

A scattering matrix formulation of the topological index of interacting fermions in one-dimensional superconductors

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We construct a scattering matrix formulation for the topological classification of one-dimensional superconductors with effective time reversal symmetry in the presence of interactions. For a closed geometry, Fidkowski and Kitaev have shown that such systems have a \mathbb{Z}_8 topological classification. We show that in the weak coupling limit, these systems retain a unitary scattering matrix at zero temperature, with a topological index given by the trace of the Andreev reflection matrix, $\text{tr } r_{\text{he}}$. With interactions, $\text{tr } r_{\text{he}}$ generically takes on the finite set of values $0, \pm 1, \pm 2, \pm 3$, and ± 4 . We show that the two topologically equivalent phases with $\text{tr } r_{\text{he}} = \pm 4$ support emergent *many-body* end states, which we identify to be a topologically protected Kondo-like resonance. The path in phase space that connects these equivalent phases crosses a non-fermi liquid fixed point where a multiple channel Kondo effect develops. Our results connect the topological index to transport properties, thereby highlighting the experimental signatures of interacting topological phases in one dimension.

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Introduction Superconducting wires exist in two topologically distinct classes, which can be distinguished by the presence or absence of a Majorana bound state at the wire's ends [1–4]. Because of their non-Abelian exchange statistics [5–7] such Majorana states have been proposed as an element for a fault-tolerant quantum computational scheme [8–11]. The topological phases have been shown to be robust against the presence of (moderate) disorder [12, 13] or interactions [14–17]. While only few rather exotic systems are believed to realize the non-trivial phase [5, 18, 19], such systems can in principle be engineered in solid state devices [20–22]. Recent experiments on semiconductor wires proximity coupled to superconductors have been reported to show indications of the existence of Majorana bound states [23, 24].

Whereas the topological classification of superconducting wires with or without Majorana states exclusively rests on the presence of particle-hole symmetry, in many of the proposals for the actual realization of topological superconducting wires an additional approximate *effective time-reversal* symmetry appears [25]. For the $p + ip$ model of a spinless superconductor wire of width W and Fermi velocity v_F , the effective time reversal symmetry requires the superconducting gap to be much smaller than the transverse quantization energy $\hbar v_F/W$ [26], a condition that is easily met in view of the generic smallness of the gap in the proposals to engineer topological superconductors. With effective time reversal symmetry, Majorana states carry a sign, such that Majorana states of the same sign coexist at the same end of the superconducting wire. As a result, the effective time reversal symmetry changes the topological classification from \mathbb{Z}_2 to \mathbb{Z} , which counts the number of Majorana end states, with sign.

Unlike the \mathbb{Z}_2 classes, the \mathbb{Z} effective time reversal symmetry classification is not stable against the presence of

interactions, even if the interactions preserve the symmetry. In a seminal work, Fidkowski and Kitaev [27] showed that interactions break the free-fermion \mathbb{Z} topological index down to \mathbb{Z}_8 (see also Refs. 28–31). To understand this result, one notes that for the seven topological classes with $n = 0, \pm 1, \pm 2$, and ± 3 there exist realizations with less than two fermions at each wire's end (as two Majoranas combine into one fermion). Hence, no local interaction term is allowed for the low-energy sector and the free-fermion classification remains valid. Fidkowski and Kitaev then showed that the two classes with $n = \pm 4$ can be adiabatically connected by a suitable interaction, so that eight inequivalent classes remain [32].

The experimental observation of Majorana states inevitably relies on the coupling of the topological superconductor to a normal-metal contact, the simplest detection scheme being through a measurement of the two-terminal Andreev conductance [23, 24]. With effective time reversal symmetry, but without interactions, the Andreev conductance G at zero bias and zero temperature takes the quantized value $G = 2|n|e^2/h$, if there are n Majorana end states [33–35]. Distinguishing positive and negative n requires a phase-sensitive measurement of the Andreev reflection matrix, using $\text{tr } r_{\text{he}} = n$ [36].

