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On some problems in Sasakian and Kähler geometry

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Introduction

It is known that Kähler structures merge together Riemannian, symplectic and complex structures. Their odd dimensional companions, Sasakian structures, do the analogue for Riemannian, contact and CR structures. These even and odd dimensional realms are closely related. Indeed a Sasakian manifold is sandwiched between its Kähler cone and its transverse Riemannian structure and, given a compact Hodge Kähler manifold one can construct a Sasakian structure on the principal circle bundle associated to the integral Kähler form.

A prominent role is played by Sasaki-Einstein manifolds, also due to their application in physics in the so-called AdS/CFT correspondence. There is a large number of examples and techniques to build Sasaki-Einstein manifolds, see e.g. [11, Chap. 5] and [71] and the references therein. As an example of interrelation between the Sasakian structure and the two Kähler structures, we mention that a manifold is Sasaki-Einstein if and only if its transverse Kähler structure is Kähler-Einstein if and only if the Riemannian cone is Ricci-flat.

Among these techniques we mention the application of deformations of known Sasakian manifolds, for which we refer to Chapter 3, in particular of the standard Sasakian spheres and and some of their Sasakian submanifolds given by the zero set of complex polynomials. These manifolds provide a large number of odd-dimensional Einstein structures on, among others, homology spheres and exotic spheres.

In this thesis we deal with four different problems that touch both the Kähler and the Sasakian setting. As there is such close correlation between Sasakian and Kähler structures, one possible line of work can be to prove for the transverse Kähler metric results known to hold for general Kähler metrics.

The first two problems are about the possible generalization of Sasaki-Einstein metrics and we follow this line of work for the first. Two possible generalizations are Sasaki-Ricci solitons (see e.g. [36] and also [69] for an introduction to the Sasaki-Ricci flow) and Sasaki-extremal metrics [13]. In the Kähler case, the presence of such metric gives information about the Lie algebra of infinitesimal transformations of the ambient manifold, due to the known results of Tian and Zhu [75] and Calabi [14]. For the Sasakian case, it has been proved by Boyer and Galicki [13] that the presence of a Sasaki-extremal metric gives a splitting of a quotient of the Lie algebra of the transversally holomorphic fields. Our work of Chapter 4, see also [63], is the solitonic analogue of this as we provide a result about the decomposition of the same quotient algebra in the case of a Sasaki-Ricci soliton.

Moreover, in the Sasaki-extremal case, van Coevering [77] has proved that a starting Sasaki-extremal metric, under certain assumptions of the relative Futaki invariant, is stable under known types of deformations of Sasakian structures. We provide also the solitonic analogue of this result by proving that starting from a Sasaki-Ricci soliton there exists a neighborhood of zero in the parameter space in which lies a *generalized* Sasaki-Ricci soliton. Our techniques, as van Coevering's, make use of an infinite dimensional implicit function argument. This generalizes a result of Li [56] for Kähler-Ricci solitons.

The second problem we treat is also related to generalization of Einstein metrics. For simplicity we state and prove the result for compact Kähler manifolds and the argument goes verbatim for Sasakian manifolds. We focus on the existence of Kähler metric which generalize a Kähler-Einstein metric in both ways, namely it is what we call an *extremal Kähler-Ricci soliton*. This is a natural question and it appears to be absent in the literature. We prove that, under the assumption of the positivity of the holomorphic sectional curvature, these metrics are Einstein. We do not know whether we can drop this assumption nor whether there are examples of such metrics. For a more detailed introduction to the problem we refer to the one of Chapter 5 or [17].

Then we move to the third problem that deals with Legendrian submanifolds of Sasaki-Einstein manifolds. Legendrian submanifolds are submanifolds of maximal dimension whose tangent space at each point is contained in the contact distribution at that point, see e.g. [64]. They can be defined in the contact setting and are of interest, for instance, in contact topology.

In the case of a compact regular Sasakian manifold M, that is a circle bundle and a Riemannian submersion onto a compact Kähler base B, to every Legendrian L of M one can take its projection in B which is a Lagrangian submanifold. Moreover, Riemannian properties such as minimality, being totally geodesic or umbilical hold for L if and only if hold for its projection, this due to the relation between the two mean curvature vectors.

The most standard example of Sasaki-Einstein manifold is the round odd dimensional sphere with its standard contact structure. Given a minimal submanifold L^n of S^{2n+1}, Lê and Wang [54] proved a characterization of L

being Legendrian in terms of the existence of a certain family of eigenfunctions of eigenvalue $2n+2$ for the induced Laplacian on L. Moreover they provide a lower bound for the multiplicity of $2n+2$ and prove that if this lower bound is attained then L is totally geodesic.

Their functions are exactly the contact moment map of the contactomorphic action of the Sasaki transformation group of the sphere and their technique makes use of the theory of minimal submanifolds in Euclidean spheres and the computation of the Laplacian by means of a local coframe.

In Chapter 6, see also [18], we construct two families of functions on the Legendrian that we prove to be eigenfunctions of eigenvalue $2n+2$. One of them is a direct generalization of the ones of Lê-Wang, indeed we take the moment map up to a constant and with different techniques we prove it to define a family of $(2n+2)$-eigenfunctions. Our techniques use the theory of deformations of minimal Legendrian submanifolds and the fact that the moduli space of such submanifolds are identified with an eigenspace of the Laplacian of a fixed base minimal Legendrian.

By means of a dimensional count, we generalize also the result about the lower bound obtaining exactly the one of Lê-Wang in the case of the sphere. If the lower bound is attained we prove, in the regular case, that L is totally geodesic in M together with a rigidity result about M which turns out to be a Sasaki-Einstein circle bundle over a complex projective space. This also proves that the converse does not hold, namely there can be totally geodesic Legendrian submanifolds that do not attain this lower bound as one can take it in an ambient manifold which is not a sphere.

We then construct another family of eigenfunctions on L, this time exploiting the embedding of M in a Ricci-flat Kähler cone $C(M)$. The family is parameterized by the Lie algebra of infinitesimal Kähler automorphisms of $C(M)$ and is defined by means of the Nomizu operator. The computations are made locally in terms of a local frame and exploits properties of the Nomizu operator together with the Ricci-flatness of $C(M)$.

The last problem we treat deals with the space of Kähler and Sasakian metrics. The idea of defining a weak Riemannian structure on the moduli space of Riemannian structures on a fixed compact Riemannian (M, g) goes back to the 1950s with the work of Ebin [32]. It the case when a compact M has a Kähler structure ω, an analogous construction has been made on the space of Kähler metrics $\mathcal{H}$ cohomologous to ω. By means of the $\partial\overline{\partial}$-Lemma this space is parameterized by smooth functions on M, namely

$$\mathcal{H} = \{\varphi \in C^\infty(M) : \omega + i\partial\overline{\partial}\varphi > 0\}.$$

The first definition of a weak Riemannian structure on $\mathcal{H}$ is due to Mabuchi [57] and then revisited by Semmes and Donaldson [67, 30] and is known as the *Mabuchi metric*. The pairing of two tangent vectors at $u \in \mathcal{H}$ is defined to be

$$(\psi_1, \psi_2)_u = \int_M \psi_1 \psi_2 d\mu_u$$

where $d\mu_u$ is the volume form of the Kähler metric $\omega + i\partial\overline{\partial}u$. It has been proved that it gives $\mathcal{H}$ formally the structure of a locally symmetric space of nonpositive curvature. Being $\mathcal{H}$ infinite dimensional, the Cauchy and Dirichlet problems need not have solution and if they do, it needs not be smooth. Indeed it has been proved by X.X. Chen [24] that the Cauchy problem does not always have solution and that the Dirichlet problem always have a $C^{1,1}$ solution and this is the best regularity possible.

Other choices of Riemannian structures on $\mathcal{H}$ are possible. The first is known as the *Calabi metric* and is defined by

$$(\psi_1, \psi_2)_u = \int_M \Delta_u \psi_1 \Delta_u \psi_2 d\mu_u$$

where Δ_u is the Laplacian induced by the metric $\omega + i\partial\overline{\partial}u$.

Its study was suggested by Calabi in the 1950s and completed by Calamai [16] in the 2010s. He proved that it gives $\mathcal{H}$ constant sectional curvature and that the Cauchy and Dirichlet problems have smooth explicit solutions. Another possible choice, also introduced by Calabi, is known as the *gradient metric* and is defined by

$$(\psi_1, \psi_2)_u = \int_M (d\psi_1, d\psi_2) d\mu_u$$

that is the L^2 product of the gradients of ψ_1 and ψ_2. It was studied for Riemann surfaces by Calamai [16] and the curvature was computed by Calamai and Zheng [20]. In complex dimension greater than one, the gradient metric was not known to have solutions for the Cauchy or Dirichlet problem.

Being Sasakian structures transversally Kähler, one can consider the space of smooth Sasakian metrics transversally cohomologous to a given one. The definition of the space $\mathcal{H}$ is analogous with the slight change that one takes basic functions. The definitions of Mabuchi, Calabi and gradient metrics are done analogously with the use of the canonical Sasakian volume form. The Mabuchi metric was proved by Guan and Zhang [45, 48] to behave analogously as the Kähler case, together with the existence of $C^{1,1}$ Dirichlet geodesics [46].

Clarke and Rubinstein [29] proved that the restriction of the Ebin metric to the space of Kähler metric is exactly twice the Calabi metric. In Chapter 7, see also [19], we compute the restriction of the Ebin metric on the space of Sasakian metrics. Being the latter metrics on M and not only transverse metrics, the presence of the extra direction along the leaves gives rise to more terms. Indeed we find the following.

Theorem 0.1. *The restriction is twice the* sum *of the Calabi and the gradient metric that we call the* sum metric*.*

We explicitly compute its covariant derivative and write down the geodesic equation. By using a fixed point technique we prove the short time existence, for any chosen Hölder regularity, of a $C^{k,\alpha}$ geodesic starting from an assigned point with an assigned velocity. We prove it for the Kähler setting and the argument works verbatim for the Sasakian one. With the same technique we are able to prove also a short time existence result for the gradient metric, in any dimension and for any chosen Hölder regularity. The existence of *smooth* geodesics is still an open problem.

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Chapter 1

Preliminaries

In this chapter we recall the definition and main facts of Kähler and contact geometry, mainly to fix notation. We refer to [11] and the references therein.

1.1 Kähler manifolds

Definition 1.1. A complex chart on a real manifold M^{2n} is a pair (U, φ) where $U \subset M$ is an open set and $\varphi : U \to \varphi(U) \subset \mathbb{C}^n$ is a diffeomorphism. The manifold M is *complex* if there an atlas of complex charts $\{(U_j, \varphi_j)\}_j$ whose transition functions $\varphi_j \circ \varphi_k : \varphi_k(U_j \cap U_k) \to \varphi_j(U_j \cap U_k)$ are biholomorphisms.

On a real manifold M, a section $J \in \Gamma(\mathrm{End}(TM))$ such that $J^2 = -\mathrm{id}$ is called an *almost complex structure* on M and the pair (M, J) an *almost complex manifold*. Such J has complex eigenvalues $\pm i$ and allows the complexified tangent bundle $TM^{\mathbb{C}}$ to split as $T^{1,0}M \oplus T^{0,1}M$, the i- (resp $-i$-) eigenspaces of J. An almost complex structure can be always defined on a complex manifold by mapping $\frac{\partial}{\partial z_j} \mapsto i\frac{\partial}{\partial z_j}$ where $(z_1, \dots, z_n)$ are local holomorphic coordinates. Conversely, when an almost complex structure admits local holomorphic coordinates that express it as the multiplication by i on $TM^{\mathbb{C}}$ it is said to be *integrable*. There is the well known theorem of Newlander and Nirenberg that gives a characterization of integrability.

Theorem 1.2. *Let J be an almost complex structure on M. Its Nijenhuis tensor N_J, defined by*

$$N_J(X, Y) = [X, Y] + J([JX, Y] + [X, JY]) + [JX, JY] \qquad X, Y \in \Gamma(TM),$$

vanishes if, and only if, J is integrable, making M an complex manifold.

So a manifold with an integrable complex structure can be defined to be a complex manifold. Let us now consider its transformations.

Definition 1.3. A map $f : (M_1, J_1) \to (M_2, J_2)$ between two (almost) complex manifolds is said to be (pseudo)holomorphic if $df \circ J_1 = J_2 \circ df$.

In particular when $M_1 = M_2 = M$ and f is bijective, we call it an *automorphism* of M. We can now introduce the group of such automorphisms.

Proposition 1.4. *The group* $\mathrm{Aut}(M, J)$ *of biholomorphisms of a compact complex manifold* (M, J)*is a complex Lie group whose Lie algebra* $\mathfrak{aut}(M, J)$ *is given by the (real) holomorphic vector fields, that is* $X \in \Gamma(TM)$ *such that* $\mathcal{L}_X J = 0$.

For any almost complex structure J, its action on the complexified tangent bundle can be extended to the complexified cotangent bundle, giving a splitting $TM^{*\mathbb{C}} = T^{1,0}M^* \oplus T^{0,1}M^*$ which in turn defines a splitting of the bundle of complex valued forms of degree k

$$\bigwedge^k (M, \mathbb{C}) = \bigoplus_{p+q=k} \bigwedge^{p,q} (M, \mathbb{C})$$

where $\bigwedge^{p,q}(M, \mathbb{C}) := \bigwedge^p T^{1,0}M^* \otimes \bigwedge^q T^{0,1}M^*$. We denote by $\Omega^{p,q}(M, \mathbb{C})$ the space of sections of such bundle and call its elements (p, q)-forms or forms of *bidegree* (p, q). Let us now focus on integrable complex structures and consider the action of the exterior derivative d. It can be proved, see [51, Chap. IX], that the exterior derivative of a (p, q)-form has exactly a $(p + 1, q)$- and a $(p, q + 1)$-component if, and only if, J is integrable. This allows to split $d|_{\Omega^{p,q}(M,\mathbb{C})}$ in two parts, namely $d = \partial + \overline{\partial}$, where $\partial := \pi^{p+1,q} \circ d$ and $\overline{\partial} := \pi^{p,q+1}$, where $\pi^{a,b}$ denotes the projection onto $\Omega^{a,b}(M, \mathbb{C})$. In this case it follows from $d^2 = 0$, considering bidegrees, the well known relations $\partial^2 = \overline{\partial}^2 = 0$ and $\partial\overline{\partial} + \overline{\partial}\partial = 0$. We can then define, from the complex $(\Omega^{p,q}, \overline{\partial})$ the cohomology groups by

$$H^{p,q}_{\overline{\partial}}(M) = \frac{\ker \overline{\partial}|_{\Omega^{p,q}(M)}}{\mathrm{Im}\, \overline{\partial}|_{\Omega^{p,q-1}(M)}}$$

called the *Dolbeault cohomology groups*.

We introduce now a notion of compatibility between an (almost) complex structure and a Riemannian metric.

Definition 1.5. Given (M, J) an (almost) complex manifold. A Riemannian metric g is called *Hermitian* if $g(JX, JY) = g(X, Y)$ for all $X, Y \in \Gamma(TM)$. Define the 2-form ω by $\omega(X, Y) = g(JX, Y)$, which is called the *Kähler form* of (J, g). The triple (g, J, ω) is called an *(almost) Hermitian manifold.*

It is enough to give two pieces of such data to determine uniquely the third one, yet sometimes we give all three of them for more emphasis. We can now give the main definition.

Definition 1.6. A Hermitian metric g on a complex manifold (M, J) is called a *Kähler metric* if its Kähler form is closed, i.e. $d\omega = 0$.

It is immediate to verify that ω has bidegree $(1,1)$ and, being $\overline{\partial}$-closed, its class defines an element of $H^{1,1}_{\overline{\partial}}(M)$. On a Kähler manifold it turns out that we can find, *locally*, a function that generates the Kähler form, i.e. $\omega = i\partial\overline{\partial}f$ for a *Kähler potential* f unique up to a constant. This is due to the following well known lemma.

Lemma 1.7 (Local $\partial\overline{\partial}$-Lemma)**.** *Let α be a real closed $(1,1)$-form on the unit disc $U \subset \mathbb{C}^n$. Then there exist a smooth function $f \in C^\infty(U)$ such that $\alpha = i\partial\overline{\partial}f$.*

1.2 Hodge Theory and Kähler identities

On a compact Hermitian manifold (M^{2n}, g, J, ω) one can define the *Hodge-$*$* as the map $\Omega^{p,q}(M, \mathbb{C}) \to \Omega^{n-p,n-q}(M, \mathbb{C})$ defined by

$$\varphi \wedge *\psi = g(\varphi, \psi)\,\mathrm{vol}_g$$

for all forms $\varphi, \psi \in \Omega^{p,q}(M, \mathbb{C})$ and where vol_g is the volume form of (M, g). By the compactness of M and using integration we can consctuct a L^2 Hermitian inner product on forms, namely

$$\langle \varphi, \psi \rangle := \int_M \varphi \wedge *\psi \qquad \text{for } \varphi, \psi \in \Omega^{p,q}(M, \mathbb{C})$$

with respect to which $*$ is an isometry.

We can define the following operators

$$\begin{aligned} \partial^* &= - * \partial * : \Omega^{p,q}(M, \mathbb{C}) \to \Omega^{p-1,q}(M, \mathbb{C}) \\ \overline{\partial}^* &= - * \overline{\partial} * : \Omega^{p,q}(M, \mathbb{C}) \to \Omega^{p,q-1}(M, \mathbb{C}), \end{aligned}$$

which turn out to be formal adjoints of ∂ and $\overline{\partial}$, i.e. such that $\langle \partial\varphi, \psi \rangle = \langle \varphi, \partial^*\psi \rangle$ and $\langle \overline{\partial}\varphi, \psi \rangle = \langle \varphi, \overline{\partial}^*\psi \rangle$. Let $d^* = - * d*$ be the adjoint of d with respect to the L^2 product. We can construct three differential operators acting on forms out of these six, namely

$$\Delta_d = dd^* + d^*d$$

$$\Delta_\partial = \partial\partial^* + \partial^*\partial$$
$$\Delta_{\overline{\partial}} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$$

called d- (resp ∂-, $\overline{\partial}$-) Laplacian.

Consider now the kernels of these operators. Let

$$\mathcal{H}^{p,q}(M) = \{\alpha \in \Omega^{p,q}(M) | \Delta_{\overline{\partial}}\alpha = 0\}$$

and call its elements $\overline{\partial}$-*harmonic* (p,q)-forms. The integers $h^{p,q}(M) := \dim \mathcal{H}^{p,q}(M)$ are called the *Hodge numbers* of M. We can now state the following important theorem even in a more general setting than the Kähler one.

Theorem 1.8 (Hodge)**.** *On a compact Hermitian manifold the numbers* $\dim \mathcal{H}^{p,q}(M)$ *are finite. Moreover we have the orthogonal decomposition*

$$\Omega^{p,q} = \mathcal{H}^{p,q}(M) \oplus \overline{\partial}\Omega^{p,q-1}(M) \oplus \overline{\partial}^*\Omega^{p,q+1}(M).$$

In particular any Dolbeault class in $H^{p,q}_{\overline{\partial}}(M)$ *has a unique harmonic representative, that is we have the isomorphism* $H^{p,q}_{\overline{\partial}}(M) \simeq \mathcal{H}^{p,q}_{\overline{\partial}}(M)$

A consequence of the Hodge Theorem is the following lemma, a key of many arguments in complex geometry.

Lemma 1.9 (Global $\partial\overline{\partial}$-Lemma)**.** *Let* (M, J) *a compact complex manifold and* β *a* d*-exact real* $(1,1)$ *form. Then there exist a smooth real function* $f \in C^\infty(M)$ *such that* $\beta = i\partial\overline{\partial}f$.

In the Kähler case with Kähler form ω we consider another operator $L : \Omega^{p,q}(M, \mathbb{C}) \to \Omega^{p+1,q+1}(M, \mathbb{C})$ defined by wedging with ω, i.e. $L\alpha = \alpha \wedge \omega$. Its adjoint with respect to the L^2 product is denoted by L^* and can be computed to be $L^* = (-1)^{p+q} * L*$ on $\Omega^{p,q}$. We have the following well known identities.

Proposition 1.10 (Kähler identities)**.** *On a compact Kähler manifold we have*

$$[L, \partial^*] = i\overline{\partial}, \qquad [L, \overline{\partial}^*] = -i\partial$$
$$[L, \partial] = i\overline{\partial}^*, \qquad [L^*, \overline{\partial}] = -i\partial^*.$$

Moreover $\Delta_d = 2\Delta_\partial = 2\Delta_{\overline{\partial}}$.

A consequence of the Kähler identities is the well known Hodge decomposition. Define $H^{p,q}(M, \mathbb{C})$ to be the space of complex d-closed (p,q)-forms modulo complex d-exact (p,q)-forms. We then have

Theorem 1.11. *On a compact Kähler manifold we have the following*

$$H^r(M,\mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(M,\mathbb{C})$$

and that $\overline{H^{p,q}(M,\mathbb{C})} = H^{q,p}(M,\mathbb{C})$.

With the additional fact given by the Serre duality $* : \mathcal{H}^{p,q}(M) \overset{\simeq}{\to} \mathcal{H}^{n-p,n-q}(M)$ we have the following corollary about Hodge numbers.

Corollary 1.12. *Let M^{2n} be a compact Kähler manifold with Hodge numbers $h^{p,q}$ and Betti numbers $h^k = \sum_{p+q=k} h^{p,q}$. For $0 \leq p,q \leq n$ we have*

1. *$h^{p,q}$ is finite;*
2. *$h^{p,p} \geq 1$ and $h^{n,n} = h^{0,0} = 1$;*
3. *$h^{p,q} = h^{q,p} = h^{n-p,n-q}$;*
4. *The odd Betti numbers are even.*

1.3 Chern classes and the Calabi-Yau theorem

We briefly recall here the notion of Chern class associated to complex vector bundles. Let $E \to M$ be a complex vector bundle of rank r on the complex manifold M. Let ∇ be a connection on E with curvature form Ω. We define the following functions f_k on the space of complex $r \times r$ matrices given by

$$\det(A + \lambda I) = f_r(A) + \lambda f_{r-1}(A) + \ldots + \lambda^{r-1} f_1(A) + \lambda^r.$$

The map f_i is a polynomial of degree i and is $\mathrm{GL}(r,\mathbb{C})$-invariant. In particular $f_r = \det$ and $f_1 = \mathrm{tr}$.

Back to the vector bundle with connection (E,∇), for each $i = 1,\ldots,r$ consider, for the $2i$-form

$$c_i(E,\nabla) := f_i\left(\frac{\sqrt{-1}}{2\pi}\Omega\right)$$

called the *i-th Chern form*. The following fact holds.

Proposition 1.13. *The form $c_i(E,\nabla)$ is d-closed hence it defines a deRham class in $H^{2i}(M)$. If $\widetilde{\nabla}$ is another connection on E then $c_i(E,\widetilde{\nabla})$ and $c_i(E,\nabla)$ are cohomologous.*

This allows to give the following definition.

Definition 1.14. The cohomology class of $c_i(E, \nabla)$, denoted by $c_i(E)$ is called the *i-th Chern class* of the vector bundle E.

If $E = TM^{\mathbb{C}}$ is the complexified tangent bundle of a complex manifold M, then the i-th Chern class is called the *i-th Chern classes of M* and is denoted by $c_i(M)$.

We will focus only on the first Chern class of vector bundles and complex manifolds.

Definition 1.15. A compact complex manifold is said to be *Fano* or *positive* (resp. *anti-Fano* or *negative*) if its first Chern class can be represented by a real positive (resp. negative) $(1,1)$-form. Recall that a $(1,1)$-form α is positive if $i\alpha(X, \overline{X}) > 0$ for all $(1,0)$-fields $X \neq 0$.

The following proposition relates the first Chern class of a manifold with its Ricci form $\rho = \mathrm{Ric}(J\cdot, \cdot)$.

Proposition 1.16. *On a Kähler manifold M with Ricci form ρ the first Chern class $c_1(M)$ is represented by $\frac{1}{2\pi}\rho$.*

One can ask whether the converse of Proposition 1.16 holds. That is, on a Kähler manifold M and for any real closed $(1,1)$ form α representing $2\pi c_1(M)$, in other words whether there exist a Kähler metric on M whose Ricci form equals α.

The following theorem states the celebrated Calabi conjecture proved by Yau.

Theorem 1.17. *Let (M, g, ω) be a compact Kähler manifold. Then any real $(1,1)$-form ρ representing $2\pi c_1(M)$ is the Ricci form of a unique Kähler metric h whose Kähler form is cohomologous to ω.*

There is the following corollary.

Corollary 1.18. *Let M be a compact Kähler manifold. If $c_1(M) = 0$ then it admits a Ricci-flat Kähler metric. If it is Fano then it admits a Kähler metric with positive Ricci curvature.*

Combining this corollary with an argument of Bochner (see e.g. [35]) one can prove the following.

Proposition 1.19. *A compact Kähler manifold with positive first Chern class has vanishing first Betti number. In particular it has no nonzero harmonic 1-forms.*

1.4 Contact structures

We now move to odd dimensional manifolds and we recall some definition and facts about (almost) contact manifolds. We follow [9, 11].

Definition 1.20. A smooth odd-dimensional manifold M^{2n+1} is said to be a *(strict) contact manifold* or to carry a *contact structure* if it admits a global differential form $\eta \in \Omega^1(M)$, called *contact form*, such that

$$\eta \wedge (d\eta)^n \neq 0 \tag{1.1}$$

everywhere on M.

Such a form defines on M a distribution of hyperplanes and clearly for any nowhere zero function f, the form $f\eta$ is also a contact form defining the same distribution of hyperplanes. Conversely every distribution of hyperplanes $D \subset TM$ on M is locally defined by the kernel of a local 1-form. It is globally defined if, and only if, the real line bundle TM/D is trivial. In some references, e.g. [40], what we call a contact structure is called *coorientable contact structure*.

Such hyperplane distribution D is called the *contact distribution*. By Frobenius theorem, condition (1.1) implies that D is not integrable. From now on, a contact manifold will be as in Definition 1.20 unless otherwise specified. We have the following Proposition/Definition.

Proposition 1.21. *On a contact manifold (M, η) there exist a unique vector field ξ such that*

$$\eta(\xi) = 1, \qquad \iota_\xi d\eta = 0 \tag{1.2}$$

which is called the Reeb vector field *of the contact structure.*

1.5 Almost contact structures

The following is a known fact about contact manifolds. We refer to [9] for its proof.

Theorem 1.22. *Let M^{2n+1} be a contact manifold. Then its structure group can be reduced from* $\mathrm{GL}(2n+1, \mathbb{R})$ *to* $\mathrm{U}(n) \times 1$.

This result leads to the definition of an almost contact structure. Namely,

Definition 1.23. A manifold of dimension $2n+1$ whose structure group can be reduced from $\mathrm{GL}(2n+1, \mathbb{R})$ to $\mathrm{U}(n) \times 1$ is said to be *almost contact*.

However, for our purpose, another structure is more convenient. We provisionally call it (ξ, η, Φ)-structure.

Definition 1.24. A (ξ, η, Φ)-structure on a differentiable manifold M^{2n+1} is a triple formed by a vector field ξ, a 1-form η and a $(1,1)$-tensor field Φ on M satisfying

$$\eta(\xi) = 1 \qquad \text{and} \qquad \Phi^2 = -\operatorname{id} + \eta \otimes \xi. \tag{1.3}$$

The following properties, sometimes inserted in the definition, actually follow.

Lemma 1.25. *On a (ξ, η, Φ)-structure the tensor fields are such that $\Phi\xi = 0$ and $\eta \circ \Phi = 0$.*

Now we introduce a compatibility notion for a Riemannian metric on M.

Definition 1.26. A Riemannian metric g on M is said to be *compatible with the (ξ, η, Φ)-structure* if

$$g(\Phi\cdot, \Phi\cdot) = g - \eta \otimes \eta. \tag{1.4}$$

Setting one of the arguments of (1.4) equal to ξ, we immediately see that η and ξ are the Riemannian dual of each other with respect to g.

We now have the following fact.

Proposition 1.27. *Compatible metrics always exists and are not unique.*

Proof sketch. Start with any Riemannian metric h' and let $h = h'(\Phi^2\cdot, \Phi^2\cdot) + \eta \otimes \eta$, that is still a Riemannian metric. Finally define

$$g = \frac{1}{2}\big(h + h(\Phi\cdot, \Phi\cdot) + \eta \otimes \eta\big) \tag{1.5}$$

that is still Riemannian and satisfies (1.4). The uniqueness obviously does not hold since one can start with any metric h'. $\square$

The relation (1.4) and the fact that $\eta^\sharp = \xi$ allow to construct the so called Φ-*bases* namely local orthonormal frames of the form $\{X_0 = \xi, X_1, \Phi X_1, \dots X_n, \Phi X_n\}$.

The following theorem explains why we introduced the (ξ, η, Φ)-structures.

Theorem 1.28. *Every $(2n+1)$-manifold admitting a (ξ, η, Φ)-structure also admits a reduction of its structure group to $\mathrm{U}(n) \times 1$. Conversely every almost contact manifold admits a (ξ, η, Φ)-structure.*

So it makes sense to say that (ξ, η, Φ) is an almost contact structure and we will say so in the following.

Let us now see how almost contact metric structures and contact structure relate. Given an almost contact metric structure, consider the bilinear form given by

$$F(X, Y) = g(\Phi X, Y)$$

which is easily seen to be a 2-form by definition of the almost contact metric condition.

Definition 1.29. An almost contact structure (ξ, η', Φ) is compatible with the contact structure (M, η) if $\eta' = \eta$, ξ is the Reeb vector field of η and $F = d\eta$.

We now have the following proposition that links the contact structures with the almost contact ones.

Proposition 1.30 ([9, p. 25])**.** *If (M, η), then there exists an almost contact metric structure (η, ξ, Φ, g) (with same η) such that the fundamental 2-form equals $d\eta$.*

Definition 1.31. We call (η, ξ, Φ, g) the *associated almost contact structure* to the contact structure η or more simply we shall call (η, ξ, Φ, g) a *contact metric structure*.

1.6 Normality and Sasakian structures

Consider the manifold $C(M) = M \times \mathbb{R}^+$ which is called the *cone* of M. Let $r > 0$ be the coordinate on the second factor. We see M identified with the submanifold $M \times \{1\} \subset C(M)$. We define on it a tensor field of type $(1, 1)$ by

$$J(X, f\partial_r) = (\Phi X - f\xi, \eta(X)\partial_r) \tag{1.6}$$

which is easily seen to be an almost complex structure, that is $J^2 = -\operatorname{id}$. We use this to give the following notion.

Definition 1.32. An almost contact manifold (M, Φ, η, ξ) is said to be *normal* if the endomorphism J defined above has vanishing Nijenhuis tensor $N_J = 0$.

The vanishing N_J can be related to the vanishing of the Nijenhuis tensor N_Φ of Φ on M. Namely, one can compute[1] that the vanishing of N_J is

[1]Using the convention $d\eta(X, Y) = X\eta(Y) - Y\eta(X) - \eta([X, Y])$, that is without the one-half coefficient.

equivalent to the vanishing of the following tensor fields on M

$$\begin{aligned} N^{(1)} &= N_\Phi + d\eta \otimes \xi \\ N^{(2)}(X,Y) &= (\mathcal{L}_{\Phi X}\eta)Y - (\mathcal{L}_{\Phi Y}\eta)X \\ N^{(3)} &= \mathcal{L}_\xi \Phi \\ N^{(4)} &= \mathcal{L}_\xi \eta. \end{aligned}$$

These tensor fields have the following properties.

Proposition 1.33. *On any almost contact structure (η, ξ, Φ) the vanishing of $N^{(1)}$ implies the vanishing of all others. Moreover, on a contact metric structure (η, ξ, Φ, g) the fields $N^{(2)}$ and $N^{(4)}$ vanish and $N^{(3)}$ vanishes if, and only if, ξ is Killing for g.*

We consider now the case of a contact metric manifold (η, ξ, Φ, g) where the field ξ is Killing for g. In this case we say the manifold has a *K-contact structure*. We have just seen, in Proposition 1.33, a characterization of this property.

Such property gives quite a rigidity for curvature. Namely

Proposition 1.34. *On a K-contact manifold $(M^{2n+1}, \eta, \xi, \Phi, g)$, every plane containing ξ has sectional curvature equal to* 1. *In particular* $\mathrm{Ric}(\xi, \xi) = 2n$.

The converse of the statement about the Ricci curvature also holds, giving a further characterization of the K-contact condition.

Theorem 1.35. *A contact metric manifold M^{2n+1} is K-contact if, and only if,* $\mathrm{Ric}(\xi, \xi) = 2n$.

We now state the contact-geometric definition of the main object of this thesis.

Definition 1.36. If a contact metric structure (η, ξ, Φ, g) is normal (i.e. if its associated almost contact metric structure is) then (η, ξ, Φ, g) is called *Sasakian*.

We prefer to state the main properties of such structures in the next chapter where a more direct definition will be given.

1.7 Foliations

In this section we briefly recall the definition and main facts about foliations. We shall follow [58] for the definition in terms of foliated atlases and distributions, then we will talk about transverse structures.

Let $q \geq 1$ be an integer. The object we will model on foliations of codimension q is $\mathbb{R}^n$ together with the family of affine subspaces parallel to $\mathbb{R}^{n-q}$. Consider the second projection $\pi : \mathbb{R}^{n-q} \times \mathbb{R}^q \to \mathbb{R}^q$ and we call *vertical leaves* the inverse images by π of points of $\mathbb{R}^q$.

A local diffeomorphism $\varphi : U \to U'$ of $\mathbb{R}^n$ is said to be a *local automorphism* of the model foliation if in components it is such that

$$\varphi(x, y) = (\varphi^1(x, y), \ldots, \varphi^{n-q}(x, y), \varphi^{n-q+1}(y), \ldots, \varphi^n(y)).$$

If $x \in U$, the connected component of x of the trace in the vertical leaves is called the *vertical plaque*. The family of all such local diffeomorphisms is a pseudogroup denoted by $\Gamma_{n,q}$. We are ready now for the main definition.

Definition 1.37. Let M be a n-dimensional manifold. A *foliated atlas* of codimension q is an atlas on M whose transition maps belong to $\Gamma_{n,q}$.

A *foliation* of codimension q on M is a foliated maximal atlas $\widehat{\mathcal{A}}$.

An element $\varphi : U \to \mathbb{R}^n$ of $\widehat{\mathcal{A}}$ is called *foliated chart* and its domain is called a *distinguished open set*. The inverse image of the vertical plaque of $\varphi(x)$ for any $x \in U$ is called *plaque* of x in U. It depends only on x and U as the coordinate changes respect the vertical plaques. The local coordinates $(x, y) \in \mathbb{R}^{n-q} \times \mathbb{R}^q$ in a foliated chart are called *foliated* or *distinguished*. In distinguished coordinates, the plaques are given by the equations $y^i = \text{const}, i = 1, \ldots, q$.

Let us now see how a codimension q foliation on a n-dimensional manifold determines a distribution of the same codimension. Let x be a point of a foliated manifold and U a distinguished open set around it with foliated coordinates (x, y). Then define a distribution by taking the subspace $E_x \subset T_xM$ spanned by the derivatives $\frac{\partial}{\partial x^1}\big|_x, \ldots, \frac{\partial}{\partial x^p}\big|_x$ where $p = n - q$.

This does not depend on the choice of the foliated chart and by construction the distribution E is smooth.

A smooth distribution is completely integrable if it is the distribution associated to a foliation in the way just defined. By the well known Frobenius theorem, a distribution is completely integrable if, and only if, it is involutive.

