    j=[ sum(map(lambda m: select(m,l+1),p))\
for l in range(k) ]
    aus = 1
    monomial = 1
    for l in range(k):
        aus = aus*(l+1)^(j[l])*factorial(j[l])
        monomial = monomial*z[l]^j[l]
    P = P + (1/aus)*monomial
sage: P([ sum([ t^((i+1)*(j+1)) for j in range(n) ]) \
for i in range(k) ])
```

For istance the cycle-index for $k = 4$ is equal to

$$P_{\mathbb{S}_4} = \frac{1}{24}z_0^4 + \frac{1}{4}z_0^2 z_1 + \frac{1}{8}z_1^2 + \frac{1}{3}z_0 z_2 + \frac{1}{4}z_3$$

and the coefficient of t^{12} in

$$P_{\mathbb{S}_4}\Big(\sum_{i\in S} t^i, \sum_{i\in S} t^{2i}, \dots, \sum_{i\in S} t^{ki}\Big)$$

is equal to 15.

Remark 16. We select $S = \mathbb{Z}$, so we use Sage PowerSeriesRing object. In the actual computation we don't need an infinite amount of terms of $\sum_{i\in\mathbb{Z}} t^{ji}$ but only $i = 1, \dots, k$.

If we want to compute the number of all the partitions of n we have to consider the coefficient of t^n in

$$\sum_{k=1}^{\infty} P_{\mathbb{S}_k}\Big(\sum_{i\in S} t^i, \sum_{i\in S} t^{2i}, \dots, \sum_{i\in S} t^{ki}\Big)$$

We have also the recursive formula for this computation, that is more efficient from a computational point of view.

Appendix B

Joyal theory

We will introduce shortly the concept of *species* in the context of *Joyal theory* as we can read in [LY89] and [BLL98].

B.1 Another categorial point of view

Consider the category **Fin** of finite sets and bijections.

Definition 54. Given a finite set U, we will call $T[U]$ the set of T-*structures* (or structure of *species* T) on U. Given a bijection $u : U \to V$ we will call $T[u] : T[U] \to T[V]$ *transport of* T-*structures along* u.

Example 18. Given a finite set of vertices U, we can consider the species of simple graphs $\mathcal{G}[U]$ as

$$\mathcal{G}[U] = \left\{ (U, E) \text{ s.t. } E \subseteq \binom{U}{2} \right\}$$

where $\binom{U}{2}$ is the set of unordered pairs of distinct elements of U. Using species transport $\mathcal{G}[\sigma]$, where $\sigma \in \mathcal{S}_U$, we can permute the labels of the vertices.

We are interested in *classes of species*, *i.e.* species that are equivalent according to a re-labelling of the elements of U. Consider then the following action of $\mathcal{S}_U$ over $T[U]$

$$\begin{aligned} \mathcal{S}_U \times T[U] &\to T[U] \\ (\sigma, t) &\mapsto \sigma \cdot t = T[\sigma](t) \end{aligned}$$

Definition 55. we will call

- *exponential generating series*

$$T(x) = \sum_{n\geq 0} |T[n]| \frac{x^n}{n!}$$

where $U = [n]$ for all $n \in \mathbb{N}$.

- *type-generating series*

$$\tilde{T}(x) = \sum_{n\geq 0} |T[n]/\mathbb{S}_n| x^n$$

where $U = [n]$ and $\mathcal{S}_{[n]} = \mathbb{S}_n$ acts over $T[n]$ as before.

- *cycle-index series*

$$Z_T(x_1, x_2, \dots) = \sum_{n\geq 0} \sum_{\sum i\lambda_i = n} Fix_T(\sigma) \frac{x_1^{\lambda_1} \cdots x_n^{\lambda_n}}{1^{\lambda_1}\lambda_1! 2^{\lambda_2}\lambda_2! \cdots n^{\lambda_n}\lambda_n!}$$

where $\sigma \in \mathbb{S}_n$ is a permutation of type $(\lambda_1, \dots, \lambda_n)$ and $Fix_T(\sigma) = |\{t \in T[n] \text{ s.t. } \sigma \cdot t = t\}|$.

