

Bilateral trading and incomplete information: Price convergence in a small market.

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Abstract

We study a model of decentralised bilateral interactions in a small market where one of the sellers has private information about her value. In addition to this informed seller, there are two identical buyers and another seller, whose valuation is commonly known to be in between the two possible valuations of the informed seller. We consider an infinite horizon game with simultaneous one-sided offers and simultaneous responses. We *characterise* one PBE of the game and show that, as the discount factor goes to 1, prices in all transactions converge to the same value. We then show that this is the case with *any* stationary equilibrium of the game. That is, the asymptotic outcome is *unique* across all stationary equilibria.

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1 Introduction

This paper studies a small market in which one of the players has private information about her valuation. As such, it is a first step in combining the literature on (bilateral) trading with

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incomplete information with that on market outcomes obtained through decentralised bilateral bargaining.

We shall discuss the relevant literature in detail later on in the introduction. Here we summarise the motivation for studying this problem.

One of the most important features in the study of bargaining is the role of alternative bargaining partners in determining the trading outcome. There have been several different approaches to modelling the effect of such alternatives on an existing negotiation, starting with treating alternatives to the current bargaining game as exogenously given and always available. Accounts of negotiation directed towards practitioners and policy-oriented academics, like Raiffa’s masterly “The Art and Science of Negotiation”, ([37]) have emphasised the key role of the “Best Alternative to the Negotiated Agreement” and mentioned the role of searching for such alternatives in preparing for negotiations. We could also take such alternatives into account by explicitly considering strategic choice of bargaining partners.

Proceeding more or less in parallel, there has been considerable work on bargaining with incomplete information. The major success of this work has been the complete analysis of the bargaining game in which the seller has private information about the minimum offer she is willing to accept. The probability distribution from which the seller’s reservation price is drawn is common knowledge. The buyer makes repeated offers which the seller can accept or reject; each rejection takes the game to another period and time is discounted at a common rate by both parties.¹ With the roles of the seller and buyer reversed, this has also been part of the development of the foundations of dynamic monopoly and the Coase conjecture.²

Our model here analyses a small market in which players make strategic choices of whom to make offers to and (for sellers), whose offer to accept. Thus, though trades remain bilateral, a buyer can choose to make an offer to a different seller than the one who rejected his last offer and the seller can entertain an offer from some buyer whom she has not bargained with before. These alternatives are internal to the model of a small market, rather than given as part of the environment. What we do is as follows: We take the basic problem of a seller with private information and an uninformed buyer and add another buyer-seller pair. The new seller’s valuation is commonly known and is different from the possible valuations of the informed seller. The buyers’ valuations are identical and commonly known. Specifically, the informed seller’s valuation can

¹Other, more complicated, models of bargaining have also been formulated (for example, [11]), with two-sided offers and two-sided incomplete information, but these have not usually yielded the clean results of the game with one-sided offers and one-sided incomplete information.

²The “Coase conjecture” relevant here is the bargaining version of the dynamic monopoly problem, namely that if an uninformed seller (who is the only player making offers) has a valuation strictly below the informed buyer’s lowest possible valuation, the unique sequential equilibrium as the seller is allowed to make offers frequently, has a price that converges almost immediately to the lowest buyer valuation. In the current work, we show that even if the bilateral bargaining is embedded in a small market, a similar conclusion holds for all stationary equilibria-hence an extended Coase conjecture holds.

either be L or H ($H > L \geq 0$) and the new seller's valuation is M such that $M \in (L, H)$. Each seller has one good and each buyer wants at most one good. This is the simplest extension of the basic model that gives rise to alternatives for each player, though only *one* buyer can deviate from the incomplete information bargaining to choose the other seller (if this other seller accepts the offer), since each seller only has one good to sell.³