What is the Andreev-reflection signature of the \mathbb{Z}_8 topological classification for a superconducting wire in the effective time reversal symmetry class, and what is the nature of the emergent end states if the wire is weakly coupled to a normal contact? In this letter we answer these questions, showing how the breakdown of the free-fermion \mathbb{Z} classification to \mathbb{Z}_8 is reflected in the Andreev reflection matrix r_{he} . Establishing the scattering properties of interacting topological phases highlights the experimental signatures of the emergent end states in conductance measurements or Josephson currents, thereby providing an important link between recent theoretical

advances and future experimental work.

Our main results can be summarized as follows: For a normal lead weakly coupled to the superconducting wire, we find (i) that generically the scattering matrix is well defined and unitary at zero temperature, in spite of the presence of interactions in the superconducting wire, and (ii) that in the presence of interactions, $\text{tr } r_{\text{he}}$ is restricted to the values $0, \pm 1, \pm 2, \pm 3$, and ± 4 . Exceptions to these rules do occur, but they form a set of measure zero and are unstable to perturbations. The topological phases with $-3 \leq \text{tr } r_{\text{he}} \leq 3$ are topologically equivalent to their free-fermion analogues with $|n| \leq 3$ uncoupled Majorana end states. Conversely, we show that the phases with $\text{tr } r_{\text{he}} = \pm 4$ are characterized by emergent *many-body* end states, which we identify as topologically protected Kondo-like resonances. These two configurations, which are both stable with respect to perturbations, are topologically *equivalent* in the presence of interactions. The path in phase space that connects these equivalent phases crosses a non-fermi liquid fixed point associated with a multiple channel Kondo problem, without closing the bulk excitation gap in the superconductor.

Model. We derive these results in the framework of a multi-channel Majorana chain [37] with an effective time reversal symmetry, $\mathcal{T}^2 = 1$, corresponding to class BDI in the Cartan classification [38]. In the absence of interactions, the Majorana chain is described by the Hamiltonian

$$H_S = \sum_{\alpha,j} +(\Delta_\alpha d_{j+1,\alpha}^\dagger d_{j,\alpha}^\dagger - t d_{j+1,\alpha}^\dagger d_{j,\alpha} + \text{h.c.}) + \sum_{\alpha,j} \mu d_{j,\alpha}^\dagger d_{j,\alpha}, \quad (1)$$

where $t > 0$ is the hopping parameter, μ the chemical potential, and $d_{j,\alpha}$ is the annihilation operator for a fermion on site j and channel α . We consider $j = 1, 2, \dots$, so that Eq. (1) describes a half-infinite wire beginning at $j = 1$. We choose $\Delta_\alpha \neq 0$ to be real, so that H_S is invariant under an effective antiunitary time-reversal symmetry operation \mathcal{T} , $\mathcal{T} d_{j,\alpha} \mathcal{T}^{-1} = d_{j,\alpha}$.

Equation (1) can be conveniently rewritten in terms of Majorana operators, $\gamma_{j,\alpha} = d_{j,\alpha} + d_{j,\alpha}^\dagger$ and $\tilde{\gamma}_{j,\alpha} = -i(d_{j,\alpha} - d_{j,\alpha}^\dagger)$. The model undergoes a topological phase transition at $|\mu| = 2|t|$. The topological phase is characterized by two zero energy Majorana end states for each channel, exponentially localized at opposite ends of the chain and separated by an energy gap from the bulk excitations. Since we consider a half-infinite chain, only the end state located near $j = 1$ is of relevance for us. For positive Δ_α the operator for this Majorana end state commutes with \mathcal{T} and we denote it by γ_α ; for negative Δ_α the end-state operator anticommutes with \mathcal{T} , and we write $\tilde{\gamma}_\alpha$. The effective time-reversal symmetry permits perturbations that couple Majorana end states of different type, $\mathcal{T} i \gamma_\alpha \tilde{\gamma}_\beta \mathcal{T}^{-1} = i \gamma_\alpha \tilde{\gamma}_\beta$, but forbids cou-

pling between Majorana end modes of the same type. The system is therefore classified by a \mathbb{Z} topological index, which counts the number N of Majorana end states of “ γ ” type minus the number \tilde{N} of end states of “ $\tilde{\gamma}$ ” type. Without loss of generality, we take the channels $\alpha = 1, 2, \dots, N$ to be of “ γ ” type, whereas the remaining channels $\alpha = N + 1, \dots, N + \tilde{N}$ are of “ $\tilde{\gamma}$ ” type.