Example 19. Consider for $n = 2$ the species of simple graphs over 2 vertices. We have only 2 graphs: the complete graph K_2 and its complement K_2^C. $\mathbb{S}_n = \{e, (12)\}$ and

$$\begin{aligned} e \cdot K_2 &= K_2 \\ (12) \cdot K_2 &= K_2 \\ e \cdot K_2^2 &= K_2^C \\ (12) \cdot K_2^C &= K_2^C \end{aligned}$$

So

$$\begin{aligned} T(x) &= 2\frac{x^2}{2} + \dots \\ \tilde{T}(x) &= 2x^2 + \dots \\ Z_T(x_1, x_2, \dots) &= 2x_1^2 + 2x_2 + \sum_{n\geq 3} \sum_{\sum i\lambda_i = n} Fix_T(\sigma) \frac{x_1^{\lambda_1} \cdots x_n^{\lambda_n}}{1^{\lambda_1}\lambda_1! 2^{\lambda_2}\lambda_2! \cdots n^{\lambda_n}\lambda_n!} \end{aligned}$$

For $n = 3$ we have more complex series, *e.g.* $(12)(3) \in \mathbb{S}_3$ fixes 4 species and permutes 2 pairs of species, thus its therm in the cycle-index series is $4x_1^1 x_2^1$.

We can compute the series $T(x)$ and $\tilde{T}(x)$ starting from $Z(x)$.

Theorem 29. *For any species T, we have*

$$\begin{aligned} T(x) &= Z_T(x, 0, 0, \dots) \\ \tilde{T}(x) &= Z_T(x, x^2, x^3, \dots) \end{aligned}$$

Proof. Substituting $x_1 = x$ and $x_i = 0$ for all $i \geq 2$ gives

$$Z_T(x, 0, 0, \dots) = \sum_{n\geq 0} \frac{1}{n!}\Big(\sum_{\sigma\in\mathbb{S}_n} Fix_T(\sigma)x^{\lambda_1}0^{\lambda_2}\cdots\Big)$$

For each $n \geq 0$ we have that $x^{\lambda_1}0^{\lambda_2}\cdots = 0$ except if $\lambda_1 = n$ and $\lambda_i = 0$ for all $i \geq 2$, *i.e.* only the identity permutation $\sigma = id$ contribute to the sum. Thus

$$\begin{aligned} Z_T(x, 0, 0, \dots) &= \sum_{n\geq 0} \frac{1}{n!} Fix_T(id)x^n \\ &= \sum_{n\geq 0} \frac{1}{n!} |T[n]| x^n \\ &= T(x) \end{aligned}$$

With theorem 9 we can prove that

$$\begin{aligned} Z_T(x, x^2, x^3, \dots) &= \sum_{n\geq 0} \frac{1}{n!} \sum_{\sigma\in\mathbb{S}_n} Fix_T(\sigma)x^{\lambda_1}x^{2\lambda_2}x^{3\lambda_3}\cdots \\ &= \sum_{n\geq 0} \frac{1}{n!} \sum_{\sigma\in\mathbb{S}_n} Fix_T(\sigma)x^n \\ &= \sum_{n\geq 0} |T[n]/\mathbb{S}_n| x^n \\ &= \tilde{T}(x) \end{aligned}$$

□

B.2 Algebra of species

We want now to define some interesting species and operations among them. From now on we will call *base set* or *label set* the finite set U.

Definition 56. We will call

- *empty species* 0 the species such that $0[U] = \emptyset$.
- *empty set species* 1 the species such that $1[\emptyset] = \{\emptyset\}$ and $1[U] = \emptyset$ if $U \neq \emptyset$.
- *set species* $\mathcal{E}$ the species such that $\mathcal{E}[U] = \{U\}$.
- *subset species* $\mathcal{P}$ the species of subsets of the label set.
- *partition species* P the species of partitions of the base set.

We easily compute

$$\mathcal{E}(x) = |\{\emptyset\}|x^0 + \sum_{n>0} |\{\{1, \dots, n\}\}| \frac{x^n}{n!} = e^x$$
$$\tilde{\mathcal{E}}(x) = \sum_{n\geq 0} x^n = \frac{1}{1-x}$$

But how can we compute the subset species and the partition species?