In our model, buyers make offers *simultaneously*, each buyer choosing only one seller.⁴ Sellers also respond simultaneously, accepting at most one offer. A buyer whose offer is accepted by a seller leaves the market with the seller and the remaining players play the one-sided offers game with or without asymmetric information. We consider the case when buyers' offers are public, so that the continuation strategies can condition on both offers in a given period and the set of players remaining.⁵ We first **characterise** a stationary equilibrium. The **main result** of our analysis shows that in the incomplete information game, **any stationary equilibrium** must have certain specific qualitative features. As agents become patient enough, these qualitative features enable us to show that **all price offers in any stationary equilibrium converge to the highest possible value of the informed seller (H)**. In the two-player case, there is a unique sequential equilibrium for the "gap"⁶ case. However, there could be non-stationary equilibria with different outcomes in the four-player, public offers case. We note that the discussion of the features of the stationary equilibrium is concerned with the properties of proposed actions occurring with positive probability on the equilibrium path. However, in order to show existence, the properties of outcome paths following deviations become important.

The equilibrium we construct to demonstrate existence is in (non-degenerate) randomized behavioural strategies. As agents become patient enough, in equilibrium competition always takes place for the seller whose valuation is commonly known. The *equilibrium* behaviour of beliefs is similar to the two-player asymmetric information game. However, the off-path behaviour sustaining any equilibrium is different and has to take into account many more possible deviations.⁷

Our asymptotic convergence result above has a flavour of the Coase Conjecture and could be considered as a contribution to the discussion of its robustness to the setting of a small market.

³What do the seller's valuations represent? (The buyers' valuations are clear enough.) We could consider a seller who can produce a good, if contracted to do so, at a private cost of H or L and pays no cost otherwise. Or we could consider the value she gets from keeping the object as H or L . Thus, supposing her value is L , if she accepts a price offer p with probability α , her payoff is $L(1 - \alpha) + (p)\alpha = (p - L)\alpha + L$. Hence, one can think of $(p - L)$ as the net expected benefit to the seller from selling the good at price p . For the purpose of making the decision on whether to accept or reject, the two interpretations give identical results.

⁴Simultaneous offers extensive forms capture best the essence of competition.

⁵We also discuss private offers in the extensions. This is the case when only the proposer and the recipient of an offer know what it is and the only public information is the set of players remaining in the game.

⁶The *gap* case implies the situation when the highest possible value of the seller is strictly below the valuation of the buyer.

⁷The result of this paper is not confined to uncertainty described by two types of seller. Even if the informed seller's valuation is drawn from a continuous distribution on $(L, H]$, it can be shown that the asymptotic convergence to H still holds as the unique limiting stationary equilibrium outcome. Please refer to [8] for a formal analysis.

Related literature: The modern interest in this approach dates back to the seminal work of Rubinstein and Wolinsky ([38], [39]), Binmore and Herrero ([5]) and Gale ([19],[20]). These papers, under complete information, mostly deal with random matching in large anonymous markets, though Rubinstein and Wolinsky (1990) is an exception. Chatterjee and Dutta ([9]) consider strategic matching in an infinite horizon model with two buyers and two sellers and Rubinstein bargaining but with complete information.

In ([7]), we have analysed markets with equal finite numbers of buyers and sellers, under complete information. The bargaining is with one-sided offers. We show that in any stationary equilibrium, as agents become patient enough, prices in all transactions converge to a single value. In the current work, we show convergence of prices in an incomplete information framework.⁸

Chatterjee and Dutta ([10]) study a similar setting as this paper but with sequential offers by buyers. They do not obtain asymptotic convergence of all stationary equilibrium outcomes to a uniform price. This is mainly because they always have an equilibrium in which prices in the two transactions are bounded away from each other. Thus, the asymptotic stationary equilibrium outcome is not unique, unlike our current model.

