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Perfect periodic scheduling for three basic cycles

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Abstract Periodic scheduling has many attractions for wireless telecommunications. It offers energy saving where equipment can be turned off between transmissions, and high-quality reception through the elimination of jitter, caused by irregularity of reception. However, perfect periodic schedules, in which each (of n) client is serviced at regular, prespecified intervals, are notoriously difficult to construct. The problem is known to be NP-hard even when service times are identical. This paper focuses on cases of up to three distinct periodicities, with unit service times. Our contribution is to derive a $O(n^4)$ test for the existence of a feasible schedule, and a method of constructing a feasible schedule if one exists, for the given combination of client periodicities. We also indicate why schedules with a higher number of periodicities are unlikely to be useful in practice. This methodology can be used to support perfect periodic scheduling in a wide range in real world settings, including machine maintenance service, wireless mesh networks and various other telecommunication networks transmitting packet size data.

Keywords Scheduling · Perfect periodicity · Feasibility testing · Wireless mesh networks

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1 Introduction

In this paper, we explore the construction of schedules for clients who are serviced on a periodic basis, where clients receive identical service durations but at different intervals. An immediate practical problem motivates our research, namely local transmission of data in Wireless Mesh Networks (WMN) which is of increasingly importance for communication worldwide (Akyildiz et al. 2005). In wireless networks, data are transmitted in packets, and this feature can be exploited to extract considerable energy saving of a factor of 5 to 10 through efficient algorithms for packet scheduling (Quintas and Friderikos 2012). Wireless Mesh Networks represent a technology that economically increases the geographical area within which mobile clients may access broadband communication.

Mesh routers facilitate multi-hop wireless transmission to relay data over extended distances without the cost, delay and disruption of installing cabled access points. The routers also act as local access points for devices or clients (e.g. for WiFi access). The routers typically have a fixed location and can be mounted on buildings, street lamps, etc. It is the provision of local access (i.e. devices connecting to the access point) that is of relevance in this paper. Local access is governed by a star topology with all devices connecting to the local access point. Bandwidth must be scheduled so that at any point in time, an access point is servicing at most one client. In this scenario, perfect periodic scheduling provides numerous benefits associated with predictability of activity time. A schedule is perfect periodic when each client is itself serviced regularly at a fixed interval, known as its periodicity. The duration and periodicity of service for each client is given. In classic machine scheduling context, processing of a job is synonymous with servicing of a client, and processing time with duration. We will use these two paradigms interchangeably throughout. The predictability allows the total power consumption of the network to be minimised and the interference from competing simultaneous transmissions to be reduced or even eliminated.

Several studies have been undertaken on perfect periodic scheduling problems. The problem of finding a feasible perfect periodic schedule has been proved by Bar-Noy et al. (2002a) to be NP-hard in general. However, there are some special cases for which simple closed form expressions offer a polynomial time solution method. For three products with three distinct periodicities, Glass (1992) derives a simple test for existence of a feasible schedule, and a method of constructing feasible schedules if one exists. In addition, Glass (1994) and Glass et al. (1994) study the feasibility problem in the context of the Economic Lot Scheduling Problem where several products with different durations are produced periodically so as to minimise holding and set up costs. For two (three) products with two (three) distinct periodicities, they develop a simple necessary and sufficient condition for the feasibility test and construction of a feasible schedule.

Due to the difficulty of finding a feasible perfect periodic schedule for any given periodicities, called as *request periodicity*, heuristics are therefore used to allocated close values, called as *allocated periodicity*, using specific criteria. Brakerski et al. (2006) study perfect periodic scheduling for multiple servers, with the objective is to minimise the maximum ratio of the allocated periodicity and requested periodicity. They develop some approximation algorithms for the problem. Patil and Garg (2006) propose an adaptive algorithm, called *Adapmin*. They find a worst case performance bound on the quality of schedules produced by *Adaptmin* and compare *Adapmin* to the algorithm of Brakerski *et al.*. Chen and Huang (2008) propose an efficient algorithm of requested periodicities with high density and large variance.

For perfect periodic scheduling with identical processing times, Bar-Noy et al. (2004) consider two objective measures of maximum and weighted average ratios between the allocated periodicity and requested periodicity. They present a few efficient heuristic algorithms using a methodology, called *tree scheduling*, based on an hierarchical round-robin approach, where the hierarchy is a form of tree. Bar-Noy et al. (2002b) develop tree-based approximation algorithms for perfect periodic scheduling with the objective of minimising weighted average ratios between the allocated periodicity and requested periodicity. Brakerski et al. (2003) study the question of dispatching in a perfect periodic schedule, namely how to find the next item to schedule, assuming that the schedule is already given somehow.

All of the above approaches to finding a perfect periodic schedule are limited in the range of schedules which they consider. For example, tree scheduling algorithms (Bar-Noy et al. 2002b; Brakerski et al. 2003) restrict their solution space to a set of periodicities with a non-trivial greatest common divisor. Moreover, Brakerski et al. (2006), Patil and Garg (2006), and Chen and Huang (2008) constrain solutions to those for which the ratio between any pair of two allocated periodicities is a power of 2. Thus, further improvements in heuristics depend upon being able to establish perfect periodic schedules with a wider range of periodicities. Additional motivation for developing our understanding of perfect period scheduling, arises in the context of wireless mesh networks where the response time variability in transmissions, known as *jitter*, is an important issue. Jitter relates to quality of reception and is measured by total response time variation as well as the ”bottleneck metric,” of maximum deviation from an ideal interval, identified by Steiner and Yeomans (1993). Corominas et al. (2007) provide effective heuristics for measures which prove to be optimal for the case of two tasks. However, they avoid attempts to find solutions with no interval variation, which are perfectly periodic, because of ”the serious practical disadvantage . . . that there may not be a feasible solution.”

The aim of this paper is to address the issue of feasibility for perfect periodic scheduling problem on a single resource. We restrict our attention to problems with identical task durations and at most three distinct periodicities in total. Observe that the focus on tasks with identical durations is generally no limitation in the packet scheduling environment of wireless networks. Current practice is to use a common cycle time to co-ordinate packet scheduling (Quintas and Friderikos 2012). Thus, any methodology which facilitates variety in periodicities infers enhanced flexibility and thus potential additional energy savings for network operators. In the context of a wireless network with homogeneous link bandwidth, transmission time for a data packet is uniform throughout the network, and time may thus be divided into unit slots accordingly for scheduling purposes. Our dual contributions are to provide a test of feasibility and a method for constructing a feasible schedule (when one exists).

This article contributes to a broader programme of research into energy efficient scheduling for WMNs using time slot models. A complementary algorithm for coordinating given periodic schedules at mesh nodes with data transmission through the routing tree of the access network is provided by Allen et al. (2012a), and demonstrated on simple networks. Then, for routing trees with a binary structure, Kim and Glass (2012) develop a perfect periodic scheduling methodology for efficiently transmitting data through the access network, while providing the local clients at a mesh node with a common scheduling cycle. For a general routing tree, Allen et al. (2012b) introduce an integer programming formulation to describe an optimised schedule with time slots, and a fast heuristic approximation which produces

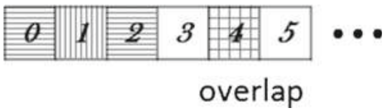


Fig. 1 The only possible, infeasible, schedule for two clients with periodicities 2 and 3, respectively

near-optimal solutions. They additionally explore the potential for increasing throughput in a wireless mesh network by varying transmission rates at individual links, taking account of signal interference.

The remainder of the paper is structured as follows. The problem itself is formulated in purely mathematical terms in the next section. Our results rely only upon classic properties in modulo arithmetic, which are included for completeness. We then analyse the case of two distinct periodicities in Sect. 3. We establish a closed form test for the existence of a perfect periodic schedule, and in doing so indicate the structure of a feasible solution. Section 4 provides similar results for the case of three periodicities. Various cases are analysed, resulting in a test for feasibility which runs in $O(n^4)$ time, where n is the total number of clients, in the last subsection. The construction of a feasible solution itself, where one exists, is presented explicitly in Sect. 6. A few concluding remarks are then offered in Sect. 7. For convenience, the remainder of the paper is framed in the telecommunications context of "service" to "clients," and not in the alternative machine scheduling terminology of "processing" of "jobs."

2 Problem formulation and background results

In order to understand the difficulty of devising perfectly period schedules consider the following small examples with only two clients. First suppose that each client requests a periodicity of 2. In this case, their requests can be accommodated by scheduling one client in the odd numbered time slots and the other in even numbered time slots, although there would be no residual capacity for any other client. If, however, one of the clients reduces their demand by a half to periodicity 3, then there would appear to be residual capacity, but in fact the problem becomes impossible to solve rather than easier, as illustrated in Fig. 1. By contract, there is no difficulty in scheduling two clients with longer periodicities of 6 and 9 time slots without overlap, as illustrated in Fig. 2. It is thus the relationship between the periodicities, and not simply their total effective capacity requirement, which determines whether a feasible schedule can be constructed.

The problem of finding a Perfect Periodic Schedule with unit service (or processing) times is generally formulated as follows. We use $R(a, b)$ to denote the remainder function of two positive integers, a and b, that is, $R(a, b) = a - b\lfloor a/b \rfloor$.

Given a set of clients with associated service periodicity $q_i \in \mathbb{N}$ for $i = 1, \ldots, n$, find time slot τ_i for which the following time slots are distinct:

$$\tau_i^{(k)} = R(\tau_i, q_i) + (k-1)q_i, \ k \in \mathbb{N}$$

for $1 \le i \le n$.

However, as we consider a restricted number of periodicities, it is more efficient for us to employ High Multiplicity encoding Brauner et al. (2005).

PPS1: Given a set of n_i clients with associated service periodicity q_i for $i = 1, \ldots, m$ (with $n_i, q_i \in \mathbb{N}$), find time slot τ_{ij} for which the following time slots are distinct:

$$\tau_{ij}^{(k)} = R(\tau_{ij}, q_i) + (k-1)q_i, \ k \in \mathbb{N}$$

for $0 \le j \le n_i - 1$ and $1 \le i \le m$.

The above high multiplicity formulation has input data: $m, q_1, \ldots, q_m, n_1, \ldots, n_m$ which may be encoded in $O(m \log q_{\max})$ steps, where $q_{\max} = \max_{1 \le i \le m} q_i$ (since instances are restricted to having $n_i < q_i$ to avoid obvious infeasibility). A drawback of this super-efficient encoding is that it does not afford a polynomial time solution algorithm since describing a solution, τ_{ij}, takes more than polynomial time in the input variables. Thus, PPS cannot belong to P nor NP under high multiplicity encoding. However, the problem in terms of the standard encoding, with input variables $n, q_1, \ldots, q_n$, is known to be NP-hard from Bar-Noy et al. (2002a) as mentioned above. It is therefore sensible to evaluate the computational complexity of a solution procedure in terms of the standard encoding.

In the PPS1 scenario, any feasible schedule is periodic and the length of the period, T, is the least common multiple of q_i for all $i = 1, \ldots, m$, $lcm(q_1, \ldots, q_m)$. Service to a client with periodicity q_i is repeated $lcm(q_1, \ldots, q_m)/q_i$ times within the overall period T, for each i. It is sufficient, therefore, to find a feasible schedule over $T = lcm(q_1, \ldots, q_m)$ time slots instead of over the infinite horizon. Many results relating to PPS1 are based on classic properties of congruences from Number Theory. Let $gcd(a, b)$ denote the greatest common divisor of integers a and b.

Chinese Remainder Theorem *Let a_1 and a_2 be positive integers, and b_1 and b_2 be any integers. Then, the simul-*

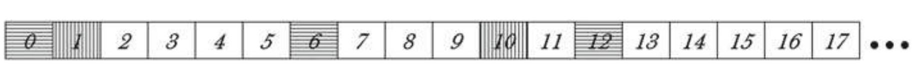


Fig. 2 A feasible schedule for two clients with periodicities 6 and 9, respectively

taneous congruences $t \equiv b_1 \bmod a_1$ *and* $t \equiv b_2 \bmod a_2$ *have a solution if and only if* $gcd(a_1, a_2)$ *divides* $b_1 - b_2$.

The following result follows directly from the Chinese Remainder Theorem as observed by Bar-Noy et al. (2002a, Lemma 12).

Lemma 1 *A solution is feasible if and only if* $\tau_{ij} \not\equiv \tau_{i'j'} \bmod gcd(q_i, q_{i'})$ *for* $i \neq i'$ *or* $j \neq j'$.

By Lemma 1, there is no feasible schedule for any set of periodicities $q_1, \ldots, q_m$ which contains a pair of periodicities, q_i and $q_{i'}$ which are coprime, i.e. $gcd(q_i, q_{i'}) = 1$. Congruence results have a direct relationship to the theory of linear Diophantine equations (eg. $a_1x + a_2y = b$). In this equation form, the following result was used as long ago as 1st century AD (eg. by Indian mathematician Brahmagupta, around AD 628) Jones and Jones (1998).

Lemma 2

$$a_1 x \equiv b \bmod a_2$$

has a solution if and only if $gcd(a_1, a_2)$ *divides* b.

3 Two distinct periodicities

In this section, we study the case when there are two distinct periodicities. A test of existence of a feasible schedule is provided in Theorem 1 and a schedule construction in its proof. Throughout this section, we use a partition $\mathbf{\Lambda}$ of time slots $\{0, \ldots, lcm(q_1, q_2) - 1\}$ into sets $\Lambda_l = \{t \mid t \equiv l \bmod gcd(q_1, q_2), 0 \leq t \leq lcm(q_1, q_2) - 1\}$ for $l = 0, \ldots, gcd(q_1, q_2) - 1$. We refer to the time slot $l + p \cdot gcd(q_1, q_2)$ in Λ_l as *position* p in Λ_l.

Note that $|\Lambda_l| = lcm(q_1, q_2)/gcd(q_1, q_2)$ for $l = 0, \ldots, gcd(q_1, q_2) - 1$, and a client with the periodicity of q_i is repeated $lcm(q_1, q_2)/q_i$ time in $T = lcm(q_1, q_2)$, from which the following observation follows.

Lemma 3 *Up to* $q_i/gcd(q_1, q_2)$ *clients with the periodicity of* q_i *for* $i = 1, 2$ *may be serviced in a single* $\mathbf{\Lambda}$*-set, and a client is serviced in just one* $\mathbf{\Lambda}$*-set.*

Proof Observe that the difference between any two consecutive time slots in each $\mathbf{\Lambda}$-set is $gcd(q_1, q_2)$. Since $gcd(q_1, q_2)$ divides both q_1 and q_2, each client has to be serviced in only one of $\mathbf{\Lambda}$-sets. Moreover, the services for a client with the periodicity of q_i occur in every $q_i/gcd(q_1, q_2)$ time slots in a single $\mathbf{\Lambda}$-set. Thus, up to $q_i/gcd(q_1, q_2)$ clients with the periodicity of q_i can be serviced in a single $\mathbf{\Lambda}$-set. □

Lemma 4 *Any two clients whose periodicities differ cannot be scheduled in the same* $\mathbf{\Lambda}$*−set.*

Proof Clients j and j' with periodicities q_1 and q_2, respectively, can be scheduled only if $\tau_{1j} \not\equiv \tau_{2j'} \bmod gcd(q_1, q_2)$ by Lemma 1, and hence any two clients with different periodicities cannot be scheduled in the same $\mathbf{\Lambda}$-set. □

Theorem 1 *An instance with two distinct periodicities,* q_1 *and* q_2 *with* n_1 *and* n_2 *clients, respectively, is perfect periodically schedulable if and only if*

$$\left\lceil \frac{n_1}{q_1/gcd(q_1, q_2)} \right\rceil + \left\lceil \frac{n_2}{q_2/gcd(q_1, q_2)} \right\rceil \leq gcd(q_1, q_2). \tag{1}$$

Proof (⇒) From Lemma 3, the number of $\mathbf{\Lambda}$-sets required for scheduling n_i clients with the periodicity of q_i is $\lceil n_i/(q_i/gcd(q_1, q_2)) \rceil$ for $i = 1, 2$. Thus, by Lemma 4, the instance is schedulable only if

$$\left\lceil \frac{n_1}{q_1/gcd(q_1, q_2)} \right\rceil + \left\lceil \frac{n_2}{q_2/gcd(q_1, q_2)} \right\rceil \leq gcd(q_1, q_2).$$

(⇐) We now construct a feasible solution for an instance satisfying the condition (1). We determine the first time slot in which client j with the periodicity of q_i is serviced, τ_{ij}, as follows. Let

$$\tau_{1j} = \left\lfloor \frac{j}{q_1/gcd(q_1, q_2)} \right\rfloor + R\left(j, \frac{q_1}{gcd(q_1, q_2)}\right) gcd(q_1, q_2) \tag{2}$$

for $j = 0, \ldots, n_1 - 1$, and

$$\tau_{2j} = \left\lceil \frac{n_1}{q_1/gcd(q_1, q_2)} \right\rceil + \left\lfloor \frac{j}{q_2/gcd(q_1, q_2)} \right\rfloor + R\left(j, \frac{q_2}{gcd(q_1, q_2)}\right) gcd(q_1, q_2) \tag{3}$$

for $j = 0, \ldots, n_2 - 1$.