Scattering matrix classification. To study the scattering properties, we couple the semi-infinite wire to a half-infinite normal lead. The Hamiltonian is given by

$$H = H_S + H_{\text{lead}} + H_T, \quad (2)$$

where H_{lead} is the Hamiltonian of the non-interacting ideal lead, $H_{\text{lead}} = \sum_{k,\alpha} \xi_k c_{k,\alpha}^\dagger c_{k,\alpha}$. Here $c_{k,\alpha}$ is the annihilation operator for an electron with momentum k measured with respect to the Fermi point and channel index α and ξ_k the corresponding kinetic energy. The term H_T describes tunneling between the lead and the superconductor, $H_T = \tilde{t}_T \sum_{k,\alpha} c_{k,\alpha}^\dagger d_{1,\alpha} + \text{h.c.}$, \tilde{t}_T being the tunneling matrix element. Considering that the superconducting wire is gapped in the bulk, we can project the tunneling Hamiltonian onto the low-energy sector consisting of the zero-energy Majorana end-states,

$$H_T = \frac{t_T}{2} \sum_k \left(\sum_{\alpha=1}^N c_{k,\alpha}^\dagger \gamma_\alpha + i \sum_{\alpha=N+1}^{N+\tilde{N}} c_{k,\alpha}^\dagger \tilde{\gamma}_\alpha \right), \quad (3)$$

where t_T is the effective tunneling coupling of the end-state Majorana to the lead. As the system is gapped, no quasiparticle excitations are transmitted through the superconductor, and scattering processes are described by a unitary reflection matrix. In the BDI class, each Majorana end mode gives rise to perfect Andreev reflection, although the sign of the reflection amplitude is opposite for the two types of Majorana modes. The topological index is $n \equiv N - \tilde{N} = \text{tr } r_{\text{he}}$ [36, 39], where r_{he} is evaluated at zero energy.

Generalization to interacting systems. We now generalize the above classification scheme to interacting systems. We consider short-range two-fermion interactions, so that, after projection to the low-energy sector of the Majorana end states, the interaction Hamiltonian involves the end-state operators γ_α and $\tilde{\gamma}_\alpha$ only [40–42]. If all Majorana end states are of the “ γ ” type, the most general interaction in the low-energy sector has the form

$$H_{\text{int}} = \sum_{\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4} W_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \gamma_{\alpha_1} \gamma_{\alpha_2} \gamma_{\alpha_3} \gamma_{\alpha_4}. \quad (4)$$

This interaction Hamiltonian commutes with the effective time-reversal operator \mathcal{T} . If there are end states of both types, additional terms with four operators $\tilde{\gamma}$ or mixed terms with two operators of each type are also allowed by the effective time-reversal symmetry.

We now discuss how the inclusion of the interaction Hamiltonian affects a Majorana chain with free-fermion

topological index n . We'll find that the cases $|n| < 4$, $|n| = 4$, and $|n| > 4$ are qualitatively different, and discuss these three cases separately.

The case $|n| < 4$. Since generic potential perturbations gap out pairs of Majorana states of opposite type, for weak interactions it is sufficient to limit our discussion to the ‘‘minimal’’ realizations of the topological phases, which have $\tilde{N} = 0$, if the topological index $n = N$ is positive, and $N = 0$, if $n = -\tilde{N}$ is negative. Since the low-energy interaction Hamiltonian $H_{\text{int}} = 0$ if $N < 4$ and $\tilde{N} = 0$, or if $\tilde{N} < 4$ and $N = 0$, see Eq. (4), we immediately conclude that the Andreev-reflection signatures of these phases are unaffected by interactions, so that the reflection matrix is unitary and $\text{tr } r_{\text{he}} = n$.