A paper analysing outside options in asymmetric information bargaining is that by Gantner([24]), who considers such outside options in the Chatterjee and Samuelson ([11]) model. Our model differs from hers in the choice of the basic bargaining model and in the explicit analysis of a small market. (There is competition for alternatives too, in our model but not in hers.) Another paper, which in a completely different setting, discusses outside options and bargaining is Atakan and Ekmekci([1]). Their model is based on the presence of inflexible behavioural types and matching over time. They consider the steady state equilibria of this model in which there are inflows of different types of agents every period. Their main result shows that there always exist equilibria where there are selective breakups and delay, which in turn leads to inefficiency in bargaining. In a recent paper, Board and Pycia([6]) have shown that in a model of bilateral bargaining with incomplete information and one-sided offers, the Coase conjecture does not hold in the presence of outside options.

Some of the main papers in one-sided asymmetric information bargaining are the well-known ones of Sobel and Takahashi([41]), Fudenberg, Levine and Tirole ([17]), Ausubel and Deneckere ([2]). The dynamic monopoly papers mentioned before are the ones by Gul and Sonnenschein

⁸ In the complete information case (see [7]), we get a similar result. This however, does not imply that the analyses are the same. In the bilateral bargaining game with complete information where the seller has valuation H , the price is H ; if it is L , the price is L . From this fact, it is non-trivial to guess that as the discount factor goes to 1 and the probability of a H seller being positive, the price goes to H . (This explains the large number of papers on this bilateral case.) With four players, even with only one seller's value being unknown, the problem is compounded by the presence of the other alternatives. We leave out the construction of the equilibrium itself, which requires some careful consideration of appropriate beliefs. Without this construction, of course, the equilibrium path cannot be known to be such, so the fact that two equilibrium paths end up looking similar doesn't mean that the *equilibria* are the same.

([25]) and Gul, Sonnenschein and Wilson([26]).⁹

There are papers in very different contexts that have some of the features of this model. For example, Swinkels [43] considers a discriminatory auction with multiple goods, private values (and one seller) and shows convergence to a competitive equilibrium price for fixed supply as the number of bidders and objects becomes large. We keep the numbers small, at two on each side of the market. Other papers that have looked into somewhat related issues but in a different environment are Fuchs and Skrzypacz ([16]), Kaya and Liu ([29]) and Horner and Vieille ([28]). Compte and Jehiel ([14]) in the context of reputational bargaining and Lee and Liu ([31]) in the context of repeated bargaining consider the role of outside options of the informed player and examine how the equilibrium outcome is altered. Also, there are several other papers on searching for outside options, for example, Chikte and Deshmukh ([13]), Muthoo ([32]), Lee ([30]), Chatterjee and Lee ([12]) (This has private information about outside options). We do not discuss these in detail because they are not directly comparable to our work.

Outline of rest of the paper: The rest of the paper is organised as follows. Section 2 discusses the model in detail. The qualitative nature of the equilibrium, its detailed derivation and the asymptotic characteristics are given in section 3, which is the heart of the paper. Section 4 discusses the possibility of other equilibria, as well as the private offers case. Finally, Section 5 concludes the paper.

2 The Model

2.1 Players and payoffs

The setup we consider has two uninformed homogeneous buyers and two heterogeneous sellers. Buyers (B_1 and B_2) have a common valuation of v for the good (the maximum willingness to pay for a unit of the indivisible good). There are two sellers. Each of the sellers owns one unit of the indivisible good. Sellers differ in their valuations. The first seller (S_M) has a reservation value of M which is commonly known. The other seller (S_I) has a reservation value that is private information to her. S_I 's valuation is either L or H , where,

$$v > H > M > L$$

It is commonly known by all players that the probability that S_I has a reservation value of L is $\pi \in [0, 1)$. It is worthwhile to mention that $M \in [L, H]$ constitutes the only interesting case. If $M < L$ (or $M > H$) then one has no uncertainty about which seller has the lowest reservation value. Although our model analyses the case of $M \in (L, H)$, the same asymptotic result will be

⁹See also the review paper of Ausubel, Cramton and Deneckere ([3])

true for $M \in [L, H]$.