We now show that the time slots allocated by (2) and (3) provide a feasible solution. From Lemma 1, it is sufficient to prove that $\tau_{ij} \not\equiv \tau_{i'j'} \bmod gcd(q_i, q_{i'})$ for $i \neq i'$ or $j \neq j'$. Take the case when $i = 1$, $i' = 2$, $0 \leq j_1 \leq n_1 - 1$ and $0 \leq j_2 \leq n_2 - 1$, and consider τ_{1j_1}, τ_{2j_2} modulo $gcd(q_1, q_2)$. From (2) and (3),

$$0 \leq \left\lfloor \frac{j_1}{q_1/gcd(q_1, q_2)} \right\rfloor < \left\lceil \frac{n_1}{q_1/gcd(q_1, q_2)} \right\rceil \leq \left\lceil \frac{n_1}{q_1/gcd(q_1, q_2)} \right\rceil + \left\lfloor \frac{j_2}{q_2/gcd(q_1, q_2)} \right\rfloor$$

and

$$0 < \left\lfloor \frac{j_2}{q_2/gcd(q_1, q_2)} \right\rfloor \leq \left\lfloor \frac{n_2 - 1}{q_2/gcd(q_1, q_2)} \right\rfloor \leq \left\lceil \frac{n_2}{q_2/gcd(q_1, q_2)} \right\rceil - 1.$$

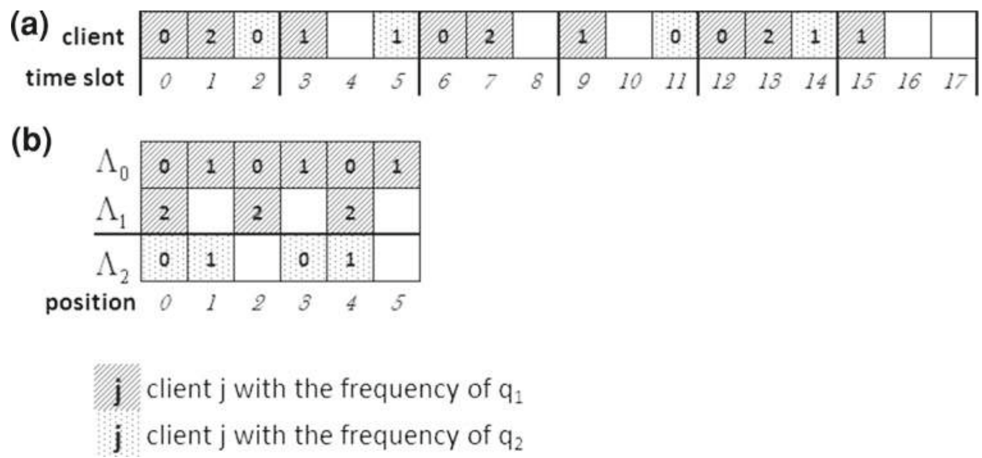


Fig. 3 A solution for Example 1 **a** schedule in $[0, T-1]$, **b** partition of schedule into **Λ**-set

Thus,

$$\begin{aligned}0 &\le \left\lfloor \frac{j_1}{q_1/gcd(q_1,q_2)} \right\rfloor \\ &< \left\lceil \frac{n_1}{q_1/gcd(q_1,q_2)} \right\rceil + \left\lfloor \frac{j_2}{q_2/gcd(q_1,q_2)} \right\rfloor \\ &< \left\lceil \frac{n_1}{q_1/gcd(q_1,q_2)} \right\rceil + \left\lceil \frac{n_2}{q_2/gcd(q_1,q_2)} \right\rceil - 1 \\ &\le gcd(q_1,q_2) - 1\end{aligned}$$

since (1) holds. Thus, $\tau_{1j_1} \not\equiv \tau_{2j_2} \mod gcd(q_1,q_2)$ as claimed.

It remains to show that $\tau_{ij} \equiv \tau_{ij'} \mod q_i$ implies that $j = j'$ for $i = 1, 2$. Consider the case with $i = 1$. Observe that

$$0 \le R\left(j, \frac{q_1}{gcd(q_1,q_2)}\right) \le \frac{q_1}{gcd(q_1,q_2)} - 1, \quad \text{and}$$

$$0 \le \left\lfloor \frac{j}{q_1/gcd(q_1,q_2)} \right\rfloor < gcd(q_1,q_2)$$

from above. Thus,

$$\begin{aligned}0 &\le \tau_{1j} \\ &= \left\lfloor \frac{j}{q_1/gcd(q_1,q_2)} \right\rfloor + R\left(j, \frac{q_1}{gcd(q_1,q_2)}\right) gcd(q_1,q_2) \\ &< gcd(q_1,q_2) + \left(\frac{q_1}{gcd(q_1,q_2)} - 1\right) gcd(q_1,q_2) \\ &\le q_1\end{aligned}$$

for each $j = 0, \ldots, n_1 - 1$, showing that distinct τ_{ij} values cannot be congruent modulo q_1. A similar argument holds for the case with $i = 2$, completing the proof. □

In the feasible solution described above, the clients with the periodicity of q_1 are scheduled in Λ_l for $l = 0, \ldots, \lceil n_1/(q_1/gcd(q_1,q_2)) \rceil - 1$, and the clients with periodicity of q_2 in Λ_l for $l = \lceil n_1/(q_1/gcd(q_1,q_2)) \rceil, \ldots, \lceil n_1/(q_1/gcd(q_1,q_2)) \rceil + \lceil n_2/(q_2/gcd(q_1,q_2)) \rceil - 1$. The following example illustrates the construction of a perfect periodic schedule described in Theorem 1.

Example 1 Consider the case of two periodicities $q_1 = 6$ and $q_2 = 9$ with $n_1 = 3$ and $n_2 = 2$, respectively. This instance satisfies criterion (1) for the schedulability, since

$$\begin{aligned}&\left\lceil \frac{n_1}{q_1/gcd(q_1,q_2)} \right\rceil + \left\lceil \frac{n_2}{q_2/gcd(q_1,q_2)} \right\rceil \\ &\quad = \left\lceil \frac{3}{2} \right\rceil + \left\lceil \frac{2}{3} \right\rceil = 3 \le 3 = gcd(q_1,q_2).\end{aligned}$$

Following the construction of the above proof, we calculate the first time slots in which each client is serviced using (2) and (3) as follows:

$$\begin{aligned}\tau_{10} &= \lfloor 0/2 \rfloor + R(0,2)3 \ = \ 0 \\ \tau_{11} &= \lfloor 1/2 \rfloor + R(1,2)3 \ = \ 3 \\ \tau_{12} &= \lfloor 2/2 \rfloor + R(2,2)3 \ = \ 1 \\ \tau_{20} &= \lceil 3/2 \rceil + \lfloor 0/3 \rfloor + R(0,3)3 \ = \ 2 \\ \tau_{21} &= \lceil 3/2 \rceil + \lfloor 1/3 \rfloor + R(1,3)3 \ = \ 5.\end{aligned}$$

Therefore, the first client with the periodicity of q_1 is serviced in the set of time slots $\{0, 6, 12\}$, and the second and third in the set of time slots $\{3, 9, 15\}$ and in the set of time slots $\{1, 7, 13\}$, respectively. Similarly, the first client with the periodicity of q_2 is serviced in the set of time slots $\{2, 11\}$, and second client with the periodicity of q_2 in the set of time slots $\{5, 14\}$. A single complete period of length ($T = 18$) of feasible schedule is depicted in Fig. 3a and its partition into **Λ**-sets in Fig. 3b.

4 Theoretical results for three distinct periodicities

In this section, we study the existence of a feasible solution where there are three distinct periodicities, q_1, q_2 and q_3. It is convenient to identify the factors of the periodicities, q_1, q_2 and q_3. Let $g = gcd(q_1,q_2,q_3)$, $g_{12} = gcd(q_1,q_2)/g$, $g_{23} = gcd(q_2,q_3)/g$, $g_{31} = gcd(q_3,q_1)/g$, $g_{10} = q_1/(gg_{12}g_{31})$, $g_{20} = q_2/(gg_{23}g_{12})$, and $g_{30} = q_3/(gg_{31}g_{23})$. Thus, the periodicities are presented as $q_1 =$

$gg_{10}g_{12}g_{31}$, $q_2 = gg_{20}g_{12}g_{23}$ and $q_3 = gg_{30}g_{23}g_{31}$. Observe that integers in sets $\{g_{10}, g_{20}, g_{30}\}$, $\{g_{12}, g_{23}, g_{31}\}$, $\{g_{10}, g_{23}\}$, $\{g_{20}, g_{31}\}$ and $\{g_{30}, g_{12}\}$ are coprime within each set but not necessarily coprime to g nor between sets. We first show that any instance can be reduced to one in which $g_{i0} = 1$ for $i = 1, 2, 3$ in Sect. 4.1. Then, we consider the case when $g_{i0} = 1$ for $i = 1, 2, 3$ and $gcd(q_1, q_2, q_3) = 1$ in Sect. 4.2, and we later extend the results to general periodicities in Sects. 4.3 and 5.

4.1 Problem reduction

The following theorem shows how an instance can be reduced to an instance with $g_{i0} = 1$ by letting $n'_i = \lceil n_i/g_{i0} \rceil$ and $q'_i = q_i/g_{i0}$ for $i = 1, 2, 3$. The proof relies on a congruence relationship given in Lemma 5 below.

Theorem 2 *An instance with n_1, n_2, n_3 clients with the periodicity of q_1, q_2, q_3, respectively, is feasible if and only if there is a feasible perfect periodic schedule for $\lceil n_i/g_{i0} \rceil$ clients with a periodicity of q_i/g_{i0} for $i = 1, 2, 3$.*

Proof ($\Rightarrow$) Take a feasible schedule. If a client with the periodicity of q_2 or q_3 is serviced in a time slot t, then it is also serviced in time slots $t + hT/g_{10}$, for $h = 0, \ldots, g_{10} - 1$, since both q_2 and q_3 divide T/g_{10}. Hence, the subschedule consisting of clients with the periodicity of q_2 or q_3 has periodicity T/g_{10} and is repeated g_{10} times within the overall period T of the three periodicities. Now by Lemma 5, a client, say j, with the periodicity of q_1 occupies time slots $\tau_{1j} + hq_1/g_{10}$ for $h = 0, \ldots, (T/q_1) - 1$ within the period $T' = \{0, \ldots, T/g_{10} - 1\}$. Moreover, no client with the periodicity of q_2 or q_3 can be serviced in these time slots. Thus, up to g_{10} clients with the periodicity of q_1 can be scheduled in these time slots, i.e. $\{t \mid t \equiv \tau_{1j} + hq_1/g_{10} \bmod T/g_{10}, 0 \leq h \leq T/q_1 - 1\}$. From this perspective, feasibility depends upon $\lceil n_1/g_{10} \rceil$, but not on n_1 itself. Therefore, repeating this argument for $i = 1, 2, 3$, any instance can be reduced to an instance with $g_{i0} = 1$ by letting $n'_i = \lceil n_i/g_{i0} \rceil$ and $q'_i = q_i/g_{i0}$ for $i = 1, \ldots, 3$.
($\Leftarrow$) Given a feasible schedule for the reduced problem, i.e. τ'_{ij} for $i = 1, 2, 3$, $j = 0, \ldots, \lceil n_i/g_{i0} \rceil - 1$, set initial time slots for clients in the original problem as follows:

$$\tau_{ij} = \tau'_{i\lfloor j/g_{i0} \rfloor} + R(j, g_{i0})(q_i/g_{i0}) \tag{4}$$

for $i = 1, 2, 3$, $j = 0, \ldots, n_i - 1$. We now show that the schedule implied by (4) provides a feasible solution.

From Lemma 1, it is sufficient to prove that $\tau_{ij} \not\equiv \tau_{i'j'} \bmod gcd(q_i, q_{i'})$ for $i \neq i'$ or $j \neq j'$. Suppose otherwise. Then there are non-identical pairs i, j and i', j' for which $\tau_{ij} \equiv \tau_{i'j'} \bmod gcd(q_i, q_{i'})$. However, if $i = i'$, then

$$\tau'_{ij} - \tau'_{ij'} \equiv \left(R\left(j', g_{i0}\right) - R\left(j, g_{i0}\right)\right)(q_i/g_{i0}) \mod q_i$$

implying that $\tau'_{ij} \equiv \tau'_{ij'} \bmod q_i$, and hence that $j = j'$, by Lemma 1 and feasible of schedule i. On the other hand, if $i \neq i'$, $i = 1$ and $i' = 2$ say, then

$$\tau'_{1\lfloor j/q_{10} \rfloor} - \tau'_{2\lfloor j'/q_{20} \rfloor} \equiv R\left(j', g_{20}\right) g_{12}g_{23} \\ -R\left(j, g_{10}\right) g_{12}g_{13} \bmod q_{12}$$

implying that $\tau'_{1\lfloor j/q_{10} \rfloor} \equiv \tau'_{2\lfloor j/q_{20} \rfloor} \bmod gcd(q'_1, g'_2)$ since $gcd(q'_1, g'_2) = gcd(q_1, g_2) = g_{12}$. This provides the required contradiction, from Lemma 1 and feasibility of i. □

Lemma 5 *The following sets are congruent modulo T/g_{i0}*

$\{kq_i \mid k{=}0, \ldots, T/q_i - 1\}$ *and* $\{k'q_i/g_{i0} \mid k'{=}0, \ldots, T/q_i{-}1\}$

for $i = 1, 2, 3$.