The case $|n| = 4$. The case $n = \pm 4$ is nontrivial, since already in its minimal realization it allows for a nontrivial interaction Hamiltonian $H_{\text{int}} = W\gamma_1\gamma_2\gamma_3\gamma_4$. Introducing the fermionic operators $f_{\uparrow} = (1/2)(\gamma_1 + i\gamma_2)$, $f_{\downarrow} = (1/2)(\gamma_3 + i\gamma_4)$, the low-energy interaction Hamiltonian for the case $n = 4$ takes the form $H_{\text{int}} = -W(2f_{\uparrow}^{\dagger}f_{\uparrow} - 1)(2f_{\downarrow}^{\dagger}f_{\downarrow} - 1)$. The interaction lifts the four-fold degeneracy of the zero energy level of the free fermion system and creates two doubly degenerate correlated states at energy $\pm W$. The situation with a degenerate level located at the end of the superconducting wire closely resembles a local impurity problem and much insight can be gained by performing a mapping to the latter. Hereto, the tunneling Hamiltonian is rewritten using the operators $g_{L,\uparrow,k} = 2^{-1/2}(c_{k,1} + ic_{k,2})$, $g_{R,\uparrow,-k} = 2^{-1/2}(-c_{k,1}^{\dagger} - ic_{k,2}^{\dagger})$, $g_{L,\downarrow,k} = 2^{-1/2}(c_{k,3}^{\dagger} + ic_{k,4})$, $g_{R,\downarrow,-k} = 2^{-1/2}(-c_{k,3}^{\dagger} - ic_{k,4}^{\dagger})$, so that the resulting Hamiltonian takes the form of the symmetric Anderson model [43]

$$H = \sum_{\beta=L,R} \sum_{k,\sigma=\uparrow,\downarrow} \left[\xi_k g_{\beta,\sigma,k}^{\dagger} g_{\beta,\sigma,k} + \frac{t_{\text{T}}}{2} \left(g_{\beta,\sigma,k}^{\dagger} f_{\sigma} + \text{h.c.} \right) \right] - W(2f_{\uparrow}^{\dagger}f_{\uparrow} - 1)(2f_{\downarrow}^{\dagger}f_{\downarrow} - 1). \quad (5)$$

In general, the scattering matrix of this Kondo-like problem is non-unitary due to inelastic spin flip processes. However, at zero temperature unitarity is recovered due to the formation of a Kondo screening cloud, and the scattering matrix takes the simple form [44]

$$S_{\sigma,\sigma'} = \delta_{\sigma,\sigma'} \left(\begin{array}{cc} r_{\text{RR}} & t_{\text{RL}} \\ t_{\text{LR}} & r_{\text{LL}} \end{array} \right) \Big|_{\sigma} = \left(\begin{array}{cc} 0 & -1 \\ -1 & 0 \end{array} \right). \quad (6)$$

Returning to the original basis with the lead operators $c_{k,\alpha}$, one finds that the normal reflection matrix $r_{\text{ee}} = 0$, whereas $r_{\text{he}} = \mathbb{1}_{4 \times 4}$. We conclude that at zero temperature the topological index follows the non-interacting formula $\text{tr } r_{\text{eh}} = n = 4$. As a physical consequence, a tunneling conductance experiment will show a zero-bias zero-temperature Andreev conductance $G = 8e^2/h$, which is the same conductance as without interactions. However, unlike in the non-interacting case, this quantized conductance peak is a consequence of a formation of a many-body Kondo-like resonance at the end of the interacting

