A dynamical systems approach to systemic risk

Piero Mazzarisi

UNIVERSITÀ DI PISA
FACOLTÀ DI SCIENZE MATEMATICHE, FISICHE E NATURALI
CORSO DI LAUREA IN FISICA

CURRICULUM TEORICO

MASTER'S DEGREE THESIS

SUPERVISOR: PROF. FABRIZIO LILLO
Scuola Normale Superiore
INTERNAL SUPERVISOR: PROF. RICCARDO MANNELLA
Università di Pisa
CO-SUPERVISOR: PROF. STEFANO MARMI
Scuola Normale Superiore

ACADEMIC YEAR 2013/2014

A Maria ,
alla Sua Bellezza che salverà il mondo

Contents

Introduction

In financial markets, systemic risk can be defined as financial *system* instability typically caused by a stochastic shock and exacerbated by relations between financial intermediaries. By definition, systemic risk involves the financial system, a collection of interconnected institutions that have mutually beneficial business relationships through which illiquidity, insolvency, and losses can quickly propagate during periods of financial distress. In fact, a systemic financial instability is related to the market structure created by financial institutions operating in the market and systemic risk refers to the possibility that a negative shock affecting a micro region of the financial system, has bad consequences at the macro level. As a consequence, the initial shock propagates throughout the whole financial system and the risk becomes *systemic*.

Systemic risk in financial markets is a problem of renewed interest after the financial crises that have affected recently the USA economy (2007-2009) and the eurozone economy (2009+). These events have highlighted the limitations of existing economic and financial models based on the assumption of intrinsic stability of efficient financial markets. These models are based on the hypothesis that a financial market is a stable system which reaches always the equilibrium, at most in the long run. The occurrence of recent financial crises breaks this hypothesis and these events have made evident the necessity of a new scientific approach, based on complex dynamical systems, in order to describe the problem of systemic risk and systemic financial stability in a more general framework [1].

Financial markets exhibit several of the properties that characterize complex systems [2]. They are open systems in which many subunits, financial institutions, interact nonlinearly in the presence of feedbacks. In this framework, systemic risk can be seen as an emergent phenomenon [3], and it can be interpreted as a phase transition from stability to instability.

Many scientific works have tried to describe systemic risk in financial markets, using the methodology of networks theory, dynamical systems theory, and statistical physics. They have shown that the (in)stability of financial markets can be associated with the network properties [4, 5] of the financial system, its dynamical properties [6], and in particular feedback effects [7, 8] arising from the impact of the trading strategy adopted by financial institutions. The recent work of Corsi et al. [9] combines these evidences to show how financial innovations[1], reducing the cost of diversification, increase the strength and coordination of feedback effects and as a

[1]Financial innovations refer to the creating and marketing of new types of financial tools, such as new securities, in order to reduce costs. For example, through the securitization of asset investments, financial institutions may reduce the risk associated with their portfolio. This practice is equivalent to create a diversified portfolio characterized by low risks without incurring costs of diversification.

consequence, they increase systemic risk in financial markets. A challenging issue concerns the fact that financial markets are complex systems characterized by a behavioral component, which describes the forecasting strategies of financial institutions. The behavioral component of the financial market dynamics reflects the fact that, differently from the weather forecasting which will not influence the future state of the atmosphere, the forecasting of future prices and risks will influence today the choices of financial agents, and as a consequence it will define the future state of financial market dynamics.

This behavioral component is defined through the expectations scheme adopted by financial agents to forecast prices and risks of assets. At the micro level, the expectations scheme adopted by a financial agent, defines its portfolio choices, and as a consequence its dynamical state during the evolution of financial market dynamics. At the macro level, the aggregate behavior of all financial agents influences the systemic financial stability, and systemic risk can be seen as the breaking of stability due to a collective phenomenon arising from the aggregate behavior of all financial agents.

Typically financial agents populating a financial market, are boundedly rational. They don't know exactly the future realization of prices and risks but they forecast it, looking at the historical time series of past realizations [1]. The forecasting of future risks is found by a financial agent according to an expectations scheme, that is a rule which uses the values of past realizations of risks in order to obtain a forecast. This rule is not fixed and a financial agent may change it in order to improve its forecasting strategy.

Hommes et al. [10, 11] have theoretically and experimentally studied how the expectations of financial agents follow statistical laws, and for this reason they can be described through mathematical models. The main point is that the today expectations of financial institutions about the future prices and risks, influence today the dynamics of financial markets. Hence, in financial markets the expectation feedbacks play a crucial role in defining the dynamical properties of the financial system.

In this thesis, we propose a dynamical systems approach to systemic risk, in order to answer the following question: What is the relationship between the role of expectation feedbacks defining the dynamical behavior of financial institutions at the micro level and the emerging macro consequences on systemic financial stability? Specifically, we propose a dynamical model to study the dynamics of a financial market when feedback effects are accounted for. We focus on the possible dynamical outcomes displayed by financial markets due to feedback effects and, through our dynamical approach, we study systemic risk as an emergent phenomenon characterized by the breaking of dynamical stability of the financial system.

In our model, the financial market is stylized through two different sets. From one side there are the investment assets whose prices are described by stochastic processes. In quantitative finance, the interesting quantity is not the price but it is the return, *i.e.* the increment of price divided by the price itself. In our approach, the return of an asset is a Gaussian stochastic variable whose variance can be considered as a measure of the risk associated with the investment asset. The stochastic process associated with the return of an asset is formed by two components. The first component is described by a discrete Wiener process and it is related to the intrinsic randomness of financial time series. It is called *exogenous* component because it represents a variable which does not depend on other variables. On the contrary, the second component of the stochastic process depends on the demand or supply for the investment asset. Typically, financial markets are illiquid, *i.e.* the buying (the selling) of an asset tends to put upward (downward) pressure on its price. Hence, this second component depends on the trading strategies adopted by financial institutions operating in the market. It is called *endogenous* component because it depends on other variables which characterize the dynamics of the financial market. When the return processes are stationary, the endogenous component of the returns has a deterministic nature because it is related to the past realizations of these variables.

All assets are assumed to be equivalent and characterized by the same values of expected return and risk.

From another side there is a collection of financial institutions which create their portfolio[2] selecting a linear combination with equal weights of investment assets. In the model, financial institutions define the values of two variables: the number of assets and the value of total investment. The number of assets in a portfolio is called *diversification*. The value of total investment of a financial institution is the total asset position of the institution, and it is related to another variable called *leverage*. The money invested in assets by a financial institution is the sum of its equity (its own capital) and its debt. The leverage of a financial institution is the ratio between the total asset position and the value of its equity. Hence, the adopted leverage by a financial institution can be interpreted also as the degree of the debt associated with the financial institution.

In real financial markets, the largest investors operating in the market typically adopt a trading strategy called target leverage strategy. In our model, we consider that all financial institutions adopt target leverage.

In fact, the value of leverage changes stochastically in time because the actual value of the total asset position of a financial institutions depends on the stochastic evolution of asset prices. The target leverage strategy adopted by financial institutions consists in maintaining a fixed leverage by buying or selling assets.

The optimal values of the diversification and the target leverage are chosen by financial institutions solving an optimization problem in the presence of diversification cost and the Value at Risk (VaR) constraint.

The cost of diversification is related to the fact that a portfolio with a larger number of assets incurs larger expenses in terms of qualified staff, physical locations, etc. When the cost of diversification decreases, financial institutions increase the diversification of their portfolios in order to reduce the portfolio risk[3].

[2]A portfolio is defined by a stochastic variable, the portfolio return, that is the sum of m stochastic variables, the asset returns, weighted by a factor $\frac{1}{m}$

[3]It is exactly true when the returns of assets are independent stochastic variables. In the model, the return of an asset is described by the sum of the exogenous component and the endogenous one. By neglecting the endogenous component, exogenous components of asset returns are independent and identically distributed Gaussian noises. In this case, a linear combination of m asset returns weighted by the factor $\frac{1}{m}$, is characterized by a value of the variance that is reduced by $\frac{1}{m}$ compared with the variance of only one asset return. When the endogenous component is present, in general diversification does not reduce the portfolio risk for two reason. The first one is related to the fact that the hypothesis of independence is not valid anymore. The second reason refers to the fact

The VaR constraint represents a debt limit for financial institutions. Hence, the VaR constraint imposes an upper bound to the leverage. This constraint depends on the risk of portfolio and specifically, when the risk associated to the portfolio of a financial institution is high, the financial market policy imposes to reduce the debt level, reducing so the value of the leverage.

In defining leverage and diversification, financial institutions try to maximize the expected return of their portfolio minimizing the risk associated with the portfolio. The values of diversification and leverage strictly depend on the risk of assets. In the model, we assume that financial institutions solve the same optimization problem in order to find the optimal values of leverage and diversification.

In illiquid financial markets, the target leverage strategy has the potential for a positive feedback effect. In fact, if the prices of assets in the portfolio increase, it follows that the leverage decreases. As a consequence, a financial institution buys in order to bring back the leverage to its optimal value, the target. This trading strategy is said portfolio rebalancing. By buying, the prices of assets increase further. The same mechanism works in reverse. The strength of this positive feedback effect is proportional to the value of the adopted target leverage. The larger is the leverage, the larger is the impact of the positive feedback. At the same time, also the coordination of these feedback effects affecting different assets plays a crucial role. When the number of assets in the portfolios of financial institutions is large, feedback effects create a correlation between initially uncorrelated assets. In fact, the larger is the diversification of portfolios, the larger is the overlap between the portfolios owned by different financial institutions. According to the movements of asset prices, financial institutions buy or sell in a similar way when they own similar portfolios. Hence, the impact of positive feedback effects increases due to the coordination of financial institutions. In this sense, the aggregate behavior of financial institutions operating in the market may have crucial consequences on systemic financial stability.

In this thesis, the endogenous component of the return describes the impact of this positive feedback due to the portfolio rebalancing made by financial institutions. The endogenous component describes the impact of feedback effects at a specific time depending on the choices of leverage and diversification made by financial institutions at a previous time. Hence, when the stochastic processes governing the returns of assets are stationary, the positive feedback has a deterministic nature. In fact, the previous choice about leverage and diversification made by financial institutions defines the future impact of feedback effects due to the collective behavior of all financial institutions. Looking at the averaged values, the dynamical properties of the financial market at a specific time depend on the portfolio optimization problem solved by financial institutions at the past time.

A important point of this thesis is that feedback effects increase also the risk of assets because typically the positive feedback due to the portfolio rebalancing amplifies the oscillations around the mean return, increasing so the variance associated with each asset. Since the portfolio choices strictly depend on the risk of assets, feedback effects will influence importantly the dynamical evolution of portfolios. For this reason, the strength (due to higher leverage) and the

that the value of the endogenous component depends on the values of other variables, such as diversification and leverage.

coordination (due to similarity of the portfolios of financial institutions) of feedback effects may lead to a collective phenomenon characterized by the breaking of systemic financial stability and we refer to it as systemic risk.

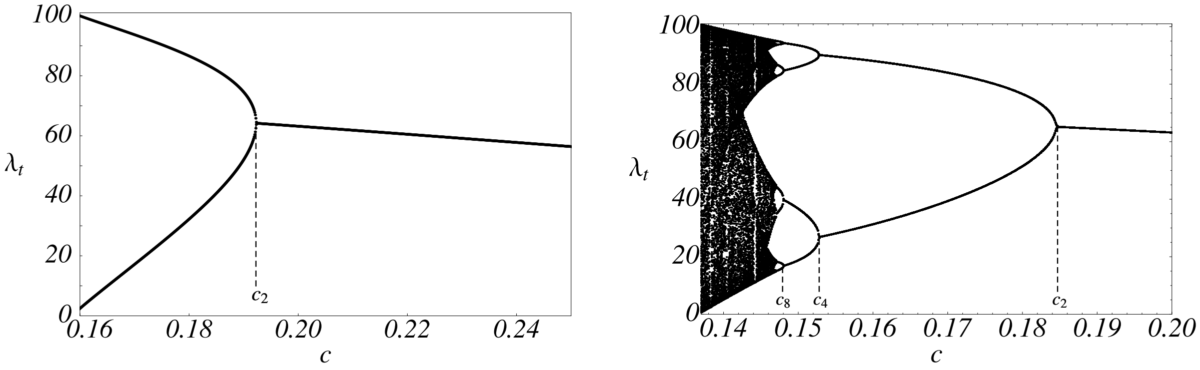
In this thesis, we investigate the dynamics of the typical portfolio of a financial institution as a consequence of the impact on risk of the feedback effects arising from the aggregate behavior of all financial institutions operating in the market. At each time, a financial institution has to solve its portfolio optimization problem in order to satisfy the VaR constraint. This constraint strictly depends on the risk of portfolio. The larger is the portfolio risk, the smaller has to be the value of total asset position, and so the leverage. On the contrary, during less risky periods, a financial institutions may choose a larger value of its total asset position.

Hence, at each time, a financial institution makes expectations of the risk associated with its portfolio and according to this it has to define the optimal values of leverage and diversification.

The crucial point is related to the fact that a financial market is an expectation feedback system, that is a system in which financial institutions know all past history of the financial market dynamics and they make choices at the present time trying to anticipate what will be the future realizations of prices and risks related to investment assets. In this thesis, we study how boundedly rational financial institutions may form their expectations about future risks using the time series of the past realizations of risk. According to the risk expectations they optimize their portfolios, defining the values of target leverage and diversification. The portfolio dynamics is strictly related to systemic risk. In fact, in financial markets systemic financial stability strictly depends on the adopted values of leverage and diversification.

We study two different schemes through which financial institutions form the risk expectations: naive expectations and adaptive expectations.

Financial institutions have naive expectations of risk when the forecast is equal to the last observed risk. In this case, when the cost of diversification is high and as a consequence, financial institutions optimize their portfolios choosing small values of leverage and diversification, the dynamics of the financial market is stable and converges asymptotically to a steady state. If diversification costs decrease, financial institutions may create a more diversified portfolio. When the impact on risk of the positive feedback is small, a more diversified portfolio is less risky and as a consequence, a financial institution may increase its investment, and so the leverage, fulfilling still the VaR constraint. The dynamics of the financial market remains stable. At a given threshold of the cost of diversification, the strength and the coordination of feedback effects trigger the breaking of financial stability of whole system. After the threshold, the dynamics of the financial system oscillates between highly risky periods and less risky periods. During these financial cycles, we can notice how the financial leverage switches from aggressive configurations (speculative periods) to cautious ones (non-speculative periods). The financial leverage cycles reflect the occurrence of periods characterized by a macro-component of risk, due to an higher impact of positive feedback effects, followed by periods in which feedback effects do not affect importantly the risk of assets. Mathematically, the breaking of systemic financial stability occurs in a very specific way through a period-doubling bifurcation. Hence, from a dynamical point of view, systemic risk refers to the occurrence of bifurcations that break the stability of the



(a) naive expectations

(b) adaptive expectations

Figure 1: Bifurcation diagrams of the leverage, λ, as a function of the cost of diversification, c. A bifurcation diagram shows the asymptotic dynamics of the system when the initial transient period is passed, for each value of the bifurcation parameter, in this case the cost of diversification. Figure shows the dynamics associated with a portfolio variable, the leverage. Panel (a) shows the case of naive expectations of risk. At the right, we can see a stable dynamics characterized by a steady state. At a given threshold c_2, a period-doubling bifurcation occurs and after the bifurcation point, the dynamical behavior of the financial system is cyclical with period 2. Panel (b) shows the case of adaptive expectations of risk. Like the naive case, at the right the dynamics is stable. At a given threshold, a period-doubling bifurcation occurs and the dynamical behavior is cyclical with period 2, again. But differently from the naive case, a period-doubling cascade occurs. There exist values of the bifurcation parameter at which the period of the cycles doubles, and finally the dynamical system becomes chaotic.

financial system. When financial cycles appear, the amplitude of cycles can be interpreted as a measure of systemic risk in the financial market.

By approaching the bifurcation, the convergence time, necessary to the system in order to reach the steady state after a perturbation, increases. In dynamical systems theory, this phenomenon is known as the critical slowing down. One important consequence is that the slowing down leads to an increase in autocorrelation related to the portfolio variables before approaching the criticality. The increased autocorrelation can be interpreted as a early-warning signal that the dynamics of the system is approaching a bifurcation. Therefore, in the model, it is a early-warning signal of systemic risk.

After the bifurcation point, the dynamical behavior of the system is cyclical. The amplitude of these cycles is strictly related to the value of the bifurcation parameter, in this case the cost of diversification. Close to the bifurcation point, the amplitude of the cycle is a square-root function of the cost of diversification. During financial cycles, financial institutions make systematic mistakes in forecasting the risk associated with the investment assets. In this situation, clear information appears, for example, in the autocorrelation function of the forecasting errors. This information may be used by financial institutions in order to improve the expectations scheme adopted in forecasting the risk.

Subsequently, we investigate a more realistic scenario in which financial institutions adopt adaptive expectations of risk. In forecasting the risk of an asset, financial institutions know all past realizations of the asset risk and the forecast is equal to the sum of all past observations weighted by exponentially decreasing factors over time. These factors are defined univocally by a parameter that can be interpreted as the memory of the expectation scheme. An adaptive expectations scheme characterized by a larger memory, gives more weight to the older realizations of risk in respect to the recent ones. On the contrary, less memory means to consider mostly the last observations. Once the memory is fixed, when the cost of diversification is high, the dynamics of the financial market is stable. Decreasing the cost of diversification, at a given threshold a breaking of systemic financial stability occurs by means of a period-doubling bifurcation. Decreasing further the cost of diversification, a cascade of period-doubling bifurcations leads the financial system towards (physical) *chaos*. Hence, under adaptive expectations, the financial system exhibits a dynamical transition from a periodic cyclical behavior to deterministic chaos. In the chaotic regime, in addition to the occurrence of highly risky periods identified by a very large macro-component of risk due to feedback effects, the chaotic dynamical behavior of financial system is characterized by positive entropy, suggesting how much an improvement of expectations scheme by financial institutions may be hard due to missing information about financial market dynamics. In fact, the positive entropy reflects the sensitive dependence on initial conditions, that is the signature of the chaotic evolution of a dynamical system. This phenomenon has the effect of reducing autocorrelation of forecasting errors, for example, and as a consequence less information is available to financial institutions in order to improve their expectations scheme.

In the case of adaptive expectations, the memory of the expectation scheme plays an important role. When the memory increases, it follows that the financial market dynamics moves towards stability.

Financial institutions operating in the market look at the time series of forecasting errors and they try to improve their forecasting strategy using the available information coming from past financial performances. Hence, we analyze the case in which the memory of the adaptive expectation scheme is a time-varying parameter during the evolution of financial market dynamics. We assume that financial institutions modify the value of the memory parameter looking at the autocorrelation function of the forecasting errors. This represents a simple case of how a financial agent can take information about the goodness of its forecasting strategy and use this information to improve its expectation scheme. In this thesis, a procedure of adaptive learning adopted by financial institutions has stabilizing effects on financial market dynamics because it leads the financial system towards systemic stability.

Finally, from the point of view of financial market *policy*, the results of this thesis highlight how the dynamical properties of financial markets may drastically change when market conditions change. In fact, once the memory of the expectations scheme adopted by financial institutions is fixed, a decrease of diversification costs in the presence of strong feedback effects may lead to the breaking of systemic financial stability. Similarly to diversification costs, also relaxing the VaR constraint will have the same consequences. In fact, the relaxing of the constraint allows to financial institutions to adopt a larger value of target leverage. This leads to an increase of the strength of feedback effects and, as a consequence, the financial system moves towards systemic instability.

The thesis is divided into five Chapters:

1. **Financial systemic risk**: we give a definition of systemic risk in financial markets and we highlight the most important and universal aspects related to the study of systemic

financial stability, from the point of view of network theory and dynamical systems theory.

2. **Dynamical systems theory**: we recall some basic elements of dynamical systems theory, in preparation for our dynamical systems approach to systemic risk.

3. **A model of systemic risk in financial markets**: we review robust empirical evidences of the presence of feedback effects in financial markets, and we introduce the model of *Corsi et al.* [9]. Specifically, in the model feedback effects are the consequence of the impact on the price and risk of assets due to the trading strategy adopted by financial institutions.

4. **The role of expectation feedbacks in systemic financial stability**: we introduce our original dynamical model in which feedback effects are considered through expectations of risk. In this Chapter we discuss in detail the dynamical behavior of the model and the role of expectation feedbacks in systemic financial stability. We describe the case of naive expectations. All the material of this Chapter is original.

5. **Adaptive expectations and adaptive learning**: we describe the case of adaptive expectations, highlighting how universal nonlinear aspects related to the feedback may trigger the dynamical transition to chaos. Finally, we discuss a possible generalization of our dynamical model, in which financial institutions may improve their expectations scheme according to a learning procedure. Hence, we discuss the case of adaptive learning. All the material of this Chapter is original.

Chapter 1

Financial systemic risk

What is *Systemic Risk*? Focusing on the case of financial markets, the answer to this question is not trivial. By definition, systemic risk involves the financial *system*, a collection of financial institutions investing in economic resources, the investment assets, whose price evolves stochastically in time. Financial markets exhibit several of the properties that characterize complex systems [2]. They are open systems in which many subunits, financial institutions, such as banks, hedge funds, pension funds etc. , interact nonlinearly in the presence of feedback. Systemic risk in financial markets refers to collective phenomena describing how in certain conditions the perturbations affect significantly the entire financial market, thus becoming *systemic*.[1]

Systemic risk is in effect an *emergent phenomenon*. Systemic risk occurs in the field of social sciences, and in particular finance, but there is a very interesting analogy with the visionary idea of *P. W. Anderson*, explained in his paper "*More Is Different*" [3]. The key point is that in the passage from the reductionist analysis, focused on the study of the elementary unit of a system, to the constructionist one, focused on the study of an extensive system, entirely new properties appear. In physics, this is essentially due to the fact that the state of a really big system has less symmetry with respect to the elementary units, which compose it. The phenomenon of superconductivity [12] is one of the most spectacular examples of the broken symmetry which ordinary macroscopic systems undergo. The essential idea is that in the $N \to \infty$ limit, the system will undergo a phase transition in which the microscopic symmetry is broken, and the macroscopic consequences are completely new. "In this case we can see how the whole becomes not only more than but very different from the sum of its parts" [3]. The physical idea of emergence describes very well the underlying aspects of systemic risk. In fact, systemic risk in financial markets can be seen as a phase transition from stability to instability. Making a parallelism with physics, the financial stability of a subunit of the financial system takes the role that symmetry has in a physical system. When we look at an aggregate of interacting subunits, which form the financial system, the stability of the entire system may be broken, similarly to what happens through a phase transition. In order to make clear how a mechanism

[1] A clarification is necessary in order to avoid confusion about the word "risk". In quantitative finance, risk refers also to an investment asset, which is described by a stochastic process that defines the price dynamics. Hence, the risk of an asset is the diffusion rate of the price. Instead, systemic risk is related to collective phenomena inducing potentially the collapse of a financial market.

of this type works in a financial system, the following example is useful: in financial markets, all subunits, financial institutions, try to maintain a stable financial state through the diversification of risk, that is a strategy based on investing in a large number of assets. Looking at only one financial subunit, the diversification of risk should lead to minimize the probability of a loss, maximizing the probability of profit. However, looking at the entire system, financial subunits interact strongly because they have several investments in common. These strong interactions in financial markets can be described as feedback effects that may induce the breaking of financial stability of whole system. This example represents the underlying intuition at the base of our dynamical systems approach.

In this thesis, we study the problem of systemic risk and systemic financial stability through a dynamical systems approach. Hence, in our case, the systemic financial stability corresponds in effect to the dynamical stability of the system.

In general, there are many definitions of systemic risk, each of these refers to a particular point of view of approaching the problem. A common factor in the various definitions of systemic risk is that a triggering event, such as a stochastic shock or an institutional failure, causes a chain of bad economic consequences involving a significant fraction of the system. For this reason, systemic risk refers to a a scenario in which the financial system moves from a stable state to an unstable one. In the latter state, a shock may trigger a collapse. In our case, we associate systemic risk with a dynamical phase transition, that is a bifurcation.

Many scientific works have tried to describe systemic risk in financial markets, using the methodology of networks theory, dynamical systems theory, and statistical physics. Important contributions come from the recent works of *Gai and Kapadia* [5], *Douady and Choi* [6] [13], and *Cont and Wagalath* [8].

In this Chapter, we describe three mechanisms that are the most important reasons behind the occurrence of systemic risk in financial markets. In describing these mechanisms, we refer to the three recent papers [5, 6, 8], which give a clear and complete description of the concerned arguments.

Before presenting the most important and universal aspects related to systemic risk of these papers, we give a schematic and simplified representation of crucial ideas behind the approach to the problem of systemic financial stability.

Assuming there are Alice, Bob, and Carol, as financial agents. Through the flow of money between them and the investments they make, Alice, Bob, and Carol form a financial system. A breakdown of financial stability may occur in different ways:

- Alice borrows money from Bob, who in turn borrows money from Carol. If Alice experiences accidentally a loss, she becomes insolvent with Bob. As a consequence, Bob is insolvent with Carol. Hence, an initial shock triggers a cascade of insolvencies. The financial system loses its stability. We can refer to this situation as *direct contagion of risk*.

- Alice borrows money from Bob, who in turn borrows money from Carol, who in turn borrows money from Alice (closed loop). If a perturbation affects the economic state of

Alice, she does not pay all debt, but she gives to Bob less money than she should. A perturbation affects the economic state of Bob and, as a consequence, a perturbation will affect the economic state of Carol, and so on. A situation of this type can be approached through a dynamical model that describes the dynamics of perturbations affecting the financial system identified by Alice, Bob, and Carol. The financial system loses its stability if perturbations within the system grow in time, causing the successive defaults of Alice, Bob, and Carol. This situation can be again represented as a direct contagion due to *perturbations*, which spread within financial systems.

- Alice, Bob, and Carol invest in initially uncorrelated assets of a financial market. If Alice's asset suffers a negative shock of price, Alice liquidates her asset position, that is she sells everything. In financial markets, when an investor sells (or buys), he moves downward (upward) the price of the traded assets, through a mechanism said *price impact*. When Alice liquidates the investment assets in common with Bob and Carol, the price impact of trading creates a positive feedback, which induces the falling of prices of these assets. As a consequence of price falling, the initially uncorrelated assets display a positive correlation, because the price movements occur simultaneously in the same direction. Finally, the positive feedback contributes to increase the risk of assets, because it induces large price movements. As a consequence, risk associated with the assets of Bob and Carol increases. We can refer to this situation as indirect contagion due to *feedback effects*.

The previous situations represent three mechanisms that may lead to the breakdown of stability of financial markets dynamics. Obviously, when the systemic stability is lost, both direct contagions and indirect ones may lead to a financial crisis.

By exploiting different models or different assumptions, the recent scientific literature about systemic risk in financial markets refers to one of these mechanisms. Our work combines a dynamical systems approach together with the role of feedback effects in financial markets, providing so a new and original contribution to the problem of systemic risk.

In the next Sections, we review the scientific works of *Gai et Kapadia* [5], *Douady and Choi* [6], and *Cont and Wagalath* [8], illustrating how the previous mechanisms of contagion of risk can be mathematically described. These three recent works explain clearly the ideas behind the problem of systemic risk in financial markets and we review them in order to make evident analogies and differences between our model and the state of the art.

1.1 Direct contagion of risk

In this Section we present some results of *Gai et Kapadia* [5] as an example highlighting a crucial aspect of the study of systemic risk in financial markets, that is the *connectivity* between financial counterparties. Many recent works [14, 15, 16, 17, 18, 4, 19] have studied extensively the problem of systemic risk through direct contagion. All these works stress the fact that the direct contagion of risk is related first of all to the topological features of the network describing the financial system, and in particular systemic risk depends strictly on the measure of connectivity introduced in these papers.

Figure 1.1: Network of large exposures between UK banks, 2008. *Source:* FSA returns. The exposure is an investment from a bank to another counterpart. A large exposure is one that exceeds 10% of a lending bank's capital. The data are correspond to the first quarter (three-month period) of 2008. Each node represents a bank in the United Kingdom. The size of each node represents the value of the total exposure of a bank. The darkness of a line is proportionate to the value of a single bilateral exposure. Figure is taken from *Gai et al.* [15].

In this thesis, differently from the approach of *Gai and Kapadia* [5], we focus on *indirect* contagion of risk due to feedback effects in financial markets. But, at the same time, the network structure of our model, represented by a bipartite graph, is always characterized by a measure of connectivity. In our approach, the connectivity will be defined by the degree of portfolio overlap between financial institutions investing in financial markets. This measure of connectivity has received great attention in recent works about the contagion of risk, see for example [18].

The advantage of the model [5] of *Gai and Kapadia* is the capacity of explaining very clearly the role of the network structure in direct contagion of risk, simplifying at most the economic framework.

In recent years, financial systems became more complex and interconnected, and as a consequence, they became more susceptible to systemic collapse. In Figure (1.1), the network of exposures between the major UK banks in the first quarter of 2008 is shown. The nodes represent banks, their size represents each bank's overall asset position in the interbank network, and the thickness of the links reflects the value of interbank exposures between institutions. The exposure represents simply an interbank lending or an investment of a bank in a financial asset owned by an other bank. For a quantitative analysis of the degree of interconnectedness in a typical banking system see [20].

The authors use network techniques developed in epidemiology and statistical physics to identify the point of instability in a financial system modeled by a random network in which individual banks are randomly linked together by their interbank exposures. They refer to the point of instability as the system condition in which a shock (liquidity default of a bank) triggers the system collapse (liquidity defaults of all banks).

The financial network consists of n financial intermediaries, banks, which are linked together

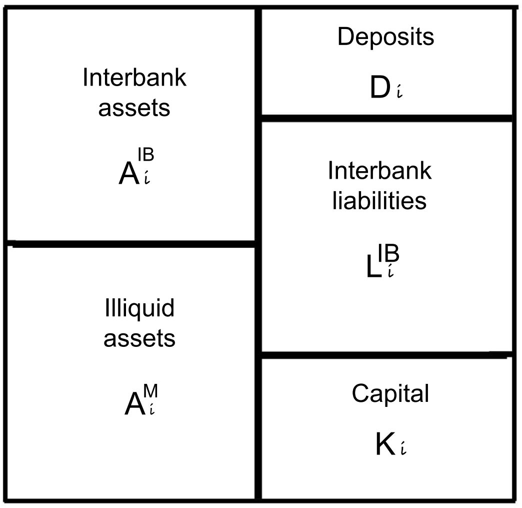


Figure 1.2: Stylized balance sheet defining the bank economic state in the model of *Gai and Kapadia* [5].

randomly by their mutual investments. Each bank is represented by a node of the network, and the interbank exposures of bank i define the links with other banks. These links are directed, reflecting the fact that interbank connections comprise both assets and liabilities. We suppose that each bank i has k_i incoming links representing its interbank assets (*i.e.* money owed to the bank by a counterparty) and j_i outgoing links representing its interbank liabilities (*i.e.* payments of debts). Since every interbank outgoing link for one bank is a incoming link for another bank, in the limit $n \to \infty$, the average number of outgoing links across all banks in the network must equal the average number of incoming links. The authors refer to this quantity as the average degree or connectivity of the system, and they denote it by z. The number of incoming links (asset investments) for each bank is randomly (according to the Poisson distribution) determined. As a consequence, the outgoing links (interbank liabilities) for each bank are automatically determined. From a mathematical point of view, the financial network just described, is a Erdös-Rényi graph [21].

The economic state of each bank of the financial network is defined by its balance sheet, see Figure (1.2). The total asset position of each bank consists of interbank assets, A_i^{IB}, and illiquid external assets, A_i^M, such as mortgages. The latter ones are called illiquid because when banks sell this type of assets in order to get cash, they obtain typically a cash value smaller than original asset value. This happens because the resale price does not coincide with the actual price before the trading. In fact, the selling of an illiquid asset lets down the price of the asset itself during the trading, as we will see subsequently. In this way, the obtained cash at the end of the trading is always smaller than the value of the illiquid asset before the trading.

In the model, the total interbank asset position of every bank is evenly distributed across its incoming links and it is independent of the number of links the bank has. Since, each interbank asset is a liability for another bank, interbank liabilities, L_i^{IB}, are automatically determined. Interbank liabilities represent for example the debts level of a bank in respect to a lender (another bank). In addition to interbank liabilities, the authors assume that the only other component of a bank's liabilities is given by deposits, D_i. Deposits represent money placed into a banking institution for safekeeping. A deposit is a liability owed by the bank to the depositor (the person or entity that made the deposit). Finally, capital, K_i, of a bank is simply the difference between the value of asset position $(A_i^{IB} + A_i^M)$ and the value of total liabilities $(L_i^{IB} + D_i)$. Typically, regulatory policies in financial markets require banks of maintaining capital above a specific level.

These assumptions on bank's balance sheet are stylized, but at the same time they provide a useful benchmark to investigate how the contagion of risk may spread within the financial system, and in particular the role of diversification in systemic risk. Diversification refers to the number of investments made by a bank, that is the connectivity characterizing the financial network.

In the financial system just described, systemic risk refers to a collapse of a macro-component of the financial network triggered by a random shock affecting only one bank (removal of a node of the network). In fact, the authors assume that initially all banks are solvent, that is the balance sheet of each bank is in equilibrium. At time $t = 1$, a bank j suffers a liquidity shock that leads the bank to default. As a consequence, bank j can not pay its debts, *i.e.* its liabilities are set equal to zero, $L_j^{IB} = 0$. Since a liability of the bank j corresponds to an interbank asset for a counterparty i and interbank assets of bank i are evenly distributed across its incoming links k_i, at time $t = 2$, bank i suffers a liquidity shortfall equal to $\frac{A_i^{IB}}{k_i}$.

The condition for bank i to remain solvent is

$$\left(1 - \frac{1}{k_i}\right) A_i^{IB} + q\, A_i^M - L_i^{IB} - D_i > 0 \tag{1.1}$$

where q is the resale price of the illiquid asset, as we have explained before. The value of q may be less than 1 in the event of asset sales by banks in default. The solvency condition (1.1) is simply a constraint on capital. On the contrary, the shock propagates within the financial network if there is at least a neighbouring bank i for which

$$\left(1 - \frac{1}{k_i}\right) A_i^{IB} + q\, A_i^M - L_i^{IB} - D_i < 0 \quad \longrightarrow \quad \frac{K_i - (1-q)A_i^M}{A_i^{IB}} < \frac{1}{k_i} \tag{1.2}$$

If the condition (1.2) holds, then contagion starts to spread. In particular, a second bank is forced to default. In turn, this second default may create liquidity shortfalls inducing other banks to default, and so on. The authors define banks that are exposed in this sense to the default of a single neighbour as vulnerable and other banks as safe. The vulnerability of a bank clearly depends on the number of its incoming links, that is a random variable. Hence, a bank i with k incoming links is vulnerable with probability

$$P_k = P\left[\frac{K_i - (1-q)A_i^M}{A_i^{IB}} < \frac{1}{k}\right] \quad \forall k \geq 1$$

In principle, in sufficient large networks, the problem of contagion can be solved analytically using an approach based on generating functions [21], and in particular on the generating function

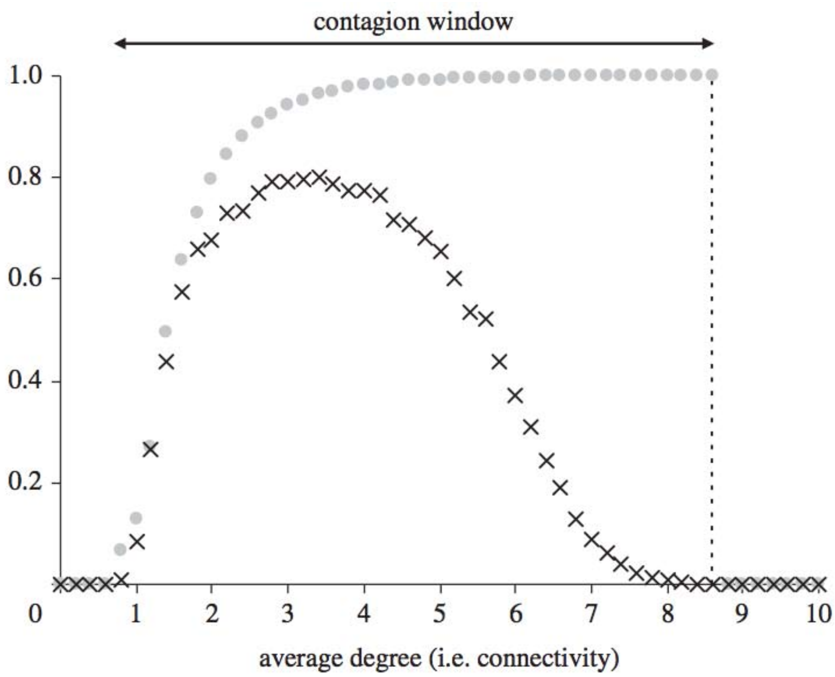


Figure 1.3: Direct contagion of liquidity shortfall in the financial network. The Figure is obtained with 1000 numerical simulations of the model. Grey circle represents the average extent of contagion as a function of the connectivity. For example, when the average connectivity is equal to 2, in the case of systemic event, after the initial liquidity shock affecting a bank randomly chosen, the direct contagion of liquidity shortfall propagates to 700 banks. Since the authors consider a financial network formed by 1000 banks, the fraction of the financial network affected by a liquidity shock is equal to 0.7, as reported in the y-axis. Cross represents the frequency of contagion as a function of connectivity. The frequency represents simply the ratio between the times in which at least the contagion is equal to 0.1 and the total number of simulations. For example, when the average connectivity is equal to 3, over a number of 1000 simulations, there are approximately 800 systemic events in which the 90% of the entire financial network suffers a liquidity shortfall. Hence, in this framework the frequency measures the probability of systemic risk due to an initial stochastic shock, as a function of the connectivity. Figure is taken from *Gai and Kapadia* [5].

for the probability distribution of vertex degrees k, see [5]. We show the most intuitive result based on numerical simulations of the model just described.

Figure (1.3) summarizes the numerical results obtained by simulating the model of financial contagion in financial networks. The randomness affects only the structure of financial network (links are created randomly according to the Poisson distribution) and the initial liquidity shock (at time $t = 1$, a randomly chosen bank suffers a liquidity shortfall). The contagion mechanism is a deterministic process according to (1.2). The authors consider a network of $n = 1000$ banks and they assume the asset side of the balance sheet, see Figure (1.2), of all banks to be identically composed: $A_i^M = 80\%$ of total asset position, $A_I^{IB} = 20\%$ of total asset position. As a consequence, the interbank liabilities are determined automatically once the interbank assets (A_i^{IB}) have been fixed. The liability side is composed of: $K_i = 4\%$ of total asset position, and deposit, D_i, are chosen so that initially the balance sheet is topped up by deposits until the total liability position equals the total asset position[2].

Figure (1.3) is obtained through 1000 simulations of the model for each value of connectivity.

[2]Notice that bank's capital K_i is listed in the liability side in order to stress that $K_i = assets - liabilities$

The extent of contagion (number of banks suffering a liquidity shock with respect to the total number of banks, $n = 1000$) as a function of connectivity is shown (grey circles). Also the frequency of contagion as a function of connectivity is shown (crosses). The frequency of systemic liquidity crises refers to the systemic events in which at least 10% of banks are forced to default in consequence of the (stochastic) default of the first bank.

This result highlights that financial system is characterized by two transitions. The first one occurs for a small value of connectivity (in Figure (1.3) $z \approx 1$). This first transition point refers to a network property, that is the appearing of connected components of increasing size according to the increase of connectivity. The spread of contagion depends strictly on the presence of network patterns from one node to the other ones. In this model, the direct contagion of risk is essentially a percolation process [22, 23, 24, 25] in which the *resilience* of the financial network is related to the connectivity, which defines the average size of the connected components of the network.

When a bank of one of these components, suffers a liquidity default, the contagion spreads to the whole connected component. When the connectivity increases further, there is only a macro-component (*giant* component), that is a component whose size is approximately equal to the size of network. In this case, when the initial shock propagates, the contagion spreads to the entire financial network.

The second transition (in Figure (1.3) $z \approx 8$) is related to the balance sheet of banks, thus it occurs for an economic reason. Beyond the transition point, the financial network is very interconnected and as a consequence, total interbank assets, A_i^{IB}, of every bank are evenly distributed over a large number of incoming links. Hence, even if a bank suffers a liquidity shock, which induces it to default, the balance sheet of all counterparties of the bank is not affected crucially, thus all other banks remain solvent. The initial shock does not propagate.

The two transition points define the contagion window in which a liquidity shock may trigger a systemic direct contagion of risk. Hence, systemic risk in a financial network strictly depends on the degree of connectivity of the network which describes financial system. The connectivity is a measure of how much the financial network is interconnected and thus how much it is fragile.

1.2 Perturbations and financial crisis dynamics

Douady and Choi [6] [13] study the direct contagion of risk through a dynamical systems approach. In a dynamical framework, the key point is to investigate how a perturbation affecting the financial market increases or decreases during the dynamical evolution of the system.

Douady and Choi describe a very general framework that is in good agreement with the financial instability hypothesis of *Minsky* [26]. Indeed, the authors find through their analysis that there are two regimes in financial markets, one stable and one unstable. In the stable regime, a perturbation can be absorbed by the system, while in the unstable regime an initial perturbation propagates within the system and its size increases during dynamical evolution, finally leading to a financial crisis.

A dynamical systems approach may capture very well this feature through the study of

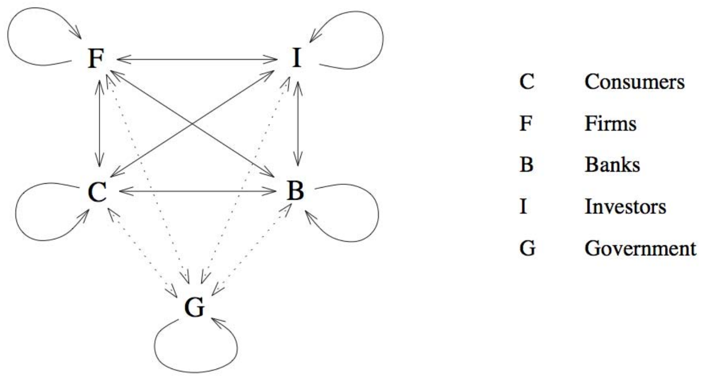


Figure 1.4: A financial system of five agents connected by flow of funds among them: consumers (C), firms (F), banks (B), investors (I), and government (G). The cash flows from an agent to an other define the financial network. For example, the connections can be represented as: C and C are connected through house investments, C and B through debt investments, C and I through pension funds, C and F through salaries and consuptions, F and F because companies invest in each other, F and B through loans and stocks, F and I through stocks and bonds, B and B through coupons and interbank lending, B and I through bonds, stocks and loans, I and I through derivatives coming from investments. Finally, G is connected to other agents through taxes, salaries, coupons of bonds, and incentive actions in case of financial recession. Figure is taken from [13].

bifurcations, which refer to structural modification of the system behavior upon a continuous change in the parameters of its equations.

Recent works of *Castellacci and Choi* [27, 28] exploit a dynamical systems approach to study the contagion of financial instabilities between different economies. For example, these works try to explain how the financial crisis that has affected the USA economy in 2007, propagated within the eurozone, leading to the euro crisis in 2009.

However, *Douady and Choi* [6] [13] describe only a general framework in order to stress the possibility of defining a *market instability indicator* as the largest eigenvalue of the Jacobian matrix which defines at the first order the evolution of perturbations in financial markets. On the contrary, in our thesis we present a dynamical model in which we can analyze the specific market conditions corresponding to the breaking of systemic financial stability. Specifically, we are able to describe what type of bifurcation is associated with the loss of stability and we can quantify in some sense the degree of systemic risk in the considered financial system.

In the model [6], the authors define the financial system as a network formed by n financial agents. Each agent represents the class of aggregates financial units operating in the market with the same strategy. Each unit is connected to counterparties through the flow of funds between them: investments and debts, as an example see Figure (1.4). As explained in the previous Section, the financial network describing an economy is interconnected.

Here, the authors stress also another fact that is financial agents typically borrow money (debts D) in order to invest in assets (A) a quantity larger than their initial equity (E). Let λ be the financial leverage

$$\lambda = \frac{A}{E}$$

Financial agents are leveraged in order to maximize their profits. This fact has also the opposite consequence. When a liquidity shock occurs, for example triggered by the falling of asset prices, investors lose a large value of their initial investment, and as a consequence it may happen that investors can not pay their debts. Hence, it follows that lenders suffer a liquidity shortfall. In turn, lenders can not give new loans, and so on. In this way, shocks propagate within the financial system.

In describing the dynamical framework in which to investigate financial crises, the authors assume that the economy is based on limited material resources and market participants, therefore the wealth of each financial agent is bounded from above. The boundedness hypothesis has the mathematical consequence that the dynamical system evolves in a compact space, allowing the authors to use some results of dynamical systems theory (in Chapter 2, some elements of dynamical systems theory are described).

The very general setting described by the authors is based on the assumption that the economy at time t is defined by a wealth vector $\mathbf{w}(t) = (w_1(t), \dots, w_n(t)) \in \mathbb{R}^n$, where $w_i(t)$ is the wealth of agent i at time t. The wealth of agent i at time t is defined as the sum of the equity, $E_i(t)$ and debt, $D_i(t)$

$$w_i(t) = E_i(t) + D_i(t)$$

At the same time, wealth can also be divided into liquid assets, $A_i^L(t)$ (cash or assets can be converted into cash), and invested assets, $A_i(t)$

$$w_i(t) = A_i^L(t) + A_i(t)$$

Essentially, the wealth state of agent i evolves in time according to the dynamics for the debt, $D_i(t)$, and the invested assets, $A_i(t)$. The debt at time $t+1$,

$$D_i(t+1) = (1 + r_{i,t}^D)D_i(t) + \Delta D_i(t+1),$$

is equal to the previous debt level considering the average interest expense ($r_{i,t}^D$), plus the new loans obtained at time $t+1$ ($\Delta D_i(t+1)$). Each agent i has a maximum level of debt,

$$D_i(t) \leq D_{i,max}(t),$$

that depends on its equity, $E_i(t)$, representing in another form the constraint on a minimum capital requirement.

The invested assets at time $t+1$,

$$A_i(t+1) = (1 + r_{i,t}^A)A_i(t) + \Delta A_i(t+1),$$

is equal to the assets value at time t considering the stochastic variations of asset prices (described by a (stochastic) return $r_{i,t}^A$), plus the new investments at time $t+1$ ($\Delta A_i(t+1)$).

The quantities $\Delta D_i(t+1)$ and $\Delta A_i(t+1)$ depend on the flow of funds, $F_{i,j}(t)$ and $F_{j,i}(t)$. $F_{i,j}(t)$ is equal to funds transferred from agent j to agent i at time t. For agent i, it represents new loans or profits from previous investments. $F_{j,i}(t)$ is equal to funds transferred from agent i to agent j. For agent i, it represents new investments or payments of previous debts.

The flows of funds together with the dynamics of the debts and assets define the dynamics of the liquidities. In fact, liquidities at time $t+1$ is equal to

$$L_i(t+1) = L_i(t) + \sum_{j=1, j\neq i}^{n} F_{i,j}(t) - \sum_{k=1, k\neq i}^{n} F_{k,i}(t) - \Delta A_i(t+1)$$

The flows of funds between financial agents are the solution of an optimization problem that provides the optimal cash flows maximizing the profits of financial agents. The authors do not describe explicitly how $F_{i,j}(t)$ are found, because it is inessential to the aim of their paper. For a discussion about this point, see [6].

According to the previous relations, the wealth dynamics of agent i is described by the following difference equation

$$w_i(t+1) = w_i(t) + \sum_{j=1}^{n} F_{i,j}(t) - \sum_{k=1, k\neq i}^{n} F_{k,i}(t) \tag{1.3}$$

where in the first sum, the diagonal component $F_{i,i}(t)$ represents the assets return due to the stochastic evolution of asset prices ($F_{i,i}(t) = r^A_{i,t} A_i(t)$).

Through the n difference equations identified by (1.3), the economy is described by the dynamics of wealth vector $\mathbf{w}(t)$. Symbolically, a dynamical system is

$$\mathbf{w}(t+1) = \mathbf{f}\left[\mathbf{w}(t)\right] \tag{1.4}$$

Mathematically, the n difference equations, (1.4), describe a n-dimensional dissipative[3] dynamical system, see Chapter 2. Specifically, $\mathbf{f}$ is characterized by nonlinear terms due to the presence of constraints, like the debt constraint ($D_i(t) \leq D_{i,max}(t)$). Finally, the dynamical system described by $\mathbf{f}$ is random, due to stochastic evolution of asset returns described by the stochastic variables $r^A_{i,t}$, $i = 1, \dots, n$. Assuming that the return processes, which describe these stochastic variables, are stationary, we can consider the average values of the stochastic variables. In this way, we can identify a deterministic dynamical system described by $\bar{\mathbf{f}}$:

$$\mathbf{w}(t+1) = \bar{\mathbf{f}}\left[\mathbf{w}(t)\right] \tag{1.5}$$

A perturbation affecting the wealth vector at time t, $\delta\mathbf{w}(t)$, reflects a liquidity shortfall affecting a financial agent. In the dynamical approach established by the authors, the propagation of an initial perturbation (direct contagion of risk) is described up to the first-order approximation through the (*local*) Jacobian matrix of system $\bar{\mathbf{f}}$, $\mathcal{D}_{\mathbf{w}(t)}\bar{\mathbf{f}}$, calculated at the actual system state $\mathbf{w}(t)$:

$$\delta\mathbf{w}(t+1) = \left[\mathcal{D}_{\mathbf{w}(t)}\bar{\mathbf{f}}\right] \cdot \delta\mathbf{w}(t) \tag{1.6}$$

The contagion spreads to the entire financial system if the size of initial perturbation increases in

[3]A financial system is typically an open system in which there is not a conserved quantity ("Hamiltonian") and there is a path dependence in the dynamical evolution. In general, this two aspects define a financial system as a dissipative structure.

time. From a mathematical point of view, the potential for a contagion is related to the largest eigenvalue of the Jacobian matrix. When the modulus of the largest eigenvalue is smaller than 1, the system absorbs the initial perturbation. On the contrary, when the modulus is larger than 1, perturbations contain at least a component that will increasingly propagate within the system. As a consequence, the contagion may lead to a system collapse. The modulus of the largest eigenvalue implicitly depends on financial leverage λ adopted by financial agents operating in the market, see [13]. Indeed, the larger is the leverage adopted by financial agents, the larger is the variation of wealth. In fact, according to a stochastic variation of asset prices, the value of earning or loss of a financial agent depends on the adopted leverage. For example, if a financial market is populated by highly leveraged agents, after a stochastic shock of prices, financial agents lose a lot of money due to their strategy based on the financial leverage. Hence, starting from a stochastic variation of wealth, the following variations increase in time if financial agents are highly leveraged, and this mechanism can be mathematically described through (1.6), where the largest eigenvalue of the Jacobian matrix is larger than 1. Hence, from an economic point of view, the larger is financial leverage, the larger is the probability of contagion.

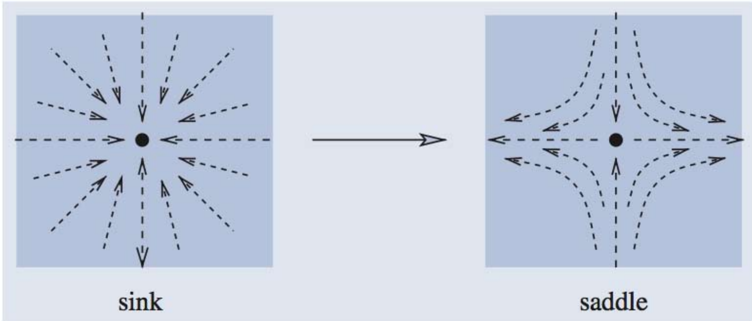
A subtle mathematical aspect is related to the fact that the Jacobian matrix is a local quantity because its entries are calculated at the actual system state, $\mathbf{w}(t)$. A potential advantage of this approach is related to the possibility of estimating empirically the largest eigenvalue through a regression analysis of flows of funds from an agent to one other, see [6].

Douady and Choi highlight that in a dynamical systems approach the stability of a financial system is related to the eigenvalues of the Jacobian. Specifically, the systemic financial stability is lost when a bifurcation occurs as consequence of changed market conditions, mathematically described by a change of model parameters (see Chapter 2 for mathematical aspects). A bifurcation refers to a dynamical transition from a stable evolution (characterized by a stable fixed point, also called a *sink*) to a more complex one.

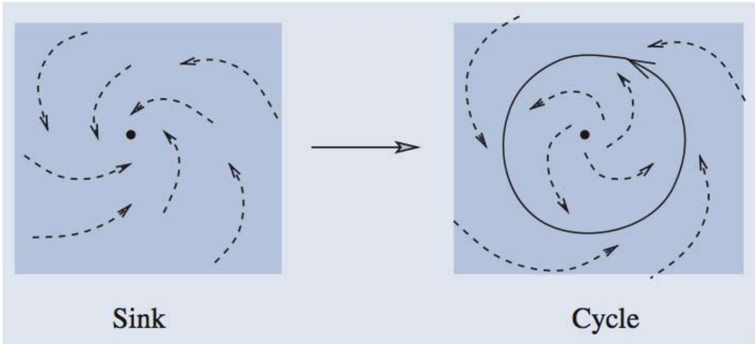
In the explained setting, two possibilities occur, see [29] for a mathematical description of bifurcations. When the largest eigenvalue crosses the critical value equal to 1, the equilibrium becomes a saddle, that is a system state characterized by at least one unstable direction in phase space, see panel (a) of Figure (1.5). The financial system moves away from the old equilibrium and typically this situation describes a financial recession and in the worst scenario, a system collapse.