Players have a common discount factor $\delta \in (0, 1)$. If a buyer agrees on a price p^j with seller S_j at a time point t , then the buyer has an expected discounted payoff of $\delta^{t-1}(v - p^j)$. The seller's discounted payoff is $\delta^{t-1}(p^j - u_j)$, where u_j is the valuation of seller S_j .

2.2 The extensive form

This is an infinite horizon, multi-player bargaining game with one-sided offers and discounting. The extensive form is as follows:

At each time point $t = 1, 2, \dots$, offers are made simultaneously by the buyers. The offers are targeted. This means an offer by a buyer consists of a seller's name (that is S_I or S_M) and a price at which the buyer is willing to buy the object from the seller he has chosen. Each buyer can make only one offer per period. Two informational structures can be considered; one in which each seller observes all offers made (*public targeted offers*) and the one (*private targeted offers*) in which each seller observes only the offers she gets. (Similarly for the buyers, after the offers have been made-in the private offers case each buyer knows his own offer and can observe who leaves the market.) In the present section we shall focus on the first and consider the latter in a subsequent section. A seller can accept at most one of the offers she receives. Acceptances or rejections are simultaneous. Once an offer is accepted, the trade is concluded and the trading pair leaves the game. Leaving the game is publicly observable. The remaining players proceed to the next period in which buyers again make price offers to the sellers. As is standard in these games, time elapses between rejections and new offers.

3 Equilibrium

We will look for *Perfect Bayes Equilibrium*[18] of the above described extensive form. This requires sequential rationality at every stage of the game given beliefs and the beliefs being compatible with Bayes' rule whenever possible, on and off the equilibrium path. We will focus on *stationary equilibria*¹⁰. These are the equilibria where strategies depend on the history only through the updated value of π (the probability that S_I 's valuation is L). Thus, at each time point, buyers' offers depend only on the number of players remaining and the value of π . The sellers' responses depend on the number of players remaining, the value of π and the offers made by the buyers.

¹⁰Please see section 5 for discussion of non-stationary equilibria

3.1 The Benchmark Case: Complete information

Before we proceed to the analysis of the incomplete information framework, we state the results of the above extensive form with complete information. A formal analysis of the complete information framework has been done in [7].

Suppose the valuation of S_I is commonly known to be H . In that case there exists a unique¹¹ stationary equilibrium (an equilibrium in which buyers' offers depend only on the set of players present and the sellers' responses depend on the set of players present and the offers made by the buyers) in which one of the buyers (say B_1) makes offers to both the sellers with positive probability and the other buyer (B_2) makes offers to S_M only. Suppose $E(p)$ represents the expected maximum price offer to S_M in equilibrium. Assuming that there exists a unique $p_l \in (M, H)$ such that,

$$p_l - M = \delta(E(p) - M)^{12}$$

, the equilibrium is as follows:

1. B_1 offers H to S_I with probability q . With the complementary probability he makes offers to S_M . While offering to S_M , B_1 randomises his offers using an absolutely continuous distribution function $F_1(\cdot)$ with $[p_l, H]$ as the support. F_1 is such that $F_1(H) = 1$ and $F_1(p_l) > 0$. This implies that B_1 puts a mass point at p_l .

2. B_2 offers M to S_M with probability q' . With the complementary probability his offers to S_M are randomised using an absolutely continuous distribution function $F_2(\cdot)$ with $[p_l, H]$ as the support. $F_2(\cdot)$ is such that $F_2(p_l) = 0$ and $F_2(H) = 1$.

It is shown in [7] that this p_l exists and is unique. Also, the outcome implied by the above equilibrium play constitutes the unique stationary equilibrium outcome and as $\delta \rightarrow 1$,

$$q \rightarrow 0, q' \rightarrow 0 \text{ and } p_l \rightarrow H$$

This means that as market frictions go away, we tend to get a uniform price in different buyer-seller matches. If S_I 's type is commonly known to be L , the asymptotic convergence is to M . Thus, one might conjecture that for high values of π , the convergence is to M and for low values to H . In this paper, however, we show convergence to H , even with incomplete information. The analysis is different, since the current value of π plays a major role.