Proof Consider the expression $kq_1 \equiv k'q_1/g_{10} \bmod T/g_{10}$. Since $g_{12}g_{31}$ divides q_1, q_1/g_{10} and T/g_{10}, this is equivalent to $kq_1/(g_{12}g_{31}) \equiv k'q_1/(g_{10}g_{12}g_{31}) \bmod T/(g_{10}g_{12}g_{31})$, that is, $kg_{10} \equiv k' \bmod g_{20}g_{30}g_{23}$ for $0 \leq k, k' \leq g_{20}g_{30}g_{23} - 1$. Thus, it is sufficient to show that $\Gamma = \{t \mid t \equiv kg_{10} \bmod g_{20}g_{30}g_{23}, \ 0 \leq k \leq g_{20}g_{30}g_{23} - 1\} = \{0, 1, \ldots, g_{20}g_{30}g_{23} - 1\}$. Suppose otherwise, then $|\Gamma| < g_{20}g_{30}g_{23}$, and there must exist k and k', $k \neq k'$, such that $g_{10}k \equiv g_{10}k' \bmod g_{20}g_{30}g_{23}$ where $0 \leq k, k' \leq g_{20}g_{30}g_{23} - 1$. Then, $g_{10}k - g_{10}k' = g_{10}(k - k') \equiv 0 \bmod g_{20}g_{30}g_{23}$. Since g_{10} and $g_{20}g_{30}g_{23}$ are coprime, this implies that $k - k' \equiv 0 \bmod g_{20}g_{30}g_{23}$, which cannot hold for the given choice of k and k', giving the required contradiction. A similar argument can be applied for $i = 2, 3$. □

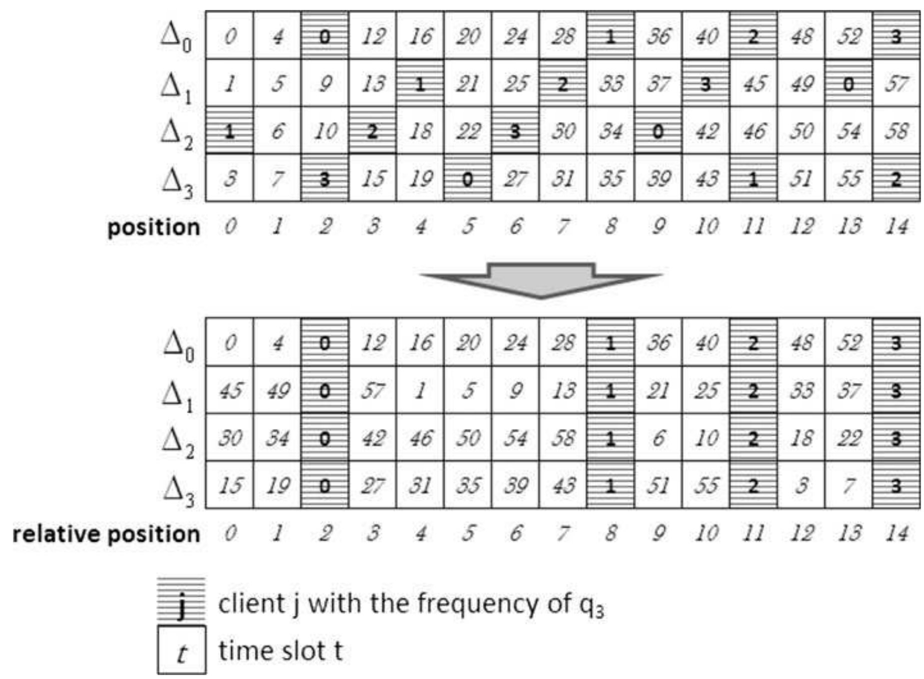
4.2 Special case of $gcd(q_1, q_2, q_3) = 1$, for the reduced problem

While the analysis in this section focuses of q_3, the choice is purely arbitrary. We assume that $g = 1$ and $g_{i0} = 1$ for $i = 1, 2, 3$, and thus $T = lcm(q_1, q_2, q_3) = g_{12}g_{23}g_{31}$. Take a partition $\boldsymbol{\Delta}$ of time slots $\{0, 1, \ldots, T - 1\}$ into sets $\Delta_l = \{t \mid t \equiv l \bmod g_{12}, 0 \leq t \leq T - 1\}$ for $l = 0, \ldots, g_{12} - 1$. We refer to the time slot $l + pg_{12}$ in Δ_l as *position* p in Δ_l. Observe that $|\Delta_l| = T/g_{12} = g_{23}g_{31}$ for $l = 0, \ldots, g_{12} - 1$.

Lemma 6 *Each client with the periodicity of q_1 or q_2 receives all its service slots in the same $\boldsymbol{\Delta}$-set. Moreover, no $\boldsymbol{\Delta}$-set accommodates both a client with the periodicity of q_1 and a client with the periodicity of q_2.*

Proof Observe that g_{12} is the difference between any two consecutive time slots in each $\boldsymbol{\Delta}$-set. Since g_{12} divides both q_1 and q_2, each client with the periodicity of q_1 or q_2 has to be serviced in a single $\boldsymbol{\Delta}$-set. Moreover, clients j and j' with the periodicities of q_1 and q_2, respectively, are simultaneously scheduled if and only if $\tau_{1j} \not\equiv \tau_{2j'} \bmod g_{12}$ by Lemma 1,

Fig. 4 Rearranging time slots in each Δ_l with respect to q_3



and hence any two clients with the periodicities of q_1 and q_2 cannot be scheduled in the same $\mathbf{\Delta}$-set. □

The partition $\mathbf{\Delta}$ of time slots thus enable us to view the schedule of service to clients with the periodicities of q_1 and q_2 independently. We now consider the interplay with clients with the periodicity of q_3. Observe that there is an integer $\bar{q}_3$, $1 \leq \bar{q}_3 \leq g_{12} - 1$ for which $q_3\bar{q}_3 \equiv 1 \bmod g_{12}$, from Lemma 2 since $gcd(q_3, g_{12}) = 1$. Let $p^{[l]}_{3j}$ denote the first position in Δ_l of service to client j with the periodicity of q_3 corresponding to time slot $l + p^{[l]}_{3j} g_{12}$.

Lemma 7 *In a feasible schedule for any instance with three distinct periodicities, q_1, q_2 and q_3, with $g = 1$ and $g_{i0} = 1$ for $i = 1, 2, 3$, any client j with the periodicity of q_3 is serviced exactly once in each set Δ_l for $l = 0, \ldots, g_{12} - 1$. Moreover, then its position in Δ_l is given by*

$$p^{[l]}_{3j} = R(\tau_{3j}/g_{12} + l(q_3\bar{q}_3 - 1)/g_{12}, q_3)$$

where $\bar{q}_3$ is the integer, $1 \leq \bar{q}_3 \leq g_{12} - 1$, for which $q_3\bar{q}_3 \equiv 1 \bmod g_{12}$, and $\tau_{3j} \equiv 0 \bmod g_{12}$.

Proof We first show that any client with the periodicity of q_3 is serviced exactly once in each $\mathbf{\Delta}$-set. To that end, it is sufficient to show that $\Gamma = \{\tau_{3j} + q_3 k \bmod g_{12} \mid k = 0, 1, \ldots, g_{12} - 1\} = \{0, 1, \ldots, g_{12} - 1\}$. Suppose otherwise, then $|\Gamma| < g_{12}$, and there must exist k and k', $k \neq k'$, $0 \leq k, k' \leq g_{12} - 1$ such that $\tau_{3j} + q_3 k \equiv \tau_{3j} + q_3 k' \bmod g_{12}$. Thus, $q_3(k - k') \equiv 0 \bmod g_{12}$. Since q_3 and g_{12} are coprime, this implies that $k - k' \equiv 0 \bmod g_{12}$, which contradicts the given choice of k and k' by $|k - k'| < g_{12}$.

Since service is cyclic and $\{\tau^{(k)}_{3j} \mid k = 0, \ldots, g_{12} - 1\}$ covers all elements in set $\{\Delta_l \mid l = 0, \ldots, g_{12} - 1\}$, we may assume without loss of generality that $\tau_{3j} \equiv 0 \bmod g_{12}$, and so $\tau_{3j} \in \Delta_0$. Thus, $\tau^{(k)}_{3j} = \tau_{3j} + kq_3 \equiv l \bmod g_{12}$ for $k \equiv l\bar{q}_3 \bmod g_{12}$. Then, the position of service to client j with the periodicity of q_3 in set Δ_l is

$$\begin{aligned} p^{[l]}_{3j} &= R((\tau^{(k)}_{3j} - l)/g_{12}, q_3) \\ &= R((\tau_{3j} + lq_3\bar{q}_3 - l)/g_{12}, q_3) \\ &= R(\tau_{3j}/g_{12} + l(q_3\bar{q}_3 - 1)/g_{12}, q_3) \end{aligned}$$

for $j = 0, \ldots, n_3$ and $l = 0, \ldots, g_{12} - 1$. □

By Lemma 7, we assume without loss of generality that $\tau_{3j} \in \Delta_0$, and hence the position of subsequent services to a client j with the periodicity of q_3 within a set Δ_l depends only on l and not on k. We therefore introduce the concept of *relative position* within a Δ_l set as

$$\widehat{p} = p - l(q_3\bar{q}_3 - 1)/g_{12}$$

taken modulo q_3, where the original position p within the set corresponds to time slot $l + pg_{12}$. Thus, we can rearrange time slots in each set Δ_l, so that any client with the periodicity of q_3 appears to be serviced in the same relative position in all Δ_l's by putting time slot $R(lq_3\bar{q}_3, T)$ in the first position of set Δ_l. Figure 4 depicts the procedure when $q_1 = 12$, $q_2 = 20$ and $q_3 = 15$ i.e. $g_{12} = 4$, $g_{23} = 5$ and $g_{31} = 3$, and $n_3 = 4$.

Corollary 1 *Each client with the periodicity of q_3 occupies the same single relative position in each of the $\mathbf{\Delta}$-sets, i.e.*

$$\widehat{p}^{[l]}_{3j} = R(\tau_{3j}/g_{12}, q_3) \text{ for } l = 0, \ldots, g_{12} - 1.$$

For any feasible schedule, a relative position $\widehat{p}$ is referred to as *free* if no client with the periodicity of q_1 or q_2 is serviced in the pth relative position of any of the $\mathbf{\Delta}$-sets. Hence, in a feasible schedule, the number of free relative positions is at least as great as n_3 because a client with the periodicity of q_3 is serviced in a particular relative position of each of the $\mathbf{\Delta}$-sets by Corollary 1. In order to establish the precise condition for accommodating n_3 clients with the periodicity of q_3 in Theorem 3, we now examine the interplay between (relative) positions occupied by clients with the periodicities of q_1 or q_2.

Lemma 8 *In a feasible schedule for any instance with three distinct periodicities, q_1, q_2 and q_3, with $g = 1$ and $g_{i0} = 1$ for $i = 1, 2, 3$, for any pair of clients, one with the periodicity of q_1 and the other with the periodicity of q_2, there exist precisely one position in which both clients are serviced.*

Proof Take a client j_1 with the periodicity of q_1 and a client j_2 with the periodicity of q_2. If there is no position which serves both clients, then both clients can be serviced in same set, which contradicts to the results of Lemma 1.

If there is more than one position within T which serves the two given clients, then there exist integers k_1, k_1', k_2 and k_2' such that $\tau_{1j_1}^{(k_1)} - l_1 = \tau_{2j_2}^{(k_2)} - l_2$ and $\tau_{1j_1}^{(k_1')} - l_1 = \tau_{2j_2}^{(k_2')} - l_2$ where $0 \le k_1, k_1' \le g_{23}, 0 \le k_2, k_2' \le g_{31}, k_1 \ne k_1', k_2 \ne k_2'$. Thus, $\tau_{1j_1}^{(k_1)} - \tau_{2j_2}^{(k_2)} = l_1 - l_2 = \tau_{1j_1}^{(k_1')} - \tau_{2j_2}^{(k_2')}$, and hence $k_1q_1 - k_2q_2 = k_1'q_1 - k_2'q_2$, and therefore $(k_1 - k_1')q_1 = (k_2 - k_2')q_2$. Since $g_{12} > 0$ and g_{23} is coprime to g_{31}, g_{23} divides $|k_1 - k_1'|$ and g_{31} divides $|k_2 - k_2'|$, this implies, for the given range of the k variables, that $k_1 = k_1'$ and $k_2 = k_2'$ completing the proof. □

Corollary 2 *Consider an instance of the PPS problem with three distinct periodicities, q_1, q_2 and q_3, with $g = 1$ and $g_{i0} = 1$ for $i = 1, 2, 3$. Suppose that α clients with the periodicity of q_1 are serviced in a single $\mathbf{\Delta}$-set and β clients with the periodicity of q_2 in another $\mathbf{\Delta}$-set. Then, there exist precisely $\alpha\beta$ positions which serve both a client with the periodicities q_1 and a client with the periodicity of q_2.*

Proof From the result of Lemma 8, any pairs of clients with the periodicities of q_1 and q_2 has precisely one position in which both clients are serviced. There are $\alpha\beta$ different pairs of clients, one with the periodicity of q_1 and the other with the periodicity of q_2. Thus, there are $\alpha\beta$ positions in which both clients with the periodicities q_1 and q_2 are serviced. □

Theorem 3 *Consider an instance with clients of each of three distinct periodicities, q_1, q_2 and q_3, with $g = 1$ and $g_{i0} = 1$ for $i = 1, 2, 3$. The instance is schedulable if and only if there exists an integer x, $1 \le x \le g_{12} - 1$ such that*

$$\left\lceil \frac{n_1}{x} \right\rceil \le g_{31} - 1, \tag{5}$$

$$\left\lceil \frac{n_2}{g_{12} - x} \right\rceil \le g_{23} - 1, \tag{6}$$

$$n_3 \le \left(g_{31} - \left\lceil \frac{n_1}{x} \right\rceil \right) \left(g_{23} - \left\lceil \frac{n_2}{g_{12} - x} \right\rceil \right). \tag{7}$$

Proof (⇒) Take a feasible solution. Let x_i denote the number of sets of $\mathbf{\Delta}$ serving clients with the periodicity of q_i, respectively, for $i = 1, 2$. Let α and β denote the maximum number of clients with the periodicity of q_1 and q_2, respectively, serviced in a single set in $\mathbf{\Delta}$. Thus, $\alpha \ge \lceil n_1/x_1 \rceil$ and $\beta \ge \lceil n_2/x_2 \rceil$. Note that $|\Delta_l| = g_{23}g_{31}$ and that a client with the periodicity of q_1 and q_2 occurs exactly g_{23} and g_{31} times, respectively, in a single set in $\mathbf{\Delta}$. Hence, from the result of Corollary 2, the maximum number of free relative positions in Δ_l is

$$\begin{aligned} &g_{23}g_{31} - (g_{23}\alpha + g_{31}\beta - \alpha\beta) \\ &\quad = (g_{31} - \alpha)(g_{23} - \beta) \\ &\quad \le \left(g_{31} - \left\lceil \frac{n_1}{x_1} \right\rceil \right) \left(g_{23} - \left\lceil \frac{n_2}{x_2} \right\rceil \right). \end{aligned}$$

Now $x_2 \le g_{12} – x_1$ ensures that $\lceil n_2/x_2 \rceil \ge \lceil n_2/(g_{12} - x_1) \rceil$. Since each $\mathbf{\Delta}$-set has at least n_3 time slots to accommodate the n_3 clients with the periodicity of q_3, inequality (7) is satisfied by setting $x = x_1$. Moreover, since a client with the periodicity of q_3 is serviced once in each $\mathbf{\Delta}$-set by Lemma 7, the maximum number of clients with the periodicity of q_1 (or q_2) in a $\mathbf{\Delta}$-set, α (or β), is at most $g_{31} - 1$ (or $g_{23} - 1$), implying conditions (5) and (6).
(⇐) Now suppose that there exists an integer x, $1 \le x \le g_{12} - 1$, satisfying the inequalities, (5), (6) and (7). Let $\alpha = \lceil n_1/x \rceil$ and $\beta = \lceil n_2/(g_{12} - x) \rceil$. We construct a feasible solution as follows. We schedule the first time slots of the clients with the periodicity of q_1 at

$$\begin{aligned} \tau_{1j} &= R(q_3\bar{q}_3 \lfloor j/\alpha \rfloor + R(j, \alpha)g_{12},\ q_1) \\ &= \left\lfloor \frac{j}{\alpha} \right\rfloor + \left(\frac{q_3\bar{q}_3 - 1}{g_{12}} \left\lfloor \frac{j}{\alpha} \right\rfloor + R(j, \alpha) \right) g_{12} \bmod q_1, \end{aligned} \tag{8}$$

for $j = 0, \ldots, n_1 - 1$. Observe that client j with the periodicity of q_1 is in position $l(q_3\bar{q}_3 - 1)/g_{12} + R(j, \alpha)$ of set Δ_l, for $l = \lfloor j/\alpha \rfloor$. Thus, there are α clients with the periodicity of q_1 in each set Δ_l for $l = 0, \ldots, x - 2$, and the remainder, i.e. $n_1 - \alpha(x - 1)$ clients, in Δ_{x-1}. The first time slots of the clients with the periodicity of q_2 is placed in time slots

$$\tau_{2j} = R(q_3\bar{q}_3 (x + \lfloor j/\beta \rfloor) + R(j, \beta)g_{12},\ q_2), \tag{9}$$

for $j = 0, \ldots, n_2 - 1$. Client j with the periodicity of q_2 is therefore in position $l(q_3\bar{q}_3 - 1)/g_{12} + R(j, \beta)$ of set Δ_l, for $l = x + \lfloor j/\beta \rfloor$. Thus, there are β clients with the periodicity of q_2 in each set Δ_l for $l = x, \ldots, g_{12} - 2$, and the remainder, i.e. $n_2 - \beta(g_{12} - x - 1)$, clients in $\Delta_{g_{12}-1}$.