When the largest eigenvalues are a pair of complex conjugate numbers and their modulus crosses the critical value equal to 1, the equilibrium evolves from a sink to a cycle, see panel (b) of Figure (1.5). In this case, the system circles along the new attracting cycle and financial agents experience a sequence of growths and contractions in their wealth. In this scenario, systemic risk can be measured as the amplitude of system cycles. In fact, the larger is the amplitude of cycles, the larger is the probability that system exceeds the natural bounds of the economy (boundedness hypothesis) in which any shock may trigger the system collapse.

The advantage of a dynamical systems approach to the problem of systemic financial stability and systemic risk, as shown in this work, is that the study of system stability naturally appears looking at the Jacobian matrix of the dynamical system. Furthermore, the distance from a



(a) saddle-node bifurcation



(b) Andronov-Hopf bifurcation

Figure 1.5: Panel (a): Saddle-node bifurcation. Nearby points move away from the saddle in the horizontal direction and drift toward another sink or an attractor. Panel (b): Andronov–Hopf bifurcation. As the equilibrium goes from attracting to repelling with a pair of complex eigenvalues with modulus greater than 1, a periodic cycle appears. Figures are taken from [6].

bifurcation, which represents the loss of systemic stability, can be "measured", estimating the eigenvalues of the Jacobian.

1.3 Feedback effects in financial markets

In the previous Sections, we have presented two approaches to systemic risk due to the direct contagion between financial institutions operating in the market. Another point of view about systemic risk in financial markets is related to *feedback* effects coming from distressed selling of asset position or rebalancing of large financial portfolios. Feedback effects have been recognized as a destabilizing factor in recent financial crises [30, 31, 32, 33, 7].

In this Section, we review recent empirical evidences about feedback effects in financial markets and we describe how the contagion between institutions can be indirectly mediated by these feedback effects, due to the overlapping investments and overlapping portfolios. For this purpose, we refer to a model of *Cont and Wagalath* [8].

The model of *Corsi et al* [9], described in Chapter 3 and constituting the starting point for work in this thesis, exploits an approach that is similar to the model of Cont and Wagalath, in order to taking into account the feedback effects due to a particular strategy of trading, adopted by financial institutions operating in financial markets. In the model of Corsi et al., feedback effects play a crucial role in defining the stability of return processes and systemic risk measured

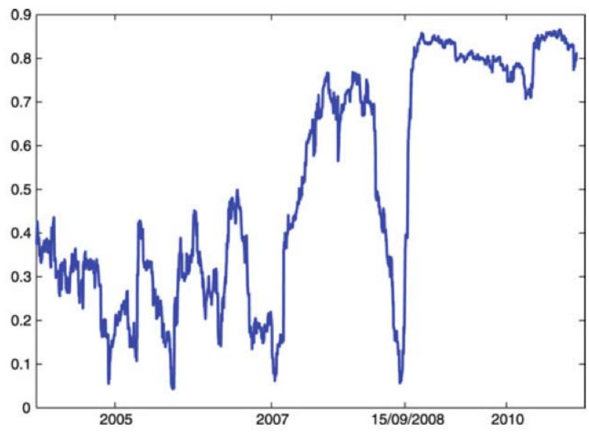


Figure 1.6: EWMA estimator of correlation of daily returns between two sector indices of the S&P 500: SPDR XLE (energy) and SPDR XLK (technology). Figure is taken from *Cont and Wagalath* [8].

as the probability of default of financial institutions.

The underlying fact is that in financial markets the selling (or the buying) of an asset moves its price (*price impact* [34, 35, 36]). When, for example, a financial institution suffers a liquidity shortfall, it sells assets, or in the worst case financial institutions may liquidate the whole asset position. The price impact of the trading leads to a fall of asset prices. When prices fall, due to the overlapping portfolios, other financial institutions begin to sell and as a consequence the mechanism of falling prices becomes self-reinforcing. This mechanism is in effect a positive feedback.

A mechanism of this type explains very well the empirical evidences associated with jumps in correlations between assets (or asset classes) following the collapse of financial institutions, as the case of Lehman Brothers on September 15, 2008, see Figure (1.6). Figure shows the exponential weighted moving average (EWMA) estimator of realized correlation, $Corr^{EWMA}$, between two sector indices of the S&P 500: SPDR XLE (energy) and SPDR XLK (technology). The latter ones can be seen as two investment assets traded in stock markets of USA economy.

Given the time series of prices $\{p_t\}$, the return is simply equal to

$$r_t = \frac{p_{t+\Delta t} - p_t}{p_t}$$

Returns are the percentage of gain or loss in a given time period Δt.

The quantity $Corr^{EWMA}$ is an estimator of realized correlation at time t between the pair of assets. It is similar to the sample correlation, in which however the centered returns are weighted through a memory factor decreasing (backwards) in time according to a geometric series. The weights are defined through a parameter, $\omega \in [0, 1]$. This parameter is called the memory of the estimator because the elements of the sum at each time step are weighted according to the value of ω. Specifically, when $\omega \to 1$, the last realizations of returns are weighted largely. On the contrary, when $\omega \to 0$, the past realizations have more weight in defining the actual value of the estimator.

In Figure (1.6), Δt corresponds to 1 day. The EWMA correlation, $Corr^{EWMA}$, is equal to

$$Corr^{EWMA}(r_t^{XLE},\, r_t^{XLK}) = \frac{Cov_\omega(r_t^{XLE},\, r_t^{XLK})}{\sqrt{V_\omega(r_t^{XLE})\, V_\omega(r_t^{XLK})}}$$

where

$$Cov_\omega(r_{1,t},\, r_{2,t}) = \frac{1-\omega}{\omega} \sum_{i=1}^{\infty} \omega^i (r_{1,t-i} - \bar{r}_1)\,(r_{2,t-i} - \bar{r}_2)$$

and

$$V_\omega(r_t) = \frac{1-\omega}{\omega} \sum_{i=1}^{\infty} \omega^i (r_{t-i} - \bar{r})^2$$

The marked quantities represent the sample averages. Hence, $\bar{r}_1$ is the sample mean of the time series of returns associated with the investment asset 1. Similarly, $\bar{r}_2$. $\omega \in [0, 1]$ is the memory factor.

At date 15/09/2008, the EWMA correlation displays a significative positive jump. The common interpretation is the following: after the collapse of the Lehman Brothers institution on September 15, 2008, the distressed liquidation of asset position has induced the falling of prices of the assets in the portfolio. As a consequence, also slightly correlated returns have showed a jump in the correlation function, due to feedback effects arising from distressed selling [32].

This is only one empirical evidence of how a positive feedback may crucially affect the systemic stability of financial markets. The positive feedback has three important consequences on investment assets, affecting *returns*, *volatilities*, and *correlations*.

In order to explain feedback effects in financial markets, we focus on the most important aspects of *Cont and Wagalath* model [8].

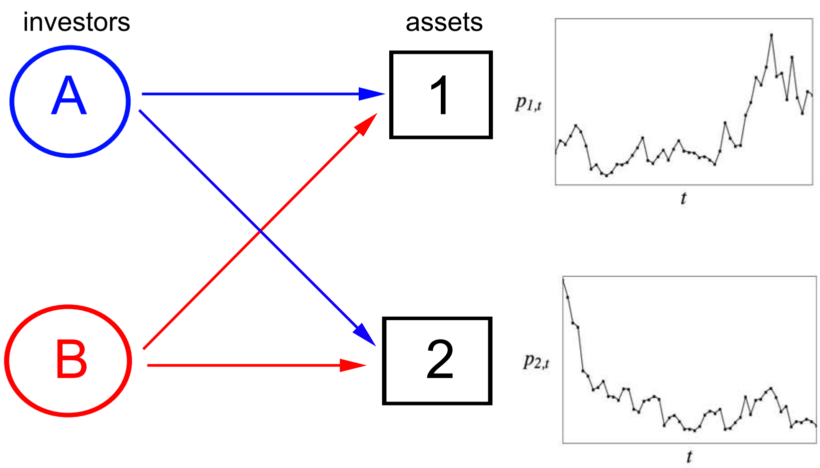


Figure 1.7: Stylized financial market formed by two financial institutions, A and B, investing in two assets, 1 and 2. Each asset is described by a stochastic variable (the price, p_1 and p_2), which evolves in time according to a discrete Wiener process. The right panels show qualitatively how the prices of investment assets may evolve in time.

Consider a market with two financial institutions, Alice (A) and Bob (B), investing in two assets, see Figure (1.7). Each asset is described by a discrete Wiener process defining its price at time t. Hence, the asset return is a normally distributed stochastic variable

$$r_{j,t} = m_j + \epsilon_{j,t} \ , \quad j = 1, 2 \tag{1.7}$$

where $\epsilon_{j,t}$ is a Gaussian noise, $\mathcal{N}(0, \sigma_j)$, and m_j represent the expected mean of return. Without loss of generality, the authors assume $m_j = 0$ for $j = 1, 2$. The two noises, $\epsilon_{1,t}$ and $\epsilon_{2,t}$, are uncorrelated, *i.e.* $\mathbb{E}[\epsilon_{1,t} \, , \, \epsilon_{2,t}] = 0$. Each Gaussian noise is characterized by a standard deviation σ_i. In Quantitative Finance, when the asset returns are assumed to be normally distributed stochastic variables, the standard deviation σ_i can be considered a risk measure, and it is called *volatility.*

Assuming for simplicity $\Delta t = 1$, the price at time t is simply equal to

$$p_{i,t} = p_{i,t-1} + p_{i,t-1} \, r_{i,t} \tag{1.8}$$

We consider Bob as an investor which has created initially his portfolio by investing in the assets 1 and 2. However, during the stochastic evolution of asset prices he does not trade assets. Hence, Bob does not influence the evolution of asset prices and he works as a spectator. Bob represents all investors in financial market, except Alice. At initial time, $t = 0$, Alice invests in the two assets and the initial investment V_0 is equal to

$$V_0 = \sum_{i=1,2} a_i \cdot p_{i,0} = \sum_{i=1,2} a \cdot p_{i,0} \tag{1.9}$$

where, for the sake of simplicity, the authors assume $a_1 = a_2 = a$. The value of Alice's investment, V_t, evolves according to (1.8). If the investment value drops below a threshold $\beta_0 V_0 < V_0$, Alice progressively may exit her position, generating a negative demand across two assets, proportionally to the positions held by Alice. The authors purpose is to model the price impact of this *distressed selling* and investigate its effect on realized volatility and correlations of two assets held by Alice.

If at time t the value of Alice's investment is $V_t < \beta_0 V_0$, Alice progressively liquidates her asset position. She sells her assets such that, at time $t+1$, the value of her investment decreases to $V^*_{t+1} < V_t$. To model the distressed selling, the authors introduce a *concave* function $f : \mathbb{R} \to \mathbb{R}$ which measures the rate at which Alice exits her position: when investment value drops from V_t to V^*_{t+1}, Alice redeems a fraction $f\left(\frac{V_t}{V_0}\right) - f\left(\frac{V^*_{t+1}}{V_0}\right)$ of her asset position, see [8] for the mathematical properties of f. Thus, the net supply in two assets due to the distressed selling is equal to

$$-a\left(f\left(\frac{V^*_{t+1}}{V_0}\right) - f\left(\frac{V_t}{V_0}\right)\right)$$

The assumptions on Alice's behavior imply that $f : \mathbb{R} \to \mathbb{R}$ is continuous, everywhere increasing, except for $[\beta_0, +\infty]$ where it is constant.

Finally, the authors assume that the price impact due to the trading, is linear in the supply of assets. This means simply that the return of an asset at the future time is affected by a negative contribution that is proportional to the net supply of the same asset due to the distressed selling. The proportionality constant is indicated as $\frac{1}{\gamma_j}$. γ_j is called the market *depth* (or asset *liquidity*). For a discussion about the assumption of linearity see [35].

As a consequence, the asset return at time $t+1$ is affected by the impact of trading

$$r_{j,t+1} = \epsilon_{j,t+1} + \frac{a}{\gamma_j}\left(f\left(\frac{V^*_{t+1}}{V_0}\right) - f\left(\frac{V_t}{V_0}\right)\right) \tag{1.10}$$

The larger is γ_j, the smaller is the price impact on the asset j. The meaning of γ_j is the following: a net supply (or demand) of $\frac{\gamma_j}{100}$ shares (asset unities) moves the price of j by 1%. The authors assume $\gamma_1 = \gamma_2 = \gamma$.

It is important to notice that in (1.10) the contribution of the second term is negative because f is concave and increasing, and $V^*_{t+1} < V_t$.

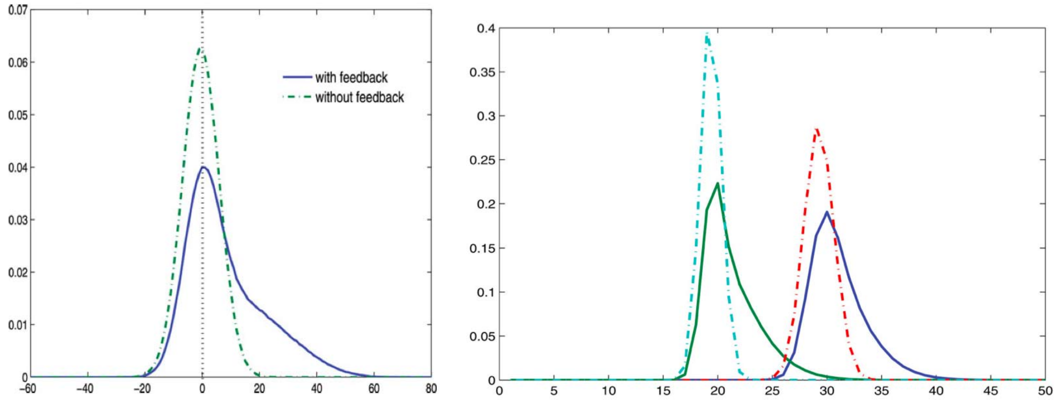
The successive selling of the assets induces feedback effects on asset returns. Once the value of Alice's investments suffers a negative shock ($V_t < \beta_0 V_0$), the distressed selling induces a cascade of negative realizations of asset returns due to the price impact of the trading (second term in (1.10)). Feedback effects arising from the distressed selling, amplify the initial shock and, as a consequence, increase systemic risk of financial markets.

In fact, in addition to trigger a cascade of negative returns, feedback effects increase correlation and volatilities of the two assets. Indeed, feedback effects have the consequence of amplifying the stochastic fluctuations of returns, specifically in this model when these fluctuations are negative. As a consequence, the amplitude of the fluctuations is larger and the volatility increases. At the same time, feedback effects induce a positive correlation between initially uncorrelated assets. In fact, when Alice's portfolio suffers a negative stochastic shock, Alice sells the entire asset position formed by the two investment assets, 1 and 2. Hence, during the cascade of negative returns, the return processes associated with the two investment assets are characterized by the same negative contribution due to the distressed selling of Alice. Hence, the distressed selling induces a positive correlation between the asset returns.

In Figure (1.8) the probability distributions of realized correlation and realized volatilities of the two assets are obtained by performing a Monte Carlo simulation of the model. The authors simulate the model just described, in a time window of 1 year. Hence, $\Delta t = 1\ day$ means that they consider *daily* returns. Finally, the standard deviations of Gaussian noises are equal respectively to $\sigma_1 = 30\%\ year^{-\frac{1}{2}}$ and $\sigma_2 = 20\%\ year^{-\frac{1}{2}}$. The other values of parameters are reported in Figure (1.8).

Differently from the initial part of this Section where the EWMA estimator is used, the authors focus on the sample covariance and the sample variance of the times series of returns, in order to find realized correlations and volatilities. For each simulated path, the realized sample covariance between asset 1 and asset 2, is

$$Cov_{1,2} = \frac{1}{T}\sum_{t=0}^{T-1}(r_{1,t} - \bar{r}_1)(r_{2,t} - \bar{r}_2)$$



(a) realized correlation

(b) realized asset volatilities

Figure 1.8: Panel (a): Probability density function of realized correlation between two initially uncorrelated assets, with and without feedback effects due to distressed selling. Panel (b): Probability density function of realized volatilities for each asset, with (continuous lines) and without (dotted lines) feedback effects due to distressed selling. In the x-axis, the values are considered in percentage. For example, a realized sample correlation equal to 40% corresponds to 0.4. The probability density functions are obtained through a Monte Carlo simulation of the model. The model parameters are fixed to: $\sigma_1 = 30\%\, year^{-\frac{1}{2}}$, $\sigma_2 = 20\%\, year^{-\frac{1}{2}}$, $\frac{a}{\gamma} = \frac{1}{10}$, $\frac{a\, p_{1,0}}{V_0} = \frac{a\, p_{2,0}}{V_0} = \frac{1}{2}$, $\beta_0 = 0.95$. The function f is chosen to be $f(x) = \frac{-1}{(\beta_{liq}-\beta_0)^4}(x-\beta_0)^4$, where $\beta_{liq} = 0.55$. Figures are taken from *Cont and Wagalath* [8].

and the realized sample variance of asset j, is

$$V_j = \frac{1}{T}\sum_{t=0}^{T-1}(r_{j,t} - \bar{r}_j)^2$$

where $\bar{r}_j = \frac{1}{T}\sum_{t=0}^{T-1} r_{j,t}$ is the sample mean of return of asset j. According to the previous relations, the realized correlation between assets 1 and 2 is $\frac{Cov_{1,2}}{(V_1 V_2)^{\frac{1}{2}}}$, while the realized volatility of asset j is $V_j^{\frac{1}{2}}$.

In Figure (1.8) the probability distributions of correlation and volatilities without feedback[4] are shown using dotted lines, while the same distributions, when feedback effects are accounted for, are shown using continuous lines. The latter ones display a heavy positive tail. This feature suggests that feedback effects are associated with a larger probability that large correlations and high volatilities occur.

The larger is the correlation between two assets, the larger is the probability that two negative realizations of asset returns occur simultaneously. The larger is the volatility, the larger is the probability of a loss. In fact, volatility gives an estimation of the amplitude of return's oscillations around the mean. Hence, a larger value of volatility is associated with more probable negative realizations of asset return. In this sense, feedback effects increase systemic risk in financial markets.

Finally, feedback effects are the cause of indirect contagion between financial institutions. In fact, while Alice liquidates her asset position, Bob suffers the negative realizations cascade of asset returns, and finally, when Alice is out from the trading, Bob's investment is characterized

[4]Feedback effects are simply not considered assuming that Alice does not trade assets like Bob.

by high volatility and increased correlation between assets in the portfolio. In this sense, feedback effects induce indirect contagion of risk between financial institutions.

In the model of *Cont and Wagalath* [8] an important aspect emerges, that is the difference between *exogenous* and *endogenous* quantities. In quantitative finance, an exogenous quantity refers to a variable coming from outside the system. For example, in (1.10) the Gaussian noise, $\epsilon_{i,t}$, is an exogenous variable in the sense that it is unexplained by the considered model. On the contrary, an endogenous quantity refers to a variable that describes a particular effect which is explained inside the model. The second term in (1.10) is the endogenous component of return and it describes the impact of positive feedback. In particular, also the risk of assets is characterized by an endogenous component describing the impact of feedback effects on risk. In Figure (1.8) the heavy positive tail (excess risk) can be interpreted as the contribution of an endogenous component.

1.4 Relation between the reviewed literature and our model

In our thesis, we propose a dynamical systems approach to systemic risk in financial markets. However, differently from *Douady and Choi* [6], we describe a specific dynamical model, in which we can analyze in detail the dynamical behavior of the financial system under investigation. In our case, the dynamical system is based on a network model, described by a bipartite graph[5]. Like the case of *Gai and Kapadia* [5], the connectivity of the financial network plays an important role in the study of systemic financial stability. The degree of overlap between portfolios of financial institutions represents the measure of connectivity (see Chapter 3). Finally, we investigate the case of indirect contagion of risk, mediated by feedback effects, similarly to the case of *Cont and Wagalath* [8]. Specifically, we look at the impact of feedback effects on the risk of assets. The strength (due to the financial leverage adopted by financial institutions) and the coordination (due to connectivity of the financial network) of feedback effects define the dynamical behavior of the financial system. In our case, systemic risk is related to the occurrence of bifurcations that lead to the breaking of systemic financial stability.

[5]A bipartite graph is a network characterized by two types of nodes and links run only between nodes of unlike types. In describing a financial system, financial institutions represent one kind of nodes. These link to the second kind's nodes that represent the investment assets.

Chapter 2

Dynamical systems theory

In this Chapter, we describe some elements of dynamical systems theory. One of the most interesting aspect related to dynamical systems is that they have led to successful results in many fields of natural sciences. One famous example comes from the pioneering work of *May* [37] about the logistic map. May highlighted the relevance of the nonlinearity in dynamical systems. Also from simple nonlinear systems, a very complex dynamics may arise and sometimes chaos may occurs. Furthermore, *Feigenbaum* [38] discovered that there are universal laws which describe the dynamical behavior of a system, defining how the transition from stable to chaotic behavior occurs. It was a surprising fact noticing how apparently different systems in natural sciences display the same dynamical features.

We are interested in dynamical systems in order to approach the problem of systemic risk in economy.

A dynamical system is simply a fixed rule which describes what future state follows from a current state of the system. We are particularly interested in dissipative dynamical systems described by difference equations, a map.

Generally, a dissipative dynamical system depends on different parameters. The dynamical behavior of the system is defined by the values of these parameters. When the value of a system parameter changes, a bifurcation may occur. A bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of the system causes a sudden qualitative or topological change in its behavior. Specifically we are interested in the period-doubling bifurcation that is a bifurcation in which the system switches to a new behavior with twice the period of the original system. In the case of the period-doubling cascade to chaos, successive period-doubling bifurcations lead the dynamical system towards deterministic chaos.

In Section 2.1, we give a mathematical definition of a dynamical system, particularly looking at discrete dynamical systems.

In Section 2.2, we will focus on the local aspects of the dynamical systems theory highlighting the theory of bifurcations from equilibrium based on center-manifold reduction and Poincaré-Birkhoff normal form. We will investigate the case of period-doubling bifurcations. We particularly refer to [29] in this Section.

In Section 2.3, we will focus on the non-local aspects of dynamical systems exhibiting chaos.

The characterization of a chaotic system requires an approach based on the ergodic theory of dynamical systems. We will describe quantities related to dimension, entropy and characteristic exponents of a dynamical system. We particularly refer to [39] in this Section.

2.1 Dynamical system

A dynamical system is the mathematical formalization of a rule which describes the time evolution of a system in its phase space. Let $\mathbf{f}$ be a function defined in a manifold X, a dynamical system is a 3-tuple $(T,\ X,\ \mathbf{f})$

$$\mathbf{f}\ :\ T \times X\ \rightarrow\ X$$

where $(T,\ +)$ defines a semigroup with the identity, describing the time variable t. In order to study our model, we focus on dynamical systems theory in $\mathbb{R}^n$ ($X \subseteq \mathbb{R}^n$) and we consider t as a discrete time variable. Hence, $(T,\ +)$ is simply the set of natural numbers with the operation of sum. Let be $\mathbf{x}_t \in X\ \forall t$, $\{\mathbf{x}_t|_{t\in T}\ :\ \mathbf{x}_{t+1} = \mathbf{f}(\mathbf{x}_t)\}$ is called the *orbit* determined by $\mathbf{f}$, that is the trajectory of the dynamical system. Since t is a discrete time variable, f is also called a *map*. Generally, $\mathbf{f}$ can depend on different model parameters. We focus on the case in which only one parameter may change while the other ones are fixed. We refer to these systems as one-parameter ones. The one-dimensional case ($X \subset \mathbb{R}$) is a trivial deduction of the general theory in $\mathbb{R}^n$.

When $\mathbf{f}$ is a nonlinear function in X and t is a discrete variable, generally the dynamical system described by $\mathbf{f}$ is a dissipative non-Hamiltonian system. A dissipative system is a dynamical system for which there is not a Hamiltonian function, conserved along the system's trajectories. Particularly, a dissipative system is not invariant under time reversal. The main aspect of a dissipative non-Hamiltonian system is related to the fact that the Liouville's theorem is not valid, that is, the volume of the phase-space is *not* conserved by the time evolution operator represented by the map describing the dynamical system. Finally, there are *not* conserved quantities generally, *i.e.* functions of system variables that are constant along each trajectory. We are particularly interested in dissipative dynamical systems.

2.2 Bifurcation Theory

The word bifurcation is used to describe a situation in which a small smooth change made to the parameter values (the bifurcation parameters) of a dynamical system causes a sudden qualitative or topological change in its behavior. Particularly, we are interested in the period-doubling (flip) bifurcation.

Let t be the discrete time variable, the dynamical evolution of a system is described by a map $\mathbf{f}$

$$\mathbf{x}_t = \mathbf{f}(c, \mathbf{x}_{t-1})\quad,\quad \mathbf{x} \in X \subseteq \mathbb{R}^n\quad,\quad c \in \mathbb{R} \tag{2.1}$$

where $t = 1, 2,\ ...$ is the index labeling successive points on the trajectory and c is a parameter of the map. Generally, $\mathbf{x}$ is a vector of dimension n defined in X. The space of all possible states

$\mathbf{x} \in X$ is called the phase space of the dynamical system. $\mathbf{f}$ is a function defined in $\mathbb{R}^n$ and in many interesting cases it is a nonlinear function. We focus on $\mathbf{f}$ as a continuous function in X. Finally, let X be invariant under the map $\mathbf{f}$, that is $\mathbf{x} \in X \longrightarrow \mathbf{f}(c, \mathbf{x}) \in X \;\; , \;\; \forall \mathbf{x} \in X$.

First of all, it is important to study the conditions under which the dynamical behavior of a system is asymptotically stable. The stability for a dynamical system means that the long-run dynamics is attracted by one stable state represented by a point in the phase space. The time evolution operator, represented by $\mathbf{f}$, leads the system to converge asymptotically to this state, also called an equilibrium point. An element (point) $\mathbf{x}^* \in X$ is an stable state for the dynamical system described by $\mathbf{f}$ if for any $\mathbf{x}$ close to $\mathbf{x}^*$ the iterated function sequence

$$\mathbf{x},\; \mathbf{f}(c, \mathbf{x}),\; \mathbf{f}(c, \mathbf{f}(c, \mathbf{x})),\; \mathbf{f}(c, \mathbf{f}(c, \mathbf{f}(c, \mathbf{x}))),\; ...$$

converges to $\mathbf{x}^*$. Hence, an equilibrium point is a state which attracts the asymptotic dynamics of the system described by the time evolution operator $\mathbf{f}$. This equilibrium point can be found by solving $\mathbf{f}(\mathbf{x}^*) = \mathbf{x}^*$. $\mathbf{x}^*$ is said a stable fixed point.

The notion of equilibrium point for a dynamical system is mathematically described through the definition of the hyperbolic stable fixed point.

2.2.1 Linear theory

$\mathbf{x}^*$ is a fixed point for the dynamical system when

$$\mathbf{x}^* = \mathbf{f}(c, \mathbf{x}^*) \tag{2.2}$$

The Brouwer theorem [40] ensures that a fixed point always exists for a dynamical system described by a continuous function $\mathbf{f}$ defined in a *compact* set X. The theorem does not ensure that the fixed point is an equilibrium point.

In order to define the stability of a fixed point, we can consider the expansion of (2.1) at the fixed point $\mathbf{x}^*$

$$\mathbf{x}^* + \delta\mathbf{x}_t = \mathbf{f}(c, \mathbf{x}^*) + \left(\mathcal{D}_{\mathbf{x}_{t-1}} \mathbf{f}(c, \mathbf{x}_{t-1})|_{\mathbf{x}_{t-1}=\mathbf{x}^*}\right) \delta\mathbf{x}_{t-1} + \mathcal{O}(\delta\mathbf{x}_{t-1}^2) \tag{2.3}$$

where $\delta\mathbf{x}_t = \mathbf{x}_t - \mathbf{x}^*$ and $\mathcal{D}_{\mathbf{x}_{t-1}} \mathbf{f}(c, \mathbf{x}_{t-1})|_{\mathbf{x}_{t-1}=\mathbf{x}^*}$ is the Jacobian matrix associated with $\mathbf{f}$ calculated at the fixed point. For the sake of notational simplicity, we refer to the Jacobian matrix as $\mathcal{D}_{\mathbf{x}^*}\mathbf{f}$. Since $\mathbf{x}^*$ is a fixed point, see (2.2), neglecting the term of second order in $\delta\mathbf{x}$, a linearized system is defined

$$\delta\mathbf{x}_t = (\mathcal{D}_{\mathbf{x}^*}\mathbf{f})\, \delta\mathbf{x}_{t-1} \tag{2.4}$$

Generally, the Jacobian matrix of a dynamical system can be diagonalized through a linear change of coordinates and the Jacobian matrix is characterized by its eigenvalues ξ_i, $i = 1, 2, ..., n$. If $|\xi_i| < 1 \;\; \forall i$, then as $t \to \infty$ the perturbation from equilibrium, $\delta\mathbf{x}_t$, decays exponentially. On the contrary, if $|\xi_i| > 1$, then the perturbation will grow.

For the linearized map (2.4) the eigenspace E_{ξ_i} is the subspace spanned by the eigenvector associated with ξ_i (conveniently defined depending on the fact that ξ_i is real or complex). By definition, these subspaces are invariant under the time evolution operator defined by the Jacobian. Starting from these ones, we can define the so-called invariant linear subspaces E^s, E^c and E^u as

$$\begin{aligned} E^s &= \mathrm{span}\{\mathbf{x} |\, \mathbf{x} \in \mathrm{E}_{\xi_i}, \quad |\xi_i| < 1\} \\ E^c &= \mathrm{span}\{\mathbf{x} |\, \mathbf{x} \in \mathrm{E}_{\xi_i}, \quad |\xi_i| = 1\} \\ E^u &= \mathrm{span}\{\mathbf{x} |\, \mathbf{x} \in \mathrm{E}_{\xi_i}, \quad |\xi_i| > 1\} \end{aligned}$$

E^s is the *stable* invariant linear subspace, E^c is the *center* one and E^u is the *unstable* one. They are invariant under the linearized map associated with $\mathbf{f}$ and $E^s \oplus E^c \oplus E^u = \mathbb{R}^n$. When the center subspace and the unstable one are empty, the fixed point $\mathbf{x}^*$ is an equilibrium point for the linearized map (2.4), because any perturbation from equilibrium decays exponentially in time.

The linear dynamics is asymptotically stable if and only if the spectrum of Jacobian matrix of the dynamical system $\mathbf{f}$ lies within the unit circle in the complex plane. The latter condition is verified depending on the value of the system parameter c.

The crucial point is to relate the linearized dynamics to the local nonlinear dynamics described by $\mathbf{f}$, so that the stability of the first one implies the stability of the second one. For this purpose, we define a fixed point as *hyperbolic* if the center subspace E^c is empty. For the sake of simplicity but without loss of generality, we assume also that the unstable subspace E^u is empty. Even if E^u is not empty, in the linear theory, it is always possible to consider E^u as a stable linear subspace for the same linearized system under a time reversal transformation. This trick leads to the following theorem, due to Hartman and Grobman:

Hartman-Grobman theorem [41]: Let $\mathbf{x}^*$ be an hyperbolic equilibrium for $\mathbf{f}$ at some fixed value of c. Then there exists a homeomorphism[1] $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ and a neighborhood U of $\mathbf{x}^*$ where

$$\mathbf{f}(c, \mathbf{x}) = \Psi^{-1} \left(\mathcal{D}_{\mathbf{x}^*} \mathbf{f} \cdot \Psi(\mathbf{x}) \right)$$

for $\mathbf{x}$ such that $\mathbf{x} \in U$ and $\mathbf{f}(c, \mathbf{x}) \in U$.

The Hartman-Grobman theorem provides a local relation between the nonlinear dynamics and the linear one. Hence, the stability (in the sense of fixed point convergence) of the latter implies the stability of the former.

We are interested in the situation in which there exist values of c such that the eigenvalues of the Jacobian matrix are smaller than 1, in absolute value (E^u is empty).

If $\mathbf{x}^*$ is an hyperbolic fixed point for $\mathbf{f}$ at c, then as c changes, the equilibrium point will shift its location, but it will remain hyperbolic stable if the eigenvalues are within the unit circle. In fact, all eigenvalues ξ_i are function of c. If an eigenvalue reaches the unit circle, then the fixed

[1] An homeomorphism is a continuous function between topological spaces (in this case $\mathbb{R}^n$) that has a continuous inverse function.

point is no longer hyperbolic and a local bifurcation can occur.

There are many possibilities according to the values assumed by the eigenvalues which reach the unit circle. We analyze the situation in which a simple real eigenvalue becomes equal to -1. This case is generally called a *period-doubling bifurcation*, or flip bifurcation.

2.2.2 Nonlinear theory: center manifold reduction

A period-doubling bifurcation is characterized by a non degenerate eigenvalue of the Jacobian matrix of $\mathbf{f}$ equal to -1, occurring at a fixed value of $c = c^*$. As a consequence, the center invariant linear subspace E^c is one-dimensional. This suggests that the interesting dynamics of a nonlinear system $\mathbf{f}$ near a period-doubling bifurcation occurs on a one-dimensional invariance subset of phase space called the *center manifold*. A mathematically precise definition of manifolds and related geometric ideas can be found in [42]. Intuitively, the center manifold corresponds to the invariant linear subspace E^c when we consider the nonlinear dynamical system (2.1). A center manifold W^c associated with E^c is the subspace tangent to E^c at the fixed point $\mathbf{x}^*$ for $c = c^*$. The tangency condition at fixed point together with the specific mathematical shape of $\mathbf{f}$ univocally define the center manifold W^c.

It is important to notice that a reduction procedure exists in order to define the map $\mathbf{f}$ on the center manifold and as a consequence also the center manifold is defined parametrically. In principle, the reduction procedure is always possible due to the invariance of the center manifold. In fact, due to the invariance, if $\mathbf{x} \in W^c$ it follows that $\mathbf{f}(\mathbf{x}) \in W^c$. Hence, since the center manifold W^c is one-dimensional, there exists a graph $\mathbf{x} = \mathbf{h}(x_c)$ which parameterizes locally W^c as a function of a variable x_c. Once that $\mathbf{h}$ is found, we can find a one-dimensional function f of the variable x_c which corresponds to the projection of $\mathbf{f}$ on the center manifold W^c.

In many cases, the procedure can be implemented only approximately. For a detailed discussion see [29].

The crucial aspect is related to the invariance of W^c which implies that in a neighborhood U of the fixed point $\mathbf{x}^*$, a system state $\mathbf{x}_t \in U \cap W^c$ evolves in $U \cap W^c$, that is $\mathbf{f}(c^*, \mathbf{x}_t) \in U \cap W^c$. Hence, assuming without loss of generality that the unstable invariant linear subspace E^u is empty, the phase space of the system is the direct sum of two subspaces, the center manifold W^c and the complement of W^c. The invariance of W^c implies the invariance of the complement.

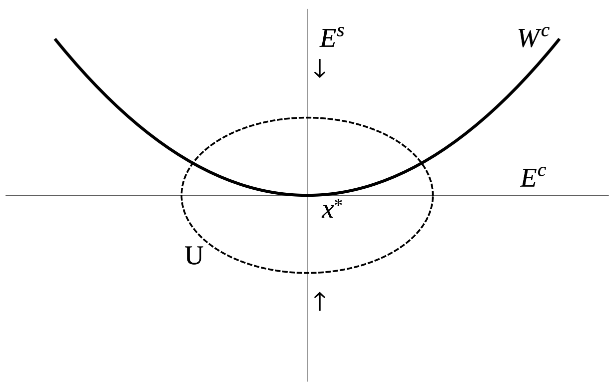


Figure 2.1: Graphical interpretation of the center manifold W^c

Figure (2.1) gives a graphical interpretation of the decomposition of the phase space into the direct sum of W^c and its complement (E^s) in 2-dimensional case.

The main intuition is that perturbations from $\mathbf{x}^*$ can be studied separately in W^c and in the complement of W^c. In the latter, the results of the linear theory can be applied and the Hartman-Grobman theorem defines that in the complement of W^c the orbit of the dynamical system is attracted by the fixed point $\mathbf{x}^*$. On the contrary, the dynamics of the map $\mathbf{f}$ in W^c is not attracted by $\mathbf{x}^*$ anymore and the nonlinearities of $\mathbf{f}$ define the asymptotic evolution of the map.

We have introduced the problem of center manifold reduction in order to give a general description of the period-doubling bifurcation. In fact, when a dynamical system is one-dimensional, that is $f : T \times X \rightarrow X, \quad X \subseteq \mathbb{R}$, if $\frac{\partial f}{\partial x}|_{c^*,x^*} = \xi = -1$, the whole phase space X coincides with the center manifold. Hence, we can study the one-dimensional map f in order to characterize the dynamics and to obtain its Poincaré-Birkhoff normal form.

Since a period-doubling bifurcation can be investigated in a one-dimensional center manifold, from now on we refer to a dynamical system described by a one-dimensional continuous function f defined in the appropriate one-dimensional space.

Finally, the results just explained are exactly valid for a specific value of $c = c^*$ for which the critical eigenvalue is exactly equal to -1. Qualitatively, a period-doubling bifurcation refers to a transition of the dynamical system from a stable dynamics characterized by an attracting fixed point to a two-period dynamics characterized by an oscillating evolution between two states of the phase space. We refer to (c^*, x^*) as the *criticality* at which the transition occurs. Unfortunately, the two-period dynamics at the criticality has zero amplitude. The new bifurcating solutions become distinct from the original equilibrium only when $c \neq c^*$. When $c \neq c^*$ there is not a center linear subspace E^c and thus the center manifold is not defined. In principle, for $c \neq c^*$ the study of the bifurcation on the center manifold through the Poincaré-Birkhoff normal form is not justified. *Ruelle and Takens* [43] pointed out that the study of the map f on the center manifold is approximately justified also in a neighborhood of criticality c^*.

In the next Section, we will deduce the Poincaré-Birkhoff normal form related to a period-doubling bifurcation. This refers to a local representation of the dynamical system described by f. This local representation has the advantage of making evident the symmetry of the original dynamical system.

2.2.3 Nonlinear theory: Poincaré-Birkhoff normal form

The occurrence of a period-doubling bifurcation is related to a local reflection symmetry which characterizes the dynamical system described by f. Without loss of generality, we assume that f is a continuous function in $X \subseteq \mathbb{R}$ and for the sake of notational simplicity we consider that criticality occurs in $(c^*, x^*) = (0, 0)$. The variable x can be intended as $x - x^* = x - 0 = x$, that is a perturbation from criticality.

When the dynamical system is one-dimensional, the critical eigenvalue corresponds to the first derivative of the map f calculated at the criticality:

$$\xi = \frac{\partial f(c,x)}{\partial x}|_{(c=0,x=0)} = -1$$

Generally, $\xi \equiv \xi(c)$, where c represents a shift from the critical value 0 in the parameter interval. Hence, the eigenvalue $\xi(c)$ strictly depends on the system parameter c.

The aim is to obtain a new map $\mathcal{N} : T \times X \to X$ that reproduces locally the dynamical behavior defined by f. $\mathcal{N}$ can be obtained through a nonlinear change of coordinate which remove some inessential nonlinearities of f at the criticality. The value $\xi = -1$ determines which nonlinear terms of f are essential or not. The local reflection symmetry of the system is related to the eigenvalue equal to -1.

At the criticality $(0,0)$, the dynamical system can be written as the Taylor series of f

$$x_t = f^{(1)}(x_{t-1}) + f^{(2)}(x_{t-1}) + \ ... \tag{2.5}$$

where the value of zero-order term of the expansion is zero. Consider then the nonlinear change of coordinate

$$z = \Theta(x) = x + \zeta_k\, x^k \tag{2.6}$$

with inverse

$$x = \Theta^{-1}(z) = z - \zeta_k\, z^k + \mathcal{O}(z^{2k-1}) \tag{2.7}$$

where Θ is a near identity change of coordinate characterized by a nonlinear term of order k times the parameter of the transformation, ζ_k. The main goal is to determine an appropriate ζ_k to remove the k-order term of (2.5).

With the new variable z, we find

$$\begin{aligned} z_t &= f(\Theta^{-1}(z_{t-1})) + \zeta_k\ \left(f(\Theta^{-1}(z_{t-1}))\right)^k \\ &= f(z_{t-1}) - L_k(z) + \mathcal{O}(z^{k+1}) \end{aligned} \tag{2.8}$$

where L is defined as

$$L_k(z) = \left(\frac{\partial f^{(1)}(z)}{\partial z}|_{z=0}\right)\ \zeta_k\, z^k - \zeta_k\ \left(f^{(1)}(z)\right)^k \tag{2.9}$$

It is important to notice that L_k is an homogenous function of degree k of z. The crucial point is that the solution of the following equation, if it exists,

$$f^k(z) - L_k(z) = 0 \tag{2.10}$$

allows to remove the k-order term of the map f at criticality. Notice that the term of order k of the transformation Θ does not affect the terms of order smaller than k of f. In principle this procedure can be iterated with successive nonlinear changes of coordinate in order to remove all possible nonlinearities. In the specialist literature [29], (2.10) is called the *homological equation*, or *Lie bracket*.

For a period-doubling bifurcation, the first order of the Taylor series (2.5) is simply equal to

$$f^{(1)}(z) = \xi z \tag{2.11}$$

with $\xi = -1$. We find that $L_k(z)$ is

$$L_k(z) = -\zeta_k\, z^k\, (1 - (-1)^{k-1}) \tag{2.12}$$

In this relation, the term $(1 - (-1)^{k-1})$ will vanish when k is odd. As a consequence, the odd terms of the Taylor series (2.5) can not be removed because the homological equation (2.10) has no solution in this case. On the contrary, terms of even degree can be removed and the Poincaré-Birkhoff normal form $\mathcal{N}$ of the map f becomes:

$$z_t = \mathcal{N}(z_{t-1}) = \xi\, z_{t-1}\, (1 + \phi_3\, z_{t-1}^2 + \phi_5\, z_{t-1}^4 + \;...\,) \tag{2.13}$$

where ϕ_3 is related to the third-order term of the Taylor series (2.5) and it depends on the solution of (2.10) with $k = 2$, ϕ_3 is related to the term of order 5 of the Taylor series (2.5) and it depends on the solutions of (2.10) with $k = 2$ and $k = 4$, and so on. As claimed before, the Poincaré-Birkhoff normal form is characterized by a local reflection symmetry.

In principle, (2.13) is exactly valid at the criticality. According to Ruelle and Takens [43], the normal form approximately describes the local behavior of the dynamical system f in a neighborhood of $c^* = 0$. As a consequence, the map $\mathcal{N}$, found with the previous procedure and calculated at c ($c \equiv c - c^* = c - 0$), is

$$z_t = \mathcal{N}(c, z_{t-1}) = \xi(c)\, z_{t-1}\, (1 + \phi_3(c)\, z_{t-1}^2 + \phi_5(c)\, z_{t-1}^4 + \;...\,) \tag{2.14}$$

where ξ, ϕ_3, ϕ_5 are now estimated for a value of c different from zero.

After the criticality at which the period-doubling bifurcation occurs, the fixed point is not attracting anymore. A mathematical characterization of the new solution can be found in [44]. Qualitatively, this solution identifies a two-period orbit oscillating between two states, x_+ and x_-. Intuitively, the trick is to notice that the twice-iterated map $f^2(c, x) = f(c, f(c, x))$ displays new solutions which correspond to the states x_+ and x_-. In fact, the equation $x = f^2(c, x)$ has three solutions. The first one corresponds to a shift in the phase space of the old fixed point which is not stable anymore. There are two new solutions due to the fact that the criticality corresponds to a *pitchfork* bifurcation [29] for f^2 and a pitchfork bifurcation is characterized by a branch of two new bifurcating solutions.

The problem of finding these states can be simplified looking at the normal form $\mathcal{N}$ in a neighborhood of the criticality. In fact, solving the following equation

$$z = \mathcal{N}(c, \mathcal{N}(c, z)) \tag{2.15}$$

we can find two states, z_+ and z_-, that describe approximately the states x_+ and x_-. Due to the reflection symmetry of the normal form, it follows that

$$z_+ = \mathcal{N}(c, z_-) \tag{2.16}$$

$$z_- = \mathcal{N}(c, z_+) \tag{2.17}$$

$$z_+ = -z_- \tag{2.18}$$

As a consequence, in a opportune neighborhood of the criticality it is not necessary to consider explicitly the second iterate of f. Indeed, the solution of

$$-z = \mathcal{N}(c, z) \tag{2.19}$$

represents the states z_+ and z_- describing approximately the two-period orbit of the original map f.

2.2.4 Feigenbaum scenario

In the previous Sections we have described analytically how, changing the system parameter c, a dynamical transition occurs from a stationary system state to a two-period orbit through a period-doubling bifurcation. When the dynamical system is characterized by a stable hyperbolic fixed point, the long-run dynamics converges asymptotically to a steady state. After the bifurcation, the long-run dynamics converges asymptotically to a two-period orbit, that is the dynamical system oscillates between two configuration. The system parameter c triggers this transition.

In many cases, if the parameter c changes in the same direction of the first bifurcation, successive period-doubling bifurcations occur. After the first bifurcation, the states x_+ and x_-, defining the two-period orbit, identify a branch of bifurcating solutions as function of the parameter c. An example of this typical scenario is shown in Figure (2.2). It represents the so-called bifurcation diagram in which the asymptotic states x_t of the dynamical evolution are plotted as a function of c.

It may happen that at some threshold for c a new period-doubling bifurcation can occur. Hence, from the first branch, another two branches appear and so on. This scenario characterized by a period-doubling cascade is the so-called *Feigenbaum scenario* [38] [45].

Typically, in an interval of c-values, the dynamical behavior is periodic and it is characterized by a specific period n. When a period-doubling bifurcation occurs, there is another interval of c-values in which the dynamical behavior is still periodic, but now the period is doubled ($2n$). This cascade of period doublings proceeds until that chaos[2] appears.

Feigenbaum showed for the first time [38] that this scenario is displayed by a large number of very different dynamical systems. Indeed, its occurrence essentially bypasses the details of the equations governing each specific system because the theory of this behavior is universal. Particularly, when f is defined in a closed interval, it is related to a mathematical shape of f characterized by a maximum and decreasing to zero at the extrema of interval. For further

[2] In the next Section, we will give the mathematical definition of deterministic chaos

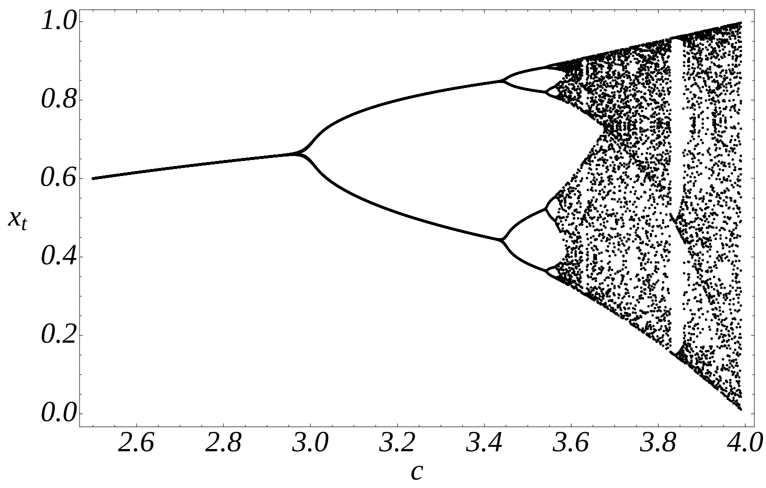


Figure 2.2: A Feigenbaum scenario. The Figure represents the bifurcation diagram of the logistic map $x_t = c\, x_{t-1}(1 - x_{t-1})$, studied for the first time by *May* [37].

details see [45].

The *Feigenbaum constant* δ reflects this universal behavior which characterizes many dynamical systems. Suppose that we refer to the value of c, at which a period-doubling bifurcation occurs, as c_n. n represents the period of system's orbit after the bifurcation. Hence, we have a series of values $(c_2, c_4, c_8, c_{16}, c_{32}, c_{64}, \ ...\ )$, each one represents in the parameter interval the value of c at which there is a period doubling.

The index series is the so-called *Sharkovsky ordering* [46] that is

$$1 \to 2 \to 2^2 \to 2^3 \to 2^4 \to 2^5 \to 2^6 \to 2^7 \to ...$$

Particularly the elements of latter series identify the indices of the series of c_n values. Let i be a index following the Sharkovsky ordering, if we define

$$\delta_i = \frac{c_{i+1} - c_i}{c_{i+2} - c_{i+1}}$$

δ_i (quickly) approaches the constant value δ

$$\delta = 4.6692016...$$

Thus, this definite number appears naturally in the bifurcation diagram of all dynamical systems exhibiting a period-doubling cascade to chaos.

2.3 Elements of Ergodic Theory of Chaos

In this Section we give a definition of deterministic chaos and we refer particularly to a dynamical system with discrete time variable, exhibiting a period-doubling cascade to chaos. As claimed in Section 4.2.2, a dynamical system of this type can be studied in one-dimensional space. Hence, we suppose that the map f is a continuous function in an interval $X \subseteq \mathbb{R}$.

In a chaotic regime, see Figure (2.2), during the dynamical evolution, the system explores many states defined in a phase space that is a subset of X. In this regime, the dynamical behavior is aperiodic. For this reason, it is useful to look at the non local properties of the dynamical system. Particularly, we will investigate non local quantities such as the Lyapunov exponent, the information dimension, and the Kolmogorov-Sinai entropy.

2.3.1 Deterministic chaos

The period-doubling cascade to chaos is an example of the phenomenon according to which nonlinear models exhibit very complex dynamics. This is called deterministic chaos. Non linear models are deterministic (*i.e.* without noise) because, given an initial state with infinite precision, the entire future path is exactly determined. However, chaotic solutions exhibit sensitivity to initial states. A small perturbation of the initial state leads to a completely different time path. Initial state uncertainty in chaotic systems leads to unpredictability in the long run.

According to this notion of chaos[3], we can define [47]

Definition The dynamics of a difference equation (map) $x_t = f(x_{t-1})$ is called **chaotic** if there exists a set Λ of positive Lebesgue measure, such that f has sensitive dependence on initial conditions with respect to Λ, that is, there exists a positive distance C such that for all initial state x_0 and any ϵ-neighborhood U of x_0, there exists an initial state $y_0 \in \Lambda \cap U$ and a time $T > 0$ such that the distance $d(x_T,\, y_T) = d(f^T(x_0),\, f^T(y_0)) > C$.

2.3.2 Dissipative systems and strange attractors

We are particularly interested in dissipative dynamical systems which exhibit chaos. We highlight that a dynamical system is dissipative in order to stress the fact that during chaotic evolution the concept of conserved quantity is replaced by the concept of *attractor*. Below we will give the definition of attractor for the dynamics. In order to introduce the tools necessary to characterize a system of this type in chaotic regime, we highlight briefly some typical properties.

A dissipative system is a dynamical system which is characterized by absence of a conserved quantity along a trajectory. Particularly, Liouville's theorem is not valid, *i.e.* a dissipative dynamical system does not conserve the volume of phase space. In fact, for example, when the dynamics converges to a steady state the determinant of Jacobian matrix (or the first derivative in one-dimensional case) of the system is smaller than 1. This means that in the stationary regime, the phase-space volume is contracted by the time evolution.

As we have seen in the previous Sections, when a stable equilibrium point exists, the study of the linearized system through the Jacobian fully characterize the problem. Also when a bifurcation occurs, a local study at the criticality leads to an approximate solution, such as the case of period-doubling bifurcation.

[3] In principle, there are many interpretations which focus on different aspects of a chaotic system. Hence, an exact mathematical definition of chaos is nontrivial and there exist a number of non-equivalent definitions in the literature.

However, according to the values of the parameters, if the dynamical behavior of the system is chaotic, it has no sense anymore to describe the system through the linearization around an equilibrium because there is not anymore a generalized equilibrium point (such as a stable fixed point or a periodic orbit). On the contrary, the dynamical behavior of the system is aperiodic.

Hence, it is necessary that the approach moves from a local point of view to a non-local one. Particularly, the tool in which we want to describe the chaos exhibited by a dissipative dynamical system is *ergodic* theory [39].

The theory treats the time average of the system state during the dynamical evolution. Typically, a dissipative dynamical system with initial condition x_0 exhibits a transient behavior followed by an asymptotic regime. Since, we are interested in time averages, transients become irrelevant. Therefore, if f^t is the t-iterated map, the point $f^t(x_0)$ representing the system state at a large time t eventually lies on an attracting compact set $A \subset X$, called attractor. The operational definition of attractor is that it is a set on which the system states $f^t(x_0)$ accumulate for large t.

Let E be a subset of X and let $f^t(E)$ be the set obtained by evolving each of the points in E forwards during time t. To give a mathematical definition of an attractor [48], we say that A is an attracting set with fundamental neighborhood U if

1. for every open set $V \supset A$ we have $f^t(U) \subset V$ when t is large enough

2. $f^t(A) = A$ for all t.

The mathematical intuition of this definition is related to the idea that a dynamical system converges asymptotically to the attracting compact set A. Hence, if U is an open set in X, and the closure of $f^t(U)$ is compact and contained in U for all sufficiently large t, then the set $A = \cap_{t>0} f^t(U)$ is a compact attractor with fundamental neighborhood U.

When the dynamical behavior of a system is chaotic, the asymptotic motion of the system is aperiodic within the attractor, that is an orbit of the system explores all points of the attractor in a infinite time. Since the attractor is compact, any system's orbit is aperiodic and bounded in the chaotic regime. Furthermore, in the chaotic regime the dynamical system exhibits sensitive dependence on initial conditions. Sensitivity to initial conditions implies divergence of nearby trajectories on the attractor.

At the same time, when we consider dissipative dynamical system, dissipation implies attraction of trajectories on the attractor.