Definition 57. For any species $\mathcal{F}, \mathcal{G}$ with $\mathcal{G}[\emptyset] = \emptyset$. We can define the *multiplication*

$$\begin{aligned}(\mathcal{F} \cdot \mathcal{G})[U] &= \cup_{S\subseteq U} \mathcal{F}[S] \times \mathcal{G}[U \setminus S] \\ &= \Big\{(s,t) \text{ s.t. there exists } S \subseteq U \\ &\quad \text{with } s \in \mathcal{F}[S] \text{ and } t \in \mathcal{G}[U \setminus S]\Big\} \\ (\mathcal{F} \cdot \mathcal{G})[\sigma](s,t) &= (\mathcal{F}[\sigma_1](s), \mathcal{G}[\sigma_2](t))\end{aligned}$$

with σ_1, σ_2 restrictions to the label sets of s, t respectively. We can also define the *composition*

$$\begin{aligned}(\mathcal{F} \circ \mathcal{G})[U] = \Big\{&(s,T) \text{ s.t. } s \in \mathcal{F}[T] \text{ and } T \text{ is a set of} \mathcal{G}\text{-structures} \\ &\text{whose label sets partition } U\Big\}\end{aligned}$$

Let $(s,T) \in (\mathcal{F} \circ \mathcal{G})[U]$. For each $t \in T$ let σ_t be the restriction of σ to the label set of t and let τ be a function from T defined by $\tau(t) = \mathcal{G}[\sigma_t](t)$. Then define

$$(\mathcal{F} \circ \mathcal{G})[\sigma](s) = \tau(s)$$

Theorem 30. *For any species $\mathcal{F}, \mathcal{G}$ with $\mathcal{G}[\emptyset] = \emptyset$ we have*

$$\begin{aligned}
(\mathcal{F} \cdot \mathcal{G})(x) &= \mathcal{F}(x)\mathcal{G}(x) \\
(\widetilde{\mathcal{F} \cdot \mathcal{G}})(x) &= \tilde{\mathcal{F}}(x)\tilde{\mathcal{G}}(x) \\
Z_{\mathcal{F}.\mathcal{G}}(x_1, \dots) &= Z_{\mathcal{F}}(x_1, \dots)Z_{\mathcal{G}}(x_1, \dots) \\
(\mathcal{F} \circ \mathcal{G})(x) &= \mathcal{F}(\mathcal{G}(x)) \\
(\widetilde{\mathcal{F} \circ \mathcal{G}})(x) &= Z_{\mathcal{F}}(\tilde{\mathcal{G}}(x), \tilde{\mathcal{G}}(x^2), \tilde{\mathcal{G}}(x^3), \dots) \\
Z_{\mathcal{F} \circ \mathcal{G}}(x_1, \dots) &= Z_{\mathcal{F}}(Z_{\mathcal{G}}(x_1, \dots), Z_{\mathcal{G}}(x_2, \dots), Z_{\mathcal{G}}(x_3, \dots), \dots)
\end{aligned}$$

B.2.1 Subset species and partition species

If $\mathcal{E}$ is the set species then an $\mathcal{E} \cdot \mathcal{E}$-structure is a partition of the label set into two parts. An element is in the second part if it is not in the first, so $\mathcal{E} \cdot \mathcal{E} = \mathcal{P}$ the subset species. Thus

$$\mathcal{P}(x) = \mathcal{E} \cdot \mathcal{E}(x) = e^{2x} = \sum_{n=0}^{\infty} \frac{2^n}{n!}$$

A partition is a pairwise disjoint set of nonempty sets, using up all the elements of the input set. Then, calling $\mathcal{E}^+$ the species of non-empty sets, we obtain $P = \mathcal{E} \circ \mathcal{E}^+$. Thus

$$\begin{aligned}
P(x) &= \mathcal{E} \circ \mathcal{E}^+(x) = e^{e^x - 1} \\
\tilde{P}(x) &= exp\Big(\frac{x_1}{1 - x_1} + \frac{x_2}{2(1 - x_2)} + \frac{x_3}{3(1 - x_3)} \dots\Big) \\
Z_P(x_1, \dots) &= exp\Big(x_1 exp(x_1 + \frac{1}{2}x_2 + \dots) + \frac{x_2}{2} exp(x_2 + \frac{1}{2}x_3 + \dots) + \dots\Big)
\end{aligned}$$

We can then perform this kind of computation with Sage.

```
sage: from sage.combinat.species.stream\
import Stream, _integers_from
sage: from sage.combinat.species.generating_series\
import CycleIndexSeriesRing
sage: CIS = CycleIndexSeriesRing(QQ)
sage: p = SymmetricFunctions(QQ).p()
#we can consider the builtin species of partitions
sage: Partitions = species.PartitionSpecies()
sage: Partitions.isotype_generating_series()\
.coefficients(13)
```

```
[1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77]
#or we can compute it with
sage: S = species.SetSpecies()
sage: NonEmptySet = S.restricted(min=1)
#composition
sage: P = S(NonEmptySet)
sage: P.isotype_generating_series().coefficients(13)
[1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77]
#or we can apply the theorem e.g. for n=6
sage: R.<t> = PolynomialRing(QQ)
sage: Z = P.cycle_index_series()\
.coefficient(6).expand(6,'t')
sage: IsoType = Z([t^(n+1) for n in range(6)])
sage: IsoType.coefficients()
[11,...]