Let us now introduce the concept of leaf. We have defined it directly for the model foliation, now we introduce a definition that allows the generalization of it to any foliation.

Definition 1.38. Let x_0 be a point on a foliated manifold M. The *leaf* L_{x_0} passing through the point x_0 is the set of points that can be reached from x_0 via piecewise differentiable paths whose tangent vector at any point belong to the correspondent subspace in the associated distribution.

One can show that any leaf admits a natural structure of p-dimensional submanifold of M.

We now introduce the concept of transverse structure. We follow [11] which in turn follows [58]. Let now M be a foliated manifold with a foliation $\mathcal{F}$. We denote by $T\mathcal{F}$ the subalgebra of $\Gamma(TM)$ of vector fields tangent to the leaves of $\mathcal{F}$.

Definition 1.39. A vector field X on a M is called *foliate* if for every field V tangent to the leaves the Lie bracket $[V, X]$ is still tangent to the leaves. The Lie algebra of such fields is denoted by $\mathfrak{fol}(M, \mathcal{F})$.

We have an exact sequence of vector bundles

$$0 \longrightarrow E \longrightarrow TM \xrightarrow{\pi_Q} Q \longrightarrow 0 \tag{1.7}$$

where E is the integrable distribution given by the foliation and Q is the quotient called *normal bundle* of the foliation. Any foliate vector field X defines, by projection, a section $\overline{X}$ of Q and it follows from the definition of foliate that this is independent of the coordinates along the leaves. The set of such sections is a Lie algebra $\mathfrak{trans}(M, \mathcal{F})$ called the Lie algebra of *transverse vector fields* which fits into the following exact sequence

$$0 \longrightarrow T\mathcal{F} \longrightarrow \mathfrak{fol}(M, \mathcal{F}) \longrightarrow \mathfrak{trans}(M, \mathcal{F}) \longrightarrow 0.$$

On $\mathfrak{trans}(M, \mathcal{F})$ the Lie bracket is defined by $[\overline{X}, \overline{Y}] = \overline{[X, Y]}$.

A *transverse frame* at $x \in M$ is a q-ple of sections $(Y_1, \ldots, Y_q)$ that gives a basis of the fiber Q_x at x. One can collect all transverse frames at all points of M and obtain a principal $\mathrm{GL}(q, \mathbb{R})$-bundle over M which we denote by $L_T(M, \mathcal{F})$.

If G is a subgroup of $\mathrm{GL}(q, \mathbb{R})$ we denote by $\pi_T : P_T(M, G, \mathcal{F}) \to M$ the corresponding principal G-subbundle. As in the classical case, we see a point $z \in L_T(M, \mathcal{F})$ lying on the fiber over $x \in M$ as an isomorphism $z : \mathbb{R}^q \to Q_x$. The bundle $L_T(M, \mathcal{F})$ and its G-subbundles have a 1-form given by

$$\langle \theta, X \rangle = z^{-1} \pi_Q \pi_{T*} X$$

which we use to define a foliation on $L_T(M, \mathcal{F})$ by means of the distribution

$$E_{Tz} = \{v \in T_z L_T(M, \mathcal{F}) : \iota_v \theta = \iota_v d\theta = 0\}. \tag{1.8}$$

It can be proved that the distribution (1.8) is integrable, so it defines a foliation $\mathcal{F}_T$ on $L_T(M, \mathcal{F})$ which is called the *lifted foliation.*

Definition 1.40. Let $G \subset \mathrm{GL}(q, \mathbb{R})$ be a Lie subgroup and $P_T(M, G, \mathcal{F})$ a principal G-subbundle of $L_T(M, \mathcal{F})$. We say that $P_T(M, G, \mathcal{F})$ is a *transverse G-structure* if for all $z \in P_T(M, G, \mathcal{F})$, it is $E_{Tz} \subseteq T_z P_T(M, G, \mathcal{F})$.

Examples of transverse G-structures are when $G = \mathrm{O}(q)$, giving rise to *Riemannian foliations* or, when $q = 2m$ and $G = \mathrm{GL}(m, \mathbb{C})$ (resp. $\mathrm{U}(m)$) giving rise to *transverse almost complex structures* (resp. *transverse Hermitian structures*).

Let us now focus on Riemannian foliations. We can define the tensor on M given by

$$g^T(X, Y) = \langle z^{-1}\pi_Q X, z^{-1}\pi_Q Y \rangle \qquad for X, Y \in T_{\pi_T(z)}M$$

which defines a nonnegative symmetric bilinear form and has kernel $E \otimes TM + TM \otimes E$. Thus it defines a Riemannian metric on Q and it is called a *transverse Riemannian metric.*

Transverse Riemannian metrics on a foliated manifold are in correspondence with transverse $\mathrm{O}(q)$-structures.

1.8 Fundamentals of orbifolds

In this section we recall a structure that arises when one deals with foliation and their leaf spaces, the notion of orbifold. These were first introduced by Satake [65] under the name of V-manifolds in 1956 and he immediately developed Riemannian geometry on them in [66]. The complex counterpart was introduced at the same time by Baily [3, 4] where he develops a Hodge theory on complex V-manifolds and proves an analogue of the Kodaira embedding theorem. The concept was retaken by Thurston in 1980 in [74] where these object, called orbifolds, were used in the study of 3-manifolds. For this treatment we follow [11]. See also [10].

Let us now state the definition. Let X be a Hausdorff paracompact space.

Definition 1.41. A *smooth local uniformizing system* or *smooth orbifold chart* (resp. *complex*) is a triple $(\widetilde{U}, \Gamma, \varphi)$ where $\widetilde{U}$ is an open subset of $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) containing the origin, Γ is a finite group of diffeomorphisms (resp. biholomorphisms) acting effectively on $\widetilde{U}$ and $\varphi : \widetilde{U} \to U$ is a continuous map onto an open set $U \subset X$ such that $\varphi \circ \gamma = \varphi$ for all $\gamma \in \Gamma$ and induces a homeomorphism $\widetilde{U}/\Gamma \simeq U$.

An *injection* between orbifold charts $(\widetilde{U}, \Gamma, \varphi)$ and $(\widetilde{U}', \Gamma', \varphi')$ is a smooth (resp. holomorphic) embedding $\lambda : \widetilde{U} \to \widetilde{U}'$ such that $\varphi' \circ \lambda = \varphi$ and $\Gamma' \leq \Gamma$.

An *smooth orbifold atlas* (resp. *complex*) is a family $\{\widetilde{U}_i, \Gamma_i, \varphi_i\}$ of local uniformizing systems such that

(i) $X = \bigcup_i U_i$ where $U_i := \varphi_i(\widetilde{U}_i)$;

(ii) Given two local uniformizing systems $(\widetilde{U}_i, \Gamma_i, \varphi_i)$ and $(\widetilde{U}_j, \Gamma_j, \varphi_j)$ and a point $x \in U_i \cap U_j$ there exist an open neighborhood U_k of x and a chart $(\widetilde{U}_k, \Gamma_k, \varphi_k)$ such that there are injections $\lambda_{ik} : (\widetilde{U}_k, \Gamma_k, \varphi_k) \to (\widetilde{U}_i, \Gamma_i, \varphi_i)$ and $\lambda_{jk} : (\widetilde{U}_k, \Gamma_k, \varphi_k) \to (\widetilde{U}_j, \Gamma_j, \varphi_j)$.

An atlas is said to be a refinement of another if any chart of the latter inject into some chart of the former. Two atlases are *equivalent* if they admit a common refinement. We have now finally the following.

Definition 1.42. A *smooth orbifold* (resp. *complex*) is a paracompact Hausdorff space together with an equivalence class of smooth (resp. complex) orbifold atlases.

Let X be an orbifold and fix an orbifold chart $(\widetilde{U}, \Gamma, \varphi)$. For any $x \in \varphi(\widetilde{U})$ we choose a point $p \in \varphi^{-1}(x)$ and consider the isotropy subgroup $\Gamma_p \subseteq \Gamma$. Its conjugacy class depends only on x so we define this class to be the *isotropy subgroup* at x and denote it by Γ_x. We call *regular* the points $x \in X$ that have trivial Γ_x or *singular* otherwise. The subset of regular points form a dense subset of X. If all points are regular we have a smooth manifold, which are trivial examples of orbifold by taking $\Gamma_i = 1$ in the local uniformizing systems.

One of the notions from the differential geometry of manifolds that holds also for orbifolds is the one of vector and principal bundles.

Let X be an orbifold with atlas $\{\widetilde{U}_i, \Gamma_i, \varphi_i\}$.

Definition 1.43. An *orbibundle* or *V-bundle* with structure group G and fiber a G-manifold F is a fiber bundle $B_{\widetilde{U}_i}$ over $\widetilde{U}_i$ with fiber F and structure group G together with homomorphisms $h_{\widetilde{U}_i} : \Gamma_i \to G$ such that

(i) if b lies in the fiber over $\widetilde{x}_i \in \widetilde{U}_i$ then $bh_{\widetilde{U}_i}(\gamma)$ lies in the fiber over $\gamma^{-1}\widetilde{x}_i$ for all $\gamma \in \Gamma_i$;

(ii) if $\lambda_{ji} : \widetilde{U}_i \to \widetilde{U}_j$ is an injection, then there is a bundle map $\lambda^*_{ji} : B_{\widetilde{U}_j}|_{\lambda_{ji}(\widetilde{U}_i)} \to B_{\widetilde{U}_i}$ such that if $\gamma \in \Gamma_i$ and $\gamma' \in \Gamma_j$ is the unique element such that $\lambda ji \circ \gamma = \gamma' \circ \lambda_{ji}$ then $h_{\widetilde{U}_i}(\gamma) \circ \lambda^*_{ji} = \lambda^*_{ji} \circ h_{\widetilde{U}_j}(\gamma')$; and if λ_{kj} is another such injection then $(\lambda_{kj} \circ \lambda_{ji})^* = \lambda^*_{ji} \circ \lambda^*_{kj}$.

Analogously to the manifold case we define the *rank* of an orbibundle and a *principal* orbibundle.

Let us now consider the total space of an orbibundle. By taking small enough orbifold charts on X we can assume that $B_{\widetilde{U}_i} = \widetilde{U}_i \times F$. The action of Γ_i extends to $\widetilde{U}_i \times F$ by $\gamma \cdot (\widetilde{x}_i, b) = (\gamma^{-1}\widetilde{x}_i, bh_{\widetilde{U}_i}(\gamma))$ and we define Γ^* to be the stabilizer of (x_i, b). The total space E of the orbibundle will then admit an orbifold structure with local uniformizing systems $(B_{\widetilde{U}_i}, \Gamma_i^*, \varphi_i^*)$.

In the case of a principal orbibundle we have the following fact.

Proposition 1.44. *Let P be the total space of a principal orbibundle on an orbifold X. Then P is a smooth manifold if, and only if, the $h_{\widetilde{U}_i}$ are injective for all i.*

At last, we mention the link between foliations and orbifolds, that is the reason why we are interested in them.

Theorem 1.45 ([58, Prop. 3.7])**.** *The leaf space of a Riemannian foliation of codimension q with compact leaves admits a q-dimensional orbifold structure that makes the natural projection an orbifold submersion.*

Let us now conclude with examples of orbifolds. We have remarked earlier that a smooth (or complex) manifold is trivially a smooth (or complex) orbifold. Let us now explore a less trivial example that will come up again later.

Example 1.46 (Weighted projective space). Fix an array $\mathbf{w} = (w_0, \ldots, w_n)$ of positive integers and consider the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ given by

$$\lambda \cdot (z_0, \ldots, z_n) = (\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n). \tag{1.9}$$

We consider the orbit space $\mathbb{CP}(\mathbf{w})$ of $\mathbb{C}^{n+1} \setminus \{0\}$ and we call it *weighted projective space* with weights $\mathbf{w}$.

We now describe a family local uniformizing systems on it that makes it a complex orbifold. Cover $\mathbb{CP}(\mathbf{w})$ with open sets

$$U_i = \{[z_0 : \ldots : z_n] \in \mathbb{CP}(\mathbf{w}) : z_i \neq 0\}$$

and consider the sets $\widetilde{U}_i = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : z_i = 1\}$ and let Γ_i be the cyclic group of ${w_i}^{\text{th}}$ roots of unity acting on $\widetilde{U}_i$ by restricting (1.9) and leaving the origin fixed. It is $\widetilde{U}_i/\Gamma_i \simeq U_i$ and the map φ_i is given by $\varphi_i(\mathbf{y}_i) = [\mathbf{y}^{w_i}]$ where we call $\mathbf{y}_i = (y_{i0}, \ldots, \widehat{y}_{ii}, \ldots, y_{in}) \in \widetilde{U}_i$ to be a generic element and $[\mathbf{y}^{w_i}] = [y_{i0}^{w_i}, \ldots, \widehat{y}_{ii}^{w_i}, \ldots, y_{in}^{w_i}]$.

An orbifold atlas is then given by these orbifold charts plus their nontrivial intersections, namely the ones of the form $(\widetilde{U}_{i_0} \cap \ldots \cap \widetilde{U}_{i_k}, \Gamma_{i_1 \ldots i_k}, \varphi_{i_1 \ldots i_k})$ where $\Gamma_{i_1 \ldots i_k} = \mathbb{Z}_{\gcd(w_{i_1}, \ldots, w_{i_k})}$. The maps are given by $\varphi_{i_1 \ldots i_k}(\mathbf{y}) = [\mathbf{y}^{\gcd(w_{i_1}, \ldots, w_{i_k})}]$ in the same notations introduced earlier.

Let us now describe the injection maps on double intersections $\widetilde{U}_i \cap \widetilde{U}_j$. There exist a local uniformizing system $(\widetilde{U}_{ij}, \Gamma_{ij}, \varphi_{ij})$. The injections are

$\lambda_{ij,i} : \widetilde{U}_{ij} \to \widetilde{U}_i$ given by $\lambda_{ij,i}(\mathbf{y}) = \mathbf{y}^{\gcd(w_i,w_j)/w_i}$, choosing the principal branch, and analogously $\lambda_{ij,j} : \widetilde{U}_{ij} \to \widetilde{U}_j$ and any other injection satisfying $\varphi_i \circ \lambda_{ij,i} = \varphi_{ij}$ differs by a different choice of branch.

Chapter 2

Sasakian manifolds and examples

2.1 Direct definition and main facts

Sasakian geometry can be seen as the intersection of Riemannian, Contact and CR geometry, just like Kähler geometry is the intersection of complex, Riemannian and symplectic.

One could start with a Riemannian manifold (M, g) and consider its Riemannian cone $(C(M) = M \times \mathbb{R}^+, \overline{g} = r^2 g + dr^2)$ where $\mathbb{R}^+$ is the open half line $(0, +\infty)$ and r a coordinate on it.

Definition 2.1. (M, g) is said to be *Sasakian* if and only if its Riemannian cone is Kähler (of complex dimension $n+1$).

So M has dimension $2n+1$. We have an integrable almost complex structure $J \in \mathrm{End}(TC(M))$. Notice that the field $R = r\partial_r$ on the cone has norm $\overline{g}(R, R) = r^2$. We can use it to define the vector field $\xi = J(r\partial_r)$ which is tangent to M. Indeed it is $\overline{g}$-orthogonal to R due to the fact that $\overline{g}$ is J-Hermitian, so it does not have a component along ∂_r (which is on its own orthogonal to TM). The g-norm of ξ is $g(\xi, \xi) = \frac{1}{r^2}\overline{g}(R, R) = 1$. A first useful set of properties that follow from O'Neill's computations for warped products are the following.

Lemma 2.2. *The Levi-Civita connection $\overline{\nabla}$ of $(C(M), \overline{g})$ has the following properties, for any X, Y vector fields on CM tangent to M.*

$$\overline{\nabla}_R R = R \tag{2.1}$$

$$\overline{\nabla}_R X = \overline{\nabla}_X R = X \tag{2.2}$$

$$\overline{\nabla}_X, Y = \nabla_X Y - g(X, Y) R \tag{2.3}$$

where ∇ is the Levi-Civita connection of (M, g). In particular $[R, TM] = 0$.

Proposition 2.3. *The fields R and ξ are real holomorphic. That can be usefully rewritten as*

$$\overline{\nabla}_A \xi = \overline{\nabla}_{JA} R = JA; \qquad \overline{\nabla}_{JA} \xi = \overline{\nabla}_A R = A. \tag{2.4}$$

Proof. Compute, using that $\overline{\nabla} J = 0$ and Lemma 2.2

$$\begin{aligned}(\mathcal{L}_R J)Y &= [R, JY] - J[R, Y] \\ &= \overline{\nabla}_R JY - \overline{\nabla}_{JY} R - J(\overline{\nabla}_R Y - \overline{\nabla}_Y R) \\ &= -\overline{\nabla}_{JY} R + J\overline{\nabla}_Y R \\ &= -JY + JY = 0.\end{aligned}$$

and $(\mathcal{L}_R J)R = [R, \xi] - J[R, R] = 0$. Moreover, using again that J is parallel and (2.4)

$$\begin{aligned}(\mathcal{L}_\xi J)(Y) &= [\xi, JY] - J[\xi, Y] \\ &= \overline{\nabla}_\xi JY - \overline{\nabla}_{JY} \xi - J(\overline{\nabla}_\xi Y - \overline{\nabla}_Y \xi) \\ &= -\overline{\nabla}_{JY} \xi - \overline{\nabla}_Y R \\ &= \overline{\nabla}_Y R - \overline{\nabla}_Y R = 0\end{aligned}$$

and we conclude. □

Let us consider the 1-form $\eta = d^c \log r$ on the cone.[1]

Proposition 2.4. *The function $f = \frac{1}{2} r^2$ is a global Kähler potential of $C(M)$, i.e. the Kähler form is $\overline{\omega} = dd^c f = d(r^2 \eta)$.*

Remark 2.5. The cone $C(M)$ is what is known as the *symplectization* of (M, η).

Proposition 2.6. *The function $f = \frac{1}{2} r^2$ is a Kähler potential for the cone metric, i.e. $dd^c f = 2i\partial\overline{\partial} f = \omega$.*

Proposition 2.7. *The field ξ is tangent to M, has unit length and is Killing, i.e. $\mathcal{L}_\xi g = 0$.*

Proof. ξ is obviously tangent to M being orthogonal to R. As $\overline{g}$ is J-Hermitian we have

$$g(\xi, \xi) = \frac{1}{r^2} \overline{g}(JR, JR) = 1.$$

[1] We use the convention $d^c f = Jdf = -df \circ J$.

We note en passant that $\mathcal{L}_\xi \overline{g} = 0$. Indeed for vector fields A, B on the cone we have

$$\overline{g}(\overline{\nabla}_A \xi, B) = \overline{g}(J\overline{\nabla}_A r\partial_r, B) = \omega(A, B)$$

which is skew-symmetric, hence $\overline{\nabla}\xi$ is $\overline{g}$-skewsymmetric so ξ is Killing for $\overline{g}$. Now, using that $\overline{g}(X, R) = 0$ for all X tangent to M,

$$\begin{aligned}(\mathcal{L}_\xi g)(X, Y) &= g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) \\ &= \frac{1}{r^2}\big(\overline{g}(JX, Y) + \overline{g}(JY, X)\big) \\ &= \frac{1}{r^2}\big(\omega(X, Y) + \omega(Y, X)\big) = 0. \qquad \square\end{aligned}$$

Consider the distribution $D = \ker \eta$ on M. It will give a splitting

$$TM = D \oplus L_\xi \tag{2.5}$$

where L_ξ is the trivial line bundle tangent to ξ. This is an orthogonal splitting, indeed if $X \in \ker \eta$ we have $0 = \eta(X) = g(\xi, X)$.

Let us now define an endomorphism Φ of TM by $\Phi\xi = 0$ and $\Phi|_D = J|_D$.

Proposition 2.8. *The endomorphism Φ has the properties*

(i) $\Phi^2 = -\operatorname{id} + \eta \otimes \xi$*;*

(ii) $g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y)$*;*

making $(D, J|_D)$ a pseudoconvex almost CR structure.

Proof. Property (i) follows from the fact thet $J^2 = -\operatorname{id}$ and that $\eta \otimes \xi(Z)$ gives the component along ξ of the vector field Z and (ii) follows from the fact that $\overline{g}$ is Hermitian. The fact that the CR structure is pseudo convex follows from the fact that its Levi form can be taken to be $\omega(J\cdot, \cdot)$ which is positive definite by definition. $\square$

Remark 2.9. The triple (Φ, η, ξ) satisfying the property (i) of Proposition 2.8 is an almost contact structure in the sense of the previous chapter. Being endowed with a metric g that satisfies also (ii), then it is a metric almost contact structure.

Let us now prove some other equivalent facts for (M, η, Φ, ξ) to be a Sasakian manifold.

Proposition 2.10. *The endomorphism Φ is such that (so could be defined as) $\Phi X = \nabla_X \xi$. Moreover*

$$(\nabla_X \Phi)Y = g(\xi, Y)X - g(X, Y)\xi. \tag{2.6}$$

Proof. Using that $JX = \overline{\nabla}_X \xi = \nabla_X \xi - g(X,\xi)R$ we can infer that if $X \in D$ then $\nabla_X \xi = JX$ and also $\nabla_\xi \xi = 0$ since $g(\nabla_\xi \xi, Y) = -g(\xi, \nabla_Y \xi) = 0$ for all Y being ξ unitary and Killing. So, on TM, the tensors Φ and $\nabla\xi$ agree. Finally, (2.6) is proved in [11, Theorem 7.3.16]. □

Theorem 2.11. *The following are equivalent on a Riemannian manifold* (M, g).

1. *There exist a Killing unitary vector field* ξ *so that the tensor* $\Phi = \nabla\xi$ *satisfies* $(\nabla_X \Phi)Y = g(\xi, Y)X - g(X,Y)\xi$*;*
2. *There exist a Killing unitary vector field* ξ *such that the Riemann curvature tensor satisfies* $R(X,\xi)Y = g(\xi,Y)X - g(X,Y)\xi$*;*
3. *The Riemannian cone of* (M, g) *is Kähler.*

Proof. Let $(\nabla_X \Phi)Y = \nabla_X \Phi Y - \Phi \nabla_X Y$. So one can compute, using that ξ is Killing (see e.g. [11, Lemma 7.3.8]), that $R(X,\xi)Y = (\nabla_X \Phi)Y$ so this establishes the equivalence of the first two conditions.

Define an endomorphism J on $TC(M)$ by

$$JR = \xi \qquad JX = \Phi X - \eta(X)R. \tag{2.7}$$

It is easy to see it is a complex structure and we compute $\overline{\nabla} J$. For X, Y fields on M it is $(\overline{\nabla}_X J)R = \overline{\nabla}_X JR - J\overline{\nabla}_X R = \nabla_X \xi - \eta(X)R - JX = 0$. Also,

$$\begin{aligned}(\overline{\nabla}_X J)Y &= \overline{\nabla}_X(\Phi Y - \eta(Y)R) - J(\nabla_X Y - g(X,Y)R) \\ &= \nabla_X \Phi Y - g(X,\Phi Y)R - \eta(Y)X - X\eta(Y)R - \Phi\nabla_X Y \\ &\quad + \eta(\nabla_X Y)R + g(X,Y)\xi \\ &= (\nabla_X \Phi)Y - (\nabla_X \eta)(Y)R - g(X,\Phi Y)R - \eta(Y)X + g(X,Y)\xi \\ &= 0.\end{aligned}$$

using (i) and Lemma 2.2. Also $(\overline{\nabla}_R J)R = (\overline{\nabla}_R J)X = 0$ hold good, so we conclude that the cone is Kähler. Finally, it has been proved above that the Kähler assumption on the cone implies (i), so the proof is complete. □

2.2 Characteristic foliation and transverse geometry

The integral curves of ξ are geodesics, i.e. $\nabla_\xi \xi = 0$. They define a foliation on M called the Reeb foliation. The shape of such leaves gives a first classification of Sasakian manifolds.

Indeed if the leaves are closed, they have to be circles. A theorem of Wadsley says that if the leaves are circles and are geodesics, then M admits a locally free S^1 action (finite isotropy). If there are non closed leaves, their closure is diffeomorphic to a k-dimensional torus.

Definition 2.12. If the circle action above is (locally) free, the Sasakian manifold is said to be *(quasi-)regular*. If the leaves are not circles, the manifold is then *irregular*.

Moreover, a general foliation theoretic result of Molino stated in the previous chapter, asserts that in the case the leaves are compact, then the leaf space carries an orbifold structure and the standard projection from the manifold is an orbifold Riemannian submersion. This fact plus the circle action will allow us to give a structure theorem for (quasi)regular Sasakian manifolds in terms of Kähler leaf spaces, Riemannian submersions and principal circle bundles.

Let $\mathcal{F}$ be the characteristic foliation on M^{2n+1}. As said earlier it is nonsingular and of codimension $2n$. Let us now build some geometry on the normal bundle of the foliation, $\nu\mathcal{F} := TM/T\mathcal{F}$.

Our aim is to give the foliation a *transverse Riemannian structure*, that is a metric on the normal bundle identified with $(D, J, d\eta)$. This is a construction that can be done starting from any foliation on M, that will split $TM = E \oplus E^{\perp}$, where E is an integrable distribution. In our case $E = L$ and $E^{\perp} = D$, that is the splitting (2.5). We write every vector field $X = X^D + \eta(X)\xi$. We then define, for X, Y fields on M,

$$g^T(X, Y) = g(X^D, Y^D) \tag{2.8}$$

Computations give the following.

Proposition 2.13. *The following relations occur*

(i) $g = g^T + \eta \otimes \eta$*;*

(ii) $g^T(X, Y) = \frac{1}{2} d\eta(X, \Phi Y)$*.*

Proof. By definition of g^T, the D-component of a vector field X is $X - \eta(X)\xi$ so by bilinearity we obtain (i). For (ii) we compute

$$\begin{aligned}
\frac{1}{2} d\eta(X, \Phi Y) &= \frac{1}{2}\Big(X\eta(\Phi Y) - \Phi Y \eta(X) - \eta([X, \Phi Y])\Big) \\
&= \frac{1}{2}\Big(g(\nabla_X \xi, \Phi Y) - g(\nabla_{\Phi Y} \xi, X)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\Big(g(\Phi X, \Phi Y) - g(\Phi^2 Y, \xi)\Big) \\
&= \frac{1}{2}\Big((g - \eta \otimes \eta)(X, Y) + g(X, Y) - g(\eta \otimes \xi(Y), X)\Big) \\
&= g^T(X, Y)
\end{aligned}
$$

□

Let us now say something about the curvature of the metric g^T.

Proposition 2.14. *The Levi-Civita connection associated to g^T is*

$$
\nabla^T_X Y = \begin{cases} (\nabla_X Y)^D & \text{if } X \in D \\ [X, Y]^D & \text{if } X \in L \end{cases}
$$

Proof. One checks that ∇^T is indeed a torsion free connection compatible with g^T. □

We can now define the O'Neill tensors A and T in the following way

$$
\begin{aligned}
T_{E_1} E_2 &= \left(\nabla_{E_1^L} E_2^D\right)^L + \left(\nabla_{E_1^L} E_2^L\right)^D \\
A_{E_1} E_2 &= \left(\nabla_{E_1^D} E_2^D\right)^L + \left(\nabla_{E_1^D} E_2^L\right)^D
\end{aligned}
$$

By the same computations that are done in the case of Riemannian submersions ([8]) we can compute the relation between curvatures and transverse curvatures. In particular we will need it for the Ricci curvatures, namely

$$
\mathrm{Ric} = \mathrm{Ric}^T - 2g. \tag{2.9}
$$

2.2.1 Basic cohomology and transverse Hodge theory

Let us now describe basic cohomology, a tool that can be defined for any foliation and, in the Sasakian setting, can be used to describe deformations of Sasakian structures and to give invariants.

Definition 2.15. Let $\mathcal{F}$ be a p-dimensional foliation on M. A k-form $\alpha \in \Omega^k(M)$ is *basic* if

$$
i_\xi \alpha = 0 \text{ and } \mathcal{L}_\xi \alpha = 0
$$

for all vector fields ξ tangent to the leaves. Let Ω^*_B be the space of basic forms on M.

On a foliated chart $(U, x^1, \ldots, x^p, y^1, \ldots, y^{n-p})$ of M^n these two conditions say that a basic k-form ω locally looks like

$$\omega|_U = \sum \omega_{i_1,\ldots,i_k} dy^{i_1} \wedge \ldots \wedge dy^{i_k} \tag{2.10}$$

with no dx^j's (contraction property) and where the coefficients do not depend on x^j (Lie derivative property).

Just writing down the Cartan formula for Lie derivatives we can infer that if α is basic, then $d\alpha$ also is. So we can consider the restriction of the exterior differential

$$d_B = d|_{\Omega_B^p} : \Omega_B^p \to \Omega_B^{p+1}.$$

The complex (Ω_B^*, d_B) is called *basic DeRham complex* and its cohomology ring $H_B^*(\mathcal{F})$ the *basic cohomology ring of the foliation* $\mathcal{F}$. A crucial property of the basic DeRham cohomology is the existence of the following exact sequence that relates it with the usual DeRham cohomology

Theorem 2.16. *The following sequence, called the Gysin sequence,*

$$\ldots \to H_B^p(M) \to H^p(M,\mathbb{R}) \to H_B^{p-1}(M) \to H_B^{p+1}(M) \to \ldots \tag{2.11}$$

is exact.

This follows from the exact sequence of complexes

$$0 \to \Omega_B^p(M) \to \Omega^p(M)^T \to \Omega_B^{p-1}(M) \to 0 \tag{2.12}$$

where T is the compact Abelian subgroup (hence torus) of the isometry group of M given by the closure of the leaves of the characteristic foliation.

We use now the fact that the transverse geometry is Kähler to define the *basic (p,q)-forms.*

As it is done in classical complex geometry, the transverse complex structure splits the normal bundle as

$$\nu(\mathcal{F}) = \nu(\mathcal{F})^{1,0} \oplus \nu(\mathcal{F})^{0,1}$$

and similarly its dual. This induces a splitting of the bundle of r-forms and consequently a splitting

$$\Omega_B^r = \bigoplus_{p+q=r} \Omega_B^{p,q}.$$

We can also construct the basic Dolbeault operators ∂_B and $\overline{\partial}_B$ that will share the properties of the usual ones on complex manifolds, namely $d_B = \partial_B + \overline{\partial}_B$ and $d_B^c = i(\overline{\partial}_B - \partial_B)$.

The cohomology of the complex $(\Omega_B^{\bullet,\bullet}, \overline{\partial}_B)$ is called *basic Dolbeault cohomology.* Both the basic deRham and the basic Dolbeault cohomologies share some of the properties of their classical counterparts. There is a transversal Hodge theory, mainly developed by El-Kacimi Alaoui and others [33]. There is a transverse Hodge star operator that we can define on a basic r-form by

$$*_B\alpha := *(\eta \wedge \alpha) = (-1)^r \iota_\xi * \alpha.$$

It allows to construct the adjoint of d_B defining $\delta_B = - *_B d_B *_B$ which together define the *basic d-laplacian*

$$\Delta_B = \delta_B d_B + d_B \delta_B$$

on basic r-forms allowing us to define the *basic harmonic* forms by taking its kernel. A first result in transversal Hodge theory states that every basic cohomology class has a unique harmonic representative, just like the in the classical setting. We recall a couple of results, holding in the wider K-contact setting.

Theorem 2.17. *Let (M, g, ξ, η) be a compact K-contact manifold and let $H_B^\bullet(M)$ be its basic cohomology. Then the following hold.*

(i) The basic deRham groups $H_B^k(M)$ are finite dimensional and vanish for $r > 2n$;

(ii) $H_B^{2n}(M) \simeq \mathbb{R}$;

(iii) The class $[d\eta]_B$ is a nontrivial element of $H_B^2(M)$;

(iv) $H_B^1(M) \simeq H_{\mathrm{dR}}^1(M)$

(v) There is a transverse Poincaré duality $H_B^r(M) \simeq H_B^{2n-r}$.

The numbers $h_B^r(M) := \dim H_B^r(M)$ are called *basic Betti numbers.* One can also define the *basic Euler characteristic* out of them.

Let us now consider the basic Dolbeault cohomology. It also has analog properties with its complex geometry counterpart.

Theorem 2.18. *Let M be a compact Sasakian manifold. Then*

(i) $H_B^{n,n}(M) \simeq \mathbb{R}$;

(ii) The class $[d\eta]_B$ is nonzero and lies in $H_B^{1,1}(M)$;

(iii) $H_B^{p,p}(M)$ has positive dimension for $p \geq 1$.

(iv) There is a transverse Hodge Decomposition

$$H_B^r(M) = \bigoplus_{p+q=r} H_B^{p,q}(M);$$

(v) There is a transverse Serre duality $H_B^{p,q}(M) \simeq H_B^{n-p,n-q}(M)$

The numbers $h_B^{p,q}(M) = \dim H_B^{p,q}(M)$ are called *basic Hodge numbers.*

One can define also the operator $L : \omega_B^k(M) \to \omega_B^{k+2}(M)$ by $L\alpha = \alpha \wedge d\eta$, and its adjoint $\Lambda = - *_B L *_B$. We also have the operators $\partial^* = - *_B \partial *_B$ and $\overline{\partial}^* = - *_B \overline{\partial} *_B$ and their corresponding Laplacians.

Lemma 2.19. *For these operators the usual Kähler identities hold.*

There also is a transverse $\partial\overline{\partial}$-Lemma, due to El Kacimi-Alaoui [33].

Lemma 2.20. *Let M be a compact Sasakian manifold, and let ω, ω' be basic real closed $(1,1)$-forms in the same basic cohomology class. Then there exists a smooth basic function f such that $\omega' = \omega + i\partial_B\overline{\partial}_B f$.*

Finally we consider the transverse Ricci form $\rho^T = \mathrm{Ric}^T(J\cdot, \cdot)$ which is of Hodge type $(1,1)$, real valued and d_B-closed. The basic cohomology class $c_1^B = [\rho^T/2\pi] \in H_B^{1,1}(\mathcal{F}_\xi)$ is called *first basic Chern class.* If c_1^B admits a positive (resp. negative)[2] representative, then M is said to be *transverse Fano* (resp. *transverse anti-Fano*).

2.3 General structure theorems

In the regular (resp. quasi-regular) case, the leaf space $Z = M/\mathcal{F} = M/S^1$ has the structure of a compact manifold (resp. orbifold). The transverse Kähler structure pushes down to a Kähler structure on Z that makes it in a Kähler manifold or orbifold and gives a Riemannian submersion. Precisely we have the following.

Theorem 2.21. *Let Z be the space of leaves of a compact regular (or quasi-regular) Sasakian manifold (M, g). Then Z admits the structure of a Kähler manifold (or orbifold) and the projection $\pi : (M, g) \to (Z, h, \omega_h)$ is a Riemannian (orbifold) submersion with totally geodesic fibers. Moreover the class $[\omega_h]$ is (proportional to) an integral class in the (orbifold) cohomology group $H^2_{(\mathrm{orb})}(Z, \mathbb{Z})$.*

[2] In the sense of real valued $(1,1)$-forms.

A few words of explanation. The manifold M is a principal circle (orbi)-bundle on Z in which the splitting $TM = D \oplus L_\xi$ together with the Lie(S^1)-valued, that is real valued 1-form η define a connection. The form ω_Z is the transverse Kähler form $\omega^T = \frac{1}{2}d\eta$, so ω_Z is proportional to the curvature form of such bundle. Since 2π times the first Chern class is represented by the curvature form, it follows that the class of ω is proportional to an integral class in $H^2(Z,\mathbb{Z})$. A Kähler (orbi)fold whose Kähler form has such property is called *Hodge (orbi)fold*.