¹¹Up to the choice of B_1 and B_2

¹²Given the nature of the equilibrium it is evident that $M(p_l)$ is the minimum acceptable price for S_M when she gets one(two) offer(s).

3.2 Equilibrium of the one-sided incomplete information game with two players

In the equilibrium of the four-player game, if a buyer-seller pair leaves the market after an agreement, then the two-player games that could arise can be of two types. Either it is one of the buyers remaining with the seller S_M or it is one of the buyers remaining with seller S_I . In the former case, the unique equilibrium is the buyer making an offer of M to S_M . In the latter case, we have a two-player game with one-sided asymmetric information, where buyer makes all the offers. In the current subsection, we discuss the equilibrium of this game.

The setting is as follows: There is a buyer with valuation v , which is common knowledge. The seller's valuation can either be H or L where $v > H > L = 0$ ¹³. At each period, conditional on no agreement being reached till then, the buyer makes the offer and the seller (informed) responds to it by accepting or rejecting. If the offer is rejected then the value of π is updated using Bayes' rule and the game moves on to the next period when the buyer again makes an offer. This process continues until an agreement is reached. The equilibrium of this game(as described in, for example, [15]) is as follows.

For a given δ , we can construct an increasing sequence of probabilities, $d(\delta) = \{0, d_1, \dots, d_t, \dots\}$ such that if on the equilibrium path the game can continue for at most t periods from now then we have the current belief $\pi_t \in [d_t, d_{t+1})$. Here π_t is the updated belief implied by the play for the game so far. The buyer in this case offers $p_t = \delta^t H$. The H type seller rejects this offer with probability 1. The L type seller rejects this offer with a probability that implies, through Bayes' Rule, that the updated value of the belief $\pi_{t-1} = d_{t-1}$. The cutoff points d_t 's are such that the buyer is indifferent between offering $\delta^t H$ and continuing the game for a maximum of t periods from now or offering $\delta^{t-1} H$ and continuing the game for a maximum of $t - 1$ periods from now. Thus, here t means that the game will last for at most t periods from now. The maximum number of periods for which the game can last is given by $N(\delta)$. It is already shown in [15] that this $N(\delta)$ is uniformly bounded above by a finite number N^* as $\delta \rightarrow 1$.

The off-path behaviour that sustains the above equilibrium is described in [8].

Given a π , the expected payoff to the buyer $v_B(\pi)$ is calculated as follows:

For $\pi \in [0, d_1)$, the two-player game with one-sided asymmetric information involves the same offer and response as the complete information game between a buyer of valuation v and a seller of valuation H . Thus we have

$$v_B(\pi) = v - H \text{ for } \pi \in [0, d_1)$$

¹³ $L = 0$ is assumed to simplify notation and calculations.

For $\pi \in [d_t, d_{t+1})$, ($t \geq 1$), we have,

$$v_B(\pi) = (v - \delta^t H)a(\pi, \delta) + (1 - a(\pi, \delta))\delta(v_B(d_{t-1})) \quad (1)$$

where $a(\pi, \delta)$ is the equilibrium acceptance probability of the offer $\delta^t H$.

These values will be crucial for the construction of the equilibrium of the four-player game. In the following subsection, we describe the characteristics of any stationary equilibrium of the four-player game with incomplete information and formally show existence of such an equilibrium.

3.3 Equilibrium of the four-player game with incomplete information.

In this subsection, we will characterise a particular stationary PBE of the four-player game with incomplete information. This is summarised in the following proposition.