These two opposite aspects may coexist only in a geometric object that is an attractor of great complexity characterized by a non-integer dimension. This object is called a *strange attractor* [49].

The study of a system on an attractor, be it strange or not, gives a global picture of the long run behavior of dynamics. Particularly, more detailed information about the system dynamics is given by the invariant probability measure ρ on the attractor A.

2.3.3 Invariant probability measure

Ergodic theory says that a time average equals a space average. As claimed before, the first one refers to the time average of the system state during the dynamical evolution, or more generally the time average of a function φ of system states

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} \varphi\left[f^t(x_0)\right] \tag{2.20}$$

where x_0 is the initial condition, f is the map describing the dynamical system and f^t is the t-iterated map.

The space average refers to the average of a function φ of system states weighted through an *invariant probability measure* ρ, defined on the attractor A in which the dynamics evolves. The space average of a function φ is

$$\int_A \rho(dx)\, \varphi(x) \tag{2.21}$$

where $x \in A$.

The probability measure ρ on A describes how frequently various parts of A are visited by the orbit of the system. Operationally [39], ρ is defined as the time average of Dirac deltas at the point x_t

$$\rho(x) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} \delta(x - x_t) \tag{2.22}$$

The probability measure is invariant when it satisfies the following relation

$$\rho\left(f^{-t}(E)\right) = \rho(E)\,, \quad t > 0 \tag{2.23}$$

where we consider E as a subset of A and $f^{-t}(E)$ as the set obtained by evolving each of the points in E backwards during time t. Essentially, a measure describing a dissipative dynamical system on an attractor is defined as invariant through (2.23) because in this way it is invariant under the time evolution operator (f^t) and this invariance may be expressed as: for all φ it follows that

$$\rho(\varphi \circ f^t) = \rho(\varphi) \tag{2.24}$$

where $\rho(\varphi)$ is

$$\rho(\varphi) = \int \rho(dx)\varphi(x) \tag{2.25}$$

If ρ satisfies (2.23), (2.24) can be proved trivially.

Finally, an invariant probability measure ρ may be decomposable into several different pieces, each of which is again invariant in its support, that will be a subset of the attractor A. If not, ρ is said to be indecomposable or *ergodic*. If ρ is ergodic, then the ergodic theorem [50] asserts that for every continuous function φ,

$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T} \varphi\left[f^t(x_0)\right] = \int_A \rho(dx)\varphi(x) \tag{2.26}$$

for almost all initial conditions x_0 with respect to the measure ρ.

This is a crucial result in ergodic theory of dynamical systems because studying some properties of system orbits, particularly in the chaotic regime, we can deduce non-local quantities, related to the ergodic measure, as Lyapunov exponent, information dimension, and entropy, which describe important non-local features of dynamical system.

Before introducing these quantities depending on the ergodic measure, we highlight two important aspects about the existence and the uniqueness of the ergodic invariant measure.

Krylov-Bogoliubov Theorem [51]: If the compact attractor A is invariant under the dynamical system (f^t), then there is a probability measure ρ invariant under f^t and with support contained in A. One may choose ρ to be ergodic.

The important assumption is that A is compact. The theorem asserts that an ergodic measure exists. Hence, operationally, this measure is defined by time averages through (2.22).

Unfortunately, when the dynamics becomes chaotic and the attractor is strange, there are in general many invariant measures in a dynamical system, but not all of them are physically relevant. For example, if x^* is an unstable fixed point, then the δ-function at x^* is an invariant measure, but it is not observed because any small perturbation leads the system to leave the unstable fixed point.

Therefore, the observed measure is only one, the so-called *physical* measure. From an experimental point of view, a reasonable physical measure is obtained according to the following idea of Kolmogorov. Considering the dynamical system with an external noise term added

$$x_t = f(x_{t-1}) + \varepsilon\eta(t) \tag{2.27}$$

where $\eta(t)$ is some noise and $\varepsilon > 0$ is a real parameter. For suitable noise and $\varepsilon > 0$, the stochastic time evolution (2.27) has a unique stationary measure ρ_ε. In this framework, the physical measure is $\rho = \lim_{\varepsilon\to 0}\rho_\varepsilon$. Again, this is an operational definition which works very well in many cases. Particularly, in a computer study, roundoff errors should play the role of small random noise and due to the sensitive dependence on initial conditions, also a very small level of noise has important effects in defining a unique physical measure.

Once that we can claim the existence of the physical measure for a dynamical system, we can introduce some non-local quantities describing the sensitive dependence on initial conditions of dynamical system, in particular in a chaotic regime.

What is the reason to define these non-local quantities? The answer to this question [52] is related to the fact that a dynamical system, evolving chaotically, continually produces dynamical information (with a rate of production called entropy), because it continually explores new states on an strange attractor. Furthermore, two orbits that initially are very close, evolve in different way after a time large enough, due to the sensitive dependence on initial conditions.

From an external point of view, an observer typically takes information about a dynamical system looking at time correlations between functions which depend on system's orbits. If an orbit is periodic, time correlations display clear structures and as a consequence the observer has complete information about the dynamical behavior. On the contrary, since the precision of measuring is always finite, in the chaotic regime characterized by aperiodic evolution and sensitive dependence on initial conditions, the observer *loses information* about system at some rate (it is important to notice that here the point of view is external).

A time correlation function can be typically written as

$$Corr_{\varphi,\psi}(t) = \int_A \rho(dx)\varphi(x)\psi(f^t(x)) - \left(\int_A \rho(dx)\varphi(x)\right)\left(\int_A \rho(dx)\psi(x)\right) \tag{2.28}$$

with φ and ψ as two functions whose one want to know the correlation. When the system is chaotic, the correlation function (2.28) will be characterized by some decay properties, of course. These decay properties are related to the rate of information production or information loss, depending on the point of view, internal or external respectively.

This rate is related to some non-local quantities that we will introduce.

2.3.4 Largest Lyapunov Exponent (LLE)

If the initial state of a time evolution is slightly perturbed, the exponential rate at which the perturbation changes with time is called a Lyapunov exponent. When the dynamical behavior is periodic, the Lyapunov exponent is negative because the orbit with different initial conditions are attracted by an asymptotic system state, so any perturbation decreases. On the contrary, when dynamical behavior is chaotic, the Lyapunov exponent is positive and it describes the rate at which the initial perturbation increases, that is the sensitive dependence on initial conditions.

There may be several Lyapunov exponents for a n-dimensional dynamical system. The most important is obviously the largest one (LLE) which describes the transition from a periodic behavior to a chaotic one.

To explain a theorem which relates the Lyapunov exponents to the ergodic measure, we consider for a moment a n-dimensional dynamical system described by a map $\mathbf{f}$ defined in a n-dimensional space X.

The exponential rate associated with a perturbation in a point of the trajectory is dictated by the Jacobian matrix of the map calculated at the same point of the trajectory. Hence, in principle, this rate may be different in different points of the attractor. When the dynamical system is ergodic, the Oseledec theorem states that the linear transformations obtained by the Jacobian matrix calculated at different points of the trajectory, are asymptotically equivalent almost everywhere to a linear transformation associated with a constant matrix, whose eigenvalues are related to the Lyapunov exponents. Specifically, the logarithms of the eigenvalues are the Lyapunov exponents associated with the dynamical system. Hence, a Lyapunov exponent assumes the same value almost everywhere on the attractor.

Multiplicative ergodic theorem of Oseledec [53]: Let ρ be a probability measure on X (eventually on an attractor $A \subset X$) and $\mathbf{f} : T \times X \to X$ a measure preserving map such that ρ

is ergodic. Let also $\mathcal{T}$ a linear map, $\mathcal{T}: X \to M(n \times n, \mathbb{R})$, the latter is space of matrices $n \times n$ whose entries are real numbers. If $\mathcal{T}$ is defined such that

$$\int \rho(dx) \ln^+ ||\mathcal{T}(\mathbf{x})|| < \infty$$

where $\ln^+ y = \max(0, \ln y)$, defining the matrix $\mathcal{T}^t(\mathbf{x}) = \mathcal{T}(\mathbf{f}^{t-1}(\mathbf{x})) \cdots \mathcal{T}(\mathbf{f}(\mathbf{x}))\mathcal{T}(\mathbf{x})$, then, for ρ-almost all $\mathbf{x}$, the following limit exists:

$$\lim_{t\to\infty} \left(\mathcal{T}^{t\,*}(\mathbf{x}) \cdot \mathcal{T}^t(\mathbf{x})\right)^{\frac{1}{2t}} = \mathcal{L}(\mathbf{x})$$

where $\mathcal{T}^{t\,*}(\mathbf{x})$ is the adjoint of $\mathcal{T}^t(\mathbf{x})$ and we have taken the $2t$-th root of the positive matrix $\mathcal{T}^{t\,*}(\mathbf{x}) \cdot \mathcal{T}^t(\mathbf{x})$.

The logarithms of the eigenvalues of $\mathcal{L}(\mathbf{x})$ are called Lyapunov exponents and they are ρ-almost everywhere constant (as the consequence of the assumption that ρ is ergodic). Particularly, we refer to the largest Lyapunov exponent as LLE. In the one-dimensional case of period-doubling cascade to chaos, there is only one Lyapunov exponent. Also in the one-dimensional case, we refer to it as LLE.

If $X \subset \mathbb{R}^n$, $\mathcal{T}$ can be considered also as the Jacobian matrix describing the linearized system associated with $\mathbf{f}$. In the one-dimensional case, $X \subset \mathbb{R}$, $\mathcal{T}$ coincides with the first derivative in $x \in A$ of $f: T \times X \to X$, where A is the attractor for the dynamics, that is

$$\frac{\partial f^t(x)}{\partial x} = \frac{\partial f(x)}{\partial x}|_{f^{t-1}(x)} \cdot \frac{\partial f(x)}{\partial x}|_{f^{t-2}(x)} \cdots \frac{\partial f(x)}{\partial x}|_{f(x)} \cdot \frac{\partial f(x)}{\partial x}|_x \tag{2.29}$$

If ρ is ergodic and f is defined in a compact support (attractor A), LLE is well-defined and in one-dimensional case it is simply

$$LLE = \lim_{t\to\infty} \frac{1}{t} \ln \left(\frac{\partial f^t(x)}{\partial x}\right) \tag{2.30}$$

for ρ-almost all $x \in A$.

In one-dimensional space and in a chaotic regime, LLE gives a measure of the rate of exponential growth of an infinitesimal δx, which can be seen as the initial distance between two orbits. Hence, LLE is a measure of the sensitive dependence on initial conditions of a dynamical system.

Until now, we have considered a dynamical system described by f (or $\mathbf{f}$) without focusing on system's parameters. For a dynamical problem depending on a bifurcation parameter c, for example a period-doubling cascade to chaos, the aim would be to estimate LLE from an experimental point of view as a function of the system's parameter c. Indeed, an estimation of LLE corresponds exactly to a measure of sensitive dependence on initial conditions of a dynamical system.

However, the situation is unfortunate because the largest Lyapunov exponent is in general a discontinuous function of the bifurcation parameter c, [54] [55]. This leads to a philosophical problem about the sense of an experimental measure that is, in principle, discontinuous when c

changes [39].

In the next Section, we define the entropy of a dynamical system. This quantity has the advantage of being easily measurable. In fact, only the knowledge of one system's orbit is necessary in order to estimate it.

2.3.5 The Kolmogorov-Sinai invariant or entropy

The intuition behind the concept of entropy is defining *a quantity which will measure, in some sense, how much information is produced by a process, or better, at what rate the information is produced* [56]. In dynamical systems theory, the word *information* is related in some sense to the number of asymptotic states that the dynamical system explores during the evolution. For example, in chaotic regime a dynamical system, see Figure (2.2), explores a larger and larger number of states when time increases, due to the aperiodic motion and the sensitive dependence on initial conditions. In fact, two dynamical states that are initially different but indistinguishable at a certain precision, will evolve into distinguishable states after a finite time.

We can define the positive rate at which the dynamical system produces information exploring new states as the entropy.

In order to give a mathematical definition of entropy, the crucial point is to describe what is the probability associated with these asymptotic states.

The ergodic measure of a dynamical system evolving asymptotically within an attractor A can be used to define the probability of finding the system in a region $\mathcal{A}_i \subset A$, as $\rho(\mathcal{A}_i)$. In some sense, we can interpret $\rho(\mathcal{A}_i)$ as the probability associated with a dynamical state whose position is within $\mathcal{A}_i$.

Assuming that the attractor A is a compact set with a given metric (when a dynamical system is defined in $\mathbb{R}^n$, the metric can be considered as induced by a norm), we can consider a finite partition of A, $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_\nu)$. A partition of A is simply a division of A as a union of non-overlapping and non-empty subsets $\mathcal{A}_i$. Through the partition $\mathcal{A}$, we can interpret exactly $\rho(\mathcal{A}_i)$ as a probability. In fact, since ρ is a measure and the dynamical system is asymptotically confined within the attractor A, it follows that

$$\sum_{i=1}^{\nu} \rho(\mathcal{A}_i) = 1$$

Following the Shannon's idea, a quantity H can be defined as

$$H(\mathcal{A}) = -\sum_{i=1}^{\nu} \rho(\mathcal{A}_i) \ \ln \ \rho(\mathcal{A}_i) \tag{2.31}$$

$H(\mathcal{A})$ is the information content of the partition $\mathcal{A}$ with respect to the ergodic measure ρ. The quantity H so defined is a good measure of information because it is additive [56]. If there are two uncorrelated sources of information, the total H is simply the sum of two types of information related to the two sources. The logarithm in (2.31) captures this additivity characteristic.

However, H described by (2.31) is not the entropy of a dissipative dynamical system. The reason comes from the fact that the subsets $\mathcal{A}_i$ contain more than one dynamical state when the partition $\mathcal{A}$ is finite. Hence, $\rho(\mathcal{A}_i)$ is not the probability associated with only one state of the dynamical evolution. In principle, it is necessary a finer and finer partition in order to find the entropy of the dynamical system. This procedure may be avoided using a *generating partition* [57].

The main reason is to find the specific partition of the phase space which allows to access to all information related to the dynamics of the system. The main aim is the decomposition of the phase space into the minimal elements which represent the dynamical states of the system. Making a parallelism with statistical mechanics, this procedure is similar in some sense to the decomposition of the phase space into cells whose dimension is $\hbar$. Since the main goal is obtaining a partition related to the dynamical behavior of the system, the generating partition is defined through the map f iterated backwards in time.

The procedure is the following. Starting from a finite partition $\mathcal{A}$, let $f^{-k}\mathcal{A}_i$ be the set of points within the attractor A mapped by f^k to $\mathcal{A}_i$. We then denote denote by $f^{-k}\mathcal{A}$ the partition $(f^{-k}\mathcal{A}_1, \dots, f^{-k}\mathcal{A}_\nu)$. Finally $\mathcal{A}^{(n)}$ is defined as [39]

$$\mathcal{A}^{(n)} = \mathcal{A} \vee f^{-1}\mathcal{A} \vee \cdots \vee f^{-n+1}\mathcal{A} \tag{2.32}$$

which is the partition whose pieces are

$$\mathcal{A}_{i_1} \cap f^{-1}\mathcal{A}_{i_2} \cap \cdots \cap f^{-n+1}\mathcal{A}_{i_n} \tag{2.33}$$

with $i_j \in \{1, 2, \dots, \nu\}$, *i.e.* $(i_1, \dots, i_n)$ are opportune combinations of indices. What is the significance of these partitions? The partition $\mathcal{A}^{(n)}$ in (2.32) is formed by elements of the type (2.33) that represents the smallest intersection between all elements of the partitions $\mathcal{A}$, $f^{-1}\mathcal{A}$, ... , $f^{-n+1}\mathcal{A}$. These sets are of course disjoint and the union gives the attractor A.

The partition $\mathcal{A}^{(n)}$ represents a refinement of the partition $\mathcal{A}$ through the map f iterated backwards in time. It is important to notice that $f^k\mathcal{A}$ is not in general a partition.

The following limits are asserted to exist [58], defining $h(\rho, \mathcal{A})$ and $h(\rho)$:

$$h(\rho, \mathcal{A}) = \lim_{n\to\infty} \left[H(\mathcal{A}^{(n+1)}) - H(\mathcal{A}^{(n)}) \right] \tag{2.34}$$

$$h(\rho) = \lim_{diam\mathcal{A}\to 0} h(\rho, \mathcal{A}) \tag{2.35}$$

where $diam\mathcal{A} = \max_i\{\text{diameter of } \mathcal{A}_i\}$.

$h(\rho, \mathcal{A})$ is the rate of expansion of information when we consider a finer partition $(\mathcal{A}^{(n+1)})$ with respect to the previous one $(\mathcal{A}^{(n)})$. $h(\rho)$ is the limit for finer and finer partitions and it is called *Kolmogorov-Sinai invariant* of a dissipative dynamical system or simply *entropy*.

The last limit may sometimes be avoided, *i.e.* $h(\rho, \mathcal{A}) = h(\rho)$. This hold in particular if $diam\mathcal{A}^{(n)} \to 0$ when $n \to \infty$, or if f is invertible and $diam\ f^n\mathcal{A}^{(2n)} \to 0$ when $n \to \infty$. Mathematically, this happens when the σ-algebra in which the ergodic measure is defined, contains all $f^{-k}\mathcal{A}_i \quad \forall k = 1, \dots, \infty$. In this case $\mathcal{A}$ is called a generating partition. For a deeper discussion about the generating partition see [57].

As claimed before, from an external point of view, $h(\rho)$ is also the rate of information loss about the system dynamics, estimated by an observer.

The relationship between entropy and Lyapunov exponents is very interesting. *Ruelle* [59] has proved that for a finite-dimensional dynamical system $\mathbf{f}$

$$h(\rho) \leq \sum \text{positive Lyapunov exponents} \tag{2.36}$$

that is, for a one-dimensional system f

$$h(\rho) \leq LLE \tag{2.37}$$

Another quantity of interest is the order-2 Renyi entropy K_2 defined as

$$H_2(\mathcal{A}) = -\ln \sum_{i=1}^{\nu} \rho(\mathcal{A}_i)^2 \tag{2.38}$$

$$K_2(\mathcal{A}) = -\lim_{diam\mathcal{A}\to 0} \lim_{n\to\infty} \frac{1}{n} H_2(\mathcal{A}^{(n)}) \tag{2.39}$$

Grassberger and Procaccia [60] have proved that K_2 is a lower bound to the entropy $h(\rho)$:

$$K_2(\rho) \leq h(\rho) \tag{2.40}$$

These various quantities can be measured from an experimental point of view, studying the correlation integral, as *Procaccia et al.* [61] [62] suggest.

Given a long-time series of orbit's points on the attractor A for a one-dimensional dynamical system, $\{x_t\ ,\ t = 1,\ \ldots\ , N\}$ (since that the time variable is discrete, the period time between a point and the successive one is simply $\Delta t = 1$), we can consider the vector $\vec{x}_i = \{x_{i+k\cdot\Delta t}\}_{k=1}^{N}$, with the opportune periodical boundary conditions. The definition of the correlation integral is

$$C(r) = \lim_{N\to\infty} \frac{1}{N^2} \sum_{i,j=1}^{N} \theta(r - d(\vec{x}_i, \vec{x}_j)) \tag{2.41}$$

where θ is the Heaviside function, d is a metric distance in $\mathbb{R}^N$.

In Chapter 5 we will explain as a numerical approximation of (2.41) is useful to estimate empirically quantities like the Kolmogorov-Sinai entropy, the order-2 Renyi entropy, and the information dimension from the time series describing an orbit of the dynamical system.

2.3.6 The dimension of an attractor

Finally, another quantity of interest in defining as chaotic a dynamical system is the dimension of attractor A. In chaotic regime, we have seen that the attractor A is strange when it displays a non-integer dimension. Intuitively, the dimension of a set is the amount of information needed to specify points on it accurately. In dynamical systems theory, the typical definition of the dimension of a strange attractor is given through the metric induced by the norm of $\mathbb{R}^n$, or $\mathbb{R}$ in one-dimensional case.

The Hausdorff dimension measures the local size of a space taking into account the distance between points, the metric. Qualitatively, the main intuition about the Hausdorff dimension is the following. Consider the number $N(r)$ of balls of radius r required to cover A completely. When r is very small, $N(r)$ grows polynomially with $\frac{1}{r}$. The Hausdorff dimension is the unique number $\dim_H A$ such that $N(r)$ grows as $\frac{1}{r^{\dim_H A}}$ as r approaches zero. More precisely, this defines the box-counting dimension, which equals the Hausdorff dimension when the limit for $r \downarrow 0$ exists.

Hausdorff dimension [63]: Given the attractor A with a metric, and $r > 0$, σ is a covering of A by a countable family of set σ_k with diameter $d_k = diam(\sigma_k) \leq r$. Given $\alpha \geq 0$, we can write

$$\mu_r^\alpha(A) = \inf_\sigma \sum_k (d_k)^\alpha$$

When $r \downarrow 0$, $\mu_r^\alpha(A)$ increases to a (possible infinity) limit $\mu^\alpha(A)$ called the Hausdorff measure of A in dimension α. Hence,

$$\dim_H A = \sup\{\alpha \, : \, \mu^\alpha(A) > 0\}$$

is the Hausdorff dimension of A.

Generally, the Hausdorff dimension of an attractor is a quantity difficult to measure because it is strictly related to the geometry of the attractor. Instead, a different definition can be given in order to connect the dimension of the attractor with the probability measure defined on the attractor itself.

Information dimension [64]: Given a probability measure ρ on an attractor A, the information dimension $\dim_I \rho$ is the smallest Hausdorff dimension of A of ρ measure 1.

It is important to notice that A is not closed in general, and therefore the Hausdorff dimension $\dim_H(supp\,\rho)$ of the support of ρ may be larger that $\dim_I \rho$. In any case, we have $supp\,\rho \subseteq A$ and therefore

$$\dim_I \rho \leq \dim_H(supp\,\rho) \leq \dim_H A$$

In many typical cases, when a dynamical system is defined in a metric space and the metric is induced by the norm, if an attractor A is compact, it follows that it is also closed. Hence, the Hausdorff dimension corresponds to the information dimension.

The advantage of using the information dimension is the fact that it can be also defined through [65]

$$\lim_{r\to 0} \frac{\ln\, \rho[B_x(r)]}{\ln\, r} = \dim_I \rho \tag{2.42}$$

for ρ-almost all x, where $B_x(r)$ is a ball of radius r centered at x. The existence of the limit implies that it is constant, by the ergodicity of ρ.

The fact that the definition of the information dimension is related to the ergodic measure ρ, leads *Grassberger and Procaccia* [61] to find that the quantity

$$\lim_{r\to 0} \frac{\ln C(r)}{\ln r} = \nu \tag{2.43}$$

where $C(r)$ is the correlation integral (2.41), is related to the information dimension. ν is called the **correlation dimension**.

Specifically, Grassberger and Procaccia find that [66]

$$\nu \leq \dim_I \rho \leq \dim_H A$$

In many cases, the three dimensions coincide. The relation (2.43) makes the correlation dimension an easily measurable quantity in order to define the degree of chaos of a dynamical system evolving within a strange attractor.

Chapter 3

A model of systemic risk in financial markets

In this Chapter, an analytical dynamical model [9] is introduced to investigate systemic risk in financial markets. The model provides a framework for studying the dynamics of an illiquid financial market populated by financial institutions investing in risky assets.

An asset is a financial resource that can be described by a stochastic process associated with a stochastic variable, that is the asset return. An asset is characterized by a risk. The variance of the stochastic process is a risk measure when the asset return is a normally distributed variable.

A financial market can be seen as a bipartite network characterized by financial institutions (representing one type of nodes) investing (*i.e.* creating links between nodes) in some risky assets (other type of nodes). In the model, the number (diversification) of investment assets and the invested capital correspond to the solution of an optimization problem which takes into account costs of diversification and the Value at Risk constraint, that is a constraint on a minimum capital requirement in order to cover a loss. The solution of this optimization problem defines the portfolio of a financial institution. The portfolio is created according to a micro-prudential strategy based on diversification in order to reduce exogenous risk.

In financial markets, financial institutions borrow from other financial institutions in order to invest an amount of money which is the sum of their initial equity and their debt. The ratio between the invested capital and the equity is called financial leverage. Financial institutions try to maintain a fixed value of leverage, buying or selling assets.

The purchase or the sell of assets moves their prices in an illiquid financial market, that is a market in which the trading affects the asset returns. Typically, when assets price grows, financial institutions buy, putting upward pressure on assets price which continues to grow, and vice-versa. This mechanism has the potential for a positive feedback. This positive feedback depends strictly on the financial leverage and the portfolio diversification adopted by financial institutions.

Once that the framework is established, the model highlights how in a financial market requiring to investors a financial coverage in the form of VaR constraint, a micro-prudential strategy, based on the portfolio diversification, may increase the systemic risk through the endogenous feedback effect induced by portfolio rebalancing in presence of assets illiquidity. Specifically, the introduction of a financial innovation that allows reducing the cost of portfolio diversification leads financial institutions to increase leverage and diversification. As a consequence, the strength (due to higher leverage) and coordination (due to similarity of portfolios) of feedback effects increase, affecting crucially the return and the risk of the investments assets and potentially triggering a transition from a stationary dynamics of price returns to a non stationary one characterized by steep growths (bubbles) and plunges (bursts) of market prices.

In Section 3.1 we describe the basic features of the dynamical model: the return process, the economic network, and the portfolio optimization problem. In Section 3.2, we describe the positive feedback and its impact on return and risk. Finally, in Section 3.3, we analyze the systemic risk in financial markets and the probability of default of financial institutions.

3.1 Basic features

In this Section, we introduce basic features of the dynamical model: the typical returns process for the risky investments, the network formed of the financial institutions and the asset classes, the portfolio optimization problem in presence of costs of diversification and VaR constraint.

3.1.1 The asset's return process

A very important field of study in economy is related to the dynamics of assets prices. The result of more than half a century of empirical studies on financial time series indicates that all investment assets share some quite non-trivial statistical properties [67]. First of all, the price dynamics of each asset is seemingly random. Beyond the specific model in order to describe it, the variations of prices exhibit shared qualitative properties, also called *stylized empirical facts* in the field of quantitative finance.

It is important to notice that the scales used to measure prices are often given in units that are fluctuating in time [2]. This aspect differentiates crucially the financial studies from physics. In order to find the more appropriate variable, financial studies involve returns, instead of prices, of assets. There are many definitions of the return. However, all of them are related to the time variation of the price.

If p_t is the price of an asset at time t, the commonly used definition of the return is the following:

$$r_t = \frac{p_{t+\Delta t} - p_t}{p_t}$$

The merit of this approach is that returns provide a direct percentage of gain or loss in a given time period. This definition works very well until that the time period Δt is not too large. The collection of the time realizations of asset returns defines a financial series.

An underlying assumption made in the study of quantitative finance is the *ergodicity* of time series. The ergodic property ensures that the time average of a quantity converges to its expectation. Ergodicity is typically satisfied by a financial series.

Some stylized empirical facts which are common to a wide set of financial series are

- **Absence of autocorrelations** [68]: linear autocorrelations of assets returns are often insignificant (the linear autocorrelation is represented by the sample linear correlation between a financial series with itself shifted by a specific time lag).
- **Heavy tails** [69]: the distribution of returns seems to display a power-law tail characterized by an excess kurtosis larger than zero (the sample kurtosis corresponds to the fourth standardized moment of a distribution, the Gaussian distribution is characterized by a value of the kurtosis equal to 3, the excess kurtosis is defined as the kurtosis in excess with respect the Gaussian case).
- **Aggregational Gaussianity** [70]: when the time scale Δt defining returns increases, the returns distribution looks more and more like a normal distribution.
- **Slow decay of autocorrelation in squared returns** [2, 71]: the autocorrelation function of squared returns decays slowly as a function of the time lag, this is interpreted as a sign of long-range dependence.
- **Volatility clustering** [72]: different measures of volatility display a positive autocorrelation over several days, which quantifies the fact that high-volatility events tend to cluster in time.

From an effective point of view, we can refer to an asset return as a stochastic variable. According to the ergodicity hypothesis, the time average corresponds to the expected value of the return, $\mathbb{E}[r_t] = \langle r_t \rangle$. The sample variance corresponds to the following expected value:

$$\mathbb{E}\left[(r_t - \mathbb{E}[r_t])^2\right] = \mathbb{E}\left[r_t^2\right] - \mathbb{E}[r_t]^2 \tag{3.1}$$

Assuming that the return r_t is a Gaussian stochastic variable, (3.1) represents a *measure of the risk*. Hence, through the assumption of Gaussianity:

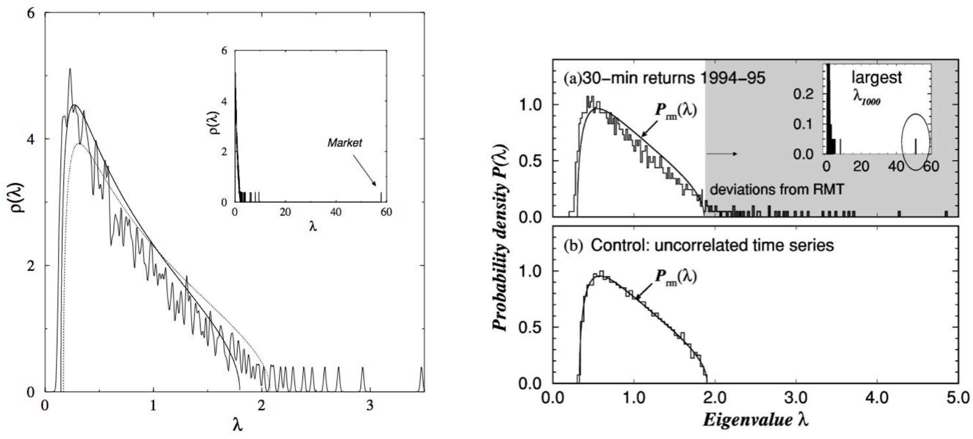
- we refer to the expected value of the square of the standardized return as the **risk** of the asset
- we refer to the standard deviation of the standardized return as the **volatility** of the asset.

A financial market is typically characterized by a large number of investments assets. The aggregate dynamics of all assets plays an important role in the risk management of a portfolio containing a large number of assets.

The main tool to analyze the interdependence of assets returns is the correlation matrix C of returns

$$Corr_{i,j} = \rho_{i,j}$$

where $\rho_{i,j} \in [-1, 1]$ is the linear correlation between the stochastic process characterizing assets i and j.



(a) (b)

Figure 3.1: Smoothed density $\rho(\lambda)$ of the eigenvalues λ_j of the correlation matrix extracted from $M = 406$ assets of the $S\&P500$ during the years 1991-1996 [73], panel (a). The theoretical density (solid line) is obtained assuming that the matrix is purely random, according to the Random Matrix Theory (RMT). In the same plot it is included an insert in order to show the largest eigenvalue corresponding to the market factor. The eigenvalue distribution $P(\lambda)$ [74] for the correlation matrix, constructed from 30-min returns for $M = 1000$ stocks of USA markets for the 2-yr period 1994–95, is shown in the upper plot of panel (b). The solid curve shows the outcome according to the Random Matrix Theory. The inset shows the largest eigenvalue. The lower plot shows $P(\lambda)$ for a random correlation matrix computed from N = 1000 computer-generated random uncorrelated time series with length $L = 6448$. In this case, the theoretical density obtained through the RMT is in agreement with the numerical simulation.

Two empirical studies [73, 74] about the correlation matrix $Corr$ of returns of the assets traded in the three major US stock exchanges (NYSE, AMEX and NASDAQ) showed that almost all eigenvalues of $Corr$ are distributed according to the eigenvalues distribution of a correlation matrix defined by uncorrelated Wiener processes, see Figure (3.1). This latter distribution can be analytically found through a Random Matrix Theory approach [75, 76]. Since the properties of these eigenvalues are consistent with the properties of random correlation matrices, it follows that a large part of matrix $Corr$ does not contain information. On the contrary, the few highest eigenvalues of $Corr$ which deviate from randomness contain information about the returns dynamics. These highest eigenvalues are associated to factors. In fact, the return as a stochastic variable can be considered as a (linear) combination of different stochastic variables. A factor can be seen as one of these stochastic variables and it is common to many assets.

Specifically, the highest one is the market factor. The market factor describes a common component in the stochastic processes of assets returns that is related to the tendency of financial market. Similarly, other factors are related to specific sectors of the economy.

In the model we present, the financial market is characterized by M investment assets following a multivariate factor model with one factor f_t representing the market factor. The return of each investment j is

$$r_{j,t} = \mu_j + b_j \, f_t + \epsilon_{j,t} \tag{3.2}$$

where from a mathematical point of view $\epsilon_{j,t}$ and f_t are Gaussian random noises. The time evolution of (3.2) is assumed to be discrete. In (3.2) μ_j is the drift of the asset j. The market factor f_t and $\epsilon_{j,t}$ are assumed to be uncorrelated and normally distributed with zero mean and constant standard deviation. In quantitative finance, $\epsilon_{j,t}$ is said a idiosyncratic noise. The drift μ_j represents the expectation value of the return of the asset j. The idiosyncratic noise refers to a component of the asset return that has no correlation with market factor.

For the sake of simplicity and for analytical tractability, let us assume that $\mu_j = \mu$, $b_j = 1$ $\forall j = 1, \ldots, M$ and the standard deviation associated with $\epsilon_{j,t}$ is the same for all assets. These assumptions reflect a hypothesis of homogeneity for investment assets.

Since the market factor and the idiosyncratic noise are normally distributed, the standard deviations are measures of volatilities. Particularly, the market factor f_t is characterized by a systematic (undiversifiable) volatility component σ_s. This component of volatility is in common to all assets and it can not be reduced through the portfolio diversification, it is systematic. The idiosyncratic noise $\epsilon_{j,t}$ is characterized by an idiosyncratic (diversifiable) volatility component σ_d. Thus, each risky investment entails both a diversifiable risk component and a undiversifiable risk component, *i.e.* the variance of the investment risky asset j can be decomposed as

$$\sigma_j^2 = \sigma_s^2 + \sigma_d^2 \tag{3.3}$$

The component $\epsilon_{j,t}$ is said idiosyncratic and as a consequence the volatility component σ_d is said diversifiable because if we consider the following linear combination of m different random variables $\epsilon_{j,t}$

$$\sum_{j=1}^{m} \frac{1}{m} \epsilon_{j,t} \tag{3.4}$$

it follows that the variance of this linear combination is simply equal to

$$\mathbb{E}\left[\left(\sum_{j=1}^{m} \frac{1}{m} \epsilon_{j,t}\right)^2\right] = \frac{\sigma_d^2}{m} \tag{3.5}$$

assuming that two different idiosyncratic noises $\epsilon_{j,t}$ and $\epsilon_{k,t}$ $(j \neq k)$ are uncorrelated, that is $\mathbb{E}[\epsilon_{j,t}\ ,\ \epsilon_{k,t}] = 0$.

From the financial point of view, it is clear that a selection of a linear combination of m different stochastic processes reduces the idiosyncratic risk associated with this linear combination through a factor $\frac{1}{m}$ with respect to consider only one of these processes. We refer to m as the *diversification*. Obviously, a linear combination of m stochastic processes described by (3.2) reduces only the idiosyncratic component of the risk because f_t is a component common to all stochastic processes. As a consequence, the systematic component of the risk associated with a linear combination can not be reduced and for this reason it is said undiversifiable.

From now on, we refer to f_t and $\epsilon_{j,t}$ as exogenous variables and to σ_d^2 and σ_s^2 as exogenous risks. In quantitative finance, there is a distinction between *exogenous* quantities and *endogenous* quantities.

- An exogenous quantity refers to a variable coming from outside the system. In fact, f_t and $\epsilon_{j,t}$ are exogenous variables in the sense that they are unexplained by the model we will describe. These variables are chosen in order to describe some stylized empirical facts about the statistical properties of returns explained before.

- An endogenous quantity refers to a variable that describes a particular effect which is explained inside the model. For example, we will see that an asset return is also affected by an endogenous component coming from the balance sheet dynamics of financial institutions.

Finally, the return process (3.2) describes some stylized facts as the absence of autocorrelations for returns, the aggregational gaussianity depending on the time scale adopted and the aggregate dynamics of all assets characterized by a market factor. On the contrary, other stylized facts as the heavy tails, the slow decay of autocorrelation in squared returns and the volatility clustering can not be deduced by (3.2). A recent work of *Thurner et al.* [77] suggests that these latter properties can be explained endogenously within a model of leveraged investors borrowing from a bank in order to buy assets. An investor is said leveraged when it borrows from a bank in order to invest in assets a value larger than its initial equity. Let E be the initial equity of an investor and let A be the total asset position of the same investor, that is simply the whole capital invested in assets, the *leverage* is the ratio

$$\lambda = \frac{A}{E}$$

In the next Sections, we will clarify the consequences of a financial market populated by leveraged investors. Thurner et al. suggest that as leverage increases price fluctuations become heavy tailed and display clustered volatility.

3.1.2 The economic network

In the study of the systemic risk and the financial contagion, the modeling of a financial market through a network emerges in natural way [14, 16, 78]. Once that M random processes describing the dynamics of returns of M investments assets are defined, naturally N financial institutions which create a portfolio investing in some selected assets can be introduced.

A portfolio represents a selection of some investments assets in which a financial institution invests its available capital represented by A. In the next Section, we will describe an optimization problem whose solution establishes how a financial institution selects the number of investment assets, m, and the financial leverage, λ. Now, we will focus on the properties of an economic network formed of N financial institutions and M risky investments.

Beyond the micro-economic foundations of models, a common recognized feature is related to the network-based measure of connectivity between counterparties. In the model, the *overlap* between portfolios plays the role of the connectivity between financial institutions operating in the market.

In the model, the economy is composed by a group of N financial institutions investing in the M risky investments. This scenario can be represented by a bipartite network. A bipartite network [21] has two types of vertices and edges run only between vertices of unlike types. In describing an economic system, financial institutions represent one kind of nodes. These link to the second kind's nodes that represent the M investment assets. Each financial institution chooses randomly and independently m investments across the set of M available ones. This fact can be represented by m links connecting the bank with its investments.

For analytical tractability, the simplifying assumption of homogeneity across financial institutions is made. This hypothesis means that all financial institutions have the same initial equity E and they choose the same number m of risky investments. m is the degree of *diversification* of a portfolio. As a consequence of the random choose about investments, the portfolios of two financial institutions are generally different. We will use also the word *bank* to refer to financial institutions.

A specific risky investment is owned statistically by n banks, where n is a random variable described by a binomial distribution. The mean value is clearly $E[n] = m\frac{N}{M}$.

The overlap, o, between two portfolios, defined as the number of risky investments in common, is a random variable and it is distributed as an hypergeometric distribution

$$\mathcal{P}(o; M, m) = \frac{\begin{pmatrix} m \\ o \end{pmatrix}\begin{pmatrix} M-m \\ m-o \end{pmatrix}}{\begin{pmatrix} M \\ m \end{pmatrix}} \qquad 0 \leq o \leq m$$

Its mean value is $E[o] = \frac{m^2}{M}$. Finally the fractional overlap of two portfolios is $o_f = \frac{o}{m}$ is a number between 0 and 1 describing the fraction of the portfolio that is in common between two financial institutions. The mean fractional overlap is $E[o_f] = \frac{m}{M}$. Therefore, the value of the diversification m of a portfolio is also a measure of the degree of connectivity between two financial institutions. In fact, a unipartite weighted network corresponds to the bipartite network just explained. The unipartite network is a network in which nodes are represented by financial institutions that are connected each other through weighted links describing the overlap degree between two portfolios owned by two financial institutions. The overlap o_f measures the connectivity between financial institutions. Ultimately, the diversification m defines the overlapping portfolios. The diversification m plays a crucial role in the contagion of risk [18]. The reason is simple. When a bank liquidates an asset, it impacts negatively the market price of the asset. When the portfolios of banks overlap each other, the asset liquidation affects the risk which spreads to other banks. It will be investigated under which conditions the risk becomes systemic.

3.1.3 The portfolio optimization problem

The model describes a scenario in which financial institutions adopt a simple investment strategy consisting in forming an equally weighted portfolio composed on m randomly selected risky investments from the whole collection of M available investment assets. If A_i is the value of total asset position of a bank i, A_i is equally distributed on the m selected investment assets. During the stochastic evolution of the market, the actual value of A_i depends on the returns dynamics of the selected assets (3.2). The actual value of A_i is a stochastic variable.

Supposing that a financial institution i selects m risky investments described by (3.2),

$\{r_{j_1}, r_{j_2}, \ldots, r_{j_m}\}$, the actual value at time t of $A_{i,t}$ depends on the following linear combination of the m stochastic variables describing the asset returns

$$r^p_{i,t} = \sum_{i=1}^{m} \frac{1}{m} (\mu + f_t + \epsilon_{j_i,t}) \tag{3.6}$$

where $r^p_{i,t}$ is the portfolio return, that is a stochastic variable formed of the stochastic variables describing the asset returns. Hence, each portfolio is characterized by an expected return equal to μ and, according to (3.5), an exogenous volatility equal to

$$\sigma_p = \sqrt{\sigma_s^2 + \frac{\sigma_d^2}{m}} \tag{3.7}$$

It is clear from (3.7) that a larger value of m reduces the portfolio volatility σ_p and as a consequence the portfolio of a financial institution becomes less risky. However, in financial markets the diversification of a portfolio requires some costs. Hence, a larger diversification m reduces the exogenous risk and at the same time reduces the profit of a financial institution.

An investment is always associated with a transaction cost in a financial market. However the latter represents a small value of the total cost necessary to diversify a portfolio. When a financial institution decides to invest in a new asset, it must develop new resources as physical structures, human capital, and financial knowledge, which may require also higher costs. We refer to these as diversification costs.

These play an important role to prevent that all institutions operating in financial markets diversify completely their portfolios. For this reason, the existence of diversification costs is assumed in the definition of the portfolio optimization problem. A pioneering work [79] of *Cooper and Kaplanis* has approached the problem of estimating the costs supported by foreign financial agents investing in equity markets of other countries. These costs summarize the transaction costs and other types of costs related to the allocation of money in foreign markets. A more recent work [80] has improved the methodology for a better estimation. The outcomes are summarized in Figure (3.2). The percentage cost for an equity investment is shown for nine countries from year 1980 to year 2004.

It is evident that the values of costs have decreased through time for all countries. This is a consequence of introduction and proliferation of financial instruments which reduce the investment costs to allow the risk diversification.

Financial institutions seek to maximize returns from the risky investments under their Value at Risk (VaR) constraints [81]. According to Basel III [1], financial institutions have to fulfill a capital requirement in order to borrow money and to invest it. The VaR is the most probable worst expected loss at a given level of confidence over a given time horizon [83]. In our case, the time horizon is simply equal to $\Delta t = 1$. Let $f(r^p)$ be the probability density for portfolio return

[1]Basel III [82] is an accord stipulated by the Basel Committee on Banking Supervision in order to give a comprehensive set of reform measures designed to improve the regulation, supervision and risk management within the banking sector.

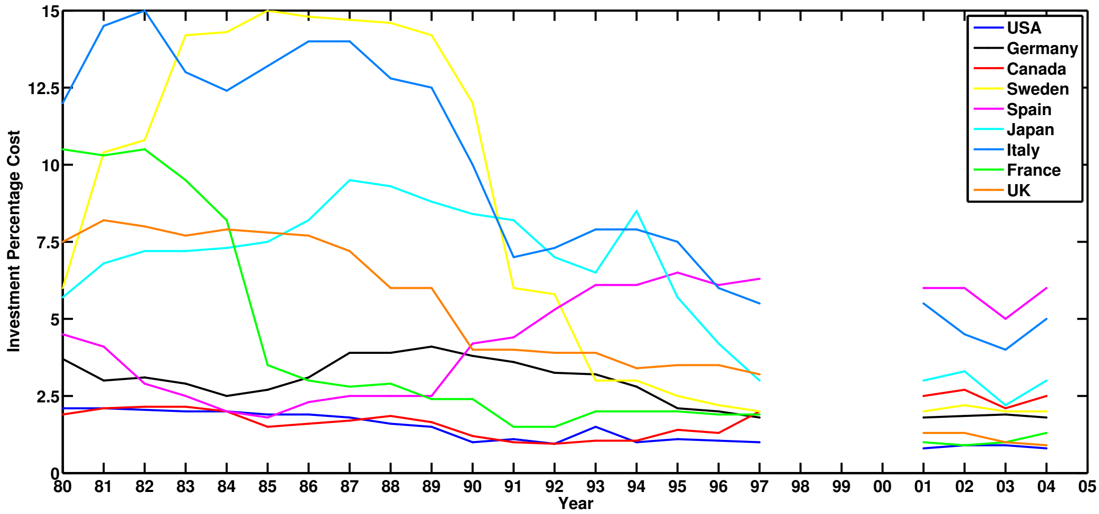


Figure 3.2: The estimated values of the percentage costs supported by investors in a foreign equity market. The costs associated with each country are estimated through a methodology proposed by *Cooper and Kaplanis* [79] and improved by *Thapa and Poshakwale* [80]. The data are taken by the more recent *Thapa*'s work [80]. Nine different countries are identified by nine different colors. The data represent the time evolution of the investment percentage costs from year 1980 to year 2004, with the exception of a time window from 1997 to 2001 in which the values of costs are not reported because the empirical data are not available.

r^p. The VaR Λ_{VaR} associated with a certain probability of loss P_{VaR} is defined implicitly by

$$\int_{-\infty}^{-\Lambda_{VaR}} f(r^p)\, dr^p = P_{VaR}$$

The capital requirement imposed to financial institutions is a consequence of the Value at Risk calculated for a specific portfolio and it is equivalent to the following constraint

$$VaR = \alpha\, \sigma_p\, A \leq E \tag{3.8}$$

where α is a scaling parameter, A is the level of total assets of a bank portfolio and E is the initial equity of the bank. $\alpha\sigma_p$ represents in another form the value of the quantile Λ_{VaR} associated with the loss probability P_{VaR}. In fact, since r^p is a Gaussian stochastic variable, σ_p is the standard deviation of the probability density function $f(r^p)$ and it represents the value at which the cumulative distribution corresponds approximately to a probability equal to 13.6%. As a consequence, α can be found in order that the cumulative distribution calculated at $\alpha\sigma_p$ is equal to P_{VaR}.

In making investments, financial institutions have to fulfill a capital requirement in the form of the constraint (3.8).

Finally, to find the optimal portfolio, financial institutions must take into account the expense coming from liabilities. Let r_L be the average interest expense related to money borrowed by a financial institution, then the Net Interest Margin (NIM) of the financial institution is $\mu - r_L$.

Summarizing, given their NIM and level of equity E, financial institutions, taking into account costs of diversification and VaR constraints, choose the level of total asset A and degree of diversification m which maximize their returns from the risky investments. That is, assuming costs of diversification proportional to m, financial institutions maximize

$$\max_{A,m} \ A(\mu - r_L) - \tilde{c}m \quad \text{s.t.} \quad \alpha A\sqrt{\sigma_s^2 + \frac{\sigma_d^2}{m}} \leq E \tag{3.9}$$

where $\tilde{c}$ is the cost for investment (assumed to be the same across all investments according to the homogeneity hypothesis). Dividing by E and defining $c = \frac{\tilde{c}}{E}$ as the cost in unit of equity, the maximization problem (3.9) is equivalent to

$$\max_{\lambda,m} \ \lambda(\mu - r_L) - cm \quad \text{s.t.} \quad \alpha\lambda\sqrt{\sigma_s^2 + \frac{\sigma_d^2}{m}} \leq 1 \tag{3.10}$$

where $\lambda = \frac{A}{E}$ is the leverage adopted by financial institutions.

The maximization problem (3.10) can be solved through the method of Lagrange multipliers. Transforming the VaR constraint by squaring both sides, the Lagrangian can be written as

$$L = \lambda(\mu - r_L) - cm - \frac{1}{2}\gamma\left(\alpha^2\lambda^2\left(\sigma_s^2 + \frac{\sigma_d^2}{m}\right) - 1\right) \tag{3.11}$$

where γ is the Lagrangian multiplier for the VaR constraint. The solution of the portfolio optimization problem can be found solving the following equations

$$m = \lambda\sigma_d\sqrt{\frac{\alpha}{2c}\frac{\mu - r_L}{\sigma_p}} \tag{3.12}$$

$$\lambda = \frac{1}{\alpha\sigma_p} \tag{3.13}$$

The solution of (3.12) (3.13) defines the optimal diversification m^* and the optimal leverage λ^*. According to the previous assumptions, m^* and λ^* completely describe the portfolio of a financial institution.

The optimal degree of diversification is inversely related to the cost of diversification c. Both portfolio variables (diversification and leverage) are inversely related to the quantile depending on α (in fact, replacing in (3.12) the value of λ given in (3.13), also the diversification m will be inversely related to α).

Figure (3.3) shows the numerical solutions for the optimal degree of diversification m^* and the optimal leverage λ^* as a function of different values of diversification costs c, panels (a) and (b). Each line corresponds to different levels of systematic to idiosyncratic noise ratio ($\frac{\sigma_s}{\sigma_d} = \{0, 0.3, 0.6\}$). When diversification costs decrease, the degree of diversification increases and as a consequence of relaxing the VaR constraint (because σ_p decreases through the risk diversification) a larger value of the leverage is allowed. Below a specific value of costs, the degree of diversification and the leverage become constant because a portfolio is fully diversified across all the M available investment assets.

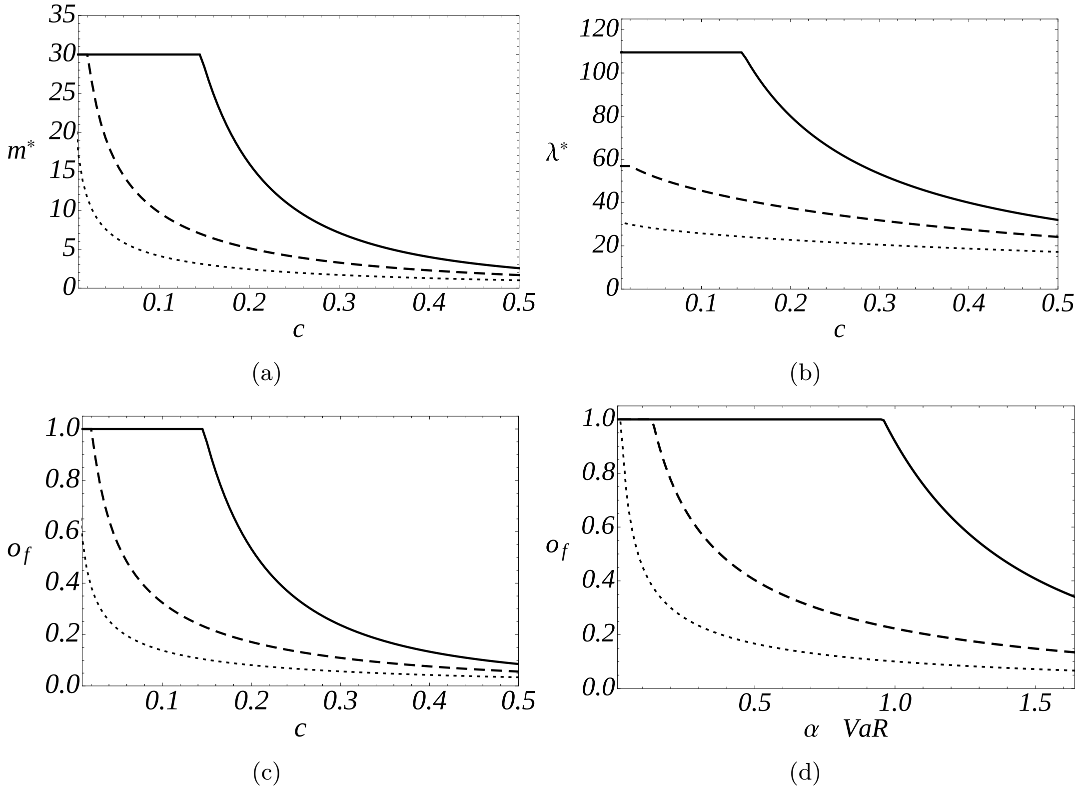


Figure 3.3: The optimal degree of diversification m^*, panel (a), and the optimal financial leverage λ^*, panel (b), as a function of the diversification cost c, obtained by solving numerically equation (3.12) (3.13). The mean fractional overlap $o_f = \frac{m}{M}$ between two portfolios versus the diversification cost c, panel (c), and versus the α parameter of the VaR constraint. The used parameters are: $M = 30$, $\mu - r_L = 0.08$, $\sigma_d = \frac{0.05}{1.64}$. In panels (a), (b) and (c) $\alpha = 1.64$, while in panel (d) $c = 0.25$. In all panels the solid line refers to $\frac{\sigma_s}{\sigma_d} = 0$, the dashed line refers to $\frac{\sigma_s}{\sigma_d} = 0.3$ and the dotted line refers to $\frac{\sigma_s}{\sigma_d} = 0.6$.

In fact, looking at the fractional overlap o_f between portfolios, panel (c) of Figure (3.3), o_f increases after a reduction of diversification costs. When the latter become smaller and smaller, o_f reaches the maximal value equal to 1 and this situation corresponds to a scenario in which all financial institutions fully diversify their portfolios which become equivalent to the market portfolio (the one containing all the M investments with equal weights).

Finally, panel (d) of Figure (3.3) shows the fractional overlap as a function of the α-value defining the quantile in the VaR constraint. In this approach, it is important to notice that a reduction of α is followed by increasing values of leverage and diversification and as a consequence the value of the fractional overlap increases. Increasing α means requiring that financial institutions have to cover negative returns associated with smaller probability. Larger values of α correspond to a more safe economy.