There is a converse of the previous theorem.

Theorem 2.22. *Let (Z, h, ω_Z) be a compact Hodge manifold or orbifold and let M be the principal circle (orbi)bundle defined by $[\omega_Z]$ and let η be a 1-form on M such that $d\eta = 2\pi^*\omega_Z$. Then the metric $\pi^*h + \eta \otimes \eta$ makes M a Sasakian manifold or orbifold. The total space M is a smooth manifold if the local uniformizing group inject into the structure group S^1.*

2.4 Examples

The first example, right from the definitions, is the $(2n + 1)$-sphere.

Example 2.23. View $S^{2n+1} \subset \mathbb{R}^{2n+2}$ and consider its standard contact structure

$$\eta = \left(\sum_{j=0}^{n} (y_j dx_j - x_j dy_j) \right) \Bigg|_{S^{2n+1}}$$

whose Reeb vector field is well known to be $\sum_j H_j$ where $H_j = y_j \partial_{x_j} - x_j \partial_{y_j}$. Moreover the standard complex structure of $\mathbb{R}^{2n+2}$ is

$$J = \sum_j (y_j dx_j \otimes \partial_{y_j} - x_j dy_j \otimes \partial_{x_j}).$$

After restricting it to the sphere and extending it to be zero on ξ we get a tensor Φ that makes $(S^{2n+1}, g_{\mathrm{round}}, \eta, \xi, \Phi)$ a Sasakian manifold. The Reeb flow is, after switching to complex coordinates,

$$\varphi_t(z_0, \ldots, z_n) = e^{2\pi i t}(z_0, \ldots, z_n). \tag{2.13}$$

The leaves are circles and the leaf space is $\mathbb{CP}^n$ with the Fubini-Study metric.

Definition 2.24. A Sasaki transformation on (M, g, η, Φ) is a diffeomorphism belonging to

$$\mathrm{Aut}(M, g, \eta, \xi, \Phi) := \{f \in \mathrm{Iso}(M, g) : f_*\xi = \xi\}.$$

From this, it follows that $f^*\eta = \eta$ from the properties of the Reeb field and contact form and $f_*\Phi = \Phi f_*$ from the equation $g(X,Y) = d\eta(X, \Phi Y) + \eta(X)\eta(Y)$.

Example 2.25. Continuing the sphere example, one sees that the its automorphism group is $\mathrm{O}(2n+2) \cap \mathrm{Sp}(n+1) = \mathrm{U}(n+1)$.

Theorem 2.26. *Every complete homogeneous Sasaki (M, g') manifold with $\mathrm{Ric}_{g'} \geq \delta > -2$ is regular and compact and admits a Sasaki-Einstein metric g compatible with the contact structure. Moreover Mit is the total space of a principal circle bundle over a generalized flag manifold with its standard Kähler-Einstein metric. By means of the previous theorem, the converse is also true.*

The regularity follows from a general well-known result of Boothby and Wang in the contact setting, namely that any homogeneous contact manifold must be regular. The compactness follows from the Ricci curvature assumption with a slight generalization of Myers' theorem for Sasaki manifolds due to Hasegawa and Seino. The generalized flag manifolds come up because there is a transitive group of isometries that preserves the Kähler structure on the leaf space, so there is a homogeneous Kähler manifold and by known results it has to be a generalized flag. Some other Sasakian manifolds can be embedded in weighted Sasakian spheres. First a definition.

Definition 2.27. Let M be a Sasakian manifold with tensors Φ and ξ and N be a submanifold. If $\Phi_p(T_pN) \subset T_pN$ and $\xi_p \in T_pN$ for every $p \in N$ then N is said to be a Sasakian submanifold of M.

Consider a polynomial $F : \mathbb{C}^{n+1} \to \mathbb{C}$ such that, with respect to the action of $\mathbb{C}^*$ defined earlier, $F(\lambda \cdot z) = \lambda^d F(z)$ for some integer d. The zero locus of F has only one singularity in the origin, so the intersection $L_F := \{F = 0\} \cap S^{2n+1}$ is a submanifold called *link*. Using the real notation and the Cauchy-Riemann equation, one can verify by computation that $\xi_{\mathbf{w}} \cdot F = 0$ and $\Phi_{\mathbf{w}}(\ker dF) \subset \ker dF$, so L_F is a Sasakian submanifold of the weighted Sasakian sphere. The leaf space Z_F of L_F is just the zero locus of F seen in the weighted projective space $\mathbb{P}(\mathbf{w})$. There is a characterization of the Fano property of Z_F, namely

Proposition 2.28. *Z_F is Fano if and only if $\sum w_j =: |\mathbf{w}| > d$.*

A particular example is when the polynomial is of the form

$$F(z) = \sum_{j=0}^{n} z_j^{a_j}$$

for some array **a** of positive integers. In this case the link is denoted by the simpler notation $L(\mathbf{a})$ and the polynomial and link are called Brieskorn-Pham.

There is a result of Brieskorn that gives necessary and sufficient condition on the array **a** for the link to be an integral or rational homology sphere. Moreover Boyer, Galicki and Kollár gave sufficient conditions by means of inequalities on the a_j for the link to be admit a quasi-regular Sasaki-Einstein metric.

Moreover, Ghigi and Kollár [41] proved that if the a_i are pairwise coprime then the link is homeomorphic to a sphere and a characterization for the existence of Sasaki-Einstein metrics.

In particular the link $L(2, 3, 7, 43, 1333)$ is diffeomorphic to the standard S^7 admits a 41 complex dimensional family of Sasaki-Einstein deformations. The other oriented diffeomorphism classes of S^7 also admit several hundred of inequivalent families of structures.

We now describe an example of family of Sasaki-Einstein 5-folds exhibited by Gauntlett, Martelli, Sparks and Waldram [39]. These can be both quasi-regular and irregular and besides they are the first example of *irregular* Sasaki metrics that up to that time (2004) were conjectured not to exist [22]. They are explicitly given locally and then it is proved that can be extended to a complete compact 5-fold which turns out to be $S^2 \times S^3$. Moreover the isometry group of these manifolds acts with cohomogeneity one. It has been proved later that they can be the only 5-folds whose isometry group acts in cohomogeneity one.

2.5 Einstein and η-Einstein metrics

In this section we consider Sasaki-Einstein metrics, we introduce the notion of η-Sasaki-Einstein metric and we prove a result that interrelates this notion about the metric on a manifold with other properties of the cone metric and of the transverse metric.

From Theorem 2.11 it follows that $R(X, \xi)\xi = X - g(X, \xi)\xi$. So we infer that the Ricci tensor[3] of M^{2n+1} is such that

$$\mathrm{Ric}(\xi, \xi) = 2n. \tag{2.14}$$

Definition 2.29. A Sasakian manifold (M, g, η, ξ, Φ) is said to be *η-Sasaki-*

[3] $\mathrm{Ric}(X, Y) := \mathrm{tr}(Z \mapsto R(X, Z)Y)$ i.e. contracting the second and forth indices and without dividing by $\dim M - 1$.

Einstein if there exist constants[4] $\lambda, \nu \in \mathbb{R}$ such that

$$\mathrm{Ric}_g = \lambda g + \nu \eta \otimes \eta. \tag{2.15}$$

By the computation at the beginning of the section, it follows that $\lambda + \nu = 2n$. So in particular, if g is Einstein, the Einstein constant has to be $2n$ hence the Ricci curvature is positive and the scalar curvature is constant equal to $2n(2n+1)$. As a corollary of Myers' theorem we get.

Corollary 2.30. *A complete Sasaki-Einstein manifold is compact with finite fundamental group.*

Proposition 2.31. *A Sasakian metric on M^{2n+1} is η-Einstein if, and only if, the transverse metric is Kähler-Einstein.*

Proof. Let M be η-Sasaki-Einstein with constant λ, i.e.

$$\mathrm{Ric} = \lambda g + (2n - \lambda)\eta \otimes \eta.$$

Then by the relation (2.9) we can easily infer that

$$\begin{aligned} \mathrm{Ric}^T &= \mathrm{Ric} + 2g = (\lambda + 2)g + (2n - \lambda)\eta \otimes \eta \\ &= (\lambda + 2)g^T + (2n + 2)\eta \otimes \eta \\ &= (\lambda + 2)g^T \end{aligned}$$

where the last equality holds because $\eta \otimes \eta$ vanishes on the subbundle D. Conversely if the transverse metric is Einstein, i.e. $\mathrm{Ric}^T = \tau g^T$, then on D it will hold that $\mathrm{Ric} + 2g = \tau(g - \eta \otimes \eta)$ that is

$$\mathrm{Ric}|_D = (\tau - 2)g|_D \tag{2.16}$$

If $X \in D$ then

$$\mathrm{Ric}(X, \xi) = 2n\eta(X) = 0 = (\tau - 2)g(\xi, X) + (2n - \tau + 2)\eta \otimes \eta(X, \xi) \tag{2.17}$$

and finally

$$\mathrm{Ric}(\xi, \xi) = 2n = (\tau - 2)g(\xi, \xi) + (2n - \tau + 2)\eta \otimes \eta(\xi, \xi). \tag{2.18}$$

Equations (2.16), (2.17) and (2.18) together prove that

$$\mathrm{Ric} = (\tau - 2)g + (2n - \tau + 2)\eta \otimes \eta$$

and we are done. □

[4] If they are functions, they are necessarily constant if $\dim M > 3$.

We can now prove the property that relates the three metrics for a Sasakian structure.

Proposition 2.32. *The following are equivalent.*

(i) The metric g is Sasaki-Einstein, i.e. $\mathrm{Ric}_g = 2ng$*;*

(ii) The cone metric is Ricci flat, i.e. $\mathrm{Ric}_{\overline{g}} = 0$.

(iii) The transverse metric is Kähler-Einstein $\mathrm{Ric}^T = 2(n+1)g^T$.

Proof. The equivalence between (i) and (ii) follows from the expression of the Ricci tensor for a warped metric. Indeed this is (see [8])

$$\begin{aligned}
&\mathrm{Ric}_{\overline{g}}(R,R) = 0\\
&\mathrm{Ric}_{\overline{g}}(X,R) = 0\\
&\mathrm{Ric}_{\overline{g}}(X,Y) = \mathrm{Ric}_g(X,Y) - \frac{2n}{r^2}\overline{g}(X,Y) = \mathrm{Ric}_g(X,Y) - 2ng(X,Y).
\end{aligned}$$

The equivalence between (i) and (iii) is the case $\lambda = 2n$ and $\tau = 2(n+1)$ of Proposition 2.31. $\square$

Chapter 3

Deformations of Sasakian structures and applications

The study of deformations of a Sasakian structure (η, ξ, Φ, g) is feasible when one keeps some of the tensors or structures fixed and varies others.

3.1 Fixed CR structure (Type I)

We start considering deformations of a Sasakian structure (η, ξ, Φ, g) with underlying CR structure $(D = \ker \eta, J = \Phi_{|}D)$ that keep the CR structure fixed but deform the Reeb foliation. So the contact subbundle has to be the same and also the restriction of Φ to it. The new tensors are of the form

$$\widetilde{\eta} = f\eta, \qquad \widetilde{\xi} = \xi + \rho \tag{3.1}$$

where f is a positive function and ρ is an infinitesimal transformation of (D, J), that is it lies in $\mathfrak{cr}(D, J) := \{X \in \mathfrak{aut}(D) : \mathcal{L}_X J = 0\}$. The Lie derivative of J makes sense as $[X, D] \subseteq D$.

The Reeb condition forces the relation

$$f = \frac{1}{1 + \eta(\rho)}.$$

As long as $\eta(\widetilde{\xi}) > 0$, the form $\widetilde{\eta}$ is still contact, its kernel D is unchanged and we require $\Phi|_D = \widetilde{\Phi}|_D$. We define also the fields

$$\begin{aligned} \widetilde{\Phi} &= \Phi - \Phi\widetilde{\xi} \otimes \widetilde{\eta} \\ \widetilde{g} &= d\widetilde{\eta} \circ (\widetilde{\Phi} \times \mathrm{id}) + \widetilde{\eta} \otimes \widetilde{\eta}. \end{aligned} \tag{3.2}$$

They both will satisfy the compatibility conditions in the definition of an almost contact structure with compatible metric.

The normality of the deformed almost contact structure obtained follows from the fact that the almost CR structure is still integrable since it was not changed and the relation $\mathcal{L}_{\widetilde{\xi}}\widetilde{\Phi} = 0$ holds since it is trivial on $\widetilde{\xi}$ and on D is equivalent to $\mathcal{L}_\rho \Phi = 0$. It is not automatic that $\widetilde{\xi}$ is Killing with respect to $\widetilde{g}$, so we assume it and the following fact follows.

Proposition 3.1. *If the deformation* (3.1) *applied to a Sasakian structure is through K-contact structures, then it is through Sasakian structures.*

For any strictly pseudoconvex almost CR structure (D, J) on a manifold M, we denote by $\mathfrak{F}(D, J)$ the set of all K-contact structures having (D, J) as underlying almost CR structure.

Definition 3.2. A deformation defined by (3.1) within $\mathfrak{F}(D, J)$ is said to be *of type I.*

Let us now relate such space with the Lie algebra of infinitesimal CR automorphisms. Fix an almost CR structure (D, J) and assume it is what is called *of Sasaki type*, i.e. there exist a Sasakian structure that admits it as underlying almost CR structure. This means that the set $\mathfrak{F}(D, J)$ is not empty. Fix a structure $\mathcal{S}_0 = (\eta_0, \xi_0, \Phi_0, g_0)$ in it.

Proposition 3.3. *A contact metric structure $\mathcal{S}$ lies in $\mathfrak{F}(D, J)$ if and only if its Reeb field $\xi \in \mathfrak{cr}(D, J)$.*

We now identify the set $\mathfrak{F}(D, J)$ with a cone of $\mathfrak{cr}(D, J)$. Namely, after fixing a Sasaki structure $S_0 \in \mathfrak{F}(D, J)$ with contact form η_0, define

$$\mathfrak{cr}^+(D, J) = \{\xi \in \mathfrak{cr}(D, J) : \eta_0(\xi) > 0\}.$$

Let us state some first properties of this set. Recall that a subset C of a vector space such that if $v \in C$ then $\lambda v \in C$ for all $\lambda > 0$ is called a cone.

Proposition 3.4. *The set $\mathfrak{cr}^+(D, J)$ is a convex cone in the Lie algebra $\mathfrak{cr}(D, J)$ and moreover it is invariant by the adjoint action of the group of CR transformations.*

This helps to give the following description of $\mathfrak{F}(D, J)$.

Proposition 3.5. *The map $(\xi, \eta, \Phi, g) \mapsto \xi$ defines a bijection $\iota : \mathfrak{F}(D, J) \to \mathfrak{cr}^+(D, J)$.*

Proof. Let $\mathcal{S}'$ and $\mathcal{S}''$ two Sasakian structures in the same type I deformation class that have the same Reeb vector field ξ, that is $\rho = 0$ hence $f = 1$. So also $\eta' = \eta''$ and $\Phi' = \Phi''$ being the CR structure fixed. Fix now $\xi \in \mathfrak{cr}^+(D, J)$. Define $\eta = \frac{1}{\eta_0(\xi)}\eta_0$. Then $\eta(\xi) = 1$ and $i_\xi d\eta = 0$ because ξ fixes D. Let $\Phi = \Phi_0 - \Phi_0\xi \otimes \eta$ and g in a compatible way. This contact metric structure belongs to $\mathfrak{F}(D, J)$ by the preceding proposition. □

It is proved in [13] that the Lie algebra $\mathfrak{cr}(D, J)$ decomposes as $\mathfrak{t}_k \oplus \mathfrak{p}$ where $\mathfrak{t}_k$ is the algebra of a maximal torus $\mathbb{T}^k$ of dimension $1 \leq k \leq n+1$ and p is a completely reducible $\mathbb{T}^k$-module. Moreover every element of $\mathfrak{cr}^+(D, J)$ is conjugate to an element in $\mathfrak{t}_k^+ := \mathfrak{t}_k \cap \mathfrak{cr}^+(D, J)$.

This allows to identify $\mathfrak{t}_k^+$ with $\mathfrak{cr}^+(D, J)$ modulo the action of the group of CR transformations. It can then be thought as the moduli space of the Sasakian structures compatible with the fixed CR structure (D, J).

We conclude mentioning that there is a very special case of type I deformation, namely when $k = 1$. In this case the above moduli space is one dimensional so ρ of definition (3.1) is a multiple of ξ. This is the case of D-homothetic deformations, defined for real $a > 0$ as

$$\begin{aligned}
\eta' &= a\eta \\
\xi' &= \frac{1}{a}\xi \\
g' &= ag + a(a-1)\eta \otimes \eta \\
\Phi' &= \Phi.
\end{aligned}$$

They were first introduced by Tanno (see [73] and references therein) and we shall see some of their applications below.

3.2 Fixed Reeb field (Type II)

We now wish to keep the Reeb field fixed and change the other pieces of data. Fix a Sasakian structure $\mathcal{S} = (\eta, \xi, \Phi, g)$ on a closed manifold M and let

$$\mathfrak{F}(\xi) = \{(\eta', \xi, \Phi', g') : \text{is a Sasakian structure on } M\}.$$

The 1-form $\zeta = \eta - \eta'$ is basic so $d\eta$ and $d\eta'$ define the same basic cohomology class in $H^2_B(M)$.[1] Fixing an almost complex structure J on the normal bundle of $\mathcal{F}_\xi$, we define $\mathfrak{F}(\xi, J)$ to be the set of Sasakian structures in $\mathfrak{F}(\xi)$ that have the same transverse complex structure, that is

$$\mathfrak{F}(\xi, J) = \{\text{Sasakian structures } (\eta', \xi, \Phi', g') : \pi_\nu \Phi' = J\pi_\nu\}$$

where $\pi_\nu : TM \to \nu(\mathcal{F}_\xi)$ is the projection onto the normal bundle of the characteristic foliation $\nu(\mathcal{F}) = TM/L_\xi$ where as above L_ξ is the line bundle singled out by the characteristic foliation.

We now state a result that tells us about a parameterization of such space. We denote by $C^\infty_B(M)/\mathbb{R}$ the space of basic smooth functions up to constants. More precisely we have

[1] Of course the basic cohomology does not change since the Reeb foliation is the same.

Proposition 3.6 ([11, 13])**.** *The space $\mathfrak{F}(\xi, J)$ is modeled on $C^\infty_B(M)/\mathbb{R} \times C^\infty_B(M)/\mathbb{R} \times H^1(M, \mathbb{Z})$. That is, if $(\eta, \xi, \Phi, g) \in \mathfrak{F}(\xi, J)$ then any other structure $(\widetilde{\eta}, \widetilde{\xi}, \widetilde{\Phi}, \widetilde{g})$ is determined by two basic functions φ, ψ and an integral form α in the following way*

$$\begin{aligned} \widetilde{\eta} &= d^c\varphi + d\psi + \alpha \\ \widetilde{\Phi} &= \Phi - (\xi \otimes (\widetilde{\eta} - \eta)) \circ \Phi \\ \widetilde{g} &= d\widetilde{\eta} \circ (\mathrm{id} \otimes \widetilde{\Phi}) + \widetilde{\eta} \otimes \widetilde{\eta}, \end{aligned} \tag{3.3}$$

where the d^c is done with respect to the fixed J and of course we see $\alpha \in H^1(M, \mathbb{R})$.

Moreover, if endowed with the compact-open topology as sections of vector bundles, the group $H^1(M, \mathbb{Z})$ labels its connected components. We can now define a second kind of Sasakian deformations.

Definition 3.7. A deformation within a connected component of $\mathfrak{F}(\xi, J)$ of the form $\eta \mapsto \eta + \zeta$ is said to be *of type II*.

Remark 3.8. The corresponding transverse Kähler forms are related by $\omega'^T = \omega^T + i\partial_B\overline{\partial}_B\varphi$ so the deformation takes place in the same basic Kähler class.

More generally we can consider the set of Sasakian structure that share the same Reeb foliation $\mathcal{F}_\xi$, clearly a superset $\mathfrak{F}(\mathcal{F}_\xi) \supset \mathfrak{F}(\xi)$. Consider now the set

$$\mathfrak{F}^+(\mathcal{F}_\xi) = \bigcup_{a \in \mathbb{R}^+} \mathfrak{F}(a^{-1}\xi)$$

and its image $\mathfrak{F}^-(\mathcal{F}_\xi)$ under the *conjugation* of Sasakian structure that has a sign changed to all tensor fields except of course the metric. We have then

Proposition 3.9. *There is a disjoint decomposition*

$$\mathfrak{F}(\mathcal{F}_\xi) = \mathfrak{F}^+(\mathcal{F}_\xi) \cup \mathfrak{F}^-(\mathcal{F}_\xi).$$

Proof. For the not obvious inclusion, let $\mathcal{S}' \in \mathfrak{F}(\mathcal{F}_\xi)$. Being the Reeb foliation unchanged, it has to be $\xi' = \frac{1}{f}\xi$ for some smooth function f and also $\eta' = f\eta$. Then

$$0 = i_{\xi'}d\eta' = \frac{1}{f}(\xi \cdot f)\eta - df$$

so df is zero on any vector in the contact subbundle hence f only depends on the leaf coordinates. Our aim is to prove that f has to be constant. Differentiating we get

$$d(\xi \cdot f) \wedge \eta + \xi \cdot f d\eta = 0.$$

It is enough to show that the first term vanishes. It is equal to $d^h(\xi f) \wedge \eta$ and every horizontal vector field kills ξf so we conclude. ∎

Definition 3.10. Two Sasakian structures are a-homologous if there exist some $a > 0$ such that $\xi' = \frac{1}{a}\xi$ and $[d\eta']_B = a[d\eta]_B$.

We report here a result about the variation formulas occurring when a type II deformation is performed. They are proved like the ones in the Kähler case (see e.g. [8, p. 93]) and we will need in Chapter 4 the variation of the scalar curvature under type II deformations.

Proposition 3.11 ([11, 13]). *Let φ be a basic function on a Sasakian manifold (M, η, ξ, Φ, g) defining the type II deformation $\eta_\varphi = \eta + \frac{1}{2}d^c\varphi$. Let $d\mu_\varphi, \rho^T_\varphi, s^T_\varphi$ be the volume element, transverse Ricci form and transverse scalar curvature of the deformed structure, respectively. Then the following variational formulas occur.*

$$\begin{aligned} D_\varphi d\mu_\varphi|_{\varphi=0} &= -\frac{1}{2}\Delta_B \varphi d\mu \\ D_\varphi \rho^T_\varphi|_{\varphi=0} &= -i\partial\overline{\partial}\left(\frac{1}{2}\Delta_B\varphi\right) \\ D_\varphi s^t_\varphi|_{\varphi=0} &= -\frac{1}{2}\Delta^2_B\varphi + 2(\rho^T, i\partial\overline{\partial}\varphi). \end{aligned}$$

3.3 Deformations of the transverse complex structure

Here we introduce a third type of deformation one can perform on Sasakian structures. Actually they are defined as deformations of a one dimensional foliation and we shall discuss the existence of compatible Sasakian metrics. We shall follow [60].

It is known from the deformation theory of Kodaira and Spencer [52] small deformations of the complex structure of a Kähler manifold still admit compatible Kähler metrics. This is not the case for deformations of the characteristic foliation of a Sasakian manifold and we shall introduce an obstruction to the existence of compatible Sasakian structures.

We start by introducing what we mean by a deformation of a one dimensional foliation, which we shall call a *flow*. Let M be a closed manifold and $V \subseteq \mathbb{R}^\ell$ an open set.

Definition 3.12. A *smooth family of flows* on M over V is a flow $\widetilde{\mathcal{F}}$ on $M \times V$ such that every level $M_t := M \times \{t\}$ is saturated by the leaves of $\widetilde{\mathcal{F}}$ for each $t \in V$. We shall call the restricted foliation $\mathcal{F}_t$ and denote the family by $\{\mathcal{F}_t\}_{t\in V}$.

It is easy to see that by the saturation property, if a vector field on $M \times V$ is tangent to the leaves then it has no component tangent to V. So the second projection $\mathrm{pr}_2 : T(M \times V) \to TV$ factors through a map $T(M \times V)/T\widetilde{\mathcal{F}} \to TV$, whose kernel we call the *family of normal bundles* of $\{\mathcal{F}_t\}_{t\in V}$.

A *family of Riemannian flows* is a family of flows $\{\mathcal{F}_t\}_{t\in V}$ as above together with a Riemannian metric $\widetilde{g}_\nu$ on the family of normal bundles such that $(\mathcal{F}_t, \widetilde{g}_\nu|_{M_t})$ is a Riemannian flow for each $t \in V$.

Analogously we define a *family of transversely holomorphic flows* by introducing a complex structure $\widetilde{J}$ on the family of the normal bundles such that $(\mathcal{F}_t, \widetilde{J}|_{M_t})$ is a transversely holomorphic flow for each $t \in V$.

Given two such structure, it is easy to define a *family of Kähler structures* by introducing a Kähler form and imposing its closedness as a form on $M \times V$.

Definition 3.13. A Riemannian flow on M is called *isometric* if there exist a metric g on M and a vector field ξ never vanishing and tangent to the leaves such that $\mathcal{L}_\xi g = 0$. The pair (g, ξ) is then called a *Killing pair*.

We now start to define the obstruction we said above.

Given a Killing pair (g, ξ) we consider the dual form $\eta = \xi^\flat$ with respect to g. Then $d\eta$ is a basic 2-form.

Definition 3.14. The *basic Euler class* of the foliation is defined to be, up to constants, $\mathbb{R}^*[d\eta]_B \in H^2_B(M)$.

We now consider the $(0,2)$-component of $d\eta$. It is $\overline{\partial}$-closed since a computation shows that $\overline{\partial}(d\eta^{0,2}) = (dd\eta)^{0,3} = 0$.

Definition 3.15. The $(0,2)$-component of the basic Euler class is defined to be $\mathbb{R}^*[d\eta^{0,2}]_B \subseteq H^2_B(M)$. If it vanishes we say that the Euler class is of $(1,1)$-type.

It must be proved that this definitions do not depend on the choice of the Killing pair, as it is done in [60, Lemma 3.18] where it is proved that two Killing pairs give rise to the same $(0,2)$-components of the basic Euler forms up to multiplication of a nonzero real number.

As an example of basic Euler class of type $(1,1)$ there are of course Sasakian structures, because $d\eta$ is a transversal Kähler form. So we have the following.

Lemma 3.16. *The basic Euler class of the transversally holomorphic flow of a Sasakian manifold is of type* $(1,1)$.

We are now ready to state one of the main results of Nozawa's paper.

Theorem 3.17. *Let V be an open neighborhood of the origin in $\mathbb{R}^\ell$ that parameterizes a smooth family of transversally holomorphic Riemannian flows $(\mathcal{F}_t, g_t, J_t)$ on a closed manifold M. Assume that $(\mathcal{F}_0, g_0, J_0)$ is the underlying transversally Kähler structure of a Sasakian metric (g, η). If the basic Euler class of the family is of type $(1,1)$ for each $t \in V$ then there exist an open neighborhood V_1 of 0 in V and a smooth family of Sasakian metrics $\{(g_t, \eta_t)\}_{t \in V_1}$ such that $(g_0, \eta_0) = (g, \eta)$ and that the Reeb foliation of (g_t, η_t) is $(\mathcal{F}_t, J_t)$ for every $t \in V_1$.*

The following corollary will be useful to us.

Corollary 3.18. *A deformation of a positive Sasakian structure is of $(1,1)$-type, in particular when the original Sasakian metric is Einstein or more generally when it is transversally Fano.*

3.4 Applications to Betti numbers and pinching results

3.4.1 Vanishing results for Betti numbers

The earliest application of deformations may be dated back in the 60s by Tanno in [73]. He applied the D-homothetic deformations to infer the vanishing of the first two Betti numbers of a Sasakian manifold under some curvature assumptions. He does that by proving the non existence of harmonic 1-forms or 2-forms on Sasakian manifolds that have some positivity requirements, exploiting the fact that if a form is harmonic with respect to some metric, then it is still harmonic with respect to a D-homothetically deformed one. More precisely the key result for the first Betti number is the following.

Theorem 3.19. *On a compact K-contact manifold M there is no harmonic 1-form w such that $\mathrm{Ric}(w^\sharp, w^\sharp) + 2g(w^\sharp, w^\sharp) \geq 0$ everywhere and with strict inequality at some point. In particular if $\mathrm{Ric} + 2g$ is positive definite then $b_1(M) = 0$.*

From which it follows the corollary

Corollary 3.20. *If a compact K-contact manifold M^{2n+1} has sectional curvature $> -\frac{3}{2n-1}$ then $b_1(M) = 0$.*

This is true because if we assume the sectional curvature to be $> -K$ for some positive K then we can compute, with respect to an orthonormal basis

$(\xi = e_0, e_1, \ldots, e_{2n})$ with respect to which the Ricci tensor is diagonal and using that the sectional curvature of any plane containing ξ is 1

$$R_{ii} = \sum_{k=0}^{2n} g(R(e_i, e_k)e_i, e_k) = \sum_{k=1}^{2n} K_{ik} + 1 > 1 - (2n-1)K;$$

and from this one can estimate $\mathrm{Ric}(w, w)+2g(w, w) > 3(1-(2n-1)K)g(w, w)$ and apply the Theorem.

Then to study the second Betti number he uses the splitting of harmonic 2-forms in harmonic forms of pure and hybrid type.

Definition 3.21. A 2-form α on a Sasakian manifold is said to be of pure type if $\alpha(\Phi\cdot, \Phi\cdot) = \alpha$ and of hybrid type if $\alpha(\Phi\cdot, \Phi\cdot) = -\alpha$.

Every harmonic 2-form can be written as a sum of a pure and a hybrid one, that is $w = w^{\mathrm{p}} + w^{\mathrm{h}}$ where

$$w^{\mathrm{p}}(X, Y) = \frac{1}{2}\Big(w(X, Y) + w(\Phi X, \Phi Y)\Big)$$
$$w^{\mathrm{h}}(X, Y) = \frac{1}{2}\Big(w(X, Y) - w(\Phi X, \Phi Y)\Big).$$

This uses that for any harmonic form w of any degree on a Sasakian manifold it is $i_\xi w = 0$. A first result is

Theorem 3.22. *A compact Sasakian manifold M^{2n+1} with strictly positive sectional curvature has no harmonic 2-forms of hybrid type. Moreover if $n \geq 2$ and the sectional curvature is $> -\frac{3}{2n-1}$ then there are no harmonic 2-forms of pure type either.*

So we have

Corollary 3.23. *If a Sasakian manifold of dimension ≥ 5 has positive Riemannian pinching, then has vanishing second Betti number.*

The methods of proof still involve some expression similar to $\mathrm{Ric}(w, w) + 2g(w, w)$ for forms of higher degree.

3.4.2 Relations with pinching

On a Sasakian manifold we consider

$$H = \sup\{K(\Phi X, X) | X \in D_x, x \in M\}$$

$$L = \inf\{K(\Phi X, X) | X \in D_x, x \in M\}.$$

If $H+3 > 0$ we set $\mu = \frac{L+3}{H+3}$ and we say that the manifold is μ-holomorphically pinched. This means that the transverse curvature is μ-holomorphically pinched, being $K(X, \Phi X) = K^T(X, \Phi X) - 3$ for a unit section X of D. This is invariant under D-homothetic deformations.

We say that a Sasakian manifold has μ-holomorphic pinching if there exist positive constants μ, K such that $\mu \leq K(\Phi X, X) \leq K$ for all x on the manifold and X in the tangent space at x. The following facts are proved

Theorem 3.24. *Let M be a Sasakian manifold. Then the following hold.*

1. *If M is μ-holomorphically pinched with $\mu > \frac{1}{2}$, then $b_2(M) = 0$.*
2. *If M is μ-holomorphically pinched with $\mu > \frac{2}{3}$, then it is D-homothetically related to a metric with has strictly positive sectional curvature.*
3. *If M is $\frac{4}{5}$-holomorphically pinched, then it is D-homothetically related to a metric with Riemannian pinching $\frac{1}{4}$. Hence, M is homeomorphic with a sphere if simply connected and complete.*
4. *If M is μ-holomorphically pinched with $\mu > \frac{1}{2}$, and has constant scalar curvature, then there is a D-homothetically related metric having curvature constantly* 1*.*

These follow from the fact that a μ-holomorphically pinched Sasakian metric is D-homothetically related to one of Riemannian pinching $\widetilde{\delta} \geq \frac{3\mu-2}{4-3\mu}$.

3.5 Weighted Sasaki spheres and links

We go back to type I deformations in their full generality. and we apply them to the standard Sasaki sphere. This dates back to Takahashi's work [72]. Fix an array $\mathbf{w} = (w_0, \ldots, w_n)$ of positive real numbers and deform the Reeb vector field of the standard Sasaki sphere by adding the vector

$$\rho_{\mathbf{w}} = \sum_{j=0}^{n} (w_j - 1)\big(y_j \partial_{x_j} - x_j \partial_{y_j}\big) \in \mathfrak{aut}(S^{2n+1})$$

which belongs to the algebra of infinitesimal automorphisms of the standard Sasaki sphere (isomorphic on its own to $\mathfrak{u}(n+1)$). So the new Reeb field becomes $\xi_{\mathbf{w}} = \xi + \rho_{\mathbf{w}}$ and the other tensors are defined by (3.2). If the weights are positive integers then the Sasaki structure will be quasi regular and has a weighted projective space $\mathbb{P}(\mathbf{w})$ as leaf space.

Now consider the weighted action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ defined by $\lambda \cdot z = (\lambda^{w_0} z_0, \dots \lambda^{w_n} z_n)$. Let F be a complex polynomial of *weighted degree* d, namely that $F(\lambda \cdot z) = \lambda^d F(z)$ and such that its only singularity is in the origin. Consider $L_{\mathbf{w},F} = \{F = 0\} \cap S^{2n+1}$ called the *link of* F *with weights* $\mathbf{w}$. This can be given the Sasakian structure as a submanifold of the weighted sphere, being $\xi_{\mathbf{w}}$ tangent to it and its tangent space invariant under $\Phi_{\mathbf{w}}$.

Now consider the particular case of a polynomial of the form $F_{\mathbf{a}}(z) = \sum z_j^{a_j}$ for a vector of positive integers $\mathbf{a}$. This will be of weighted degree $d = \operatorname{lcm}(a_j)$ with respect to the weighted action with weights $w_j = d/a_j$. Its link, denoted more conveniently by $L(\mathbf{a})$ is called *Brieskorn-Pham link*.

The following theorem holds

Theorem 3.25. *Let* $|\mathbf{w}| = \sum w_j$. *Then the link* $L_{\mathbf{w},F}$ *of the polynomial* F *of* $\mathbf{w}$*-degree* d *is*

1. *Fano if and only if* $|\mathbf{w}| > d$*;*
2. *antiFano if and only if* $|\mathbf{w}| < d$*;*
3. *of vanishing first Chern class if and only if* $|\mathbf{w}| = d$*.*

So in particular we have the following.

Corollary 3.26. *The Brieskorn-Pham link* $L(\mathbf{a})$ *is*

1. *Fano iff* $\sum \frac{1}{a_j} > 1$*;*
2. *antiFano iff* $\sum \frac{1}{a_j} < 1$*;*
3. *of vanishing first Chern class iff* $\sum \frac{1}{a_j} = 1$*.*

Some of these manifolds provide examples of non-round Einstein metrics on spheres, Einstein metrics for other odd-dimensional manifolds among which homology and exotic spheres. For more details see [71] and references therein.