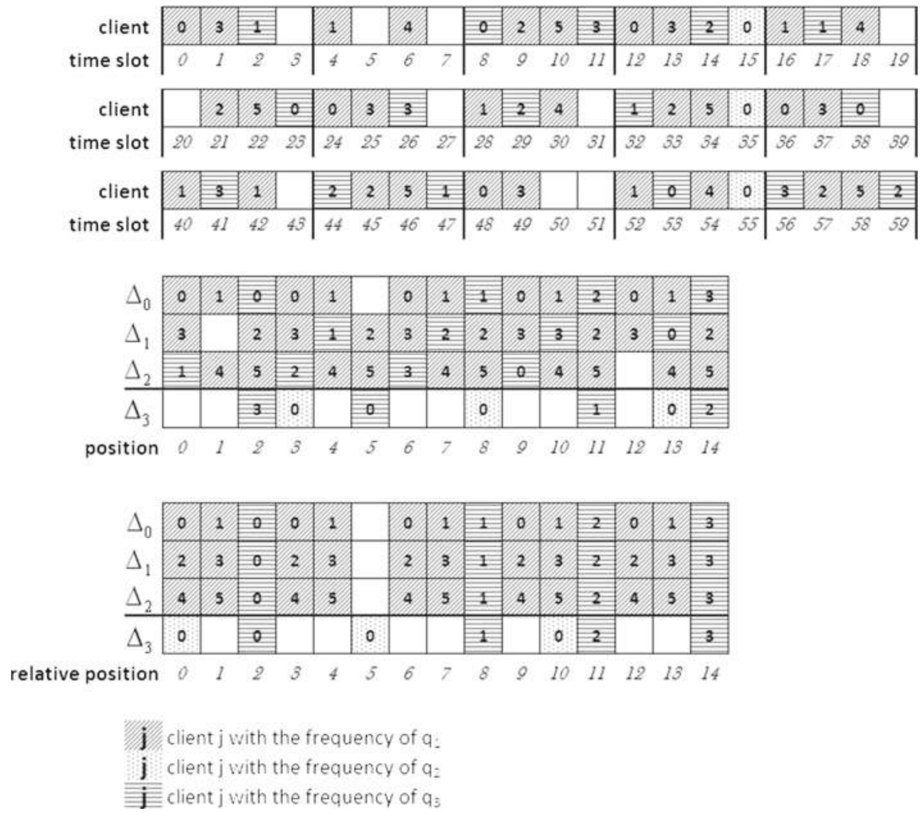


Fig. 5 A solution for Example 2 **a** schedule in $[0, T-1]$, **b** partition of schedule into $\mathbf{\Delta}$-set

Then each client with the periodicity of q_1 in Δ_0 occupies $g_{23} = T/q_1$ positions, and the each client with the periodicity of q_2 in Δ_x occupies $g_{31} = T/q_2$ positions. Moreover, the above construction ensures that all clients with the periodicity of q_1 are scheduled in the same relative positions within a set as those in set Δ_0, and all clients with the periodicity of q_2 are in the relative positions used in set Δ_x. It thus follows from Corollary 2 that $g_{23}\alpha + g_{31}\beta - \alpha\beta$ relative positions are occupied by clients with the periodicities of q_1 and q_2. The remaining $g_{23}g_{31} - (g_{23}\alpha + g_{31}\beta - \alpha\beta) = (g_{31} - \alpha)(g_{23} - \beta)$ relative positions remain for clients with the periodicity of q_3. From (7), there are at least n_3 such free relative positions and the clients with the periodicity of q_3 may therefore be scheduled as follows. Schedule the first time slot of the client j with the periodicity of q_3 at

$$\tau_{3j} = R(\widehat{p}_j g_{12}, q_3)$$

for $j = 0, \ldots, n_3 - 1$, where $\widehat{p}_j$ denote $(j+1)$th free relative position in Δ_0. □

Observe that the conditions for feasibility given in Theorem 3 are asymmetric in the three periodicities, which at first sight seems odd. However, the conditions are in fact symmetric in the periodicities as proved in Appendix 1. The following Lemma studies the computational complexity of finding an integer x referred to in Theorem 3.

Lemma 9 *For an instance with three distinct periodicities with $g = 1$ and $g_{i0} = 1$ for $i = 1, 2, 3$, there is a test for feasibility whose computational complexity is $O(n)$.*

Proof From the result of Theorem 3, the feasibility test can be done in $O(g_{12})$ steps by examining each integer x, $1 \leq x \leq g_{12} - 1$ and testing inequality (7). If $g_{12} \leq 2\max\{n_1, n_2\}$, then there are $O(n)$ operations as claimed since $\max\{n_1, n_2\} \leq n$. On the other hand, if $g_{12} > 2\max\{n_1, n_2\}$, then set $x = \max\{n_1, n_2\}$. Then, the inequality (7) reduces to $n_3 \leq (g_{31} - 1)(g_{23} - 1)$. As a result, the complexity of feasibility test is $O(n)$. □

The following example illustrates the construction of a perfect periodic schedule described in Theorem 3.

Example 2 Consider the case of three periodicities $q_1 = 12$, $q_2 = 20$ and $q_3 = 15$ (ie. $g_{12} = 4$, $g_{23} = 5$ and $g_{31} = 3$) with $n_1 = 6$, $n_2 = 1$ and $n_3 = 4$, respectively. This instance is schedulable since taking $x = 3$ satisfies $x \leq 3 = g_{12} - 1$ and conditions (5)–(7) of Theorem 3. The construction of the above proof then results in the feasible schedule presented in Fig. 5.

4.3 Case of $gcd(q_1, q_2, q_3) \geq 2$, for the reduced problem

In this section, we consider the more general case of three distinct periodicities when $g = gcd(q_1, q_2, q_3) \geq 1$, while maintaining the restriction that $g_{i0} = 1$ for $i = 1, 2, 3$. Our focus is that of describing the feasible solution space, which forms the theoretical basis of the test for feasibility presented in Sect. 5 and the algorithm for constructing a feasible solution in Sect. 6.

Throughout this section, we use a partition, $\boldsymbol{\Lambda}$, of time slots $\{0, 1, \ldots, T-1\}$ into sets $\Lambda_l = \{t \mid t \equiv l \bmod g, 0 \le t \le T-1\}$ for $l = 0, \ldots, g-1$. We denote the number of sets serving clients with just one periodicity of q_i by λ_i, the number serving two separate periodicities, i and i', by $\lambda_{ii'}$, for $i \ne i'$ and $i, i' \in \{1, 2, 3\}$ and the number serving all three periodicities by λ_{123}. The number of clients with the periodicity of q_1 in set $\lambda_1, \lambda_{12}, \lambda_{31}, \lambda_{123}$ are denoted by n_{10}, n_{12}, n_{13}, μ_1, respectively, and similar notation is used for the other periodicities. Observe that $n_1 = n_{10} + n_{12} + n_{13} + \mu_1$, $n_2 = n_{20} + n_{21} + n_{23} + \mu_2$ and $n_3 = n_{30} + n_{31} + n_{32} + \mu_3$, and denote the distribution of clients to sets in $\boldsymbol{\Lambda}$ by $\underline{n} = (n_{10}, n_{20}, n_{30}, n_{12}, n_{21}, n_{23}, n_{32}, n_{31}, n_{13}, \mu_1, \mu_2, \mu_3)$.

We now analyse the relationship between the number of the sets of each type, the λs, and the number of clients of each periodicity distributed to these sets, in Lemmas 10–12. We will then return to the issue of testing for the existence of a feasible solution in Theorem 4.

Lemma 10 *A feasible PPS instance with three distinct periodicities, q_1, q_2 and q_3, and $g_{i0} = 1$ for $i = 1, 2, 3$ has a solution in which λ_{12}, λ_{23} and λ_{31} are each at most one.*

Proof Take an instance for which any such feasible schedule has at least one of λ_{12}, λ_{23} and λ_{31} greater than 1. Take a solution with smallest possible $\max(\lambda_{12}, \lambda_{23}, \lambda_{31}) = \lambda_{12}$ say. Let Λ and Λ' denote two of the sets serving clients with the periodicities of q_1 and q_2.

Suppose that n_1 and n_2 clients with the periodicity of q_1 and q_2, respectively, are serviced together in Λ_l, and that n_1' and n_2' clients with the periodicity of q_1 and q_2, respectively, are serviced together in $\Lambda_{l'}$. The schedule with the set Λ_l (or $\Lambda_{l'}$) may be viewed as a single schedule for the clients n_1 and n_2 (or n_1' and n_2') alone with the periodicity of $q_1' = q_1/g$ and $q_2' = q_2/g$, respectively. Thus, $gcd(q_1', q_2') = g_{12}$, and by Theorem 1,

$$\left\lceil \frac{n_1}{q_1/gg_{12}} \right\rceil + \left\lceil \frac{n_2}{q_2/gg_{12}} \right\rceil \le g_{12} \quad \text{and}$$
$$\left\lceil \frac{n_1'}{q_1/gg_{12}} \right\rceil + \left\lceil \frac{n_2'}{q_2/gg_{12}} \right\rceil \le g_{12}.$$

If $\lceil n_2/(q_2/gg_{12}) \rceil \ge \lceil n_1'/(q_1/gg_{12}) \rceil$, then let $\widetilde{n}_2 = \min\{q_2/gg_{12} \lceil n_1'/(q_1/gg_{12}) \rceil, n_2\}$, and thus

$$\left\lceil \frac{n_1 + n_1'}{q_1/gg_{12}} \right\rceil + \left\lceil \frac{n_2 - \widetilde{n}_2}{q_2/gg_{12}} \right\rceil \le g_{12} \quad \text{and}$$
$$\left\lceil \frac{\widetilde{n}_2 + n_2'}{q_2/gg_{12}} \right\rceil \le \left\lceil \frac{\widetilde{n}_2}{q_2/gg_{12}} \right\rceil + \left\lceil \frac{n_2'}{q_2/gg_{12}} \right\rceil \le g_{12}.$$

Hence, exchanging $\widetilde{n}_2$ clients of the n_2 clients with the periodicity of q_2 in Λ_l with all n_1' clients with the periodicity of q_1 in $\Lambda_{l'}$ results in time slots in a set $\Lambda_{l'}$ serving only clients with the periodicity of q_2 and Λ_{12} has been reduced by 1. If $\lceil n_2/(q_2/gg_{12}) \rceil < \lceil n_1'/(q_1/gg_{12}) \rceil$, then the same argument applies with clients in Λ_l exchanged with clients in $\Lambda_{l'}$, providing the required contradiction. □

A similar result holds for λ_{123}, as captured in Lemma 11 below, but its proof is more lengthy and therefore relegated to Appendix 2.

Lemma 11 *A feasible PPS instance with three distinct periodicities, q_1, q_2 and q_3, and $g_{i0} = 1$ for $i = 1, 2, 3$ has a solution in which $\lambda_{123} \le 1$.*

Lemma 12 *A feasible PPS instance with three distinct periodicities, q_1, q_2 and q_3, and $g_{i0} = 1$ for $i = 1, 2, 3$ has a solution in which λ_{12}, λ_{23}, λ_{31} and λ_{123} are each at most one.*

Proof The result follows by taking a feasible solution which satisfies Lemma 11, i.e. with $\lambda_{123} \le 1$, and applying the exchange described in the proof of Lemma 10. Since no new $\boldsymbol{\Lambda}$-set with three periodicities is created by the exchange, λ_{123} is not increased, and the resultant solution satisfies Lemma 12. □

Theorem 4 *For an instance with three distinct periodicities, q_1, q_2 and q_3, and $g_{i0} = 1$ for $i = 1, 2, 3$ which has a feasible solution, there exist an (non-negative) integer vector $\underline{n} = (n_{10}, n_{20}, n_{30}, n_{12}, n_{21}, n_{23}, n_{32}, n_{31}, n_{13}, \mu_1, \mu_2, \mu_3)$, and non-negative integers, λ_1, λ_2, λ_3, λ_{12}, λ_{23}, λ_{31} and λ_{123} for which*

$$\sum_{i'=0, i' \ne i}^{3} n_{ii'} + \mu_i = n_i \quad \text{for } i = 1, 2, 3 \tag{10}$$

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_{12} + \lambda_{23} + \lambda_{31} + \lambda_{123} \le g \tag{11}$$

$$\left\lceil \frac{n_{i0}}{q_i/g} \right\rceil = \lambda_i \quad \text{for } i = 1, 2, 3 \tag{12}$$

$$\lambda_{12}, \lambda_{23}, \lambda_{31} \le 1 \tag{13}$$

$$\left\lceil \frac{n_{12}}{q_1/(gg_{12})} \right\rceil + \left\lceil \frac{n_{21}}{q_2/(gg_{12})} \right\rceil \le g_{12}\lambda_{12} \tag{14}$$

$$\left\lceil \frac{n_{23}}{q_2/(gg_{23})} \right\rceil + \left\lceil \frac{n_{32}}{q_3/(gg_{23})} \right\rceil \le g_{23}\lambda_{23} \tag{15}$$

$$\left\lceil \frac{n_{31}}{q_3/(gg_{31})} \right\rceil + \left\lceil \frac{n_{13}}{q_1/(gg_{31})} \right\rceil \le g_{31}\lambda_{31} \tag{16}$$

and

$$\lambda_{123} \le 1 \tag{17}$$

$$(1 - \lambda_{123})(\mu_1 + \mu_2 + \mu_3) = 0 \tag{18}$$

$$\left\lceil \frac{\mu_1}{x} \right\rceil \le g_{31} - 1 \tag{19}$$

$$\left\lceil \frac{\mu_2}{g_{12}-x} \right\rceil \le g_{23}-1 \qquad (20)$$

$$\left(g_{31}-\left\lceil \frac{\mu_1}{x} \right\rceil\right)\left(g_{23}-\left\lceil \frac{\mu_2}{g_{12}-x} \right\rceil\right) \ge \mu_3 \qquad (21)$$

for some integer x, $1 \le x \le g_{12}-1$.

Proof Consider a feasible schedule for an instance of PPS problem with three distinct periodicities, with the notation of $\boldsymbol{\Lambda}$-sets and number of clients, $n_{ii'}$, as defined at the beginning of this section. For the solution to be feasible, the total number of $\boldsymbol{\Lambda}$-sets required can be no more than g, as captured by (11). We consider each type of $\boldsymbol{\Lambda}$-set in turn: those with clients of one, two and then three different periodicities. Without loss of generality, we may assume from Lemma 12 that there is no more than one $\boldsymbol{\Lambda}$-set for clients with each possible combination of two or three periodicities within the set, implying (13) and (17).

The $\boldsymbol{\Lambda}$-sets with just clients of a single periodicity, q_i say, can each service up to q_i/g clients by Lemma 3. The number of sets required to schedule the clients associated with n_{i0}, λ_{i0}, is therefore $\lceil n_{i0}/(q_i/g)\rceil$ as given by (12) for $i=1,2,3$.

Consider $\boldsymbol{\Lambda}$-sets with clients of two different periodicities. If $\lambda_{12}=0$, then (14) holds for $n_{12}=n_{21}=0$. Otherwise, $\lambda_{12}=1$, and n_{12} and n_{21} clients with the periodicity of q_1 and q_2, respectively, are serviced together in a single $\boldsymbol{\Lambda}$-set. From the result of Theorem 1 applied for periodicities of q_1/g and q_2/g,

$$\left\lceil \frac{n_{12}}{q_1/(gg_{12})} \right\rceil + \left\lceil \frac{n_{21}}{q_2/(gg_{12})} \right\rceil \le g_{12}\lambda_{12}$$

as given by (14). Similar expressions hold for the other two pairs of periodicities, and are given by (15) and (16), respectively.