As claimed before, a larger value of the mean overlap between portfolios induces more correlation between financial institutions and as a consequence risk shocks can propagate from one of them to the others. At the same time, also another effect can appear. Coordinated operations in the financial market induced by the correlation between portfolios can impact the risk of assets. Also when each financial institution adopts a micro-prudential strategy (risk diversification and

VaR constraint), a systemic effect due to a financial strategy adopted by all banks may induce a large impact on risk.

3.2 Endogenous dynamics

In this Section, we describe how the dynamics of balance sheet of financial institutions creates feedback effects affecting the returns of investments and as a consequence an endogenous risk component appears. These feedback effects can be described by introducing an endogenous component in the return processes. Depending on the values of model parameters, the strength (related to the value of leverage) and the coordination (due to similarity of bank portfolios) of feedback effects may trigger a transition from a stationary dynamics of price returns to a non stationary one.

3.2.1 Marked-to-market financial institutions

A financial market is composed by financial institutions adopting specific trading strategies in order to realize an expected portfolio return through a risk management strategy. Typically, large financial institutions operating in financial markets are

- **highly leveraged**: a large part of their investments comes from loans obtained by other institutions. Neglecting the liquidities that are cash or cash assets producing no incomes, the level of all investments A is equal to the initial equity E plus the level of debts. This means that leverage $\lambda = \frac{A}{E}$ is larger than 1, sometimes it is much larger than 1.

- **marked-to-market**: depending on adopted strategy, they modify their balance sheet according to actual prices of owned assets.

Commonly, the adopted strategy by large financial institutions is to maintain a chosen value of the leverage (target leverage). For this purpose, banks borrow from or lend to another banks in order to adjust their balance sheet and maintain fixed the ratio $\frac{A}{E}$.

Adrian and Shin [7] have shown the relation between leverage and balance sheet size for different types of financial institutions of USA economy. In order to make clear how a target leverage strategy works, we refer to Figure (3.4). Let us consider the behavior of a bank that manages its balance sheet actively to maintain a target leverage of 10. If initial equity is equal to 10, the bank borrows 90 to reach a value of $\lambda = \frac{A}{E} = 10$, where A represents the total asset position at initial time. Suppose the price of securities increases by 1% to 101. If we assume that the value of the debt does not change approximately, leverage then falls to $\frac{101}{11} = 9.18$. The bank targets leverage of 10. It takes on additional debt equal to 9 and with this money it purchases assets for a value equal to 9. After the purchase, leverage is now back up to 10. The mechanism of maintaining target leverage works in reverse when the assets price falls. We refer to this mechanism as portfolio rebalancing.

Panel (a) of Figure (3.5) shows the leverage variations corresponding to the total assets growth for major Commercial Banks of the USA economy from 1963 to 2006. The data are

Assets	Liabilities
100	Equity, 10
	Debt, 90

The price of assets increases by 1%

Assets	Liabilities
101	Equity, 11
	Debt, 90

Banks increase debt to maintain target leverage

Assets	Liabilities
110	Equity, 11
	Debt, 99

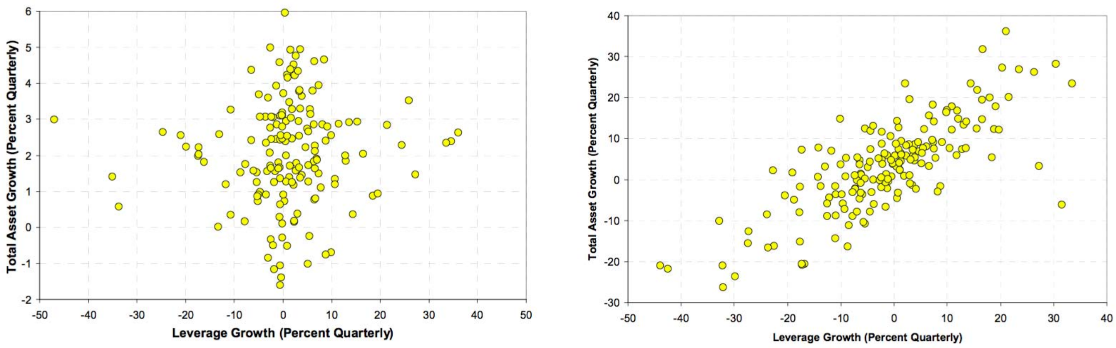
Figure 3.4: An illustrative example of a balance sheet dynamics which maintains the target leverage. At first time, the target leverage is chosen equal to 10 (the ratio between assets and equity). At a second time, the price of assets varies, we suppose that it increases by 1%. As a consequence, the value of the equity increases and the ratio between assets and equity decreases, *i.e.* leverage decreases. In order to maintain fixed to 10 the value of leverage, a financial institution increases the debt level, leaving unchanged the equity level, and it invests further to reach a value of leverage equal to the target.

consistent with the target leverage mechanism just explained because no changes in the value of leverage correspond to variations of total assets. Furthermore, panel (b) of Figure (3.5) shows equivalent data related to Security Brokers and Dealers, that include the major Wall Street investment banks. In this case, there is a strongly *positive* relationship between changes in total assets and changes in leverage. The leverage is procyclical.

- **Definition**: a quantity is defined *procyclical* when it is positive related to the tendency of the financial market, *i.e.* when the price of all assets increases, a procyclical quantity increases and vice-versa.

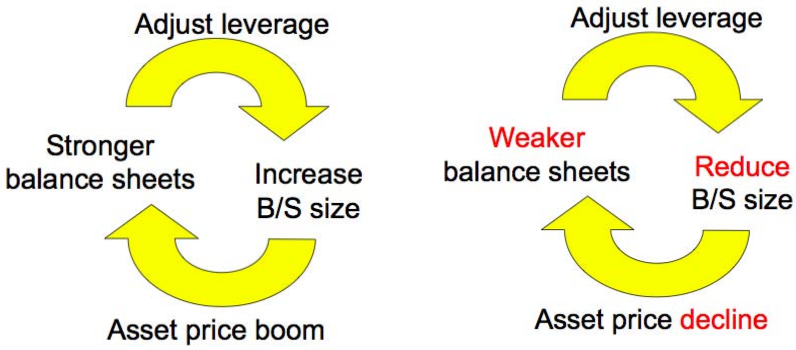
- **Definition**: a financial market is *illiquid* when the buying (the selling) of an asset is followed by a positive (negative) change of the asset price, *i.e.* when an agent operating in the market purchases an asset, it creates a positive pressure which puts upward the asset price, and vice-versa.

If financial markets are illiquid such that greater demand for the asset tends to put upward pressure on its price, then there is the potential for a feedback effect in which stronger balance sheets feed greater demand for the asset, which in turn raises the asset's price and lead to stronger balance sheets. The mechanism works exactly in reverse in downturns. If financial markets are illiquid such that greater supply of the asset tends to put downward pressure on its price, then there is the potential for a feedback effect in which weaker balance sheets lead to greater sales of the asset, which depresses the asset's price and lead to even weaker balance sheets. In Figure (3.6) this potential feedback effect is graphically summarized.



(a) Commercial Banks (b) Security Brokers and Dealers

Figure 3.5: The change of total assets versus the change in leverage for the largest commercial banks, left panel, and the largest security brokers and dealers, right panel, of the United States economy. Both types of financial institutions represent the main actors operating in a financial market. The plotted empirical data are obtained by *Adrian and Shin* [7] through the flow of funds data (*e.g* data from the Federal Reserve) from 1963 to 2006. The empirical outcomes can be interpreted as the result of active management of balance sheets by financial intermediaries who respond to changes in prices and measured risk.



(a) raising of asset price (b) falling of asset price

Figure 3.6: A stylized scheme of the rebalancing feedback effects triggered by positive changes of asset price, left panel, and negative changes of assets price, right panel.

The feedback effect coming from portfolio rebalancing affects crucially the risk of the assets. Particularly, these feedback effects have larger impact on risk if they arise from coordinated actions of financial institutions due to high correlation between them in consequence of portfolios highly diversified.

In many recent papers [84, 85, 86, 87, 88, 89] the attention was been focused on this feedback effect which can potentially trigger the contagion of risk between financial institutions and eventually lead the economic system towards a financial crisis.

Below, we describe how the feedback effect is taken into account in the model [9].

3.2.2 Asset demand from portfolio rebalancing

Having identified their optimal portfolio choosing the values of leverage and diversification, financial institutions periodically rebalance their portfolios in order to maintain the desired target leverage.

In fact, due to the stochastic evolution of the market , a bank i at time t holds an actual asset position equal to $A_{i,t}$. In order to maintain λ fixed, the optimal value of the asset position will be $A^*_{i,t} = \lambda\, E_{i,t}$. As shown previously, the financial behavior of banks is to enlarge (reduce) the asset position according to the increase (decrease) of assets prices without affecting the equity level. The rebalancing of the portfolio of individual bank i at time t is given by the difference between the desired asset position and the actual one, *i.e.* $\Delta R_{i,t} = A^*_{i,t} - A_{i,t}$. The latter can be rewritten as [9]

$$\Delta R_{i,t} = (\lambda - 1)\, r^p_{i,t}\, A^*_{i,t-1} \tag{3.14}$$

where $r^p_{i,t} = \sum_{j=1}^{M} I_{j\in i}\, \frac{r_{j,t}}{m}$ is the portfolio return associated with bank i at time t. $I_{j\in i}$ is the indicator function which takes value 1 if investment j is in the portfolio of bank i and zero otherwise.

$\Delta R_{i,t}$ in (3.14) represents a change in the asset value amplified by the excess leverage $(\lambda - 1)$ as a consequence of the profit or the loss from the investments in the chosen portfolio $(r^p_{i,t} A^*_{i,t-1})$. In fact, when $r^p_{i,t}$ at time t is different from zero, its value multiplied to the previous optimal choice of total assets, $A^*_{i,t-1}$, gives the amount of profit (loss).

The relation (3.14) reflects exactly the balance sheet dynamics described in the previous Section. Hence, VaR constrained financial institutions will have a positive feedback effect on the prices of the assets in their portfolios.

In order to explain the positive feedback effect on the prices of the assets, it is necessary to compute the total demand of a generic risky investment j at time t. This will be simply the sum of the individual demand of the financial institutions who picked investment j in their portfolio.

$$D_{j,t} = \sum_{i=1}^{N} I_{j\in i} \frac{1}{m} \Delta R_{i,t} = \sum_{i=1}^{N} I_{j\in i} (\lambda - 1)\, r^p_{i,t}\, \frac{A^*_{i,t-1}}{m} \tag{3.15}$$

The total demand of investment j is proportional to the amount of portfolio rebalancing of each bank i divided by the value of diversification m, according to the assumption that banks distribute equally the value of total asset position among all selected investment assets.

Assuming that total assets are approximately the same for all the banks, *i.e.* $A^*_{i,t-1} \simeq A^*_{t-1}$, it follows that

$$D_{j,t} \approx (\lambda - 1) \frac{A^*_{t-1}}{m} \left(\sum_{i=1}^{N} I_{j\in i}\, r^p_{i,t} \right)$$

Since $r^p_{i,t} = \sum_{k=1}^{M} I_{k\in i}\, \frac{r_{k,t}}{m}$, the sum in the above expression for $D_{j,t}$ can be rewritten as

$$\frac{1}{m} \sum_{i=1}^{N} I_{j\in i} \sum_{k=1}^{M} I_{k\in i}\, r_{k,t}$$

This term can be calculated considering all the banks which have investment j in their portfolio and then, for each investment k (including j) it is necessary to count the number of them having investment k in their portfolio. This is a random variable as seen before. On average there are $\frac{Nm}{M}$ banks having investment j in the portfolio. All of them have (by definition) investment j in the portfolio. On the other hand, for an investment $k \neq j$, we can consider the restricted bipartite network of $\frac{Nm}{M}$ banks, each having $m-1$ investments among a universe of $M-1$. Therefore the number of banks having investment j in their portfolio and having also investment k is $\frac{Nm}{M}\frac{m-1}{M-1}$. Thus, on the average, the term can be rewritten as

$$\frac{N}{M}\left(r_{j,t} + \frac{m-1}{M-1}\sum_{k\neq j} r_{k,t}\right)$$

and an approximate form of the demand of asset j is

$$D_{j,t} \approx (\lambda-1)\frac{A^*_{t-1}}{m}\frac{N}{M}\left(r_{j,t} + \frac{m-1}{M-1}\sum_{k\neq j} r_{k,t}\right) \tag{3.16}$$

Particularly, the demand of an asset j strictly depends on the excess leverage $\lambda - 1$. If financial institutions are fully leveraged under VaR constraint, periodically there is a large demand of assets. When the demand is large, there is a large impact on the prices of assets. In the next Section, the price impact function is defined and the consequences on the assets returns and risk are investigated.

3.2.3 The dynamics of asset returns with rebalancing feedback

As claimed before, the demand of an asset tends to put upward pressure on its price and vice-versa. Some recent works [90, 35] suggest that the change of the price of an asset j due to the demand impact is approximately proportional to the number of traded shares s_j and the asset volatility σ_j normalized by the average daily volume of asset j, V_j. In financial markets, an investment asset is divided into portions which are offered for sale. Thus, a share is an indivisible unit of an asset. According to [35], if p_t corresponds to the price of asset j before the trading and $p_{t+\Delta t}$ is the one after the trading, the price impact is approximately

$$\frac{p_{t+\Delta t} - p_t}{p_t} = \pm\sigma_j \frac{s_j}{V_j} \tag{3.17}$$

where the sign $\pm$ depends on the direction of the order. A set of empirical studies [91, 36] indicates that the price impact function is better described by a concave shape. Particularly a square-root impact function

$$\frac{p_{t+\Delta t} - p_t}{p_t} = \pm\sigma_j \sqrt{\frac{s_j}{V_j}} \tag{3.18}$$

fits better the empirical data. The main point is that the price impact function depends on the number of traded shares that is related to the asset demand. The variation of the price in (3.17) or (3.18) induces an endogenous component in the asset return.

For the sake of analytical tractability, in the model [9] a linear price impact function (3.17) is assumed. However, we replace $\frac{\sigma_j}{V_j}$ with a different constant, that is the *liquidity* of the investment asset j. Furthermore, there are many definitions of liquidity. In this framework, the liquidity of an asset is defined as the ability of the asset to sustain the demand (or the supply) without moving the price. The liquidity of an asset j can be considered through a finite parameter γ_j. The price impact function is inversely related to the liquidity γ_j. The larger is γ_j, the smaller is the price impact of a trading on asset j. The limit $\gamma_j \to \infty \ \forall j$ coincides with the ideal case of a perfectly liquid market in which the trading does not move the price.

In rebalancing their portfolio at time $t-1$, financial institutions induce an endogenous component $e_{j,t-1}$ of the return of investment j at time t that is

$$e_{j,t} = \frac{1}{\gamma_j}\frac{D_{j,t}}{C_{j,t}} \tag{3.19}$$

where $D_{j,t-1}$ is defined in (3.16) and $C_{j,t-1}$ is the market capitalization of investment j. The ratio $\frac{D_{j,t}}{C_{j,t}}$ corresponds to the number of traded shares s_j.

Again, assuming that $A^*_{i,t-1} \simeq A^*_{t-1}$ the market capitalization of investment j is

$$C_{j,t} = \sum_{i=1}^{N} I_{j\in i}\frac{A^*_{i,t-1}}{m} \approx \frac{A^*_{t-1}}{m}\sum_{i=1}^{N} I_{j\in i} = \frac{N}{M}A^*_{t-1} \tag{3.20}$$

because on average there are $m\frac{N}{M}$ banks having investment j in their portfolio.

The endogenous component of the return of asset j, $e_{j,t-1}$, defines the price impact function due to the portfolio rebalancing. When the total demand of an asset j is large with respect to the total capitalization of the asset, it follows that the portfolio rebalancing impacts largely the price of the asset if the market is illiquid.

This effect reflects the feedback mechanism explained before. In fact, in presence of rebalancing feedbacks, the return process of an asset j at time t will now be made of two components:

$$r_{j,t} = f_t + \epsilon_{j,t} + e_{j,t-1} \tag{3.21}$$

where $\mu_j = 0$ for notational simplicity. The exogenous component $f_t + \epsilon_{j,t}$ is related to the stochastic evolution of the price of asset j. The endogenous component $e_{j,t-1}$ describes the feedback effect coming from the previous period portfolio rebalancing of the financial institutions. It can affect critically the stability of the dynamics of returns. In order to make evident this feature, substituting equations (3.16), (3.21) and (3.20) in (3.19) and using matrix notation, the dynamics of the vector of the endogenous components follows a Vector Autoregressive (VAR) process

$$\mathbf{e}_t = \mathbf{\Phi}\mathbf{r}_t = \mathbf{\Phi}(\mathbf{e}_{t-1} + \boldsymbol{\varepsilon}_t) \tag{3.22}$$

where $\varepsilon_{j,t} = f_t + \epsilon_{j,t}$, $\mathbf{e}_t = \{e_{j,t}\}_{j=1,\dots,M}$ and $\boldsymbol{\varepsilon}_t = \{\varepsilon_{j,t}\}_{j=1,\dots,M}$.

The linear operator $\mathbf{\Phi} \equiv (\lambda - 1)\mathbf{\Gamma}^{-1}\mathbf{\Psi}$ with

$$\mathbf{\Gamma} = \begin{pmatrix} \gamma_1 & 0 & \cdots & 0 \\ 0 & \gamma_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \gamma_M \end{pmatrix}$$

$$\mathbf{\Psi} = \begin{pmatrix} \frac{1}{m} & \frac{1}{m}\frac{m-1}{M-1} & \cdots & \frac{1}{m}\frac{m-1}{M-1} \\ \frac{1}{m}\frac{m-1}{M-1} & \frac{1}{m} & \cdots & \frac{1}{m}\frac{m-1}{M-1} \\ \vdots & & \ddots & \vdots \\ \frac{1}{m}\frac{m-1}{M-1} & \frac{1}{m}\frac{m-1}{M-1} & \cdots & \frac{1}{m} \end{pmatrix}$$

where $\mathbf{\Gamma}$ and $\mathbf{\Psi}$ are $M \times M$ matrices.

Thus the endogenously determined component of returns follows a VAR process of order one. The dynamics of such VAR(1) process is dictated by the eigenvalues of the matrix $\mathbf{\Phi} \equiv (\lambda - 1)\mathbf{\Gamma}^{-1}\mathbf{\Psi}$. In particular, since the maximum eigenvalue of the matrix $\mathbf{\Psi}$ is always equal to 1 [9, 76], the maximum eigenvalue of the VAR(1) process becomes

$$\Lambda_{max} \approx (\lambda - 1)\overline{\gamma^{-1}} \tag{3.23}$$

where $\overline{\gamma^{-1}}$ is the average of γ^{-1}. Hence, the maximum eigenvalue depends on the degree of leverage and on the average illiquidity of the investments. When the maximum eigenvalue is greater than one, the return processes become non-stationary and explosively accelerating. It is important to remark that even a reduction in the liquidity of only one risky investment (by changing the average illiquidity of the investments) impacts the dynamics of all the traded investments and can potentially drive the whole financial system towards instability. In fact, depending on the average of the $\frac{1}{\gamma_j}$ the maximum eigenvalue (and thus the dynamical properties of the whole system) will be highly sensitive to illiquid investments, *i.e.* to investment having a small γ. For the sake of the analytical tractability, $\gamma_j = \gamma \ \forall j$ is assumed, that is all investments have the same liquidity.

Figure (3.7) shows the largest eigenvalue Λ_{max} of the operator $\mathbf{\Phi}$ as a function of the diversification cost c (left panel) and as a function of the mean of the fractional portfolio overlap, $E[o_f] = \frac{m}{M}$, see Section 3.1.1.

A reduction of the diversification cost induces banks to choose a larger value of leverage, see Figure (3.3). As a consequence, the larger leverage reinforces the feedback effect due to the portfolio rebalancing which can lead to dynamics instability of the assets returns (for $\Lambda_{max} > 1$) when c decreases below a certain threshold (which is higher for smaller ratio of systematic to idiosyncratic volatility). In this case, rebalancing feedback effects trigger a transition from a stationary dynamics of price returns to a non-stationary one characterized by steep growths and plunges of market prices. Of course, the instability of the economic system corresponds to the instability of returns dynamics.

Together with the strength of feedback effects related to the value of leverage, an higher

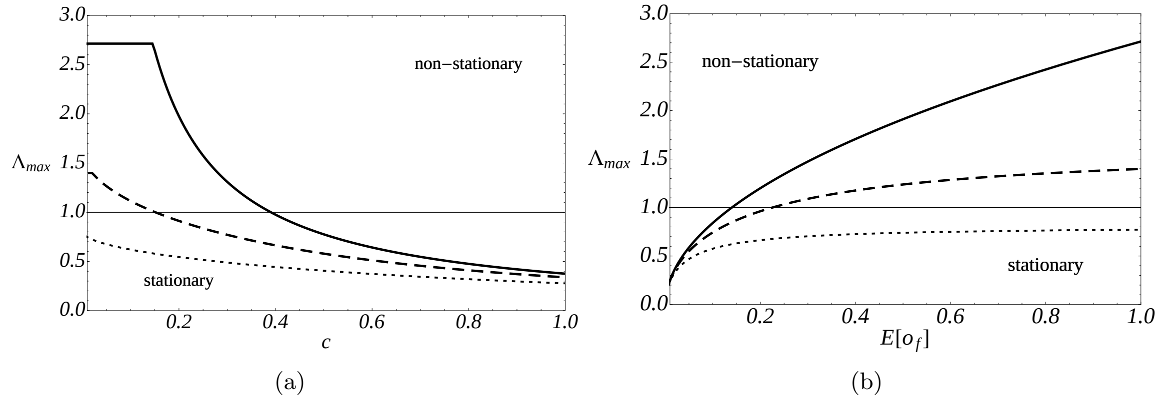


Figure 3.7: The largest eigenvalue Λ_{max} of the linear operator $\mathbf{\Phi}$ describing the dynamics of endogenous components, under the assumption $\gamma_j = \gamma \; \forall j$. In the left panel, the relation between Λ_{max} and the value of diversification cost c. In the right panel, the relation between Λ_{max} and the mean fractional overlap between portfolios, $E[o_f]$. When Λ_{max} reaches the critical value equal to 1, a transition from stationarity to non-stationarity occurs for the dynamics of endogenous components and as a consequence for the dynamics of asset returns. The used parameters are: $M = 30$, $\mu - r_L = 0.08$, $\sigma_d = \frac{0.05}{1.64}$ and $\alpha = 1.64$. The solid line refers to $\frac{\sigma_s}{\sigma_d} = 0$, the dashed line refers to $\frac{\sigma_s}{\sigma_d} = 0.3$ and the dotted line refers to $\frac{\sigma_s}{\sigma_d} = 0.6$.

level of coordination in portfolio rebalancing, due to similarities in the portfolio compositions, also reinforces the feedback. This aspect is related to the coordination of the demands of assets. When the overlap between portfolios increases, many financial institutions own the same investment assets. Hence, according to the tendency of asset returns, they require to purchase (to sell) the same assets.

According to this, the demand (3.16) of an asset will be the sum of similar contributions. The feedback effect described by endogenous components will have a large impact on the asset returns. Also the degree of diversification m plays a role in triggering a transition from stability to instability of economic system.

Summarizing, larger values of leverage λ and diversification m increase the impact of rebalancing feedback on asset prices and eventually they may put the system towards the region of instability ($\Lambda_{max} > 1$).

The endogenous components describing the rebalancing feedback in (3.21) affects also the risk of the assets. This effect is a consequence of the balance sheet dynamics of financial institutions which modify their asset position in order to maintain the target leverage but at the same time the purchasing or the selling of assets makes them more volatile. In fact, in addition to the exogenous risk component (3.3), the price impact function induces an endogenous risk component that takes into account the excess volatility induced by trading. Under the hypothesis of assets homogeneity, the endogenous risk component affects largely all investment assets when there is coordination in portfolio rebalancing due to similar portfolios and when financial institutions are over-leveraged. In the next Section, the variance-covariance matrix of returns will be computed in a closed form.

3.2.4 Endogenous feedbacks and endogenous risks

The homogeneity assumptions allow to give an exact description of the variance-covariance matrix of returns. In fact, the linear operator $m\mathbf{\Psi}$ can be written as

$$m\mathbf{\Psi} = (1-b)\mathbb{1} \; + \; b\,\imath\,\imath' \tag{3.24}$$

where the scalar $b = \frac{m-1}{M-1}$, identity matrix $\mathbb{1}$ and the column vector of ones $\imath$. According to this decomposition, the VAR for the vector of endogenous components in equation (3.22) can be rewritten as

$$\mathbf{e}_t = (1-b)\mathcal{A}(\mathbf{e}_{t-1} + \boldsymbol{\varepsilon}_t) + bM\mathcal{A}\,\imath\,(\bar{e}_{t-1} + \bar{\varepsilon}_t) \tag{3.25}$$

with matrix $\mathcal{A} \equiv \frac{\lambda-1}{m}\mathbf{\Gamma}^{-1}$ and scalars $\bar{e}_t \equiv \frac{1}{M}\sum_{k=1}^{M} e_{k,t}$ and $\bar{\varepsilon}_t \equiv \frac{1}{M}\sum_{k=1}^{M} \varepsilon_{k,t}$. The scalar $\bar{e}_t$ can be interpreted as the endogenous return of the market portfolio. Thus, the endogenous component of an individual investment becomes

$$e_{j,t} = (1-b)a_j(e_{j,t-1} + \varepsilon_{j,t}) + bMa_j(\bar{e}_{t-1} + \bar{\varepsilon}_t) \tag{3.26}$$

with scalar $a_j = \frac{\lambda-1}{m\gamma_j}$. Coherently with the assumption that all investments have the same liquidity $\gamma_j = \gamma$, it follows that $a_j = \frac{\lambda-1}{m\gamma} \equiv a \; \forall j$.

Therefore, the process for $e_{j,t}$ can be rewritten as a linear combination of a standard univariate AR(1) process and a dynamic process depending on the averages of previous period endogenous components and shocks. In this way, $e_{j,t}$ is a mixture of a perfectly idiosyncratic process (*i.e.* uncorrelated with the others investment processes) receiving weight $1-b$ and a perfectly correlated process with weight b. Since $b = \frac{m-1}{M-1}$, the higher is the overlap between portfolios (related to the value of m), the the higher is the weight given to the perfectly correlated component of mixture and, hence, the higher the correlations among the endogenous components of the different investments. As claimed before, a larger value of diversification m induces a stronger correlation in portfolio rebalancing implemented by financial institutions and as a consequence the impact of feedback effects is stronger.

Averaging over investments (that is summing over all j and dividing by M) in (3.26), the process for $\bar{e}_t$ becomes

$$\bar{e}_t = a(1-b+Mb)(\bar{e}_{t-1} + \bar{\varepsilon}_t) \equiv \phi(\bar{e}_{t-1} + \bar{\varepsilon}_t) \tag{3.27}$$

where $\phi \equiv a(1-b+Mb)$. Therefore, the dynamics of the average process $\bar{e}_t$ is also an autoregressive of order one.

Finally, defining the distance of the endogenous component of investment j from the average as $\Delta e_{j,t} \equiv e_{j,t} - \bar{e}_t$, and similarly the distance of the exogenous component of investment j from the average as $\Delta\varepsilon_{j,t} \equiv \varepsilon_{j,t} - \bar{\varepsilon}_t$, it follows that

$$\Delta e_{j,t} = (1-b)a(\Delta e_{j,t-1} + \Delta\varepsilon_{j,t}) \tag{3.28}$$

The dynamics of the individual distance of the endogenous component of investment j from the average value $\bar{e}_t$ is also an autoregressive process of order one.

Therefore, the return dynamics of the endogenous components of each individual investment (3.26) can be interpreted as an autoregressive AR(1) process (3.28) around a common process for the average value (3.27) also following an AR(1) and where the amplitude of (3.28) ($\propto (1-b)$) is decreasing when the portfolio diversification m increases.

This situation can be interpreted as a stochastic scenario corresponding to the decomposition of a many-body physical system into the motion of the center of mass and the motion around the center of mass.

Thanks to this representation, the computation of the variance and the covariances of the process for the endogenous components $e_{j,t}$ can be obtained [9]. In particular, this computation makes sense in the stationary regime, that is when the largest eigenvalue of the operator $\mathbf{\Phi}$ is smaller than 1 ($\Lambda_{max} < 1$). When Λ_{max} crosses the critical value equal to 1, the return dynamics becomes non-stationary, and the feedback can trigger steep growths and plunges of market prices.

Taking into account the feedback introduces a new endogenous component in the variance of the investment asset given by the variance of the endogenous component

$$V(r_{j,t}) = \sigma_j^2 + V(e_{j,t-1}) \tag{3.29}$$

where $\sigma_j^2 = \sigma_d^2 + \sigma_s^2$ can be interpreted as the "bare" level of the risk of asset j. On the contrary, the endogenous risk component $V(e_{j,t-1})$ can be interpreted as a *variable* component because it depends on leverage λ and diversification m chosen by financial institutions in optimizing their portfolio. Leverage and diversification can change through time in relation with the change of market conditions (described by model parameters).

This risk component is due to the finite liquidity of the investments and it becomes equal to zero when the liquidity $\gamma \to \infty$.

Analogously, the covariance between the returns of two investments increases due to a new contribution coming from the covariance between the endogenous components.

$$Cov(r_{j,t}, r_{k,t}) = \sigma_s^2 + Cov(e_{j,t-1}, e_{k,t-1}) \tag{3.30}$$

Defining the excess leverage as $\tilde{\lambda} = (\lambda - 1)$, the variance and the covariance of endogenous components as a function of m and $\tilde{\lambda}$ are (see [9])

$$\begin{aligned} V(e_j) &= -\tilde{\lambda}^2 \{ \frac{m^2(\sigma_d^2(\tilde{\lambda}^2 - \gamma^2(M-1)) + \sigma_s^2(\tilde{\lambda}^2 - \gamma^2(M-1)^2))}{(\gamma^2 - \tilde{\lambda}^2)(m^2(\gamma^2(M-1)^2 - \tilde{\lambda}^2) + 2\tilde{\lambda}^2 mM - \tilde{\lambda}^2 M^2)} + \\ &+ \frac{2m(M(\sigma_d^2(\gamma^2 - \tilde{\lambda}^2) - \tilde{\lambda}^2\sigma_s^2) - \gamma^2\sigma_d^2)}{(\gamma^2 - \tilde{\lambda}^2)(m^2(\gamma^2(M-1)^2 - \tilde{\lambda}^2) + 2\tilde{\lambda}^2 mM - \tilde{\lambda}^2 M^2)} + \\ &+ \frac{M(M(\sigma_d^2(\tilde{\lambda}^2 - \gamma^2) + \tilde{\lambda}^2\sigma_s^2) + \gamma^2\sigma_d^2)}{(\gamma^2 - \tilde{\lambda}^2)(m^2(\gamma^2(M-1)^2 - \tilde{\lambda}^2) + 2\tilde{\lambda}^2 mM - \tilde{\lambda}^2 M^2)} \} \end{aligned} \tag{3.31}$$

$$\begin{aligned} Cov(e_j, e_k) &= -\tilde{\lambda}^2\{\frac{m^2(\sigma_s^2(\tilde{\lambda}^2-\gamma^2(M-1)^2)-\gamma^2(M-2)\sigma_d^2)}{(\gamma^2-\tilde{\lambda}^2)(m^2(\gamma^2(M-1)^2-\tilde{\lambda}^2)+2\tilde{\lambda}^2 mM-\tilde{\lambda}^2M^2)}+ \\ &+\frac{-2m(\tilde{\lambda}^2M\sigma_s^2+\gamma^2\sigma_d^2)}{(\gamma^2-\tilde{\lambda}^2)(m^2(\gamma^2(M-1)^2-\tilde{\lambda}^2)+2\tilde{\lambda}^2 mM-\tilde{\lambda}^2M^2)}+ \\ &+\frac{M(\tilde{\lambda}^2M\sigma_s^2+\gamma^2\sigma_d^2)}{(\gamma^2-\tilde{\lambda}^2)(m^2(\gamma^2(M-1)^2-\tilde{\lambda}^2)+2\tilde{\lambda}^2 mM-\tilde{\lambda}^2M^2)}\} \end{aligned} \quad (3.32)$$

The return of the portfolio of bank i is

$$r_{i,t}^p = \sum_{j=1}^{M} I_{j\in i}\frac{r_{j,t}}{m} \quad (3.33)$$

with m the chosen value of diversification by bank i. As a consequence of (3.29) and (3.30), the variance of the portfolio return of a typical financial institution i in presence of rebalancing feedback effects becomes

$$V(r_{i,t}^p) = \sigma_s^2 + \frac{\sigma_d^2}{m} + \frac{V(e_{j,t-1})}{m} + \frac{m-1}{m}Cov(e_{j,t-1}, e_{k,t-1}) \quad (3.34)$$

being σ_d^2 the variance of the exogenous idiosyncratic (diversifiable) noise and σ_s^2 the variance of the exogenous (undiversifiable) common factor.

This means that the rebalancing feedback increases the volatility of portfolio due to the illiquidity of investment assets through the third term and the fourth one in (3.34). Particularly, it can be shown that a larger leverage increases both variance and covariances of $e_{j,t-1}$, while an higher degree of diversification reduces the variance and increase the covariances. Finally, the correlations among the endogenous returns tend to one as $m \to M$.

In Figure (3.8) the variance of returns divided by the exogenous diversifiable volatility (panel (a)) and the correlation between the endogenous component of returns of two investments (panel (b)) are shown as a function of diversification cost c. The panel (c) of Figure (3.8) shows the variance of portfolio returns divided by the exogenous diversifiable volatility as a function of c while panel (d) shows the correlation between the portfolio returns of two different financial institutions as a function of c. The outcomes refer to different values of the ratio $\frac{\sigma_s}{\sigma_d}$.

By decreasing cost, variance of returns increases as well as correlations. If the market factor is not strong enough, there is a value of c for which variance diverges, corresponding to the case where the maximum eigenvalue Λ_{max} becomes equal to one. In this limit, correlations become closer and closer to one. Since a reduction of diversification cost corresponds to an increase of leverage and diversification, see Figure (3.3), it is evident that larger values of these reinforce the rebalancing feedback which affects the risk of investment assets, $V(r_{j,t})$.

Looking at panel (c), it is important to notice that by reducing diversification cost, the variance of portfolio returns initially declines. This corresponds to a regime of the financial market dynamics in which a diversified portfolio is less risky. However, after that the variance of portfolio reaches a minimum for a given value of c, by reducing further diversification cost, the variance of portfolio suddenly increases because the economic system is closer and closer to

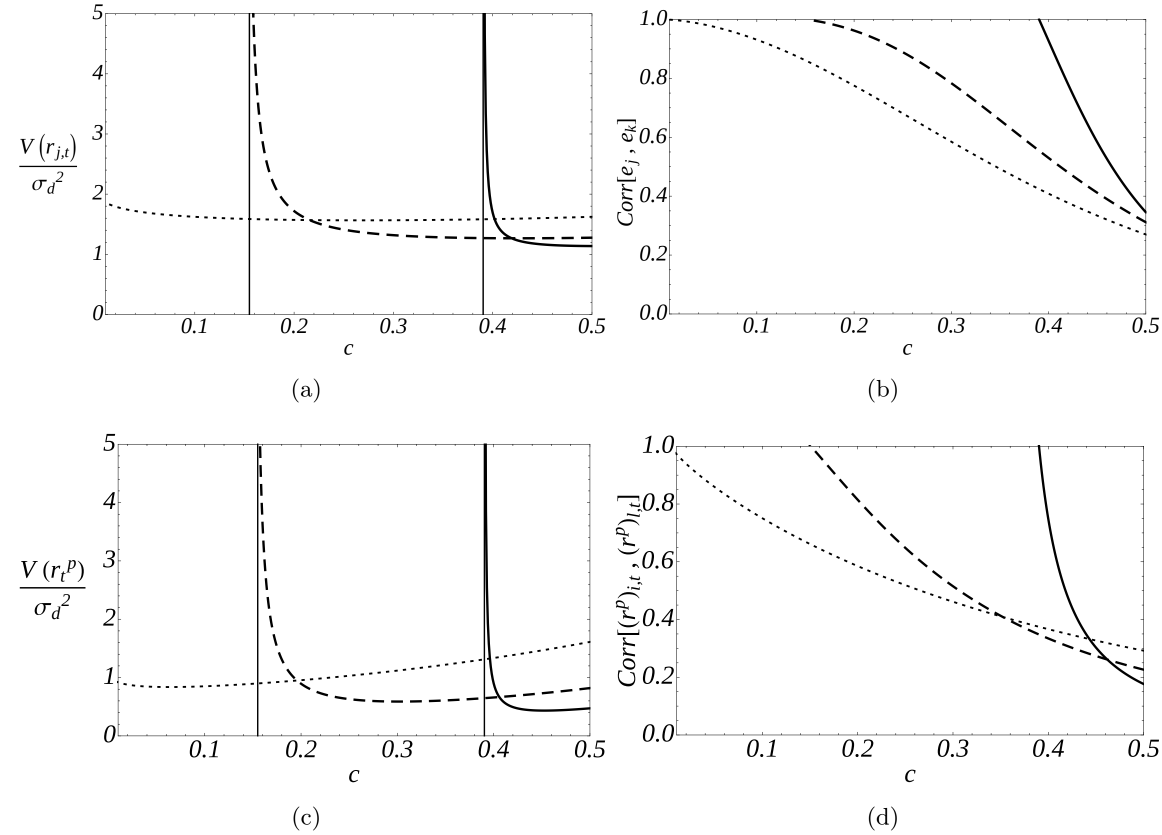


Figure 3.8: The variance of investment returns, $V(r_{j,t})$, divided by the value of the exogenous diversifiable volatility component, σ_d, as a function of the diversification cost c, panel (a). The correlation of the endogenous component of returns between two investments, $Corr\ (e_{j,t}, e_{k,t})$, as a function of c, panel (b). The variance of portfolio return, $V(r_t^p)$, divided by the value of the exogenous diversifiable volatility component, σ_d, as a function of the diversification cost c, panel (c). Finally, the correlation of portfolio returns between two financial institutions, $Corr\ (r^p_{i,t},\ r^p_{l,t})$, as a function of c. The used parameters are: $M = 30$, $\mu - r_L = 0.08$, $\sigma_d = \frac{0.05}{1.64}$ and $\alpha = 1.64$. The solid line refers to $\frac{\sigma_s}{\sigma_d} = 0$, the dashed line refers to $\frac{\sigma_s}{\sigma_d} = 0.3$ and the dotted line refers to $\frac{\sigma_s}{\sigma_d} = 0.6$.

the non-stationary condition $\Lambda_{max} = 1$. In this last regime, even small variations of the cost lead to large increases of the riskiness of the portfolios.

Finally, correlation between portfolios steadily increases by reducing diversification costs essentially because the overlap between portfolios increases, see panel (d) of Figure (3.8). It is important to notice that the condition of divergence of the variance does not imply a perfect overlap between portfolio, see panel (c) of Figure (3.3). For example, with the given parameters the transition to infinite variance and non stationary portfolios occurs at $E[o_f] = 0.21$ when $\frac{\sigma_s}{\sigma_d} = 0$ and at $E[o_f] = 0.34$ when $\frac{\sigma_s}{\sigma_d} = 0.3$. Thus correlation between portfolios can become very close to one even if the portfolio overlap is relatively small.

Below, we will propose a model in which the endogenous risk induced by rebalancing feedback effects is taken into account in solving the portfolio optimization problem by financial institutions. We will investigate how the feedback effects stress the dynamical evolution of the portfolio adjusted to the past measures of risk.

3.3 Systemic Risk

In the model, the systemic risk can be measured as the probability of default of financial institutions. At the same time, the systemic risk can be approached from a dynamical point of view.

3.3.1 The probability of default

From a static point of view and assuming for notational simplicity that the mean value of the portfolio returns is zero, that is $\mu = 0$, the portfolio return at time t of a bank i conditioned on a systematic shock $s_t^{shock} = \bar{e}_{t-1}^{shock} + f_t^{shock}$ (exogenous or endogenous or both of them) is

$$r_{i,t}^p|_{s_t^{shock}} = \sum_{j=1}^{M} I_{j\in i} \frac{s_t^{shock} + \epsilon_{j,t} + \Delta e_{j,t-1}}{m} \tag{3.35}$$

Since $\epsilon_{j,t}$ are normally distributed, the portfolio return distribution conditioned on this systematic shock is

$$r_{i,t}^p|_{s_t^{shock}} \sim N\left(s_t^{shock}, \frac{\tilde{\sigma}_d^2}{m}\right) \tag{3.36}$$

where N indicates the normal distribution and $\tilde{\sigma}_d$ includes the endogenous diversifiable risk contribution deriving from $\Delta e_{j,t}$ in addition to the exogenous one.

A bank i is in default if it suffers a loss larger than the value of its equity. Consequently, the probability of default of a financial institution i given a systematic shock s_t^{shock} is [9]

$$\begin{aligned} PD_i &= P\left(r_{i,t}^p|_{s_t^{shock}} \leq -\alpha\sqrt{\tilde{\sigma}_s^2 + \frac{\tilde{\sigma}_d^2}{m}}\right) \qquad (3.37) \\ &= CDF\left(\frac{-\alpha\sqrt{\tilde{\sigma}_s^2 + \frac{\tilde{\sigma}_d^2}{m}} - s_t^{shock}}{\sqrt{\frac{\tilde{\sigma}_d^2}{m}}}\right) \xrightarrow[m\to M,\ M\to\infty]{} 1 \quad \forall s_t^{shock} < -\alpha\tilde{\sigma}_s \end{aligned}$$

where CDF is the standard normal cumulative distribution function and $\tilde{\sigma}_s$ contains the endogenous systematic risk contribution deriving from $\bar{e}_t$ in addition to the exogenous one.

Therefore, for any negative shock larger than VaR, *i.e.* $\alpha\tilde{\sigma}_s$, the probability of default increases with the degree of diversification m. Further, as claimed before, a large overlap between portfolios of the financial institutions increases the systemic risk of whole economic system because it will play a crucial role in the contagion of risk. In fact, from a dynamical point of view, the endogenous feedback effects will trigger a sequence of portfolio rebalances causing the falling of price of all risky assets, when the mean portfolio overlap is large.

Since $\mathbf{e}_t = \mathbf{\Phi r}_t$, see equation (3.22), the vector of investments returns (3.21) at time t, after a shock $f_t = s_t^{shock}$ occurring at time t, can be rewritten as

$$\mathbf{r}_t = \mathbf{\Phi r}_{t-1} + \imath s_t^{shock} + \boldsymbol{\epsilon}_t \tag{3.38}$$

The total future impact of the shocks over the next h periods will be given by the sum of h contribution, each of which is equal to the mean investment return conditioned on the systematic shock s_t^{shock} that is approximately [9] (for h sufficiently large)

$$E[\mathbf{r}_{t+h}|s_t^{shock}] = (\mathbf{\Phi})^h \, \imath s_t^{shock} \quad \longrightarrow \quad E\left[\left(\sum_h \mathbf{r}_{t+h}\right)|s_t^{shock}\right] \approx (\mathbb{1} - \mathbf{\Phi})^{-1} \, \imath \, s_t^{shock} \tag{3.39}$$

Hence, the larger the maximum eigenvalue of $\mathbf{\Phi}$ the larger will be the magnitude and persistence of future adjustments leading to a larger cumulative impact that the financial system will have to absorb. The relation (3.39) holds approximately when the return dynamics is stationary, that is when the largest eigenvalue of $\mathbf{\Phi}$ is smaller than 1. On the contrary, any perturbation s_t^{shock} increases through time and the economic system can not be able to absorb it. The largest eigenvalue strictly depends on the leverage λ. When financial institutions are over-leveraged, any shock deeply propagates within the economic system.

3.3.2 Transition from stationarity to non-stationarity

In the last years, new financial instruments were introduced in financial markets apparently allowing financial institutions to diversify the risk of the portfolio. In this sense, they permit to reduce the cost of diversification c. *Marsili et al.* [92, 93] have investigated how the proliferation of financial instruments can erode the stability of an economic system. In fact it can drive the economic system towards a state characterized by strong fluctuations and strong correlations among risks.

In the model [9] we have presented, the introduction of financial innovation has several crucial consequences.

1. When a financial innovation reduces the cost c, financial institutions increases the optimal degree of diversification m. At first time, the portfolio volatility is reduced and financial institutions can fulfill the VaR constraint using a larger value of leverage λ. As seen in the previous Section, the probability of default of a financial institution increases according to larger values of diversification and leverage.

2. As a consequence of larger diversification m, the overlap between different portfolios increases and the economic system is more interconnected. The correlations among portfolio returns increase. Any shock can propagate easily within the system.

3. An increase in leverage will heavily affect the dynamics of the risky investments by increasing both their variances, covariances and correlations, through the impact of rebalancing feedback effects.

As a consequence of these effects, financial institutions operate in market in aggressive (due to higher leverage) and coordinated (due to correlations among portfolio returns) way. Through the rebalancing feedback, they can drive endogenously the asset return dynamics pushing up the assets price. The same feedback mechanism works also in the opposite direction during market crisis.

Below a given threshold of the diversification cost c, a transition from a stationary dynamics to a non-stationary one for the asset returns occurs. In this regime, the feedback mechanism reinforces the fluctuation amplitudes of asset prices, leading to steep growth (bubble) and plunge (burst) of market prices.

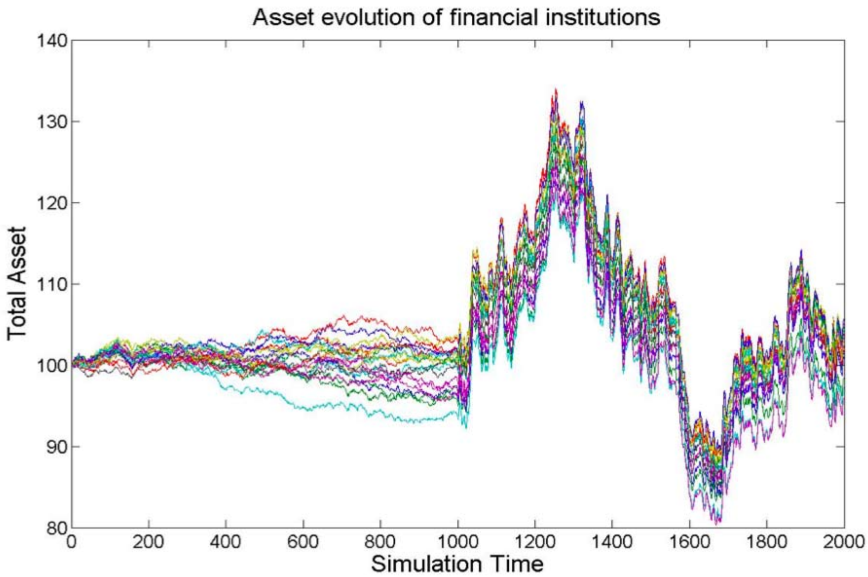


Figure 3.9: Numerical simulation of the dynamics of individual total asset of financial institutions before and after a structural break (at simulation time 1000) on the diversification costs that induces an increase of leverage and diversification. Leverage goes from 10 to 60 and fractional overlap from 0.1 to 0.8.

Figure (3.9) shows the impact on the dynamics of financial intermediaries total asset of a shift in the degree of leverage and diversification induced by a reduction in the diversification costs. The model is numerically simulated to show how the bank assets dynamics changes with a structural break represented by a sudden increase (at simulation time 1000) in the degree of leverage and diversification. The bank assets dynamics reflects the portfolio return dynamics. In fact, the total asset return for a bank is simply the return of the portfolio times the chosen value of leverage λ. After the reduction of diversification costs at time 1000, rebalancing feedback effects trigger bubbles and bursts of assets price.

From a mathematical point of view, the transition from stationarity to non-stationarity for the return dynamics occurs when the largest eigenvalue of the linear operator $\mathbf{\Phi}$, describing the dynamics of endogenous components, becomes equal to 1.

$$\Lambda_{max} = 1$$

Assuming that all assets have the same liquidity γ, the previous condition is equivalent to

$$\Lambda_{max} = \frac{\lambda - 1}{\gamma} = 1 \quad \longrightarrow \quad \lambda = \gamma + 1$$

As a consequence, in the model, the stationarity of the return dynamics means an upper bound for the leverage λ defined by the liquidity γ.

Chapter 4

The role of expectation feedbacks in systemic financial stability

In Chapter 3 we have introduced an analytical model to investigate the effects of micro-prudential changes on macro-prudential outcomes. Specifically, the model investigates the consequence of reducing the costs of portfolio diversification in a financial system populated by homogeneous financial institutions having capital requirement in the form of VaR constraint and following standard mark-to-market and risk management rules. In particular it highlights how the strategy of maintaining a fixed *target leverage ratio*, through the portfolio rebalancing in presence of asset illiquidity, induces a positive feedback which increases the assets volatility by its impact on returns. This latter feature is modeled by the introduction of an endogenous component in the return process which is proportional to the assets demand and it is inversely related to the asset liquidity parameter γ. The feedback effects strongly affect the dynamics of traded assets.

In Chapter 3, the problem of the portfolio optimization is solved once that the exogenous diversifiable volatility, σ_d, and the exogenous systematic volatility, σ_s, have been determined. However, in the original formalization of the model, financial institutions optimize their portfolio using the exogenous risk parameters without considering the endogenous component or learning from past time series.

Here we present a dynamical systems approach to the problem of portfolio optimization when endogenous feedback effects are accounted for. Specifically, we will focus on how financial institutions respond to an increase of risk due to the portfolios rebalancing. In this scenario, the expectations of financial institutions about the future realizations of the asset risk play a crucial role in defining the dynamical outcomes. In particular, the adopted expectations scheme may condition the financial stability of the economic system. We assume that all financial institutions use the same expectations scheme. We consider two expectations scheme about the forecasting of risk: naive expectations and adaptive expectations. The latter ones are described in the next Chapter.

The key point is that the combined effect of portfolio choices, made by financial institutions, drives endogenously the dynamics of financial market. Under naive expectations, at a given threshold the endogenous feedback triggers the appearing of financial cycles characterized by a sequence of speculative periods and non-speculative ones. During financial cycles, we can notice how the financial leverage of a bank portfolio switches from aggressive configurations (speculative periods) to cautious ones (non-speculative periods). The financial leverage cycles reflect the occurrence of periods characterized by a macro-component of risk, due to an higher impact of the feedback, followed by periods in which feedback effects do not affect importantly the asset risk.

When financial cycles appear, the risk can not be reduced through diversification. In fact, although a diversified portfolio reduces the exogenous component of risk, a larger degree of portfolios overlap increases the endogenous impact on volatility. During financial cycles there is not an equilibrium between the latter effects.

All the material in this Chapter is original.

4.1 The dynamical model with expectations

At the end of Chapter 3, we have found an analytical expression for the variance of portfolio return when the impact of the positive feedback is considered (see (3.34) in Section 3.2).

$$V(r^p) = V(e^p) + \sigma_s^2 + \frac{\sigma_d^2}{m} \tag{4.1}$$

where m is the number of assets in the portfolio, r^p is the portfolio return, $V(e^p)$ is the endogenous risk due to the feedback effects. Since for hypothesis the portfolio return r^p is a normally distributed stochastic variable, the variance of the portfolio, $V(r^p)$, can be considered as a risk measure. In order to explain our outcomes from an economic point of view, we adopt the convention of referring to the variance of the portfolio as the risk while, we refer to the square root of the variance as the volatility.

Once that financial institutions optimize their portfolio choosing the optimal configuration of m (diversification) and λ (leverage), the price impact of rebalancing strategy, in order to maintain the target leverage λ, affects the risk of the portfolio through $V(e^p)$.

The analytical expression of $V(e^p)$ is (see (3.34) in Section 3.2)

$$V(e^p) = \frac{V(e_j)}{m} + \frac{m-1}{m} Cov(e_j, e_k) \tag{4.2}$$

where e_j is the endogenous component describing the effect of rebalancing feedback on the return of an asset j, $V(e_j)$ is the variance of the endogenous component e_j and $Cov(e_j, e_k)$ is the covariance of e_j and e_k. Both the variance $V(e_j)$ and the covariance $Cov(e_j, e_k)$ are functions of the diversification m and the leverage λ. Defining the excess leverage as $\tilde{\lambda} = (\lambda - 1)$, the variance and the covariance of endogenous components, see Section 3.2, are

$$\begin{aligned} V(e_j) \quad = \quad & -\tilde{\lambda}^2 \{ \frac{m^2(\sigma_d^2(\tilde{\lambda}^2 - \gamma^2(M-1)) + \sigma_s^2(\tilde{\lambda}^2 - \gamma^2(M-1)^2))}{(\gamma^2 - \tilde{\lambda}^2)(m^2(\gamma^2(M-1)^2 - \tilde{\lambda}^2) + 2\tilde{\lambda}^2 mM - \tilde{\lambda}^2 M^2)} + \\ & + \frac{2m(M(\sigma_d^2(\gamma^2 - \tilde{\lambda}^2) - \tilde{\lambda}^2\sigma_s^2) - \gamma^2\sigma_d^2)}{(\gamma^2 - \tilde{\lambda}^2)(m^2(\gamma^2(M-1)^2 - \tilde{\lambda}^2) + 2\tilde{\lambda}^2 mM - \tilde{\lambda}^2 M^2)} + \end{aligned}$$

$$+\frac{M(M(\sigma_d^2(\tilde{\lambda}^2-\gamma^2)+\tilde{\lambda}^2\sigma_s^2)+\gamma^2\sigma_d^2)}{(\gamma^2-\tilde{\lambda}^2)(m^2(\gamma^2(M-1)^2-\tilde{\lambda}^2)+2\tilde{\lambda}^2 mM-\tilde{\lambda}^2M^2)}\} \tag{4.3}$$

$$\begin{aligned} Cov(e_j,e_k) \quad = \quad & -\tilde{\lambda}^2\{\frac{m^2(\sigma_s^2(\tilde{\lambda}^2-\gamma^2(M-1)^2)-\gamma^2(M-2)\sigma_d^2)}{(\gamma^2-\tilde{\lambda}^2)(m^2(\gamma^2(M-1)^2-\tilde{\lambda}^2)+2\tilde{\lambda}^2 mM-\tilde{\lambda}^2M^2)}+ \\ & +\frac{-2m(\tilde{\lambda}^2M\sigma_s^2+\gamma^2\sigma_d^2)}{(\gamma^2-\tilde{\lambda}^2)(m^2(\gamma^2(M-1)^2-\tilde{\lambda}^2)+2\tilde{\lambda}^2 mM-\tilde{\lambda}^2M^2)}+ \\ & +\frac{M(\tilde{\lambda}^2M\sigma_s^2+\gamma^2\sigma_d^2)}{(\gamma^2-\tilde{\lambda}^2)(m^2(\gamma^2(M-1)^2-\tilde{\lambda}^2)+2\tilde{\lambda}^2 mM-\tilde{\lambda}^2M^2)}\} \end{aligned} \tag{4.4}$$

When in a time interval Δt all financial institutions are marked-to-market rebalancing continuously their portfolios, they impact the risk of the portfolio through an endogenous feedback effect that induces the endogenous risk component (4.2). As a consequence, constrained by VaR, at the end of time interval Δt, financial institutions have to solve their portfolio optimization problem according to the observed value of the risk.