3.6 Transverse Calabi problem

El Kacimi-Alaoui [33] proved that several typical results in complex and Kähler geometry also hold for the transverse geometry of a Riemannian foliation, other than the transverse $\partial\bar{\partial}$-Lemma 2.20 we have also a transverse Yau theorem.

Theorem 3.27 (Transverse Yau theorem)**.** *If c_1^B is represented by a real basic $(1,1)$-form, then it is $\frac{1}{2\pi}$ times the Ricci form of a unique transverse Kähler metric ω^T in the same basic cohomology class.*

Now recall that the Ricci form for a Sasakian manifold is defined as $\rho_g(X,Y) = \mathrm{Ric}_g(X, \Phi Y)$ and is related to the transverse one by $\rho_g = \rho^T - 2d\eta$. Then the Sasakian setting the transverse Yau theorem reads as follows.

Theorem 3.28. *Let (M, η, ξ, Φ, g) be a Sasakian structure whose first basic Chern class is represented by the real basic $(1,1)$-form $\frac{\rho}{2\pi}$. Then there exist a unique Sasakian structure in $\mathfrak{F}(\xi)$ cohomologous to the old one such that $\rho - 2d\eta_1$ is its Ricci form and $\eta_1 = \eta + d^c\varphi$ for some basic function φ and the other tensor fields are given by* (7.5)*.*

Indeed, by the transverse Yau theorem and the transverse $\partial\overline{\partial}$-Lemma, there exist a transverse metric with Kähler form ω_1^T and a smooth basic function φ such that $\omega_1^T = d\eta + dd^c\varphi$. So, since a contact form η is related to its transverse Kähler form by $\omega^T = d\eta$, we see that a choice that works is $\eta_1 = \eta + d^c\varphi$.

Boyer, Galicki and Matzeu [12] apply it to find η-Sasaki-Einstein metrics on certain manifolds, starting from their transverse Kähler-Einstein metric found as solution, when exists, of a transverse Calabi problem. They pose the following.

Problem. *Given a manifold M with Sasakian structure (ξ, η, Φ, g) and with a basic first Chern class c_1^B that is represented by either a positive definite, negative definite real basic $(1,1)$ form ρ^T , or if c_1^B vanishes, does there exist a Sasakian structure in $\mathfrak{F}(\mathcal{F}_\xi)$ with an η-Einstein metric g'?*

This boils down to solving a transverse Monge-Ampère equation for a basic φ

$$\det\big(g^T_{i\bar{\jmath}} + \partial_i\partial_{\bar{\jmath}}\varphi\big) = \det(g^T_{i\bar{\jmath}})\exp(-k\varphi + F) \tag{3.4}$$

which admits solutions in the $k < 0$ and $k = 0$ case. Recalling that a metric is η-S-E with constant λ if and only if its transverse metric is KE with constant $\lambda + 2$ we have an analogue of the Aubin and Yau theorem.

Theorem 3.29. *If the class c_1^B is zero or can be represented by a negative definite $(1,1)$- form, then there exists a Sasakian structure in $\mathfrak{F}(\mathcal{F}_\xi)$ on M with an η-Sasaki-Einstein metric g with constant $\lambda = -2$ in the first case and $\lambda < -2$ in the second.*

So the case $c_1^B \leq 0$ is unobstructed. The analogue of the theorem in Kähler geometry stating that every Fano manifold admits a metric with positive Ricci curvature reads as follows in the Sasakian setting.

Theorem 3.30. *Let $\mathcal{S} = (g, \eta)$ be a positive Sasakian structure on M^{2n+1}. Then M admits a Sasakian structure $\mathcal{S}'$ which is a-homologous to $\mathcal{S}$ and has positive Ricci curvature.*

Proof. Let $\rho/2\pi$ be a positive $(1,1)$-form that represents $c_1^B(M)$. Applying the transverse Yau theorem we can find a metric g_1 whose Ricci form is $\rho - 2d\eta$ and whose transverse Ricci curvature is positive. Now perform a D-homothetic deformation for some $a > 0$ and get $(g_2^T, \eta_2) = (\frac{1}{a}g_1^T, \frac{1}{a}\eta_1)$. Now the Ricci tensor of g_2 satisfies

$$\begin{aligned}\mathrm{Ric}_{g_2}|_{D_2} &= \mathrm{Ric}^T_{g_2} - 2g_2^T \\ &= \mathrm{Ric}^T_{g_1} - \frac{2}{a}g_1^T\end{aligned}$$

that is positive for large enough a, being M compact and $\mathrm{Ric}^T_{g_1}$ positive. We are set on D. Now from known curvature properties of Sasaki metrics we have $\mathrm{Ric}_2(X, \xi_2) = 2n\eta_2(X)$ which proves that Ric_2 is positive. By construction g_2 is a-homologous to g. □

The Sasakian structure also lets us have a variation of Myers' theorem, namely.

Theorem 3.31. *Let g be a complete Sasakian metric on M^{2n+1} with $c_1^B > \delta > 0$. Then M is compact with finite fundamental group.*

Proof. Apply the transverse Yau theorem and a D-homothetic deformation like in the proof of the previous theorem, to get a Sasakian structure (g_2, η_2) and estimate

$$\begin{aligned}\mathrm{Ric}_{g_2}|_{D_2} &= \mathrm{Ric}^T_{g_2} - 2g_2^T \\ &= \mathrm{Ric}^T_{g_1} - \frac{2}{a}g_1^T \\ &> \delta - \frac{2}{a} > 0\end{aligned}$$

for large enough a. Moreover from $\mathrm{Ric}_2(X, \xi_2) = 2n\eta_2(X)$ we obtain that $\mathrm{Ric}_2 > \delta$ and we can apply the classical Myers theorem to conclude. □

Chapter 4

Sasaki-Ricci solitons and their deformations

4.1 Definitions and examples

Definition 4.1. A Sasakian metric g on M is a Sasaki-Ricci soliton if there exist $c \in \mathbb{R}$ and a transversely holomorphic complex vector field X commuting with ξ such that

$$\operatorname{Ric}^T + cg^T = \mathcal{L}_X g^T. \tag{4.1}$$

As in the Riemannian case we call it steady if $c = 0$, shrinking if $c < 0$ and expanding if $c > 0$.

The requirement for X to commute with ξ comes from being consistent with [36] which requires the function $\eta(X)$ to be basic and with the requirement for a soliton to be a self similar solution of the Sasaki-Ricci flow, that is for X to be integrated to an automorphism of the foliation. So both X foliate and $[\xi, X] \in D$ imply $[\xi, X] = 0$.

As in the Kähler case we see that the assumption for X to be transversally holomorphic is redundant. Taking the imaginary part of (4.1) we see that $\operatorname{Im} \overline{X}$ is transversally Killing. We then apply the following result holding in the more general setting of transversally Kähler harmonic foliations on closed manifolds.

Theorem 4.2 ([49, 59]). *Let Y be a foliate vector field of a foliation as above. Then $\overline{Y}$ it is transversally Killing if, and only if, it is transversally holomorphic and transversally divergence-free.*

The definition of [36] forces X to be also Hamiltonian to force the existence of a function whose gradient has $(1,0)$-part equal to the D-component of X.

Let us now see how it is straightforward to construct examples from Kähler-Ricci solitons by applying the typical construction of circle bundles in Sasakian geometry.

Proposition 4.3. *Let (W, h, ω) a compact Kähler-Ricci soliton with gradient function f such that $[\omega]$ is integral. Then there exist a principal circle bundle M on W that admits a Sasaki-Ricci soliton.*

Proof. Being ω a Hodge form, we can perform the standard construction in Sasakian geometry and obtain the Sasakian principal circle bundle $p : (M, g, \eta, \xi) \to W$. By definition we have that

$$\mathrm{Ric}^T = p^* \mathrm{Ric}_h, \qquad p^* h = g^T, \qquad \mathcal{L}_{\widetilde{X}} g^T = p^* \big(\mathcal{L}_X h\big) \tag{4.2}$$

where in the last equation $p_* \widetilde{X} = X$ and because g^T is a basic tensor.

Let us also define $\widetilde{f} = f \circ p$, basic by construction. The complex field $\partial^\sharp \widetilde{f}$ is then the horizontal lift of $\partial^\sharp f$.

Up to normalization, we take the complex Hamiltonian holomorphic field $\widetilde{X} = -i\widetilde{f}\xi + \partial^\sharp \widetilde{f}$, that commutes with ξ and satisfies the SRS equation because of (4.2). □

In some cases this can be done also in the non-compact and not necessarily Hodge situation.

Example 4.4 (Riemannian submersion onto the Hamilton cigar soliton). There exist a steady Sasaki-Ricci soliton on $\mathbb{R}^3$ which is a Riemannian submersion of the steady gradient Kähler-Ricci soliton given by the Hamilton cigar metric on $\mathbb{R}^2$ (see e.g. [21]).

Let (x, y, z) be coordinates on $\mathbb{R}^3$ and $p : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection onto the first two coordinates. The Kähler form of the cigar is $\omega = \frac{2}{1+x^2+y^2} dx \wedge dy$. Let us start by finding a contact form η on $\mathbb{R}^3$ such that $\frac{1}{2} d\eta = p^*\omega$. A possible η is found by assuming it to be of the form

$$\eta = adx + B(x, y)dy + \lambda dz \tag{4.3}$$

for an appropriate function B and $\lambda \neq 0$. By solving the PDE obtained by imposing $\frac{1}{2} d\eta = p^*\omega$ one can take

$$B(x, y) = \frac{4}{\sqrt{1 + y^2}} \arctan\left(\frac{x}{\sqrt{1 + y^2}}\right).$$

Since $\lambda \neq 0$ and $\frac{\partial B}{\partial x}$ we have that $d\eta \wedge \eta \neq 0$. The Reeb vector field is $\xi = \frac{1}{\lambda}\partial_z$. We take the metric to be $g = p^* g_0 + \eta \otimes \eta$ where g_0 is the cigar metric. We then define

$$\frac{1}{2}\Phi = \left(\partial_x - \frac{a}{\lambda}\partial_z\right) \otimes dy + \left(\frac{B(x, y)}{\lambda}\partial_z - \partial_y\right) \otimes dx$$

which satisfies all compatibility conditions with (g, η) to have a contact metric structure. Its torsion tensor is

$$N_\Phi = -\frac{8}{\lambda(1+x^2+y^2)}\partial_z \otimes (dx \wedge dy)$$

which equals $-\xi \otimes d\eta$, so the tensor $N_\Phi + \xi \otimes d\eta$ vanishes and the contact metric structure is normal, hence g is Sasakian. The gradient function on the cigar soliton is $f = \log(1 + x^2 + y^2)$ which we lift to a basic function on $\mathbb{R}^3$ and call f such lift as well. The Hamiltonian holomorphic field on $\mathbb{R}^3$ will be, in transverse complex coordinates,

$$X = -if\xi + z\frac{\partial}{\partial z}$$

as $z\frac{\partial}{\partial z} = \partial^\sharp f$ is the horizontal lift of the $(1, 0)$-part of $\operatorname{grad} f$. It is such that $[\xi, X] = 0$ and $\mathrm{Ric}^T = \frac{1}{2}\mathcal{L}_X g^T$, so that g is a steady Sasaki-Ricci soliton.

4.2 Normalized Hamiltonian holomorphic vector fields

We make the same assumption as in [36]. Namely we start with a positive compact Sasakian manifold $(M^{2n+1}, g, \eta, \xi, \Phi)$. If we assume $c_1(D) = 0$ and normalize, we have $2\pi c_1^B = (2n + 2)[\frac{1}{2}d\eta]_B$.

Let h be a Ricci potential, that is a real basic function such that $\rho^T - (2n+2)\frac{1}{2}d\eta = i\partial_B\overline{\partial}_B h$ and consider the operator Δ^h acting on basic functions as

$$\Delta^h u = \Delta_{\overline{\partial}_B} u - (\overline{\partial} u, \overline{\partial} h).$$

Here we have dropped the B subscript and we will do the same in the following as it will be clear from the context.

Remark 4.5. This is the $\overline{\partial}$-Laplacian on functions, with respect to the weighted product $\langle f, g\rangle_h = \int_M f\overline{g}e^h\mu$. Indeed

$$\begin{aligned}
\langle \overline{\partial}^*\overline{\partial} u, v\rangle_h &= \langle \overline{\partial} u, \overline{\partial} v\rangle_h \\
&= \int_M \partial_{\overline{a}} u \overline{\partial_{\overline{b}} v} g^{\overline{a}b} e^h \mu \\
&= -\int_M \left(\nabla_b \nabla^b u \cdot \overline{v} + \nabla^b u \nabla_b h \cdot \overline{v}\right) e^h \mu \\
&= \int_M \Delta^h u \cdot \overline{v} e^h \mu
\end{aligned}$$

with volume form $\mu = (\frac{1}{2}d\eta)^n \wedge \eta$.

We now consider a class of vector fields introduced in [27, 36].

Definition 4.6. A complex vector field X on M commuting with ξ is called *Hamiltonian holomorphic* if its projection onto the normal bundle $\overline{X}$ is transversally holomorphic and the basic function, sometimes called *potential*, $u = i\eta(X)$ is such that

$$\iota_X \omega^T = i\overline{\partial}_B u.$$

It is *normalized* if $\int_M u e^h \mu = 0$.

It must then have the form

$$X = -iu\xi + \partial^\sharp u = -iu\xi + \nabla^j u e_j,$$

where $e_j = \frac{\partial}{\partial z^i} - \eta_i \xi$ generate $D^{1,0} \simeq \nu(\mathcal{F}_\xi)^{1,0}$.

We recall a widely known fact.

Lemma 4.7. *The subset $\mathfrak{h} = \{X$: is Hamiltonian holomorphic$\}$ is a Lie subalgebra of the algebra of the algebra of vector fields on M.*

Proof. Let X, Y be Hamiltonian holomorphic with potentials u, v.

Their bracket is

$$[X, Y] = -i(Xv - Yu)\xi + [\partial^\sharp u, \partial^\sharp v] \tag{4.4}$$

using the facts that u, v are basic, that the e_j's commute among each other and with ξ and that $d\eta$ has basic type $(1,1)$ so it must vanish when evaluated on two $(1,0)$ fields. If we let $w := Xv - Yu$ it is (dropping the B's and the T's for simplicity)

$$\begin{aligned}
\iota_{[X,Y]}\omega &= \mathcal{L}_X \iota_Y \omega - \iota_Y \mathcal{L}_X \omega \\
&= \mathcal{L}_X (i\overline{\partial} v) - \iota_Y (i\partial\overline{\partial} u) \\
&= \iota_X (i\partial\overline{\partial} u) - \iota_Y (i\partial\overline{\partial} u) \\
&= i\overline{\partial}(Xv - Yu) \\
&= i\overline{\partial} w.
\end{aligned}$$

So $[X, Y]$ is Hamiltonian holomorphic with potential w. □

Let Λ_1 be the first eigenspace of Δ^h with eigenvalue λ_1.

Theorem 4.8 ([36])**.** *We have*

1. $\lambda_1 \geq 2m + 2$.

2. *Equality holds if and only if there exists a nonzero Hamiltonian normalized holomorphic vector field.*

3. *The correspondence $\Lambda_1 \to \mathfrak{h}_N$ given by*

$$u \mapsto -iu\xi + \partial^\sharp u$$

where $\mathfrak{h}_N$ denotes the space of normalized Hamiltonian holomorphic fields, is an isomorphism.

Proof. 1. We can replicate the computation made in Futaki's book [35] to conclude that

$$(\lambda - (2m+2))\|\overline{\partial}u\|_h^2 = \|D_g u\|_h^2 \geq 0$$

for every u in the eigenspace of eigenvalue λ and the norms are taken using the weighted L^2 product and $D_g : C_B^\infty(M)^\mathbb{C} \to \Gamma(\nu(\mathcal{F}_\xi)^{1,0} \otimes \Omega_B^{0,1}(M))$ is the operator such that $\ker D_g = \mathcal{H}_g$.

2. It means that the map in item 3 is surjective. Let X be a Hamiltonian holomorphic vector field with potential function u. Then the g^T-dual of the $(1,0)$-part of X is a $\overline{\partial}$-closed form α such that $\alpha = \overline{\partial}u$, which is the same as $\iota_X\omega^T = i\overline{\partial}u$. This acts as a function u in the Hodge decomposition $\alpha = \alpha_H + \overline{\partial}u$ would in the Kähler setting. The same computation in Futaki's book shows that

$$\nabla_{\bar{j}}(-\Delta^h u + (2m+2)u) = 0$$

which means that the function $\Delta^h u - (2m+2)u$ equals some constant c. Integrating this equality it we get

$$\int_M \Delta^h u \cdot e^h \mu - (2m+2)\int_M u e^h \mu = c\,\mathrm{vol}_h(M)$$

which implies $c = 0$ if we start with a normalized vector field.

3. Being eigenspaces finite dimensional, we have also injectivity, hence isomorphism. □

We can use this correspondence to prove the following.

Proposition 4.9. *The subspace $\mathfrak{h}_N$ is a Lie subalgebra of $\mathfrak{h}$.*

Proof. Let $X, Y \in \mathfrak{h}_N$ be the image of functions u, v via the correspondence. Then from the proof of Lemma 4.7 we have that the potential of $[X, Y]$ is $w = Xv - Yu$. Its integral is

$$\begin{aligned}\int_M we^h\mu &= \int_M (\overline{\partial}v, \overline{\partial}\overline{u})e^h\mu - \int_M (\overline{\partial}u, \overline{\partial}\overline{v})e^h\mu \\ &= \int_M \Delta^h v \cdot ue^h\mu - \int_M \Delta^h u \cdot ve^h\mu \\ &= 0\end{aligned}$$

where in the last equality we use that u and v are eigenfunctions of Δ^h with the same eigenvalue $2n+2$ and in the penultimate the self-adjointness of Δ^h (see e.g. [36, Eq. (33)]). □

4.3 A Lie algebra of infinitesimal transformations and its decomposition

Let there exist a Sasaki-Ricci soliton (SRS for short) as in [36], i.e. a Sasakian metric such that

$$\rho^T - (2n+2)\tfrac{1}{2}d\eta = \mathcal{L}_X \tfrac{1}{2}d\eta \tag{4.5}$$

with Hamiltonian holomorphic normalized vector field X and potential θ_X which, by an easy computation (e.g. [36]), is equal to the Ricci potential h up to a constant. The field X can be written as

$$X = -i\theta_X\xi + \partial^\sharp h \tag{4.6}$$

with $\int_M \theta_X e^h\mu = 0$. Let the section of $D^{1,0}$ given by $\partial^\sharp h = \partial^\sharp\theta_X$ decompose as $\widetilde{X} - iJ\widetilde{X}$, where J is the transverse complex structure.

Consider the following operators L and $\overline{L}$ acting on basic functions.

$$\begin{aligned}Lu &= \Delta u - (\overline{\partial}u, \overline{\partial}h) - (2n+2)u = \Delta u - \overline{X} \cdot u - (2n+2)u \\ \overline{L}u &= \Delta u - (\overline{\partial}h, \overline{\partial}\overline{u}) - (2n+2)u = \Delta u - X \cdot u - (2n+2)u.\end{aligned}$$

Lemma 4.10. *The operators L and $\overline{L}$ have the following properties.*

1. $\overline{L\overline{u}} = \overline{L}u$;
2. *Each of them is self-adjoint with respect to the weighted L^2-product on the space of basic functions.*
3. *L and $\overline{L}$ commute, so $\overline{L}$ maps* $\ker L$ *into itself.*

Proof. The first item is just a computation using that the pairing $(,)$ is Hermitian and that the Laplacian is a real operator.

For the second, notice that $L + (2n+2)\,\mathrm{id} = \Delta^h$ is self-adjoint because it is the $\overline{\partial}$-Laplacian of the weighted metric as shown in Remark 4.5.

For the commutativity, it is enough to show $[L - \overline{L}, \overline{L}] = 0$. We have

$$(L - \overline{L})u = (X - \overline{X})u = 2i\,\mathrm{Im}(X)u = -2iJ\widetilde{X}u.$$

This operator commutes with $\overline{L}$ if and only if $J\widetilde{X}$ commutes with the Laplacian (it is a general fact that Killing fields commute with Laplacians) and that $[J\widetilde{X}, X] = 0$ being $\widetilde{X}$ transversally real holomorphic. $\square$

Let E_λ be the eigenspace of $\overline{L}|_{\ker L}$ of eigenvalue $-\lambda$. If $u \in E_\lambda \subset \ker L$ then it lies in the $(2n+2)$-eigenspace of Δ^h so u defines a normalized Hamiltonian vector field $Y = -iu\xi + \partial^\sharp u$ by the correspondence in Theorem 4.8.

Now we compute the adjoint action of X on $\mathfrak{h}_N$.

Proposition 4.11. *For Y in the image of E_λ it is $[X, Y] = \lambda Y$.*

Proof. The action of X is given by (4.4), namely

$$[X, Y] = -i(Xu - Yh)\xi + [\partial^\sharp h, \partial^\sharp u]. \tag{4.7}$$

Consider the two summands separately. Compute, for $u \in \ker L$,

$$\begin{aligned}
\partial^\sharp(\overline{L}u) &= 2i\partial^\sharp(J\widetilde{X} \cdot u) \\
&= 2i[J\widetilde{X}, \partial^\sharp u] \\
&= 2iJ[\widetilde{X}, \partial^\sharp u]
\end{aligned}$$

where the second equality is due to the fact that $\mathrm{grad}(Kf) = [K, \mathrm{grad}\, f]$ for any Riemannian manifold, Killing vector field K and function f on it. So we obtain, if $u \in E_\lambda$,

$$\begin{aligned}
[\partial^\sharp h, \partial^\sharp u] &= [\widetilde{X}, \partial^\sharp u] - i[J\widetilde{X}, \partial^\sharp u] \\
&= -\frac{1}{2i}J\partial^\sharp(\overline{L}u) - \frac{1}{2}\partial^\sharp(\overline{L}u) \\
&= -\partial^\sharp(\overline{L}u) \\
&= \lambda\partial^\sharp u.
\end{aligned} \tag{4.8}$$

Now note that $Yh = \partial^\sharp u \cdot h = \nabla^i u \nabla_i h = \overline{X}u$. So

$$-i(Xu - Yh) = -i(X - \overline{X})u = -i(L - \overline{L})u = -i\lambda u.$$

Hence $[X, Y] = \lambda Y$. $\square$

Consider now the zero eigenspace $E_0 = \ker L \cap \ker \overline{L}$ of $\overline{L}|_{\ker L}$. Mimicking Tian and Zhu's argument [75] we get that for $u \in \ker L \cap \ker \overline{L}$ it is $L(\operatorname{Re} u) = L(\operatorname{Im} u) = 0$, so E_0 splits as $E_0' \oplus E_0''$, the space of real valued and purely imaginary functions in $\ker L \cap \ker \overline{L}$. This corresponds to a splitting of the image of E_0 as $\mathfrak{h}_0 = \mathfrak{k}_0 \oplus \mathfrak{k}_0'$. We have

From now on, if $\mathfrak{p} \subseteq \mathfrak{h}$ is a Lie subalgebra containing ξ, we let $\overline{\mathfrak{p}}$ denote the quotient $\mathfrak{p}/\xi$. The following lemma is basically [77, Lemma 2.11].

Lemma 4.12. *The space $\overline{\mathfrak{k}_0}$ is formed by the fields whose real part is transversally Killing.*

Our goal now is to write a decomposition of some algebra of transversally holomorphic vector fields, analogously to the case of extremal Sasakian metrics. A natural Lie algebra to consider would be $\mathfrak{fol}(M, \xi, J)$. This is infinite dimensional as it contains the space of sections of the foliation distribution. So its projection onto the normal space of the foliation has been considered in [13, 77]. On the other hand, in analogy of the Kähler-Ricci soliton case, we are interested only in *Hamiltonian* fields which in particular are transversally holomorphic, that is $\mathfrak{h} \subset \mathfrak{fol}(M, \xi, J)$. More precisely, we consider the projection $\overline{\mathfrak{h}}$ onto the normal space, which also is finite dimensional.

As it is said in [36], given a Hamiltonian holomorphic vector field, one can obtain a normalized one by adding a constant multiple of ξ, so the space of normalized fields is a set of representatives for the classes of $\overline{\mathfrak{h}}$.

We have already computed the action of X on normalized vector fields, so we can get also the adjoint action of its class $\overline{X} \in \overline{\mathfrak{h}}$. Recall that the Lie algebra of infinitesimal Sasaki transformations is defined by

$$\mathfrak{aut}(\mathcal{S}) = \{X \in \Gamma(TM) : \mathcal{L}_X g = 0, \mathcal{L}_X \eta = 0\}.$$

Of course ξ is central in it, so it makes sense to consider the quotient $\mathfrak{aut}(\mathcal{S})/\xi$. Finally, let $\mathfrak{aut}^T = \{\overline{Y} : \mathcal{L}_{\overline{Y}} J = 0, \mathcal{L}_{\overline{Y}} g^T = 0\}$. We have the following result.

Theorem 4.13. *On a compact Sasaki-Ricci soliton $\mathcal{S}$, the finite dimensional Lie algebra $\overline{\mathfrak{h}}$ admits the decomposition*

$$\overline{\mathfrak{h}} = \overline{\mathfrak{k}_0} \oplus J\overline{\mathfrak{k}_0} \oplus \bigoplus_{\lambda > 0} \overline{\mathfrak{h}}_\lambda$$

where $\overline{\mathfrak{k}_0}$ is the space in Lemma 4.12 and $\overline{\mathfrak{h}}_\lambda = \{\overline{Y} \in \overline{\mathfrak{h}} : [\overline{X}, \overline{Y}] = \lambda \overline{Y}\}$.

Moreover $\overline{\mathfrak{k}_0} \oplus J\overline{\mathfrak{k}_0}$ is the centralizer in $\overline{\mathfrak{h}}$ of $\overline{X}$ and the space $\overline{\mathfrak{k}_0}$ can be identified with $\mathfrak{aut}(\mathcal{S})/\xi$ and with $\mathfrak{aut}^T$.

Remark 4.14. In contrast with the similar decomposition in the case of extremal Sasakian metrics, here we do not have any summand corresponding to transversally parallel fields. In fact, SRS are Fano and therefore there are no basic harmonic 1-forms, hence no transversally parallel fields.

Proof of the theorem. The eigenspace decomposition follows from the adjoint action of X on $\mathfrak{h}_N$ computed above.

For the last statement, on one hand it is clear that a function in E_0 induces a field that commutes with X in $\mathfrak{h}_N$ so its class belongs to the centralizer of $\overline{X}$. On the other, a class $\overline{Y}$ induced by a normalized function $u \in \ker L$ centralizes X if and only if $[\partial^\sharp h, \partial^\sharp u] = 0$. From (4.8) we see that in this case we need to have

$$\overline{L}u = \overline{L\overline{u}} = \Delta^h \overline{u} - (2n+2)u = \mathrm{const}\,.$$

Integrating it with the weighted measure we see that the constant has to be zero, so $u \in E_0$.

Let us now prove the statement about the Lie algebra of infinitesimal Sasaki transformations. In [77, Lemma 2.11] it is proved that for a purely imaginary basic function f the field $V = \mathrm{Re}\,\partial^\sharp f$ lifts to a vector field $\widetilde{V} \in \mathfrak{aut}(\mathcal{S})$ and conversely a vector field $\widetilde{V} \in \mathfrak{aut}(\mathcal{S})$ is such that its projection is the real part of $\partial^\sharp f$ for the purely imaginary function $f = i\eta(\widetilde{V})$. The $\partial^\sharp$-image of purely imaginary functions, followed by the projection onto the normal bundle is exactly $\overline{\mathfrak{k}}_0$ of Lemma 4.12.

There is a well known exact sequence, see e.g. [11],

$$0 \to \{\xi\} \to \mathfrak{g}' \to \mathfrak{aut}^T \to H^1_B(M) \simeq H^1(M,\mathbb{R}).$$

that means that the first (basic) cohomology group is an obstruction to the identification $\mathfrak{aut}(\mathcal{S})/\xi \simeq \mathfrak{aut}^T$ and in the transversal Fano case there is no such obstruction. □

Remark 4.15. In order to be consistent with analogue results in the literature for the Sasaki extremal case [13, 77] or more generally transversely Kähler harmonic foliations as in [59] we have stated Theorem 4.13 for the quotient algebra $\mathfrak{h}/\xi$. The computation of the adjoint action together with Lemma 4.12 prove that a similar decomposition holds for the finite dimensional Lie algebra $\mathfrak{h}_N$ as well although the Lie algebra of infinitesimal Sasaki transformations cannot fit in the picture since it is not contained in $\mathfrak{h}_N$.

4.4 Deformations of Sasaki-Ricci solitons

4.4.1 Generalized Sasaki-Ricci solitons

Here we extend to the Sasakian setting the result obtained for Kähler-Ricci solitons by Li in [56].

There is a wider class of metric that includes Sasaki-Ricci solitons. In the following let θ_X be the potential (up to constant) of a Hamiltonian holomorphic vector field and let Δ_B denote the d_B-Laplacian acting on basic functions.

Definition 4.16. A *generalized Sasaki-Ricci soliton* (generalized SRS for short) on compact M^{2n+1} is a Sasakian metric whose transverse scalar curvature satisfies

$$s^T - s_0^T = -\Delta_B \theta_X \tag{4.9}$$

for a Hamiltonian holomorphic vector field X and where $s_0^T = \frac{1}{\mathrm{vol}(M)} \int_M s^T \mu$ is the average transverse scalar curvature of g and $\mu = (\frac{1}{2} d\eta)^n \wedge \eta$ is the volume form as before.

This is of course a generalization of Sasaki-Ricci solitons.

An "imaginary" version of Lemma 4.12 can be stated as follows. See [55] for the Kählerian counterpart.

Lemma 4.17. *Let $(M, \mathcal{S})$ be a Sasakian manifold. The transverse field X can be expressed as $X = \partial^\sharp f$ for a real basic function f if, and only if, $V = \operatorname{Im} \partial^\sharp f$ is Killing for g^T. In this case V lifts to $\widetilde{V} \in \mathfrak{aut}(\mathcal{S})$. Conversely, if $\widetilde{V} \in \mathfrak{aut}(\mathcal{S})$ then its projection is $\operatorname{Im} \partial^\sharp f$ for the real function $f = -\eta(\widetilde{V})$.*

Proof. Let $X = \partial^\sharp f$ with f real and let $V = \operatorname{Im} X$. Then we notice, since ω^T real, that V has $f/2$ as Hamiltonian function with respect to the transverse symplectic form. Indeed

$$\iota_V \omega^T = \frac{1}{2i}(\iota_{\partial^\sharp f} - \iota_{\overline{\partial^\sharp f}})\omega^T = \frac{1}{2}(\overline{\partial} f + \partial f) = \frac{1}{2} df.$$

So $\mathcal{L}_V \omega^T = 0$. Conversely, let $X = \partial^\sharp f = iV + JV$ with V transversally Killing and $f = u + iv$. Then taking the imaginary part of the equation $\iota_{\partial^\sharp f} \omega^T = i(\overline{\partial} u + i \overline{\partial} v)$ we have $\iota_V \omega^T = \overline{\partial} u$ and hence $\partial\overline{\partial} u = 0$ so u is constant. In this case, to extend V to a $\widetilde{V} \in \mathfrak{aut}(\mathcal{S})$ we need to find a function a such that $\widetilde{V} = a\xi + V$ is contact. This means

$$\mathcal{L}_{\widetilde{V}} \eta = da + \iota_V d\eta = da + df = 0$$

so we see that we can lift V to $\widetilde{V}$ if, $a = -f$. Conversely $\widetilde{V} = -f\xi + V$ being contact means that $0 = d(\eta(\widetilde{V})) + \iota_V d\eta = -df + \iota_V d\eta$ hence $V = \operatorname{Im} \partial^\sharp f$. $\square$

4.4.2 Main result

As in [77], let us now fix a compact connected $G \subseteq \mathrm{Aut}(\mathcal{S})^0$ with Lie algebra $\mathfrak{g}$ with center $\mathfrak{z}$ and such that $\xi \in \mathfrak{z} \subseteq \mathfrak{g}$. Then it makes sense to consider $\overline{\mathfrak{z}}$, whose elements are transversally Killing and imaginary parts of projected Hamiltonian holomorphic fields whose potentials are G-invariant.

We want to apply to $\mathcal{S}$ a deformation parameterized by $(t, \alpha, \varphi) \in \mathcal{B} \times \mathfrak{z} \times C^\infty_B(M)$. That is, a combination of type I, type II and Nozawa deformations.

Start with a basis $\{v_0 = \xi, v_1, \dots, v_d\}$ of $\mathfrak{z}$ and let $X_j = i\overline{v}_j + J\overline{v}_j$ in a way that $\operatorname{Im} X_j = \overline{v}_j$. Consider the functions (depending on the Sasakian structure)

$$p^0_{t,\alpha,\varphi} = 1 \text{ and } p^j_{t,\alpha,\varphi} = -\eta_{t,\alpha,\varphi}(v_j). \tag{4.10}$$

Let $Y_j = -ip^j_g \xi + \partial^\sharp p^j_g$. It is Hamiltonian holomorphic and the functions p^j_g acts as a holomorphy potential as in the Kähler case.

Let $\mathcal{H}^{\mathfrak{p}}_g$ be, for any Lie algebra $\xi \in \mathfrak{p} \subseteq \mathfrak{aut}(\mathcal{S})$, the space of functions u such that $\partial^\sharp u$ lies in the complexified quotient $\overline{\mathfrak{p}}^{\mathbb{C}}$.

A metric defines an orthogonal splitting of $H^k(M)^G$ as

$$H^k(M)^G = \mathcal{H}^{\mathfrak{z}}_g \oplus W_g$$

Let $\Pi^\perp_g$ be the projection onto W_g. We will consider the function

$$S(t, \alpha, \varphi) := \Pi^\perp_g \Pi^\perp_{t,\alpha,\varphi} G_{t,\alpha,\varphi}(s^T_{t,\alpha,\varphi} - s^0_{t,\alpha,\varphi}) \tag{4.11}$$

where $G_{t,\alpha,\varphi}$ is the Green operator of d_B with respect to the metric $g_{t,\alpha,\varphi}$. For the metric $g_{t,\alpha,\varphi}$ to be a generalized SRS we need $G_{t,\alpha,\varphi}(s^T_{t,\alpha,\varphi} - s^0_{t,\alpha,\varphi})$ to lie in $\mathcal{H}^{\mathfrak{z}}_{t,\alpha,\varphi} := \mathcal{H}^{\mathfrak{z}}_{g_{t,\alpha,\varphi}}$, so $S(t, \alpha, \varphi) = 0$. Since $\ker(\Pi^\perp_g \circ \Pi^\perp_{t,\alpha,\varphi}) = \ker \Pi^\perp_{t,\alpha,\varphi}$ if the deformation is small enough, we have

$$S \colon \mathcal{V} \subseteq \mathcal{B} \times \mathfrak{z} \times H^k(M)^G \to W_g$$

for $\mathcal{V}$ a neighborhood of $(0, 0, 0)$. Let us compute the derivatives of S. The derivative along φ behaves as in the Kähler case.

Lemma 4.18 ([13, 36]). *As in the Kähler case, the variation of the scalar curvature under type II deformations is*

$$D_\varphi s^T_\varphi|_{\varphi=0}(\psi) = -\frac{1}{2}\Delta^2_B \psi - 2(\rho^T, i\partial_B \overline{\partial}_B \psi). \tag{4.12}$$

Moreover, the average scalar curvature is constant.