Finally, consider a $\boldsymbol{\Lambda}$-set with clients of three different periodicities. If $\lambda_{123}=0$, then there is no set with clients of each of the three periodicities, $\mu_i=0$ for $i=1,2,3$, and inequalities (18), (19), (20) and (21) hold for $x=1$. Otherwise, $\lambda_{123}=1$, (18) holds, and the μ_1, μ_2 and μ_3 clients with the periodicity of q_1, q_2 and q_3, respectively, are serviced in a single $\boldsymbol{\Lambda}$-set. From Theorem 3 applied for periodicities of $q_i'=q_i/g$ for $i=1,2,3$, the inequalities (19), (20) and (21) hold for some integer x, $1\le x\le g_{12}-1$, completing the proof. □

5 Feasibility test for three distinct periodicities

In this section, we develop a polynomial test for feasibility, based upon the results of Theorems 3 and 4. The feasibility test systematically searches for a vector $\underline{n}$ to satisfy constraints (10)–(21) of Theorem 4. In doing so, it reveals information about a feasible solution, where one exists. However, it is convenient to reserve the lengthy construction of the feasible solution itself for a separate algorithm presented in the next section. The feasibility test, FeasTest, is developed in two parts. We first present a subprocedure, RedFeas, for finding an integer vector $\underline{n}$ with $\mu_1=\mu_2=\mu_3=0$, for reduced instances. Then, we extend this algorithm to cover the general case, where g_{i0} and μ_{i0} for $i=1,2,3$ may take non-zero values.

Algorithm RedFeas: *Test for the existence of a feasible schedule with* $\mu_1=\mu_2=\mu_3=0$, *for the reduced problem with three distinct periodicities*

Input: $q_1, q_2, q_3, n_1, n_2, n_3$

Output: $(\lambda^A, \underline{n}^A) = (\lambda^A, n_{10}, n_{20}, n_{30}, n_{12}, n_{21}, n_{23}, n_{32}, n_{31}, n_{13})$

Code:

1. *Initialization*
   $g \leftarrow gcd(q_1,q_2,q_3)$, $g_{12} \leftarrow gcd(q_1,q_2)/g$, $g_{23} \leftarrow gcd(q_2,q_3)/g$, $g_{31} \leftarrow gcd(q_3,q_1)/g$,
   $\underline{n}^A \leftarrow (n_1,n_2,n_3,0,0,0,0,0,0)$,
   $\lambda^A \leftarrow \sum_{i=1}^{3}\lceil n_i/(q_i/g)\rceil$.
2. *Establish a solution with minimum number of sets with clients of one or two periodicities only*
   For $\delta=0,1,\dots,\min\{\lceil n_1/(q_1/gg_{12})\rceil,\ g_{12}-1\}$,
   - 2.1 *Set* n_{12}, λ_{12} *and* n_{21}
     $n_{12} \leftarrow \min\{n_1,\ \delta q_1/(gg_{12})\}$.
     If $n_{12}=0$, then $\lambda_{12}\leftarrow 0$ and $n_{21}\leftarrow 0$.
     Otherwise $\lambda_{12}\leftarrow 1$ and
     $n_{21} \leftarrow \min\{n_2,\ q_2/g - q_2/(gg_{12})\lceil n_{12}/(q_1/gg_{12})\rceil\}$.
   - 2.2 *Set* n_{23}, λ_{23} *and* n_{32}
     $n_{23} \leftarrow R(n_2-n_{21}, q_2/g)$.
     If $n_{23}=0$ or $\lceil n_{23}/(q_2/gg_{23})\rceil = g_{23}$ or $n_3=0$, then $\lambda_{23}\leftarrow 0$, and $n_{23}\leftarrow 0$ and $n_{32}\leftarrow 0$.
     Otherwise $\lambda_{23}\leftarrow 1$ and
     $n_{32} \leftarrow \min\{n_3,\ q_3/g - q_3/(gg_{23})\lceil n_{23}/(q_2/gg_{23})\rceil\}$.
   - 2.3 *Set* n_{31}, λ_{31} *and* n_{13}
     $n_{31} \leftarrow R(n_3-n_{32}, q_3/g)$.
     If $n_{31}=0$ or $\lceil n_{31}/(q_3/gg_{31})\rceil = g_{31}$ or $n_1=n_{12}$, then $\lambda_{31}\leftarrow 0$, and $n_{31}\leftarrow 0$ and $n_{13}\leftarrow 0$.
     Otherwise $\lambda_{31}\leftarrow 1$ and
     $n_{13} \leftarrow \min\{n_1-n_{12},\ q_1/g - q_1/(gg_{31})\lceil n_{31}/(q_3/gg_{31})\rceil\}$.
   - 2.4 *Set* n_{10}, n_{20} *and* n_{30}
     $n_{10} \leftarrow n_1-n_{12}-n_{13}$.
     $n_{20} \leftarrow n_2-n_{21}-n_{23}$.
     $n_{30} \leftarrow n_3-n_{31}-n_{32}$.
   - 2.5 *Identify an improved solution*
     $\underline{n} \leftarrow (n_{10}, n_{20}, n_{30}, n_{12}, n_{21}, n_{23}, n_{32}, n_{31}, n_{13})$.
     $\lambda \leftarrow \lceil n_{10}/(q_1/g)\rceil + \lceil n_{20}/(q_2/g)\rceil + \lceil n_{30}/(q_3/g)\rceil + \lambda_{12}+\lambda_{23}+\lambda_{31}$.
     If $\lambda<\lambda^A$, then $\lambda^A\leftarrow\lambda$ and $\underline{n}^A\leftarrow\underline{n}$.
3. **Return** $(\lambda^A, \underline{n}^A)$ and **stop**.

Observe that the computational complexity of Algorithm RedFeas is determined by the number of iterations of Step 2, which is itself bounded by n because $n_1 \le n$ and $q_1/(gg_{12}) = g_{31} \ge 1$. Therefore, the overall complexity of Algorithm RedFeas is $O(n)$. We now verify that RedFeas is indeed a feasibility test.

Lemma 13 *Consider an instance with $g_{i0} = 1$ for $i = 1, 2, 3$, for which there exists a feasible solution with $\lambda_{123} = 0$. Then, Algorithm RedFeas returns an integer vector $\underline{n}$ with $\mu_1 = \mu_2 = \mu_3 = 0$ corresponding to a feasible solution.*

Proof Observe that Algorithm RedFeas produces a solution of a particular form. It considers all possible values for n_{12} which satisfies (22). Then, all other values are determined iteratively as follows: $n_{ii'}$ for $i, i' = 1, 2, 3, i \neq i'$ in Step 2.1–2.3 as in (23)–(27) below, followed by n_{10}, n_{20}, n_{30} in Step 2.4 as in (28)–(30). Therefore, Algorithm RedFeas tests for all solutions of this form.

$$n_{12} = n_1 \text{ or } \delta q_1/gg_{12} \text{ for some } \delta \in \{0, 1, \ldots, g_{12} - 1\} \tag{22}$$

$$n_{21} = \begin{cases} 0 & \text{if } n_{12} = 0 \\ \min\{n_2,\ q_2/g - q_2/(gg_{12}) \lceil n_{12}/(q_1/gg_{12}) \rceil\} & \text{otherwise} \end{cases} \tag{23}$$

$$n_{23} = \begin{cases} 0 & \text{if } R(n_2 - n_{21}, q_2/gg_{23}) > (g_{23} - 1)q_2/(gg_{23}) \\ R(n_2 - n_{21}, q_2/gg_{23}) & \text{otherwise} \end{cases} \tag{24}$$

$$n_{32} = \begin{cases} 0 & \text{if } n_{23} = 0 \\ \min\{n_3,\ q_3/g - q_3/(gg_{23}) \lceil n_{23}/(q_2/gg_{23}) \rceil\} & \text{otherwise} \end{cases} \tag{25}$$

$$n_{31} = \begin{cases} 0 & \text{if } R(n_3 - n_{32}, q_3/gg_{31}) > (g_{31} - 1)q_3/(gg_{31}) \text{ or } n_1 = n_{12} \\ R(n_3 - n_{32}, q_3/gg_{31}) & \text{otherwise} \end{cases} \tag{26}$$

$$n_{13} = \begin{cases} 0 & \text{if } n_{31} = 0 \\ \min\{n_1 - n_{12},\ q_1/g - q_1/(gg_{31}) \\ \lceil n_{31}/(q_3/gg_{31}) \rceil\} & \text{otherwise} \end{cases} \tag{27}$$

$$n_{10} = n_1 - n_{12} - n_{13} \tag{28}$$

$$n_{20} = n_2 - n_{21} - n_{23} \tag{29}$$

$$n_{30} = n_3 - n_{31} - n_{32} \tag{30}$$

Now take a feasible solution $\underline{n}' = (n'_{10}, n'_{20}, n'_{30}, n'_{12}, n'_{21}, n'_{23}, n'_{32}, n'_{31}, n'_{13}, 0, 0, 0)$ with $\lambda_{123} = 0$, for which without loss of generality, $\lambda_{12} \leq 1$, $\lambda_{23} \leq 1$ and $\lambda_{31} \leq 1$ by Lemma 12, and λ_{23} is as small as possible and for this minimum λ_{23} value, λ_{31} is as small as possible. We now show that this feasible solution with $\lambda_{123} = 0$ can be modified to one which satisfies (22)–(30).

We first modify n'_{12} and n'_{21}, if required. If $\lambda_{12} = 0$, then n'_{12} and n'_{21} remain unchanged at 0 (i.e. we set $n_{12} = n_{21} = 0$). Otherwise, $\lambda_{12} = 1$ (and $n'_{12} \geq 1$ and $n'_{21} \geq 1$). By (14) of Theorem 4, we may increase n'_{12} up to $n_{12} = \min\{n_1, (q_1/gg_{12})\lceil n_{12}/(q_1/gg_{12})\rceil\}$ and n'_{21} to $n_{21} = \min\{n_2,\ q_2/g - q_2/(gg_{12}) \lceil n_{12}/(q_1/(gg_{12}))\rceil\}$, respectively, without increasing λ. Observe that n_{12} and n_{21} satisfies (22) and (23), respectively.

We now modify n'_{23} and n'_{20}, having determined n_{21}. Suppose that $\lambda_{23} = 0$ (and $n'_{23} = 0$). If $R(n_2 - n_{21}, q_2/g) > (g_{23} - 1)q_2/(gg_{23})$ (and $\lambda_{23} = 0$), then n'_{23} is left unchanged at 0 and n'_{20} modified to $n_{20} = n_2 - n_{21}$, satisfying (29) without increasing λ_2 (since $n_{21} \geq n'_{21}$). If on the other hand $R(n_2 - n_{21}, q_2/g) \leq (g_{23} - 1)q_2/(gg_{23})$, then we modify n'_{23} to $n_{23} = R(n_2 - n_{21}, q_2/g)$ and n'_{20} to $n_{20} = n_2 - n_{21} - n_{23}$ preserving (29), which results in $\lambda_{23} = 1$ and λ_2 is decreased by one, leaving λ unchanged. Thus, (24) is satisfied for the modified solution in the case when $\lambda_{23} = 0$, for the original solution. Now suppose that $\lambda_{23} = 1$ (and $n'_{23} \geq 1$). Then, by hypothesis, there is no feasible solution for given λ_{12} value, with $\lambda_{23} = 0$. The λ_2 value is unaffected by increasing n'_{20}, if necessary, to $n_{20} = q_2/g\lceil n'_{20}/(q_2/g)\rceil$ from (12) of Theorem 4. Thus, n'_{23} may be decreased to $n_{23} = n_2 - n_{21} - n_{20} = R(n_2 - n_{21}, q_2/g) > 0$ satisfying (24) and reserving (29), without increasing λ.

We now modify n'_{32}. Recall that $n_{23} = 0$ only in the case when $n'_{23} = 0$ (and $n'_{32} = 0$) and we then leave n'_{32} unmodified, i.e. we set $n_{32} = 0$, without increasing λ. Now suppose that $n_{23} \geq 1$. By (15) of Theorem 4, we modify n'_{32} to $n_{32} = \min\{n_3,\ q_3/g - q_3/(gg_{23}) \lceil n_{23}/(q_2/gg_{23})\rceil\}$ without increasing λ. Thus, n_{32} satisfies (25).

We now modify n'_{31} and n'_{30} given n_{32} to satisfy (26) and then (30), respectively. The validity of the modification follows by the same argument as used above for n'_{23} and n'_{20}, but for the permitted indices, since λ_{31} (as well as λ_{23}) is selected by as small as possible.

Finally, we modify n'_{13} and n'_{10}. In the case that $n_{31} = 0$, $n'_{31} = 0$ and $n'_{13} = 0$ in the original feasible solution, and we set $n_{13} = 0$ and $n_{10} = n_1 - n_{12} - n_{13}$ without increasing λ. Now suppose that $n_{31} \geq 1$. By (16) of Theorem 4, we may modify n'_{13} to $n_{13} = \min\{n_1 - n_{12},\ q_1/g - q_1/(gg_{31}) \lceil n_{31}/(q_3/gg_{31})\rceil\}$ and n'_{10} to $n_{10} = n_1 - n_{12} - n_{13}$ without increasing λ. Thus, n_{13} and n_{10} satisfies (27) and (28), respectively. □

We now develop an algorithm allowing a $\mathbf{\Lambda}$-set with three periodicities. By Lemma 11, it is sufficient to explore all combinations of constructing at most one $\mathbf{\Lambda}-$set with three periodicities. We now give a formal description of the algorithm.

Observe that μ_1, μ_2 and μ_3 are each bounded by n. By Lemma 9, finding an integer x in Step 4 can be done in $O(n)$. Therefore, the overall complexity of FeasTest is $O(n^4)$.

Lemma 14 *Consider an instance for which there exists a feasible solution. Then, Algorithm FeasTest returns an integer vector $\underline{n}$, for which λ_{12}, λ_{23}, λ_{31} and λ_{123} are each at most one for the corresponding reduced problem.*

Proof Taking a feasible solution for any given instance. By Theorem 2, there is a corresponding feasible solution to the reduced problem instance, $\underline{n}$ say. We make another instance by letting $\widetilde{n}_i = n_i - \mu_i$ for $i = 1, 2, 3$. Then, since there exists a feasible solution for the original (reduced) instance, there must be a feasible solution with $\lambda_{123} = 0$ for the new (reduced) instance. By Lemma 13, a feasible solution with $\lambda_{12} \leq 1, \lambda_{23} \leq 1$ and $\lambda_{31} \leq 1$ for the new (reduced) instance will be identified by Algorithm RedFeas. Since Algorithm FeasTest explores all combination of μ_i for $i = 1, 2, 3$, it will

Algorithm FeasTest: *Test for the existence of a feasible schedule allowing a* $\boldsymbol{\Lambda}$*-set with three periodicities, for the reduced problem with three distinct periodicities*
Input: $q_1, q_2, q_3, n_1, n_2, n_3$
Output: $\underline{n}^B = (n_{10}, n_{20}, n_{30}, n_{12}, n_{21}, n_{23}, n_{32}, n_{31}, n_{13}, \mu_1, \mu_2, \mu_3)$ *or infeasible*
Code:

1. *Test for density*
   If $\sum_{i=1}^{3} n_i/q_i > 1$, then **return** *infeasible* and **stop**.
2. *Initialisation and problem reduction*
   $g \leftarrow gcd(q_1, q_2, q_3)$, $g_{12} \leftarrow gcd(q_1, q_2)/g$,
   $g_{23} \leftarrow gcd(q_2, q_3)/g$, $g_{31} \leftarrow gcd(q_3, q_1)/g$,
   $g_{10} \leftarrow q_1/(gg_{12}g_{31})$, $g_{20} \leftarrow q_2/(gg_{12}g_{23})$, $g_{30} \leftarrow q_3/(gg_{23}g_{31})$.
   For $i = 1, \ldots, 3$,
   $q_i \leftarrow q_i/g_{i0}$ and $n_i \leftarrow \lceil n_i/g_{i0} \rceil$.
3. *Search for solution of sets with clients of one or two periodicities only*
   Run Algorithm RedFeas$(q_1, q_2, q_3, n_1, n_2, n_3 : \lambda^A, \underline{n}^A)$.
   If $\lambda^A \leq g$, then **return** $\underline{n}^B = (\underline{n}^A, 0, 0, 0)$ and **stop**.
4. *Search remaining solution domain*
   For $\mu_1 = 1, \ldots, \min\{n_1, (q_1/g) - 1\}$,
   $\mu_2 = 1, \ldots, \min\{n_2, (q_2/g) - 1\}$ and $\mu_3 = 1, \ldots, \min\{n_3, (q_3/g) - 1\}$
   For $x = 1, \ldots, g_{12} - 1$
   If $\lceil \mu_1/x \rceil \leq g_{31} - 1$, and $\lceil \mu_2/(g_{12} - x) \rceil \leq g_{23} - 1$, and $\mu_3 \leq (g_{31} - \lceil \mu_1/x \rceil)(g_{23} - \lceil \mu_2/(g_{12} - x) \rceil)$, then
   Run Algorithm RedFeas$(q_1, q_2, q_3, n_1 - \mu_1, n_2 - \mu_2, n_3 - \mu_3 : \lambda^A, \underline{n}^A)$
   If $\lambda^A \leq g - 1$, then **return** $\underline{n}^B = (\underline{n}^A, \mu_1, \mu_2, \mu_3)$ and **stop**.
5. **Return** *infeasible*.

find an integer vector $\underline{n}$ corresponding to a feasible solution for the given (reduced) instance if one exists, and hence, equivalently, to the original instance by Theorem 2. □

6 Construction of a feasible schedule for three distinct periodicities

In this section, we develop a construction for a feasible perfect periodic schedule with three distinct periodicities, when one exists. We build upon the feasibility tests of the last section, using the structure of the solution which it provides in the form of the $\underline{n}$ vector. The construction of a feasible schedule in algorithm 3PPS-Sched follows that indicated in the proof of Theorem 2. It builds upon a feasible solution to the reduced instance produced by subroutine RedSched. We now outline the algorithmic construction, 3PPS-Sched and RedSched, of a feasible solution. The validity and computational complexity of the algorithm(s) are then considered in Theorems 5 and 6. An illustrative example is offered at the end of this section.