Proposition 1 *There exists a $\delta^* \in (0, 1)$ such that if $\delta > \delta^*$, then for all $\pi \in [0, 1)$, there exists a stationary PBE of the four-player game with incomplete information. The equilibrium has the following features.*

(i) *One of the buyers (say B_1) will make offers to both S_I and S_M with positive probability. The other buyer B_2 will make offers only to S_M .*

(ii) *B_2 while making offers to S_M , will put a mass point at $p'_l(\pi)$ and will have an absolutely continuous distribution of offers from $p_l(\pi)$ to $\bar{p}(\pi)$ where $p'_l(\pi)$ ($p_l(\pi)$) is the minimum acceptable price to S_M when she gets one(two) offer(s). For a given π , $\bar{p}(\pi)$ is the upper bound of the price offer S_M can get in the described equilibrium ($p'_l(\pi) < p_l(\pi) < \bar{p}(\pi)$). B_1 , when making offers to S_M , will have an absolutely continuous (conditional) distribution of offers from $p_l(\pi)$ to $\bar{p}(\pi)$, putting a mass point at $p_l(\pi)$.*

(iii) *B_1 , when making offers to S_I on the equilibrium path, makes the same offer, as a function of the current belief about the type of the informed seller, as in the two-player game with one-sided asymmetric information.*

(iv) *S_I 's response behaviour on the equilibrium path is identical to that in the two-player game. S_M accepts the largest offer with a payoff at least as large as the expected continuation payoff from rejecting all offers.*

(v) *Each buyer in equilibrium obtains a payoff of $v_B(\pi)$.*

(vi) Strategies off the equilibrium path that sustain the on-path play are described in appendix (A.2)

The mass points and the distribution of buyers' offers will depend upon π though we show that these distributions will collapse in the limit (as $\delta \rightarrow 1$).

The descriptions above imply that if the discount factor is high enough, then irrespective of the probability with which the informed seller has valuation L , in equilibrium buyers always compete for the seller whose valuation is known to be M . In the later sub-section, we will show that this will be the case for any stationary equilibrium of the four-player game with incomplete information.

We provide here an informal account of the proof, the details of which are in appendix (A.1). The first step is to consider the effect of buyer competition for the sellers. Our complete information analysis suggests that if the sellers have values H and M , competition for the M seller will drive prices to H , whilst if these values are M and L , competition will be for the L seller. One might therefore expect, in an environment where one seller value is unknown to the buyers, that competition will be for S_I if the probability of S_I 's value being L is sufficiently high. As it turns out, this is not the case for high enough δ . This is a fundamental difference between the complete and incomplete information settings. We formalise this by proving a *competition lemma*, terminology borrowed from a similar result in the different model of Chatterjee and Dutta [10]. To state it more formally, let \bar{p}_t be the price offer such that the payoff to the buyer from getting it accepted is the same as the buyer's equilibrium expected payoff in the two-player game with S_I , when on the equilibrium path the game will last for at most t more periods. Hence,

$$\bar{p}_t = v - [(v - \delta^t H)\alpha + (1 - \alpha)\delta(v - \bar{p}_{t-1})]$$

Let p'_t be the upper bound of the minimum acceptable price offer to S_M , when she gets only one offer in equilibrium. The *competition lemma* shows that if δ is high enough then $\bar{p}_t > p'_t$.

The second step is to derive the equilibrium. We fix a $\delta > \delta^*$ and derive the sequence $d_\tau(\delta) = \{0, d_1, d_2, \dots, d_t, \dots\}$, which is identical with the similar sequence in the two-player game with asymmetric information. We can find a $t \geq 0$ ($t \leq N^*$) such that $\pi \in [d_t, d_{t+1})$. Next, we compute $v_B(\pi)$, which is the expected payoff to a buyer in a two-player game with the informed seller. We define $\bar{p}(\pi)$ as

$$\bar{p}(\pi) = v - v_B(\pi)$$

Let $p'_l(\pi)$ be the minimum acceptable price offer to seller S_M in the event she gets only one offer in equilibrium. From the *competition lemma* we know that $\bar{p}(\pi) > p'_l(\pi)$. Assuming existence of a $p_l(\pi) \in (p'_l(\pi), \bar{p}(\pi))$ such that $p_l(\pi)$ is the minimum acceptable price to S_M in the event she gets *two* offers in equilibrium, we derive the distribution of offers and the probabilities at the mass

points from the indifference condition of the buyers.