The derivative of S along φ is

$$D_\varphi S|_{(0,0,0)} = \Pi_g^\perp (D_\varphi \Pi_{t,\alpha,\varphi})|_{(0,0,0)}\theta_X + \Pi_g^\perp D_\varphi A_{t,\alpha,\varphi}|_{(0,0,0)} \tag{4.13}$$

where $A_{t,\alpha,\varphi} = G_{t,\alpha,\varphi}(s^T_{t,\alpha,\varphi} - s^0_{t,\alpha,\varphi})$.

Let $\{f^j_{t,\alpha,\varphi}\}$ be obtained from (4.10) via the Gram-Schmidt procedure with respect to the weighted L^2 product

$$\langle f, h\rangle_{t,\alpha,\varphi} = \int_M f h e^{\theta_X} \mu_{t,\alpha,\varphi}.$$

In particular it is $f^0_{t,\alpha,\varphi} = \mathrm{vol}^{-1/2}_{t,\alpha,\varphi}$ and

$$f^1_{t,\alpha,\varphi} = \frac{p^1_{t,\alpha,\varphi} - \langle p^1_{t,\alpha,\varphi}, 1\rangle_{t,\alpha,\varphi}\frac{1}{\mathrm{vol}_{t,\alpha,\varphi}}}{\|p^1_{t,\alpha,\varphi} - \langle p^1_{t,\alpha,\varphi}, 1\rangle_{t,\alpha,\varphi}\frac{1}{\mathrm{vol}_{t,\alpha,\varphi}}\|_{t,\alpha,\varphi}}. \tag{4.14}$$

Now we have

$$\begin{aligned}(D_\varphi \Pi_{t,\alpha,\varphi})|_{(0,0,0)}\theta_X &= \sum_{j=0}^d \langle f^j_{0,0}, \theta_X\rangle_g D_\varphi f^j_{t,\alpha,\varphi}|_{(0,0,0)} \pmod{\ker \Pi_g^\perp}\\ &= D_\varphi f^1_{t,\alpha,\varphi}|_{(0,0,0)} \pmod{\ker \Pi_g^\perp}\end{aligned}$$

as $\langle f^j_{0,0}, \theta_X\rangle_g = \delta_{1,j}$. Deriving (4.14) we have

$$D_\varphi f^1_{t,\alpha,\varphi}|_{(0,0,0)}(\psi) = D_\varphi p^1_{t,\alpha,\varphi}|_{(0,0,0)}(\psi) = X\psi \pmod{\ker \Pi_g^\perp} \tag{4.15}$$

because $\Pi_g^\perp$ kills the constants.

Now, deriving the relation $(-2i\partial\overline{\partial} A_{t,\alpha,\varphi}, \frac{1}{2}d\eta_{t,\alpha,\varphi}) = s^T_{t,\alpha,\varphi} - s^0_{t,\alpha,\varphi}$ we have

$$(-2i\partial\overline{\partial} D_\varphi A_{t,\alpha,\varphi}|_{(0,0,0)}(\psi), \frac{1}{2}d\eta) + (-2i\partial\overline{\partial} A_{t,\alpha,\varphi}, \frac{1}{2}dd^c\psi) = -\frac{1}{2}\Delta^2\psi - 2(\rho^T, i\partial\overline{\partial}\psi).$$

So we have

$$\begin{aligned}\Delta_g D_\varphi A_{t,\alpha,\varphi}|_{(0,0,0)}(\psi) &= 2(i\partial\overline{\partial}\theta_X, i\partial\overline{\partial}\psi) - \frac{1}{2}\Delta_g^2\psi - 2(\rho^T, i\partial\overline{\partial}\psi)\\ &= (2n+2)\Delta_g\psi - \frac{1}{2}\Delta_g^2\psi \end{aligned} \tag{4.16}$$

where we have used that $2(i\partial\overline{\partial}\theta_X, i\partial\overline{\partial}\psi) = 2(\rho^T, i\partial\overline{\partial}\psi) + (2n+2)\Delta_g\psi$ from the SRS equation. So we get

$$D_\varphi A_{t,\alpha,\varphi}|_{(0,0,0)}(\psi) = G_g\Big(-\frac{1}{2}\Delta_g^2\psi + (2n+2)\psi\Big). \tag{4.17}$$

Using (4.17) in (4.13) becomes

$$\begin{aligned} D_\varphi S|_{(0,0,0)}(\psi) &= -\Pi_g^\perp\left(\frac{1}{2}\Delta_g\psi - X\psi - (2n+2)\psi\right) \\ &= -\Pi_g^\perp(\overline{L}\psi) \end{aligned} \tag{4.18}$$

where $\overline{L}$ is the operator defined in Section 4.3. The derivative with respect to α can be computed similarly for the first summand but the computation (4.17) cannot be repeated as the foliation changes. We then have

$$(D_\alpha\Pi_{t,\alpha,\varphi})|_{(0,0,0)}(\beta) = \Pi_g^\perp\left(\eta(\beta)\theta_X + D_\alpha A_{t,\alpha,\varphi}|_{(0,0,0)}(\beta)\right). \tag{4.19}$$

Remark 4.19. A case where we can compute also the second term is given when $\dim\mathfrak{z} = 1$, that is when there is room only for Tanno D-homothetic deformations [73]. In this case $\alpha = a\xi$, so

$$\begin{aligned} \xi_a &= (a+1)\xi \\ \eta_a &= \frac{1}{a+1}\eta \\ g_a^T &= \frac{1}{a+1}g^T \\ \Phi_a &= \Phi. \end{aligned}$$

We have $\mu_a = (a+1)^{-(n+1)}\mu$ and so $s_a^T - s_a^0 = (a+1)(s^T - s^0)$.

Being the characteristic foliation unchanged it makes sense to replicate the computation (4.17) and we get $\Delta_a A_a = (a+1)(s^T - s^0)$ so $\Delta_g(D_a A_a)|_{(0,0,0)} - \Delta_g\theta_X = s^T - s^0$. Using the generalized SRS equation we have that $(D_a A_a)|_{(0,0,0)}$ is a constant and hence killed by $\Pi_g^\perp$. So finally

$$(D_\alpha S)|_{(0,0,0)}(\beta) = \Pi_g^\perp\left(\eta(\beta)\theta_X\right).$$

In any case the derivative will assume the form

$$DS|_{(0,0,0)}(\beta,\psi) = -\Pi_g^\perp\left(\overline{L}\psi - \eta(\beta)\theta_X - D_\alpha A_{t,\alpha,\varphi}|_{(0,0,0)}(\beta)\right). \tag{4.20}$$

Proposition 4.20. *The map $DS|_{(0,0,0)}(0,0,\cdot) : H^k(M)^G \to W_g$ is surjective.*

Proof. Let ψ', orthogonal to the image of $DS|_{(0,0,0)}$, be a representative of a class in the cokernel. It must be then $\langle\psi', \Pi_g^\perp\overline{L}\psi\rangle = \langle\psi', \overline{L}\psi\rangle = 0$ for all $\psi \in H^k(M)^G$ and the scalar product is the weighted one. It follows that $\overline{L}^*\psi' = \overline{L}\psi' = 0$.

This means that $\partial_g^\sharp\overline{\psi'}$ is a transversally holomorphic G-invariant vector field, so it belongs to $\mathfrak{z}^{\mathbb{C}}$, a contradiction. □

Let now $K = \ker DS|_{(0,0,0)} \subseteq \mathcal{B} \times \mathfrak{z} \times H^k(M)^G$, let π be the projection onto it and consider the map

$$G := S \times \pi : \mathcal{V} \to W_g \times K \tag{4.21}$$

which is such that $DG|_{(0,0,0)}(0, \cdot)$ is invertible. We can now state the SRS analogue of [77, Thm. 4.7].

Theorem 4.21. *Let $\mathcal{S} = (\eta, \xi, \Phi, g)$ be a Sasaki-Ricci soliton, $G \subseteq \mathrm{Aut}(\mathcal{S})^0$ be a fixed compact connected subgroup of Sasaki transformations of $(M, \mathcal{S})$ and such that $\xi \in \mathfrak{z} \subseteq \mathfrak{g}$. Let $(\mathcal{F}_\xi, J_t)_{t \in \mathcal{B}}$ be a G-equivariant deformation.*

Then there is a neighborhood $\mathcal{V}$ of $(0,0,0) \in \mathcal{B} \times \mathfrak{z} \times C^\infty_B(M)^G$ such that

$$\mathcal{E} = \left\{ (t, \alpha, \varphi) \in \mathcal{V} : g_{t,\alpha,\varphi} \textit{ is a generalized SRS} \right\}$$

is a smooth manifold of dimension $\dim \mathcal{B} + \dim \mathfrak{z}$.

Proof. We start with a SRS so M is positive and the deformation is of $(1,1)$-type.

The map G of (4.21) is under the assumptions of [43, Thm. 17.6] so we have a neighborhood $\mathcal{N}$ of zero in $\mathcal{B} \times \mathfrak{z}$ and a function $f : \mathcal{N} \to C^\infty_B(M)^G$ such that $S(t, \alpha, f(t, \alpha)) = 0$ for all $(t, \alpha) \in \mathcal{N}$. So the space of solutions of $S = 0$ is parameterized by (t, α) and hence it has dimension $\dim \mathcal{B} + \dim \mathfrak{z}$. $\square$

Chapter 5

Extremal Kähler-Ricci solitons

We pointed out that our work in Chapter 4 is the solitonic counterpart of the Sasaki-extremal metrics and that Sasaki-Ricci solitons and Sasaki-extremal metrics are two different generalizations of Sasaki-Einstein metrics. In this appendix we discuss the results obtained in [17] jointly with S. Calamai. For simplicity we work in the Kähler setting and we assume that a Kähler metric is both a Ricci soliton and an extremal metric. Under a curvature assumption we prove that it must be Einstein.

5.1 Introduction to the problem

Let M^{2n} be a compact complex manifold. A Kähler metric g on M is said to be *Kähler-Einstein* if it is Einstein as a Riemannian metric, i.e. it is proportional to its Ricci tensor or, equivalently, if there exists $c \in \mathbb{R}$ such that

$$\rho_g = c\omega_g \tag{5.1}$$

where ρ_g (resp. ω_g) denotes the Ricci form (resp. Kähler form) of g.

There are two possible generalizations of this notion. The first is the notion of *extremal metric* introduced by Calabi in [14, 15] (see also [38]) in the attempt to find a canonical representative in a given Kähler class $\Omega \in H^{1,1}(M) \cap H^2(M, \mathbb{R})$. These metrics are defined to be the critical points of the Riemannian functional $\mathcal{M}_\Omega \to \mathbb{R}$ defined by

$$g \longmapsto \int_M s_g^2 \omega^n$$

where $\mathcal{M}_\Omega$ is the space of the Kähler metrics on M in the class Ω and s_g is the scalar curvature of g. He also showed that a metric is extremal if, and only if, the gradient of s_g is a holomorphic vector field. Constant scalar curvature

Kähler metrics (cscK), hence in particular Kähler-Einstein, are examples of extremal metrics, but there are extremal metrics of non constant scalar curvature (see again [14]).

Another direction to generalize the Einstein condition (5.1) is the following. A Kähler metric g is called a *Kähler-Ricci soliton* (KRS) if there exist $c \in \mathbb{R}$ and a holomorphic vector field X such that

$$\rho_g + c\omega_g = \mathcal{L}_X \omega_g.$$

These metrics give rise to special solutions of the Kähler-Ricci flow (see e.g. [28]), namely they evolve under biholomorphisms. It is known that on a compact manifold, if $c \geq 0$ then g is Einstein (see e.g. again [28]), so in the compact Kähler case one only considers the so-called *shrinking* Kähler-Ricci solitons ($c < 0$) whose equation, after a scaling, can be written as

$$\rho_g - \omega_g = \mathcal{L}_X \omega_g. \tag{5.2}$$

The Hodge decomposition for the dual of X allows us to introduce a holomorphy potential with respect to g, i.e. a complex-valued function θ_X such that $\iota_X \omega_g = i\bar{\partial}\theta_X$. By means of this function we can infer that the Kähler form ω_g belongs to $2\pi c_1(M)$, making M a Fano manifold.

The first examples of non-Einstein compact Kähler-Ricci solitons go back to the constructions of Koiso [53] and independently Cao [21] of Kähler metrics on certain $\mathbb{CP}^1$-bundles over $\mathbb{CP}^n$. Koiso himself remarks that this Kähler-Ricci soliton metric is not Calabi extremal and proves that if it were, it would be Einstein.

There is a class of manifolds for which there are existence results for both kinds of metrics, namely toric manifolds (see e.g. [1]). For extremal metrics we mention for instance the existence result of Donaldson [31] for toric surfaces. For the KRS we refer to the existence result, in all dimensions, of Wang and Zhu [78]. Finally, the existence of a Kähler-Einstein metric on a compact Fano manifold is equivalent to the notion of *K-stability* stated by Chen, Donaldson and Sun in [25] and subsequent papers.

It is then natural to ask, and it appears to be absent in the literature, what happens when a metric generalizes a Kähler-Einstein metric in both these ways. In Theorem 5.15 we prove, under the assumption of positivity of the holomorphic sectional curvature, that an extremal KRS is in fact Kähler-Einstein.

It would be interesting to prove or disprove this result without the assumption on the holomorphic sectional curvature, i.e. to solve the following.

Problem 1. *Prove that every extremal Kähler-Ricci soliton is Einstein or find a counterexample.*

Going back to our result, it is not too restrictive to assume positive holomorphic sectional curvature provided it does not exceed a certain numerical bound. Indeed it has been proved by Futaki and Sano [37] that if the diameter of a Ricci soliton is $< \frac{10}{13}\pi$ then the soliton is trivial. On the other hand, Tsukamoto [76] proved that if a Kähler metric g has holomorphic sectional curvature $> \varepsilon$, then the diameter of the manifold is bounded from above by $\frac{\pi}{\sqrt{\varepsilon}}$.

From these results we can infer that if the holomorphic sectional curvature is greater than $(\frac{13}{10})^2$ then the KRS must be Einstein, so our result is non trivial when the holomorphic sectional curvature does not exceed this number. The authors do not know whether there are any connections between positive holomorphic sectional curvature and the extremality condition or whether the extremality gives conditions on the diameter .

The paper is organized as follows. We start recalling some notation and conventions of Kähler geometry. This level of detail seems necessary in order to avoid confusion among different conventions. We then go on proving, for a Kähler-Ricci soliton, the characterization of being extremal in terms of the length of the complex Hessian of its potential function and in terms of certain contractions of the Riemann curvature tensor. We then use this to prove our main result in Theorem 5.15 and we also give a condition about the isometry group of a non-Einstein extremal KRS. We finally make a remark about the replicability of the argument in the Sasakian setting.

5.2 Definitions and preliminary results

Notation. Let (M, g, J) be a smooth, compact, without boundary Kähler manifold of real dimension $2n$, with its Riemannian metric g and compatible integrable complex structure J. The corresponding Kähler form is $\omega(\cdot,\, \cdot) = g(J\cdot,\, \cdot)$. We also denote as Ric the Riemannian Ricci tensor corresponding to the Riemannian metric g; and its Ricci form as $\rho(\cdot,\, \cdot) = \mathrm{Ric}(J\cdot, \cdot)$. We label s the Riemannian scalar curvature of the metric g. We let δ be adjoint of the exterior differential d with respect to g and $\Delta_d = \delta d + d\delta$ be the d-Laplacian acting on differential forms.

We let $\sharp$ and $\flat$ denote the musical isomorphisms between fields and 1-forms. For a 1-form α we denote as $|\alpha|^2 = (\alpha,\, \alpha) = (\alpha^\sharp,\, \alpha^\sharp)$, and as well as $|Z|^2 = (Z,\, Z) = (Z^\flat,\, Z^\flat)$ the metric pairing on by means of the Riemannian metric g. Similarly, if a real $(1,1)$-form β and a 2-tensor B correspond each other via $\beta(\cdot,\, \cdot) = B(J\cdot,\, \cdot)$, then we have for the metric pairings $|B|^2 = (B,\, B) = 2|\beta|^2 = 2(\beta,\, \beta)$. For example, $|\mathrm{Ric}|^2 = (\mathrm{Ric},\, \mathrm{Ric}) = 2|\rho|^2 = 2(\rho,\, \rho)$. Notice also that for any smooth real valued function on M there holds $(\omega, dd^c u) = -\Delta_d u$.

Given any tensor T and any vector field V on a smooth manifold, we label as $\mathcal{L}_V T$ the Lie derivative of T along V.

For any smooth, real valued function u on M, we label as ∇u the Riemannian gradient of u; namely, $\nabla u = (du)^\sharp$. We also denote its $(1,0)$-part as $\partial^\sharp f = \frac{1}{2}(\nabla f - iJ\nabla f)$. We let $\operatorname{Hess} u = \frac{1}{2}\mathcal{L}_{\nabla u} g$ be the real Hessian of u.

We also label as $\mathfrak{h}(M)$ the algebra of (complex) holomorphic vector fields of M.

The first definition is very classical and tracks back Hamilton [47].

Definition 5.1. Let (M, g, J) and Ric be as above; let f be a smooth, real valued function on M. We say that $(g,\ f)$ is a Kähler-Ricci soliton when the following equation is satisfied

$$\operatorname{Ric} - g = \frac{1}{2}\mathcal{L}_{\nabla f} g. \tag{5.3}$$

Remark 5.2. It is a general fact that ∇f is real holomorphic, although this is often stated in the definition. Indeed, equation (5.3) implies that $\nabla_i \nabla_j f = 0$ for all $i,\ j \in \{1, \cdots, n\}$.

The next definition is due to Calabi [14].

Definition 5.3. Let (M, g, J) and s be as above. We say that the metric g is *Calabi extremal*, or simply extremal, when the Riemannian gradient of s is holomorphic, i.e. if $\overline{\partial}\partial^\sharp s = 0$.

In this paper we consider metrics which satisfy both these definitions.

Definition 5.4. Let (M, g, J) be as above and let (g, f) be a Kähler-Ricci soliton as in Definition 5.1. Moreover, let g be Calabi extremal as in Definition 5.3. Then, we call (g, f) an extremal Kähler-Ricci soliton.

Remark 5.5. We chose to label the pairs $(g,\ f)$ in Definition 5.4 as extremal Kähler-Ricci solitons although a similar name was given by Guan in [44] to different objects.

Not all complex valued smooth functions v on M give rise to holomorphic vector fields. The ones which do are solutions of the equation $\overline{\partial}\partial^\sharp v = 0$, they lie in the kernel of the fourth order differential operator $L_g = (\overline{\partial}\partial^\sharp)^* \overline{\partial}\partial^\sharp$ (see [35]).

The presence of an extremal metric gives information about the algebra of holomorphic vector fields $\mathfrak{h}(M)$. Namely the following theorems hold.

Theorem 5.6 ([15, 35])**.** *Let g be an extremal Kähler metric on M with scalar curvature s. Then the Lie algebra $\mathfrak{h}(M)$ has a semidirect sum decomposition*

$$\mathfrak{h}(M) = \mathfrak{a}(M) \oplus \mathfrak{h}'(M), \tag{5.4}$$

where $\mathfrak{a}(M)$ is the complex Lie subalgebra of $\mathfrak{h}(M)$ consisting of all parallel holomorphic vector fields of M, and $\mathfrak{h}'(M)$ is an ideal of $\mathfrak{h}(M)$ consisting of the image under $\partial^\sharp$ of the kernel of L_g.

Moreover $\mathfrak{h}'(M)$ has a decomposition

$$\mathfrak{h}'(M) = \bigoplus_{\lambda \geq 0} \mathfrak{h}_\lambda(M)$$

where $[\partial^\sharp s, Y] = \lambda Y$ for any $Y \in \mathfrak{h}(M)$. Furthermore the centralizer $\mathfrak{h}_0(M)$ of $\partial^\sharp s$ is the complexification of the Lie algebra consisting of Killing vector fields of M.

In the case of a Kähler-Ricci soliton a similar theorem holds.

Theorem 5.7 ([75])**.** *If g is a Kähler-Ricci soliton with $(1,0)$-vector field X. Then $\mathfrak{h}(M)$ admits the decomposition*

$$\mathfrak{h}(M) = \mathfrak{k}_0(M) \oplus \bigoplus_{\lambda > 0} \mathfrak{k}_\lambda(M)\,, \tag{5.5}$$

where $\mathfrak{k}_\lambda(M) = \{Y \in \mathfrak{h}(M) : [X,Y] = \lambda Y\}$. Moreover the centralizer $\mathfrak{k}_0(M)$ of X splits as $\mathfrak{k}_0' \oplus \mathfrak{k}_0''$ where $\mathfrak{k}_0'$ is the $\partial^\sharp$-image of real functions and $\mathfrak{k}_0''$ is the $\partial^\sharp$-image of purely imaginary functions.

The following result is due to Lichnerowicz (see [8, Proposition 2.140]).

Proposition 5.8. *On a compact Kähler manifold a (real) vector field X is holomorphic if and only if*

$$\Delta_d X^\flat - 2\,\mathrm{Ric}(X, \cdot) = 0\,. \tag{5.6}$$

5.3 Statements and proofs

The function f in Definition 5.1 has, by means of Remark 5.2, holomorphic gradient so it satisfies, applying δ on both sides of (5.6) (cf. [38, (1.17.5)])

$$\frac{1}{2}\Delta_d^2 f + (dd^c f, \omega + \frac{1}{2}dd^c f) + \frac{1}{2}(ds, df) = 0\,. \tag{5.7}$$

Tracing the KRS equation (5.3) we get $2n - s = \Delta_d f$, we substitute it into (5.7) to get

$$\frac{1}{2}\Delta_d(2n-s) - \Delta_d f + \frac{1}{2}|dd^c f|^2 + \frac{1}{2}(ds, df) = 0\ .$$

So

$$-\Delta_d s + 2(s-2n) + |dd^c f|^2 + (ds, df) = 0\ . \tag{5.8}$$

Differentiating we get

$$-\Delta_d ds + 2ds + d|dd^c f|^2 + d(ds, df) = 0\ . \tag{5.9}$$

The last term in (5.9) can be substituted with the following two lemmas.

Lemma 5.9. *On an extremal Kähler-Ricci soliton $(g,\, f)$ the holomorphic fields ∇f and ∇s commute.*

Proof. Since (g, f) is an extremal Kähler-Ricci soliton, then both $\partial^\sharp s$ and $\partial^\sharp f$ are holomorphic vector fields, i.e. $\partial^\sharp s,\, \partial^\sharp f \in \mathfrak{h}(M)$. Also, by means of Theorem 5.6, $\mathfrak{h}(M)$ splits as $\mathfrak{h}(M) = \mathfrak{a}(M) \oplus \bigoplus_{\lambda \geq 0} \mathfrak{h}_\lambda(M)$. The summand $\mathfrak{h}_0(M)$, the centralizer of $\partial^\sharp s$ in $\mathfrak{h}(M)$, contains $\partial^\sharp f$. Hence we have

$$\begin{aligned} 0 = [\partial^\sharp s,\, \partial^\sharp f] &= \frac{1}{4}\left([\nabla s,\, \nabla f] - [J\nabla s,\, J\nabla f] - i[\nabla s,\, J\nabla f] - i[J\nabla s,\, \nabla f]\right) \\ &= \frac{1}{2}\left([\nabla s,\, \nabla f] - iJ[\nabla s,\, \nabla f]\right)\ , \end{aligned}$$

and we take its real part to conclude. □

Lemma 5.10. *Whenever two real functions u, v satisfy $[\nabla u\, , \nabla v] = 0$, then there holds*

$$d(g(\nabla u, \nabla v)) = (\nabla_{\nabla u}\nabla v + \nabla_{\nabla v}\nabla u)^\flat = 2(\nabla_{\nabla u}\nabla v)^\flat\ .$$

Proof. For any vector field Y we have

$$\begin{aligned} Y \cdot g(\nabla v, \nabla u) &= g(\nabla_Y \nabla v, \nabla u) + g(\nabla v, \nabla_Y \nabla u) \\ &= g(\nabla_{\nabla u}\nabla v + \nabla_{\nabla v}\nabla u, Y) \\ &= g(2\nabla_{\nabla u}\nabla v, Y)\ . \end{aligned}$$

This completes the proof of the lemma. □

For a (Kähler-)Ricci soliton there are some quantities that are constant, see e.g. [28]. One of them is, in our notation,

$$s + |\nabla f|^2 + 2f = \mathrm{const}\,. \tag{5.10}$$

From this together with Lemma 5.10 it is easy to infer the following.

Lemma 5.11. *Let g be a KRS with real holomorphic field X and let $Z = X^{1,0}$. Then g is extremal if, and only if, $\nabla_X X$ is real holomorphic (or $\nabla_Z Z$ is holomorphic or $\nabla_{\overline{Z}}\overline{Z}$ is antiholomorphic).*

At this point it is worth noticing the following.

Proposition 5.12. *For an extremal KRS g with field X and scalar curvature s, if $X = c\nabla s$ then g is Einstein.*

Proof. We first notice that c has to be constant on M. Indeed if it were a function on M it would be holomorphic since the two fields are. Consider the function $p \mapsto |X_p|^2$ and a local maximum $q \in M$. At q we would have

$$g_q(\nabla_X X|_q, X_q) = 0.$$

Under the proportionality assumption (5.10) becomes

$$(c+2)X + 2\nabla_X X = 0. \tag{5.11}$$

At q we would have then $\frac{c+2}{2} g_q(X, X) = 0$ which implies $X = 0$ if $c \neq -2$.

If $c = -2$ we have from (5.11) and Lemma 5.10 that $\nabla_X X = \nabla|X|^2 = 0$ implying $X = 0$ as well. □

By means of the decomposition theorems 5.6 and 5.7, the fields JX and $J\nabla s$ belong to the center of the isometry algebra and are linearly independent for a non-Einstein extremal KRS. This gives us the following corollary.

Corollary 5.13. *If g is a non-Einstein extremal KRS, then the center of the isometry group of g has dimension at least 2.*

We now present a characterization of extremal Kähler-Ricci solitons.

Proposition 5.14. *Let (M^{2n}, g, ω, f) be a compact Kähler-Ricci soliton with Riemannian scalar curvature s. Let $X = \nabla f$. Then the following are equivalent.*

1. *the function $|dd^c f|^2$ is constant and $[\nabla f, \nabla s] = 0$;*
2. *g is extremal;*
3. *$\mathrm{Rm}(\cdot, \overline{\partial^\sharp f})\overline{\partial^\sharp f} = 0$;*
4. *The tensor*
$$T_X := \mathrm{Rm}(\cdot, X)X$$
commutes with J and $\alpha : (A, B) \mapsto \mathrm{Rm}(A, JX, X, B)$ is a $(1,1)$-form.

Proof. Let us first prove the equivalence between (1) and (2). By means of Proposition 5.8, the condition on g being extremal is equivalent to require the Riemannian scalar curvature s to satisfy the tensorial Lichnerowicz equation (see [8, Proposition 2.140])

$$\Delta_d ds - 2\,\mathrm{Ric}(\nabla s, \cdot) = 0\ . \tag{5.12}$$

Let us assume g to be extremal. Equation (5.12), together with the Kähler-Ricci soliton assumption $\mathrm{Ric} = g + \mathrm{Hess}\, f$, reads

$$\begin{aligned} 0 &= \Delta_d ds - 2ds - 2\,\mathrm{Hess}_f(\nabla s, \cdot) \\ &= \Delta_d ds - 2ds - 2g(\nabla_{\nabla s}\nabla f, \cdot)\ . \end{aligned} \tag{5.13}$$

By means of Lemma 5.10, formula (5.13) differs from (5.9) by the term $d|dd^c f|^2$ which has to be zero.

Conversely, assuming $[\nabla s, \nabla f] = 0$, then Lemma 5.10 holds. Also, in (5.9) the term $d|dd^c f|^2$ vanishes and then (5.9) is simply (5.12), which says, by means of Proposition 5.8, that s has holomorphic gradient. This completes the equivalence between (1) and (2).

Let g be extremal, then by means of the previous Lemma, the field $\nabla_Z Z$ where $Z = \partial^\sharp f$ is holomorphic. Then compute for any $(1,0)$-field A,

$$\begin{aligned} \mathrm{Rm}(A, \overline{Z})\overline{Z} &= \nabla_A \nabla_{\overline{Z}} \overline{Z} - \nabla_{\overline{Z}} \nabla_A \overline{Z} - \nabla_{[\overline{Z}, A]} \overline{Z} \\ &= 0 \end{aligned}$$

by using the fact that $\overline{Z}$ and $\nabla_{\overline{Z}}\overline{Z}$ are antiholomorphic (hence killed by ∇_A) that kills the first two terms and that $[\overline{Z}, A]$ is $(1,0)$ that kills the last.

Conversely, the generic form of the Riemann tensor for A of type $(1,0)$ is

$$\mathrm{Rm}(A, \overline{Z})\overline{Z} = \nabla_A \nabla_{\overline{Z}} \overline{Z}.$$

If this is zero, it means that the field $\nabla_{\overline{Z}}\overline{Z}$ is killed by ∇_A for any A of type $(1,0)$ implying that it is antiholomorphic. Indeed, for any W we have

$$0 = \nabla_{Y - iJY}(W + iJW) = \nabla_Y W + J\nabla_{JY} W + i(J\nabla_Y W - \nabla_{JY} W)$$

that is, W satisfies $\nabla_{JY} W = J\nabla_Y W$ for all Y, that is W is real holomorphic.

Hence we conclude g is extremal by means of the previous Lemma.

Let us now assume (3). We notice that its real formulation is given by the system

$$\begin{cases} \mathrm{Rm}(A, X)JX = -\,\mathrm{Rm}(A, JX)X \\ \mathrm{Rm}(A, X)X = \mathrm{Rm}(JA, X)JX. \end{cases} \tag{5.14}$$

and the second equation means exactly that $[T_X, J] = 0$. We have now, using (5.14) for the second equality,

$$\begin{aligned}\alpha(B, A) &= \operatorname{Rm}(B, JX, X, A)\\ &= -\operatorname{Rm}(B, X, JX, A)\\ &= -\operatorname{Rm}(JX, A, B, X)\\ &= -\alpha(A, B).\end{aligned}$$

To prove that α is $(1,1)$ is equivalent to prove that it is J-invariant. This follows again from (5.14) since

$$\begin{aligned}\alpha(JA, JB) &= \operatorname{Rm}(JA, JX, X, JB)\\ &= -\operatorname{Rm}(A, X, JX, B)\\ &= \operatorname{Rm}(A, JX, X, B)\\ &= \alpha(A, B).\end{aligned}$$

Conversely let $[T_X, J] = 0$ and let α be J-invariant. These assumption are exactly (5.14). □

We can use this to prove our main result.

Theorem 5.15. *Any extremal Kähler-Ricci soliton with positive holomorphic sectional curvature is Einstein.*

Proof. Let f be the soliton function. Assume it is not a constant and Z be the normalized $\partial^\sharp f$.

By assumption we have, in the direction Z, that the holomorphic sectional curvature is

$$K(Z) := \operatorname{Rm}(Z, \overline{Z}, Z, \overline{Z}) > 0.$$

By means of the previous proposition we are lead to the contradiction $K(Z) = 0$ as the above Riemann tensor vanishes. □

Remark 5.16. There is no loss of generality to assume the positivity of the holomorphic sectional curvature instead of just requiring it to have a sign. Indeed, by a theorem of Berger [6, Lemme (7.4) pag. 50] prescribing the sign of the holomorphic sectional curvature gives the same sign to the scalar curvature, which in case of Ricci solitons is always positive by means of general results (see e.g. again [28]).

The argument exposed in this paper can be replicated verbatim to prove the following result about Sasaki manifolds. We refer to [36, 13] for the notions of Sasaki-Ricci solitons, to and Sasaki-extremal metrics and for transverse curvature. Recall that, for a Sasaki manifold, being transversally Kähler-Einstein is equivalent to being η-Sasaki-Einstein.

Theorem 5.17. *Any extremal Sasaki-Ricci soliton with positive transverse holomorphic sectional curvature is η-Sasaki-Einstein.*

Indeed there are Sasakian analogues of Theorem 5.4 done by Boyer and Galicki and an extension of Theorem 5.7 discussed in Chapter 4. Moreover, the Lichnerowicz equations hold as well for the transverse quantities, see again [13].

Chapter 6

Legendrian submanifolds

In this chapter we discuss minimal Legendrian submanifolds in Sasaki-Einstein manifolds and a generalization of a theorem of Lê-Wang about minimal Legendrian submanifolds of the standard Sasakian sphere. This is an exposition of the results obtained in [18] jointly with S. Calamai.

6.1 Introduction to the problem

Let (M, η, g) be a Sasakian manifold of dimension $2n+1$. A minimal Legendrian submanifold is an n-dimensional submanifold $i : L \to M$ on which the contact form vanishes, $i^*\eta = 0$ and is minimal in the sense of Riemannian geometry with respect to the metric induced from g.

In the case where the minimal Legendrian L is embedded in the standard Sasakian round $(2n+1)$-sphere, Lê and Wang [54] constructed a family of functions on L which are eigenfunctions of the Laplacian on L of the induced metric. They give also a lower bound of the dimension of the relative eigenspace and if it is attained then the submanifold is totally geodesic. Conversely they prove that a minimal submanifold of the standard sphere admitting that certain family of functions as Laplacian eigenfunctions is necessarily Legendrian.

Although their techniques make a heavy use of the particular situation, namely the theory of minimal immersion in spheres and the presence of an ambient Euclidean space, we prove that some of their ideas can be generalized to any Sasaki-Einstein manifold.

Let L be a minimal Legendrian submanifold of a Sasaki-Einstein M. The aim of this paper is to prove that two certain families of functions on L, one of which constructed in terms of the contact moment map of the action of the Sasaki automorphism group, are eigenfunctions of the Laplacian of L and we give a lower bound for the dimension of the eigenspace.

Theorem 6.1. *Let L^n be a minimal Legendrian submanifold of an η-Sasaki-Einstein manifold $(M^{2n+1}, \eta, \xi, g, \Phi)$ with algebra of infinitesimal Sasaki automorphisms $\mathfrak{g}$. Then, for each $X \in \mathfrak{g}$, the function*

$$\eta(X) - \frac{1}{\mathrm{vol}(L)} \int_L \eta(X) dv,$$

where dv is the volume form of L of the induced metric, is an eigenfunction of the Laplacian Δ_L with eigenvalue $2n+2$. Moreover the dimension of the $(2n+2)$-eigenspace is at least $\dim \mathfrak{g} - \frac{1}{2}n(n+1) - 1$.

Moreover we prove, like in the sphere case although with totally different techniques, that if the lower bound is attained then the submanifold is totally geodesic together with a rigidity result about the ambient M, in the case of a *regular* Sasaki-Einstein manifold over a base Kähler manifold.

Theorem 6.2. *If M is a regular Sasaki-Einstein manifold and the multiplicity of the eigenvalue $2n+2$ of Δ_L is exactly $\dim \mathfrak{g} - \frac{1}{2}n(n+1) - 1$ then M is a Sasaki-Einstein circle bundle over the complex projective space endowed with the Fubini-Study metric. In particular if M is simply connected then $M = S^{2n+1}$.*

Among the techniques we use we mention the theory of deformations of minimal Legendrian submanifolds, for which we refer to [62, 61] and, in the case of regular manifolds, the correspondence between Legendrian submanifolds of Sasakian manifolds and Lagrangian submanifolds of Kähler manifolds, see [64].

This result makes use of the geometry of Legendrian submanifolds of the Kähler-Einstein base, which exists by the regularity assumption. It would be interesting to drop this assumption and prove the result for quasi-regular or irregular Sasaki-Einstein manifolds.