From now on, we assume that the exogenous systematic risk component σ_s is equal to zero. As we will see, this assumption will lead to an important simplification.

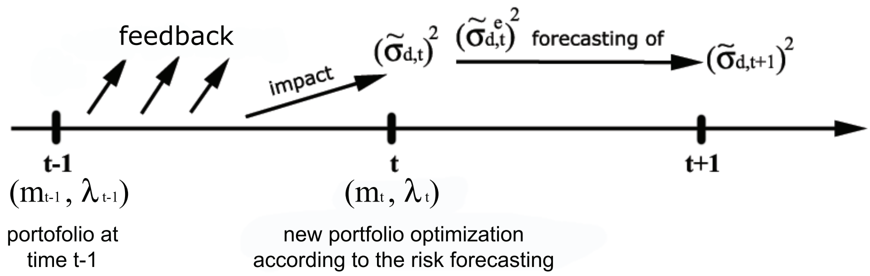


Figure 4.1: A symbolic diagram describing the consecutive steps that characterize the evolution of the portfolio. At time $t-1$ financial institutions choose the diversification m_{t-1} and the leverage λ_{t-1}. In the time interval $[t-1,t]$, financial institutions rebalance their portfolio in order to maintain fixed the value of leverage. This mechanism induces a positive feedback which affects the risk, $\tilde{\sigma}^2_{d,t}$, at time t. After a measure of the risk at time t, financial institutions form an expectation, $(\tilde{\sigma}^e_{d,t})^2$, about the future risk, and according to this forecasting, they optimize their portfolio at time t choosing new values of diversification, m_t, and leverage, λ_t.

In Figure (4.1) we show the consecutive steps occurring during the evolution of the portfolio, in order to stress the rebalancing feedback effect.

We suppose that at time $t-1$ banks choose the optimal values of the diversification, m_{t-1}, and the financial leverage, λ_{t-1}. When banks maintain the optimal target leverage λ_{t-1} rebalancing their portfolio in the time period between $t-1$ and t, the portfolio risk realized at time t is

$$(\tilde{\sigma}_{p,t})^2=\frac{\sigma_d^2}{m_{t-1}}+\frac{V(e_{j,t-1})}{m_{t-1}}+\frac{m_{t-1}-1}{m_{t-1}}Cov(e_{j,t-1},e_{k,t-1}) \tag{4.5}$$

where $\frac{\sigma_d^2}{m_{t-1}}$ is the exogenous component of the risk. In fact, at the end of the time interval $[t-1,\, t]$, before a new portfolio optimization occurs, the actual degree of diversification is m_{t-1}.

The quantity $\frac{V(e_{j,t-1})}{m_{t-1}}+\frac{m_{t-1}-1}{m_{t-1}}Cov(e_{j,t-1},e_{k,t-1})$ is the endogenous component of the risk representing the impact of the rebalancing feedback. In fact, between $t-1$ and t the target leverage for financial institutions is λ_{t-1}, and as a consequence the endogenous risk component strictly depends on m_{t-1} and λ_{t-1}. There is both an explicit dependence on m_{t-1} and an implicit dependence on m_{t-1} and λ_{t-1} (see (4.3) (4.4)).

The risk of the portfolio is the *observable* of our dynamical model. We use the convention to mark with a tilde the risk that is the sum of an exogenous component and an endogenous one.

For the hypothesis of assets homogeneity, the risk of the portfolio is simply equal to the risk associated with an asset divided by the actual degree of diversification. Hence the risk of the portfolio, realized at time t, is

$$(\tilde{\sigma}_{p,t})^2 = \frac{(\tilde{\sigma}_{d,t})^2}{m_{t-1}} \tag{4.6}$$

where $(\tilde{\sigma}_{d,t})^2$ is the diversifiable risk of an asset as a consequence of the impact of rebalancing feedback.

Then, from (4.5) and (4.6) we find

$$(\tilde{\sigma}_{d,t})^2 = \sigma_d^2 + V(e_{j,t-1}) + (m_{t-1}-1)Cov(e_{j,t-1}, e_{k,t-1}) \tag{4.7}$$

At time t the realized risk associated with any asset j is the contribution of an exogenous divesifiable component, σ_d^2 , plus an endogenous component equal to the sum of the variance, $V(e_{j,t-1})$, of the endogenous component e_j of the asset j and the covariance, $Cov(e_{j,t-1}, e_{k,t-1})$, of e_j and e_k, $k \neq j$. Both endogenous components depend on the choice of portfolio variables made at time $t-1$.

Thanks to the hypothesis of assets homogeneity, the asset diversifiable risk, $(\tilde{\sigma}_{d,t})^2$, can be obtained by only one measure of portfolio risk, $(\tilde{\sigma}_{p,t})^2$, at time t.

Under bounded rationality hypothesis, financial agents know the realization of their portfolio volatility at every past times but they do not know the exact “law of motion” of the financial market. Instead, they form expectations about the future risk using statistical models of past time series. In order to solve the problem of the portfolio optimization at time t, they measure the actual risk position, $(\tilde{\sigma}_{p,t})^2$, and they deduce the actual asset diversifiable risk, $(\tilde{\sigma}_{d,t})^2$, from Eq. (4.6). Depending on their degree of rationality, financial agents form at time t an expectation, $(\tilde{\sigma}^e_{d,t})^2$, about the future $(t+1)$ diversifiable asset risk, $(\tilde{\sigma}_{d,t+1})^2$. Hence, at time t a financial institution, constrained by VaR, solves the portfolio optimization problem according to its forecast about the expected risk position.

Summarizing schematically, the consecutive steps, that define the portfolio dynamics between $t-1$ and t, are:

- at time $t-1$, financial institutions choose m_{t-1} and λ_{t-1}
- between $t-1$ and t, financial institutions adopt a rebalancing strategy in order to maintain the target leverage λ_{t-1}
- at time t, financial institutions estimate their actual risk position $(\tilde{\sigma}_{p,t})^2$

- at time t, financial institutions deduce $(\tilde{\sigma}_{d,t})^2$ from Eq. (4.6) and they form the expectation $(\tilde{\sigma}^e_{d,t})^2$ about the future $(t+1)$ risk
- at time t, according to their expectation $(\tilde{\sigma}^e_{d,t})^2$, financial institutions solve the problem of portfolio optimization choosing m_t and λ_t

The difference equations (3.12) (3.13) fulfill the portfolio optimization problem at time t, once the expectation about the future diversifiable risk, $(\tilde{\sigma}^e_{d,t})^2$, is formed.

$$m_t = \lambda_t\, \tilde{\sigma}^e_{d,t} \sqrt{\frac{\alpha}{2c}\, \frac{\mu - r_L}{\sqrt{\frac{(\tilde{\sigma}^e_{d,t})^2}{m_t}}}} \tag{4.8}$$

$$\lambda_t = \frac{1}{\alpha \sqrt{\frac{(\tilde{\sigma}^e_{d,t})^2}{m_t}}} \tag{4.9}$$

$$\tilde{\sigma}^e_{d,t} \equiv \tilde{\sigma}^e_d\,(m_{t-1},\, \lambda_{t-1}) \tag{4.10}$$

where $\mu - r_L$ is the Net Interest Margin (NIM) and $c = \frac{\tilde{c}}{E}$ is the cost for investment (assumed to be the same across all investments) in unit of equity E. The relation (4.10) expresses in general form the expected diversifiable volatility. m_t and λ_t represent the *new* choice at time t made by financial institutions about the portfolio diversification and the financial leverage.

Given the expectation at time t about future diversifiable risk, $(\tilde{\sigma}^e_{d,t})^2$, financial institutions expect the portfolio volatility, that it will be realized at time $t+1$, is

$$\tilde{\sigma}^e_{p,t} = \sqrt{\frac{(\tilde{\sigma}^e_{d,t})^2}{m_t}} \tag{4.11}$$

The difference equations (4.8) (4.9) (4.10) define a discrete dynamical system as a *map*. The dynamics of the map describes how the portfolio of a typical financial institution evolves in time in an ideal market modeled by a bipartite network of N financial institutions and M illiquid risky investments. The dynamical aspect of our approach is related to the fact that the system state at time t depends on the past state of the system at time $t-1$, through (4.10).

The domain in which the portfolio variables can evolve is:

$$\{m,\, \lambda,\, \tilde{\sigma}^e_d\} \in [1, M] \otimes [1, \gamma + 1) \otimes \mathbb{R}^+ \tag{4.12}$$

By definition, $m \in [1, M]$ because M is the total number of investment assets and $\tilde{\sigma}^e_d > 0$. Most importantly, λ is confined in the interval $[1, \gamma + 1)$. In fact $\lambda < \gamma + 1$ expresses in another form the condition $\Lambda_{max} < 1$, see Section 3.3 of Chapter 3. When $\lambda < \gamma + 1$, the dynamics of the return processes for the investment assets is stationary. $\lambda = \gamma + 1$ represents the transition point from stationarity to non-stationarity for the returns dynamics. When $\lambda \to (\gamma+1)^-$, the variance (4.3) and the covariance (4.4) of the endogenous components in the return processes become larger and larger, since $\gamma + 1$ is an asymptote for these. If $\lambda \notin [1, \gamma + 1]$, $V(e_j)$ (see (4.3)) is a negative-valued function, and for this reason in non-stationary regimes it fails to describe the excess volatility due to the portfolio rebalancing. As a consequence, the map based on difference equations (4.8) (4.9) (4.10) fails in describing the portfolio dynamics in the non-stationary regime for the return processes.

From now on, we refer to (4.12) as the *economic domain* of the map. Only in this domain, the dynamical behavior of the map has an economic interpretation.

Since $\sigma_s = 0$, the map based on difference equations (4.8) (4.9) (4.10) is equivalent to

$$m_t = \left(\frac{\mu - r_L}{2\alpha c}\right)^2 \frac{1}{(\tilde{\sigma}_d^e(m_{t-1}, \lambda_{t-1}))^2} \tag{4.13}$$

$$\lambda_t = \left(\frac{\mu - r_L}{2\alpha^2 c}\right) \frac{1}{(\tilde{\sigma}_d^e(m_{t-1}, \lambda_{t-1}))^2} \tag{4.14}$$

In explicit form, it is evident that the degree of diversification and the leverage are not independent variables. At any time, the following linear relation exists between them

$$m_t = \frac{\mu - r_L}{2c}\lambda_t \tag{4.15}$$

From a mathematical point of view, the map is a one-dimensional dynamical system. Nevertheless, we will consider separately the two variables, in order to make evident the economic interpretation of the outcomes.

In the map there are two fundamental parameters, α and c. The parameter α summarizes all the capital requirements imposed by regulatory institutions and standard market practice. The parameter c represents the costs associated with the diversification of a portfolio. The diversification costs have decreased in the last thirty years, as a consequence of the introduction of financial innovations (see Figure 3.2). A reduction of diversification costs is followed by a larger overlapping between different portfolios due to the fact that financial institutions prefer a diversified portfolio in order to reduce the risk. In this scenario, the endogenous feedback plays a crucial role in defining the portfolio dynamics.

We will investigate particularly the consequences of a reduction of diversification costs.

Furthermore, depending on the expectation strategies for $\tilde{\sigma}_d^e$, there could be another parameter which characterizes the choice of a specific expectation scheme. It is the case of the adaptive expectations.

Finally, it is useful to note that the map described by the equations (4.8) (4.9) (4.10) is the (*deterministic*) *skeleton* [47] of the corresponded *stochastic map* in which an exogenous noise added to $V(e_{t-1}^p)$ is present. This stochastic noise would be necessary to include the stochastic fluctuations and the exogenous shocks of the assets volatility. A well-know property is that the volatility itself must be described by processes containing a stochastic term. The randomness characterize the financial market dynamics.

4.2 Expectations in financial markets

One of the most important difference between economics and natural sciences is that the dynamics of an economic system is characterized by a behavioral component which reflects the capacity of financial agents operating in the system of making decisions which influence dynamics itself. Because the procedure with which financial agents make decisions can change, as a consequence the behavioral component of an economic system can not be considered as a fixed rule of the dynamics.

The key point is to describe this behavioral component characterizing the dynamics of an economic system into a mathematical framework. For this purpose it is important to notice that decisions of economic agents today are based upon their *expectations* about the future. A mathematical formalization of the concept of expectations plays a crucial role in describing the dynamics of an economic system as a financial market.

The need for an empirically grounded behavioral theory of expectations for economic dynamics has been stressed for the first time by *Herbert Simon* [94] in 1984: "A very natural step for economics is to maintain expectations in the strategic position they have come to occupy, but to build an empirically validated theory of how attention is in fact directed within a social system, and how expectations are, in fact, formed. Taking the next step requires that empirical work in economics take a new direction, the direction of micro-level investigation proposed by Behavioralism."

The main aspect of the expectations is the fact that they influence the dynamics of an economic system, differently from what happens for a physical system [47]. To illustrate this difference, weather forecasts for tomorrow will not affect today's weather, but investors' predictions about future prices or risks may affect financial market dynamics today. The expectations of financial agents about the future state of the economy are part of the "law of motion".

In many recent works, *Hommes et al.* have experimentally studied [10, 95, 11] how groups of financial agents form their expectations and have expressed mathematically [96, 97, 98, 99] some aspects of the expectations themselves.

In describing a model of an economic system, typically two hypotheses about expectations can be considered: the *rational expectations hypothesis* (REH) and the *bounded rationality* one [47]. According to the first one, rational agents have perfect knowledge about underlying market equations, as a consequence they do not make systematic mistakes and their expectations coincide with the future realizations. A scenario of this type is characterized by a Rational Expectation Equilibrium (REE) for the economic system.

In principle, this assumption may seem very strong, especially since the "law of motion" of an economy depends on the expectations of *all* other agents. In fact, Hommes et al. have highlighted through evidences from laboratory experiments with human subjects that a situation in which financial agents know exactly the dynamics of an economic system is unrealistic. On the contrary, from laboratory experiments strong evidences are in agreement with the bounded rationality hypothesis.

According to this one, financial agents do not know the actual "law of motion" of the economy, but instead they form their expectations using statistical models of observed past time series, the so-called *backward-looking expectations* [100]. Looking at the past realizations of risk, for example, and according to their degree of rationality, financial agents forecast the future realization of the risk.

In principle, the bounded rationality hypothesis does not exclude the possibility of an equilibrium point for the dynamics of an economic system [101]. In particular, financial agents can adapt their forecasting strategy over time by updating the parameters of their statistical models according to some learning scheme as additional observations become available (*adaptive learning*) [102, 103], in order to avoid systematic mistakes. When the expectations of financial agents become in equilibrium with the future realizations, typically the economic system reaches an equilibrium point.

4.2.1 Naive expectations and adaptive expectations

In a backward-looking expectations scenario, two types of expectations can be particularly considered in studying financial markets dynamics: *naive expectations* and *adaptive expectations*. The first ones refer to a situation in which the forecast of the financial agents is equal to the last observation. Naive expectations describe in some sense irrational financial agents because they may lead to systematic mistakes of forecasting.

Instead, adaptive expectations correspond to a more realistic scenario in which financial agents forecasting is a weighted sum through a memory factor of the most recently observation and the previous expectation, that is the actual expectation is *adapted* in the direction of the last observation. In the adaptive scenario, the memory plays an important role in defining the dynamical outcomes.

In this Chapter and in the next one, we assume that financial institutions are boundedly rational and they use an expectations scheme to forecast risk. In this Chapter we analyze the dynamical features of the model with naive expectations.

In the next Chapter, we will focus on the dynamical model with adaptive expectations. At the end of the next Chapter, we describe an adaptive learning scheme that financial institutions may use to improve their forecasting strategy.

4.3 Naive expectations

To close the model described by equations (4.8) (4.9) (4.10), it is necessary to specify how financial institutions form their expectation about the asset volatilities and as a consequence about the portfolio volatility. Now we assume that financial institutions solve the portfolio optimization problem assuming that the last period volatility of the risky investments will coincide to the future one. They follow the so-called *naive expectations* scheme.

4.3.1 Dynamical portfolio optimization under naive expectations

We suppose that financial institutions at time t estimate the risk associated with an asset and they find a value equal to $(\tilde{\sigma}_{d,t})^2$. $(\tilde{\sigma}_{d,t})^2$ is the risk measured at time t but it depends on the choice about m_{t-1} and λ_{t-1} that financial institutions have made at time $t-1$ because of the rebalancing feedback. The feedback effect induces the endogenous risk component. Naive expectation strategy means that financial institutions, after an estimation of asset risk at time

t, expect that the future realization of risk of the same asset will be coincide to the actual one. We call this expected asset (diversifiable) risk as $(\tilde{\sigma}^e_{d,t})^2$. The subscript t stresses the fact that the risk expectation for the future time is matured at time t. Instead, the exponent e indicates that the quantity $(\tilde{\sigma}^e_{d,t})^2$ is the expectation of the future risk.

Financial institutions believing in naive expectations about their risk position, use the following relation to estimate the diversifiable risk of an asset

$$(\tilde{\sigma}^e_{d,t})^2 = (\tilde{\sigma}_{d,t})^2 \tag{4.16}$$

where $(\tilde{\sigma}_{d,t})^2$ is found through the relation (4.7).

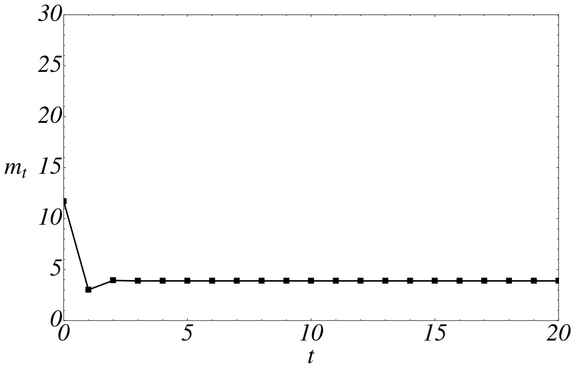
Hence, financial institutions use the relation (4.16) in order to estimate their portfolio volatility. The estimated risk position of the portfolio allows financial institutions to calculate their VaR constraint and as a consequence they can make the optimal choice of diversification m_t and leverage λ_t at time t. From now on we will consider the diversification m as a *continuous* variable. In principle m is a discrete variable because it indicates the number of investment assets in the portfolio of a financial institution. Hence, a choice of a fractional value for m has no sense. But at the same time, the qualitative features of the portfolio dynamics does not change when we use continuous values of m. For the sake of simplicity we make this assumption to test numerically our model. This choice does not affect the typical dynamical behavior of the map.

Figure (4.2) shows the dynamical behavior of the system (4.8) (4.9) (4.10) when financial institutions adopt an expectation scheme about diversifiable risk described by (4.16), for two different values of the diversification costs. The initial point is chosen randomly among all possible configurations of the system in the economic domain.

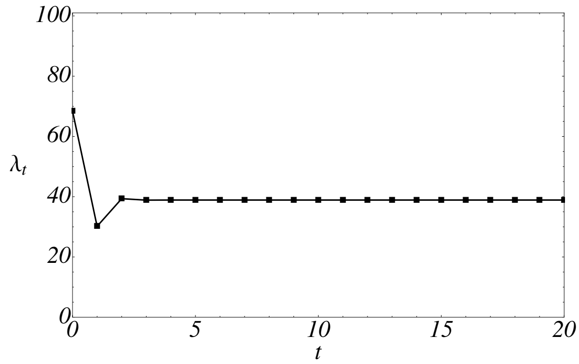
When the costs of diversification are high, the impact of supply and demand generated by financial institutions in rebalancing their portfolio does not affect critically the system stability. The dynamics of the portfolio converges to a stable configuration (m^*, λ^*) which optimizes the return under the VaR constraint without affecting criticality the future risk by the price impact. In fact, when the costs are high a bank solves its optimization problem choosing a small value of the optimal diversification, m^*. It follows that the overlap between portfolios is incomplete and as a consequence the demands of an asset, which generate the price impact on the asset itself, add together approximately to zero. There is not a significant endogenous risk component.

On the contrary when the diversification costs become lower than a threshold c_2, 2-period orbits appear for m_t and λ_t. When c is smaller than c_2, the mean overlap between the portfolios spreads and the consequence is that during a period of high target leverage the coordinated portfolios rebalancing produces a remarkable impact on the diversifiable volatility. Conditioned by the VaR constraint, financial institutions will reduce their diversification and their financial leverage. A period of small price impact will follow. Hence, all portfolios oscillate between two configurations: an aggressive configuration characterized by high leverage and large diversification and a cautious configuration characterized by small values of leverage and diversification.

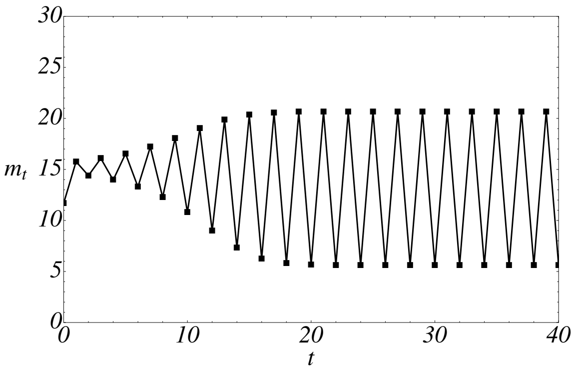
Figure (4.3) shows the *orbit diagrams*. It plots the system's attractor as function of c, that is the asymptotic dynamics when the initial transient period is passed. In the Figure (4.3) (a)



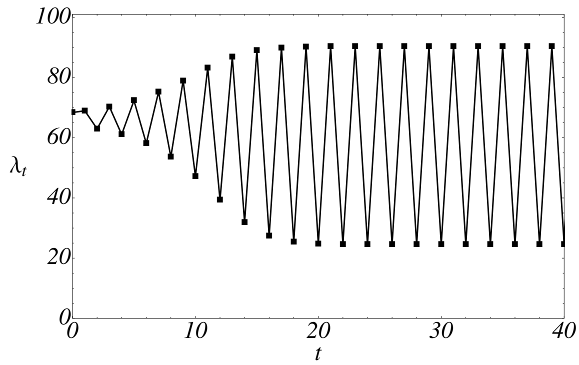
(a) $c = 0.4$



(b) $c = 0.4$



(c) $c = 0.0175$



(d) $c = 0.0175$

Figure 4.2: Dynamic evolution of the diversification m_t and the financial leverage λ_t changing the value of diversification cost c. In the panels (a) and (b) we set a higher value compared to the panels (c) and (d). The other parameters are fixed at $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, the exogenous volatility $\sigma_d = \frac{0.05}{1.64}$, the liquidity $\gamma = 100$.

(b) we show 2-period bifurcations for m_t and λ_t. The amplitude of the 2-period orbits becomes larger decreasing the parameter c. In Figure (4.3) (c) we report the ratio $\frac{\Delta(\tilde{\sigma}_d)^2}{(\tilde{\sigma}_d)^2} = \frac{(\tilde{\sigma}_d)^2 - (\sigma_d)^2}{(\tilde{\sigma}_d)^2}$ that is the fraction of squared volatility due to the endogenous feedback. When c is higher than the threshold c_2, this measure of the endogenous risk is of the order of 20% of the total risk comes from the endogenous feedback. When the value of c decreases below the threshold it begins to swing between high and small values following the swinging choices of the financial institutions. In fact for small diversification costs, at the 2-period dynamics for m_t corresponds a Pearson correlation coefficient for two endogenous components, (see Figure (4.3) (d)) which shows high linear correlation followed by smaller one (uncorrelation).

Under the hypothesis of assets homogeneity, the Pearson correlation coefficient realized at time t is simply the ratio between the covariance (4.4) of two endogenous components, $e_{j,t-1}$ and $e_{k,t-1}$, and the variance (4.3):

$$\rho_t = \frac{Cov(e_{j,t-1}, e_{k,t-1})}{V(e_{j,t-1})} \tag{4.17}$$

In panel (d) of Figure (4.3), ρ_t depends on the values of diversification and leverage at time $t-1$, m_{t-1} and λ_{t-1}. When ρ_t is close to 1, endogenous components are highly correlated due to

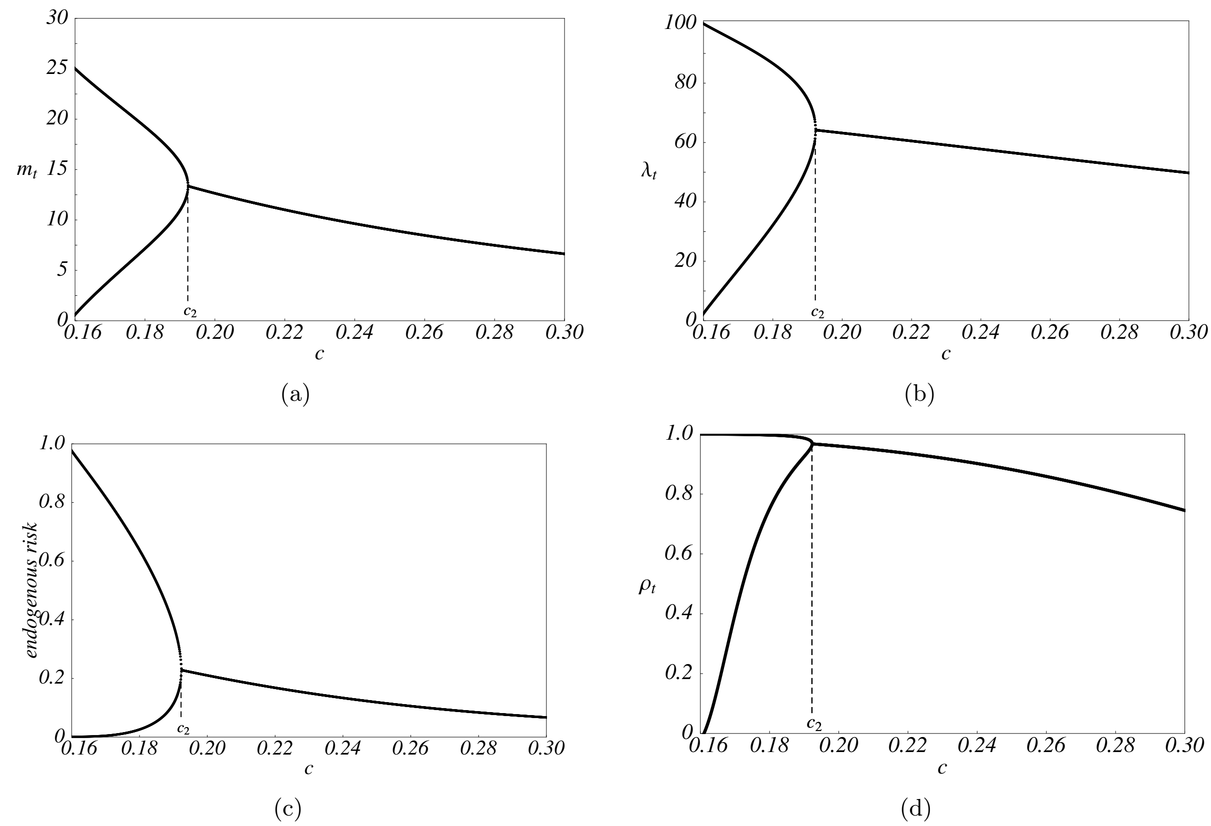


Figure 4.3: Orbit plots as function of the continuous parameter c. The panels (a) and (b) show the Orbit Plots for m_t and λ_t. The panel (c) shows the normalized endogenous risk $\frac{\Delta(\tilde{\sigma}_d)^2}{(\tilde{\sigma}_d)^2} = \frac{(\tilde{\sigma}_d)^2-(\sigma_d)^2}{(\tilde{\sigma}_d)^2}$, *i.e.* the endogenous risk component divided by the total risk for an investment j. The panel (d) shows the Pearson Correlation Coefficient between two endogenous components, e_j and e_k. The used parameters are: $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, $\sigma_d = \frac{0.05}{1.64}$, $\gamma = 100$.

a large overlap of portfolios and the impact on risk due to the rebalancing feedback represents a significative fraction of total risk. On the contrary, when ρ_t is smaller than 1, the impact of feedback effects is not important. This fact is the underlying reason for the Figure (4.3) (c).

It is evident in Figure (4.3) that below the threshold c_2, financial cycles appear. These are characterized by cyclical orbits for the portfolio variables of time period equal to 2. Financial cycles of the degree of diversification m and the leverage λ are due to the feedback effect induced by the target leverage strategy adopted by financial institutions. Cycles of m and λ reflect the cyclical realizations of the endogenous risk component, panel (c) of Figure (4.3).

There is also a second threshold c^* (in Figure (4.3) $c^* \approx 0.1589$) at which the amplitude of oscillations of leverage and diversification become comparable with the dimension of the economic domain (in particular the oscillation amplitude of leverage is exactly equal to the length of interval of possible values). This threshold corresponds to the transition from stationarity to non-stationarity occurring for the return dynamics, see Section 3.3. The reason is simply related to the fact that financial leverage reaches the critical value $\gamma + 1$ for which the transition occurs.

In principle, in our dynamical system the endogenous risk component is not an observable quantity. The observable quantity is the portfolio volatility which includes an exogenous component. Hence, a financial institution can not estimate directly the impact on risk of the rebalancing strategy of all financial institutions. On the contrary, a more appropriate quantity which can be estimated in order to have informations about the cyclical behavior of endogenous risk component, is the cycle amplitude of financial leverage λ (or similarly of degree of diversification m).

Definition The cycle amplitude of financial leverage estimates the systemic risk of financial market.

The cycle amplitude of financial leverage (or degree of diversification) depends on some model parameters and there is not a universal scale of measure. However, the appearing of financial cycles reflects the occurrence of periods characterized by a *macro*-component of risk. With macro we refer to a large risk component due to the rebalancing feedback arising from a comprehensive behavior of all agents operating in financial market. For this reason, it can be difficult to control through a regulatory policy, although it is easy to recognize.

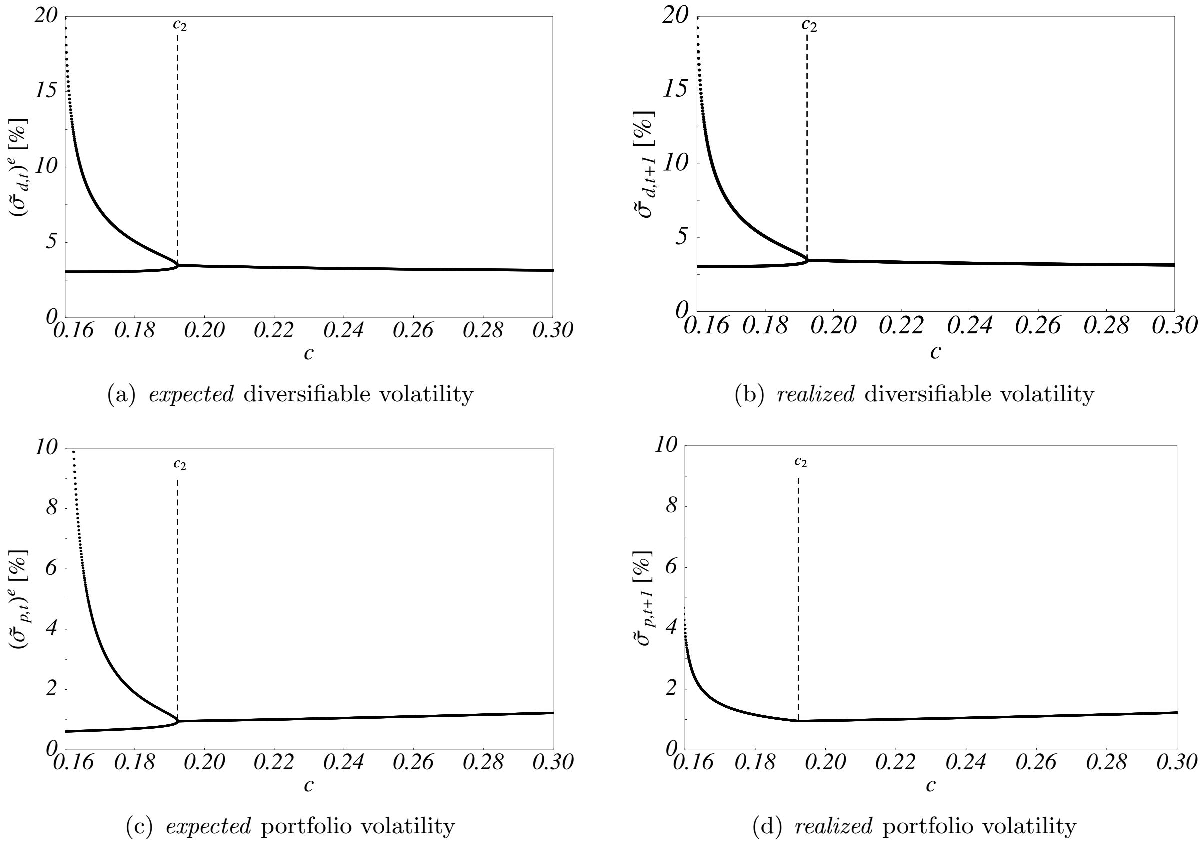
In panel (b) of Figure (4.4) we show the realizations of the asset diversifiable volatility, $\tilde{\sigma}_{d,t+1}$, at time $t+1$ after that financial institutions optimize their portfolio at time t with an expectation about asset volatility equal to $(\tilde{\sigma}_{d,t})^e$. The orbit plot for $(\tilde{\sigma}_{d,t})^e$ is shown in panel (a) of Figure (4.4). At the threshold c_2, also a transition occurs for the returns dynamics associated with the investment assets. For a value of c smaller than c_2, we can noticed that in relation to the two-period dynamics of portfolio variables, any investment asset is characterized by periods of high risk followed by periods of small risk. When diversification costs c decrease further and so the amplitude of financial leverage cycles increases, during the risky periods any asset is characterized by a even larger value of diversifiable volatility.

Even before reaching the value of diversification cost whereby the investment return processes become non-stationary (c^*: $\Lambda_{max} = \frac{\lambda^*-1}{\gamma} \geq 1$, in Figure (4.3) $c^* \approx 0.1589$), macro-risk outcomes correspond to the appearing of financial cycles.

It is important to notice how the macro-component of assets risk affects the portfolio volatility. Panel (c) and panel (d) of Figure (4.4) show the orbit plot for the expected portfolio volatility, $\tilde{\sigma}^e_{p,t}$, and the orbit plot for the realized portfolio volatility, $\tilde{\sigma}_{p,t+1}$.

At the right of the dashed line, when an equilibrium asymptotic point exists, $\tilde{\sigma}^e_{p,t}$ coincides with $\tilde{\sigma}_{p,t+1}$, *i.e.* the expectations about volatility by financial institutions are equal to their future realizations. Also both of them decrease slightly reducing the diversification costs. A reduction of c corresponds to a growth of the diversification m. This region represents a market in which a bigger diversification carries away the idiosyncratic risk. However, for values of c smaller than the threshold c_2 (the dashed line) the two-period dynamics appears, as we can see in the panel (a). The portfolio dynamics displays consecutive overestimations and underestimations of the expected portfolio volatility. Counter-intuitively, always for the same fixed value of c, the realized risk of the bank's portfolio is constant in time and it does not oscillate between two values, one larger than the other one. In fact, if the expected variance of a portfolio at time t is

$$(\tilde{\sigma}^e_{p,t})^2 = \frac{\sigma_d^2}{m_t} + \frac{V_{e_{j,t-1}}(m_{t-1},\lambda_{t-1})}{m_t} + \frac{m_{t-1}-1}{m_t} Cov_{e_{j,t-1},e_{k,t-1}}(m_{t-1},\lambda_{t-1}) \tag{4.18}$$



(a) *expected* diversifiable volatility

(b) *realized* diversifiable volatility

(c) *expected* portfolio volatility

(d) *realized* portfolio volatility

Figure 4.4: Orbit plots of the diversifiable asset volatility $(\tilde{\sigma}_{d,t})^e$ expected at time t and the realized one, $\tilde{\sigma}_{d,t+1}$, at time $t+1$, panel (a) and panel (b). Orbit plots of the portfolio volatility $(\tilde{\sigma}_{p,t})^e$ expected at time t and the realized one, $\tilde{\sigma}_{p,t+1}$, at time $t+1$, panel (c) and panel (d). The dashed line indicates the value of c at which the two-period dynamics appears. The used parameters are: $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, $\sigma_d = \frac{0.05}{1.64}$, $\gamma = 100$.

then, the realized one, after a transient $\Delta t = 1$, is

$$(\tilde{\sigma}_{p,t+1})^2 = \frac{\sigma_d^2}{m_t} + \frac{V_{e_{j,t}}(m_t, \lambda_t)}{m_t} + \frac{m_t - 1}{m_t} Cov_{e_{j,t},e_{k,t}}(m_t, \lambda_t) \tag{4.19}$$

If at time t, historical measures reveal large values for returns volatilities, due to the high impact of endogenous feedback (related to time $t-1$) then all financial institutions reduce the diversification and the leverage choosing smaller values for m_t and λ_t. These coordinated actions restrict the impact of the portfolios rebalancing on risk, but after a time interval Δt all banks' portfolios are poorly diversified and therefore they are exposed to a high idiosyncratic risk $\frac{\sigma_d^2}{m_t}$.

On the contrary, if at time t the impact of feedback effects is not important, financial institutions forecast low returns volatilities $\tilde{\sigma}_{d,t}^e$. According to their forecasting, they increase m_t and λ_t and a great impact of endogenous feedback (the second term and the third one of (4.19)) follows. Anyway, during these economic 2-period cycles, the banks micro-prudencial policy, characterized by naive expectations, does not reduce the portfolio risk. Moreover it grows as c decreases.

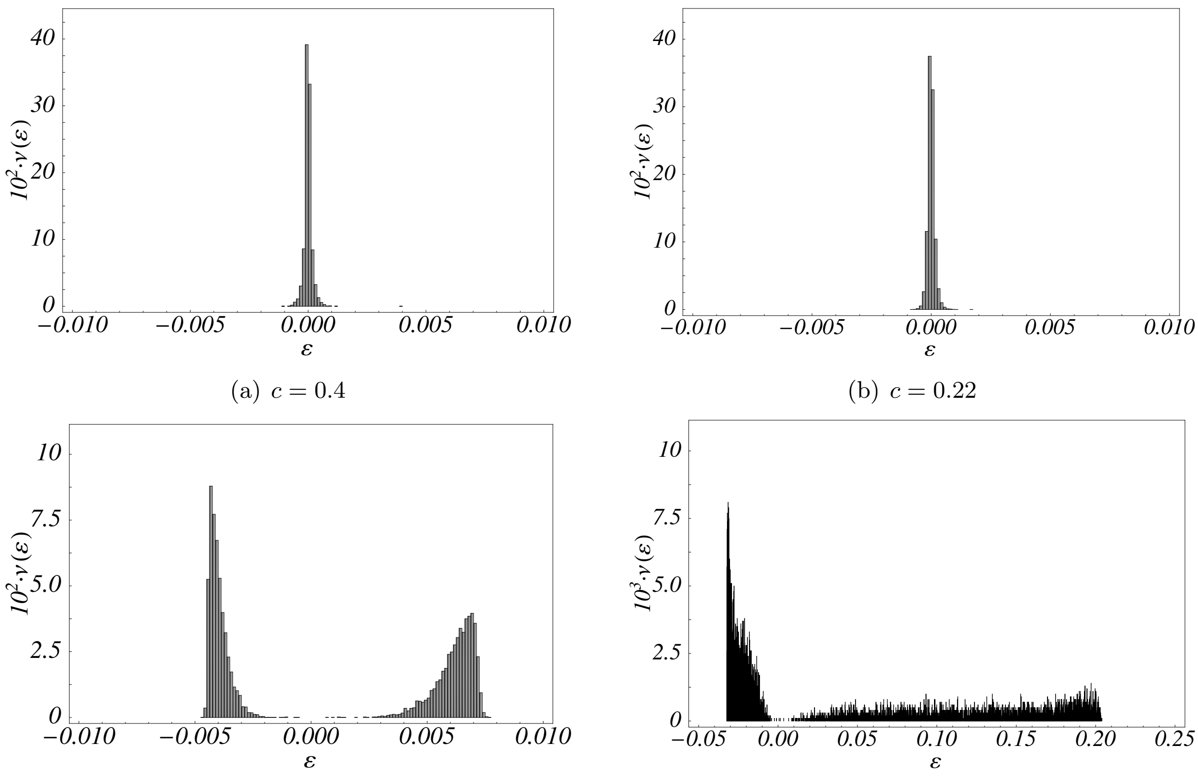
By a mathematical point of view, it is important to note that for a fixed value of c smaller than c_2 the dynamical evolution of a portfolio is characterized by a realized risk constant in time. The realized portfolio volatility, $\tilde{\sigma}_{p,t}$, is a *conserved quantity* of the discrete dynamical system.

From the previous analysis about the portfolio volatility, it is evident that financial institutions make forecasting mistakes at every times. Hence, starting from the errors made, financial institutions will try to improve their forecasting strategy. For this purpose, we will investigate the dynamical information which can be extrapolated by past realizations of risk.

If we look at the errors of the naive expectations made by financial institutions, clear informations appear. Let us define the forecast error at time t as

$$\mathcal{E}_t = \tilde{\sigma}^e_{p,t} - \tilde{\sigma}_{p,t+1} \tag{4.20}$$

The (4.20) can be estimated at each step of the discrete dynamical evolution of the system providing to the bank a measure of the accuracy of its last performance. Histograms in Figure



(a) $c = 0.4$

(b) $c = 0.22$

(c) $c = 0.18$

(d) $c = 0.16$

Figure 4.5: Histograms of the errors $\mathcal{E}$ for different values of diversification costs. In the panels (a) and (b), the value of c is smaller than the threshold c_2, in the other panels it is larger. The size of each bin is 1.25×10^{-4}. The Gaussian fluctuation introduced for the variance of the investment return is characterized by a standard deviation equal to $\sigma_{noise} = \frac{\sigma_d}{10}$ .The other parameters are: $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, $\sigma_d = \frac{0.05}{1.64}$, $\gamma = 100$.

(4.5) show the errors committed by a financial institution during the evolution of the *stochastic* system for different values of diversification costs. For stochastic system we mean the *skeleton* map in which a Gaussian fluctuation is introduced for the variance of the investment return. The fluctuation is of an order of magnitude smaller than the exogenous risk component.

When the dynamics is asymptotical stable, see panels (a) and (b) of Figure (4.5), the distribution of errors made by financial institutions is centered approximately around to zero. The statistical moments are shown in the Table (4.1). The *Jarque-Bera* test rejects with a 0.05 degree of significance (see the p-values) the Null Hypothesis H_0 that the sample data, obtained with these values of c, come from a normal distribution.

In statistics, the Jarque-Bera test is a goodness-of-fit test of whether the sample data have the skewness and kurtosis consistent with a normal distribution. The test statistics JB is defined as

$$JB = \frac{n}{6}\left(S^2 + \frac{1}{4}(K-3)^2\right)$$

where n is the number of observations, S is the sample skewness and K is the sample kurtosis ($K-3$ is the excess kurtosis). The Jarque-Bera test verifies simultaneously if the skewness and the kurtosis of the sample data are statistical consistent with the Null Hypothesis of normality, H_0.

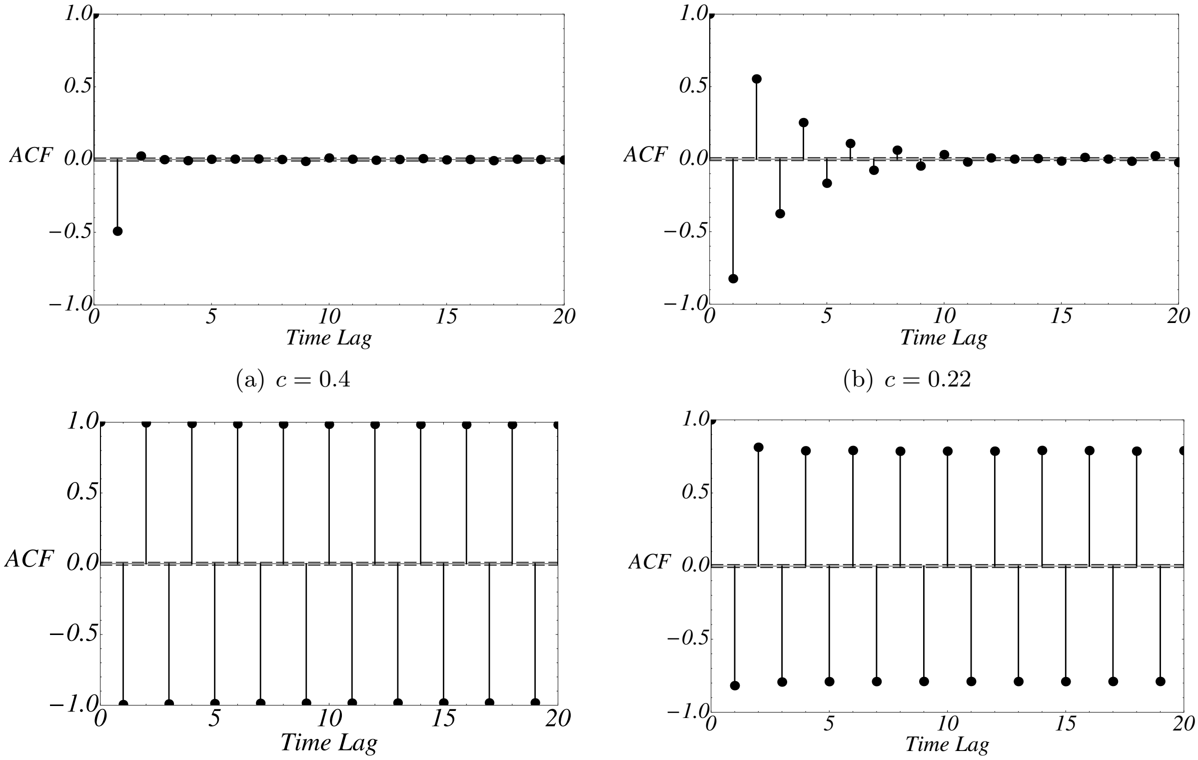
	mean	variance	skewness S	kurtosis K	Jarque-Bera test	p-value
$c = 0.4$	1.17×10^{-6}	2.68×10^{-8}	1.71	42.57	H_0 rejected	$< 1 \times 10^{-3}$
$c = 0.22$	1.14×10^{-6}	2.08×10^{-5}	0.69	8.10	H_0 rejected	$< 1 \times 10^{-3}$
$c = 0.18$	0.001	26.28	0.03	1.05	H_0 rejected	$< 1 \times 10^{-3}$
$c = 0.16$	0.051	7.03×10^{3}	0.51	1.65	H_0 rejected	$< 1 \times 10^{-3}$

Table 4.1: Statistical moments of forecasting errors series shown in Figure (4.5). Table shows the JB-test outcomes and the correspondent p-value associated with the Null Hypothesis of normality.

In particular when the value of c is larger than c_2, the JB-test rejects the normality hypothesis, being that the sample kurtosis is statistically larger than 3 suggesting the existence of heavy tails in the distribution of expectations errors. Furthermore, the positive values of skewness, see Table (4.1), suggests that the overestimations are larger in absolute value than the underestimations.

When the 2-period dynamics appears, the distribution of errors is characterized by the existence of two maxima which highlights the consecutive overestimations and underestimations of the portfolio volatility. Very close to the instability region, this distribution spreads over a large interval of $\mathcal{E}$-values.

If we look at the linear autocorrelation of error time series in Figure (4.6), it is evident this aspect. If ρ_τ is the autocorrelation coefficient at time lag τ, ρ_1 is negative significantly different from zero for all values of diversification costs, suggesting that an overestimation is followed by an underestimation, and vice versa. When the deterministic map's evolution converges to an equilibrium point, the forecasting errors are uncorrelated after few time lags, see the panels (a) and (b) of Figure (4.6). In fact, when $c = 0.4$, $\rho_1 \approx -0.5$. This is due to the fact that the eigenvalues of the map's Jacobian suggest that the system converges to the steady state swinging around it, as we can intuitively understand by Figure (4.2). This means that also the time evolution of expectations about risk oscillates around to the realized risk. As consequence



(a) $c = 0.4$ (b) $c = 0.22$

(c) $c = 0.18$ (d) $c = 0.16$

Figure 4.6: The linear autocorrelation function of the errors time series for different values of diversification costs. To test the level of significance, the upper and lower bounds are equal to $\pm\frac{z_{1-\frac{\beta}{2}}}{\sqrt{N}}$, N is the time series length, $z_{1-\frac{\beta}{2}}$ is the quantile function of the normal distribution and β is the significance level. At 5% significance level (the dashed lines) the bounds are $\frac{1.96\ \sigma_{data}}{\sqrt{N}}$, where σ_{data} is the standard deviation of the data. The amplitude of the gaussian noise is $\sigma_{noise} = \frac{\sigma_d}{10}$. The used parameters are: $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, $\sigma_d = \frac{0.05}{1.64}$, $\gamma = 100$.

of this, for all values of diversification costs the autocorrelation coefficient at first time lag is negative.

When the 2-period orbits appear, the linear autocorrelation function shows jumps between negative values at odd time lags and positive values at even time lags, as clearly appears in the panels (c) and (d) of Figure (4.6). Two forecasting errors are correlated at any time scale. In the pure deterministic evolution the linear autocorrelation function makes a zig-zag path jumping between $+1$ and -1. The stochastic noise has the effect of decreasing the correlation between the forecasting errors. In fact, the absolute amplitude of autocorrelation coefficients is a slowly decaying function of time lags. The number of time lags necessary to reach 0, depends obviously on the noise amplitude. A larger noise amplitude reduces the number of time lags. But at the same time a larger noise amplitude could hide the information which can be extrapolated by the portfolio dynamics. Anyway, financial institutions might recognize this structure in the autocorrelation function and they might improve their forecast.

4.4 Mathematical aspects of the naive expectations

In this Section, we will focus on the loss of hyperbolic equilibrium of the system and we will investigate what type of bifurcation occurs. The main interest is a deeper understanding of the *2-period* stationary state and in particular of the oscillations amplitude during the evolution of the diversification m and the financial leverage λ. The amplitude of the two-period orbit can be seen as a measure of market systemic risk. In fact, a growth of the amplitude means that the system is closer to the region of instability when the returns dynamics becomes non-stationary and any exogenous shock can trigger a plunge of market prices.

4.4.1 Linearized map

The system of equations (4.8) (4.9) (4.10) with naive expectation (4.16) can be symbolically written in *implicit* form as

$$\mathbf{F}(\mathbf{x}_t, \mathbf{x}_{t-1}, c) = 0 \tag{4.21}$$

where $\mathbf{x} = (m, \lambda)$, c is the continuous diversification costs parameter.
If $\mathbf{x}^*$ is a fixed point for (4.21), the linearized system in a neighborhood of $\mathbf{x}^*$ is

$$\delta\mathbf{x}_t = -(\mathcal{D}_{\mathbf{x_t}}\mathbf{F} \mid_{\mathbf{x}^*})^{-1}(\mathcal{D}_{\mathbf{x_{t-1}}}\mathbf{F} \mid_{\mathbf{x}^*})\delta\mathbf{x}_{t-1} \tag{4.22}$$

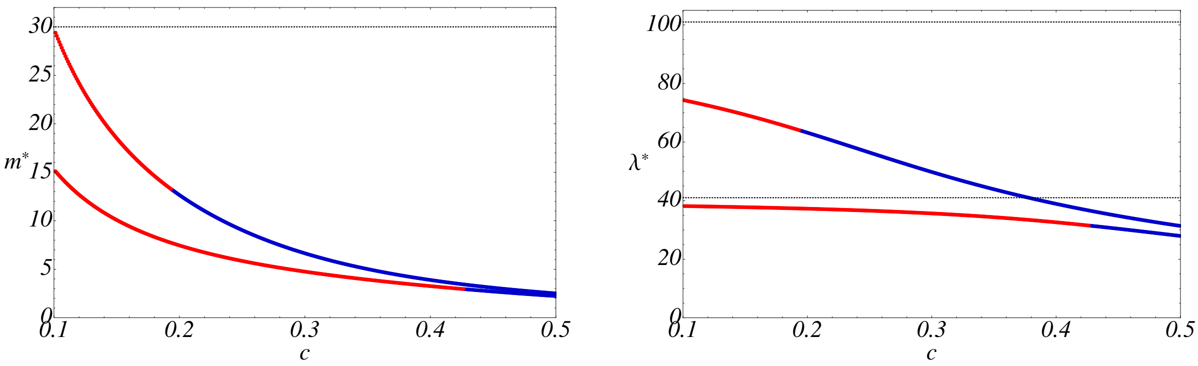
where $\delta\mathbf{x} = \mathbf{x} - \mathbf{x}^*$, $-(\mathcal{D}_{\mathbf{x_t}}\mathbf{F} \mid_{\mathbf{x}^*})^{-1}(\mathcal{D}_{\mathbf{x_{t-1}}}\mathbf{F} \mid_{\mathbf{x}^*})$ is the Jacobian matrix of the map (4.21) at the fixed point.