We now develop an algorithm for constructing a feasible schedule from the composition of $\boldsymbol{\Lambda}$-sets provided by integer vector $\underline{n}$ from Algorithm FeasTest, for the reduced

Algorithm 3PPS-Sched: *Construction of a schedule for a general problem with three distinct periodicities*
Input: $q_1, q_2, q_3, n_1, n_2, n_3$
Output: τ_{ij} *for* $i = 1, \ldots, 3$, $j = 0, \ldots, n_i - 1$, *or infeasible*
Code:

1. *Initialisation*
   $g \leftarrow gcd(q_1, q_2, q_3)$, $g_{12} \leftarrow gcd(q_1, q_2)/g$,
   $g_{23} \leftarrow gcd(q_2, q_3)/g$, $g_{31} \leftarrow gcd(q_3, q_1)/g$,
   $g_{10} \leftarrow q_1/(gg_{12}g_{31})$, $g_{20} \leftarrow q_2/(gg_{12}g_{23})$,
   $g_{30} \leftarrow q_3/(gg_{23}g_{31})$.
2. *Construction of a feasible schedule for the reduced instance, if one exists*
   For $i = 1, \ldots, 3$,
   $q_i' \leftarrow q_i/g_{i0}$ and $n_i' \leftarrow \lceil n_i/g_{i0} \rceil$.
   Run Algorithm RedSched$(f_1', f_2', f_3', n_1', n_2', n_3' : \tau_{ij}')$.
3. *Extension of a feasible solution from the reduced problem, to the original problem*
   If Algorithm RedSched returns infeasible, then **return** *infeasible* and **stop**.
   Otherwise,
   For $i = 1, \ldots, 3$, $j = 0, \ldots, n_i - 1$,
   $\tau_{ij} \leftarrow \tau'_{i\lfloor j/g_{i0} \rfloor} + R(j, g_{i0})\,(q_i/g_{i0})$.
   Return τ_{ij} for $i = 1, \ldots, 3$, $j = 0, \ldots, n_i - 1$.

problem. The construction, for $\boldsymbol{\Lambda}$-sets with two and three different periodicities, is based upon the proofs of Theorems 1 and 3.

Theorem 5 *An instance with three distinct periodicities,* q_1*,* q_2 *and* q_3 *with* $g_{i0} = 1$ *for* $i = 1, 2, 3$ *is schedulable if and only if Algorithm RedSched returns a feasible schedule.*

Proof Algorithm RedSched returns a set of first service time slots τ_{ij} for $i = 1, 2, 3$, $j = 0, \ldots, n_i - 1$, based upon the integer vector $\underline{n}$ returned by Algorithm FeasTest. Now by Lemma 14, $\underline{n}$ associates clients with $\boldsymbol{\Lambda}$-sets (n_{i0} of periodicity q_i with λ_i sets, $n_{ii'}$ and $n_{i'i}$ with $\lambda_{ii'}$ sets and μ_1, μ_2, μ_3 with λ_{123} sets, for $i, i' = 1, 2, 3$, $i \neq i'$), and λ_{12}, λ_{23}, λ_{31} and λ_{123} are each no greater than one. We shall show that algorithm RedSched constructs a solution based upon this prescribed division of $\boldsymbol{\Lambda}$-sets. The index l keeps track of the $\boldsymbol{\Lambda}$-set(s) under consideration at each step of the algorithm. To see this, observe that l is initialized to 0 and incremented in the last statement of each step. The increment in step 3 is $\lceil n_{i0}/(q_i/g) \rceil$ which is indeed λ_i for $i = 1, 2, 3$. Each of steps 4–7 is skipped if the corresponding $\boldsymbol{\Lambda}$-set is empty, since $\lambda_1 = 0$ when $n_{12} + n_{21} = 0$, etc and $\lambda_{123} = 0$ when $\mu_1 + \mu_2 + \mu_3 = 0$. Moreover, when any of steps 4–7 is implemented, l is increased by 1, in accordance with Lemma 14. It remains only to show that the construction within each $\boldsymbol{\Lambda}$-set is feasible.

Observe that in Step 3 of Algorithm RedSched, the first q_i/g positions are used for the first service of clients with the periodicity of q_i in a $\boldsymbol{\Lambda}$-set. Thus, no service to these clients is overlapped by Lemma 3.

Now, the time slots within a Λ_l-set are of the form $l + gp$ where p is the position within the set. Consider the $\boldsymbol{\Lambda}$-set, Λ'

Algorithm RedSched: *Schedule for the reduced problem with three distinct periodicities*
Input: $q_1, q_2, q_3, n_1, n_2, n_3$
Output: τ_{ij} *for* $i = 1, \ldots, 3,\ j = 0, \ldots, n_i - 1$, *or infeasible*
Code:

1. *Test for feasibility*
   Run Algorithm FeasTest($q_1, q_2, q_3, n_1, n_2, n_3$: $\underline{n}^B$).
   If Algorithm FeasTest returns infeasible, then **return** *infeasible* and **stop**.
2. *Initialisation*
   $g \leftarrow gcd(q_1, q_2, q_3)$, $g_{12} \leftarrow gcd(q_1, q_2)/g$, $g_{23} \leftarrow gcd(q_2, q_3)/g$, $g_{31} \leftarrow gcd(q_3, q_1)/g$,
   $l \leftarrow 0$, $j_1 \leftarrow 0$, $j_2 \leftarrow 0$, $j_3 \leftarrow 0$.
3. *Schedule the first time slot for each client associated with* n_{i0} *for* $i = 1, \ldots, 3$
   For $i = 1, \ldots, 3$,
      For $j = 0, \ldots, n_{i0} - 1$,
         $\tau_{ij_i} \leftarrow l + \lfloor j/(q_i/g) \rfloor + gR\,(j, q_i/g)$ and $j_i \leftarrow j_i + 1$.
      $l \leftarrow l + \lceil n_{i0}/(q_i/g) \rceil$.
4. *Schedule the first time slot for each client associated with* n_{12} *and* n_{21}
   If $n_{12} + n_{21} > 0$, then
      For $j = 0, \ldots, n_{12} - 1$,
         $\tau_{1j_1} \leftarrow l + g\,(\lfloor j/g_{31} \rfloor + R\,(j, g_{31})\,g_{12})$ and $j_1 \leftarrow j_1 + 1$..
      $\widetilde{l} \leftarrow l + g\,\lceil n_{12}/g_{31} \rceil$.
      For $j = 0, \ldots, n_{21} - 1$,
         $\tau_{2j_2} \leftarrow \widetilde{l} + g\,(\lfloor j/g_{23} \rfloor + R\,(j, g_{23})\,g_{12})$ and $j_2 \leftarrow j_2 + 1$.
      $l \leftarrow l + 1$.
5. *Schedule the first time slot for each client associated with* n_{23} *and* n_{32}
   If $n_{23} + n_{32} > 0$, then
      For $j = 0, \ldots, n_{23} - 1$,
         $\tau_{2j_2} \leftarrow l + g\,(\lfloor j/g_{12} \rfloor + R\,(j, g_{12})\,g_{23})$ and $j_2 \leftarrow j_2 + 1$.
      $\widetilde{l} \leftarrow l + g\,\lceil n_{23}/g_{12} \rceil$.
      For $j = 0, \ldots, n_{32} - 1$,
         $\tau_{3j_3} \leftarrow \widetilde{l} + g\,(\lfloor j/g_{31} \rfloor + R\,(j, g_{31})\,g_{23})$ and $j_3 \leftarrow j_3 + 1$.
      $l \leftarrow l + 1$.
6. *Schedule the first time slot for each client associated with* n_{31} *and* n_{13}
   If $n_{31} + n_{13} > 0$, then
      For $j = 0, \ldots, n_{31} - 1$,
         $\tau_{3j_3} \leftarrow l + g\,(\lfloor j/g_{23} \rfloor + R\,(j, g_{23})\,g_{31})$ and $j_3 \leftarrow j_3 + 1$.
      $\widetilde{l} \leftarrow l + g\,\lceil n_{31}/g_{23} \rceil$.
      For $j = 0, \ldots, n_{13} - 1$,
         $\tau_{1j_1} \leftarrow \widetilde{l} + g\,(\lfloor j/g_{12} \rfloor + R\,(j, g_{12})\,g_{31})$ and $j_1 \leftarrow j_1 + 1$.
      $l \leftarrow l + 1$.
7. *Schedule the first time slot for each client associated with* μ_1, μ_2 *and* μ_3
   If $\mu_1 + \mu_2 + \mu_3 > 0$, then
   7.1 *Find* $\bar{f}'_3$ *and a suitable values for* α *and* β.
      $f'_3 \leftarrow q_3/g$.
      For $x = 1, \ldots, g_{12} - 1$,
         If $R(xf'_3, g_{12}) = 1$, then $\bar{f}'_3 \leftarrow x$.
         If $\lceil \mu_1/x \rceil \le g_{31} - 1$, and $\lceil \mu_2/(g_{12} - x) \rceil \le g_{23} - 1$,
         and $\mu_3 \le (g_{31} - \lceil \mu_1/x \rceil)\,(g_{23} - \lceil \mu_2/(g_{12} - x) \rceil)$,
         then
            $\alpha \leftarrow \lceil \mu_1/x \rceil$ and $\beta \leftarrow \lceil \mu_2/(g_{12} - x) \rceil$.
   7.2 *Allocate time slots to clients with the periodicity of* q_1 *and* q_2.
      For $j = 0, \ldots, \mu_1 - 1$,
         $\tau_{1j_1} \leftarrow R\left(l + g\left(f'_3 \bar{f}'_3 \lfloor j/\alpha \rfloor + R(j, \alpha) g_{12}\right), q_1\right)$ and $j_1 \leftarrow j_1 + 1$.
      $\widetilde{l} \leftarrow l + g f'_3 \bar{f}'_3\, \lceil \mu_1/\alpha \rceil$
      For $j = 0, \ldots, \mu_2 - 1$,
         $\tau_{2j_2} \leftarrow R(\widetilde{l} + g\left(f'_3 \bar{f}'_3\, \lfloor j/\beta \rfloor / g + R(j, \beta) g_{12}\right),\ q_2)$ and $j_2 \leftarrow j_2 + 1$.
   7.3 *Find free relative positions in* Λ_{g-1}.
      $P \leftarrow \{0, \ldots, T/(gg_{12}) - 1\}$.
      $P_1 \leftarrow \{j + k(q_1/gg_{12}) \mid j = 0, \ldots, \alpha - 1, k = 0, \ldots, T/q_1 - 1\}$.
      $P_2 \leftarrow \{j + k(q_2/gg_{12}) \mid j = 0, \ldots, \beta - 1, k = 0, \ldots, T/q_2 - 1\}$.
      $P_3 \leftarrow P \setminus (P_1 \cup P_2)$.
   7.4 *Allocate clients with the periodicity of* q_3 *to free relative positions.*
      For $j = 0, \ldots, \mu_3 - 1$,
         Select a free position p from P_3.
         $\tau_{3j_3} \leftarrow R(l + gg_{12}p, q_3)$ and $j_3 \leftarrow j_3 + 1$.
         $P_3 \leftarrow P_3 \setminus \{p\}$.

Return τ_{ij} for $i = 1, \ldots, 3$, $j = 0, \ldots, n_i - 1$.

say, which contains the clients with the periodicities of both q_1 and q_2, when $\lambda_{12} = 1$. The interval between consecutive positions in Λ' of service to a client with the periodicity of q_1 (or q_2) is $q_1/g = g_{12}g_{31}$ (or $q_2/g = g_{12}g_{23}$). Thus, the positions given to τ_{1j} and τ_{2j} in Step 4 are precisely those prescribed by (2) and (3) in the proof of Theorem 1. The validity of Step 5 and 6 for λ_{23} and λ_{31}, respectively, follows by analogy.

Now consider the case when $\lambda_{123} = 1$ (and $\mu_1 + \mu_2 + \mu_3 > 0$). The construction of x, α, β and $\bar{f}$ in Step 7.1 accords with that in Sect. 4.2 for the reduced problem with periodicities q_i/g for $i = 1, 2, 3$. Time slot t in the reduced problem corresponds to time slot in Λ_l in the original problem with value of $l + gt$. Thus, the allocation of the first time slot to clients with the periodicity of q_1 and q_2 described in Step 7.2 corresponds to (8) and (9) in the proof of Theorem 3. The above allocation is designed to leave as many sets of relative free positions available for clients with the periodicity of q_3 as possible, as described in Theorem 3. The set of occupied relative positions is given by P_1 and P_2 in Step 7.3 for the periodicity of q_1 and q_2, and the allocation of the clients with the periodicity of q_3 to the free relative positions performed in Step 7.4. □

Theorem 6 *Algorithm 3PPS-Sched constructs a feasible perfect periodic schedule for an instance with three distinct periodicities in* $O\left(\max\{n^4,\ q_{\max}\}\right)$ *time if one exists, where* $n = \sum_{i=1}^{3} n_i$ *and* $q_{\max} = \max_i q_i/g_{i0}$.