Then in Theorem 6.13 we provide a generalization of the family of eigenfunctions by making use of the immersion of the Sasaki-Einstein manifold M into its Ricci-flat Kähler cone $C(M)$. This family is parameterized by the Lie algebra of the infinitesimal Kähler automorphisms of $C(M)$, which is in general bigger than the Sasaki automorphism group of M. The family is defined by means of the Nomizu operator on $C(M)$. This time our arguments are similar to the ones of Lê and Wang for the sphere and they rely on the Ricci-flatness of $C(M)$ and properties of the Nomizu operator.

It would be interesting to provide sufficient conditions for the Legendrianity of a minimal submanifold by means of any of these families of functions.

Problem 2. *Let M^{2n+1} be a Sasaki-Einstein manifold with big enough automorphism group G, let L^n be a minimal submanifold such that for each $X \in \mathfrak{g}$, the family of functions* (6.2) *or* (6.8) *are eigenfunctions of Δ_L with eigenvalue $2n+2$. Can we conclude that L is Legendrian?*

Also, it would be interesting to relate the second family with the moment map of the symplectic action on $C(M)$ of its Kähler automorphism group.

6.2 Legendrian immersions and their deformations

We recall some notions about minimal Legendrian submanifolds and their deformations. We will consider some special submanifolds of Sasakian manifolds, known as *Legendrian* (or *horizontal*), see [64].

A Legendrian submanifold of a $(2n+1)$-dimensional contact manifold (M, η) is an n-dimensional submanifold $i : L \to M$ such that for all $p \in L$ we have $i_*(T_pL) \subseteq \ker \eta_{i(p)}$.

We will consider Legendrian submanifolds which are also *minimal* in the sense of Riemannian geometry, i.e. their mean curvature field vanishes.

If we have a Legendrian submanifold L in a Sasakian manifold we can identify the space of sections of the normal bundle NL with $C^\infty(L) \oplus \Omega^1(L)$ via the isomorphism

$$\begin{aligned} \chi : \Gamma(NL) &\longrightarrow C^\infty(L) \oplus \Omega^1(L) \\ V &\longmapsto \left(\eta(V), -\frac{1}{2} i^*(\iota_V d\eta)\right) \end{aligned}$$

see [62].

In the case of a compact regular Sasakian manifold M with contact structure η that fibers over a compact Kähler manifold (B, ω) we can take the projection $\pi(L) \subseteq B$ of a Legendrian L. Following Reckziegel [64] we have that $\pi(L)$ is a Lagrangian submanifold of B, i.e. $(\pi \circ i)^*\omega = 0$ and is finitely covered by L.

Conversely, given a Lagrangian submanifold $j : N \to B$, a point $q \in N$, for any choice of p in the fiber of q there exists a neighborhood U of q and a Legendrian immersion $i : U \to M$ such that $\pi \circ i = j|_U$.

Moreover, Riemannian properties of L hold as well for $\pi(L)$ and conversely. Namely we have the following.

Proposition 6.3 ([64])**.** *The Legendrian L is minimal, or totally geodesic, if and only if the Lagrangian $\pi(L)$ is.*

A smooth family of minimal Legendrian immersions $i_t : L \to M$ is a family of maps $F : [0,1] \times L \to M$ such that for each t the map $i_t = F(t, \cdot) : L \to M$ is a minimal Legendrian immersion. Every smooth family points out a vector field W_t on L given at p by

$$W_t|_p = F_*\left(\frac{\partial}{\partial t}\bigg|_{(t,p)}\right).$$

It is known, e.g. [61, 62], that a family of immersions is Legendrian if and only if the normal component V_t of W_t satisfies

$$V_t = \chi^{-1}\left(\eta(V_t), \frac{1}{2}d\eta(V_t)\right), \tag{6.1}$$

i.e. $d\eta(V_t) = -i^*(\iota_{V_t} d\eta)$. Normal fields satisfying (6.1) are called *infinitesimal Legendrian deformations*.

We are interested in minimal Legendrian deformations of a Legendrian $i : L \to M$, that are smooth families $i_t : L \to M$ of minimal Legendrian immersions such that $i_0 = i$.

A trivial family of deformations of a minimal Legendrian submanifold is given by one-parameter families of ambient transformations. We will denote by $\mathrm{Aut}(M)$ the group of such transformations, i.e. diffeomorphisms $M \to M$ which are isometric contactomorphisms.

If we let $\varphi_t \in \mathrm{Aut}(M)$ be one of such families. Then $i_t = \varphi_t|_{i(L)} : i(L) \to M$ is a minimal Legendrian deformation, see [61].

In particular, the normal component of every field in the Lie algebra $\mathfrak{aut}(M)$ of $\mathrm{Aut}(M)$ defines an infinitesimal Legendrian deformation. This is also minimal as we are taking the normal component of a Killing vector field, see [68, Sec. 3].

When we restrict ourselves to η-Sasaki-Einstein manifolds with constant A, we have a characterization of the space of infinitesimal minimal Legendrian deformations.

Proposition 6.4 ([61]). *Let $i : L \to M$ be a minimal Legendrian submanifold in an η-Sasaki-Einstein manifold with constant A. Then the vector space of infinitesimal minimal Legendrian deformations is identified with*

$$\mathrm{Def}(L) = \mathbb{R} \oplus \{f \in C^\infty(L) : \Delta_L f = (A+2)f\}$$

where Δ_L denotes the Laplacian of L with the induced metric.

This result is obtained by combining the copy of the space of infinitesimal Legendrian deformations of L given by

$$\left\{\left(f, \frac{1}{2}df\right) : f \in C^\infty(L)\right\}$$

and the space of minimal deformation given by the kernel of the Jacobi operator $\mathcal{J}$, for which we refer to [68].

6.2.1 Contact moment maps

We finally recall the notion of contact moment map, we follow [11, Sec. 8.4.2]. In our setting the group $G = \mathrm{Aut}(M)$ is a compact group acting on M. We can extend this action to the symplectic cone $(C(M), d(r^2\eta))$ by requiring that it leaves the $\{r = \mathrm{const}\}$ levels unchanged, i.e. the action is given by $g(p, r) = (gp, r)$. Being G a contactomorphism group it is easy to see that the action on $C(M)$ is by symplectomorphisms and, being the symplectic form on the cone exact, this action is Hamiltonian. So there exists a map $\varphi : C(M) \to \mathfrak{g}^*$, such that

$$d(\varphi(X)) = -\iota_X d(r^2\eta) = d(r^2\eta(X)).$$

Hence, up to a constant, one can take the map $\varphi(p, r)(X) = r^2\eta_p(X)$. Seeing M as the $\{r = 1\}$ level set, we consider the restriction $\mu : M \to \mathfrak{g}^*$ of φ which we call the *contact moment map* for the G-action on M.

6.3 Eigenfunctions using the contact moment map

In this section we construct one possible generalization of the functions given by Lê-Wang [54]. We briefly recall their setting. They consider the standard Sasakian sphere S^{2n+1} immersed in its Kähler cone $\mathbb{C}^{n+1} \setminus \{0\}$ with respectively the round metric g and the Euclidean metric $\langle \cdot, \cdot \rangle$. It is known that the both the Sasaki transformation group of the sphere and the Kähler automorphism group of the cone is $G = \mathrm{U}(n+1)$. Let $M \in \mathfrak{u}(n+1)$. Then the moment map for the G-action on the cone is given, up to a constant, by

$$\varphi(p, r)(M) = r^2\eta_p(M_p) = r^2 g(\xi_p, M_p) = \langle \xi_p, M_p \rangle$$

We see an infinitesimal Sasaki automorphism $M \in \mathfrak{u}(n+1)$ as a linear vector field whose value at $x \in S^{2n+1}$ is Mx. Then, using that ξ at x is Jx, where J is the standard complex structure, the contact moment map $\mu : S^{2n+1} \to \mathfrak{u}(n+1)^*$ is given by

$$\mu(x)(M) = \langle Mx, Jx \rangle$$

which is exactly the function of Lê-Wang.

Back to the general setting of the Sasaki group $G = \mathrm{Aut}(M)$ with Lie algebra $\mathfrak{g}$ acting on the η-Sasaki-Einstein M, we have the contact moment map that is given by $\mu(p)(X) = \eta_p(X_p)$.

We then consider for each $X \in \mathfrak{g}$ the map $p \mapsto \eta(X)$ restricted to a minimal Legendrian submanifold and up to a constant.

We prove the generalization of one of the implications of [54, Thm. 1.1].

Theorem 6.5. *Let (M, g, η, ξ) be a $(2n+1)$-dimensional η-Sasaki-Einstein manifold with* $\mathrm{Ric} = Ag + (2n - A)\eta \otimes \eta$ *and let $L^n \subset M$ be a minimal Legendrian submanifold. Then for all $X \in \mathfrak{aut}(M)$ the function on L given by*

$$f_X = \eta(X) - \frac{1}{\mathrm{vol}(L)} \int_L \eta(X) dv, \tag{6.2}$$

where dv is the volume form on L of the induced metric, is en eigenfunction of the Laplacian Δ_L on L with eigenvalue $A + 2$. Moreover this eigenspace has dimension $\geq \dim \mathfrak{aut}(M) - \frac{1}{2}n(n+1) - 1$.

Proof. We recalled above that the map $\chi : \Gamma(NL) \to C^\infty(L) \oplus \Omega^1(L)$ given by $\chi(V) = (\eta(V), -\frac{1}{2}\iota_V d\eta)$ is an isomorphism if L is Legendrian and that the space of infinitesimal deformations of a minimal Legendrian L is

$$\mathrm{Def}(L) = \mathbb{R} \oplus \{f \in C^\infty(L) : \Delta_L f = (A+2)f\}.$$

Let $X \in \mathfrak{g} = \mathfrak{aut}(M)$ and let $X|_L = X_1 + X_2 \in \Gamma(TL) \oplus \Gamma(NL)$ be its decomposition.

From [61] it follows that X_2 defines a Legendrian deformation of L and it is known, e.g. [68], that the normal part of a Killing field defines an infinitesimal minimal deformation. Hence $\chi(X_2) \in \chi(\ker \mathcal{J})$, where $\mathcal{J}$ denotes the Jacobi operator, and so, following Ohnita [61] we have

$$\Delta_L f - (A+2)f = \mathrm{const} = C.$$

and the pair $(C, f - \overline{f}) \in \mathbb{R} \oplus \{f \in C^\infty(L) : \Delta_L f = (A+2)f\}$, where $\overline{f} = \frac{1}{\mathrm{vol}(L)} \int_L \eta(X) dv$. So the first claim follows.

Every $X \in \mathfrak{g} = \mathfrak{aut}(M)$ defines a trivial deformation of L, hence there is a linear map $\alpha : \mathfrak{g} \to \mathrm{Def}(L)$ given by $\alpha(X) = \chi(X_2)$.

Its kernel is $\ker \alpha = \{X \in \mathfrak{g} : X|_L \in \Gamma(TL)\} \subseteq \mathfrak{iso}(L)$. So we have

$$\begin{aligned} 1 + \dim E_{A+2} &\geq \dim \alpha(\mathfrak{g}) \\ &= \dim \mathfrak{g} - \dim \ker \alpha \\ &\geq \dim \mathfrak{g} - \dim \mathfrak{so}(n+1) \end{aligned} \tag{6.3}$$

$$= \dim \mathfrak{g} - \frac{n(n+1)}{2}.$$

So we have the second claim in the statement. □

Let us specialize to Sasaki-Einstein manifolds and assume that M is regular, so it is a principal circle bundle $\pi : M \to B$ over a Kähler-Einstein base manifold B and consider the case when the equality holds in the previous theorem. We prove the following, generalizing [54, Thm. 1.2] together with a rigidity result.

Theorem 6.6. *If M is regular and the eigenvalue $2n+2$ of Δ_L has multiplicity exactly $\dim \mathfrak{aut}(M) - \frac{1}{2}n(n+1) - 1$ then L is totally geodesic in M and M is a principal circle bundle over the complex projective space.*

Proof. The projection $\widetilde{L} = \pi(L) \subseteq B$ is a Lagrangian submanifold of a Kähler-Einstein manifold and it is known that $\widetilde{L}$ is covered by L [64].

To have equality one needs to have equality in (6.3) so we conclude that the isometry group of $\widetilde{L}$ is the largest possible, i.e. its Lie algebra is $\mathfrak{so}(n+1)$. Let this isometry group be denoted by K. The group K, being a subgroup of the Sasaki transformation group of M, sends leaves into leaves and thus acts on B. We claim that the action has cohomogeneity one.

Indeed it is known, see [50], that if $\dim K = \dim \mathfrak{so}(n+1)$ then L^n is either a n-sphere or $\mathbb{RP}^n$, written as $\mathrm{SO}(n+1)/H$, where $H = \mathrm{SO}(n)$ or $H = \mathbb{Z}_2 \cdot \mathrm{SO}(n)$ is the stabilizer of a $q \in \widetilde{L}$. In any case the isotropy representation of H acts transitively on the unit sphere $T_q\widetilde{L}$. Being $\widetilde{L}$ Lagrangian, the action is transitive also on the unit sphere in the normal space at q and this action has cohomogeneity one, hence also the action of $\mathrm{SO}(n+1)$ on B does.

Let $p \in \widetilde{L}$. Being $\widetilde{L}$ homogeneous under K, it is also known from [5] that the orbit $\Omega = K^{\mathbb{C}} \cdot p$ is open dense in B and Stein, hence in particular affine, and that $B \setminus \Omega$ has complex codimension 1.

Let $x \in B$ be a principal point. Being Ω open dense, the $K^{\mathbb{C}}$-orbit through x is open as well and intersects Ω, then they coincide. So B is a two-orbit Kähler manifold, i.e. is acted on by a complex algebraic group admitting exactly two orbits Ω and A.

They were classified, as complex manifolds, by Ahiezer [2, Table 2] in the case of Ω affine and A of codimension 1. The occurrences of a group K with Lie algebra $\mathfrak{so}(n+1)$ can be one of the following:

1. $\widetilde{L} = \mathrm{SO}(n+1)/\mathrm{SO}(n) = S^n$ and $B = Q_n$,
2. $\widetilde{L} = \frac{\mathrm{SO}(n+1)}{\text{center}}/S(\mathrm{O}(1) \times \mathrm{O}(n)) = \mathbb{RP}^n$ and $B = \mathbb{CP}^n$,

3. $\widetilde{L} = \mathrm{Spin}(7)/\,\mathrm{G}_2 = S^7$ and $B = Q_7$,
4. $\widetilde{L} = \mathrm{SO}(7)/\,\mathrm{G}_2 = \mathbb{RP}^7$ and $B = \mathbb{CP}^7$;

where the projective spaces and the complex hyperquadrics are endowed with the unique Kähler-Einstein metric of constant $2n+2$. This proves that the possible B are only complex hyperquadrics or complex projective spaces and M is a Sasaki-Einstein principal circle bundles over B.

Being the pairs in this list symmetric subspaces of B, we have that $\widetilde{L}$ is totally geodesic in B. By Proposition 6.3 of Reckziegel, this is equivalent to say that L is totally geodesic in M.

We want now to exclude the case $B = Q_n$. So far we have the following diagram of immersions and submersions.

$$\begin{tikzcd}
(S^n, g) \arrow[r, hook, "\widetilde{\imath}"] \arrow[d, "="'] & (M, g_{\mathrm{SE}}) \arrow[d, "\pi"] & (S^{2n+3}, g_c) \arrow[r, hook] \arrow[d, "p"] & (\mathbb{C}^{n+2}, \frac{4}{c}\langle\cdot,\cdot\rangle) \\
(S^n, g) \arrow[r, hook, "i"] & (Q_n, g_Q) \arrow[r, hook, "j"] & (\mathbb{CP}^{n+1}, g_c^{\mathrm{FS}}) &
\end{tikzcd}$$

For the metric point of view, we have the Fubini-Study metric g_c^{FS} on $\mathbb{CP}^{n+1}$ with constant holomorphic curvature c. This rescaling of the Fubini-Study metric on $\mathbb{CP}^{n+1}$ is defined by the metric given by $\frac{4}{c}$ times the round metric on S^{2n+3}, which we denote by g_c [51, vol. II, p. 273]. The choice of c in g_c^{FS} is such that $g_Q = j^* g_c^{\mathrm{FS}}$ is Kähler Einstein of Einstein constant $2n+2$ and this happens exactly for $c = \frac{4n+4}{n}$ from [70].

By [23] the only totally geodesic spheres in the quadric are immersions $i : x \mapsto [x]$ for $x \in S^n \subset \mathbb{R}^{n+1}$. The restriction of the quadric metric on it is $\frac{n}{2n+2}$ times the round metric. Being S^n simply connected for $n > 1$, we have that the Legendrian L is isometric to its projection in Q_n.

Let Δ be the Laplacian on S^n associated to the metric $\frac{n}{2n+2} g_{\mathrm{round}}$. An eigenfunction of Δ with eigenvalue $2n+2$ is an eigenfunction of the round Laplacian with eigenvalue n.

It is known from [7] that the round sphere admits the eigenvalue n with multiplicity $n(n+1)$.

To compute the lower bound, we observe that, since every Sasaki automorphisms induces by projection a Kähler automorphism of the base, that $\dim \mathfrak{aut}(M) \leq \dim \mathfrak{aut}(B) + 1 = \frac{1}{2}(n+2)(n+1) + 1$ since the automorphism group of the hyperquadric is $\mathrm{SO}(n+2)$.

In order not to attain the lower bound we need to have

$$\dim \mathfrak{aut}(M) < \frac{3}{2}n(n+1)+1$$

and this is always true since for $n > 1$ we have $\frac{1}{2}(n+2)(n+1)+1 < \frac{3}{2}n(n+1)+1$.

In the case $n = 1$ the quadric $Q_1 = \mathbb{CP}^1$ is a complex projective space, so we are left with the only case $B = \mathbb{CP}^n$. □

6.4 Eigenfunctions using the Nomizu operator

In this section we define another family of eigenfunctions on a Legendrian L of M by making use of the geometry of the Kähler cone and its group of Kähler automorphisms.

Let (M, g) be a Sasakian manifold of dimension $2n+1$ and let $(C(M), \overline{g})$ be its Kähler cone. We let e_A for $A \in \{1, \ldots, 2n+1\}$ be a local orthonormal frame at some point of M and let θ_A be its dual.

Then the set $\{\frac{1}{r}e_1, \ldots, \frac{1}{r}e_{2n+1}, \partial_r\}$ is an orthonormal frame for the cone metric $\overline{g} = r^2 g + dr^2$ and its dual is $\{r\theta_1, \ldots, r\theta_{2n+1}, dr\}$.

Let $\overline{\nabla}$ be the Levi-Civita connection of the cone metric. From the well known relations [71] we have

$$\begin{aligned}\overline{\nabla}\partial_r &= \frac{1}{r}e_A \otimes \theta_A\\ \overline{\nabla}e_B &= \frac{1}{r}e_B \otimes dr + \theta_{BC} \otimes e_C - r\partial_r \otimes \theta_B.\end{aligned}$$

Lemma 6.7. *Let $L^n \to M$ be an immersion and let $e_1, \ldots, e_n$ be an orthonormal frame of L. Let ∇ be the Levi Civita connection on M. Then, for any smooth function $f : M \to \mathbb{R}$, we have*

$$\Delta_L f|_L = -\sum_{i=1}^{n} \nabla df(e_i, e_i)|_L - H \cdot f|_L. \tag{6.4}$$

where Δ_L is the Hodge Laplacian and H is the mean curvature field of the immersion.

In particular, when the immersion is minimal, we have

$$\Delta_L f|_L = -\sum_{i=1}^{n} \nabla df(e_i, e_i)|_L. \tag{6.5}$$

Proof. Label as ∇^L the induced connection on L; by definition we have

$$\begin{aligned}\sum_{i=1}^n \nabla df(e_i,e_i)|_L &= \sum_i e_ie_if|_L - \sum_{i=1}^n \nabla_{e_i}e_if|_L \\ &= \sum_{i=1}^n e_ie_if|_L - \sum_{i=1}^n \nabla^L_{e_i}e_if|_L - \sum_{i=1}^n(\nabla_{e_i}e_if|_L - \nabla^L_{e_i}e_if|_L) \\ &= -\Delta_L f|_L - H\cdot f|_L,\end{aligned}$$

which is precisely the claimed (6.4). Since the assumption on minimality corresponds to the vanishing of H, we also get the claimed (6.5). This completes the proof of the lemma. □

Lemma 6.8. *Let $L^n \to M$ be a* minimal *immersion in a Sasaki manifold. Let f be a function on the Kähler cone $C(M)$ which does not depend on r and let Δ_L be the Hodge Laplacian on L; finally, let $e_1,\cdots,e_n$ be an orthonormal frame of L. Then we have*

$$\Delta_L f|_L = -\sum_{i=1}^n \overline{\nabla} df(e_i,e_i)|_L.$$

Proof. In view of Lemma 6.7, it suffices to show that for any $i,j\in\{1,\cdots,n\}$, then

$$\overline{\nabla} df(e_i,e_j)|_L = \nabla df(e_i,e_j)|_L, \tag{6.6}$$

where as usual ∇ is the Levi Civita of the Sasaki metric g, while $\overline{\nabla}$ is the Levi Civita connection of the metric $\overline{g} = r^2g + dr^2$. By the very definition we have

$$\begin{aligned}\overline{\nabla} df(e_i,e_j)|_L &= e_ie_jf|_L - \overline{\nabla}_{e_i}e_j\cdot f|_L \\ &= e_ie_jf|_L - \big(\nabla_{e_i}e_j\cdot f|_L - \delta_{ij}r\partial_r f|_L\big) \\ &= \nabla df(e_i,e_j)|_L,\end{aligned}$$

where at the second equality we applied [71, (1.1)]. This completes the proof of the lemma. □

Let us now construct a family of operators. For an infinitesimal Kähler automorphism K on the cone, i.e. Killing and holomorphic, we define the operator on sections of $TC(M)$ given by

$$M_K = \overline{\nabla} K + \frac{1}{2n+2}\operatorname{div}(JK)J. \tag{6.7}$$

Lemma 6.9. *Let $C(M)$ be the Kähler cone over a Sasaki-Einstein manifold and let K as above. Then*

(i) $\operatorname{div}(JK) = \mathrm{const}$*;*

(ii) M_K *is skew-symmetric and* $M_K J = J M_K$*;*

(iii) $\operatorname{tr}(JM_K) = 0$*;*

(iv) $\overline{\nabla} M_K = \overline{\mathrm{Rm}}(\cdot, K)$ *where* $\overline{\mathrm{Rm}}$ *is the Riemann* $(3,1)$*-tensor of* $\overline{g}$*.*

Proof. Let A_K be the associated Nomizu operator, i.e. $A_K = \overline{\nabla} K$. Then since K is Killing, its covariant derivative is known to be $\overline{\nabla}\,\overline{\nabla} K = \overline{\mathrm{Rm}}(\cdot, K)$.

(i) Fix $p \in C(M)$ and let v_i be a geodesic frame at p and let Y be any vector field on $C(M)$. Then

$$\begin{aligned} Y \cdot \operatorname{div}(JK)|_p &= \overline{g}(\overline{\nabla}_Y \overline{\nabla}_{v_i} JK, v_i) \\ &= -\overline{g}(\overline{\nabla}_Y \overline{\nabla}_{v_i} K, Jv_i) \\ &= -\overline{g}((\overline{\nabla}_Y A_K) v_i, Jv_i) \\ &= \overline{\mathrm{Rm}}(Y, K, Jv_i, v_i) \\ &= 2\overline{\mathrm{Ric}}(Y, K) \\ &= 0 \end{aligned}$$

since $C(M)$ is Ricci-flat (see [71]), where we have used the well known fact that $\mathrm{Ric}(X, Y) = \frac{1}{2} \operatorname{tr}(\mathrm{Rm}(X, Y) \circ J)$.

(ii) Since K is holomorphic it is $\overline{\nabla}_{J\cdot} K = J \overline{\nabla}_{\cdot} K$ so $M_K J = J M_K$. Since K is Killing, $\overline{\nabla} K$ is skew-symmetric and also J, so (ii) follows.

(iii) Let v_i be an orthonormal frame of $C(M)$. Then

$$\begin{aligned} \operatorname{tr}(JM_K) &= \overline{g}(JM_K v_i, v_i) \\ &= \overline{g}\left(\overline{\nabla}_{v_i} JK - \frac{1}{2n+2} (\operatorname{div}(JK)) v_i, v_i \right) \\ &= \operatorname{div}(JK) - \operatorname{div}(JK) \\ &= 0. \end{aligned}$$

(iv) By (i) and the fact that J is parallel, (iv) follows. ∎

We will use the following lemma.

Lemma 6.10. *Let X be any field on M. Then $\overline{\mathrm{Rm}}(r\partial_r, Jr\partial_r)K$ and $\overline{\mathrm{Rm}}(r\partial_r, JX)K$ vanish.*

Proof. We notice that $\overline{\nabla}_{r\partial_r}K$ is holomorphic. Indeed, using that $r\partial_r$ is holomorphic, it is

$$\begin{aligned}\overline{\nabla}_{r\partial_r}K &= [r\partial_r, K] + \overline{\nabla}_K r\partial_r \\ &= [r\partial_r, K] + K\end{aligned}$$

using that $\overline{\nabla} r\partial_r = \mathrm{id}$. Hence $\overline{\nabla}_{r\partial_r}K$ is holomorphic being the sum of two holomorphic fields. Then we compute

$$\begin{aligned}\overline{\mathrm{Rm}}(r\partial_r, Jr\partial_r)K &= \overline{\nabla}_{r\partial_r}\overline{\nabla}_{Jr\partial_r}K - \overline{\nabla}_{Jr\partial_r}\overline{\nabla}_{r\partial_r}K - \overline{\nabla}_{[r\partial_r, Jr\partial_r]}K \\ &= J\overline{\nabla}_{r\partial_r}\overline{\nabla}_{r\partial_r}K - J\overline{\nabla}_{r\partial_r}\overline{\nabla}_{r\partial_r}K - \overline{\nabla}_{J[r\partial_r, r\partial_r]}K \\ &= 0.\end{aligned}$$

Similarly $\overline{\mathrm{Rm}}(r\partial_r, JX)K = 0$. □

Now consider the family of functions on $f_K : C(M) \to \mathbb{R}$ defined as

$$f_K = \overline{g}(M_K\partial_r, J\partial_r). \tag{6.8}$$

We exploit the fact that $\mathrm{tr}(JM_K) = 0$ for the following lemma, that also uses that L is Legendrian.

Lemma 6.11. *Let e_i be a frame of the Legendrian L. Then*

$$\sum_{i=1}^{n} \overline{g}(M_K e_i, Je_i) = -r^2 f_K.$$

Proof. Since L is Legendrian, we can extend $\{e_i\}$ to an *orthonormal* frame $\{\frac{1}{r}e_i, J\frac{1}{r}e_i, \partial_r, \frac{1}{r}\xi = J\partial_r\}$ of $C(M)$. Then

$$\begin{aligned}0 = \mathrm{tr}(JM_K) = \frac{1}{r^2}\sum_{i=1}^{n}\Big[\overline{g}(JM_K e_i, e_i) + \overline{g}(M_K Je_i, Je_i)\Big] \\ + \overline{g}(JM_K J\partial_r, J\partial_r) + \overline{g}(JM_K\partial_r, \partial_r)\end{aligned}$$

and from Lemma 6.9.(ii) we infer

$$2r^2 f + \sum_{i=1}^{n} 2\overline{g}(M_K e_i, Je_i) = 0.$$ □

Lemma 6.12. *For any Killing and holomorphic vector field $K \in \Gamma(TC(M))$, the function f_K is constant along the direction ∂_r.*

Proof. Since $\overline{\nabla}_{\partial_r}\partial_r = 0$, we have

$$\begin{aligned}
\partial_r f_K &= \overline{g}((\overline{\nabla}_{\partial_r} M_K)\partial_r, J\partial_r) \\
&= \frac{1}{r^3}\overline{\mathrm{Rm}}(r\partial_r, K, r\partial_r, Jr\partial_r) \\
&= -\frac{1}{r^3}\overline{\mathrm{Rm}}(r\partial_r, Jr\partial_r, K, r\partial_r) \\
&= 0
\end{aligned}$$

by Lemma 6.10. □

We prove the following.

Theorem 6.13. *For any Legendrian minimal immersion $L^n \to M$ in a Sasaki-Einstein manifold, and for any both holomorphic and Killing vector field on the Kähler cone $K \in \Gamma(TC(M))$, then the functions f_K defined by (6.8) are eigenfunctions of Δ_L with eigenvalue $2n+2$.*

Proof. We fix a vector field K as in the statement and we set $f = f_K$. In order to compute $\Delta_L f$, we notice that Lemma 6.12 allows us to apply Lemma 6.8. Thus, let $\{e_1, \cdots, e_n\}$ be a local frame of L.

We begin with observing that, for any such vector field e_i, then there holds

$$e_i f = \frac{2}{r}\overline{g}(M_K\partial_r, Je_i). \tag{6.9}$$

In fact,

$$\begin{aligned}
e_i f &= \overline{g}((\overline{\nabla}_{e_i} M_K)\partial_r, J\partial_r) + \overline{g}(M_K\overline{\nabla}_{e_i}\partial_r, J\partial_r) + \overline{g}(M_K\partial_r, J\overline{\nabla}_{e_i}\partial_r) \\
&= \overline{g}(\overline{\mathrm{Rm}}(e_i, K)\partial_r, J\partial_r) + 2\overline{g}(M_K\partial_r, J\overline{\nabla}_{e_i}\partial_r) \\
&= \frac{2}{r}\overline{g}(M_K\partial_r, Je_i),
\end{aligned}$$

where at the second equality we applied Lemma 6.9.(ii) and (iv), at the third equality we applied Lemma 6.10 and [71, (1.1)]. Similarly as for (6.9), we also get

$$\nabla_{e_i} e_i f = \frac{2}{r}\overline{g}(M_K\partial_r, J\nabla_{e_i} e_i). \tag{6.10}$$

Now we compute

$$\begin{aligned}
e_ie_if &= e_i\left(\frac{2}{r}\overline{g}(M_K\partial_r, Je_i)\right)\\
&= \frac{2}{r}\left(\overline{g}((\overline{\nabla}_{e_i}M_K)\partial_r, Je_i) + \overline{g}(M_K\overline{\nabla}_{e_i}\partial_r, Je_i) + \overline{g}(M_K\partial_r, J\overline{\nabla}_{e_i}e_i)\right)\\
&= \frac{2}{r}\left(\overline{g}(\overline{\mathrm{Rm}}(e_i, K)\partial_r, Je_i) + \frac{1}{r}\overline{g}(M_Ke_i, Je_i)\right.\\
&\qquad\qquad \left. + \overline{g}(M_K\partial_r, J\nabla_{e_i}e_i) - \overline{g}(M_K\partial_r, Jr\partial_r)\right)\\
&= \frac{2}{r^2}\overline{g}(M_Ke_i, Je_i) + \frac{2}{r}\overline{g}(M_K\partial_r, J\nabla_{e_i}e_i) - 2\overline{g}(M_K\partial_r, J\partial_r),
\end{aligned}$$

where at the third equality we applied Lemma 6.9.iv and [71, (1.1)], at the third equality we applied Lemma 6.10 and [71, (1.1)].

Finally we compute

$$\begin{aligned}
\Delta_Lf|_L &= -\sum_{i=1}^n \overline{\nabla}df(e_i, e_i)|_L\\
&= -\sum_{i=1}^n (e_ie_if - \overline{\nabla}_{e_i}e_if)|_L\\
&= -\sum_{i=1}^n (e_ie_if - \nabla_{e_i}e_if + r\partial_rf)|_L\\
&= -\sum_{i=1}^n \left(\frac{2}{r^2}\overline{g}(M_Ke_i, Je_i) + \frac{2}{r}\overline{g}(M_K\partial_r, J\nabla_{e_i}e_i)\right.\\
&\qquad\qquad \left.\left. - 2\overline{g}(M_K\partial_r, J\partial_r) - \frac{2}{r}\overline{g}(M_K\partial_r, J\nabla_{e_i}e_i)\right)\right|_L\\
&= (2n+2)f|_L,
\end{aligned}$$

where at the third equality we applied [71, (1.1)], at the fourth equality we applied Lemma 6.12 and (6.10), at the fifth equality we applied Lemma 6.11. This completes the proof of the theorem. □

Remark 6.14. Let us see how to recover the functions of Lê-Wang in this setting. Let $M \in \mathfrak{su}(n+1)$ and consider it as a real $(2n+2) \times (2n+2)$ matrix. It is skew-symmetric and such that $\mathrm{tr}(JM) = 0$. Consider the vector field on $\mathbb{C}^{n+1}$ given at x by $K_x = Mx$, which is Killing and holomorphic. We claim that the function f_K is exactly the function $\langle Mx, Jx\rangle$. Indeed, if $\overline{\nabla}$ is the flat connection on $\mathbb{C}^{n+1}$, it is $\overline{\nabla}_yK = My$ for $y \in \mathbb{C}^{n+1}$. Moreover $\mathrm{div}(JK) = \mathrm{tr}(JU) = 0$. So $f_K = \langle Mx, Jx\rangle$ after identifying x with $\partial_r|_{(x,1)}$.

Let us now see the connection between our two different generalizations. It is known that there is an inclusion $\mathfrak{aut}(M) \subseteq \mathfrak{aut}(C(M))$ of the algebra of infinitesimal Sasaki automorphisms of M into the algebra of infinitesimal Kähler automorphisms of the cone $C(M)$. It consists in seeing a field $V \in \mathfrak{aut}(M)$ trivially extended to the cone and it turns out to be holomorphic and Killing with respect to the cone metric.

We proved in Theorem 6.5 that for $X \in \mathfrak{aut}(M)$ the functions on L given by $\eta(X) - \frac{1}{\mathrm{vol}(L)} \int_L \eta(X) dv$ are eigenfunctions of Δ_L with eigenvalue $2n+2$, in the Sasaki-Einstein assumption. By seeing X as an infinitesimal Kähler automorphism of $C(M)$ we compute

$$M_X \partial_r = \frac{1}{r}\left[\overline{\nabla}_{r\partial_r} X + \frac{1}{2n+2}\,\mathrm{div}(JX)\xi\right]$$

and $J\partial_r = \frac{1}{r}\xi$. Taking their inner product we have

$$f_X = \overline{g}(M_X\partial_r, J\partial_r) = \frac{1}{r^2}\overline{g}(X,\xi) + \frac{1}{2n+2}\,\mathrm{div}(JX) = \eta(X) + \frac{1}{2n+2}\,\mathrm{div}(JX). \tag{6.11}$$

Using Theorem 6.5 together with Theorem 6.13 we have, after applying the Laplacian to (6.11), that

$$f_X = \eta(X) - \frac{1}{\mathrm{vol}(L)} \int_L \eta(X) dv. \tag{6.12}$$

Hence our second generalization extends the first.

In the Lê-Wang setting, we reobtain the fact that $\int_L \eta(X) dv = 0$, which is a fortiori true being $\eta(X)$ an eigenfunction of Δ_L.

Chapter 7

Space of Kähler and Sasakian metrics

In this chapter we deal with spaces of Kähler and Sasakian metrics. We answer a question by Calabi about the existence of geodesics of the *gradient metric* for the space of Kähler potentials for any initial position and velocity. Moreover we compute that the Ebin metric restricted to the space of type II deformations of a fixed Sasakian manifold is the sum of the *Calabi* metric and the *gradient metric*. We study its geometric properties and we prove the existence of short time geodesics as well. This work was done jointly with S. Calamai and K. Zheng [19].