The fixed point $\mathbf{x}^*$ will be asymptotically stable if and only if the eigenvalues of the Jacobian lie within the unit circle in the complex plane. If $\xi = (\xi_1, \xi_2)$ are the eigenvalues, the above condition reads $\mid \xi_i \mid < 1$ for each eigenvalue.

When the center subspace (see Section 2.2 of Chapter 2) is empty, the fixed point is said hyperbolic and for the Hartman-Grobman theorem then there exists a homeomorphism which relates locally the linearized dynamics to the original map dynamics, and the stability of the first one implies the stability of the second one.

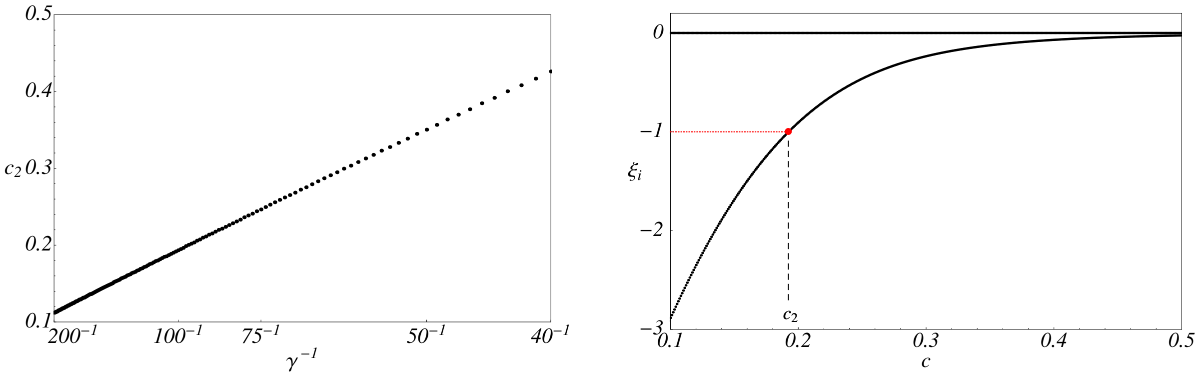
In Figure (4.7) we report the fixed points $\mathbf{x}^* = (m^*, \lambda^*)$ of the map (4.21) as function of the continuous parameter c. When the asymptotic stability condition is satisfied we mark the solution in blue, otherwise in red. It is evident that a threshold c_2 in the range of c-values exists. If $c > c_2$, the equilibrium solution is hyperbolic. For $c = c_2$ the invariant center subspace for the linearized system is not empty and a loss of hyperbolicity follows. It occurs in one specific way as we will see below. c_2 is the value of control parameter c at which there is a transition from an hyperbolic equilibrium to a stationary *2-period* dynamics. When $c < c_2$, the solution of $F(x^*, x^*, c) = 0$ is not an attractor for the dynamics anymore.

As seen above, the endogenous feedback affects critically the features of the dynamics by its impact on risk. To highlight the latter aspect, we show in Figure (4.7) the fixed points for two different values of market liquidity γ. The set of fixed points, as a function of the parameter c, defines a continuos curve in the domain of the map. In Figure (4.7) the upper curve is found with an higher value of γ, equal to 100, compared to the lower curve which is related to a value



(a) (b)

Figure 4.7: The fixed points of the map as a function of the diversification cost c. A fixed point is marked in blue if it is an asymptotic stable point, in red if it is not. The panels (a) and (b) show the solutions for the diversification and the leverage, m^* and λ^* respectively. The upper curve corresponds to a value of the liquidity parameter γ equal to 100, the lower one, instead, corresponds to $\gamma = 40$. The dotted lines represent the bounds of the domain for m and λ. In the panel (b) the two dotted lines indicate two different bounds for λ being that for the latter the domain interval depends on the value of the liquidity parameter. The other parameters are: $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, $\sigma_d = \frac{0.05}{1.64}$.



(a) (b)

Figure 4.8: The relation between the threshold c_2, at which transition in dynamics occurs, and the impact parameter γ^{-1}, in panel (a). The Jacobian eigenvalues ξ_j as function of diversification costs, c, when the market liquidity is $\gamma = 100$, in panel (b). The other parameters are: $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, $\sigma_d = \frac{0.05}{1.64}$.

of γ equal to 40. The range of c-values, for which the dynamics has an attractor of period 1, changes for different values of market liquidity. In fact, since the endogenous feedback is inversely related to the market liquidity, a larger value of the latter is related to a smaller value of the threshold c_2 at which the loss of hyperbolicity occurs. In Figure (4.7), the dotted lines represent the bounds of the economic domain defines in the previous Section. In panel (a), the dotted line establishes the bound for m at a value $M = 30$, that is the number of the available investment assets. In panel (b), the domain of λ depends on the value of liquidity parameter, as we have noticed previously. The upper dotted line defines the domain bounds for λ when $\gamma = 100$. The lower dotted line defines the domain bounds for λ when $\gamma = 40$.

In the panel (a) of Figure (4.8) the relation between γ^{-1} and the threshold c_2, at which the transition in dynamics occurs, is shown. A financial market, characterized by a smaller value of liquidity of the traded assets, loses the hyperbolic stability for an higher value of diversification costs.

Finally, when the diversification cost reaches the threshold value, at least one of the Jacobian's eigenvalue is on the unit circle and the hyperbolic equilibrium is lost. The eigenvalues of the Jacobian as function of control parameter c, for the linearized map (4.21) in the case $\gamma = 100$, are shown in Figure (4.8). At the threshold (the black dashed line), one eigenvalue is equal to -1, *i.e.* a *period-doubling (flip) bifurcation* occurs. The other eigenvalue is always equal to zero for all diversification costs. This suggests that from a mathematical point of view the system of equation is redundant. In fact, in Section 4.1 we have noticed that the dynamical system is one-dimensional. With an opportune linear change of variables, it can be reduced to a single difference equation containing all the information about the portfolio variables.

4.4.2 One-dimensional map and qualitative aspects of the flip bifurcation

In the economic domain the map described by equations (4.8) (4.9) (4.10) with naive expectation (4.16) is equivalent to the map in explicit form

$$m_t = \left(\frac{\mu - r_L}{2\alpha c}\right)^2 \frac{1}{\Sigma(m_{t-1}, \lambda_{t-1})} \tag{4.23}$$

$$\lambda_t = \left(\frac{\mu - r_L}{2\alpha^2 c}\right) \frac{1}{\Sigma(m_{t-1}, \lambda_{t-1})} \tag{4.24}$$

where $\Sigma(m_{t-1}, \lambda_{t-1})$ is the quadratic form of naive expected volatility $\tilde{\sigma}_d$

$$\Sigma(m_{t-1}, \lambda_{t-1}) = \sigma_d^2 + V_{e_{j,t-1}}(m_{t-1}, \lambda_{t-1}) + (m_{t-1} - 1)Cov_{e_{j,t-1},e_{k,t-1}}(m_{t-1}, \lambda_{t-1}) \tag{4.25}$$

being $V_{e_{j,t-1}}$ and $Cov_{e_{j,t-1},e_{k,t-1}}$ respectively the variance and the covariance of the endogenous components.

In explicit form it is evident that the difference equations (4.23) and (4.24) have the same functional dependence from $\Sigma(m_{t-1}, \lambda_{t-1})$ up to a multiplicative constant. The panel (b) of Figure (4.8) is the logical outcome. There exists an orthogonal transformation of variables, $\mathcal{O}$, which decomposes the phase space into the direct sum of the eigenspaces of linearized map

$$\mathcal{O} = \begin{pmatrix} \frac{1}{\sqrt{1+\frac{4c^2}{(\mu-r_L)^2}}} & \frac{1}{\sqrt{1+\frac{(\mu-r_L)^2}{4c^2}}} \\ \frac{1}{\sqrt{1+\frac{(\mu-r_L)^2}{4c^2}}} & \frac{-1}{\sqrt{1+\frac{4c^2}{(\mu-r_L)^2}}} \end{pmatrix}$$

Notice that the transformation $\mathcal{O}$ is also symmetric, it depends on the control parameter c but it does not depend on the parameter α. Since an eigenvalue of the 2-dimensional map is always equal to zero, the orthogonal transformation $\mathcal{O}$ allows us to study an one-dimensional map provided that an initial time transient is passed. The latter is the time step $\Delta t = 1$ to let pass so that any initial random chosen eigenvector associated with the null eigenvalue converges to the invariant stable subspace E^s (see Section 2.2 of Chapter 2). In other words, the eigenspace associated with the null eigenvalue corresponds to the *Ker* of the 2-dimensional map. The *Ker* (or null space) is the subset of all elements of the economic domain for which the image coincides with the null space of the codomain.

Furthermore, at criticality occurring at $c = c_2$, due to the presence of the null eigenvalue, the center manifold W^c coincides with the eigenspace associated with the eigenvalue equal to -1. The linear transformation $\mathcal{O}$ is equivalent to the center-manifold reduction procedure for the 2-dimensional map.

We can change the original basis (m, λ) into the eigenbasis by the linear transformation $\mathcal{O}$ and we can tune the initial conditions to project the system into the eigenspace associated with the non-vanishing eigenvalue, without loss of generality.

Let u be the eigenvector associated with the non-vanishing eigenvalue, the one-dimensional map containing all informations about dynamics is the following

$$u_t = f(c, u_{t-1}) = \left(\frac{1}{\sqrt{1 + \frac{4c^2}{(\mu - r_L)^2}}} \left(\frac{\mu - r_L}{2\alpha c} \right)^2 + \frac{1}{\sqrt{1 + \frac{(\mu - r_L)^2}{4c^2}}} \frac{\mu - r_L}{2\alpha^2 c} \right) \frac{1}{\Sigma(\mathcal{O}^{-1}\mathbf{x}_{t-1})} \tag{4.26}$$

where symbolically we write $\mathcal{O}^{-1}\mathbf{x}_{t-1}$ for the inverse transformation applied to $\mathbf{x} = (m, \lambda)$. In principle $\Sigma(\mathcal{O}^{-1}\mathbf{x}_{t-1})$ depends on u_{t-1} and the eigenvector related to the null eigenvalue. When we set the eigenvector related to the null eigenvalue equal to zero, we obtain

$$\Sigma(\mathcal{O}^{-1}\mathbf{x}_{t-1}) \equiv \Sigma(u_{t-1})$$

Since the eigenvector u is the linear combination of m and λ weighted by two positive components of the orthogonal transformation $\mathcal{O}$, the domain of the map (4.26) is simply the interval

$$\mathcal{I} = \left[\frac{1}{\frac{1}{\sqrt{1+\frac{4c^2}{(\mu - r_L)^2}}}} \quad ; \quad \frac{\gamma}{\frac{1}{\sqrt{1+\frac{(\mu - r_L)^2}{4c^2}}}} \right) \tag{4.27}$$

for all values of the diversification costs c such that the return dynamics is stationary.

Figure (4.9) shows the analytical form of the map f, (4.26), and of the second-iterate map, f^2. A shifting and a rescaling of the variable u is made with the intent that the domain $\mathcal{I}$ coincides with the unit interval. In this way, it is clear that the domain is positively invariant under the map (4.26), *i.e.* $f(c, u_r) \subseteq (0, 1)\ \forall u_r \in (0, 1)$, where u_r is the rescaled variable. The prefix "positively" is used to highlight that we are only considering forward iterations of f.

The panel (a) of Figure (4.9) shows the map's mathematical shape in the unit interval for a value of c larger than the threshold c_2. The stable fixed point is graphically represented by the black dot which is the intersection between the map f and the identity function (the dashed line). The panel (b) shows the second-iterate map, f^2, for the same value of the parameter. It is clear that there are no new solutions other than the fixed point of the map f. The panels (c) and (d) of the Figure (4.9) show the same results for a value of c smaller than the threshold. For $c < c_2$ the fixed point of f is not stable and a new attractor for dynamics appears. In fact, the second-iterate map, f^2, displays new stable solutions which are marked with other two

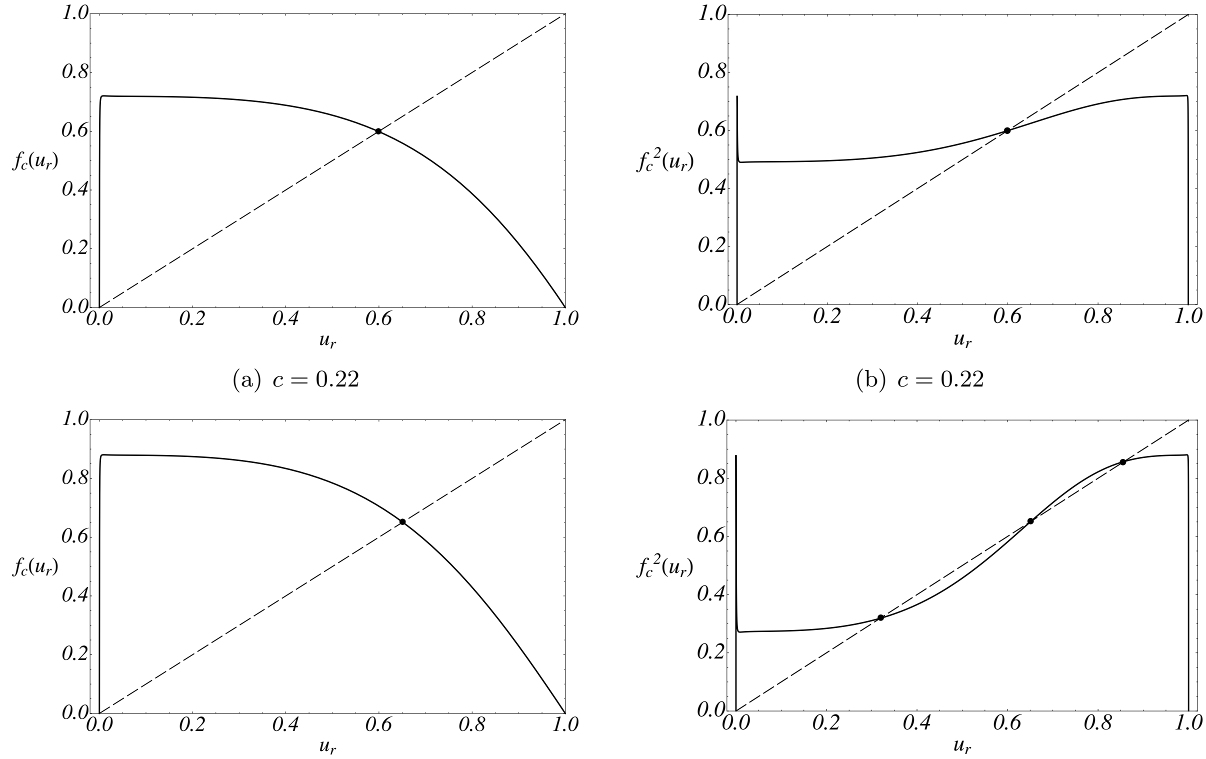


Figure 4.9: The analytical shape of the one-dimensional map, f, and of its second-iterate map, f^2, are shown after the rescaling of the domain into the unit interval. In the panels (a) and (b) the value of the parameter c is larger than the threshold c_2, the fixed point is graphically displayed as the intersection between the map f (the black line) and the identity function (the dashed line) and it is marked with a black point. In the panel (b), only one solution exists for the second-iterate map which corresponds to the fixed point of the map f. In the panel (c) and (d) the value of the parameter c is chosen smaller than the threshold c_2, wherein the flip bifurcation occurs. In the panel (c) the fixed point of the map f is always marked with a black point but it is no longer a stable equilibrium point. In fact, the second-iterate map, f^2, explains new stable asymptotic solutions (the another two point marked in black) which correspond to the attractor of period 2 for the dynamics. The used parameters are: $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, $\sigma_d = \frac{0.05}{1.64}$, $\gamma = 100$.

black dots. These new solutions are stable fixed points for f^2. They represent the attractor of period 2 for the asymptotic swinging dynamics of the map f. To find analytically these ones, in principle, we need to solve the equation $f^2(c, u_r) = u_r$. This task is no trivial because of the complex analytical form of the second-iterate map. Below we will give an approximate analytical solution of the problem in the neighborhood of the critical point.

Finally, in this framework, the loss of invariance property of the unit interval reflects the nonstationary dynamics of the endogenous component which takes place when c is smaller than c^*. When $c \in (c^*, c_2)$, there are only 2-period orbits. If there was an orbit of period larger than 2, there should be the orbit of period 4, due to the *Sharkovsky ordering*, see Chapter 2. However for all values of the control parameter c such that the returns dynamics is stationary, the solutions of the equation $f^4(u_r) = u_r$ are equivalent to those of the equation $f^2(u_r) = u_r$. Hence, in the domain $\mathcal{I}$, the parameter c does not trigger any period-doubling cascade.

Finally, we look at the time period, in units of iteration steps, required for the convergence of the dynamics to the fixed point, once the value of parameter c is determined. When $c > c_2$,

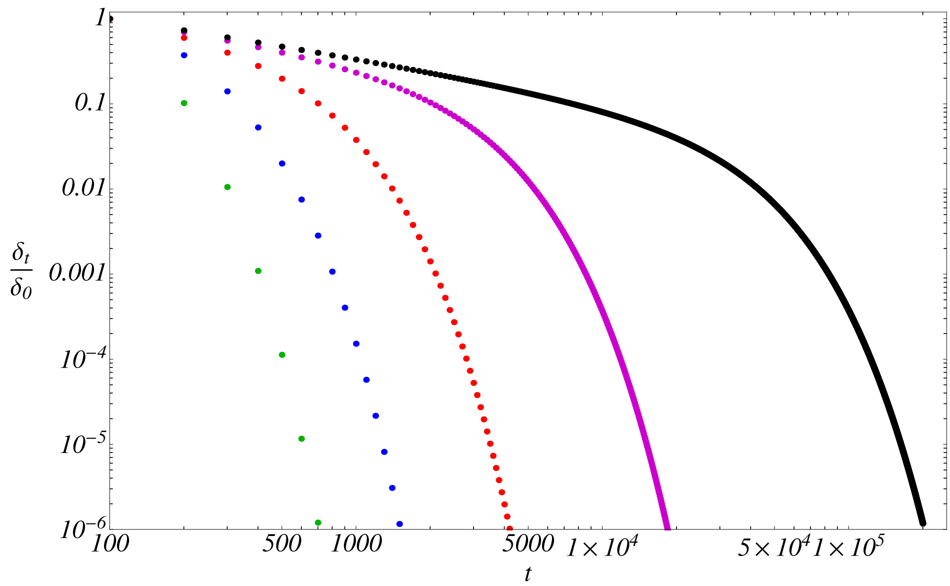


Figure 4.10: The ratio $\frac{\delta_t}{\delta_0}$ in function of the iteration steps t, where δ_t is the absolute value of u_t-values at different time steps t minus the value of the fixed point u^*, found solving numerically the equation $u^* = f(c, u^*)$. δ_0 is the value of δ_t at the iteration step $t = 100$, after that the time transient has passed. The used values for diversification costs c are all larger than the threshold c_2. In decreasing order, the parameter c takes the values: 0.194 (green), 0.193 (blue), 0.1925 (red), 0.1923 (magenta), 0.19226 (black). The used parameters are: $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, $\sigma_d = \frac{0.05}{1.64}$, $\gamma = 100$.

the convergence time increases when approaching the criticality.

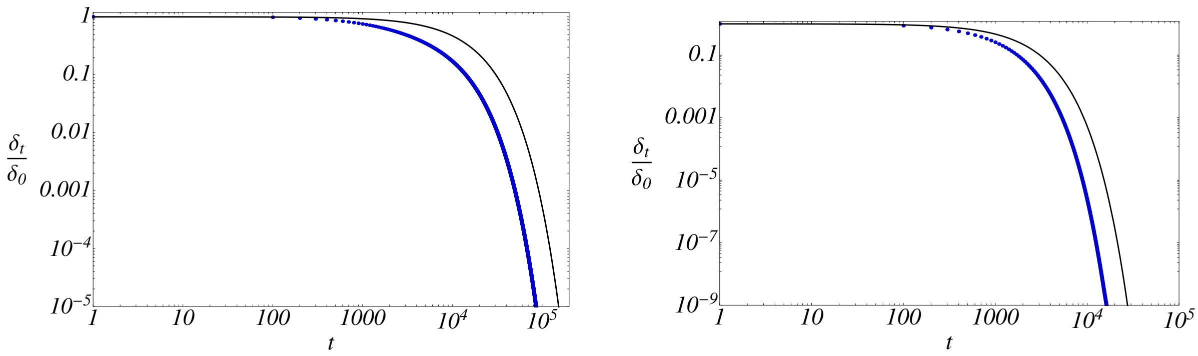
Figure (4.10) shows the ratio between δ_t and δ_0:

$$\delta_t = |u_t - u^*| \tag{4.28}$$

where u^* is the value of the fixed point of the map for the determined value of c. u^* is found solving numerically the equation $u^* = f(c, u^*)$. δ_0 is the value of δ_t at time $t = 100$. We consider a time transient equal to 100 in order to remove possible initial fluctuations due to the random choice of the initial condition. For all values of diversification costs larger than the threshold c_2, the dynamics exponentially converges with time to the equilibrium point. Approaching the criticality, $c \to c_2$, we observe an exceptionally slow convergence to the fixed point because the time scale increases.

From an economic point of view, since the diversification and the financial leverage are proportional to the eigenvector u, a slow decaying of the amplitude related to the oscillations of the economic variables may suggest that a period-doubling bifurcation can occur in economic system. In fact, the exceptionally slow convergence near a criticality is the signature of a bifurcation as a consequence of one Jacobian's eigenvalue on the unit circle [45]. It is evident in (4.22) that, in the basin of attraction of the fixed point, the absolute value of increments decays to zero very slowly due to the presence of the eigenvalue in modulus equal to one.

In our model, when we approach the criticality from above, the linear expansion in δc of the



(a) $\delta c = 1 \times 10^{-5}$ (b) $\delta c = 1 \times 10^{-4}$

Figure 4.11: The normalized deviations $\frac{\delta_t}{\delta_0}$ from the fixed point for the map f (the points marked in blue) and the analytical approximation obtained with the linearized map (the black line). In the panel (a) the value of c is chosen equal to $c_2 + \delta c$ where $\delta c = 1 \times 10^{-5}$, in the panel (b) $\delta c = 1 \times 10^{-4}$. The used parameters are: $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, $\sigma_d = \frac{0.05}{1.64}$, $\gamma = 100$. For this choice of parameters, we find $c_2 = 0.19224554$, the numerical uncertainty is on the last digit.

first derivative of the map f calculated at the fixed point u^* is

$$\Delta_{\delta c} f'(\delta c, u^*) = -1 + \theta \delta c \tag{4.29}$$

where θ is larger than zero and its specific value can be determined once that the values of all parameters are fixed. Hence in the basin of attraction of the fixed point we can study the linearized map to find an approximate solution for $\delta_t = |u_t - u^*|$. Starting from $u^* + \delta_0$ and applying t-times the linearized map, we find for δ_t

$$\delta_t = (-1 + \theta \delta c)^t \delta_0 \tag{4.30}$$

For the values of c close to $c2$, calculating the Taylor expansion in δc and summing the obtained series, finally

$$\frac{\delta_t}{\delta_0} = e^{-\frac{t}{\tau}} \tag{4.31}$$

where $\tau = \frac{1}{\theta \delta c}$ is the characteristic time scale for a specific value of the parameter c.

In Figure (4.11) the dynamical evolution of $\frac{\delta_t}{\delta_0}$ (the points marked in blue) and the approximate solution (4.31) (the black line) are shown for two different values of δc. It is evident that when c approaches c_2, the characteristic time scale τ becomes larger and larger because the dynamics is dictated by the eigenvalue of f which is closer and closer to -1. Equivalently, when $\delta c \to 0$, it follows that $\tau \to \infty$. This divergence of τ is due to the fact that the linear approximation fails in describing the convergence time at the criticality. In fact, when $c = c_2$ the exponential law in (4.31) is replaced by a power law. The power law describing the time evolution of the deviations from u^* is the inverse of the square root of the number of iterations [38].

In dynamical systems theory, this phenomenon is known as the critical slowing down [104]. One important consequence is that the slowing down leads to an increase in autocorrelation related to the portfolio variables before approaching the criticality. For example, see panel (b) of Figure (4.6). The increased autocorrelation can be interpreted as a early-warning signal that the dynamics of the system is approaching a bifurcation. In real financial market, the crucial point is to study if the increased correlation is above the noise level, just before a dynamical transition like a financial crisis. Results from elaborate and relatively realistic climate models [105] or from analysis of empirical data in physiology [106], suggest that some signals might be robust in the sense that they arise despite high complexity and noisiness.

4.4.3 Poincaré-Birkhoff normal form and flip bifurcation's amplitude

In a neighborhood of the criticality, it is possible to solve analytically the problem about how the 2-period orbit's amplitude changes as a function of the diversification cost c.

At the threshold c_2, the interval $\mathcal{I}$ (see (4.27)) corresponds to the one-dimensional center manifold W^c, defined in the previous Chapter. The spectrum at criticality determines which non-linear terms of the map f are essential. These nonlinearities are unchanged after that inessential nonlinearities can be removed by a smooth near-identity nonlinear change of coordinates. Since the eigenvalue ξ_u is equal to -1, the normal form of the map (4.26) has a reflection symmetry (see Section 2.2) and this is the underlying reason why we can find an analytical form, even if approximate, for the 2-period orbit's amplitude.

In order to solve this problem, we move the critical point (c_2, u^*) to the origin by a linear change of coordinates so that

$$\tilde{f}(0,0) = 0$$

Let us define the increments $z = u - u^*$ and $\delta c = c - c_2$ and from now on we consider negative increments δc. We refer to $\tilde{f}$ as the map f after the coordinate shift towards the origin:

$$z_{t+1} = \tilde{f}(\delta c, z_t) \tag{4.32}$$

Let us define $\xi_z(\delta c)$ as

$$\xi_z(\delta c) \doteq \frac{\partial \tilde{f}(\delta c, z)}{\partial z}|_{z=0} \tag{4.33}$$

From the pervious analysis about the eigenvalues of the Jacobian, it follows that

$$\xi_z(\delta c = 0) = -1 \tag{4.34}$$

At criticality, it is possible to remove the second-order term of the Taylor expansion of $\tilde{f}$ by performing a near-identity nonlinear coordinate change

$$\tilde{z} = \Psi(z) = z + \zeta_2 z^2 \tag{4.35}$$

where ζ_2 is the solution of *homological equation* (*Lie bracket*) (2.10) if the center manifold is one-dimensional, see Section 2.2

$$\zeta_2|_{\delta c=0} = \frac{\frac{\partial^2 \tilde{f}(\delta c, z)}{\partial^2 z}|_{(\delta c=0, z=0)}}{\frac{\partial \tilde{f}(\delta c, z)}{\partial z}|_{(\delta c=0, z=0)} - \left(\frac{\partial \tilde{f}(\delta c, z)}{\partial z}|_{(\delta c=0, z=0)}\right)^2} \tag{4.36}$$

By performing the trasformation (4.35) in a neighborhood of the origin, the map $\tilde{f}$ assumes the following form:

$$\tilde{z}_{t+1} = \xi_z(\delta c)\tilde{z}_t \left(1 + \phi_3(\delta c)\tilde{z}^2\right) + \mathcal{O}(\tilde{z}^4) \tag{4.37}$$

where

$$\frac{\partial \xi_z(\delta c)}{\partial(\delta c)}|_{\delta c=0} < 0 \tag{4.38}$$

$$\xi_z(\delta c)\phi_3(\delta c) = \frac{1}{3!}\frac{\partial^3 \tilde{f}(\delta c, \tilde{z} - \zeta_2 \tilde{z}^2)}{\partial \tilde{z}^3}|_{\tilde{z}=0} \tag{4.39}$$

The relation (4.38) refers to the fact that for negative values of δc the non-zero eigenvalue of the Jacobian of f calculated at the fixed point, see Figure (4.8), decreases and it becomes smaller than -1. In our model, the third-order term of the Taylor expansion (4.39) assumes approximately a constant value $\frac{\Phi_3}{3!}$ in the left neighborhood of δc, $[-r,\ 0]$. Generally, the radius of the left neighborhood depends on all model parameters. The fact that (4.39) assumes a constant value in a left neighborhood of the origin, is related to the analytical form of f. The value of Φ_3 can be find once that all model parameters are fixed.

In principle, we can perform further nonlinear coordinate changes and in iterative way we can remove the even-degree expansion's terms, see Chapter 2. As claimed before, removing all even-degree terms, the normal form of the map $\tilde{f}$ will be invariant under reflection. If we drop all terms from the fourth-order on, an approximate normal form is obtained in a neighborhood of the origin:

$$\tilde{z}_{t+1} \approx \xi_z(\delta c)\tilde{z}_t \left(1 + \phi_3(\delta c)\tilde{z}_t^2\right) \tag{4.40}$$

The reflection symmetry allows us to find analytically the 2-period solutions of the map's dynamics near the origin. In fact, when $\delta c < 0$ the fixed point of the map f is unstable and the new attractor of the long run dynamics is represented by the solutions of the second-iterated map, f^2, as we have noticed before. Near the origin, an analytical approximation of these 2-period solutions can be found solving the following equation

$$-\tilde{z}_{\pm} = \xi_z(\delta c)\tilde{z}_{\pm} \left(1 + \phi_3(\delta c)(\tilde{z}_{\pm})^2\right) \tag{4.41}$$

where $\tilde{z}_{\pm} = u_{\pm} - u^*$ and $u_{\pm}$ represent an analytical approximation of the stable solutions of f^2.

We refer to $\mathcal{N}$ as the normal form of the map f at all orders and we refer to $\mathcal{T}$ as the reflection operator, *i.e* the discrete operator which associates at every element of the domain of $\mathcal{N}$ its opposite value in respect to the origin. The mathematical intuition of this approach is related to the fact that at criticality $\delta c = 0$ the solutions of

$$\tilde{z} = \mathcal{N}(\mathcal{N}(\tilde{z}))$$

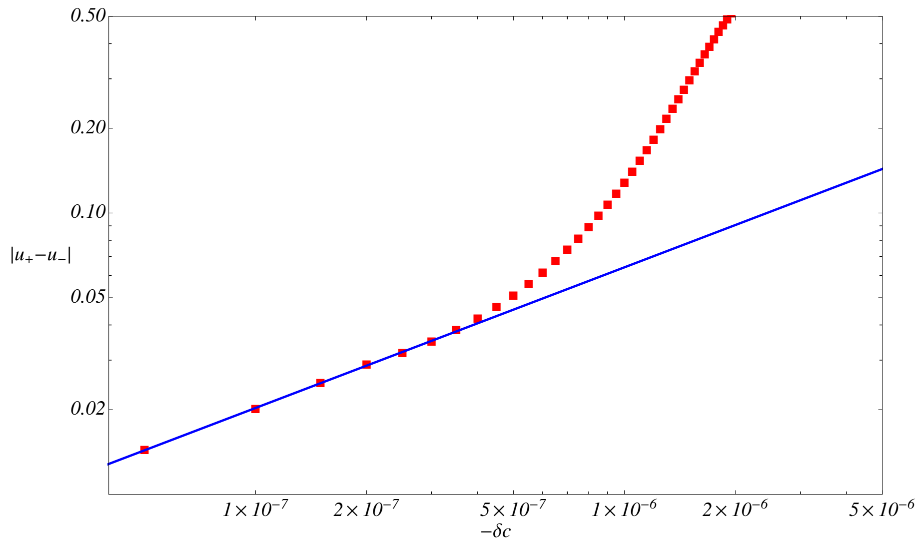


Figure 4.12: The log-log-plot of the amplitude of 2-period orbit of the map $\tilde{f}$ (red) and its analytical approximation (blue), as a function of the variation of diversification cost, $-\delta c$. The analytical function is found by the normal form truncated at the third order. The Figure is obtained with 2×10^5 iteration steps. The used parameters are: $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, $\sigma_d = \frac{0.05}{1.64}$, $\gamma = 100$. With this choice of model parameters, $\Phi_3 \approx 4.39 \times 10^{-3}$.

are equivalent to the solutions of

$$\tilde{z} = \mathcal{T}(\mathcal{N}(\tilde{z}))$$

Let be $\tilde{z}_\pm$ the solutions of the first equation. We will have $\tilde{z}_+ = \mathcal{N}(\tilde{z}_-)$ and $\tilde{z}_- = \mathcal{N}(\tilde{z}_+)$, see Section 2.2. Moreover, the reflection symmetry of $\mathcal{N}$ requires $\tilde{z}_- = -\tilde{z}_+$. Hence, the solution of the second equation is equivalent to the solution of the first equation.

In principle, this analytical approach is exactly justified at criticality. Ruelle and Takens [43] pointed out that the approach is approximately justified not only at $\delta c = 0$ but on a neighborhood of criticality.

The absolute difference between $\tilde{z}_+$ and $\tilde{z}_-$ gives an approximation of the 2-period orbit's amplitude

$$|\tilde{z}_+ - \tilde{z}_-| \approx 2\sqrt{\frac{3!\frac{\partial(-\xi_z-1)}{\partial(\delta c)}|_{\delta c=0}}{\Phi_3}\delta c} \propto \sqrt{-\delta c} \tag{4.42}$$

In a neighborhood of the origin, (4.40) describes in a approximate way the dynamical behavior of the center manifold variable u. In the asymptotic limit we will have

$$|u_+ - u_-| \approx |\tilde{z}_+ - \tilde{z}_-| \propto \sqrt{-\delta c} \tag{4.43}$$

The Figure (4.12) displays the outcome: very closely to the origin, the analytical result (4.42) fits very well the amplitude of 2-period orbit of the center manifold map (4.26). Moving away from the origin, other contributions of the Taylor expansion become non-negligible and they generate significant corrections to the fitting curve. Very closely to the criticality, the bifurcation's amplitude is a square root function of the variation of diversification cost.

Chapter 5

Adaptive expectations and adaptive learning

The expectations scheme adopted by financial institutions in the forecasting of the risk, may condition the dynamical behavior of the economic system. In this Chapter we describe the case of *adaptive expectations*.

Under adaptive expectations, the dynamical system exhibits a very complex dynamics characterized by a period-doubling cascade to chaos. When financial institutions adopt an adaptive expectations scheme, the positive feedback triggers successive dynamical transitions which induce financial cycles increasingly complex. During financial cycles, financial market is characterized by highly risky periods due to a macro-component of the risk induced by rebalancing feedback effects.

Under particular conditions, the dynamical system is characterized by a dynamical transition from a periodic behavior to a chaotic one. In the chaotic regime, the financial market dynamics can show periods characterized by higher risk compared with the risky periods of financial cycles. Furthermore, since the dynamical behavior is aperiodic, financial agents may not recognize how the economic system evolves and the improvement of the forecasting strategy may be hard. In some sense, a risky scenario occurs due to the missing information about the dynamics of financial market.

When financial cycles or chaotic regimes appear, financial agents typically try to improve their forecasting strategy in order to avoid systematic mistakes. They adapt the parameters of their forecasting strategy according to some learning scheme (*adaptive learning*). One possible scheme is the SAC learning scheme, which uses the autocorrelation function of forecasting errors series in order to improve future financial performances of forecasting.

In Section 5.1, we will describe the dynamical system under adaptive expectations and we will compare the two cases represented by adaptive and naive expectations. In Section 5.2, we will particularly focus on the adaptive expectations scheme described by a parameter representing the memory of the dynamical system. In Section 5.3 we will analyze some mathematical aspects related to the map that describes the dynamical system. In this Section, we will apply some results of the ergodic theory of dynamical systems in order to relate some dynamical features to the entropy of the system. Finally, in Section 5.4, we will describe an adaptive learning procedure (SAC learning) that financial institutions may use in order to improve their forecasting strategy.

All the material in this Chapter is original.

5.1 Adaptive expectations

An important aspect highlighted in the previous Chapter concerns the information observable by a financial institution during the evolution of market dynamics. In case of naive expectations two possibilities may essentially occur. When the dynamics has one-period orbits, the errors of expectations are uncorrelated and they can be neglected. Financial institutions make no substantial errors in forecasting the future volatility. If the flip bifurcation occurs after a change of the parameters values which characterize the financial market (particularly after the costs of diversification are reduced), naive expectations lead financial institutions to fail in forecasting their portfolio volatilities, specifically with overestimates followed by underestimates. Looking at the linear autocorrelation (ACF) of expectations errors, a clear information appears.

In fact, in case of two-period dynamics regarding the portfolio variables, the linear autocorrelation of the errors can be naturally described by an autoregressive process of order one, $AR(1)$. Let us define $\mathcal{E}_t = \tilde{\sigma}^e_{p,t} - \tilde{\sigma}_{p,t+1}$: the difference between the future portfolio volatility expected at time t and the realized one, at time $t+1$. The ACF of the errors time series is well approximated by the zero-mean $AR(1)$ process

$$\mathcal{E}_t = \rho_1 \, \mathcal{E}_{t-1} + \epsilon_t$$

where ϵ_t is a white noise and ρ_1 is the autocorrelation coefficient at the first time lag. For $\rho_1 \in (-1, 0)$, the $AR(1)$ process exhibits a linear autocorrelation characterized by a zig-zag pattern. If $\rho_1 = -1$ and there is no noise, the autocorrelation coefficients, related to the $AR(1)$ process, match exactly those related to the deterministic two-period dynamics of the naive case. Such pattern in the linear autocorrelation is evident to financial institutions which in this way will try to improve their expectations about volatility. If the naive expectations conduce to errors following an $AR(1)$ process, the banks behavior will be to base the forecast upon longer time observations. In fact, we could expect that during 2-period cycles, when the expected portfolio volatility follows a swinging dynamics that assumes consecutive large and small values, a forecasting scheme which looks at two past periods will have a stabilizing effect on the portfolio dynamics.

The economic intuition is that the longer backward observations may avoid the systematic mistakes of the naive expectations. Hence, we consider the case of *adaptive expectations*.

The following underlying assumptions are the same as the ones in the previous Chapter, particularly the exogenous systematic risk component is not considered, $\sigma_s = 0$. Taking advantage of the assets homogeneity, from an estimate of the realized portfolio volatility we can find the realized asset risk depending on the rebalancing feedback, as in (4.7).

The adaptive expectation about the volatility of a typical asset is given by

$$(\tilde{\sigma}^e_{d,t})^2 = (\tilde{\sigma}^e_{d,t-1})^2 + \omega[\, (\tilde{\sigma}_{d,t})^2 - (\tilde{\sigma}^e_{d,t-1})^2 \,] \tag{5.1}$$

where $(\tilde{\sigma}^e_{d,t})^2$ is the *expected* diversifiable risk of the asset at time t, $(\tilde{\sigma}_{d,t})^2$ is the *realized* difersifiable risk of the same asset at the same time and ω is the expectations weight factor, $0 \leq \omega \leq 1$. The quantity $(\tilde{\sigma}_{d,t})^2$ is the sum of the exogenous diversifiable risk σ^2_d plus the excess endogenous component due to the rebalancing feedback which occurs in the time interval between $t-1$ and t but it depends on the choice of portfolio variables made at time $t-1$, as in (4.7).

The expected risk of the investment asset is "adapted" in the direction of the most recently observed risk $(\tilde{\sigma}_{d,t})^2$, with expectations weight factor ω.

The expectations weight factor ω defines the memory parameter characterizing the adaptive expectations scheme. In fact, the larger is the value of ω, the larger is the contribution of the most recently observation of risk. On the contrary, decreasing ω, it follows that the largest contribution in the forecasting of the risk is given by the value of the previous expectation.

Indeed, an equivalent form for adaptive expectations is

$$(\tilde{\sigma}^e_{d,t})^2 = (1-\omega)\, (\tilde{\sigma}^e_{d,t-1})^2 + \omega\, (\tilde{\sigma}_{d,t})^2 \tag{5.2}$$

that is, expected risk at time t is a weighted average of the most recently observed risk and the most recently expected one. Naive expectations are just a special case of adaptive expectations with $\omega = 1$. Using (5.2) repeatedly, adaptive expectations can be written as a weighted sum, with geometrical declining weights, of all past observed risk:

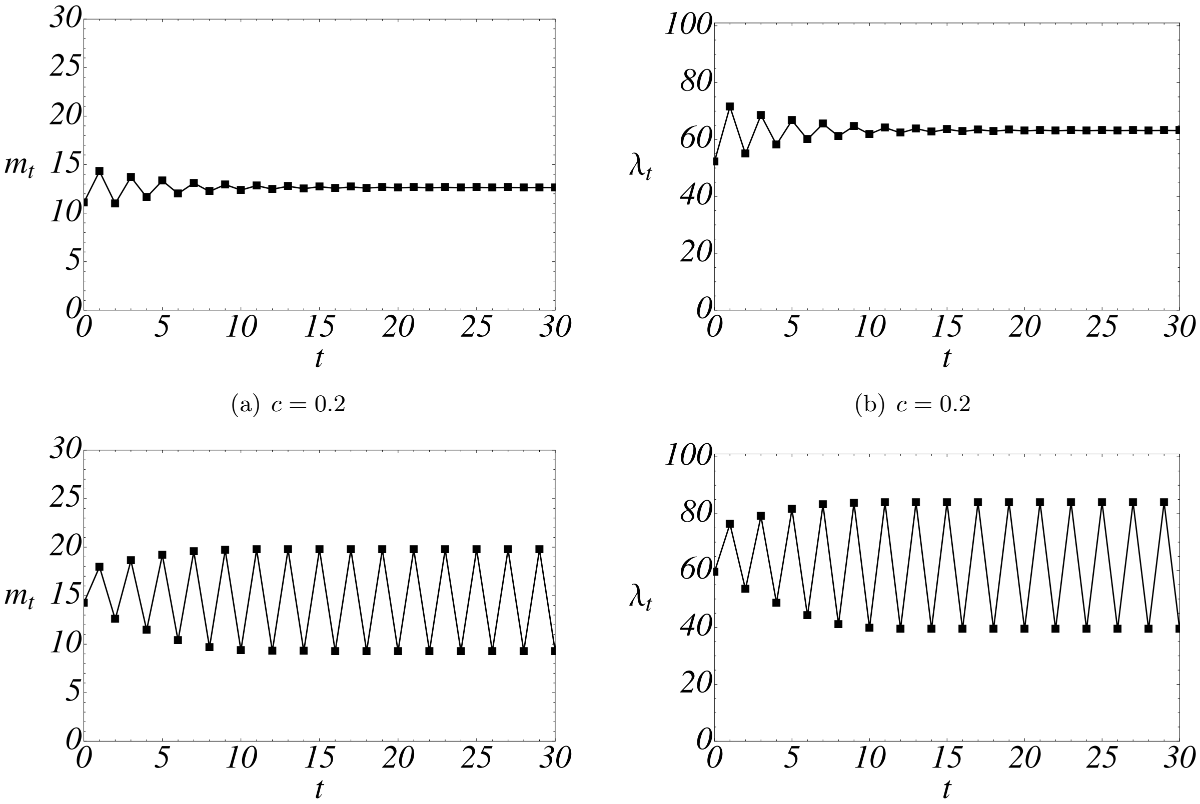
$$(\tilde{\sigma}^e_{d,t})^2 = \omega\, (\tilde{\sigma}_{d,t})^2 + \omega(1-\omega)\, (\tilde{\sigma}_{d,t-1})^2 + \omega(1-\omega)^2\, (\tilde{\sigma}_{d,t-2})^2 + \ldots = \sum_{i=0}^{\infty} \omega(1-\omega)^i\, (\tilde{\sigma}_{d,t-i})^2 \tag{5.3}$$

The version (5.3) of the (5.1) highlights that the adaptive expectation is equivalent to an *Exponential Weighted Moving Average* ($EWMA$) process for the expected asset risk as function of all past observed values of the realized risk. The weights with which the past realizations of risk are considered, depend on the memory parameter, ω.

It is important to notice that the introduction of adaptive expectations of risk is a very realistic hypothesis to study the financial market dynamics because typically, in real financial markets investors use extensions of the adaptive scheme in forecasting the risk.

The expected portfolio volatility (4.11) where $(\tilde{\sigma}^e_{d,t})^2$ is found through (5.1), together with the difference equations (4.8) (4.9) (4.10), determine the map which describes the typical dynamics of the diversification m and the leverage λ. The general considerations about the domain of definition explained in Section 4.1 and the conventions adopted in the previous Chapter remain unchanged.

Taking advantage of the absence of exogenous systematic risk component ($\sigma_s = 0$), the map



(a) $c = 0.2$ (b) $c = 0.2$

(c) $c = 0.17$ (d) $c = 0.17$

Figure 5.1: Dynamic evolution of the diversification m_t and the financial leverage λ_t for a fixed value of ω equal to 0.95 and for two "large" values of the diversification cost c. The other parameters are fixed at $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, the exogenous $\sigma_d = \frac{0.05}{1.64}$, the liquidity $\gamma = 100$.

can be rewritten in explicit form, like in the previous Chapter, and it becomes:

$$m_t = \left(\frac{\mu - r_L}{2\alpha c}\right)^2 \frac{1}{(\tilde{\sigma}^e_{d,t})^2} \quad (5.4)$$

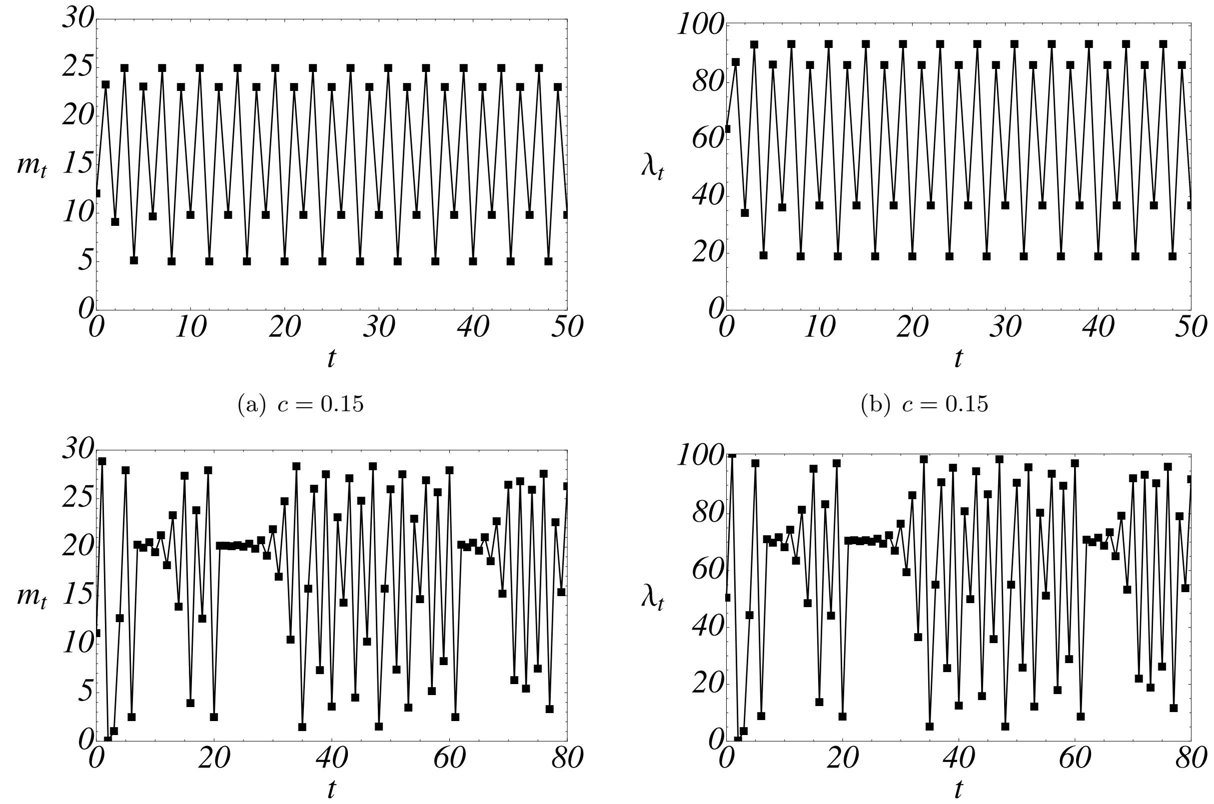
$$\lambda_t = \frac{\mu - r_L}{2\alpha^2 c} \frac{1}{(\tilde{\sigma}^e_{d,t})^2} \quad (5.5)$$

$$(\tilde{\sigma}^e_{d,t})^2 = (\tilde{\sigma}^e_{d,t-1})^2 + \omega[\, (\tilde{\sigma}_{d,t})^2 - (\tilde{\sigma}^e_{d,t-1})^2 \,] \quad (5.6)$$

Like before, the difference equations (5.4) (5.5) may be reduced to only one equation containing all information about m and λ, through a linear coordinate change. Under adaptive expectations there are many possibilities concerning the portfolio dynamics, depending on the eigenvalues of the map's Jacobian.

- When the absolute values of all eigenvalues are smaller than one, from the Hartman-Grobman theorem, an hyperbolic equilibrium point exists and as a consequence the portfolio variables converge to a stable configuration (m^* , λ^*), see the panels (a) and (b) of Figure (5.1).

- When an eigenvalue crosses the critical value -1, a period-doubling bifurcation occurs. If



(a) $c = 0.15$ (b) $c = 0.15$

(c) $c = 0.14$ (d) $c = 0.14$

Figure 5.2: Dynamic evolution of the diversification m_t and the financial leverage λ_t for a fixed value of ω equal to 0.95 and for two "small" values of the diversification cost c. The other parameters are fixed at $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, the exogenous $\sigma_d = \frac{0.05}{1.64}$, the liquidity $\gamma = 100$.

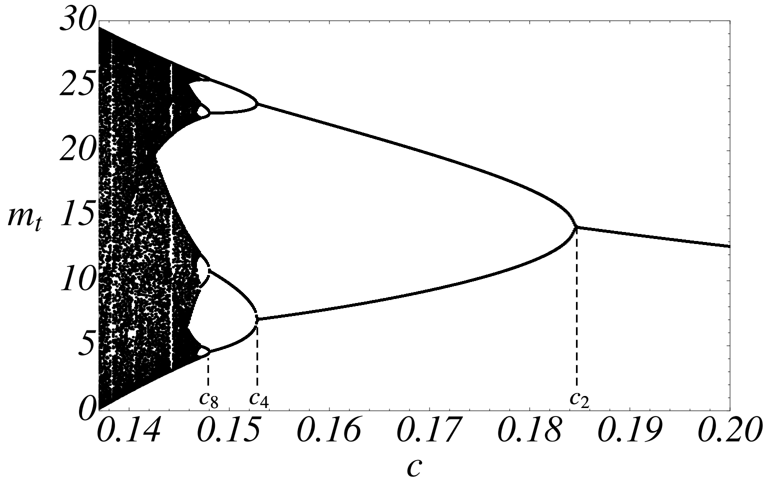
the diversification costs decrease, the stable 2-cycles are created, see the panels (c) and (d) of Figure (5.1).

- When the diversification costs decrease further, the *period-doubling cascade* to **chaos** follows.

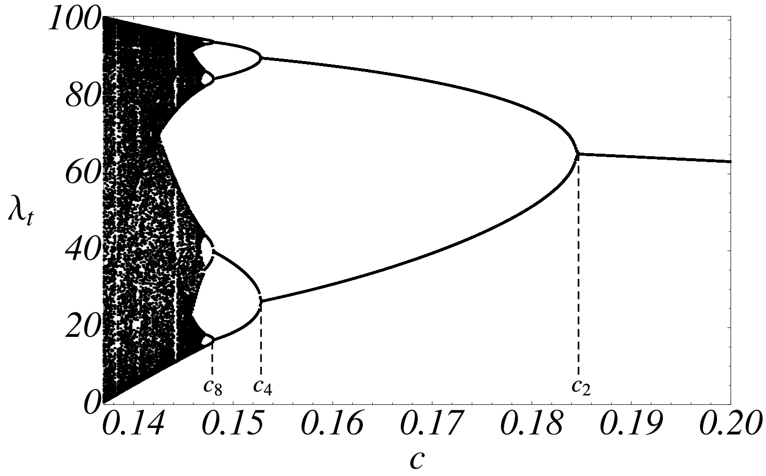
Figure (5.2) displays a 4-period orbit (panels (a) and (b)) and a chaotic one (panels (c) and (d)). The second one is related to a smaller value of diversification costs when compared to the first one. The period-doubling cascade takes place decreasing the parameter c. Both orbits are obtained for a value of the expectations weight factor ω close to one.

Figure (5.3) shows the bifurcation diagrams for the diversification m and the leverage λ. The bifurcation diagram illustrates the long run dynamics after that a time transient has passed. The value of ω is fixed close to one and the model parameter c changes accordingly. All the other parameters in Figure (5.3) are set equal to those used in Figure (4.3) in case of the naive expectations.