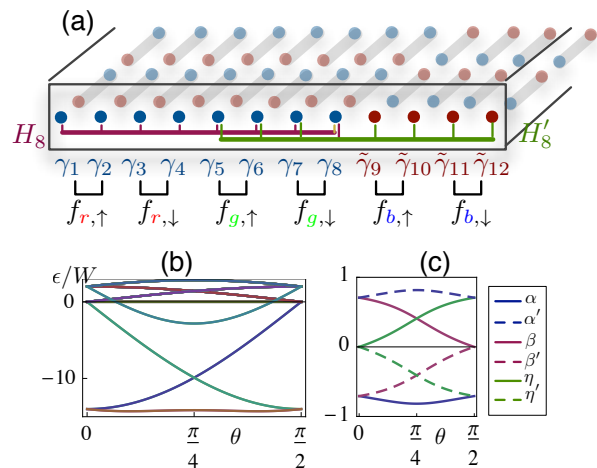


FIG. 1. (color online) (a) Schematic picture of a twelve Majorana chains that interpolate between effective free-fermion classes $n = 4$ and $n = -4$. The interactions terms coupling different Majorana end-states are indicated by thick lines. The ‘‘colored’’ fermions constructed with Majorana fermions are indicated explicitly. (b) Energy spectrum as a function of the interpolation parameter θ , showing that the spectrum remains gapped through the interpolation. The ground state is two-fold degenerate. (c) Amplitudes $\alpha, \alpha', \beta, \beta', \gamma$, and γ' for the ground state wavefunctions, as a function of the interpolation parameter θ , see Eq. (9).

superconductor, and it no longer signals the presence of four Majorana states. We can interpret this collective state, which is pinned to the Fermi energy, as the emergent edge state of the interacting superconductor.

The same analysis can be applied to the $n = -4$ case. It gives $r_{\text{he}} = -\mathbb{1}_{4 \times 4}$ and $\text{tr } r_{\text{he}} = n = -4$. This implies that $\text{tr } r_{\text{he}}$ takes different quantized values for $n = 4$ and $n = -4$. How should this result be interpreted in light of the knowledge that, with interaction, classes with $n = 4$ and with $n = -4$ are topologically equivalent [27]?

To find a scenario that resolves this paradox we monitor the Andreev reflection matrix along a path that connects the two classes $n = 4$ and $n = -4$. The minimum channel number for a continuous interpolation between $n = 4$ and $n = -4$ is twelve. For definiteness, we consider eight channels with positive Δ , labeled $\alpha = 1, 2, \dots, 8$, and four channels with negative Δ , labeled $\alpha = 9, 10, 11, 12$, so that $N = 8$ and $\tilde{N} = 4$, see Fig. 1. For the interaction Hamiltonian we choose

$$H_{\text{int}}(\theta) = W(H_8 \sin \theta + H'_8 \cos \theta) \quad (7)$$

where

$$H_8 = \gamma_1\gamma_2\gamma_3\gamma_4 + \gamma_5\gamma_6\gamma_7\gamma_8 + \gamma_1\gamma_2\gamma_5\gamma_6 + \gamma_3\gamma_4\gamma_7\gamma_8 \\ - \gamma_2\gamma_3\gamma_6\gamma_7 - \gamma_1\gamma_4\gamma_5\gamma_8 + \gamma_1\gamma_3\gamma_5\gamma_7 + \gamma_3\gamma_4\gamma_5\gamma_6 \\ + \gamma_1\gamma_2\gamma_7\gamma_8 - \gamma_2\gamma_3\gamma_5\gamma_8 - \gamma_1\gamma_4\gamma_6\gamma_7 + \gamma_2\gamma_4\gamma_6\gamma_8 \\ - \gamma_1\gamma_3\gamma_6\gamma_8 - \gamma_2\gamma_4\gamma_5\gamma_7, \quad (8)$$