7.1 Introduction to the problem

The idea of defining a Riemannian structure on the space of all metrics on a fixed manifold goes back to the sixties with the work of Ebin [32]. His work concerns the pure Riemannian setting and, among other things, defines a weak Riemannian metric on the space $\mathcal{M}$ of all Riemannian metrics on a fixed Riemannian manifold (M, g_0). The geometry of the Hilbert manifold $\mathcal{M}$ was later studied by Freed and Groisser in [34] and Gil-Medrano and Michor in [42]. In particular the curvature and the geodesics of $\mathcal{M}$ were computed.

When the underlying M has additional structure, some subsets of $\mathcal{M}$ were also studied. For instance when (M, ω) is Kähler of complex dimension n, the space of all Kähler metrics cohomologous to ω is of interest. By the dd^c-Lemma it can be parameterized by Kähler potentials, namely one considers the space $\mathcal{H}_K$ of all smooth real-valued φ such that $\omega_\varphi := \omega + dd^c\varphi > 0$ and satisfying a certain normalization condition. The tangent space of $\mathcal{H}$ at φ is then given by

$$T_\varphi \mathcal{H}_K = \left\{ \psi \in C^\infty(M) : \int_M \psi d\mu_\varphi = 0 \right\}$$

where $d\mu_\varphi = \omega_\varphi^n / n!$ is the Kähler volume form.

The first attempt to define a weak metric on $\mathcal{H}$ is first due to Mabuchi, Semmes and Donaldson, in [57, 67, 30] and defines a pairing on the tangent space of $\mathcal{H}_K$ at φ given by

$$\langle \psi_1, \psi_2 \rangle_\varphi = \int_M \psi_1 \psi_2 d\mu_\varphi.$$

We shall refer to this metric as the *Mabuchi metric*.

The geometry of the Mabuchi metric was studied in [30] where its curvature was computed and it was proved that it gives, formally, $\mathcal{H}_K$ the structure of a locally symmetric space.

Unlike finite dimensional Riemannian geometry, the Cauchy problem for the geodesics, i.e. the problem to find a geodesic leaving an assigned point with assigned direction, is not always solvable and it was proved by X. X. Chen [24] that there is $C^{1,1}$ regularity for the Dirichlet problem, namely the problem to find a geodesic joining two assigned points.

The space $\mathcal{H}_K$ can also be endowed with two different other metrics, known as the Calabi metric and the gradient (or Dirichlet) metric.

The former goes back to Calabi and it was later studied by Calamai in [16] where its Levi-Civita covariant derivative is computed, it is proved that it is of constant sectional curvature, that $\mathcal{H}$ is then isometric to a portion of a sphere and that both the Cauchy and Dirichlet problems admit smooth explicit solutions.

The latter was again proposed by Calabi and later worked again in [16] and later in [20] by Calamai and Zheng. Its Levi-Civita connection and curvature are known but the solvability of the geodesic equation was not, except for Riemann surfaces.

Since $\mathcal{H}_K$ naturally embeds in the Ebin space $\mathcal{M}$, it is natural to ask what the restriction of the Ebin metric is. It was proved by Clarke and Rubinstein [29] that such restriction is exactly twice the Calabi metric.

In this chapter we consider on $\mathcal{H}_K$ the metric given by (twice) the sum of the Calabi and the gradient metric and we will refer to it as the *sum metric*. Its study is justified by the fact that it arises when restricting the Ebin metric to the space of *Sasakian* metrics, introduced and endowed with the Sasakian analogue of the Mabuchi metric, in [45, 46] (see also [48]).

One of our results is the following.

Proposition 7.1. *The restriction of the Ebin metric to the space of Sasakian metrics is twice the sum metric.*

Then we move on to the Cauchy problem for the geodesics of the sum metric in the Riemann surfaces setting proving, after a Riemann surfaces warmup, the following.

Theorem 7.2. *On a Kähler manifold, for every initial smooth Kähler potential φ_0, smooth $\psi_0 \in T_{\varphi_0}\mathcal{H}_K$, integer $k \geq 2$ and $\alpha \in (0,1)$ there exists, for small enough time, a unique $C^{k,\alpha}$ geodesic for the sum metric, starting from φ_0 with initial velocity ψ_0.*

This result of local existence is achieved with a metric space contraction technique. Such approach is also used to prove in the same way the short time existence for geodesics of the Dirichlet metric.

Theorem 7.3. *On a Kähler manifold, for any initial data (φ_0, ψ_0) and k, α as above, there exists for small enough time a unique $C^{k,\alpha}$ geodesic for the Dirichlet metric, starting from φ_0 with initial velocity ψ_0.*

We stress that the existence of *smooth* geodesics for either the Dirichlet metric or the sum metric is still an open problem.

These estimates done can be generalized to the Sasakian setting, leading to the corresponding statements for the restriction of the Ebin metric to the space of Sasakian metrics and for the Dirichlet metric on the same space.

7.2 Preliminaries

In this section we recall the definitions of space of metrics and we define several weak Riemannian structures on them. As anticipated in the introduction, the first one goes back to Ebin [32] and starts with a closed Riemannian manifold (M, g). The space $\mathcal{M}$ is defined to be the space $S^2_+(T^*M)$ of all symmetric positive $(0,2)$-tensors on M. The formal tangent space at $g_0 \in \mathcal{M}$ is then given by all symmetric $(0,2)$-tensors $S^2(T^*M)$. This makes $\mathcal{M}$ parallelizable. For $a, b \in T_{g_0}\mathcal{M}$ Ebin defined the pairing

$$\langle a, b\rangle_{g_0} = \int_M g_0(a,b)dv_{g_0}$$

where $g_0(a,b)$ is the metric g_0 extended to $(0,2)$-tensors and dv_{g_0} is the volume form of g_0.

Moving on to Kähler manifolds, let (M, ω_0) be a closed Kähler manifold of complex dimension n and let $\mathcal{H}_K$ be the space defined in the introduction.

We will consider on $\mathcal{H}_K$ the following weak Riemannian structures, other than the Mabuchi metric already introduced above.

Let $\varphi \in \mathcal{H}_K$ and $\psi_1, \psi_2 \in T_\varphi \mathcal{H}_K$. We define the pairing

$$g^C(\psi_1, \psi_2)_\varphi = \int_M \Delta_\varphi \psi_1 \Delta_\varphi \psi_2 d\mu_\varphi \tag{7.1}$$

where, here and in the rest of the chapter, the laplacian is defined as $\Delta_\varphi f = (dd^c f, \omega_\varphi)_\varphi$. This is the opposite of the Hodge-deRham laplacian.

This metric has been introduced by Calabi and hence known as the *Calabi metric*. Its geometry has been studied by Calamai in [16]. In such paper the author exploited the Calabi volume conjecture which establish a bijection between $\mathcal{H}_K$ and the space of *conformal volume forms*

$$\mathcal{C} = \left\{ u \in C^\infty(M) : \int_M e^u d\mu_0 = \mathrm{vol} \right\} \tag{7.2}$$

that is the space of positive smooth functions on M whose integral with respect to the initial measure is equal to the volume of M (which is constant for all metrics in $\mathcal{H}_K$). The map is given by $\mathcal{H}_K \ni \varphi \mapsto \log \frac{\omega_\varphi^n}{\omega_0^n}$, where $\frac{\omega_\varphi^n}{\omega_0^n}$ represents the unique positive function f such that $\omega_\varphi^n = f\omega_0^n$. The tangent space $T_u\mathcal{C}$ is then given by

$$T_u\mathcal{C} = \left\{ v \in C^\infty(M) : \int_M v e^u d\mu_0 = 0 \right\}$$

where the Calabi metric assumes the simpler form

$$g^C(v_1, v_2)_u = \int_M v_1 v_2 e^u d\mu_0. \tag{7.3}$$

The geometry studied in [16] is actually the one of $\mathcal{C}$ with the metric (7.3). It is proved that it admits a Levi-Civita covariant derivative, that its sectional curvature is constant equal to $\frac{1}{4\,\mathrm{vol}}$ and that the Cauchy and Dirichlet problems for its geodesics admit a smooth explicit solution.

Note that this space can be defined for all Riemannian M forgetting the Kähler structure.

Another possible pairing on $T_\varphi \mathcal{H}_K$ is the following

$$g^G(\psi_1, \psi_2)_\varphi = \int_M (d\psi_1, d\psi_2)_\varphi d\mu_\varphi \tag{7.4}$$

that is, the global $L^2(d\mu_\varphi)$-product of the gradients of ψ_1 and ψ_2. This is known as the *gradient (or Dirichlet) metric*. This was also introduced by

Calabi and later studied for Riemann surfaces by Calamai in [16] and then by Calamai and Zheng in [20] in more generality.

Recall from above that the *sum metric* is defined to be $g = g^C + g^G$.

Given an initial Sasakian manifold $(M, \eta_0, \xi_0, \Phi_0, g_0)$, as explained in Chapter 3, basic functions parameterize a family of other Sasakian structures on M which share the same characteristic foliation and are in the same transverse Kähler class, in the following way.

Let $\varphi \in C^\infty_B(M)$ and define $\eta_\varphi = \eta_0 + d^c\varphi$. The space of all φ's is

$$\widetilde{\mathcal{H}} = \{\varphi \in C^\infty_B(M) : \eta_\varphi \wedge d\eta_\varphi \neq 0\}$$

and, in analogy of the Kähler case, we consider normalized "potentials"

$$\mathcal{H} = \{\varphi \in \widetilde{\mathcal{H}} : I(\varphi) = 0\}.$$

The equation $I = 0$ is a normalization condition, similar to the one in [30]. We refer to [45] for the definition of I in our case, which is such that

$$T_\varphi\mathcal{H} = \left\{\psi \in C^\infty_B(M) : \int_M \psi \frac{1}{n!}\eta_\varphi \wedge d\eta_\varphi^n = 0\right\}.$$

Every $\varphi \in \mathcal{H}$ defines a new Sasakian structure where the Reeb field and the transverse holomorphic structure are the same and, cf. Chapter 3,

$$\begin{aligned} \eta_\varphi &= \eta_0 + d^c\varphi \\ \Phi_\varphi &= \Phi_0 - (\xi \otimes d^c\varphi) \circ \Phi_0 \\ g_\varphi &= d\eta_\varphi \circ (\mathrm{id} \otimes \Phi_\varphi) + \eta_\varphi \otimes \eta_\varphi. \end{aligned} \tag{7.5}$$

Note that one could write $g_\varphi = d\eta_\varphi \circ (\mathrm{id} \otimes \Phi_0) + \eta_\varphi \otimes \eta_\varphi$ since the endomorphism $\Phi_\varphi - \Phi_0$ has values parallel to ξ and $d\eta_\varphi$ is basic. Indeed, the range of Φ_φ is the one of Φ_0 plus a component along ξ, so if we contract it with $d\eta$ the latter vanishes. As in the Kähler case, these deformations keep the volume of M fixed, which will be denoted by vol throughout the chapter.

On the space $\mathcal{H}$ one can define the Calabi metric and the gradient metric in the same ways as in formulae (7.1) and (7.4) by using the so called *basic laplacian* which acts on basic functions in the same way as in the Kähler case and by using the volume form $\frac{1}{n!}\eta_\varphi \wedge d\eta_\varphi^n$ in the integrals.

In this setting, it is easy to see that the map $\mathcal{H} \ni \varphi \mapsto \log \frac{\eta_\varphi \wedge d\eta_\varphi^n}{\eta_0 \wedge d\eta_0^n}$ maps basic functions to basic functions. The *transverse Calabi-Yau theorem* of [12] allows to prove the surjectivity of this map as in the Kähler case, more precisely between $\mathcal{H}$ and the space of *basic* conformal volume forms

$$\mathcal{C}_B = \left\{u \in C^\infty_B(M) : \int_M e^u \frac{1}{n!}\eta_0 \wedge d\eta_0^n = \mathrm{vol}\right\}.$$

As noted above, the space $\mathcal{C}$ can be defined also for Sasakian manifolds by just taking the Sasakian volume form $\frac{1}{n!}\eta_0 \wedge d\eta_0^n$ instead of the Kähler one. One might ask how the spaces $\mathcal{C}_B$ and $\mathcal{C}$ are related. Obviously $\mathcal{C}_B \subseteq \mathcal{C}$ but we can say more.

Proposition 7.4. *$\mathcal{C}_B$ is totally geodesic in $\mathcal{C}$.*

Proof. It is straightforward to verify that for any curve in $\mathcal{C}_B$ and section along it, the covariant derivative defined in [16] is still basic, meaning that the (formal) second fundamental form of $\mathcal{C}_B$ vanishes. □

7.3 The Ebin metric on the space of Sasakian metrics

Let $\mathcal{M}$ be the Ebin space of all Riemannian metrics on (M, g_0, ξ_0, η_0) Sasakian of dimension $2n+1$.

We define an immersion $\Gamma : \mathcal{H} \to \mathcal{M}$ that maps $\varphi \mapsto g_\varphi$ as defined in (7.5).

As in the Kähler case, it is injective. Indeed if two basic function $\varphi_1, \varphi_2 \in \mathcal{H}$ give rise to the same Sasakian metric, taking the corresponding transverse structures we would have $dd^c(\varphi_1 - \varphi_2) = 0$ forcing $\varphi_1 - \varphi_2 = \text{const}$. The normalization $I(\cdot) = 0$ then implies $\varphi_1 = \varphi_2$.

Let us compute the differential of Γ. Let $\varphi(t)$ be a curve in $\mathcal{H}$ with $\varphi(0) = \varphi$ and $\varphi'(0) = \psi \in T_\varphi\mathcal{H}$. Then

$$\Gamma_{*\varphi}\psi = \left.\frac{d}{dt}\right|_{t=0} g_{\varphi(t)} = dd^c\psi(\Phi_0 \otimes \mathrm{id}) + 2d^c\psi \odot \eta_\varphi \tag{7.6}$$

with the convention $a \odot b = \frac{1}{2}(a \otimes b + b \otimes a)$. For easier notation we call $\beta_\psi := dd^c\psi(\Phi_0 \otimes 1)$.

The differential of Γ is also injective. Indeed if ψ is in its kernel, then

$$0 = \Gamma_{*\varphi}\psi(\xi, \cdot) = d^c\psi,$$

forcing ψ to be zero, as it has zero integral.

On $T_g\mathcal{M}$ the Ebin metric is given by, for $a, b \in T_g\mathcal{M} = \Gamma(S^2M)$,

$$g_{\mathrm{E}}(a, b)_g = \int_M g(a, b) dv_g$$

where g is extended on $(0,2)$-tensors.

We want to compute the restriction of the Ebin metric on the space $\mathcal{H}$.

Proposition 7.5. *The restriction of the Ebin metric to $\mathcal{H}$ is twice the sum of the Calabi metric with the gradient metric*

$$\frac{1}{2}\Gamma^* g_{\mathrm{E}} = g_{\mathrm{C}} + g_{\mathrm{G}}$$

which we call the sum metric*.*

Proof. Computing the length with respect to g_φ of the tensor in (7.6) we get

$$\begin{aligned} |\beta_\psi + 2d^c\psi \odot \eta_\varphi|^2_{g_\varphi} &= g_\varphi(\beta_\psi, \beta_\psi) + 2g_\varphi(d^c\psi \otimes \eta_\varphi, d^c\psi \otimes \eta_\varphi) + 2g_\varphi(\beta_\psi, 2d^c\psi \odot \eta_\varphi) \\ &= g_\varphi(\beta_\psi, \beta_\psi) + 2g_\varphi(d^c\psi, d^c\psi)g_\varphi(\eta_\varphi, \eta_\varphi) + 2\beta_\psi((d^c\psi)^\sharp, \xi) \\ &= g_\varphi(\beta_\psi, \beta_\psi) + 2g_\varphi(d^c\psi, d^c\psi) \end{aligned}$$

using the fact that the g_φ-dual of η_φ is ξ, that the $\sharp$ is done with respect to g_φ and finally the fact that the tensor β_ψ is transverse, i.e. vanishes when evaluated on ξ.

Integrating with respect to $d\mu_\varphi$ we have

$$\langle \Gamma_{*\varphi}\psi, \Gamma_{*\varphi}\psi\rangle_\varphi = \|\beta_\psi\|^2_\varphi + 2\|d^c\psi\|^2_\varphi$$

where the right hand side are L^2 norms with respect to the metric g_φ. The second summand is twice the gradient metric on $\mathcal{H}$ given by

$$g_{\mathrm{G}}(\psi, \psi) = \int_M g_\varphi(d\psi, d\psi)\frac{1}{n!}\eta_\varphi \wedge d\eta^n_\varphi.$$

(It does not make a difference to take the gradient of the transverse gradient for basic functions).

We now want to establish a useful formula that we will need in a while. Fix $\varphi \in \mathcal{H}$ and $h \in T_\varphi\mathcal{H}$ we consider the curve $\varphi(t) = \varphi + th$ which is in $\mathcal{H}$ for small t. We then compute for every curve $f(t) \in T_\varphi\mathcal{H}$,

$$0 = \frac{d}{dt}|_{t=0}\int_M \Delta_{\varphi(t)} f\frac{1}{n!}\eta_{\varphi(t)}\wedge d\eta^n_{\varphi(t)} = \int_M (\Delta_\varphi f'(t) - (dd^c f, dd^c h)_\varphi + \Delta_\varphi f \Delta_\varphi h)\frac{1}{n!}\eta_\varphi \wedge d\eta^n_\varphi.$$

which means that $g_C(f,h)_\varphi = \int_M (dd^c f, dd^c h)_\varphi \frac{1}{n!}\eta_\varphi \wedge d\eta^n_\varphi$.

Then we have, since β_ψ is the (transverse) 2-tensor associated to the basic form $dd^c\psi$, whose pointwise norms are related by $|\beta_\psi|^2 = 2|dd^c\psi|^2$,

$$g_C(\psi, \psi) = \int_M (\Delta_\varphi\psi)^2\frac{1}{n!}\eta_\varphi \wedge d\eta^n_\varphi = \int_M (dd^c\psi, dd^c\psi)_{\omega_\varphi}\frac{1}{n!}\eta_\varphi \wedge d\eta^n_\varphi = \frac{1}{2}\|\beta_\psi\|^2_\varphi.$$

where Δ_φ was defined earlier. □

7.3.1 Metric space structure on $\mathcal{H}$

Let $g = \Gamma^* g_{\mathrm{E}}$ and consider the associated function for $p, q \in \mathcal{H}$

$$d(p,q) = \inf\{\mathrm{length}_g(\gamma) : \ \gamma \text{ is a curve in } \mathcal{H} \text{ from } p \text{ to } q\}$$

Let γ be a curve in $\mathcal{H}$ joining p and q. Its length with respect to our new metric is, by concavity of the square root: $\sqrt{a+b} \geq \frac{\sqrt{2}}{2}(\sqrt{a}+\sqrt{b})$,

$$\begin{aligned} \ell(\gamma) &= \int_0^1 \sqrt{2g_C(\dot\gamma,\dot\gamma) + 2g_G(\dot\gamma,\dot\gamma)}dt \\ &\geq \int_0^1 \sqrt{g_C(\dot\gamma,\dot\gamma)}dt + \int_0^1 \sqrt{g_G(\dot\gamma,\dot\gamma)}dt \\ &\geq d_C(p,q) + d_G(p,q). \end{aligned}$$

So $d_C(p,q) + d_G(p,q)$ is a lower bound of the set of lengths of γ as γ varies between the curves joining p to q. So we have

$$d_{\mathrm{C}} \leq d_{\mathrm{C}} + d_{\mathrm{G}} \leq d \tag{7.7}$$

We have the following.

Proposition 7.6. *$(\mathcal{H}, d)$ is a metric space and*

$$d_C + d_G \leq d \tag{7.8}$$

Proof. We need to prove that d is a distance. The symmetry and the triangle inequality are obvious from the definition. If $p, q \in \mathcal{H}$ are such that $d(p,q) = 0$ then by inequality (7.7) we would have $d_{\mathrm{C}}(p,q) = 0$ leading to $p = q$ as d_{C} is a distance on $\mathcal{H}$. □

7.4 The sum metric and its Levi-Civita connection

In this section we call Δ_φ the d-laplacian acting as $\Delta_\varphi = (dd^c f, \omega_\varphi)_\varphi$. Notice that it has *nonpositive* eigenvalues. With this laplacian the integration by parts reads

$$\int_M (df, dg)_\varphi \frac{1}{n!}\eta_\varphi \wedge d\eta_\varphi^n = -\int_M f\Delta_\varphi g \frac{1}{n!}\eta_\varphi \wedge d\eta_\varphi^n.$$

Consider on $\mathcal{H}$ the metric $2g_C + 2g_G$. It can be written, for $\varphi \in \mathcal{H}$ and $\alpha, \beta \in T_\varphi\mathcal{H}$,

$$g(\alpha,\beta) = 2\int_M \Delta_\varphi\alpha\Delta_\varphi\beta\frac{1}{n!}\eta_\varphi \wedge d\eta_\varphi^n - 2\int_M \alpha\Delta_\varphi\beta\frac{1}{n!}\eta_\varphi \wedge d\eta_\varphi^n$$

$$= 2\int_M \Delta_\varphi(\alpha - G_\varphi\alpha)\Delta_\varphi\beta\frac{1}{n!}\eta_\varphi \wedge d\eta_\varphi^n$$
$$= g_C(L_\varphi\alpha, \beta)$$

where $L_\varphi = 2(I - G_\varphi)$ with G_φ the Green operator associated to Δ_φ.

Note that the G_φ acting on functions with zero integral with respect to $d\mu_\varphi$ is the inverse of Δ_φ, since the projection on the space of harmonic functions is $H_\varphi : f \mapsto \frac{1}{\mathrm{vol}_\varphi}\int_M f\frac{1}{n!}\eta_\varphi \wedge d\eta_\varphi^n = 0$ and because of the known relation $I = H_\varphi + \Delta_\varphi G_\varphi$.

We have the first property.

Lemma 7.7. *For all $\varphi \in C_B^\infty$, the operator $L_\varphi \in \mathrm{End}(T_\varphi\mathcal{H})$ is bijective.*

Proof. Since L_φ defines a metric, it must have no kernel. The solvability of the problem $u - G_\varphi u = f$ is equivalent to the solvability of $\Delta_\varphi u - u = h$. We then consider the operator $\Delta_\varphi - I$ which is elliptic and self-adjoint with respect to the L^2 product. The space of smooth functions is then split as (by the results of [33] about transversally elliptic operators)

$$C_B^\infty(M) = \ker(\Delta_\varphi - I) \oplus \mathrm{Im}(\Delta_\varphi - I).$$

By the remark about eigenvalues done at the beginning we see that 1 does not belong to the spectrum of Δ_φ so in particular $C_B^\infty(M) \cap T_\varphi\mathcal{H} = \mathrm{Im}(\Delta_\varphi - I) \cap T_\varphi\mathcal{H}$. □

We are now ready to write down the Levi-Civita covariant derivative of g using the invertibility of L_φ.

Theorem 7.8. *For any curve φ in $\mathcal{H}$ and any section v on φ, the only solution $D_t v$ of*

$$\frac{1}{2}L_\varphi D_t v = D_t^C v - G_\varphi D_t^G v \tag{7.9}$$

is the Levi-Civita covariant derivative of g, i.e. it is torsion free and

$$\frac{d}{dt}g(v,v) = 2g(D_t v, v). \tag{7.10}$$

Proof. The fact that this covariant derivative is torsion-free is evident from its definition. To prove (7.10) we notice that the gradient metric satisfies $g_G = -g_C(G_\varphi\cdot, \cdot)$. We compute for a curve φ and a section v on it

$$\begin{aligned}\frac{d}{dt}g(v,v) &= 4g_C(D_t^C v, v) + 4g_G(D_t^G v, v)\\ &= 4g_C(D_t^C v - G_\varphi D_t^G v, v)\\ &= 2g_C(L_\varphi D_t v, v)\\ &= 2g(D_t v, v)\end{aligned}$$

by using the implicit definition of $D_t v$. □

7.4.1 The geodesic equation

The geodesic equation is then

$$\Delta_\varphi^2 D_t^C \varphi' - \Delta_\varphi D_t^G \varphi' = 0 \tag{7.11}$$

which is rewritten as

$$(\Delta_\varphi - I)\left((\Delta_\varphi \varphi')' + \frac{1}{2}(\Delta_\varphi \varphi')^2\right) - \frac{1}{2}|dd^c \varphi'|_\varphi^2 = 0. \tag{7.12}$$

Remark 7.9. It is important to notice here that if we consider the sum metric $g^C + g^G$ on the space of Kähler potentials $\mathcal{H}_K$ (although we do not get that it is the pull back of the Ebin metric) we find exactly the same equation (7.12) for the geodesics. This is because the variation of volume form follows the same rule in both cases and the basic Laplacian corresponds to the Laplacian in the Kähler setting.

Remark 7.10. It is clear that a curve φ which is a geodesic for both the Calabi and the gradient metric would be a geodesic for our metric as well. Unfortunately there are no such nontrivial curves, indeed one of them would satisfy

$$\begin{cases} (\Delta_\varphi \varphi')' + \frac{1}{2}(\Delta_\varphi \varphi')^2 - \frac{1}{2V} g_C(\varphi', \varphi') = 0 \\ (\Delta_\varphi \varphi')' + (\Delta_\varphi \varphi')^2 + \Delta_\varphi \varphi'' = 0 \end{cases}$$

Subtracting the first to the second we get

$$\frac{1}{2}(\Delta_\varphi \varphi')^2 + \Delta_\varphi \varphi'' + \frac{1}{2V} g_C(\varphi', \varphi').$$

Integrating it we would infer $g_C(\varphi', \varphi') = 0$ meaning that φ is the constant curve.

7.5 The Cauchy problem for the geodesic equation of the sum metric

7.5.1 The case of Riemann surfaces

Let us recall some notation. Let (M, ω) be a smooth Riemann surface endowed with a Kähler metric ω; let $\mathcal{H}_K = \{\varphi \in C^\infty(M) : \omega + dd^c\varphi > 0, I(\varphi) = 0\}$, where the condition $I(\varphi) = 0$ is the Donaldson normalization (see [30]), be the space of Kähler potentials and let us introduce also the following function spaces $\mathcal{H}_K^{k,\alpha} = \{\varphi \in C^{k,\alpha}(M) : \omega + dd^c\varphi > 0, \int_M \psi d\mu_\varphi = 0\}$ and

$\mathcal{H}^{k,\alpha}_{K,\delta} = \{\varphi \in C^{k,\alpha}(M) : \omega + dd^c\varphi \geq \delta, I(\varphi) = 0\}$, where $k \geq 2$ and $\alpha \in (0,1)$; at the present stage we already proved that the geodesic equation for the sum metric $g_C + g_G$, where the underlying manifold is n-dimensional Kähler, is the following equation:

$$(\Delta_\varphi - I)\Big((\Delta_\varphi\varphi')' + \frac{1}{2}(\Delta_\varphi\varphi')^2\Big) - \frac{1}{2}|dd^c\varphi'|^2_\varphi = 0, \tag{7.13}$$

where we recall here that the elliptic operator Δ_φ has non-positive spectrum. Our main result is the following

Theorem 7.11. *For every $\varphi_0 \in \mathcal{H}_K$, $\psi_0 \in T_{\varphi_0}\mathcal{H}_K$ and integer $k \geq 2$ and $\alpha \in (0,1)$ there exists a positive ε and a curve $\varphi \in C^2([-\varepsilon,\varepsilon], \mathcal{H}^{k,\alpha}_K)$ which is the unique solution of* (7.13) *with initial data (φ_0, ψ_0).*

Remark 7.12. It is an open question whether we can find geodesics which solve the Cauchy problem and belong to the function space $C^2([-\varepsilon,\varepsilon], \mathcal{H}_K)$, that is, at any time they are *smooth* Kähler potentials. We also ask whether the long time existence of our geodesics holds.

The first remark that we want to make is about the simplification of the equation (7.13) when the underlying manifold is a Riemann surface.

Lemma 7.13. *When the underlying manifold is a Riemann surface (M, ω), then the geodesic equation* (7.13) *simplifies into*

$$\varphi'' = \frac{1}{2}(\Delta_\varphi - I)^{-1}(\Delta_\varphi\varphi')^2 + \frac{1}{2V}\int_M (\Delta_\varphi\varphi')^2 d\mu_\varphi - \frac{1}{V}\int_M \varphi'\Delta_\varphi\varphi' d\mu_\varphi = 0 \tag{7.14}$$

where V is the volume of (M, ω).

Proof. For any Riemann surface and any Kähler potential φ, we have that

$$(\Delta_\varphi\varphi')^2 = |dd^c\varphi'|^2_\varphi, \tag{7.15}$$

so that we can first rewrite (7.13) as

$$(\Delta_\varphi - I)\Big(\Delta_\varphi\varphi'' - (\Delta_\varphi\varphi')^2 + \frac{1}{2}(\Delta_\varphi\varphi')^2\Big) - \frac{1}{2}(\Delta_\varphi\varphi')^2 = 0,$$

which is

$$\Delta_\varphi\Big(\Delta_\varphi\varphi'' - \frac{1}{2}(\Delta_\varphi\varphi')^2 - \varphi''\Big) = 0; \tag{7.16}$$

now, using that M is compact and without boundary, we get by integration that

$$\Delta_\varphi \varphi'' - \frac{1}{2}(\Delta_\varphi \varphi')^2 - \varphi'' + \frac{1}{2V}\int_M (\Delta_\varphi \varphi')^2 d\mu_\varphi - \frac{1}{V}\int_M \varphi' \Delta_\varphi \varphi' d\mu_\varphi = 0. \tag{7.17}$$

Finally, using that the elliptic operator Δ_φ has non-positive spectrum, we get that $\Delta_\varphi - I$ is invertible and

$$\varphi'' = \frac{1}{2}(\Delta_\varphi - I)^{-1}(\Delta_\varphi \varphi')^2 + \frac{1}{2V}\int_M (\Delta_\varphi \varphi')^2 d\mu_\varphi - \frac{1}{V}\int_M \varphi' \Delta_\varphi \varphi' d\mu_\varphi. \tag{7.18}$$

In fact, since the integral terms do not depend upon spatial variables, we have

$$(\Delta_\varphi - I)^{-1}\left(-\frac{1}{2V}\int_M (\Delta_\varphi \varphi')^2 d\mu_\varphi\right) = \frac{1}{2V}\int_M (\Delta_\varphi \varphi')^2 d\mu_\varphi$$

and similarly

$$(\Delta_\varphi - I)^{-1}\left(\frac{1}{V}\int_M \varphi' \Delta_\varphi \varphi' d\mu_\varphi\right) = -\frac{1}{V}\int_M \varphi' \Delta_\varphi \varphi' d\mu_\varphi.$$

This completes the proof of the lemma. □

Now that we have the geodesic equation in the form (7.14), we observe that it can be written as a system of two PDE as follows

$$\begin{cases} \varphi' = \psi; \\ \psi' = \frac{1}{2}(\Delta_\varphi - I)^{-1}(\Delta_\varphi \psi)^2 + \frac{1}{2V}\int_M (\Delta_\varphi \psi)^2 d\mu_\varphi - \frac{1}{V}\int_M \psi \Delta_\varphi \psi d\mu_\varphi. \end{cases} \tag{7.19}$$

In particular the integral form of the above system is telling us that a geodesic (φ, ψ) with initial data (φ_0, ψ_0) has the property that

$$\begin{cases} \varphi = \varphi_0 + \int_0^t \psi(s) ds; \\ \psi = \psi_0 + \int_0^t \left(\frac{1}{2}(\Delta_\varphi - I)^{-1}(\Delta_\varphi \psi)^2 + \frac{1}{2V}\int_M (\Delta_\varphi \psi)^2 d\mu_\varphi - \frac{1}{V}\int_M \psi \Delta_\varphi \psi d\mu_\varphi \right) ds. \end{cases} \tag{7.20}$$

The structure of the system (7.20) suggests to consider the following complete metric space

$$C^0([-\varepsilon, \varepsilon], \mathcal{H}^{k,\alpha}_{K,\delta}) \times C^0([-\varepsilon, \varepsilon], C^{k,\alpha}(M)) \tag{7.21}$$

as the function space where we are going to look for solutions of our system. The norm that we consider is defined for $\psi \in C^0([-\varepsilon,\varepsilon], C^{k,\alpha})$ as

$$|\psi|_{k,\alpha} := \sup_{t\in[-\varepsilon,\varepsilon]} \|\psi(t,\cdot)\|_{C^{k,\alpha}(M)},$$

and in the product space, the norm of any element $(\varphi,\psi) \in C^0([-\varepsilon,\varepsilon], \mathcal{H}^{k,\alpha}_{K,\delta}) \times C^0([-\varepsilon,\varepsilon], C^{k,\alpha}(M))$ is

$$|(\varphi,\psi)|_{k,\alpha} := |\varphi|_{k,\alpha} + |\psi|_{k,\alpha}.$$

Before getting into more specific arguments about (7.20) we need some lemmas.

Lemma 7.14. *Let $\chi,\varphi,\varphi_0 \in C^0([-\varepsilon,\varepsilon], \mathcal{H}^{k,\alpha}_K)$, with $k \geq 2$, $|\varphi-\varphi_0|_{k,\alpha} < r$ and $|\chi-\varphi_0|_{k,\alpha} < r$. Let $\psi \in C^0([-\varepsilon,\varepsilon], C^{k,\alpha}(M))$. Then there exists $\nu > 0$ such that, if $|\varphi-\chi|_{k,\alpha} < \nu$, then*

$$|\Delta_\varphi\psi - \Delta_\chi\psi|_{k-2,\alpha} < C(\omega,\varphi_0,r,M,k,\alpha)\cdot|\varphi-\chi|_{k,\alpha}\cdot|\psi|_{k,\alpha}. \tag{7.22}$$

In particularly,

$$|\Delta_\varphi\psi|_{k-2,\alpha} < C(\omega,\varphi_0,r,M,k,\alpha)\cdot|\varphi|_{k,\alpha}\cdot|\psi|_{k,\alpha}.$$

Proof. Let us fix an open coordinate chart on the Riemann surface. Then we can express the Laplacians as

$$\begin{aligned}
&\Delta_\varphi\psi - \Delta_\chi\psi\\
&= \left(\frac{1}{g_{1\bar 1}+\varphi_{1\bar 1}} - \frac{1}{g_{1\bar 1}+\chi_{1\bar 1}}\right)\psi_{1\bar 1}\\
&= \left(\frac{\chi_{1\bar 1}-\varphi_{1\bar 1}}{(g_{1\bar 1}+\varphi_{1\bar 1})(g_{1\bar 1}+\chi_{1\bar 1})}\right)\psi_{1\bar 1},
\end{aligned}$$

where subscripts 1 and $\bar 1$ stand for, respectively, $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar z}$. Next we observe that for a $k-2$-differentiable function f, the directive of f^{-1} is combination of the derivatives of f up to $k-2$-orders, from which we infer that

$$\begin{aligned}
\left|\frac{1}{g_{1\bar 1}+\varphi_{1\bar 1}}\right|_{k-2,\alpha} &\leq C(\omega, |g_{1\bar 1}+\varphi_{1\bar 1}|_{C^{k-2,\alpha}})\\
&= C(\omega, |g_{1\bar 1}+\varphi_{01\bar 1}|_{C^{k-2,\alpha}} + |\varphi_{01\bar 1}-\varphi_{1\bar 1}|_{C^{k-2,\alpha}})\\
&= C(\omega,\varphi_0,r).
\end{aligned} \tag{7.23}$$

Now we conclude, by means of [43, (4.7) page 53], that

$$\begin{aligned}&\Big|\Delta_\varphi\psi - \Delta_\chi\psi\Big|_{k-2,\alpha}\\&\leq |\chi-\varphi|_{k,\alpha}\cdot\left|\frac{1}{g_{1\bar{1}}+\varphi_{1\bar{1}}}\right|_{k-2,\alpha}\cdot\left|\frac{1}{g_{1\bar{1}}+\chi_{1\bar{1}}}\right|_{k-2,\alpha}\cdot|\psi|_{k,\alpha},\end{aligned}$$

which, arguing on an open covering of M and using (7.23), gives the claimed formula (7.22). This completes the proof of the lemma. □

The following preliminary result is a direct consequence of a well-known result in elliptic operators theory.