Under adaptive expectations the portfolio dynamics is quantitatively more stable than under naive expectations. Although, for a fixed value of diversification costs, oscillations of the portfolio variables occur, adaptive expectations have a stabilizing effect upon portfolio in the sense that the amplitude of oscillations decreases compared to the naive case. For example, we look at a



(a)



(b)

Figure 5.3: Bifurcation diagrams of the diversification m_t and the financial leverage λ_t as function of the parameter c. We have fixed the value of ω equal to 0.95. The other parameters are fixed at $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, the exogenous $\sigma_d = \frac{0.05}{1.64}$, the liquidity $\gamma = 100$.

value of c equal to 0.16. In panel (b) of Figure (4.3) the cycle amplitude of the financial leverage is slightly smaller than the dimension of entire domain of λ (in the specific case the economic domain of λ is $[1, 101]$). Instead, if we look at panel (b) of Figure (5.3) we can noticed that under adaptive expectations the cycle amplitude of financial leverage at $c = 0.16$ is smaller than the amplitude related to the naive case. This happens for all values of diversification costs. Similarly for the degree of diversification m.

In the previous Chapter, we have defined c^* as the value of diversification costs at which the return dynamics become non-stationary. This is the consequence of exploring values outside the economic domain. In fact, outside the domain of definition the endogenous component dynamics, which describes the rebalancing feedback, is characterized by an eigenvalue larger than 1. Under adaptive expectations a second stabilizing effect follows. The occurrence of non-stationarity of return dynamics is verified for a c-value smaller than the naive case. In fact, in Figure (4.3) we have found $c^* \approx 0.157$, while in Figure (5.3) we find $c^* \approx 0.137$.

Finally, similarly to the latter stabilizing effect, a third one is present. In case of the adaptive expectations, the occurrence of the first flip bifurcation takes place for a value of diversification costs shifted toward smaller values compared to naive criticality. If we indicate as c_2 the value of the parameter c at which the first period-doubling bifurcation occurs, c_2 strictly depends on the adaptive strategy adopted by financial institutions which is identified by their choice about the weight factor ω. In this Chapter, we adopt the convention of associating the subscripts that follow the Sharkovsky ordering (see Section 2.2 of Chapter 2) to the model parameters, to indicate the values of c and ω at which period-doubling bifurcations occur, *e.g.* see Figure (5.3). The relation between c_2 and ω is shown in Figure (5.4). It highlights the crucial aspect that

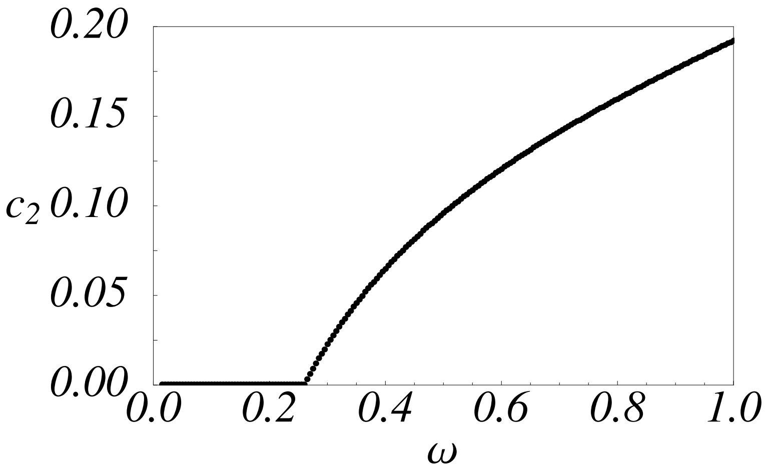


Figure 5.4: The relation between the value of the parameter c at which the first period-doubling bifurcation occurs, c_2, and the expectations weight factor, ω. The other parameters are fixed at $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, $\sigma_d = \frac{0.05}{1.64}$, $\gamma = 100$.

a cautious adapting (when the expectations weight factor is small) shifts away the criticalities toward smaller values of diversification costs. Also when model parameters change values, a numerical analysis suggests that the qualitative features of Figure (5.4) remain unchanged. Especially, there exist values of ω different from zero for which the portfolio dynamics always converges to an equilibrium state. Especially, there are always values of ω close to 0 for which financial cycles never appear when diversification costs decrease (in Figure (5.4) this happens in the interval $(0, \sim 0.25]$ of ω-values).

The limit $\omega \to 0$ corresponds to non-adapting the expected risk with the most recently observed one, see (5.1). It can be interpreted as the specular case as opposed to the naive expectations. We can refer to it as *rational expectations*. In this ideal limit, financial institutions will not make errors about the volatility forecast and the financial system will have a unique rational expectations equilibrium (REE), see Section 4.2. From a mathematical point of view, $\omega = 0$ indicates the transition from a dynamical system approach to an equilibrium model. In fact, if $\omega = 0$, the introduction of a new equation for $\tilde{\sigma}_d^e$ is necessary to close the system of equations (5.4) (5.4). The solution of the new equation defines the expected risk at time t. This means that financial institutions know the law of motion of the complex market dynamics and they trivially realize a self-fulfilling prophecy making no systematic mistakes. This approach provides the equilibrium solution (REE) but it excludes the bounded rationality evidences of financial markets. Below, we will focus on the expectations weight factor ω, that is the memory of adaptive expectations scheme, as the continuous parameter of the map and we will investigate the dynamical features of the system. In fact, although the adaptive expectations have stabilizing effects, at the same time the nature of the system dynamics is more complex and any information, useful to financial institutions to improve their forecasts, can be more difficult to detect.

Finally, for adaptive expectations the qualitative features related to the linearized map remain unchanged. In fact, the Jacobian of the map described by the equations (5.4) (5.4) (5.6) is characterized by two eigenvalues equal to zero and a negative one. The qualitative aspects are exactly the same as shown in panel (b) of Figure (4.8). The new zero eigenvalue is related to the equation (5.6). It is equal to zero because in principle knowing the orbits of m and λ, the orbit of $(\tilde{\sigma}_d)^2$ can be found through (5.3), once that the value of ω is know.

Also the qualitative features of the first period-doubling bifurcation remain the same as the naive case, examined in the previous Chapter. We do not repeat the results about the topological features of the flip bifurcations, the bifurcation amplitude and the convergence time.

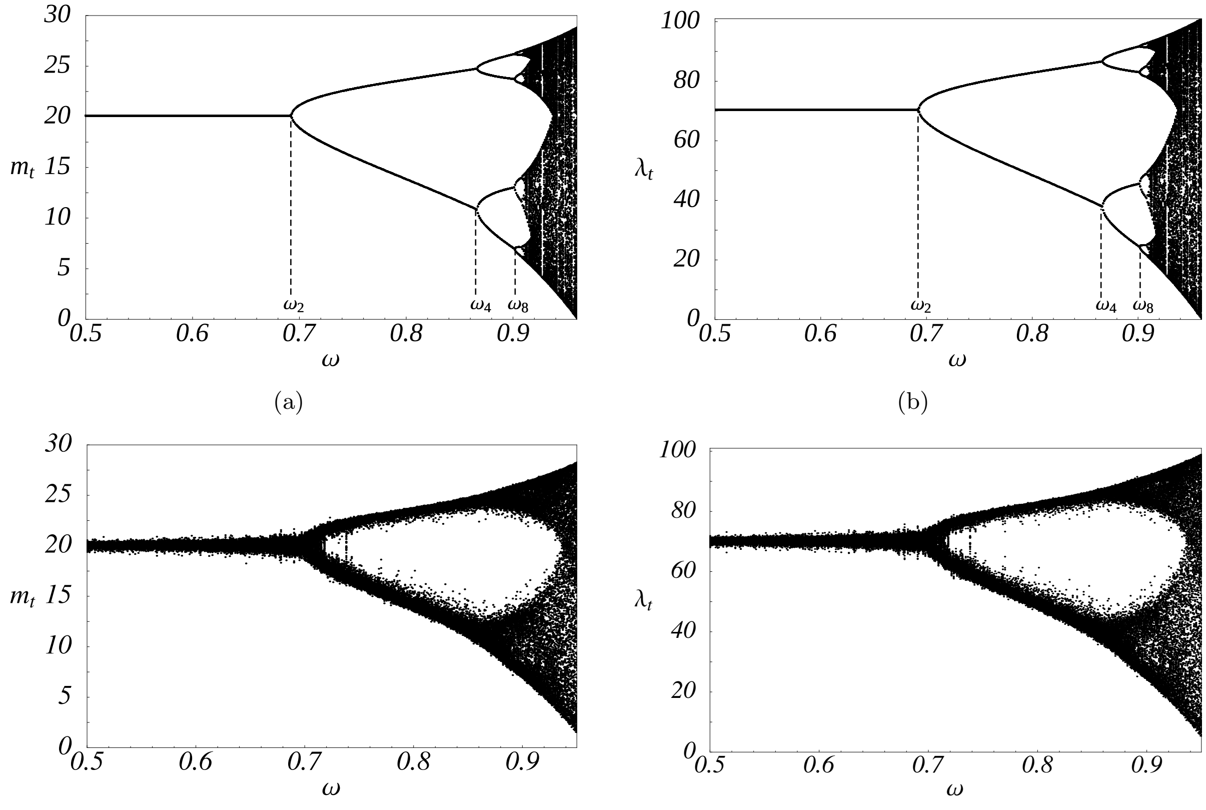
5.2 The role of memory in systemic financial stability

In this Section we will investigate the consequences of adaptive expectations on financial market dynamics when the weight factor ω, that is the memory of adaptive expectations scheme, varies once that the value of diversification costs is fixed. In this framework, the degree of rationality characterizing financial institutions is related to the weight factor ω. The latter univocally defines the typical dynamical outcomes. As seen in the previous Section, chaotic orbits may arise. We will focus especially on the period-doubling cascade to chaos. The latter is a true deterministic chaos because the dynamical system displays sensitive dependence on initial conditions, see Chapter 2. This latter property will be evident below when we will look at the Largest Lyapunov Exponent (LLE) of the map.

In our model, through the endogenous feedback, due to the balance sheet dynamics characterized by the portfolio rebalancing in order to maintain in a time period the desired target leverage, financial institutions may affect the portfolio dynamics in very different ways depending on the adaptive scheme adopted. When they are cautious in updating their expectations about risk through the most recently observed one, the portfolio dynamics converges to a steady state which corresponds to the hyperbolic fixed point of the map described by difference equations (5.4) (5.5) (5.6). The convergence of the portfolio dynamics corresponds to the stability of financial market dynamics because the portfolios of financial institutions are diversified and leveraged without affecting crucially the risk of investment assets through the positive rebalancing feedback due to their target leverage strategy. The positive feedback is in equilibrium with the portfolio dynamics.

Instead, increasing the value of the weight factor ω and adapting the expectations of risk even more in the direction of the latest realized one, attractors of period n for the dynamics appear. As a consequence, the financial stability is lost.

The n index follows the Sharkovsky ordering, *i.e.* when the value of ω increases, the 2-period orbit appears first, then the 4-period one and so on in a period-doubling way. In Figure (5.5) the value of weight factor at which the nth period-doubling bifurcation occurs is indicated for the first three criticalities.



(a) (b)

(c) $\sigma_{noise} = \frac{\sigma_d}{10}$ (d) $\sigma_{noise} = \frac{\sigma_d}{10}$

Figure 5.5: Bifurcation diagrams of the diversification m_t and the financial leverage λ_t for a fixed value of c equal to 0.14, varying the value of the expectations weight factor ω. The other parameters are fixed at $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, the exogenous $\sigma_d = \frac{0.05}{1.64}$, the liquidity $\gamma = 100$.

In principle, under adaptive expectations the rebalancing impact on volatility may trigger a very complex portfolio dynamics due to the intrinsic non-linearity which characterizes the procyclical feedback effects. Furthermore, beyond the specific model we consider, in some sense we may expect that in a illiquid financial market, the combined effect of an endogenous positive feedback together with the bounded rationality of agents in forming their expectations about the risk, has intrinsically the potential to create financial cycles best described by period-doubling bifurcations in a discrete dynamical approach. In this context, the appearance of a chaotic region naturally emerges.

In the chaotic region shown in the bifurcation diagram of Figure (5.5), two different chaotic attractors of the portfolio dynamics can be identified. We refer to ω_∞ as the value of the weight factor for which the period-doubling bifurcations have doubled the orbital period ad infinitum, so that the dynamical behavior is not longer periodic. For a value of ω larger than ω_∞, first of all we notice a two-piece chaotic attractor, *i.e.* an attractor having as support two disjoint subsets of the economic domain. When ω increases further, the dynamical behavior is characterized by an one-piece chaotic attractor, *i.e.* an attractors that is dense in one subset of the economic domain. As we will see below, there is one important detectable dynamical feature which distinguishes the first attractor from the second one. It is related to the fact that the aperiodic evolution of the portfolio dynamics on the two-piece attractor is characterized by regular jumps from one piece to the other. The same is not valid for the one-piece chaotic attractor.

When $\omega \to 1$ the dimension of the support for the one-piece chaotic attractor increases. There exists a value of $\omega = \omega^*$ for which this dimension is equal to the dimension of the economic domain. In Figure (5.5) ω^* is equal to 0.96. Particularly, the portfolio variable λ explores all possible states in the economic domain during the dynamical evolution. This means that the economic system oscillations reach their maximal amplitude. If ω increases further ($\omega > \omega^*$), the leverage λ can assume values larger than the upper bound of economic domain. When $\lambda > \gamma + 1$, see (4.12), the return processes of investment assets are non-stationary. In this case, there occurs a transition from stability to instability for the returns dynamics. When it happens, the relations (4.3) (4.4) do not describe the variance and the covariance of the endogenous components anymore, see Section 4.1. When $\omega > \omega^*$, our model fails in describing the dynamical behavior of system. The value of ω^* strictly depends on the value of the diversification costs parameter c. Referring to Figure (5.4), if the parameter c is larger than the value of criticality $c_2|_{\omega=1}$, in Figure $c_2|_{\omega=1} \approx 0.19$, period doubling bifurcations do not occur for any values of ω. As a consequence, chaotic orbits do not occur too.

In previous Chapter, we have defined c^* as the value of c at which the returns dynamics under naive expectations becomes non-stationary. So, when $c > c^*$, under adaptive expectations only two-period orbits can occur depending on the value of the weight factor ω. Furthermore, if $\omega \neq 1$, the amplitude of a two-period orbit is smaller than the naive case. Finally, if $c < c^*$ a period-doubling cascade to chaos occurs when $\omega \to 1$ and there exists ω^* close to one.

It is important to notice the role of noise in the dynamical evolution of portfolio variables. The panels (c) and (d) of Figure (5.5) show the bifurcation diagrams for the portfolio variables in the case that a Gaussian fluctuation is introduced as we have done in the previous Chapter. Although the detailed bifurcation structure has disappeared in the presence of noise, the regions of the stable steady state (for small values of ω), a stable 2-cycle and the one-piece chaotic attractor, are clearly visible[1]. The dynamical features we have just highlighted, persist approximately also in case of the stochastic map.

In the deterministic evolution, we can recognize a property arising from the universality of the period-doubling bifurcations. We can notice, in Figure (5.5), the values of the weight factor for which the doubling of the period occurs. We have marked only three transitions, ω_2, ω_4 and ω_8. In principle and in absence of numerical computing problems, in the period-doubling cascade to chaos, there is some interval of ω-values in which the system's behavior is periodic. Any interval of ω-values is characterized by the fact that the orbital period of the system is fixed. In the direction of increasing ω, at the transition point from one interval to the following

[1]The noisy bifurcation diagram has been created by adding a normally distributed random noise term with standard deviation $\sigma_{noise} = \frac{\sigma_d}{10}$ to the realized variance of investment return at time $t-1$, see (4.7), which is equivalent to adding small shocks to the endogenous component of the risk.

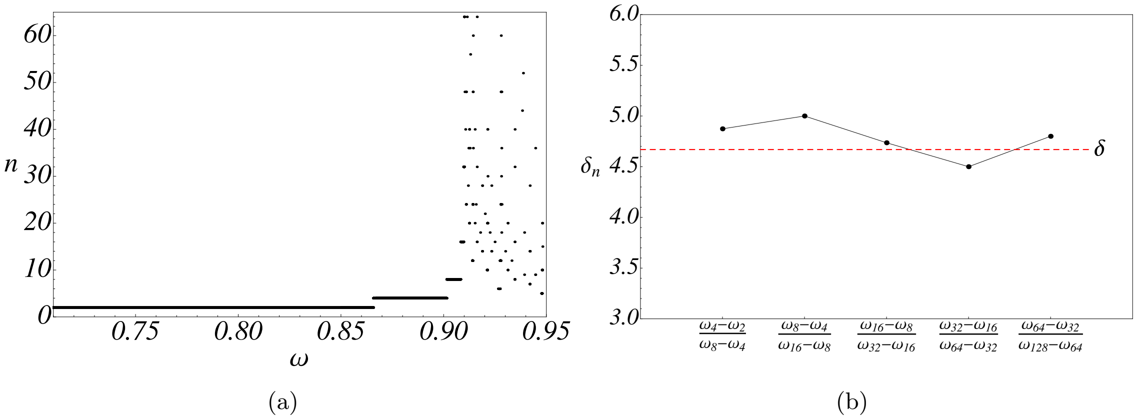


Figure 5.6: The windows of ω-values corresponding to a steady state of period n, panel (a). The values of the ratio between two consecutive periodic windows, δ_n, panel (b). The red dashed line corresponds to the value of the Feigenbaum constant. The error bar for any point, related to the numerical precision in defining periodic windows, is on the order of 10^{-4} (in the plot the error bars are hidden by black points). The used parameters are fixed at $c = 0.14$, $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, the exogenous $\sigma_d = \frac{0.05}{1.64}$, the liquidity $\gamma = 100$.

one, the time period necessary to close a cyclical orbit is double compared with the precedent orbital period. This mechanism recurs continuously until, at a certain value of the parameter ω, the period has doubled ad infinitum. Hence, the dynamics becomes aperiodic and chaotic. In the region of periodicity of the system's behavior, it is interesting to know how large is the ω-interval in which the system dynamics is characterized by a specific orbital period and what is the ratio between two consecutive ω-intervals. Beyond the mathematical point of view, the last information can be useful to recognize the conditions under which the dynamics of the economic system evolves.

In the panel (a) of Figure (5.6) the index n associated with the orbit of nth-period is shown as function of ω. In the interval of values of the weight factor for which the dynamics is chaotic, we can recognize some periodic windows partially hidden in the noise. In the steady region this outcome can be used to find the ratio between consecutive periodic windows. Let us define δ_n as

$$\delta_n = \frac{\omega_{n+1} - \omega_n}{\omega_{n+2} - \omega_{n+1}} \tag{5.7}$$

where the sum for the index n is interpreted according to the Sharkovsky ordering. In the theory of bifurcations, see Chapter 2, δ_n asymptotically approaches the constant value $\delta = 4.6692016...$, the so-called *Feigenbaum constant*. In the panel (b) of Figure (5.6), we show the outcome for our model computing δ_n up to the steady period 128. The results are consistent with the theory. Knowing the amplitude of only one periodic window and knowing that $\delta_n \approx \delta$, boundedly rational agents might approximately reconstruct the structure of the bifurcations and then, exploiting the latter, they may improve their forecast choosing the value of ω which allows the convergence to the one-period steady state.

Obviously, this learning process is more theoretical than realistic. However it highlights a crucial feature: in the periodic region this type of information can be extrapolated by the portfolio dynamics while in the chaotic regime the learning procedure just explained loses importance

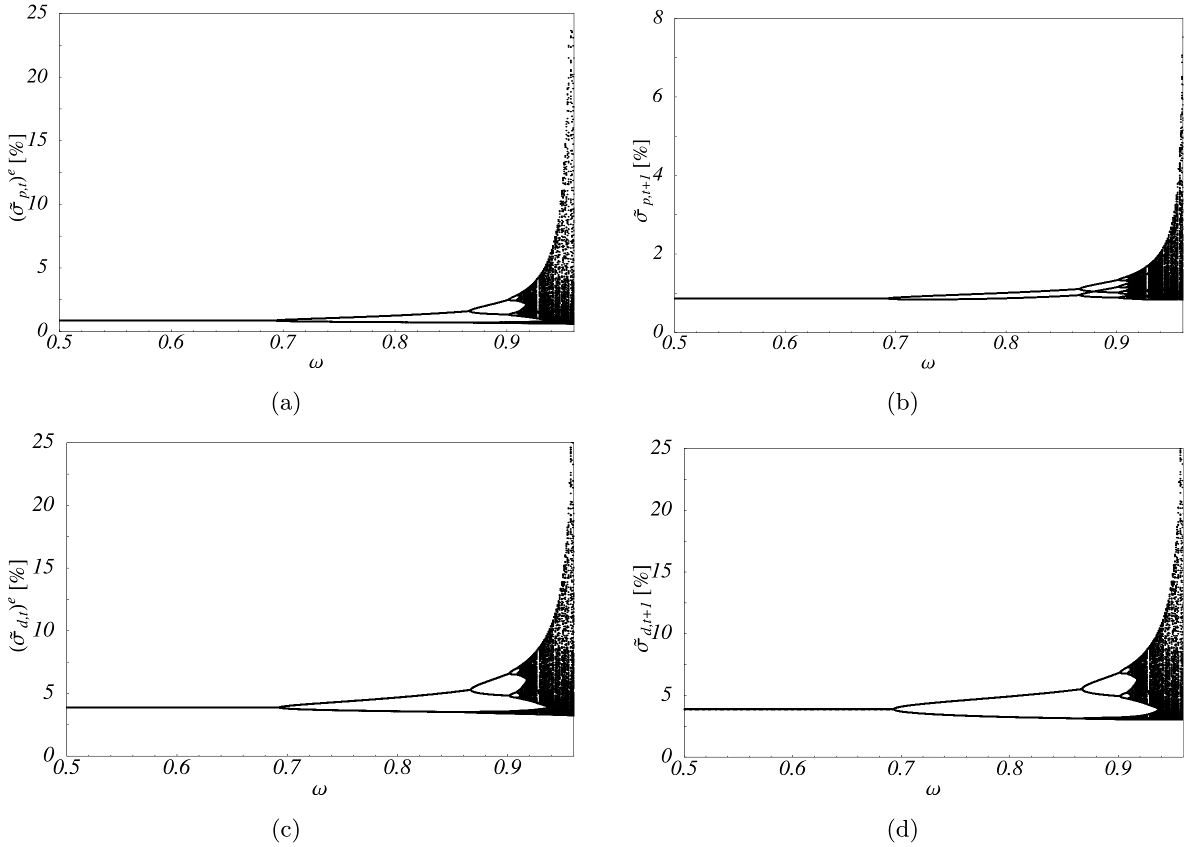


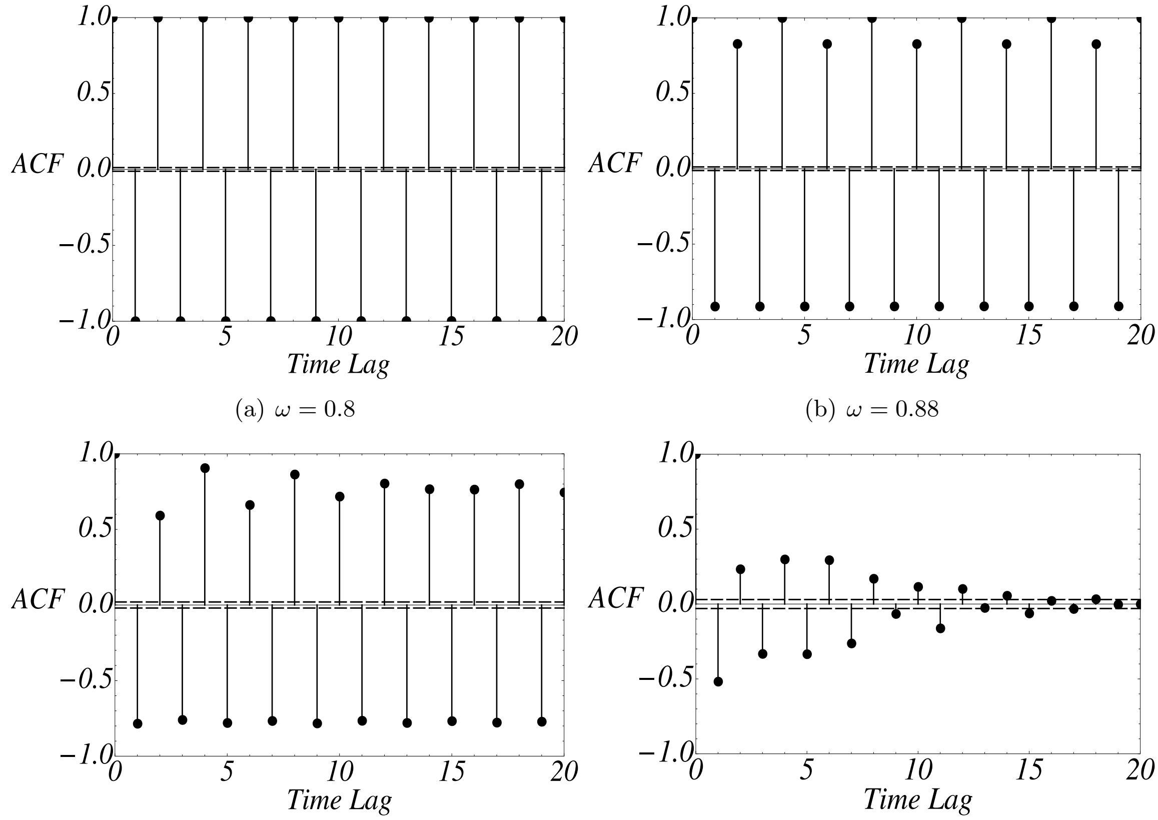
Figure 5.7: Bifurcation diagrams, respectively, for the expected portfolio volatility and the realized one, the expected asset volatility and the realized one. We have fixed the value of c equal to 1.4 and we change the value of the expectations weight factor ω. The other parameters are fixed at $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, the exogenous $\sigma_d = \frac{0.05}{1.64}$, the liquidity $\gamma = 100$.

and the information is lost. Now we will show a more realistic way to extract information about dynamics from which financial institutions can really take advantage.

Figure (5.7) shows the bifurcation diagrams for the expected portfolio volatility and the realized one, respectively $\tilde{\sigma}_p^e$ and $\tilde{\sigma}_p$, and for the expected asset diversifiable volatility and the realized one, respectively $\tilde{\sigma}_d^e$ and $\tilde{\sigma}_d$. These bifurcations diagrams display the asymptotic dynamics of the four volatilities in correspondence to the deterministic dynamical evolution of the diversification m and the leverage λ, see Figure (5.5). If we look at the realized portfolio volatility and at the expected one, it is evident that when $\omega \to 1$ the chaotic $\tilde{\sigma}_p$ oscillations have moderate amplitude compared to the large oscillations occurring for $\tilde{\sigma}_p^e$. From this we can expect that financial institutions make mistakes in forecasting their portfolio volatility also under adaptive expectations.

When the portfolio dynamics converges to the one-period steady state, $\omega < \omega_2$, financial institutions make no systematic mistakes and the forecasting errors are statistically equal to zero and uncorrelated.

On the contrary, when the system behavior is periodic or chaotic, financial institutions make systematic mistakes. We will investigate the features of the forecasting errors about the portfolio volatility that financial institutions make when they adopt *static* adaptive expectations. With



(a) $\omega = 0.8$ (b) $\omega = 0.88$

(c) $\omega = 0.92$ (d) $\omega = 0.94$

Figure 5.8: Sample autocorrelation function (ACF) of the forecasting errors about the portfolio volatility, $\mathcal{E}_t = \tilde{\sigma}^e_{p,t} - \tilde{\sigma}_{p,t+1}$, for different values of the expectations weight factor ω. To test the level of significance, the upper and lower bounds are equal to $\pm \frac{z_{1-\frac{\alpha}{2}}}{\sqrt{N}}$, N is the time series length, $z_{1-\frac{\alpha}{2}}$ is the quantile function of the normal distribution and α is the significance level. At 5% significance level (the dashed lines) the bounds are $\frac{1.96\ \sigma_{data}}{\sqrt{N}}$, where σ_{data} is the standard deviation of the data. The other parameters are fixed at $c = 0.14$, $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, the exogenous $\sigma_d = \frac{0.05}{1.64}$, the liquidity $\gamma = 100$.

static we mean the situation wherein banks maintain always the same adaptive scheme, *i.e.* when the expectations weight factor ω is constant in time.

Figure (5.8) illustrates the ACF of forecasting errors about portfolio volatility, $\tilde{\sigma}^e_{p,t} - \tilde{\sigma}_{p,t+1}$. Four typical dynamical outcomes are been considered, each of these related to a different value of ω: the two-period orbit ($\omega = 0.8$), the four-period orbit ($\omega = 0.88$), the dynamics on two-piece chaotic attractor ($\omega = 0.92$) and finally the dynamics on one-piece chaotic attractor ($\omega = 0.94$).

For $\omega = 0.8$ the portfolio dynamics converges asymptotically to a 2-period orbit and a similar situation to the naive case is obtained. Consecutive underestimations and overestimations of the risk associated with the portfolio are evident in the ACF function, exhibiting an exact zig-zag pattern between -1 at odd time lags and $+1$ at even time lags. For $\omega = 0.88$, in case of the 4-period orbit, an ordered structure appears in the ACF, which shows a pattern oscillating from a value close to -1 to two other values close to $+1$. When the dynamical system evolves in the two-piece chaotic attractor, the ACF function exhibits a significant zig-zag pattern. Even when the volatility fluctuations are chaotic, we may find a clear regularity in historical data. Financial institutions may use this information to avoid future mistakes. Instead, there are less regularities when the volatility fluctuations are on the one-piece chaotic attractor. Only for the first lags there is a significative autocorrelation showing a zig-zag pattern even though characterized by a smaller amplitude compared to the previous outcomes. If we consider that in presence of noise, the information could be hidden, for boundedly rational agents it would be hard to improve their forecasts using past errors.

It is important to notice that the period-doubling cascade to chaos corresponds to a progressive loss of the information which can be extracted from the past performances about risk forecasting.

5.3 Mathematical aspects of adaptive expectations

In this Section, we will focus on some mathematical aspects of the dynamical system describing the portfolio evolution when financial institutions use adaptive expectations in the forecasting of the risk.

From a mathematical point of view, the dynamical system with adaptive expectations is one-dimensional. We will highlight some qualitative features of the map which describes the one-dimensional system. Subsequently, we will focus on the chaotic behavior exhibited by dynamical system and we will analyze some ergodic quantities which characterize the chaotic behavior.

5.3.1 The dynamical system with adaptive expectations

In the economic domain, the dynamical system is described by the difference equations (5.4) (5.5) (5.6). A linear relation (4.15) relates m_t and λ_t.

$$m_t = \frac{\mu - r_L}{2c} \lambda_t$$

As a consequence, starting from difference equations (5.4) (5.5) we can obtain only one difference equation substituting, for example, m_t as a function of λ_t. Let us define $s_t \equiv \frac{(\tilde{\sigma}^e_{d,t})^2}{\sigma_d^2}$, the map based on difference equations (5.4) (5.5) (5.6) is mathematically equivalent to the following map

$$\begin{cases} \lambda_t & = \frac{\mu - r_L}{2c\,\alpha^2\,\sigma_d^2}\ \frac{1}{(1-\omega)s_{t-1} + \omega\,\Sigma(\lambda_{t-1})} \\ \\ s_t & = (1-\omega)s_{t-1} + \omega\,\Sigma(\lambda_{t-1}) \end{cases} \tag{5.8}$$

where the mathematical shape of $\Sigma(\lambda)$ is defined as

$$\Sigma(\lambda) =$$

$$1 + \frac{1}{\left(\frac{M-1}{M}\right)^2 \left(\frac{\lambda}{\lambda-1}\right)^2 \frac{\gamma^2}{\left(\frac{\lambda}{M} - \frac{1}{A}\right)^2} - 1} \left[1 + \frac{1}{1 - \frac{(\lambda-1)^2}{\gamma^2}} \frac{A\lambda}{M\left(\frac{A}{M}\lambda - 1\right)^2} \left(A^2\lambda^2 \left(1 - \frac{2}{M}\right) + 2\frac{A}{M}\lambda - 1 \right) \right]$$

where for notational simplicity we have defined $A = \frac{\mu - r_L}{2c}$.

Finally, noticing that

$$\lambda_t = \frac{A}{\alpha^2 \sigma_d^2} \frac{1}{s_t} \quad \longrightarrow \quad s_{t-1} = \frac{A}{\alpha^2 \sigma_d^2} \frac{1}{\lambda_{t-1}}$$

we can substitute s_{t-1} in the first difference equation of the map (5.8), thus obtaining an one-dimensional dynamical system f which corresponds mathematically to the dynamical system described by difference equations (5.4) (5.5) (5.6).

A shifting and a rescaling of the variable $\lambda \to u = \frac{\lambda - 1}{\gamma}$ can be made with the intent that the economic domain for λ coincides with the unit interval.

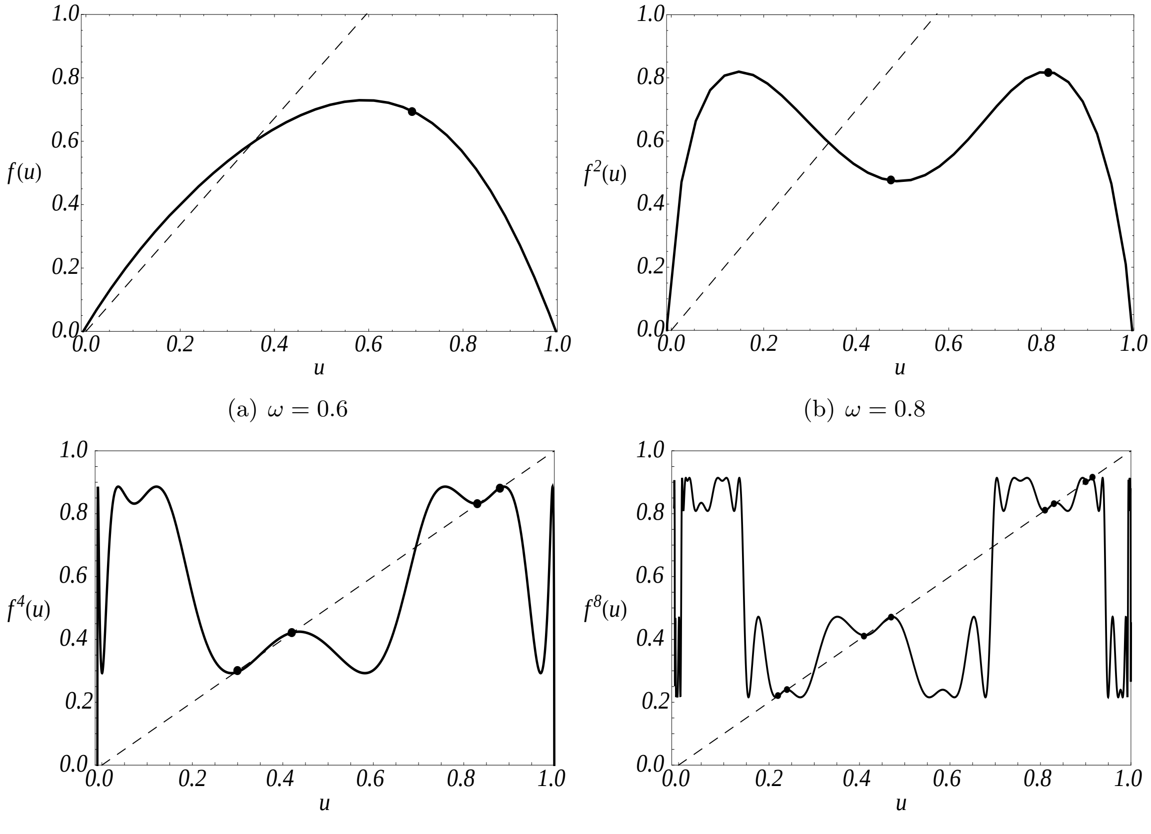


Figure 5.9: The mathematical shape of f (panel (a)), f^2 (panel (b)), f^4 (panel (c)), and f^8 (panel (d)). The values of ω are chosen such that the intersection points marked in black represent the periodic orbits of the dynamical system, cfr Figure (5.5). The model parameters are fixed at $c = 0.14$, $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, the exogenous $\sigma_d = \frac{0.05}{1.64}$, the liquidity $\gamma = 100$.

Symbolically, the one-dimensional dynamical system described by the map f, is

$$u_t = f(u_{t-1}) \tag{5.9}$$

where $u \in [0,\ 1]$, that is the unit interval is positively invariant under f.

In Figure (5.9) we show the mathematical shape of map f, see panel (a). Particularly, f is characterized by a maximum and it decreases to zero at the extrema of interval. In other words, from a qualitative point of view, the mathematical shape of f is similar to the mathematical shape of the logistic map [37]. In fact, both of them show the same universal features , like the Feigenbaum scenario, see Figures (5.3) (5.5) (5.6).

The dynamics arising from the map has some *universal* aspects due to its analytical shape, which is characterized by a maximum and it goes to zero at the extrema of interval, rather than the specific values which it assumes in the domain. In fact, "the ability of f to have complex behaviors is precisely the consequence of its double-valued inverse, which is in turn a reflection of its possession of an extremum", [45]. With "double-valued inverse" we mean the extrema of f^2 that can be find by constructing the inverse iterate of f at its maximum.

In other panels of Figure (5.9), we show the mathematical shape of the n-iterated map f^n, in particular f^2, f^4, and f^8, for different values of the parameter ω. We have chosen the ω-values such that the intersection points marked in black between the identity function (the dotted line) and the n-th iteration of the map, represent the orbits of period n that we have found in Figure (5.5). Indeed, as we have noticed in the previous Chapter, a system's orbit of period equal to n is represented by the solution of the following equation

$$u^* = f^n(u^*)$$

where f depends on some model parameters as c and ω. The stability of the solution depends on the values of these parameters.

5.3.2 Entropy of the dynamical system

The information taken from the performances of risk forecasting, for example through the autocorrelation of errors time series, represents a way with which financial institutions can extrapolate information from system orbits. This type of information strictly depends on the periodicity of the dynamical evolution. When the system repeats itself every orbital period, clear structures appear in the correlation functions as the ACF. If we know the trajectory of one orbit in the long run dynamical behavior, all information about system can be extracted. Instead, a system with sensitive initial conditions produces *dynamical information*, see Section 2.3. With dynamical information we mean the property of exploring always new states in phase space during the dynamical evolution. In the chaotic regime all dynamical information is not accessible because the knowledge of a trajectory in a past time window does not allow the extraction of all dynamical features of the system. Since any type of correlation function (*e.g.* ACF) is characterized by decay properties when the dynamical system has sensitive dependence on initial conditions, during the chaotic evolution of the financial market dynamics, financial agents lose information about the dynamical behavior of economic system. The best way to investigate this loss of information is the study of the entropy of dynamical system.

In this section, we will focus on the chaotic regime of the dynamical system described by map f (5.9). The latter is mathematically equivalent to the map based on difference equations (5.4) (5.5) (5.6), as we have seen in the previous Section. The dynamical state of the system is represented by the variable u defined in the unit interval [0, 1].

However, in Figure (5.10) we show the attractors of dynamics of u and $(\tilde{\sigma}^e_{d,t})^2$, in order to stress an economic aspect of dynamical system in the chaotic regime. An attractor is simply the attracting set in which dynamical variables accumulate during the long-run evolution of dynamical system. In the panel (a) of Figure (5.10) we show the attractor for the dynamical variables u_t and $(\tilde{\sigma}^e_{d,t})^2$. It highlights that the values of u_t near to 1 are associated with small values of $(\tilde{\sigma}^e_{d,t})^2$. Vice-versa when u_t tends to zero, we find higher values of $(\tilde{\sigma}^e_{d,t})^2$. This means that also in the chaotic regime, aggressive configurations of portfolio (high leverage and large diversification) correspond to forecasts of low risk, while cautious configurations of portfolio (small leverage and small diversification) correspond to forecasts of high risk. The panel (b) of Figure (5.10) shows the attractor for the realized portfolio volatility $\tilde{\sigma}_p$. This attractor shows

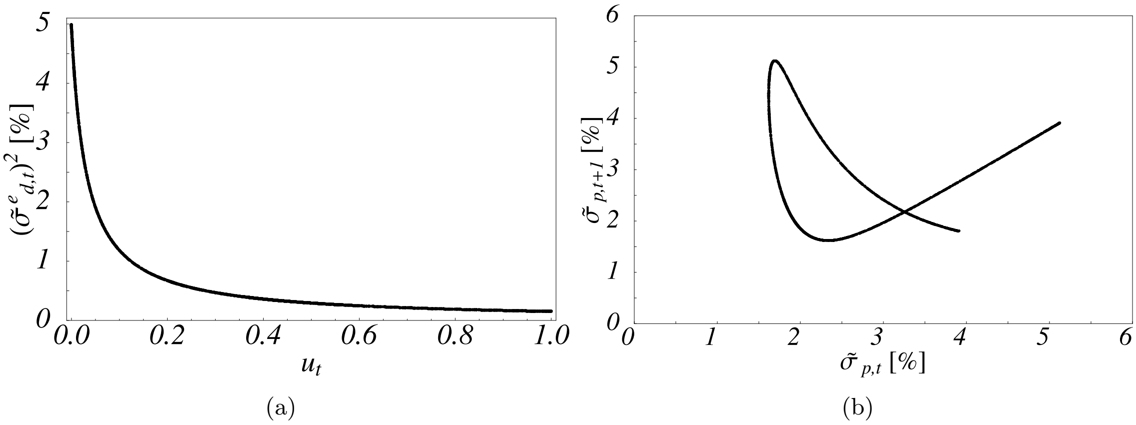


Figure 5.10: Attractors of dynamics. The attractor for the dynamical variables u_t and $\tilde{\sigma}^e_d$ in panel (a) while the attractor for the realized portfolio volatility, $\tilde{\sigma}_{p,t}$, in panel (b). Both of them are obtained with 5000 iterations. The value of c is chosen equal to 0.14 and the value of ω is 0.94. For different values of c and ω the qualitative aspects of attractors remain unchanged. The other parameters are fixed at $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, the exogenous $\sigma_d = \frac{0.05}{1.64}$, the liquidity $\gamma = 100$.

the correspondence between realized portfolio volatilities in two successive times, t and $t+1$. When we have studied in the previous Chapter the dynamics of the realized portfolio volatility, we have noticed that $\tilde{\sigma}_{p,t}$ is a conserved quantity for the map under naive expectations. Instead, under adaptive expectations the portfolio volatility is not a conserved quantity anymore and its dynamics is described by an attractor.

In the chaotic regime, the attractors of the dynamics of portfolio variables are *strange* (for a definition see Section 2.3 of Chapter 2). The strangeness of an attractor is defined by its fractal dimensionality, commonly named Hausdorff dimension. For computational reasons, we use another measure of the dimensionality, the so-called correlation dimension, introduced by *Grassberger and Procaccia*, see Section 2.3 . They[61] found that the correlation dimension is in several cases very close to the Hausdorff dimension.

The underlying idea is the following. We start with a time series of the dynamical variable u, $\{u_t\}_{t=0,\dots,N}$. We can obtain the vector $\mathbf{x}_i = \{u_{i+1}, \dots, u_{i+d}\}$, it is a sample characterized by a dimension equal to d. In this way, we obtain a series of samples, $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{N-d}$. The dynamical variable u is defined in the unit interval by definition, so every sample of data is defined in $[0, 1]^d$. Through this series of samples, we can obtained a numerical approximation of the correlation integral (2.41):

$$C^d(r) = \frac{1}{N^2} \sum_{i,j=0}^{N-d} \theta(r - ||\mathbf{x}_i - \mathbf{x}_j||_\infty) \tag{5.10}$$

where θ is the Heaviside function, $||...||_\infty$ is the norm defined as

$$||\mathbf{x}' - \mathbf{x}||_\infty = \mathrm{Max}_q |u'_q - u_q| \,, \qquad q = 1, \dots, d$$

and r is simply a parameter defining the numerical approximation of the correlation integral, (5.10).

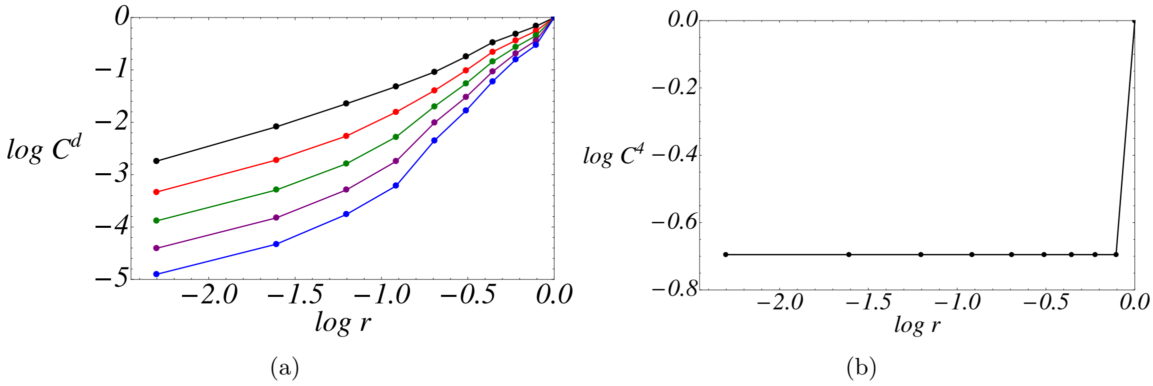


Figure 5.11: The relation between $\log C^d(r)$ and $\log r$ when $c = 0.14$. In the panel (a) $\omega = 0.94$ and in the panel (b) $\omega = 0.8$. The number of iterations is equal to 5000. Let be d the dimension of the vector $\mathbf{x}_i$, in panel (a) the points marked in black are obtained with $d = 4$, those marked in red with $d = 6$, those marked in green with $d = 8$, those marked in purple with $d = 10$ and those marked in blue with $d = 12$. In panel (b) the points (marked in black) are obtained for a value of d equal to 4. When $r \to 0$ the ratio $\frac{\log C^d(r)}{\log r}$ gives a numerical estimate of the correlation dimension for the chosen value of ω. A numerical estimate of the ratio $\frac{\log C^d(r)}{\log r}$ means to find the gradient of lines in panel (a) when $r \to 0$ through the best fit to a series of data points. The other parameters are fixed at $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, the exogenous $\sigma_d = \frac{0.05}{1.64}$, the liquidity $\gamma = 100$.

When the dimension d of samples $\mathbf{x_i}$ is sufficiently large, the limit

$$\lim_{r\to 0}\lim_{N\to\infty}\frac{\log\ C^d(r)}{\log\ r}$$

gives an appropriate estimate of the correlation dimension, see Section 2.3 of Chapter 2.

In our model, correlation dimension becomes independent from d also for d larger than 3, see Figure (5.11). When $N = 5000$, numerical estimates of the mathematical limit $\lim_{r\to 0}$ yield an correlation dimension of 0.87 ± 0.02 for $\omega = 0.94$ and $c = 0.14$ (values of parameters for which the orbit is on the one-piece chaotic attractor). The fractional dimension defines, in some sense, the strangeness of the attractor of dynamical system. When the dynamical behavior is periodic, the same numerical estimates yield an correlation dimension equal to zero, *e.g.* see panel (b) of Figure (5.11) that shows the relation between $log\ C^4(r)$ and $log\ r$ for $c = 0.14$ and $\omega = 0.8$.

So we can conclude that in chaotic regime the attractors of the system dynamics are strange. But only the knowledge of this dynamical feature does not ensure that the system displays a true deterministic chaos. The latter occurs when a system exhibits sensitive dependence on initial conditions [47], see definition of deterministic chaos in Section 2.3. Quantitative measures of this sensitive dependence can be found by studying some ergodic quantities like the Largest Lyapunov Exponent (LLE), the order-two Renyi entropy (K_2) and the Kolmogorov-Sinai entropy (h, KS-invariant) [39].

A system is ergodic when the time average is equal to the space average. The weight with which the space average has to be taken is an invariant measure. In principle there are many invariant measures in a dynamical system, but only one of them is physically relevant if the system is ergodic. Since the system evolves in a compact domain (in fact, $u \in [0, 1]$), for the Krylov-Bogoliubov theorem an ergodic measure ρ_u exists and according to the idea of

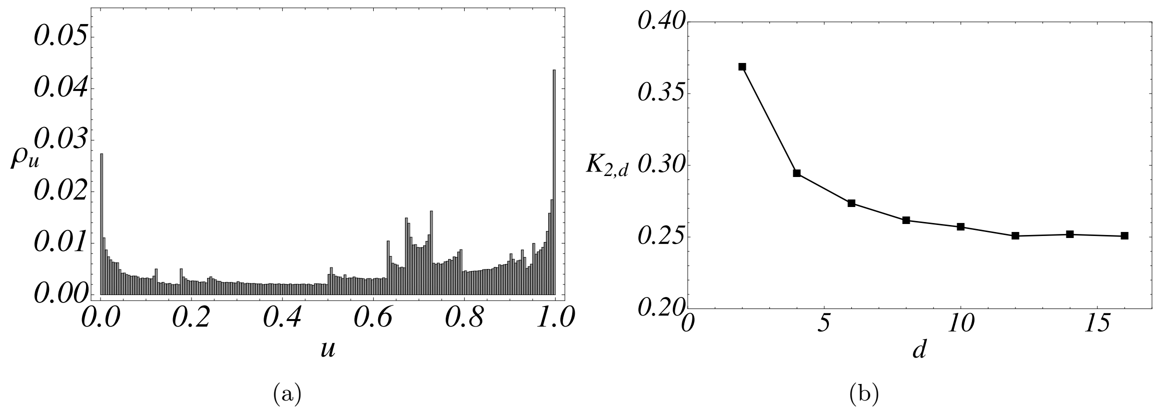


Figure 5.12: Panel (a): A numerical approximation of the physical invariant measure with a number of iterations equal to 3×10^5. Panel (b): The numerical estimation of the order-two Renyi entropy as function of the length of sample $\mathbf{x}_i$, d, panel (b). The parameters are fixed at $c = 0.14$, $\omega = 0.94$, $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, the exogenous $\sigma_d = \frac{0.05}{1.64}$, the liquidity $\gamma = 100$.

Kolmogorov, ρ_u can be considered as the physical measure of the dynamical system, see Section 2.3. A measure is said physical when space averages, obtained through the measure, coincide with experimental time averages. An experiment can be also a numerical simulation.

In the panel (a) of Figure (5.12), we show a numerical approximation of the physical measure, ρ_u, associated with the dynamical variable u of the map when $c = 0.14$, $\omega = 0.94$ and the number of iterations is 3×10^5. If the number of iterations grows, a better estimate of ρ_u can be obtained.

Ergodic quantities like the Largest Lyapunov Exponent LLE, the order-two Renyi entropy K_2, and the Kolmogorov-Sinai entropy h, are all related to the physical invariant measure ρ_u. But at the same time a numerical estimate of these quantities is simpler through algorithms that involve the numerical estimate of the correlation integral, see (5.10). An exhaustive dissertation about the ergodic measure, the largest Lyapunov exponent, the order-two Renyi entropy and the Kolmogorov-Sinai entropy can be found in Section 2.3 of Chapter 2.

The largest Lyapunov exponent gives a measure of the separation in time of two orbits that are characterized by two infinitely close initial points. For one-dimensional dynamical systems, LLE can be simply obtained through the numerical estimation of (2.30), see panel (a) of Figure (5.13). We compute LLE also using the original definition which exploits the separation in time of two orbits [49]. Let f_ω be the map describing the one-dimensional dynamical system with a fixed value of ω and let u the dynamical variable defined in the unit interval $[0, 1]$. Considering two different orbits whose initial conditions u_0^1 and u_0^2 differ by $\epsilon \ll 1$, $|u_0^1 - u_0^2| = \epsilon$. Iterating once the map f, we find

$$u_{t=1}^1 = f(u_0^1) \text{ and } u_{t=1}^2 = f(u_0^2)$$

we can describe how the absolute distance between orbits changes at the first time, computing the absolute difference D_1 at time $t = 1$ as

$$D_1 = |u_{t=1}^1 - u_{t=1}^2|$$

The ratio $\frac{D_1}{\epsilon}$ gives a measure of the rate of increasing (or decreasing) distance between orbits

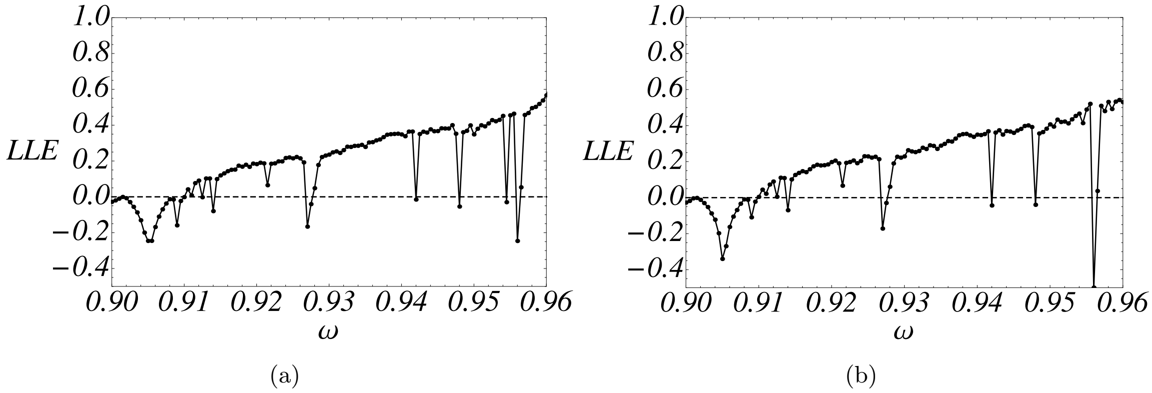


Figure 5.13: Largest Lyapunov Exponent (LLE) as function of the parameter ω. Panel (a) shows the outcome obtained through the numerical estimation of (2.30). Panel (b) shows the same outcome obtained through the algorithm based on the separation in time of orbits. The parameters are fixed at $c = 0.14$, $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, the exogenous $\sigma_d = \frac{0.05}{1.64}$, the liquidity $\gamma = 100$.

at time $t = 1$. Since, the dynamical system evolves on a bounded interval, at time $t = 1$ we make a shift $u^2_{t=1} \longrightarrow \tilde{u}^2_{t=1}$ such that $|u^1_{t=1} - \tilde{u}^2_{t=1}| = \epsilon$ again. The underlying idea is that the Lyapunov exponent gives a measure of the rate of separation of two orbits at *every* point of the map's domain. Hence, for example in the chaotic regime, we explore the points of the unit interval through the orbit described by u^1, while we measure the rate of separation represented by $\frac{D_t}{\epsilon}$ through the orbit described by u^2.