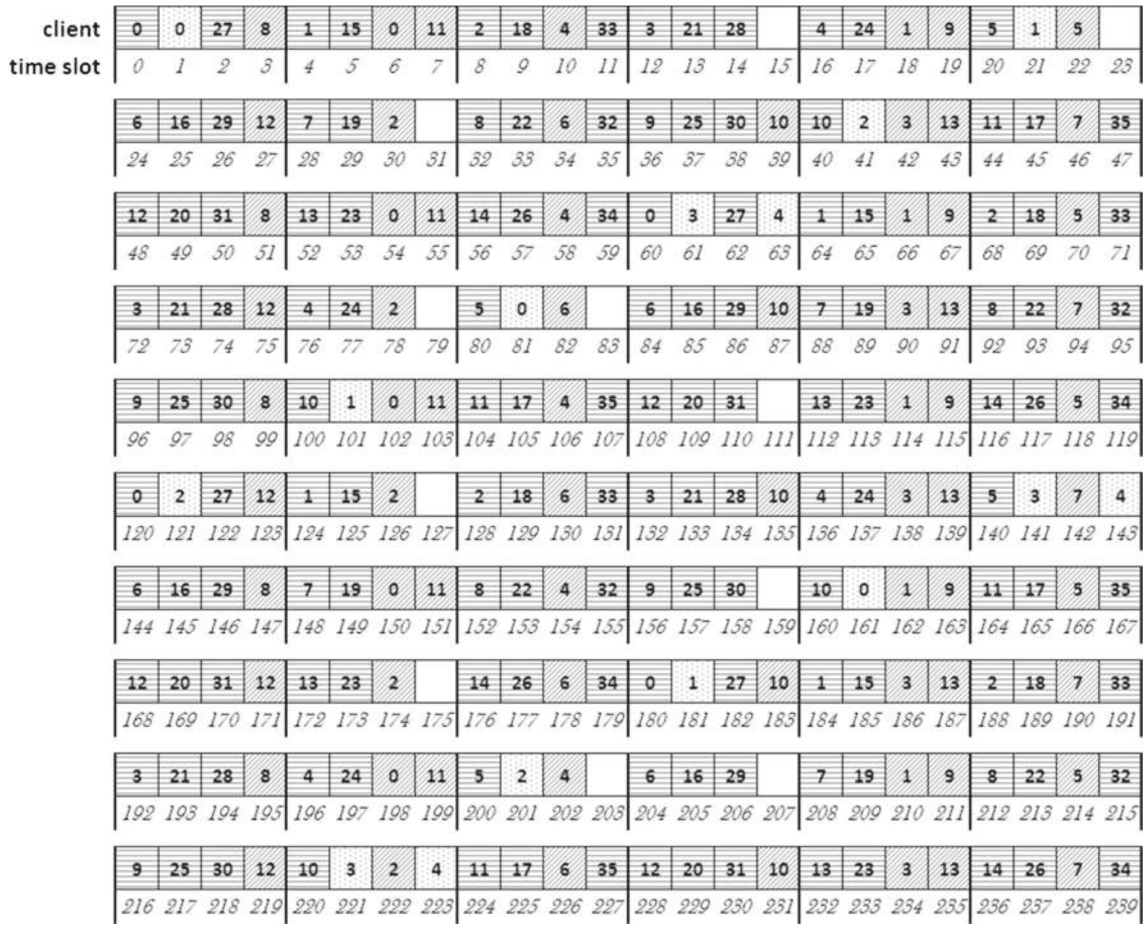


Fig. 6 A feasible schedule for Example 3

Proof Take an instance of PPS which has a feasible solution. Then by Theorem 2, the corresponding reduced instance is as constructed in Step 2 of Algorithm 3PPS-Sched, and is itself feasible by Theorem 5. Moreover, the corresponding schedule for the original instance given in Step 3, is as constructed in (4) of the proof of Theorem 2 and is therefore feasible.

Observe that Step 1 of Algorithm RedSched can be done in $O(n^4)$. Step 7 of Algorithm RedSched can be done in $O(q_{\max})$ because Steps 7.1 and 7.4 are bounded by $O(g_{12})$ and $O(g_{23}g_{31})$, respectively. All other steps of Algorithm RedSched can be done in $O(n)$. Therefore, the overall computational complexity of Algorithm 3PPS-Sched is $O\left(\max\{n^4,\ q_{\max}\}\right)$. □

The following example illustrates the construction of Algorithm 3PPS-Sched.

Example 3 Consider the case of three periodicities $q_1 = 48$, $q_2 = 80$ and $q_3 = 60$ (ie. $g = 4$, $g_{12} = 4$, $g_{23} = 5$, $g_{31} = 3$, and $g_{i0} = 1$ for $i = 1, 2, 3$) with $n_1 = 14$, $n_2 = 5$ and $n_3 = 36$, respectively. For this instance, Algorithm FeasTest gives rise to values $\lambda_1 = \lambda_2 = \lambda_{12} = 0$ and $\lambda_3 = \lambda_{23} = \lambda_{31} = \lambda_{123} = 1$ and returns an integer vector $\underline{n} = (n_{10}, n_{20}, n_{30}, n_{12}, n_{21}, n_{23}, n_{32}, n_{31}, n_{13}, \mu_1, \mu_2, \mu_3) = (0, 0, 15, 0, 0, 4, 12, 5, 8, 6, 1, 4)$. Algorithm 3PPS-Sched then produces a feasible schedule as depicted in Fig. 6. The underlying structure of the $\mathbf{\Lambda}$-sets is given by Algorithm RedSched in shown in Fig. 7.

7 Conclusions

This paper examines the perfect periodic scheduling problem with unit service time. We have extended the range of perfect periodic scheduling problem which are amenable to polynomial time algorithms from one periodicity to three distinct periodicities. The extent of such special cases is likely to be limited since the existence of perfect periodic schedule problem for the general case is NP-hard. The only models commonly used in the literature are based upon a single common periodicity: either the single periodicity itself, its powers (eg. power of 2), or based upon repeated divisibility (depicted as a tree structure).

We have presented an algorithm for testing for the existence of a feasible schedule for any combination of clients

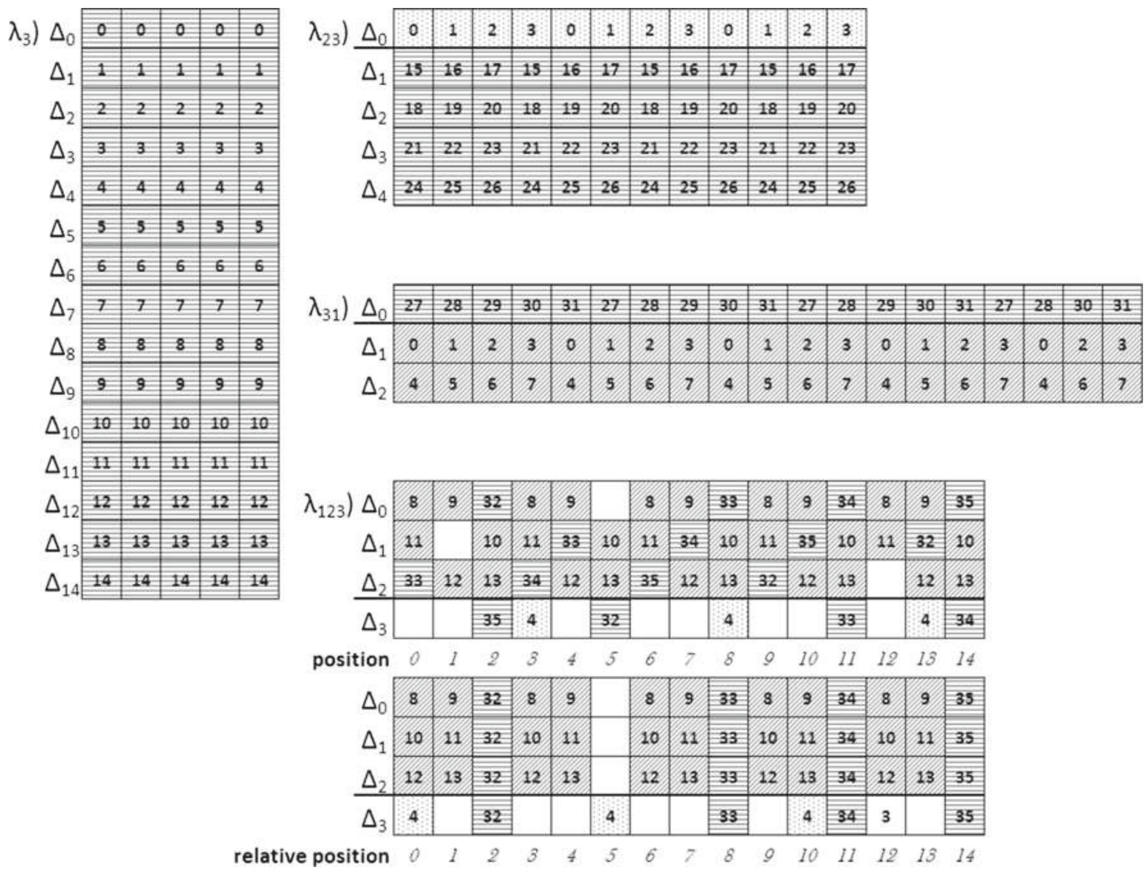


Fig. 7 The structure of each **Λ**-set in the feasible schedule for Example 3, as a **Δ**-partition

with up to three distinct periodicities, which runs in $O(n^4)$. In addition, we have provided the construction of a feasible schedule where one exists. We have thus extended the range of feasible perfect periodic schedules available for operating Wireless Mesh Networks beyond those with a common factor.

The next obvious step is to embed these algorithms within heuristics for meeting the requests of mobile telecommunication clients, according to different service level quality measures, see Bar-Noy et al. (2004), Brakerski and Patt-Shamir (2006). There is potential for accommodating a wide range of combinations of periodicities, by incorporating multiples of the three basic periodicities in particular power of 2. The addition of a fourth distinct periodicity might be investigated. However, it is unlikely to be an attractive option due to the large number of idle time slots imposed in the schedule. Future research will evaluate the advantage of the wider solution spaces made accessible through the current article. In particular, efficient methodologies will be sought for combining perfect periodic schedules for local clients at mesh nodes, with approaches to data transmission through a routing tree, such as Allen et al. (2012a,b), as is applicable for Internet access through Wireless Mesh Networks.

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Appendix 1

The following three statements are equivalent to each other.

Statement 1: there exists an integer x, $1 \leq x \leq g_{23} - 1$ *satisfying*

$$\lceil n_2/x \rceil \leq g_{12} - 1 \tag{31}$$

$$\lceil n_3/(g_{23} - x) \rceil \leq g_{31} - 1 \tag{32}$$

$$n_1 \leq (g_{12} - \lceil n_2/x \rceil)\,(g_{31} - \lceil n_3/(g_{23} - x) \rceil) \tag{33}$$

Statement 2: there exists an integer y, $1 \leq y \leq g_{31} - 1$ *satisfying*

$$\lceil n_3/y \rceil \leq g_{23} - 1 \tag{34}$$

$$\lceil n_1/(g_{31} - y) \rceil \leq g_{12} - 1 \tag{35}$$

$$n_2 \leq (g_{23} - \lceil n_3/y \rceil)\,(g_{12} - \lceil n_1/(g_{31} - y) \rceil) \tag{36}$$

Statement 3: *there exists an integer z*, $1 \le z \le g_{12} - 1$ *satisfying*

$$\lceil n_1/z \rceil \le g_{31} - 1$$
$$\lceil n_2/(g_{12} - z) \rceil \le g_{23} - 1$$
$$n_3 \le (g_{31} - \lceil n_1/z \rceil)(g_{23} - \lceil n_2/(g_{12} - z) \rceil)$$

Proof By symmetry, it is sufficient to show that *Statement* 1 implies *Statement* 2. Take an integer x satisfying the inequalities, (31), (32) and (33). Let $y = \lceil n_3/(g_{23} - x) \rceil$. Observe that $1 \le y \le g_{31} - 1$ from (32). Then, $n_3/(g_{23} - x) \le y$, and hence $1 \le x \le g_{23} - n_3/y$ which implies that the inequality (34) holds. From (33),

$$\left\lceil \frac{n_1}{g_{31} - y} \right\rceil \le \left\lceil \frac{(g_{12} - \lceil n_2/x \rceil)(g_{31} - y)}{g_{31} - y} \right\rceil = g_{12} - \left\lceil \frac{n_2}{x} \right\rceil,$$

and hence,

$$n_2 \le x \left(g_{12} - \left\lceil \frac{n_1}{g_{31} - y} \right\rceil \right) \le \left(g_{23} - \left\lceil \frac{n_3}{y} \right\rceil \right) \left(g_{12} - \left\lceil \frac{n_1}{g_{31} - y} \right\rceil \right).$$

Thus, the inequality (36) holds. Moreover, since $n_2 > 0$ and the inequalities (34) and (36) hold, $0 < g_{12} - \lceil n_1/(g_{31} - y) \rceil$, which implies that the inequality (35) also holds completing the proof. □

Appendix 2

In order to prove Lemma 11, a simple arithmetic argument is required as presented in Lemma 15.

Lemma 15 *For any positive integers, a, a', b and b', if $\lceil a/b \rceil \ge a'/b'$, then $\lceil a/b \rceil \ge \lceil (a + a')/(b + b') \rceil$.*

Proof It holds because

$$\left\lceil \frac{a}{b} \right\rceil = \left\lceil \frac{\lceil a/b \rceil (b + b')}{b + b'} \right\rceil \ge \left\lceil \frac{\lceil a/b \rceil b + (a'/b')b'}{b + b'} \right\rceil \ge \left\lceil \frac{a + a'}{b + b'} \right\rceil.$$

□

Now we give a proof for Lemma 11 as follows.

Proof of Lemma 11 Suppose otherwise, then there is an instance in which any feasible solution $\mathcal{S}$ has $\lambda_{123} \ge 2$. Select $\mathcal{S}$ to have the minimum value of λ_{123}, λ^*_{123} say, which affords a feasible solution. We prove the Lemma by reducing $\mathcal{S}$ to a feasible solution with $\lambda_{123} = \lambda^*_{123} - 1$. Take two of the $\mathbf{\Lambda}$-sets in $\mathcal{S}$ serving all three periodicities, Λ_l and $\Lambda_{l'}$ say. Suppose that μ_i, and μ'_i, clients with the periodicity of q_i for $i = 1, 2, 3$, are serviced together in Λ_l and $\Lambda_{l'}$, respectively. The schedule with the set Λ_l (or $\Lambda_{l'}$) may be viewed as a single schedule for the clients μ_1, μ_2 and μ_3 (or μ'_1, μ'_2 and μ'_3) alone with the periodicities of $q'_1 = q_1/g$, $q'_2 = q_2/g$ and $q'_3 = q_3/g$, respectively. Thus, $gcd(q'_1, q'_2, q'_3) = 1$, and from the result of Theorem 3, there exist x_1 and x'_1 satisfying

$$\frac{\mu_1}{x_1} \le g_{31} - 1, \tag{37}$$
$$\frac{\mu_2}{x_2} \le g_{23} - 1, \tag{38}$$
$$\frac{\mu'_1}{x'_1} \le g_{31} - 1, \tag{39}$$
$$\frac{\mu'_2}{x'_2} \le g_{23} - 1, \tag{40}$$
$$1 \le \mu_3 \le \left(g_{31} - \left\lceil \frac{\mu_1}{x_1} \right\rceil \right) \left(g_{23} - \left\lceil \frac{\mu_2}{x_2} \right\rceil \right), \tag{41}$$
$$1 \le \mu'_3 \le \left(g_{31} - \left\lceil \frac{\mu'_1}{x'_1} \right\rceil \right) \left(g_{23} - \left\lceil \frac{\mu'_2}{x'_2} \right\rceil \right), \tag{42}$$

where $x_2 = g_{12} - x_1$ and $x'_2 = g_{12} - x'_1$. Without loss of generality, we assume throughout the proof that

$$\frac{\mu_2}{x_2} \le \frac{\mu'_2}{x'_2}, \tag{43}$$

that is, $(g_{23} - \lceil \mu_2/x_2 \rceil) \ge (g_{23} - \lceil \mu'_2/x'_2 \rceil)$. There are three cases to consider.