and H'_8 is obtained from H_8 by the substitution $\gamma_\alpha \rightarrow \gamma_{\alpha+4}$ for $\alpha = 1, 2, 3, 4$ and $\gamma_\alpha \rightarrow \tilde{\gamma}_{\alpha+4}$ for $\alpha = 5, 6, 7, 8$. As shown by Fidkowski and Kitaev, the Hamiltonian H_8 drives the eight Majorana end states in the first eight channels into a nondegenerate topologically trivial ground state [27]. Similarly, the Hamiltonian H'_8 drives the eight Majorana states in the last eight channels into a nondegenerate trivial ground state. (Since the last eight channels have four Majorana states of each type, a potential term coupling the last eight channels would have the same effect.) For $\theta = 0$ the system is effectively in the free-fermion $n = 4$ class analyzed previously, with low-energy interaction Hamiltonian $H_{\text{int}} = W\gamma_1\gamma_2\gamma_3\gamma_4$, while for $\theta = \pi/2$ it is in the opposite case $n = -4$, with low-energy interaction Hamiltonian $W\tilde{\gamma}_9\tilde{\gamma}_{10}\tilde{\gamma}_{11}\tilde{\gamma}_{12}$. Thus, the family of interaction Hamiltonians $H_{\text{int}}(\theta)$ with $0 \leq \theta \leq \pi/2$ smoothly interpolates between the (free-fermion) classes $n = 4$ and $n = -4$.

To continue our analysis, we construct six fermionic operators out of the twelve Majorana operators, which we group into three “colors” (r,g,b), $f_{r,\uparrow} = (1/2)(\gamma_1 + i\gamma_2)$, $f_{r,\downarrow} = (1/2)(\gamma_3 + i\gamma_4)$, $f_{g,\uparrow} = (1/2)(\gamma_5 + i\gamma_6)$, $f_{g,\downarrow} = (1/2)(\gamma_7 + i\gamma_8)$, $f_{b,\uparrow} = (1/2)(\tilde{\gamma}_9 + i\tilde{\gamma}_{10})$, $f_{b,\downarrow} = (1/2)(\tilde{\gamma}_{11} + i\tilde{\gamma}_{12})$, see Fig. 1. Diagonalization of the interaction Hamiltonian (7) reveals that the many-particle ground state is two-fold degenerate for all $0 < \theta < \pi/2$ and confirms that the excitation spectrum remains gapped otherwise, see Fig. 1. The two-fold degenerate ground state is spanned by the two states of the form

$$\begin{aligned} |\psi\rangle &= \alpha |111111\rangle + \beta |110000\rangle + \eta |000011\rangle \\ |\psi'\rangle &= \alpha' |000000\rangle + \beta' |001111\rangle + \eta' |111100\rangle, \end{aligned} \quad (9)$$

where we use the basis of the occupations numbers $|n_{r\uparrow}n_{r\downarrow}n_{g\uparrow}n_{g\downarrow}n_{b\uparrow}n_{b\downarrow}\rangle$. The excitation gap and the real coefficients $\alpha, \alpha', \beta, \beta', \gamma$, and γ' as a function of θ are shown in Fig. 1. For $\theta \rightarrow 0^+$, we have $\eta = \eta' = 0$, and a transition between the two ground states is possible only by the exchange of two “red” fermions with two lead fermions in the first four channels $\alpha = 1, 2, 3, 4$. In this case the system can be mapped to the symmetric Anderson model, as discussed above. Hence, at zero temperature, the reflection matrix is unitary and satisfies $\text{tr } r_{\text{eh}} = 4$. Conversely, for $\theta \rightarrow \pi/2^-$, we find $\beta = \beta' = 0$, and a transition between the two ground states is possible only by the exchange of two “blue” fermions with lead channels in the last set of four channels, $\alpha = 9, 10, 11, 12$. In this case the system is again mapped to a symmetric Anderson model, but with $\text{tr } r_{\text{eh}} = -4$ at zero temperature. For $0 < \theta < \pi/2$, generically all six amplitudes in Eq. (9) are nonzero. In this case transitions between the two ground states can take place by the exchange of two fermions of arbitrary but *equal* colors. Since there are three colors in total, the model is mapped to a three-channel Kondo problem. Such a multi-channel Kondo problem is, however, unstable, and the system flows to