```

Example 20. The cycle-index for $n = 6$ has 31 coefficients, and looks like

$$11t_0^6+19t_0^5t_1+29t_0^4t_1^2+\cdots+29t_4^2t_5^4+19t_0t_5^5+19t_1t_5^5+19t_2t_5^5+19t_3t_5^5+19t_4t_5^5+11t_5^6$$

and we can find interesting information about particular shapes, *e.g.*

```
sage: Partitions.cycle_index_series().coefficient(6)\
.expand(6,'t').coefficient(t0*t1^2*t2^3)
52
```

Bibliography

[Ala10] Domenico Alaleona. Teoria della divisione dell'ottava in parti uguali. *Rivista Musicale Italiana*, 1(18):382–420, 1910.

[And03] Moreno Andreatta. *Méthodes algébrique en musique et musicologie du XX^e siécle.* 2003. Aspects théoriques, analytiques et compositionnels.

[BLL98] F. Bergeron, G. Labelle, and P. Leroux. *Combinatorial species and tree-like structures*, volume 67 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1998. Translated from the 1994 French original by Margaret Readdy, With a foreword by Gian-Carlo Rota.

[Can78] E. Rodney Canfield. On a problem of Rota. *Adv. in Math.*, 29(1):1–10, 1978.

[Can03] E. Rodney Canfield. Integer partitions and the Sperner property. *Theoret. Comput. Sci.*, 307(3):515–529, 2003. Selected papers in honor of Lawrence Harper.

[Eng97] Konrad Engel. *Sperner theory*, volume 65 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1997.

[Gro08] Jonathan L. Gross. *Combinatorial methods with computer applications*. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2008.

[Hal14] Bruce E. Sagon Hallam, Joshua. Factorization of the characteristic polynomial of a lattice using quotient posets. *Discrete Math. Theor. Comput. Sci.*, pages 125–136, 2014.

[HP73] Frank Harary and Edgar M. Palmer. *Graphical enumeration.* Academic Press, New York-London, 1973.

[HR00] G. H. Hardy and S. Ramanujan. Asymptotic formulæ in combinatory analysis [Proc. London Math. Soc. (2) **17** (1918), 75–115]. In *Collected papers of Srinivasa Ramanujan*, pages 276–309. AMS Chelsea Publ., Providence, RI, 2000.

[Joc14] Katharina Jochemko. Order polynomials and Pólya's enumeration theorem. *Electron. J. Combin.*, 21(2):Paper 2.52, 11, 2014.

[Koz08] Dmitry Kozlov. *Combinatorial algebraic topology*, volume 21 of *Algorithms and Computation in Mathematics.* Springer, Berlin, 2008.

[Lew87] David Lewin. *Generalized Musical Intervals and Transformations.* Yale University Press, 1987.

[LY89] J. Labelle and Y. N. Yeh. The relation between Burnside rings and combinatorial species. *J. Combin. Theory Ser. A*, 50(2):269–284, 1989.

[Maz02] Guerrino Mazzola. *The Topos of Music.* Birkhäuser Science, 2002. Geometric Logic of Concepts, Theory, and Performance.

[Pak14] Stephan DeSalvo Pak, Igor. Log-concavity of the partition function. *Ramanujan J.*, 55(130), 2014.

[Pen14] Marcus Pendergrass. Two musical orderings. *J. Math. Music*, 8(1):59–72, 2014.

[RS77] Gian-Carlo Rota and David A. Smith. Enumeration under group action. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 4(4):637–646, 1977.

[Sta91] Richard P. Stanley. Some applications of algebra to combinatorics. *Discrete Appl. Math.*, 34(1-3):241–277, 1991. Combinatorics and theoretical computer science (Washington, DC, 1989).

[Sta13] Richard P. Stanley. *Algebraic combinatorics.* Undergraduate Texts in Mathematics. Springer, New York, 2013. Walks, trees, tableaux, and more.

[Ton14] Tito Tonietti. *And Yet It Is Heard.* Science Networks. Historical Studies. Birkhäuser Science, 2014. Musical, Multilingual and Multicultural History of the Mathematical Sciences.

[Wac07] Michelle L. Wachs. Poset topology: tools and applications. In *Geometric combinatorics*, volume 13 of *IAS/Park City Math. Ser.*, pages 497–615. Amer. Math. Soc., Providence, RI, 2007.

[WC05] Xiaojun Wu and Chong-Yun Chao. An application of Pólya's enumeration theorem to partitions of subsets of positive integers. *Czechoslovak Math. J.*, 55(130)(3):611–623, 2005.

[Zie86] M. Günter Ziegler. On the poset of partitions of an integer. *Comb. Theo. J.*, A(42):215–222, 1986.