In fact, at time $t = 2$

$$u^1_{t=2} = f(u^1_{t=1}) \text{ and } u^2_{t=2} = f(\tilde{u}^2_{t=1})$$

and it follows that D_2 is equal to

$$D_2 = |u^1_{t=2} - u^2_{t=2}|$$

and so on. Using this algorithm, the Lyapunov exponent (LLE) is simply equal to

$$LLE = \lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^{t} \log \frac{D_i}{\epsilon} \tag{5.11}$$

The outcome is shown in panel (b) of Figure (5.13). This is in good agreement with the classical computation of LLE through the numerical estimation of (2.30). From the computational point of view, the algorithm that exploits the orbits separation is faster than the other one.

A positive value of LLE describes the degree of sensitive dependence on initial conditions of a dynamical system. Hence, positive values of LLE identify the chaotic regime. On the contrary, a negative value of LLE is associated with a periodic behavior of the system. We find that also in the chaotic regime there exist parameter values for which the dynamical behavior is periodic and the correspondent value of LLE is negative. Since there are chaotic regions associated with a positive value of LLE, the true deterministic chaos can be claimed for the map described by difference equations (5.4) (5.5) (5.6) with adaptive expectations.

In principle, the computation of the LLE is an hard task in the laboratory experiments. In fact, LLE is not accessible only by measuring the past realizations of a system trajectory. For this purpose, a more useful ergodic quantity is the entropy because also only the knowledge of the past realizations of a system trajectory is sufficient in order to estimate it numerically.

The entropy h of dynamical systems is defined as the mean rate of creation of dynamical information. It is related to the concept that in chaotic regime a dynamical system explores always new states during the evolution. Because the dynamical behavior is aperiodic, only a partial information about dynamics can be extrapolated from measures of past realizations and so the dynamics is essentially unpredictable. The entropy h is also known as Kolmogorov-Sinai invariant. For dissipative dynamical systems the Kolmogorov-Sinai entropy is the most important invariant quantity.

In order to estimate numerically the entropy h, let us define C_i^d as, cfr (5.10),

$$C_i^d(r) = \frac{1}{N} \sum_{j=0}^{N-d} \theta(r - ||\mathbf{x}_i - \mathbf{x}_j||_\infty) \tag{5.12}$$

Given (5.12), we can define $H^d(r)$ as

$$H^d(r) = -\frac{1}{N} \sum_{i=0}^{N-d} \log\, C_i^d(r) \tag{5.13}$$

Procaccia et al. [62] have numerically proved that the following quantity

$$\lim_{r\to 0} \lim_{d\to\infty} \lim_{N\to\infty} [H^{d+1}(r) - H^d(r)] = \Delta t\, h \tag{5.14}$$

gives a numerical estimation of the Eq. (2.35) which defines analytically the entropy h. Δt is the time step of dynamical evolution, in our case it is equal to 1.

The mathematical intuition is simple. When the length of a generic sample $\mathbf{x}_i$ increases from d to $d+1$, we collect more information about the dynamics. So the difference between $H^{d+1}(r)$ and $H^d(r)$ gives a numerical estimate of the rate of creation of dynamical information.

We use in addition an entropy concept different from the KS-invariant. It is the order-two Renyi entropy, K_2 [39]. *Grassberger and Procaccia* [60] show that the K_2 entropy is a lower bound to the entropy h, $K_2 \leq h$. Its numerical estimation can be found through the following limit

$$\lim_{r\to 0} \lim_{d\to\infty} \lim_{N\to\infty} \log \frac{C^d(r)}{C^{d+1}(r)} = \Delta t\, K_2 \tag{5.15}$$

Finally, the entropy h is related to the largest Lyapunov exponent (LLE) describing the sensitive dependence on initial conditions of the one-dimensional dynamical system through the following inequality (*Ruelle* [59], see Section 2.3)

$$h \leq LLE \tag{5.16}$$

By estimating numerically a positive-valued entropy (KS-invariant or order-two Renyi entropy) we can conclude that there exists a positive Lyapunov exponent and the dynamical system exhibits a true deterministic chaos.

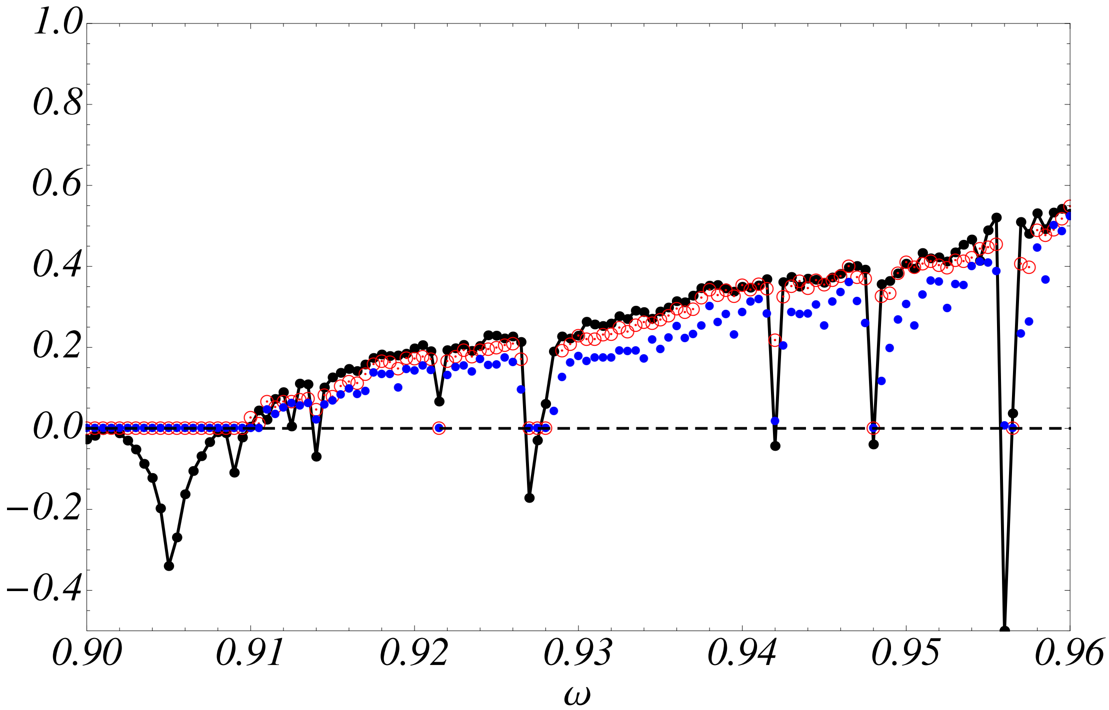


Figure 5.14: LLE (black circle), KS-entropy (red circle), K_2-entropy (blue circle) as a function of the parameter ω. The parameters are fixed at $c = 0.14$, $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, the exogenous $\sigma_d = \frac{0.05}{1.64}$, the liquidity $\gamma = 100$.

The numerical estimates of these ergodic quantities are shown in Figure (5.14). The main problem of the numerical computation is related to the convergence of mathematical limits in (5.14) (5.15). While we have chosen a value of N reasonable large ($N = 5000$) and a value of r reasonable small ($r = 0.01$), the crucial aspect is to find a value of the sample dimension, d, for which the truncated limit approaches reasonably well the mathematical limit. Panel (b) of Figure (5.12) shows that for $\omega = 0.94$ and $c = 0.14$ the K_2-entropy converges numerically to a plateau for $d > 10$. We have therefore chosen a d-value equal to 12 in the numerical estimates of entropies. In fact, numerical estimates of (5.14) as a function of d suggest that the truncated limit reaches a plateau for values of d smaller than 10. As a consequence, the choice $d = 12$ for the numerical computation of the entropies is reasonable.

In Figure (5.14) all the ergodic quantities are functions of the parameter ω. The largest Lyapunov exponent is marked in black, the Kolmogorov-Sinai entropy is marked in red, and the order-two Renyi entropy is marked in blue. It is important to notice that when the dynamical behavior is periodic (negative-valued LLE), the KS-entropy (or the K_2-entropy) is zero. In the chaotic regime the KS-entropy is positive and it increases when the parameter ω increases. This is in agreement with the profile of the LLE. The K_2-entropy is smaller than the KS-entropy for all the values of ω. Similarly the KS-entropy computed numerically is slightly smaller than the LLE.

The Kolmogorov-Sinai invariant is defined as the rate of creation of dynamical information which characterizes the system. At the same time, this entropy can be seen as the missing information which can not be extrapolated by the past states of a trajectory. In fact, when the dynamical behavior is periodic, the knowledge of a closed orbit corresponds to the knowledge of the system dynamics. This is not true anymore when the dynamical behavior is aperiodic. For example, this is the underlying reason for Figure (5.8) in which autocorrelation functions (ACF) of forecasting errors are shown for different value of ω: when the system dynamics is periodic (panel (a) and (b)) the entropy is zero and there are periodic structures in the ACF; in chaotic regimes (panel (c) and (d)) the periodic structures are lost, for $\omega = 0.94$ after few time lags, and the ACF coefficients are smaller than the bounds of significance. In other words, missing information corresponds to non-zero entropy.

In the next Section we will investigate how financial agents can improve their expectations strategy about risk using the available dynamical information.

5.4 Adaptive learning

In Section 5.2 we have described the dynamics of a typical portfolio when financial institutions use a forecasting strategy characterized by adaptive expectations about risk. Through bifurcation diagrams as a function of the expectations weight factor ω, we have found that the dynamics can be characterized by a steady state when the adaptive strategy is cautious. Moving towards naive expectations, *i.e.* $\omega \to 1$, first of all we find financial periodic cycles of the portfolio variables. Related to the value of ω, these cycles may have different orbital period. Then the chaotic regime appears. Depending on the value of diversification costs c, there exists a threshold ω^* for which the largest amplitude of leverage cycles is equal to the dimension of the economic domain of λ. In this condition, the dynamics of asset returns is non-stationary and the dynamical model fails in describing the evolution of the financial market.

In our model, the portfolio dynamics defines the financial market stability through the positive rebalancing feedback. In fact, in Figure (5.7) we can see that the periodicity or the non-periodicity of the portfolio dynamics affects the periodic or aperiodic realizations of the volatility of assets.

In Section 5.2, the expectations weight factor ω is the parameter of the model defining the strategy about the forecasting of risk. Particularly, ω represents the memory of the adaptive expectations scheme. This parameter is fixed during the dynamical evolution. Here, we consider a more flexible situation in which financial institutions modify their adaptive strategy after some considerations about their past forecasting performances. We refer to this situation as *adaptive learning.*

Adaptive learning thus means that the expectations weight factor ω is time varying, ω_t. During financial market evolution, financial institutions can estimate the quality of their adaptive expectations scheme looking at the forecasting errors about the diversifiable risk

$$\mathcal{E}_{d,t} = (\tilde{\sigma}^e_{d,t})^2 - (\tilde{\sigma}_{d,t+1})^2$$

In fact, at time t financial institutions expect that in the future time the risk associated with an asset will be equal to $(\tilde{\sigma}^e_{d,t})^2$. At time $t+1$, the realized risk of the same asset is $(\tilde{\sigma}_{d,t+1})^2$. Therefore, $\mathcal{E}_{d,t}$ is the forecasting error about the diversifiable risk made at time t.

In order to take information about the quality of past forecasting performances, financial institutions can look at the autocorrelation function ACF of the time series of forecasting errors $\mathcal{E}_{d,t}$. The ACF for the forecasting errors about the diversifiable risk is qualitatively similar to the ACF for the forecasting errors about the portfolio volatility, see Figure (5.8). Especially, the autocorrelation coefficient at the first time lag is always negative when the dynamical behavior is periodic or chaotic. This means that during financial cycles the agents operating in the market make systematic overestimations following by underestimations of risk.

Our underlying assumption in adaptive learning scheme is that agents behave like statisticians and they estimate the parameter of their forecast from past observations about their performances. In particular we assume that they look at the first autocorrelation coefficient, ρ_1, of the ACF of the errors time series, $\{\mathcal{E}_{d,t}\}$.

A natural and simple learning scheme is based upon the sample autocorrelation coefficient ρ_1. Suppose that financial institutions at time t' use an adaptive strategy defined by $\omega_{t'}$. For a time period of length T, they adapt their expectations about risk with the weight factor $\omega_{t'}$. If at the end of the time period, $t = t' + T$, financial institutions have made systematic mistakes in forecasting the risk, they will try to improve their adaptive scheme. In order to improve the forecasting strategy, financial institutions can estimate the autocorrelation coefficient at first time lag, using the series of forecasting errors made during the previous time window. A value $(\rho_1)_{t'}^{t}$ is found. If financial institutions maintain the same strategy, they will make a forecasting mistake at time $t+1$ statistically equal to the following expected value

$$\mathbb{E}\left[\mathcal{E}_{d,t+1} | \{\mathcal{E}_{d,i}\}_{i=t',\dots,t}\right] = (\rho_1)_{t'}^{t}\, \mathcal{E}_{d,t} \tag{5.17}$$

In (5.17) we have assumed that the average of the forecasting errors series is equal to zero in the time window $[t'\ ,\ t]$. This is consistent with the results of previous Sections.

In order to avoid future mistakes, the adaptive expectations about the risk of an asset (5.1) can be (slowly) modified. Until the time t , (5.1) with the time-varying parameter $\omega_{t'}$, defined at time t', is

$$(\tilde{\sigma}^e_{d,t})^2 = (\tilde{\sigma}^e_{d,t-1})^2 + \omega_{t'}[\,(\tilde{\sigma}_{d,t})^2 - (\tilde{\sigma}^e_{d,t-1})^2\,] \tag{5.18}$$

Forecasting a future forecasting mistake (5.17), financial institutions expect the diversifiable risk at time $t+1$ will be

$$(\tilde{\sigma}_{d,t+1})^2 = (\tilde{\sigma}^e_{d,t})^2 + \mathbb{E}\left[\mathcal{E}_{d,t+1} | \{\mathcal{E}_{d,i}\}_{i=t',\dots,t}\right] \tag{5.19}$$

that is equal to the sum of two contributions: the expected risk using the past strategy, $(\tilde{\sigma}^e_{d,t})^2$, and the future error statistically forecasted.

So, $(\tilde{\sigma}_{d,t+1})^2$ in (5.19) will be considered as the *new* expected diversifiable risk at time t, $(\tilde{\sigma}^e_{d,t})^2$, and coherently the adaptive scheme becomes

$$(\tilde{\sigma}^e_{d,t})^2 = (\tilde{\sigma}^e_{d,t-1})^2 + (\omega_{t'} + (\rho_1)_{t'}^{t})[\,(\tilde{\sigma}_{d,t})^2 - (\tilde{\sigma}^e_{d,t-1})^2\,] \tag{5.20}$$

and we find that the expectations weight factor evolves in time according to

$$\omega_t = \omega_{t'} + (\rho_1)_{t'}^{t} \tag{5.21}$$

When the dynamical behavior is periodic or chaotic and as a consequence $(\rho_1)_{t'}^{t} < 0$, the relation (5.21) suggests that financial institutions, adopting the Sample Autocorrelation (SAC) Learning procedure, becomes more cautious in their forecast. Improving in this way the forecasting strategy, the system dynamics could be shifted towards a steady state in which financial institutions make no errors in forecasting the risk.

To avoid drastic and non-realistic changes in the expectations scheme of financial institutions, in place of (5.21) we use

$$\omega_t = \omega_{t'} + \beta\,(\rho_1)_{t'}^{t} \tag{5.22}$$

where $0 < \beta < 1$. β is a scale factor which describes a slow change of the forecasting strategy adopted by financial institutions. Introducing the scale factor β, we assume for hypothesis that the SAC learning procedure occurs slowly. There are two reason. The first one is related to a more realistic scenario in which financial institutions are cautious in the upgrade of their past strategy. The second one is a theoretical reason related to the meaning of decreasing ω. The limit $\omega \to 0$ means increasing the memory of the adaptive expectations scheme, and as a consequence the memory of the dynamical system. This corresponds to make financial institutions more and more rational in forecasting the risk, see the previous Section for a comment about this aspect.

Hence, if at the beginning financial institutions are boundedly rational, the evolution towards rationality can not be a fast process because it requires investments in terms of time and costs [47].

The difference equations (5.4) (5.5) with the adaptive scheme (5.20) characterized by the time varying parameter ω_t, (5.22), describe the portfolio dynamics of financial institutions which use adaptive expectations in forecasting the risk and the SAC learning procedure to improve their forecasting strategy.

In Figure (5.15) we show a numerical simulation of the portfolio dynamics. In panel (a) the dynamical evolution of the diversification m_t is shown starting from an initial condition characterized by a value of ω equal to 0.94. This initial value imposes that the initial dynamical behavior is chaotic. Because the system is essentially one-dimensional, the orbit of the variable λ_t can be found starting from the orbit of m_t. It is clear that the SAC learning procedure drives the portfolio dynamics towards a steady state. In fact, the time evolution of the expectation weight factor ω_t displays a convergence to a value of ω smaller than the value at which the first flip bifurcation occurs, for the specific choice of model parameters we have made. In Figure (5.4) we have shown the values of the parameters c_2 and ω at which the first bifurcation occurs. In Figure (5.15), where diversification costs are fixed at $c = 0.14$ and the assets liquidity is $\gamma = 100$, the deterministic map shows the first dynamical transition for $\omega \approx 0.69$ (the dotted line). This means that SAC learning procedure makes the portfolio dynamics stable.

The stability of the portfolio dynamics has a stabilizing effect on the financial market dynamics, that is the positive feedback effects are in equilibrium with the dynamical evolution of

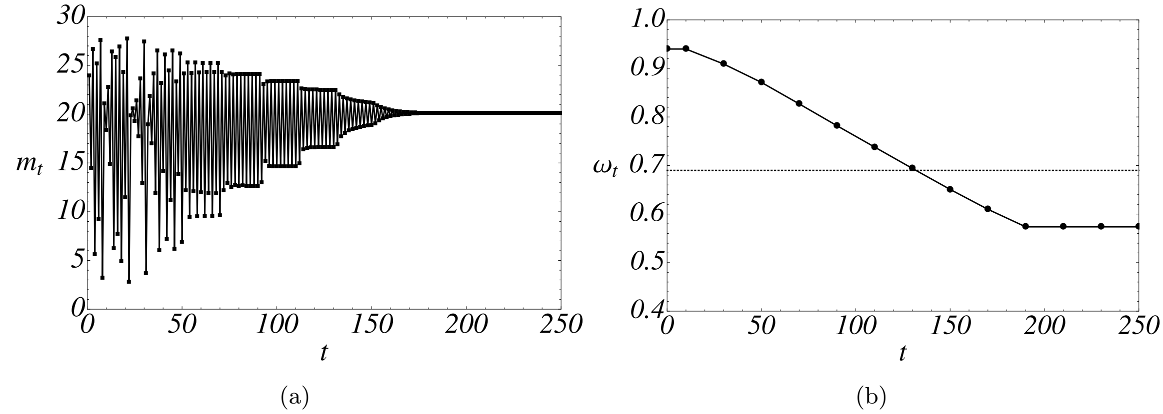


Figure 5.15: Panel (a): dynamical evolution of the degree of diversification m_t. Panel (b): dynamical evolution of the expectation weight factor ω_t. The dotted line in panel (b) represents the value numerically estimated, see Figure (5.4), at which the first period-doubling bifurcation occurs. Below the dotted line, the economic system converges to a steady state. The length T of time period, between two different upgrades of the parameter ω, is chosen equal to 20. β is equal to $\frac{1}{T}$. The value of the parameter ω at initial time is chosen to be equal to 0.94. The value of diversification costs is $c = 0.14$. This means that initially dynamical behavior is chaotic. The other parameters are fixed at $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, the exogenous $\sigma_d = \frac{0.05}{1.64}$, the liquidity $\gamma = 100$.

portfolios and as a consequence the macro-risk components (systemic risk) due to the rebalancing feedback effects, disappear.

In Figure (5.16) we can see the dynamical evolution of the expected volatility, $\tilde{\sigma}^e_{d,t}$, and the realized volatility, $\tilde{\sigma}_{d,t}$, in correspondence to the portfolio dynamics shown in Figure (5.15).

It is important to notice that a correct forecasting of risk through the choice of the optimal adaptive strategy plays a crucial role in order to make the market dynamics stable. In principle, also when financial institutions start with a non-optimal forecasting strategy, an adaptive learning procedure (slowly) induces financial institutions to improve their expectations scheme in order to use a better one. This situation, in our model, is always possible if changes in the values of model parameters happen gradually. In fact, numerical estimations suggest that the analysis made about the occurrence of the first criticality, produces always the qualitative outcomes of Figure (5.4) for all possible values of the liquidity parameter γ. In fact, γ defines the impact on risk of the rebalancing feedback effects. Especially, it can be numerically shown that in all cases a finite value of the memory, $\omega \neq 0$, exists in order to ensure that the portfolio dynamics converges to a steady state. In our approach, the convergence of the portfolio dynamics to a steady state induces the stability of the financial market dynamics.

Summarizing, the systemic financial stability depends strictly on the memory of the adaptive expectations scheme. In our case, more memory is always stabilizing: with more memory the first bifurcation (as a function of the other model parameters) towards instability comes later. Finally there always exists a finite value of the memory for which the financial market dynamics is stable. This result is in agreement with a recent work [107] of *Hommes et al.* .

In conclusion, we investigate the effects of the randomness on the dynamical evolution of portfolios. Similarly to the previous Sections, we introduce a stochastic Gaussian noise in the realized diversifiable risk. This noise is characterized by a standard deviation σ_{noise} smaller than the value of exogenous diversifiable volatility σ_d.

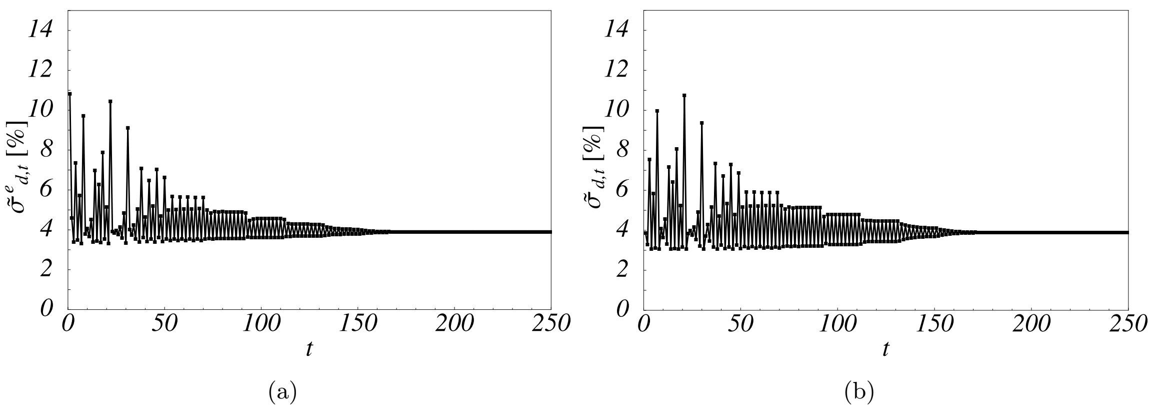


Figure 5.16: Panel (a): dynamical evolution of the expectation about diversifiable volatility of an asset. Panel (b): the correspondent realized diversifiable volatility of the same asset. The length T of time period, between two different upgrades of the parameter ω, is chosen equal to 20. β is equal to $\frac{1}{T}$. The value of the parameter ω at initial time is chosen to be equal to 0.94. The value of diversification costs is $c = 0.14$. This means that initially dynamical behavior is chaotic. The other parameters are fixed at $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, the exogenous $\sigma_d = \frac{0.05}{1.64}$, the liquidity $\gamma = 100$.

Figure (5.17) shows the dynamical evolution of the weight factor ω_t under SAC learning procedure when the Gaussian noise is accounted for. The portfolio dynamics is associated with the time evolution of ω_t like in Figure (5.15). The points marked in black correspond exactly to the deterministic evolution of ω_t in Figure (5.15). The other points correspond to different choices of σ_{noise} values. The dotted line correspond to the value of ω at which the first period-doubling bifurcation occurs. We can notice that in the stochastic dynamical evolution the convergence to a steady state is slower compared with the deterministic evolution. This agrees with what we have claimed before about the role of noise. In fact, dynamical information (for example about the autocorrelation coefficient ρ_1) can be partially hidden by noise. As a consequence, the system converges more slowly. At the same time, the noise has a stabilizing effect on portfolio dynamics because it induces financial institutions to become more cautious in forecasting the risk. In fact, in presence of noise, in the asymptotic limit financial institutions use an adaptive strategy identified by a value of ω_t that is smaller than the asymptotic value which characterizes the deterministic evolution.

However, in both cases the SAC learning strategy and adaptive expectations in forecasting the risk can drive the dynamics of the portfolio of boundedly rational agents towards a stable state. And so, they drive financial market dynamics towards stability.

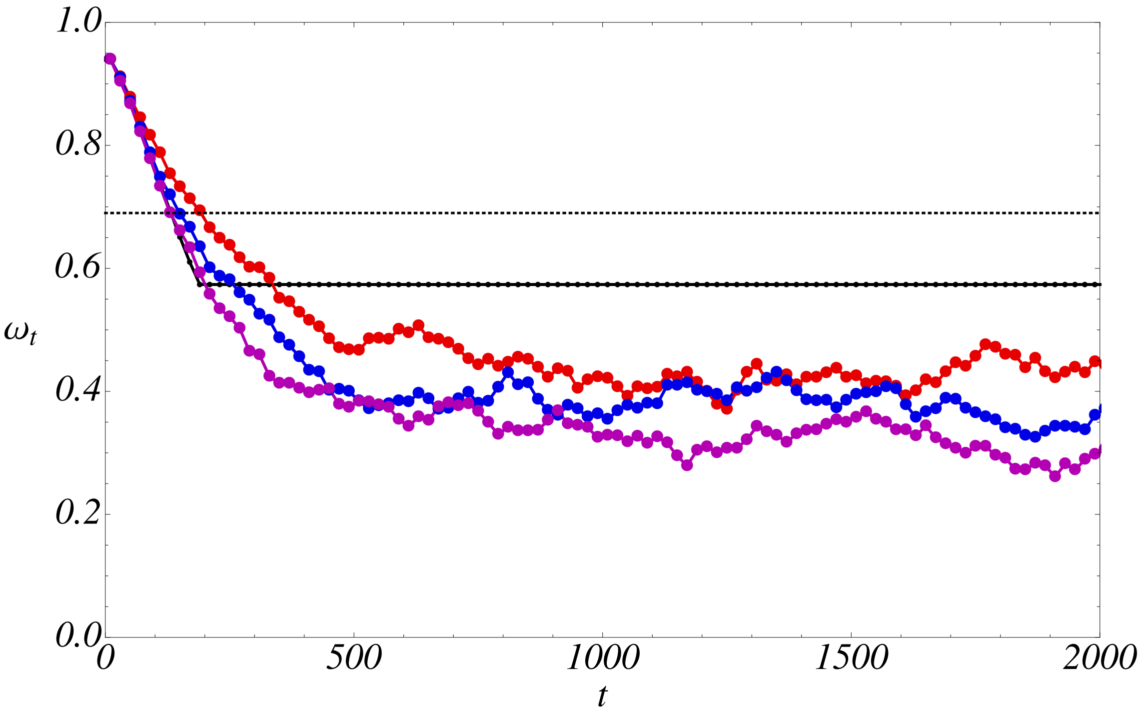


Figure 5.17: Dynamical evolution of the expectation weight factor ω_t. The dotted line in panel (b) represents the value numerically estimated, see Figure (5.4), at which the first period-doubling bifurcation occurs. Below the dotted line, the economic system converges to a steady state. The black circles represents the deterministic evolution of ω_t. The other circles correspond to the stochastic evolution when a Gaussian noise in introduced. In particular, red circles refer to $\sigma_{noise} = \frac{\sigma_d}{20}$, blue circles refer to $\sigma_{noise} = \frac{\sigma_d}{30}$, and purple circles refer to $\sigma_{noise} = \frac{\sigma_d}{50}$. The length T of time period, between two different upgrades of the parameter ω, is chosen equal to 20. β is equal to $\frac{1}{T}$. The value of the parameter ω at initial time is chosen to be equal to 0.94. The value of diversification costs is $c = 0.14$. This means that initially dynamical behavior is chaotic. The parameters are fixed at $M = 30$, $\alpha = 1.64$, $\mu - r_L = 0.08$, the exogenous $\sigma_d = \frac{0.05}{1.64}$, the liquidity $\gamma = 100$.

Conclusions and Perspectives

By exploiting a dynamical systems approach, we propose a dynamical model to describe the dynamics of a financial market populated by financial institutions investing in risky assets, when endogenous feedback effects are accounted for. Financial institutions optimize their portfolio according to the value of diversification cost, having capital requirement in the form of VaR constraint and they follow a trading strategy based on diversification and target leverage. In particular, the strategy of maintaining a fixed target leverage, through the portfolio rebalancing in presence of asset illiquidity, induces a positive feedback which increases the volatility of assets by its impact on returns. Specifically, we investigate how financial institutions respond to an increase of risk due to the positive feedback. Periodically, they estimate the actual risk associated with their portfolio and they form expectations about future risk of assets through estimates of observed past risks. According to the expectations scheme, they define the optimal portfolio configuration. Due to the impact of rebalancing feedback, the dynamical evolution of financial institution portfolios reflects the dynamical behavior of the whole financial system. In our dynamical approach, the key point is that the portfolio choices based on the expectations scheme adopted by financial institutions together with the impact on risk due to the target leverage strategy, drive the dynamics of financial market.

In this dynamical framework, we find that systemic risk is in effect an emergent phenomenon. While at a micro level, financial institutions adopt a strategy based on diversification and target leverage, in order to minimize the risk and to maximize the profit respectively, the emerging macro outcomes may be identified by the breakage of systemic financial stability. Specifically,

1. when financial institutions forecast future risk to be equal to the last observed one (*naive expectations*), depending on diversification costs, at a given threshold the positive feedback triggers a period doubling bifurcation and the consequent appearance of financial cycles characterized by a sequence of speculative periods and non-speculative ones. During financial cycles, we can notice how the financial leverage switches from aggressive configurations (speculative periods) to cautious ones (non-speculative periods). The financial leverage cycles reflect the occurrence of periods characterized by a macro-component of risk, due to an higher impact of feedback effects, followed by periods in which feedback effects do not affect importantly the asset risk. When financial cycles appear, the amplitude of cycles can be interpreted as a measure of systemic risk in financial market.

2. When financial institutions forecast future risk to be equal to a weighted average, with

geometrically declining weights, of all past risks (*adaptive expectations*), a very complex dynamics appears. When diversification costs decrease, the positive feedback triggers successive period doubling bifurcations which induce financial cycles increasingly complex. During financial cycles, the financial market is characterized by highly risky periods due to a macro-component of the risk induced by rebalancing feedback effects. At a given threshold, the dynamical system is characterized by a dynamical transition from a periodic behavior to a chaotic one. In the chaotic regime, the financial market dynamics can show periods characterized by higher risk compared with the risky periods of financial cycles.

3. Under adaptive expectations, the chaotic dynamical behavior of financial system is characterized by positive entropy, suggesting how much an improvement of expectations scheme by financial institutions may be hard due to missing information about the financial market dynamics. In the case of adaptive expectations, the memory of the expectations scheme plays a crucial role. In fact, we find that weighting mainly the past risks rather than the last observed one, has a stabilizing effect on financial market dynamics.

4. Finally, a procedure of adaptive learning based on the autocorrelation function of forecasting errors, may have a stabilizing effect on the financial market dynamics. It induces financial institutions to adopt an adaptive expectations scheme characterized by a larger value of memory, driving thus financial market dynamics towards systemic stability.

5. From the point of view of financial market **policy**, we believe that our original results deserve attention because they highlight how the dynamical properties of financial markets may drastically change when market conditions change. In fact, once the memory of the expectations scheme adopted by financial institutions is fixed, a decrease of diversification costs in the presence of strong feedback effects may lead to the breaking of systemic financial stability. Similarly to diversification costs, also a decrease of the quantile of VaR constraint will have the same consequences. Decreasing the quantile of VaR constraint is translated to allowing financial institutions to adopt a larger value of target leverage. This leads to an increase of the strength of feedback effects and, as a consequence, the financial system moves towards instability regimes. When feedback effects are strong, a decrease of the value of the quantile may break systemic financial stability.

The obtained results naturally open two main lines of investigation.

1. From a theoretical point of view, many open questions remain. Specifically, an obvious point is to consider the systematic component of risk different from zero. From a mathematical point of view, this modification leads to study a two dimensional dynamical system and, as a consequence, new dynamical effects may arise. Furthermore, a very challenging issue is related to the introduction of a more realistic price impact function described by a square-root function. This generalization introduces an extra source of nonlinearity in the feedback effects due to the rebalancing strategy adopted by financial institutions. Last but not least, relaxing some homogeneity assumptions regarding investment assets and/or financial institutions, allows to approach the problem of heterogeneity in financial markets.

2. From an empirical point of view, the necessity of testing the model on empirical bases naturally appears. For this purpose, the main problem is related to identify the appropriate time scale according to which, the dynamical effects, described by model, arise. Once the time scale is determined, the comparison of our original results with the time series of historical portfolio data, particularly before the occurrence of a financial crisis, may validate our dynamical systems approach. Specifically, analyzing the historical portfolio data, a measure of positive entropy may be a strong evidence of deterministic chaos in financial markets.

In conclusion, although feedback effects have been recognized as an important source of systemic risk in financial markets, this thesis represents an original dynamical systems approach to the considered problem. Particularly, our work focuses on the possible dynamical outcomes displayed by a financial system due to the positive feedback, and not only on the consequences on asset prices and risks due to the feedback effects. We believe that our original results, which especially suggest the possibility that chaos may occur in financial markets, indicate that our model deserves attention. Furthermore, a result of this type bypasses the specific dynamical model under consideration, since the occurrence of chaos in financial markets may be the consequence of universal aspects related to the nonlinearity of the feedback effects.

Acknowledgements

I would like to express my sincere gratitude to my supervisor, Prof. Fabrizio Lillo, whose guidance, deep knowledge, understanding, and patience, added considerably to my graduate experience. His support helped me in all the time of research and writing of this thesis. I could not have imagined to have a better mentor in my thesis work.

I would like to thank my internal supervisor, Prof. Riccardo Mannella, for his patience and important suggestions to complete my work. I am most grateful to Prof. Stefano Marmi, for his illuminating advices, to Fulvio Corsi, for the encouragement and interest in my research work, and to Davide Pirino, for our illuminating discussions about finance and his precious advices.

I am very very grateful to Prof. Giuseppe Angilella, first of all for our friendship. And for his encouragement and support.

And I would like to acknowledge my colleagues, Andrea, Alessandro, Piga, Sebastiano, Faco, Guido, Federica, Tella, Domenico, Stara, Giorgio, Gabriele, Chiara, and Alessia, for their support and friendship.

I am very grateful to MdF. There are not words to express the beauty of their friendship. Thanks from the bottom of my heart.

A special word of thanks goes to my brother Piergiorgio and my friends, Gigi, Aurelia, Luca, Antonio, Fiona, Rita, Matilde, Grace, Giorgina, Giuseppe, Michele, Federica, Enrico, Budda, Marianna, Fabiana, Roberta, Junio, and M&M. You all are special for me.

And I thank so much all my family, whose presence, support and love never lack.

A lovely word of thanks to Giulia, for her love and her thimble which make my life so lively.

Finally and most of all, I thank *mamma*, *papà*, and *zio Nino*, for their love.

You are my stable attractors in a complex world.

Bibliography

[1] J Doyne Farmer, Mauro Gallegati, Cars Hommes, Alan Kirman, Paul Ormerod, Silvano Cincotti, Anxo Sanchez, and Dirk Helbing. A complex systems approach to constructing better models for managing financial markets and the economy. *The European Physical Journal-Special Topics*, 214(1):295–324, 2012.

[2] Rosario N Mantegna and H Eugene Stanley. *Introduction to econophysics: correlations and complexity in finance*. Cambridge university press, 2000.

[3] Philip W Anderson. More is different. *Science*, 177(4047):393–396, 1972.

[4] Monica Billio, Mila Getmansky, Andrew W Lo, and Loriana Pelizzon. Econometric measures of connectedness and systemic risk in the finance and insurance sectors. *Journal of Financial Economics*, 104(3):535–559, 2012.

[5] Prasanna Gai and Sujit Kapadia. Contagion in financial networks. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science*, 466(2120):2401–2423, 2010.

[6] Youngna Choi and Raphael Douady. Financial crisis dynamics: attempt to define a market instability indicator. *Quantitative Finance*, 12(9):1351–1365, 2012.

[7] Tobias Adrian and Hyun Song Shin. Liquidity and leverage. *Journal of financial intermediation*, 19(3):418–437, 2010.

[8] Rama Cont and Lakshithe Wagalath. Running for the exit: distressed selling and endogenous correlation in financial markets. *Mathematical Finance*, 23(4):718–741, 2013.

[9] F. Corsi, S. Marmi, and F. Lillo. When micro prudence increases macro risk: The destabilizing effects of financial innovation, leverage, and diversification. *papers.ssrn*, June 2013.

[10] Cars Hommes, Joep Sonnemans, Jan Tuinstra, and Henk Van De Velden. Learning in cobweb experiments. *Macroeconomic Dynamics*, 11(S1):8–33, 2007.

[11] Te Bao, John Duffy, and Cars Hommes. Learning, forecasting and optimizing: An experimental study. *European Economic Review*, 61:186–204, 2013.

[12] John Bardeen, Leon N Cooper, and J Robert Schrieffer. Microscopic theory of superconductivity. *Physical Review*, 106(1):162–164, 1957.

[13] Youngna Choi, Raphael Douady, et al. Financial crisis and contagion: A dynamical systems approach. *Handbook on systemic risk*, page 453, 2012.

[14] Franklin Allen and Douglas Gale. Financial contagion. *Journal of political economy*, 108(1):1–33, 2000.

[15] Prasanna Gai, Andrew Haldane, and Sujit Kapadia. Complexity, concentration and contagion. *Journal of Monetary Economics*, 58(5):453–470, 2011.

[16] Rama Cont, Amal Moussa, and Edson Santos. Network structure and systemic risk in banking systems. *Edson Bastos e, Network Structure and Systemic Risk in Banking Systems (December 1, 2010)*, 2011.

[17] Fabio Caccioli, Thomas A Catanach, and J Doyne Farmer. Heterogeneity, correlations and financial contagion. *Advances in Complex Systems*, 15(supp02), 2012.

[18] Fabio Caccioli, Munik Shrestha, Cristopher Moore, and J Doyne Farmer. Stability analysis of financial contagion due to overlapping portfolios. *arXiv preprint arXiv:1210.5987*, 2012.

[19] Hamed Amini, Rama Cont, and Andreea Minca. Resilience to contagion in financial networks. *Mathematical Finance*, 2013.

[20] Leonardo Bargigli, Giovanni Di Iasio, Luigi Infante, Fabrizio Lillo, and Federico Pierobon. The multiplex structure of interbank networks. *arXiv preprint arXiv:1311.4798*, 2013.

[21] Mark EJ Newman, Steven H Strogatz, and Duncan J Watts. Random graphs with arbitrary degree distributions and their applications. *Physical Review E*, 64(2):026118, 2001.

[22] Mark EJ Newman and Duncan J Watts. Scaling and percolation in the small-world network model. *Physical Review E*, 60(6):7332, 1999.

[23] Duncan J Watts. A simple model of global cascades on random networks. *Proceedings of the National Academy of Sciences*, 99(9):5766–5771, 2002.

[24] Mark EJ Newman. The structure and function of complex networks. *SIAM review*, 45(2):167–256, 2003.

[25] Stefano Boccaletti, Vito Latora, Yamir Moreno, Martin Chavez, and D-U Hwang. Complex networks: Structure and dynamics. *Physics reports*, 424(4):175–308, 2006.

[26] Hyman Minsky. The financial instability hypothesis. *The Jerome Levy Economics Institute Working Paper*, (74), 1992.

[27] Giuseppe Castellacci and Youngna Choi. Financial instability contagion: a dynamical systems approach. *Quantitative Finance*, 14(ahead-of-print):1–13, 2014.

[28] Giuseppe Castellacci and Youngna Choi. Modeling contagion in the eurozone crisis via dynamical systems. *Journal of Banking & Finance*, 2014.

[29] John David Crawford. Introduction to bifurcation theory. *Reviews of Modern Physics*, 63(4):991, 1991.

[30] Jean-Philippe Bouchaud. The endogenous dynamics of markets: price impact and feedback loops. *arXiv preprint arXiv:1009.2928*, 2010.

[31] Markus K Brunnermeier and Lasse Heje Pedersen. Predatory trading. *The Journal of Finance*, 60(4):1825–1863, 2005.

[32] Markus K Brunnermeier. Deciphering the liquidity and credit crunch 2007-08. Technical report, National Bureau of Economic Research, 2008.

[33] Lasse Heje Pedersen. When everyone runs for the exit. *International journal of central banking*, 5(4):177–199, 2009.

[34] Jean-Philippe Bouchaud. Price impact. *Encyclopedia of quantitative finance*, 2010.

[35] Fabrizio Lillo and Davide Pirino. The impact of systemic and illiquidity risk on financing with risky collateral. *Available at SSRN 2417244*, 2014.

[36] Fabrizio Lillo, J Doyne Farmer, and Rosario N Mantegna. Econophysics: Master curve for price-impact function. *Nature*, 421(6919):129–130, 2003.

[37] Robert M May. Simple mathematical models with very complicated dynamics. *Nature*, 261(5560):459–467, 1976.

[38] Mitchell J Feigenbaum. Quantitative universality for a class of nonlinear transformations. *Journal of statistical physics*, 19(1):25–52, 1978.

[39] J-P Eckmann and David Ruelle. Ergodic theory of chaos and strange attractors. *Reviews of modern physics*, 57(3):617, 1985.

[40] Luitzen Egbertus Jan Brouwer. Über abbildung von mannigfaltigkeiten. *Mathematische Annalen*, 71(1):97–115, 1911.

[41] Philip Hartman. A lemma in the theory of structural stability of differential equations. *Proceedings of the American Mathematical Society*, 11(4):610–620, 1960.

[42] Mariana Haragus and Gérard Iooss. *Local bifurcations, center manifolds, and normal forms in infinite-dimensional dynamical systems.* Springer, 2010.

[43] David Ruelle and Floris Takens. On the nature of turbulence. *Communications in mathematical physics*, 20(3):167–192, 1971.

[44] John David Crawford, Edgar Knobloch, and Hermann Riecke. Period-doubling mode interactions with circular symmetry. *Physica D: Nonlinear Phenomena*, 44(3):340–396, 1990.

[45] Mitchell J Feigenbaum. Universal behavior in nonlinear systems. *Physica D: Nonlinear Phenomena*, 7(1):16–39, 1983.

[46] AN Sharkovskii. Coexistence of cycles of a continuous map of the line into itself. *International journal of bifurcation and chaos*, 5(05):1263–1273, 1995.

[47] Cars Hommes. *Behavioral rationality and heterogeneous expectations in complex economic systems*. Cambridge University Press, 2013.

[48] David Ruelle. Small random perturbations of dynamical systems and the definition of attractors. *Communications in Mathematical Physics*, 82(1):137–151, 1981.

[49] Michael Tabor. *Chaos and integrability in nonlinear dynamics*. Wiley, 1989.

[50] Ricardo Mané and Silvio Levy. *Ergodic theory and differentiable dynamics*, volume 8. Springer, 1987.

[51] Nicolas Kryloff and Nicolas Bogoliouboff. La théorie générale de la mesure dans son application à l'étude des systèmes dynamiques de la mécanique non linéaire. *Annals of Mathematics*, pages 65–113, 1937.

[52] David Ruelle. Sensitive dependence on initial condition and turbulent behavior of dynamical systems. *Annals of the New York Academy of Sciences*, 316(1):408–416, 1979.

[53] Madabusi S Raghunathan. A proof of oseledec's multiplicative ergodic theorem. *Israel Journal of Mathematics*, 32(4):356–362, 1979.

[54] Pierre Collet and J-P Eckmann. On the abundance of aperiodic behaviour for maps on the interval. *Communications in Mathematical Physics*, 73(2):115–160, 1980.

[55] Pierre Collet and J-P Eckmann. Positive liapunov exponents and absolute continuity for maps of the interval. *Ergodic Theory and Dynamical Systems*, 3(01):13–46, 1983.

[56] Claude Elwood Shannon. A mathematical theory of communication. *ACM SIGMOBILE Mobile Computing and Communications Review*, 5(1):3–55, 2001.

[57] Patrick Billingsley. *Ergodic theory and information*. Wiley, 1965.

[58] Jérôme Buzzi. A minicourse on entropy theory on the interval. *arXiv preprint math/0611337*, 2006.

[59] David Ruelle. An inequality for the entropy of differentiable maps. *Bulletin of the Brazilian Mathematical Society*, 9(1):83–87, 1978.

[60] Peter Grassberger and Itamar Procaccia. Estimation of the kolmogorov entropy from a chaotic signal. *Physical review A*, 28(4):2591–2593, 1983.

[61] Peter Grassberger and Itamar Procaccia. Characterization of strange attractors. *Physical review letters*, 50(5):346–349, 1983.

[62] Aviad Cohen and Itamar Procaccia. Computing the kolmogorov entropy from time signals of dissipative and conservative dynamical systems. *Physical review A*, 31(3):1872, 1985.

[63] Peter Grassberger. On the hausdorff dimension of fractal attractors. *Journal of Statistical Physics*, 26(1):173–179, 1981.

[64] HGE Hentschel and Itamar Procaccia. The infinite number of generalized dimensions of fractals and strange attractors. *Physica D: Nonlinear Phenomena*, 8(3):435–444, 1983.

[65] Lai Sang Young. Dimension, entropy and lyapunov exponents. *Ergodic Theory Dynam. Systems*, 2(1):109–124, 1982.

[66] Peter Grassberger and Itamar Procaccia. Measuring the strangeness of strange attractors. In *The Theory of Chaotic Attractors*, pages 170–189. Springer, 2004.

[67] R. Cont. Empirical properties of asset returns: stylized facts and statistical issues. *Quantitative Finance*, 1(2):223–236, 2001.

[68] Rama Cont, Marc Potters, and Jean-Philippe Bouchaud. Scaling in stock market data: stable laws and beyond. In *Scale invariance and beyond*, pages 75–85. Springer, 1997.

[69] Benoit B Mandelbrot. Scaling in financial prices: I. tails and dependence. *Quantitative Finance*, 1(1), 2001.

[70] Rosario N Mantegna and H Eugene Stanley. Scaling behaviour in the dynamics of an economic index. *Nature*, 376(6535):46–49, 1995.

[71] Jean-Philippe Bouchaud and Marc Potters. *Theory of financial risk and derivative pricing: from statistical physics to risk management.* Cambridge university press, 2003.

[72] Rama Cont. Volatility clustering in financial markets: empirical facts and agent-based models. In *Long memory in economics*, pages 289–309. Springer, 2007.

[73] Laurent Laloux, Pierre Cizeau, Jean-Philippe Bouchaud, and Marc Potters. Noise dressing of financial correlation matrices. *Physical review letters*, 83(7):1467, 1999.

[74] Vasiliki Plerou, Parameswaran Gopikrishnan, Bernd Rosenow, Luis A Nunes Amaral, Thomas Guhr, and H Eugene Stanley. Random matrix approach to cross correlations in financial data. *Physical Review E*, 65(6):066126, 2002.

[75] AM Sengupta and Partha P Mitra. Distributions of singular values for some random matrices. *Physical Review E*, 60(3):3389, 1999.

[76] F Lillo and RN Mantegna. Spectral density of the correlation matrix of factor models: A random matrix theory approach. *Physical Review-Series E-*, 72(1):016219, 2005.

[77] Stefan Thurner, J Doyne Farmer, and John Geanakoplos. Leverage causes fat tails and clustered volatility. *Quantitative Finance*, 12(5):695–707, 2012.

[78] Prasanna Gai, Andrew Haldane, and Sujit Kapadia. Complexity, concentration and contagion. *Journal of Monetary Economics*, 58(5):453–470, 2011.

[79] Ian A Cooper and Evi Kaplanis. Costs to crossborder investment and international equity market equilibrium. *Recent Developments in Corporate Finance*, 1986.

[80] Chandra Thapa and Sunil S Poshakwale. International equity portfolio allocations and transaction costs. *Journal of Banking & Finance*, 34(11):2627–2638, 2010.

[81] Bank for international settlements. Basel iii: A global regulatory framework for more resilient banks and banking systems. June 2011.

[82] Basel Committee et al. Basel iii: A global regulatory framework for more resilient banks and banking systems. *Basel Committee on Banking Supervision, Basel*, 2010.

[83] Fabrizio Lillo and Rosario N Mantegna. Dynamics of a financial market index after a crash. *Physica A: Statistical Mechanics and its Applications*, 338(1):125–134, 2004.

[84] Fulvio Corsi and Didier Sornette. Follow the money: The monetary roots of bubbles and crashes. *International Review of Financial Analysis*, 2014.

[85] Paolo Tasca and Stefano Battiston. Market procyclicality and systemic risk. *Munich Personal RePEc Archive*, 2013.

[86] Fabio Caccioli, Jean-Philippe Bouchaud, and J Doyne Farmer. A proposal for impact-adjusted valuation: Critical leverage and execution risk. *arXiv preprint arXiv:1204.0922*, 2012.

[87] Paolo Tasca and Stefano Battiston. Diversification and financial stability. Technical report, SRC systemic risk center, 2011.

[88] Markus K Brunnermeier and Lasse Heje Pedersen. Market liquidity and funding liquidity. *Review of Financial studies*, 22(6):2201–2238, 2009.

[89] Moritz Schularick and Alan M Taylor. Credit booms gone bust: monetary policy, leverage cycles and financial crises, 1870–2008. Technical report, National Bureau of Economic Research, 2009.

[90] Jean-Philippe Bouchaud, J Doyne Farmer, and Fabrizio Lillo. How markets slowly digest changes in supply and demand. *Handbook of financial markets: dynamics and evolution*, 1:57, 2009.

[91] Vasiliki Plerou, Parameswaran Gopikrishnan, Xavier Gabaix, and H Eugene Stanley. Quantifying stock-price response to demand fluctuations. *Physical Review E*, 66(2):027104, 2002.

[92] Fabio Caccioli, Matteo Marsili, and Pierpaolo Vivo. Eroding market stability by proliferation of financial instruments. *The European Physical Journal B*, 71(4):467–479, 2009.

[93] Matteo Marsili. Complexity and financial stability in a large random economy. *Quantitative Finance*, (ahead-of-print):1–13, 2013.

[94] Herbert A Simon. On the behavioral and rational foundations of economic dynamics. *Journal of Economic Behavior & Organization*, 5(1):35–55, 1984.

[95] Cars Hommes. The heterogeneous expectations hypothesis: Some evidence from the lab. *Journal of Economic Dynamics and Control*, 35(1):1–24, 2011.

[96] Cars H Hommes. Dynamics of the cobweb model with adaptive expectations and nonlinear supply and demand. *Journal of Economic Behavior & Organization*, 24(3):315–335, 1994.

[97] Cars Hommes. Cobweb dynamics under bounded rationality. In *Optimization, Dynamics, and Economic Analysis*, pages 134–150. Springer, 2000.

[98] Cars Hommes. Interacting agents in finance. Technical report, Tinbergen Institute Discussion Paper, 2006.

[99] Cars Hommes, Gerhard Sorger, and Florian Wagener. Consistency of linear forecasts in a nonlinear stochastic economy. In *Global Analysis of Dynamic Models in Economics and Finance*, pages 229–287. Springer, 2013.

[100] Cars H Hommes. On the consistency of backward-looking expectations: The case of the cobweb. *Journal of Economic Behavior & Organization*, 33(3):333–362, 1998.

[101] Cars Hommes and Gerhard Sorger. Consistent expectations equilibria. *Macroeconomic Dynamics*, 2(03):287–321, 1998.

[102] Cars Hommes. Bounded rationality and learning in complex markets. in rosser, j.b. (ed.). *Handbook of research on complexity*, pages 87–123, 2009.

[103] Cars Hommes and Mei Zhu. Behavioral learning equilibria. *Journal of Economic Theory*, 2013.

[104] Marten Scheffer, Jordi Bascompte, William A Brock, Victor Brovkin, Stephen R Carpenter, Vasilis Dakos, Hermann Held, Egbert H van Nes, Max Rietkerk, and George Sugihara. Early-warning signals for critical transitions. *Nature*, 461(7260):53–59, 2009.

[105] Timothy M Lenton, Richard J Myerscough, Robert Marsh, Valerie N Livina, Andrew R Price, and Simon J Cox. Using genie to study a tipping point in the climate system. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 367(1890):871–884, 2009.

[106] Patrick E McSharry, Leonard A Smith, and Lionel Tarassenko. Prediction of epileptic seizures: are nonlinear methods relevant? *Nature medicine*, 9(3):241–242, 2003.

[107] Cars Hommes, Tatiana Kiseleva, Yuri Kuznetsov, and Miroslav Verbic. Is more memory in evolutionary selection (de) stabilizing? *Macroeconomic Dynamics*, 16(03):335–357, 2012.