a single-channel Kondo fixed point determined by the strongest coupling constant, and correspondingly to one of the two extreme limits discussed above. Upon increasing θ from 0 to $\pi/2$ invariably there must be a point at which the coupling to the “blue” channels and the “red” or “green” channels is equal. This point is associated with a sharp phase boundary that exhibits a non-fermi liquid behavior, due to the formation of a multiple (generically: two) channel Kondo state at the wire’s end. It follows that the scattering matrix goes through a non-unitary point along the path connecting these two phases. It is at this point that the transition between the quantized values $\text{tr } r_{\text{he}} = \pm 4$ can take place. We note that this observation is consistent with recent studies that show that the crossover between topologically distinct non-interacting classes that become equivalent when interactions are present is associated with zeros of the Green’s function indicating the formation of a non-fermi liquid state [31].

The case $|n| > 4$. Upon including interactions, the free-fermion cases with $|n| = 5, 6, 7$, or 8 can be continuously connected to free-fermion classes with $|n| = 3, 2, 1$, and 0, respectively. An explicit example of a generic low energy Hamiltonian that interpolates between the free-fermion phases of $n = 7$ and $n = -1$ is presented in the supplementary material. An analysis of the transitions between the two degenerate ground states reveals that the free-fermion $n = 7$ configuration is unstable to interactions, and that in the weak-coupling limit $|t_{\text{T}}| \ll |t|$ the interacting system flows to an effectively non-interacting configuration with $\text{tr } r_{\text{he}} = -1$. A similar analysis can be applied to $|n| = 5$, $|n| = 6$, and $|n| = 8$.

Conclusions. We have constructed a scattering matrix formulation for the topological index of interacting fermions in one dimension with an effective time reversal symmetry. The scattering matrix of the interacting system is unitary at zero temperature and zero energy, and the topological index can be calculated from the trace $\text{tr } r_{\text{he}}$. With interactions, the topological index is restricted to nine possible values, $\text{tr } r_{\text{he}} = 0, \pm 1, \pm 2, \pm 3$, and ± 4 . Whereas the phases with $|\text{tr } r_{\text{he}}| < 4$ are effectively single-particle phases, we have shown that the two topologically equivalent phases with $\text{tr } r_{\text{he}} = \pm 4$ are characterized by emergent *many-body* end states, which we identify to be a topologically protected Kondo-like resonance. The path in phase space that connects these equivalent phases crosses a non-fermi liquid fixed point where a multiple channel Kondo effect develops.

Although the main motivation for our work is fundamental, a theory of implications of the topological classification for scattering properties is essential for a theoretical description of experimental geometries, where topological superconductors necessarily need to be connected to normal-metal probes. The effective time-reversal symmetry that is behind the \mathbb{Z}_8 topological classification is relevant for some of the recent proposals to realize topo-

logical superconductors in one dimension. Whereas interactions are believed to be of minor importance if the topological superconductivity derives from the proximity of a bulk superconductor, experimental efforts to minimizing the electric screening of nearby superconductors in order to allow for local gating of the device inevitably lead to a larger role of interactions in the topological superconductor.

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SUPPLEMENTARY MATERIAL

Scattering matrix of a Majorana chain with topological index $|n| > 4$

Upon including interactions, the free-fermion cases with $|n| = 5, 6, 7$, or 8 can be continuously connected to free-fermion classes with $|n| = 3, 2, 1$, and 0, respectively. For the cases $n = \pm 5, \pm 6$, or ± 7 this requires the addition of extra channels (at least, locally, near the end of the Majorana chain), similarly to the transition from $n = 4$ to $n = -4$ discussed in the main text. For presentation purposes we focus on the case $n = 7$; the other seven cases have similar phenomenology.