Case 1: $x_1 \le x'_1$
We construct a new schedule by exchanging $\widetilde{\mu}_1$ clients with the periodicity of q_1 from Λ_l with $\widetilde{\mu}_3$ clients with the periodicity of q_3 from $\Lambda_{l'}$, where $\widetilde{\mu}_1 = \min\{\mu_1, x_1\}$ and $\widetilde{\mu}_3 = \min\{\mu'_3, g_{23} - \lceil \mu'_2/x'_2 \rceil\}$. Consider the set Λ_l after the interchange. Observe that $1 \le \widetilde{\mu}_1 \le x_1$, and from (43), $\widetilde{\mu}_3 \le g_{23} - \lceil \mu'_2/x'_2 \rceil \le g_{23} - \lceil \mu_2/x_2 \rceil$. Thus, applying inequality (41),

$$\begin{aligned} \mu_3 + \widetilde{\mu}_3 &\le \left(g_{31} - \left\lceil \frac{\mu_1}{x_1} \right\rceil + 1 \right) \left(g_{23} - \left\lceil \frac{\mu_2}{x_2} \right\rceil \right) \\ &= \left(g_{31} - \left\lceil \frac{\mu_1}{x_1} \right\rceil + \left\lceil \frac{\widetilde{\mu}_1}{x_1} \right\rceil \right) \left(g_{23} - \left\lceil \frac{\mu_2}{x_2} \right\rceil \right) \\ &= \begin{cases} g_{31} \left(g_{23} - \left\lceil \frac{\mu_2}{x_2} \right\rceil \right) & \text{if } \widetilde{\mu}_1 = \mu_1 \\ \left(g_{31} - \left\lceil \frac{\mu_1 - \widetilde{\mu}_1}{x_1} \right\rceil \right) \left(g_{23} - \left\lceil \frac{\mu_2}{x_2} \right\rceil \right) & \text{if } \widetilde{\mu}_1 \ne \mu_1. \end{cases} \end{aligned}$$

The former expression implies that

$$\left\lceil \frac{\mu_2}{g_{12}} \right\rceil + \left\lceil \frac{\mu_3 + \widetilde{\mu}_3}{g_{31}} \right\rceil \le \left\lceil \frac{\mu_2}{x_2} \right\rceil + \left\lceil \frac{\mu_3 + \widetilde{\mu}_3}{g_{31}} \right\rceil \le g_{23},$$

and hence μ_2 and $\mu_3 + \widetilde{\mu}_3$ clients with the periodicity of q_2 and q_3, respectively, fit into a single $\mathbf{\Lambda}$−set, by Theorem 1. The latter expression implies that $\mu_1 - \widetilde{\mu}_1$, μ_2 and $\mu_3 + \widetilde{\mu}_3$ satisfy condition (7) of Theorem 3. Moreover, $\mu_1 - \widetilde{\mu}_1$ and μ_2 satisfy conditions (5) and (6) of Theorem 3 from (37) and (38) above. Thus, $\mu_1 - \widetilde{\mu}_1$, μ_2 and $\mu_3 + \widetilde{\mu}_3$ clients with the periodicity of q_1, q_2 and q_3, respectively, fit into a single $\mathbf{\Lambda}$-set, by Theorem 3.

Now consider the set $\Lambda_{l'}$ after the interchange. Suppose first that $\widetilde{\mu}_3 \neq \mu'_3$. Then $\widetilde{\mu}_3 = g_{23} - \lceil \mu'_2/x'_2 \rceil$, and from (42),

$$\mu'_3 - \widetilde{\mu}_3 \leq \left(g_{31} - \left\lceil \frac{\mu'_1}{x'_1} \right\rceil\right)\left(g_{23} - \left\lceil \frac{\mu'_2}{x'_2} \right\rceil\right) - \left(g_{23} - \left\lceil \frac{\mu'_2}{x'_2} \right\rceil\right) = \left(g_{31} - \left\lceil \frac{\mu'_1}{x'_1} \right\rceil - 1\right)\left(g_{23} - \left\lceil \frac{\mu'_2}{x'_2} \right\rceil\right).$$

Hence, since $1 \leq \widetilde{\mu}_1 \leq x_1 \leq x'_1$,

$$\mu'_3 - \widetilde{\mu}_3 \leq \left(g_{31} - \left\lceil \frac{\mu'_1 + \widetilde{\mu}_1}{x'_1} \right\rceil\right)\left(g_{23} - \left\lceil \frac{\mu'_2}{x'_2} \right\rceil\right).$$

Thus, $\mu'_1 + \widetilde{\mu}_1$, μ'_2 and $\mu'_3 - \widetilde{\mu}_3$ satisfy condition (7) of Theorem 3. Moreover, $\mu'_1 + \widetilde{\mu}_1$ and μ'_2 satisfy conditions (5) and (6) of Theorem 3 from (39) and (40) above. Thus, $\mu'_1 + \widetilde{\mu}_1$, μ'_2 and $\mu'_3 - \widetilde{\mu}_3$ clients with the periodicity of q_1, q_2 and q_3, respectively, fit into a single $\mathbf{\Lambda}$−set, by Theorem 3. On the other hand, suppose that $\widetilde{\mu}_3 = \mu'_3$. Recall that $1 \leq \widetilde{\mu}_1 \leq x_1 \leq x'_1$ in this case, and hence by (39)

$$\frac{\mu'_1 + \widetilde{\mu}_1}{x'_1} \leq \left\lceil \frac{\mu'_1}{x'_1} \right\rceil + 1 \leq g_{31},$$

which combines with (40) to give

$$\left\lceil \frac{\mu'_1 + \widetilde{\mu}_1}{g_{31}} \right\rceil + \left\lceil \frac{\mu'_2}{g_{23}} \right\rceil \leq x'_1 + x'_2 = g_{12}.$$

Thus, $\mu'_1 + \widetilde{\mu}_1$ and μ'_2 clients with the periodicity of q_1 and q_2, respectively, fit into a single $\mathbf{\Lambda}$-set, by Theorem 1.

Therefore, the feasibility of the schedule is preserved. Repeating this argument, we obtain a feasible schedule with $\lambda_{123} \leq 1$.

Case 2: $x_1 > x'_1$ and $\lceil \mu_1/x_1 \rceil \leq \lceil \mu'_1/x'_1 \rceil$
From $x_1 > x'_1$ and $\lceil \mu_1/x_1 \rceil \leq \lceil \mu'_1/x'_1 \rceil$, we have that $x_2 < x'_2$ and $(g_{31} - \lceil \mu_1/x_1 \rceil) \geq (g_{31} - \lceil \mu'_1/x'_1 \rceil)$. These conditions are covered by Case 1 when the periodicities 1 and 2 are interchanged.

Case 3: $x_1 > x'_1$ and $\lceil \mu_1/x_1 \rceil > \lceil \mu'_1/x'_1 \rceil$
First consider the case when $x'_1 \geq x_2$. We construct a new schedule by exchanging μ_2 clients with the periodicity of q_2 from Λ_l with $\widetilde{\mu}_1$ clients with the periodicity of q_1 from $\Lambda_{l'}$, where $\widetilde{\mu}_1 = \min\{\mu'_1, \lceil \mu'_1/x'_1 \rceil x_2\}$. Consider the set Λ_l after the interchange. Since $\lceil \mu_1/x_1 \rceil > \lceil \mu'_1/x'_1 \rceil \geq \widetilde{\mu}_1/x_2$, from Lemma 15, $\lceil \mu_1/x_1 \rceil \geq \lceil (\mu_1 + \widetilde{\mu}_1)/(x_1 + x_2) \rceil$, and hence, from (41),

$$\mu_3 \leq \left(g_{31} - \left\lceil \frac{\mu_1}{x_1} \right\rceil\right)\left(g_{23} - \left\lceil \frac{\mu_2}{x_2} \right\rceil\right) \leq \left(g_{31} - \left\lceil \frac{\mu_1 + \widetilde{\mu}_1}{x_1 + x_2} \right\rceil\right) g_{23}.$$

Thus,

$$\left\lceil \frac{\mu_1 + \widetilde{\mu}_1}{g_{12}} \right\rceil + \left\lceil \frac{\mu_3}{g_{23}} \right\rceil \leq \left\lceil \frac{\mu_1 + \widetilde{\mu}_1}{x_1 + x_2} \right\rceil + \left(g_{31} - \left\lceil \frac{\mu_1 + \widetilde{\mu}_1}{x_1 + x_2} \right\rceil\right) = g_{31}.$$

This implies that $\mu_1 + \widetilde{\mu}_1$ and μ_3 clients with the periodicity of q_1 and q_3, respectively, fit into a single $\mathbf{\Lambda}$-set by Theorem 1. Now consider the set $\Lambda_{l'}$ after the interchange. Observe that

$$\left\lceil \frac{\mu'_1}{x'_1} \right\rceil = \left\lceil \frac{\left\lceil \frac{\mu'_1}{x'_1} \right\rceil (x'_1 - x_2)}{x'_1 - x_2} \right\rceil \geq \left\lceil \frac{\mu'_1 - \left\lceil \frac{\mu'_1}{x'_1} \right\rceil x_2}{x'_1 - x_2} \right\rceil$$

and from (43) and Lemma 15, $\lceil \mu'_2/x'_2 \rceil \geq \lceil (\mu'_2 + \mu_2) / (x'_2 + x_2) \rceil$. Thus, substituting the above two constraints into (42),

$$\mu'_3 \leq \left(g_{31} - \left\lceil \frac{\mu'_1}{x'_1} \right\rceil\right)\left(g_{23} - \left\lceil \frac{\mu'_2}{x'_2} \right\rceil\right) \leq \begin{cases} g_{31}\left(g_{23} - \left\lceil \frac{\mu'_2 + \mu_2}{x'_2 + x_2} \right\rceil\right) & \text{if } \widetilde{\mu}_1 = \mu_1 \\ \left(g_{31} - \left\lceil \frac{\mu'_1 - \widetilde{\mu}_1}{x'_1 - x_2} \right\rceil\right)\left(g_{23} - \left\lceil \frac{\mu'_2 + \mu_2}{x'_2 + x_2} \right\rceil\right) & \text{if } \widetilde{\mu}_1 \neq \mu_1. \end{cases}$$

The latter expression implies that $\mu'_1 - \widetilde{\mu}_1$, $\mu'_2 + \mu_2$ and μ'_3 satisfy condition (7) of Theorem 3. Moreover, $\lceil \mu'_1/x'_1 \rceil \geq \lceil (\mu'_1 + \widetilde{\mu}_1)/(x'_1 - x_2) \rceil$ when $\widetilde{\mu}_1 \neq \mu_1$ and $\lceil \mu'_2/x'_2 \rceil \geq \lceil (\mu'_2 + \mu_2)/(x'_2 + x_2) \rceil$, and hence by letting $x = x'_1 - x_2$, $\mu'_1 - \widetilde{\mu}_1$ and $\mu'_2 + \mu_2$ satisfy conditions (5) and (6) of Theorem 3 from (39) and (40) above. Thus, $\mu'_1 - \widetilde{\mu}_1$, $\mu'_2 + \mu_2$ and μ'_3 clients with the periodicity of q_1, q_2 and q_3, respectively, fit into a single $\mathbf{\Lambda}$-set by Theorem 3 when $\widetilde{\mu}_1 \neq \mu_1$. On the other hand, when $\widetilde{\mu}_1 = \mu_1$, the former expression implies that

$$\left\lceil \frac{\mu'_2 + \mu_2}{g_{12}} \right\rceil + \left\lceil \frac{\mu'_3}{g_{31}} \right\rceil \leq \left\lceil \frac{\mu'_2 + \mu_2}{x'_2 + x_2} \right\rceil + \left(g_{23} - \left\lceil \frac{\mu'_2 + \mu_2}{x'_2 + x_2} \right\rceil\right) = g_{23},$$

since $x'_2 + x'_2 \leq x'_2 + x'_1 = g_{12}$. Hence, by Theorem 1, $\mu'_2 + \mu_2$ and μ'_3 clients with the periodicity of q_2 and q_3, respectively, fit into a single $\mathbf{\Lambda}$-set.

Now consider the alternative case, when $x'_1 < x_2$. We construct a new schedule by exchanging $\widetilde{\mu}_2$ clients with the periodicity of q_2 from Λ_l with μ'_1 clients with the periodicity of q_1 from $\Lambda_{l'}$, where $\widetilde{\mu}_2 = \min\{\mu_2, \lceil \mu_2/x_2 \rceil x'_1\}$. Consider the set $\Lambda_{l'}$ after the interchange. Since (43), $\lceil \mu'_2/x'_2 \rceil \geq \lceil \mu_2/x_2 \rceil \geq \widetilde{\mu}_2/x'_1$. From Lemma 15, $\lceil \mu'_2/x'_2 \rceil \geq \lceil (\mu'_2 + \widetilde{\mu}_2)/(x'_2 + x'_1) \rceil$, and hence,

$$\mu_3' \le \left(g_{31} - \left\lceil \frac{\mu_1'}{x_1'} \right\rceil\right)\left(g_{23} - \left\lceil \frac{\mu_2'}{x_2'} \right\rceil\right)$$
$$\le g_{31}\left(g_{23} - \left\lceil \frac{\mu_2' + \widetilde{\mu}_2}{x_2' + x_1'} \right\rceil\right).$$

Thus,

$$\left\lceil \frac{\mu_2' + \widetilde{\mu}_2}{g_{12}} \right\rceil + \left\lceil \frac{\mu_3'}{g_{31}} \right\rceil \le \left\lceil \frac{\mu_2' + \widetilde{\mu}_2}{x_2' + x_1'} \right\rceil + \left(g_{23} - \left\lceil \frac{\mu_2' + \widetilde{\mu}_2}{x_2' + x_1'} \right\rceil\right) = g_{23},$$

which implies that $\mu_2' + \widetilde{\mu}_2$ and μ_3' clients with the periodicity of q_2 and q_3, respectively, fit into a single $\mathbf{\Lambda}$-set by Theorem 1. Now consider the set Λ_l after the interchange. Since $\lceil \mu_1/x_1 \rceil > \lceil \mu_1'/x_1' \rceil \ge \mu_1'/x_1'$, from Lemma 15, $\lceil \mu_1/x_1 \rceil \ge \lceil (\mu_1 + \mu_1')/(x_1 + x_1') \rceil$, and

$$\left\lceil \frac{\mu_2}{x_2} \right\rceil = \left\lceil \frac{\left\lceil \frac{\mu_2}{x_2} \right\rceil (x_2 - x_1')}{x_2 - x_1'} \right\rceil \ge \left\lceil \frac{\mu_2 - \left\lceil \frac{\mu_2}{x_2} \right\rceil x_1'}{x_2 - x_1'} \right\rceil.$$

Thus, substituting the above two constraints into (41),

$$\mu_3 \le \left(g_{31} - \left\lceil \frac{\mu_1}{x_1} \right\rceil\right)\left(g_{23} - \left\lceil \frac{\mu_2}{x_2} \right\rceil\right)$$
$$\le \begin{cases} \left(g_{31} - \left\lceil \frac{\mu_1+\mu_1'}{x_1+x_1'} \right\rceil\right) g_{23} & \text{if } \widetilde{\mu}_2 = \mu_2 \\ \left(g_{31} - \left\lceil \frac{\mu_1+\mu_1'}{x_1+x_1'} \right\rceil\right)\left(g_{23} - \left\lceil \frac{\mu_2-\widetilde{\mu}_2}{x_2-x_1'} \right\rceil\right) & \text{if } \widetilde{\mu}_2 \ne \mu_2. \end{cases}$$

The latter expression implies that $\mu_1 + \mu_1'$, $\mu_2 - \widetilde{\mu}_2$ and μ_3 satisfy condition (7) of Theorem 3. Moreover, $\lceil \mu_1/x_1 \rceil \ge \lceil (\mu_1+\mu_1')/(x_1+x_1') \rceil$ and $\lceil \mu_2/x_2 \rceil \ge \lceil (\mu_2-\widetilde{\mu}_2)/(x_2-x_1') \rceil$ when $\widetilde{\mu}_2 \ne \mu_2$, and hence by letting $x = x_1 + x_1'$, $\mu_1+\mu_1'$ and $\mu_2-\widetilde{\mu}_2$ satisfy conditions (5) and (6) of Theorem 3 from (37) and (38) above. Thus, $\mu_1 + \mu_1'$, $\mu_2 - \widetilde{\mu}_2$ and μ_3 clients with the periodicity of q_1, q_2 and q_3, respectively, fit into a single $\mathbf{\Lambda}$-set by Theorem 3 when $\widetilde{\mu}_2 \ne \mu_2$. On the other hand, when $\widetilde{\mu}_2 = \mu_2$, the former expression implies that

$$\left\lceil \frac{\mu_1 + \mu_1'}{g_{12}} \right\rceil + \left\lceil \frac{\mu_3}{g_{23}} \right\rceil \le \left\lceil \frac{\mu_1 + \mu_1'}{x_1 + x_1'} \right\rceil + \left(g_{31} - \left\lceil \frac{\mu_1 + \mu_1'}{x_1 + x_1'} \right\rceil\right) = g_{31},$$

since $x_1+x_1' < x_1+x_2 = g_{12}$. Hence, by Theorem 1, $\mu_1+\mu_1'$ and μ_3 clients with the periodicity of q_1 and q_3, respectively, fit into a single $\mathbf{\Lambda}$-set.

Therefore, the feasibility of the schedule is preserved. Repeating this argument, we obtain a feasible schedule with $\lambda_{123} \le 1$. □

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