Lemma 7.15. *Let $\varphi, \in C^0([-\varepsilon,\varepsilon],\mathcal{H}_K^{k,\alpha})$, with $k\geq 2$. For a sufficiently small $r>0$ for which there holds $|\varphi-\varphi_0|_{k,\alpha}<r$, then on any $\psi\in C^0([-\varepsilon,\varepsilon],C^{k,\alpha}(M))$ we have the following estimate*

$$|(\Delta_\varphi - I)^{-1}\psi|_{k,\alpha}\leq C_S(\omega,\varphi_0,r)|\psi|_{k-2,\alpha}. \tag{7.24}$$

Proof. The statement is a version of the Schauder estimates as stated in [43, Theorem 6.19, page 111]. When r is sufficiently small, the condition $|\varphi-\varphi_0|_{k,\alpha}<r$ guarantees that the smallest eigenvalue of the family of elliptic operators $\Delta_{\varphi(t,\cdot)}$ for every $t\in[-\varepsilon,\varepsilon]$ is bounded by a positive λ. Moreover, all the coefficients of the whole operator $\Delta_\varphi - I$ are bounded in the norm $C^{k-2,\alpha}$ as required in [43, Theorem 6.19, page 111]. That theorem gives us that

$$|(\Delta_\varphi - I)^{-1}\psi|_{k,\alpha}\leq C_S(\omega,\varphi_0,r)\cdot(|(\Delta_\varphi - I)^{-1}\psi|_{L^\infty}+|\psi|_{k-2,\alpha}).$$

We can get rid of the term $|(\Delta_\varphi - I)^{-1}\psi|_{k-2,\alpha}$ by means of [8, 27 Theorem, page 463] and the normalisation condition of ψ i.e. $\int_M\psi\frac{1}{n!}\eta_\varphi\wedge d\eta_\varphi^n=0$; in fact the operator $\Delta_\varphi - I$ has zero kernel since the spectrum of Δ_φ is non-positive. Thus we find the claimed formula (7.24) and this completes the proof of the lemma. □

Let us isolate the next result

Proposition 7.16. *For any $(\varphi_0,\psi_0)\in\mathcal{H}_{K,\delta}^{k,\alpha}\times C^{k,\alpha}(M)$ there exist positive numbers $\varepsilon>0$ and $r>0$ such that the closed metric ball of center (φ_0,ψ_0) and radius r*

$$B_r(\varphi_0,\psi_0)\subseteq C^0([-\varepsilon,\varepsilon],\mathcal{H}_{K,\delta}^{k,\alpha})\times C^0([-\varepsilon,\varepsilon],C^{k,\alpha}(M)) \tag{7.25}$$

is mapped into itself by the application

$$
\begin{aligned}
&T(\varphi,\psi) \qquad (7.26)\\
&=\bigg(\varphi_0+\int_0^t\psi(s)ds,\psi_0+\int_0^t\bigg(\frac{1}{2}(\Delta_\varphi-I)^{-1}(\Delta_\varphi\psi)^2\\
&\qquad\qquad+\frac{1}{2V}\int_M(\Delta_\varphi\psi)^2d\mu_\varphi-\frac{1}{V}\int_M\psi\Delta_\varphi\psi d\mu_\varphi\bigg)ds\bigg).
\end{aligned}
$$

Proof. Let us remark that we look at the functions φ_0,ψ_0 as constant in time and being defined for any time-interval; in particular, it makes sense to write $|\varphi_0|_{k,\alpha}$, which is equal to $\|\varphi_0\|_{C^{k,\alpha}(M)}$. Let us estimate

$$
\begin{aligned}
&\left|\varphi_0+\int_0^t\psi(s)ds-\varphi_0\right|_{k,\alpha}=\sup_{t\in[-\varepsilon,\varepsilon]}\left\|\int_0^t\psi(s)ds\right\|_{C^{k,\alpha}(M)}\leq\sup_{t\in[-\varepsilon,\varepsilon]}\int_0^t\|\psi(s)\|_{C^{k,\alpha}(M)}ds\\
&\leq\sup_{t\in[-\varepsilon,\varepsilon]}\int_0^t\sup_{s\in[-\varepsilon,\varepsilon]}\|\psi(s)\|_{C^{k,\alpha}(M)}ds\leq\varepsilon\cdot(|\psi_0|_{k,\alpha}+|\psi-\psi_0|_{k,\alpha})\leq\varepsilon\cdot(|\psi_0|_{k,\alpha}+r).
\end{aligned}
$$

We now move on to the estimate of the second component of the map T; in order to estimate the following

$$
\left|\psi_0+\int_0^t\bigg(\frac{1}{2}(\Delta_\varphi-I)^{-1}(\Delta_\varphi\psi)^2+\frac{1}{2V}\int_M(\Delta_\varphi\psi)^2d\mu_\varphi-\frac{1}{V}\int_M\psi\Delta_\varphi\psi d\mu_\varphi\bigg)ds-\psi_0\right|_{k,\alpha}, \tag{7.27}
$$

and having in mind just to perform triangular inequalities, we specialize to specific estimates of its addenda. Namely, we first estimate

$$
\left|\int_0^t\frac{1}{2V}\int_M(\Delta_\varphi\psi)^2d\mu_\varphi ds\right|_{k,\alpha}=\int_0^t\frac{1}{2V}\int_M(\Delta_\varphi\psi)^2d\mu_\varphi ds.
$$

The above equality just means that being the quantity to estimate independent from spatial variables, then its $C^{k,\alpha}$ norm is just its absolute value. This is useful in what follows

$$
\begin{aligned}
\frac{1}{2V}\int_M(\Delta_\varphi\psi)^2d\mu_\varphi&\leq\frac{1}{2V}\int_M|(\Delta_\varphi\psi)^2|_{k-2,\alpha}d\mu_\varphi\\
&\leq\frac{1}{2V}\int_M|(\Delta_\varphi\psi)|^2_{k-2,\alpha}d\mu_\varphi=\frac{1}{2}|(\Delta_\varphi\psi)|^2_{k-2,\alpha},
\end{aligned}
$$

where at the second inequality we used the formula [43, (4.7) page 53]. Now, by means of Lemma 7.14, we conclude

$$
\left|\int_0^t\frac{1}{2V}\int_M(\Delta_\varphi\psi)^2d\mu_\varphi ds\right|_{k,\alpha}\leq\varepsilon\cdot C(\omega,\varphi_0,r,\psi_0,k)(|\psi_0|_{k,\alpha}+r)^2. \tag{7.28}
$$

A similar argument works for the following addendum of (7.27)

$$\left|-\frac{1}{V}\int_0^t\int_M \psi\Delta_\varphi\psi d\mu_\varphi ds\right| \leq \varepsilon\cdot\frac{1}{V}\int_M |\psi\Delta_\varphi\psi|_{k-2,\alpha}d\mu_\varphi$$
$$\leq \varepsilon\cdot|\psi|_{k-2,\alpha}|\Delta_\varphi\psi|_{k-2,\alpha} \leq \varepsilon\cdot C(\omega,\varphi_0,r,\psi_0,k)(|\psi_0|_{k,\alpha}+r)^2,$$

where again we used Lemma 7.14 and [43, (4.7) page 53]. Now, we establish an estimate for the term of (7.27)

$$\left|\int_0^t \frac{1}{2}(\Delta_\varphi - I)^{-1}(\Delta_\varphi\psi)^2 ds\right|_{k,\alpha} \leq \varepsilon\cdot C_S(\omega,\varphi_0,r,k)|(\Delta_\varphi\psi)^2|_{k-2,\alpha}$$
$$\leq \varepsilon\cdot C_S(\omega,\varphi_0,r,k)|(\Delta_\varphi\psi)|^2_{k-2,\alpha}$$
$$\leq \varepsilon\cdot C_S(\omega,\varphi_0,r,k)\cdot C(r,\omega,\varphi_0,\psi_0,k)^2\cdot(|\psi_0|_{k,\alpha}+r)^2,$$

where C_S is the Schauder constant of Lemma 7.15. So, we can choose any r such that the two constants $C_S(r,\omega,\varphi_0,k)$, $C(r,\omega,\varphi_0,\psi_0,k)$ are finite, and then choose a suitable ε such that the estimates we provided bound the image of T by r. This completes the proof of the proposition. □

The next ingredient we need is the following.

Proposition 7.17. *For any $(\varphi_0,\psi_0)\in\mathcal{H}^{k,\alpha}_{K,\delta}\times C^{k,\alpha}(M)$ there exist positive numbers $\varepsilon>0$ and $r>0$ such that the closed metric ball $B_r(\varphi_0,\psi_0)$ as above is mapped into itself by the application defined in* (7.26) *and moreover the map T is a contraction.*

Proof. By Proposition 7.16 we already have a radius r_0 for which T maps the ball of that radius in itself. Moreover, for any other $r<r_0$ there is an ε such that the pair r,ε still does the job. Let us now fix any two pairs (φ,ψ) and (χ,ξ) in the ball centered at (φ_0,ψ_0), whose distance is, say, η, and let us estimate the quantity

$$|T(\varphi,\psi)-T(\chi,\xi)|_{k,\alpha}. \tag{7.29}$$

The estimate for the first component of the above expression is

$$\left|\int_0^t(\psi-\xi)ds\right|_{k,\alpha} \leq \varepsilon\cdot|\psi-\xi|_{k,\alpha}\leq\varepsilon\cdot\eta. \tag{7.30}$$

The second component of $|T(\varphi,\psi)-T(\chi,\xi)|_{k,\alpha}$ is

$$\left|\int_0^t\left(\frac{1}{2}(\Delta_\varphi-I)^{-1}(\Delta_\varphi\psi)^2+\frac{1}{2V}\int_M(\Delta_\varphi\psi)^2d\mu_\varphi-\frac{1}{V}\int_M\psi\Delta_\varphi\psi d\mu_\varphi\right.\right. \tag{7.31}$$

$$-\frac{1}{2}(\Delta_\chi - I)^{-1}(\Delta_\chi\xi)^2 - \frac{1}{2V}\int_M (\Delta_\chi\xi)^2 d\mu_\chi + \frac{1}{V}\int_M \xi\Delta_\chi\xi d\mu_\chi\Big)ds\Big|_{k,\alpha}.$$

The analysis of the estimates for (7.31) break into pairs of its addenda. We begin with the following estimate

$$\begin{aligned}
&\Big|\int_0^t \frac{1}{2V}\int_M ((\Delta_\varphi\psi)^2 d\mu_\varphi - (\Delta_\chi\xi)^2 d\mu_\chi)ds\Big| \\
&\le \Big|\int_0^t \frac{1}{2V}\int_M ((\Delta_\varphi\psi)^2 d\mu_\varphi - (\Delta_\varphi\psi)^2 d\mu_\chi + (\Delta_\varphi\psi)^2 d\mu_\chi - (\Delta_\chi\xi)^2 d\mu_\chi)ds\Big| \\
&\le \varepsilon\frac{1}{2V}\Big|\int_M (\Delta_\varphi\psi)^2 dd^c(\varphi-\chi)\Big| \\
&\qquad + \varepsilon\frac{1}{2V}\int_M |(\Delta_\varphi\psi)^2 - (\Delta_\chi\xi)^2|_{k-2,\alpha} d\mu_\chi \\
&\le \varepsilon C(\omega,\varphi_0,r,\psi_0,k)|\varphi-\chi|_{k-2,\alpha} \\
&\qquad + \varepsilon\frac{1}{2V}\int_M |(\Delta_\varphi\psi)^2 - (\Delta_\varphi\xi)^2 + (\Delta_\varphi\xi)^2 - (\Delta_\chi\xi)^2|_{k-2,\alpha} d\mu_\chi \\
&\le \varepsilon C(\omega,\varphi_0,r,\psi_0,k)|\varphi-\chi|_{k,\alpha} \\
&\qquad + \varepsilon\frac{1}{2V}\int_M \big(|(\Delta_\varphi(\psi-\xi))(\Delta_\varphi(\psi+\xi))|_{k-2,\alpha} + |(\Delta_\varphi\xi)^2 - (\Delta_\chi\xi)^2|_{k,\alpha}\big)d\mu_\chi \\
&\le \varepsilon C(\omega,\varphi_0,r,\psi_0,k)|\varphi-\chi|_{k,\alpha} + \varepsilon C(\omega,\varphi_0,r,\psi_0,k)^2|\psi-\xi|_{k,\alpha} \\
&\qquad + \varepsilon\frac{1}{2V}\int_M |(\Delta_\varphi\xi)^2 - (\Delta_\chi\xi)^2|_{k-2,\alpha}\big)d\mu_\chi \\
&\le \varepsilon C(\omega,\varphi_0,r,\psi_0,k)|\varphi-\chi|_{k,\alpha} + \varepsilon C(\omega,\varphi_0,r,\psi_0,k)^2|\psi-\xi|_{k,\alpha} \\
&\qquad + \varepsilon C(r,\omega,\varphi_0,\psi_0,k)\frac{1}{2V}\int_M |(\Delta_\varphi\xi) - (\Delta_\chi\xi)|_{k-2,\alpha}\big)d\mu_\chi \\
&\le \varepsilon C(\omega,\varphi_0,r,\psi_0,k)|\varphi-\chi|_{k,\alpha} + \varepsilon C(\omega,\varphi_0,r,\psi_0,k)^2|\psi-\xi|_{k,\alpha} \\
&\qquad + \varepsilon C(\omega,\varphi_0,r,\psi_0,k)^2|\varphi-\chi|_{k,\alpha}.
\end{aligned}$$

The next addendum to estimate behaves similarly. It is

$$\begin{aligned}
&\Big|\int_0^t \Big(-\frac{1}{V}\int_M \psi\Delta_\varphi\psi d\mu_\varphi + \frac{1}{V}\int_M \xi\Delta_\chi\xi d\mu_\chi\Big)ds\Big| \\
&\le \frac{\varepsilon}{V}\Big|\int_M -\psi\Delta_\varphi\psi d\mu_\varphi + \xi\Delta_\varphi\psi d\mu_\varphi - \xi\Delta_\varphi\psi d\mu_\varphi + \xi\Delta_\chi\xi d\mu_\varphi - \xi\Delta_\chi\xi d\mu_\varphi + \xi\Delta_\chi\xi d\mu_\chi\Big| \\
&\le \frac{\varepsilon}{V}\Big(\int_M |-\psi\Delta_\varphi\psi + \xi\Delta_\varphi\psi|_{k-2,\alpha} d\mu_\varphi
\end{aligned}$$

$$+\int_M |-\xi\Delta_\varphi\psi+\xi\Delta_\chi\xi|_{k-2,\alpha}d\mu_\varphi+\int_M|-\xi\Delta_\chi\xi dd^c(\varphi-\chi)|\Big)$$

$$\leq\frac{\varepsilon}{V}\Big(\int_M|-\psi+\xi|_{k-2,\alpha}\cdot|\Delta_\varphi\psi|_{k-2,\alpha}d\mu_\varphi+\int_M|\xi|_{k-2,\alpha}\cdot$$

$$|-\Delta_\varphi\psi+\Delta_\varphi\chi-\Delta_\varphi\chi+\Delta_\chi\xi|_{k-2,\alpha}d\mu_\varphi+\int_M|-\xi\Delta_\chi\xi dd^c(\varphi-\chi)|\Big)$$

The third pair of addenda of (7.31) to estimate is

$$\Big|\int_0^t\Big(\frac{1}{2}(\Delta_\varphi-I)^{-1}(\Delta_\varphi\psi)^2-\frac{1}{2}(\Delta_\chi-I)^{-1}(\Delta_\chi\xi)^2\Big)ds\Big|_{k,\alpha}$$

$$\leq\frac{\varepsilon}{2}\cdot\Big|(\Delta_\varphi-I)^{-1}(\Delta_\varphi\psi)^2-(\Delta_\varphi-I)^{-1}(\Delta_\chi\xi)^2$$

$$+(\Delta_\varphi-I)^{-1}(\Delta_\chi\xi)^2-(\Delta_\chi-I)^{-1}(\Delta_\chi\xi)^2\Big|_{k,\alpha}$$

$$\leq\frac{\varepsilon}{2}\cdot\Big|(\Delta_\varphi-I)^{-1}\big((\Delta_\varphi\psi)^2-(\Delta_\chi\xi)^2\big)\Big|_{k,\alpha}$$

$$+\frac{\varepsilon}{2}\cdot\Big|\big((\Delta_\varphi-I)^{-1}-(\Delta_\chi-I)^{-1}\big)(\Delta_\chi\xi)^2\Big|_{k,\alpha}$$

$$\leq\frac{\varepsilon}{2}C_S(r,k,\omega,\varphi_0,\psi_0)\cdot\Big|(\Delta_\varphi\psi)^2-(\Delta_\chi\xi)^2\Big|_{k-2,\alpha}$$

$$+\frac{\varepsilon}{2}\cdot\Big|\big((\Delta_\varphi-I)^{-1}-(\Delta_\chi-I)^{-1}\big)(\Delta_\chi\xi)^2\Big|_{k,\alpha}$$

$$\leq\frac{\varepsilon}{2}C_S(r,k,\omega,\varphi_0,\psi_0)C(\omega,\varphi_0,r,k,\psi_0)\cdot\big(\big|\psi-\xi\big|_{k,\alpha}+\big|\varphi-\chi\big|_{k,\alpha}\big)$$

$$+\frac{\varepsilon}{2}\cdot\Big|\big((\Delta_\varphi-I)^{-1}-(\Delta_\chi-I)^{-1}\big)(\Delta_\chi\xi)^2\Big|_{k,\alpha}$$

$$\leq\frac{\varepsilon}{2}C_S(\omega,\varphi_0,r,k,\psi_0)C(r,k,\omega,\varphi_0,\psi_0)\cdot\big(\big|\psi-\xi\big|_{k,\alpha}+\big|\varphi-\chi\big|_{k,\alpha}\big)$$

$$+\frac{\varepsilon}{2}\cdot C_S(\omega,\varphi_0,r,k,\psi_0)\Big|(\Delta_\chi\xi)^2-(\Delta_\varphi-I)(\Delta_\varphi-I)^{-1}(\Delta_\chi\xi)^2\Big|_{k-2,\alpha}$$

$$\leq\frac{\varepsilon}{2}C_S(\omega,\varphi_0,r,k,\psi_0)C(r,k,\omega,\varphi_0,\psi_0)\cdot\big(\big|\psi-\xi\big|_{k,\alpha}+\big|\varphi-\chi\big|_{k,\alpha}\big)$$

$$+\frac{\varepsilon}{2}\cdot C_S(r,k,\omega,\varphi_0,\psi_0)\Big|\Big((\Delta_\chi-I)-(\Delta_\varphi-I)\Big)(\Delta_\varphi-I)^{-1}(\Delta_\chi\xi)^2\Big|_{k-2,\alpha}$$

$$\leq\frac{\varepsilon}{2}C_S(\omega,\varphi_0,r,k,\psi_0)C(r,k,\omega,\varphi_0,\psi_0)\cdot\big(\big|\psi-\xi\big|_{k,\alpha}+\big|\varphi-\chi\big|_{k,\alpha}\big)$$

$$+\frac{\varepsilon}{2}\cdot C_S(\omega,\varphi_0,r,k,\psi_0)\cdot C(r,k,\omega,\varphi_0,\psi_0)\cdot\big|\chi-\varphi\big|_{k,\alpha}|(\Delta_\varphi-I)^{-1}(\Delta_\chi\xi)^2|_{k-2,\alpha}$$

$$\le \frac{\varepsilon}{2} C_S(\omega, \varphi_0, r, k, \psi_0) C(r, k, \omega, \varphi_0, \psi_0) \cdot (\left|\psi - \xi\right|_{k,\alpha} + \left|\varphi - \chi\right|_{k,\alpha}) \\ + \frac{\varepsilon}{2} \cdot C_S(r, k, \omega, \varphi_0, \psi_0) \cdot C(\omega, \varphi_0, r, k, \psi_0) \cdot \left|\chi - \varphi\right|_{k,\alpha}.$$

Thus, when ε is sufficiently small, we can conclude that the norm of $T(\varphi, \psi) - T(\chi, \xi)$ is smaller than the norm of $(\varphi, \psi) - (\chi, \xi)$. This concludes the proof of the proposition. □

Proof of Theorem 7.11. We fix a pair of initial conditions (φ_0, ψ_0) as in the statement, then we have a δ corresponding to φ_0 and thus we can make the argument of Proposition 7.16 work; together with Proposition 7.17, it says that we can apply the fixed point theorem for complete metric spaces. The fixed point provided is precisely the unique wanted solution. By means of the continuity of the solution, we can consider it as long as it is non-degenerate, that is as long as $\omega + dd^c\varphi > 0$.

Notice that we worked in the product space (7.25) while the space $\mathcal{H}_K$ is not parallelizable. Since our solution (φ, ψ) is a solution of the fixed point problem (7.20) we get *a posteriori* that for any t, $\psi(t)$ lies in $T_{\varphi(t)}\mathcal{H}_K$. □

7.5.2 The general case

In the present subsection let (M, g) be a smooth Kähler manifold with $\dim_{\mathbb{C}} M = n$. About all the other notation, we stick to the one already introduced.

Our aim is to prove the following result, which corresponds to Theorem 7.11

Theorem 7.18. *For every $\varphi_0 \in \mathcal{H}_K$ (respectively $\varphi_0 \in \mathcal{H}$), $\psi_0 \in T_{\varphi_0}\mathcal{H}_K$ (resp. $\psi_0 \in T_{\varphi_0}\mathcal{H}$), integer $k \ge 2$ and $\alpha \in (0,1)$ there exists a positive ε and a curve $\varphi \in C^2([-\varepsilon, \varepsilon], \mathcal{H}_K^{k,\alpha})$ (resp. $\varphi \in C^2([-\varepsilon, \varepsilon], \mathcal{H}_K^{k,\alpha})$) which is the unique solution of* (7.12) *with initial data (φ_0, ψ_0).*

Proof. We argue as in the case of a Riemann surface of the previous subsection, but we have to take care of some more details. Our first claim is that (7.12) is equivalent to the following system

$$\begin{cases} \varphi' = \psi; \\ \psi' = L_\varphi \psi := \Delta_\varphi^{-1}\left[\frac{1}{2}(\Delta_\varphi - I)^{-1}|dd^c\psi|_\varphi^2 + |dd^c\psi|_\varphi^2 + \frac{1}{2}(\Delta_\varphi\psi)^2\right]. \end{cases} \tag{7.32}$$

This is achieved just by inverting the elliptic operators $\Delta_\varphi - I$ first and Δ_φ then, and getting thus an PDE of the form $\varphi'' = L_\varphi \varphi'$.

The second step is to define a map T as in (7.26), and to find an ε and a r such that the complete metric space (7.25) is mapped by T into itself. The explicit expression of the map T in the present case is the following

$$
\begin{aligned}
&T(\varphi,\psi) \\
&= \Big(\varphi_0 + \int_0^t \psi(s)ds, \psi_0 + \int_0^t L_\varphi \psi ds\Big).
\end{aligned} \tag{7.33}
$$

The estimate on the first component is exactly the same as in the proof of Proposition 7.16. About the estimate of the second component we compute

$$
\begin{aligned}
&\left|\psi_0 + \int_0^t L_\varphi \psi ds - \psi_0\right|_{k,\alpha} = \left|\int_0^t L_\varphi \psi ds\right|_{k,\alpha} \\
&\leq \varepsilon \cdot C_S(r,\varphi_0,\omega)\Big(\big|(\Delta_\varphi - I)^{-1}|dd^c\psi|^2_\varphi\big|_{k-2,\alpha} + \big||dd^c\psi|^2_\varphi\big|_{k-2,\alpha} + \left|\frac{1}{2}(\Delta_\varphi\psi)^2\right|_{k-2,\alpha}\Big) \\
&\leq \varepsilon \cdot C_S(r,\varphi_0,\omega)\Big(C_S(r,\varphi_0,\omega)\big||dd^c\psi|^2_\varphi\big|_{k-4,\alpha} + \big||dd^c\psi|^2_\varphi\big|_{k-2,\alpha} + \left|\frac{1}{2}(\Delta_\varphi\psi)^2\right|_{k-2,\alpha}\Big) \\
&\leq \varepsilon \cdot C(r,\varphi_0,\omega) \cdot |\psi|_{k,\alpha},
\end{aligned} \tag{7.34}
$$

where we used the Schauder estimates which generalized Lemma 7.15 twice, and we estimated both $\big||dd^c\psi|^2_\varphi\big|_{k-2,\alpha}$ and $|(\Delta_\varphi\psi)^2|_{k-2,\alpha}$ arguing in the same vein of Lemma 7.14.

Then, we have to prove that the map T is a contraction for some ε and some r. Again, the first component of T is estimated precisely as in Proposition 7.17. So, sticking to the notation in Proposition 7.17, we have to estimate

$$
\begin{aligned}
&|L_\varphi\psi - L_\chi\xi|_{k,\alpha} \leq \big|(\Delta_\varphi)^{-1}\big(\Delta_\varphi L_\varphi\psi + \Delta_\chi L_\chi\xi - \Delta_\chi L_\chi\xi - \Delta_\varphi L_\chi\xi\big)\big|_{k,\alpha} \\
&\leq C_S(r,\varphi_0,\omega)\big(\big|\Delta_\varphi L_\varphi\psi - \Delta_\chi L_\chi\xi\big|_{k-2,\alpha} + \big|\Delta_\chi L_\chi\xi - \Delta_\varphi L_\chi\xi\big|_{k-2,\alpha}\big),
\end{aligned} \tag{7.35}
$$

where we performed a Schauder estimate of the same fashion as Lemma 7.15. About the second addendum we argue as in Lemma 7.14 and moreover noticing that by (7.34) we have $|L_\chi\xi|_{k-2,\alpha} \leq C(r,\varphi_0,\psi_0,g)$; thus we can estimate

$$
\big|\Delta_\chi L_\chi\xi - \Delta_\varphi L_\chi\xi\big|_{k-2,\alpha} \leq C(r,\varphi_0,\omega,\psi_0)\cdot|\chi-\varphi|_{k,\alpha}, \tag{7.36}
$$

while for the first addendum we start noticing

$$
\big|\Delta_\varphi L_\varphi\psi - \Delta_\chi L_\chi\xi\big|_{k-2,\alpha} \tag{7.37}
$$

$$= \left| \frac{1}{2}(\Delta_\varphi - I)^{-1}|dd^c\psi|^2_\varphi - \frac{1}{2}(\Delta_\chi - I)^{-1}|dd^c\xi|^2_\chi \right.$$
$$\left. + |dd^c\psi|^2_\varphi - |dd^c\xi|^2_\chi + \frac{1}{2}(\Delta_\varphi\psi)^2 - \frac{1}{2}(\Delta_\chi\xi)^2 \right|_{k-2,\alpha}.$$

In order to handle the above expression, we begin with the estimate

$$\left|(\Delta_\varphi - I)^{-1}|dd^c\psi|^2_\varphi - (\Delta_\chi - I)^{-1}|dd^c\xi|^2_\chi\right|_{k-2,\alpha} \tag{7.38}$$
$$\leq \left|(\Delta_\varphi - I)^{-1}\big(|dd^c\psi|^2_\varphi - |dd^c\xi|^2_\chi + |dd^c\xi|^2_\chi - (\Delta_\varphi - I)(\Delta_\chi - I)^{-1}|dd^c\xi|^2_\chi\big)\right|_{k-2,\alpha}$$
$$\leq C_S(\varphi_0, r, \omega)\left||dd^c\psi|^2_\varphi - |dd^c\xi|^2_\chi + |dd^c\xi|^2_\chi - (\Delta_\varphi - I)(\Delta_\chi - I)^{-1}|dd^c\xi|^2_\chi\right|_{k-4,\alpha}.$$

Now we notice that

$$\left||dd^c\xi|^2_\chi - (\Delta_\varphi - I)(\Delta_\chi - I)^{-1}|dd^c\xi|^2_\chi\right|_{k-4,\alpha} = \left|(\Delta_\varphi - \Delta_\chi)(\Delta_\chi - I)^{-1}|dd^c\xi|^2_\chi\right|_{k-4,\alpha} \tag{7.39}$$
$$\leq C(r, \varphi_0, \omega, \psi_0) \cdot |\varphi - \chi|_{k,\alpha}$$

having in mind the arguments of Lemmata 7.14 and 7.15. Next we estimate other addendum of (7.38), which is the same as the second addendum of (7.37). We put ourselves in a coordinate chart, and we label the Riemannian metrics corresponding to φ, χ as g_φ, g_χ; we compute

$$\left||dd^c\psi|^2_\varphi - |dd^c\xi|^2_\chi\right|_{k-4,\alpha} = \left|g^{i\bar{j}}_\varphi g^{k\bar{l}}_\varphi \psi_{i\bar{l}}\psi_{k\bar{j}} - g^{i\bar{j}}_\chi g^{k\bar{l}}_\chi \xi_{i\bar{l}}\xi_{k\bar{j}}\right|_{k-4,\alpha} \tag{7.40}$$
$$\leq \left|(g^{i\bar{j}}_\varphi - g^{i\bar{j}}_\chi)g^{k\bar{l}}_\varphi \psi_{i\bar{l}}\psi_{k\bar{j}} + g^{i\bar{j}}_\chi(g^{k\bar{l}}_\varphi - g^{k\bar{l}}_\chi)\psi_{i\bar{l}}\psi_{k\bar{j}}\right.$$
$$\left. + g^{i\bar{j}}_\chi g^{k\bar{l}}_\chi(\psi_{i\bar{l}} - \xi_{i\bar{l}})\psi_{k\bar{j}} + g^{i\bar{j}}_\chi g^{k\bar{l}}_\chi \xi_{i\bar{l}}(\psi_{k\bar{j}} - \xi_{k\bar{j}})\right|_{k-4,\alpha}$$
$$\leq C(r, \varphi_0, \omega, \psi_0) \cdot \big(|\varphi - \chi|_{k-2,\alpha} + |\psi - \xi|_{k-2,\alpha}\big).$$

Finally we have to take care of the third addendum of (7.37). We compute

$$\left|(\Delta_\varphi\psi)^2 - (\Delta_\chi\xi)^2\right|_{k-2,\alpha} \leq \left|\big(\Delta_\varphi\psi - \Delta_\chi\xi\big) \cdot \big(\Delta_\varphi\psi + \Delta_\chi\xi\big)\right|_{k-2,\alpha} \tag{7.41}$$
$$\leq C(r, \varphi_0, \omega, \psi_0)\left|\Delta_\varphi\psi - \Delta_\chi\psi + \Delta_\chi\psi - \Delta_\chi\xi\right|_{k-2,\alpha}$$

$$\leq C(r, \varphi_0, \omega, \psi_0)\big(|\psi - \xi|_{k,\alpha} + |\chi - \varphi|_{k,\alpha}\big).$$

Putting together our estimates we find small enough ε and r which make T a contraction. By means of the fixed point theorem, we find a unique solution of our geodesic equation. This completes the proof of the theorem. □

7.6 The Cauchy geodesics for the Dirichlet metric

In the present section we answer a question by Calabi about the Riemannian structure on the space of Kähler potentials of a closed manifold M named after Dirichlet, and also present in the literature as *the gradient metric*. Namely, we are able to show that for any Kähler potential φ of class $C^{k,\alpha}(M)$, with $k \geq 2$ and $\alpha \in (0,1)$ and for any *initial velocity* $\psi \in C^{k,\alpha}(M)$ there is a positive time existence $\varepsilon > 0$ and a geodesic curve of class C^2 mapping the interval $[-\varepsilon, \varepsilon]$ into the space Kähler potentials φ of class $C^{k,\alpha}(M)$, which has (φ, ψ) as position and velocity at time zero.

The Dirichlet metric was studied by X. X. Chen, Calamai and Zheng already in [16, 20, 26]. In [16] it was proved that for Riemann surfaces it has zero sectional curvature and its geodesics are straight segments connecting any two Kähler potentials.

In [26] it was proved that the pseudo-Calabi flow is a gradient flow of the K-energy when the space of Kähler potentials is endowed with the Dirichlet metric.

In [20] it was partially confirmed a conjecture of Calabi, stating that the sectional curvature of the Dirichlet metric sits in between those of Mabuchi and of Calabi. Moreover it was proved that in the space of Kähler metrics equipped with the Dirichlet metric, the K-energy is convex at a cscK metric (see [20, Proposition 3.23]).

We put ourselves in the Kähler environment; by the way our argument runs as well in the case when the underlying manifold is Sasaki.

Let (M, ω) be a compact n-dimensional Kähler manifold without boundary. The geodesic equation for the gradient metric is (see [16] page 400)

$$2\Delta_\varphi \varphi'' - |dd^c \varphi'|^2_\varphi + (\Delta_\varphi \varphi')^2 = 0, \tag{7.42}$$

where φ is a curve with values in $\mathcal{H}_K$.

Our result is

Theorem 7.19. *For every $\varphi_0 \in \mathcal{H}_K$ and $\psi_0 \in T_{\varphi_0}\mathcal{H}_K$, fixing $k \geq 2$ and $\alpha \in (0,1)$ there exists a ε and a unique solution of* (7.42) *in the function space $C^2([-\varepsilon, \varepsilon], \mathcal{H}_K^{k,\alpha})$, satisfying $\varphi(t = 0) = \varphi_0$ and $\frac{d}{dt}\varphi(t = 0) = \psi_0$.*

Proof. We rewrite (7.42) as

$$\varphi'' = \frac{1}{2}\Delta_{\varphi}^{-1}\big(|dd^c\varphi'|^2_{\varphi} - (\Delta_{\varphi}\varphi')^2\big).$$

From the above equation we get the first order PDE system

$$\begin{cases} \varphi' = \psi \\ \psi' = \frac{1}{2}\Delta_{\varphi}^{-1}\big(|dd^c\psi|^2_{\varphi} - (\Delta_{\varphi}\psi)^2\big). \end{cases} \tag{7.43}$$

The argument now follows closely the one of Theorem 7.18. Given φ_0, ψ_0, we fix $0 < \delta := \frac{1}{2}\min(\omega + dd^c\varphi_0)$. We consider a complete metric space $B_r(\varphi_0, \psi_0) \subseteq C^0([-\varepsilon, \varepsilon], \mathcal{H}^{k,\alpha}_{K,\delta}) \times C^0([-\varepsilon, \varepsilon], C^{k,\alpha}(M))$ centered at (φ_0, ψ_0) seen as time-constant functions. for some $r, \varepsilon > 0$, $k \geq 2$ and $\alpha \in (0, 1)$. (We are sticking to the notation of Subsection 7.5.1.) Then we define a map T having as domain of definition that ball of radius r and having as components the integral of the system (7.43). Then we fix first a small r and then a small ε such that the map T ranges in $B_r(\varphi_0, \psi_0)$ itself; notice that once r is fixed, we have the freedom to move ε.

Then we find suitable r and ε, possibly smaller, such that the map T is also a contraction.

Finally we apply the fixed point theorem and we get that by construction T gives the wanted geodesic solution of (7.42), with initial conditions φ_0, ψ_0. This completes the proof of the theorem. $\square$

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