Consider the nine-channel setup depicted in Fig. 2. The channels $\alpha = 1, 2, \dots, 8$ have positive Δ , whereas channel $\alpha = 9$ has negative Δ . Without interactions, the topological index $n = N - \tilde{N} = 7$. To the non-interacting Hamiltonian H of Eq. 2 in the main text we add the per-

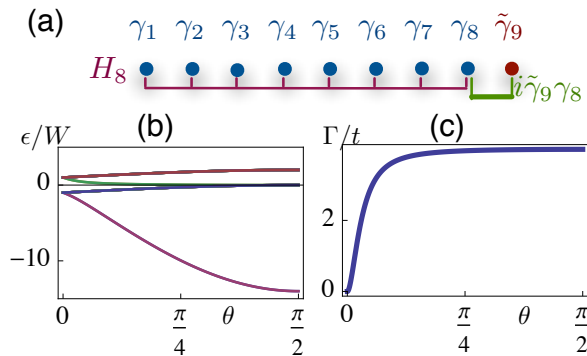


FIG. 2. (a) Schematic picture a nine channel setup that allows for an interpolation between the topologically equivalent effective free-fermion phases with $n = 7$ and $n = -1$. (b) Energy spectrum of the low energy excitations as a function of the interpolation parameter θ , showing that the system remains gapped throughout the transition. (c) Modulus square of single-particle tunneling matrix elements between the two degenerate ground states, as a function of θ .

turbation:

$$H'(\theta) = W(i \cos \theta \tilde{\gamma}_9 \gamma_8 + H_8 \sin \theta) \quad (10)$$

where the interaction H_8 is given by Eq. 8 in the main text. For $0 < \theta \leq \pi/2$ the ground state is twofold degenerate. For $\theta = 0$ the two Majorana end states γ_8 and $\tilde{\gamma}_9$ gap out, and the system is effectively in the free-fermion $n = 7$ configuration. At this special point one expects $\text{tr } r_{\text{he}} = 7$. Conversely, when $\theta = \pi/2$, the interaction term H_8 renders the eight first channels trivial, leaving a single Majorana state $\tilde{\gamma}_9$, corresponding to the free-fermion $n = -1$ configuration with $\text{tr } r_{\text{he}} = -1$. An explicit diagonalization of Eq. (10) shows that the gap remains open throughout the interpolation between these two extremes, see Fig. 2, indicating that the two are indeed in the same topological phase.

We now consider how r_{he} goes between its two extreme values if θ is taken from 0 to $\pi/2$. For θ small but positive the interaction H_8 lifts the ground state degeneracy of the free-fermion $n = 7$ configuration, up to a twofold degeneracy involving two states with opposite fermion parity, which are simultaneously located at both ends of the superconducting wire. An explicit calculation shows that single-particle tunneling events do not couple the two ground states in the limit $\theta \rightarrow 0^+$, see Fig. 2(c), indicating that the end states are not of single particle nature in this limit. Instead, in the limit $\theta \rightarrow 0^+$ transitions between the two ground states require the exchange of multiple fermions with the leads. In contrast, for generic $0 < \theta \leq \pi/2$ the end spectrum is essentially of single particle nature, and the exchange of a single fermion with the leads is sufficient for a transition between the two degenerate ground states. The degeneracy, fermion parity, and transition matrix elements between the states at generic $0 < \theta \leq \pi/2$ evolve continuously into the free-fermion

state at $\theta = \pi/2$, which has a single Majorana state of “ $\tilde{\gamma}$ ” type at the end of the Majorana chain. In the weak coupling limit $|t_{\text{T}}| \ll |t|$ such single particle tunneling events dominate over the higher order processes which derive from the limit $\theta \rightarrow 0$, so that for weak coupling the system behaves as a $n = -1$ free fermion configuration along the entire path $0 < \theta \leq \pi/2$, with the exception of the special point $\theta = 0$. We conclude that in a generic setting the free-fermion $n = 7$ configuration is unstable to interactions, and that in the weak-coupling limit the interacting system flows to an effectively non-interacting configuration with $\text{tr } r_{\text{he}} = -1$.

A similar analysis can be applied to the remaining cases $|n| = 5$, $|n| = 6$, and $|n| = 8$, and shows that upon inclusion of interactions these configurations are dominated by effectively free-fermion configurations with $|n| = 3$, $|n| = 2$, and $|n| = 0$, respectively.