The third step involves proving a lemma which we term as the *basis lemma*. This lemma exploits the fact that for any $\delta \in (0, 1)$, d_1 is independent of δ and if $\pi \in [0, d_1)$, the two player game with a buyer and the informed seller is identical to the two-player game of a buyer with valuation v and the seller with valuation H . The benchmark results now allow us to conclude that for $\pi \in [0, d_1)$, an equilibrium as conjectured in the current proposition exists.

In the final step, we prove a *recursion lemma*. Suppose for a T ($1 \leq T < N^*$), there exists a $\bar{\delta}$ such that for all δ exceeding $\bar{\delta}$, the conjectured equilibrium exists for all $\pi \in [0, d_T)$. The *recursion lemma* argues that we can find another threshold for δ , namely $\delta_T^* \geq \bar{\delta}$ such that if δ exceeds this threshold then the conjectured equilibrium exists for all $\pi \in [d_T, d_{T+1})$.

Since N^* is finite, we can find a maximum of the thresholds of δ defined in the *recursion lemma*. We denote this threshold by δ^* . Thus, if the discount factor exceeds δ^* , the *basis lemma* and the *recursion lemma* together imply that for all $\pi \in [0, 1)$, the equilibrium as conjectured in the proposition exists. This completes the informal account of the proof of the proposition.

We can show that as $\delta \rightarrow 1$, all price offers in the equilibrium characterised converge to H . To show this we argue that as $\delta \rightarrow 1$, probabilities at all mass points converge to 0, and $p_l(\pi)$ converges to $\bar{p}(\pi)$. Thus, all distributions of offers collapse. As $\delta \rightarrow 1$, from the two-player game with one-sided asymmetric information, we know that $v_B(\pi)$ goes to $v - H$. This implies $\bar{p}(\pi) \rightarrow H$. Hence, as $\delta \rightarrow 1$, all price offers converge to H . This asymptotic result is formally proved in appendix (B).

In the following sub-section, we show that any stationary equilibrium of the four-player game with incomplete information has the same characteristics as described above and as $\delta \rightarrow 1$, all price offers in equilibrium converge to H .

3.4 Uniqueness of the asymptotic equilibrium outcome

In this subsection, we show that prices in all stationary equilibrium outcomes must converge to the same value as $\delta \rightarrow 1$. This shows that there exists a unique limiting stationary equilibrium outcome.

It turns out to be convenient if we adopt the following sequence of stages in proving this result. Let Π be the set of beliefs such that for any belief belonging to this set, a stationary equilibrium exists where, on the equilibrium path, both buyers offer only to the informed seller. Let $\Phi(\Pi)$ be the set of such equilibria, in case there is more than one with this feature. Let Π^C therefore be the set of values of π , such that no stationary equilibrium strategies entail both buyers making such offers only to S_I . Let the set of stationary equilibria for $\pi \in \Pi^C$ be denoted by $\Psi(\Pi^C)$.

The first stage proves the main result for any equilibrium belonging to the set $\Psi(\Pi^C)$. In the next stage, we show that the set Π is empty. Thus, these two stages together imply the main

result of this subsection.

In stage 1, we restrict ourselves to the set $\Psi(\Pi^C)$. The argument follows from the following steps.

First, we rule out the possibility of having any stationary equilibrium with the feature that both buyers, on the equilibrium path, offer only to seller S_M . We show that if such an equilibrium exists, then it involves both buyers putting a mass point at the lower bound of the common support. This is not possible in equilibrium.

In the next step, we show that in any stationary equilibrium, S_M always accepts an equilibrium offer immediately. This is irrespective of whether S_M gets one offer or two offers on the equilibrium path. The intuition for these results is as follows. If S_M gets two offers, then rejection of both does not lead to any change in the belief and hence, the expected equilibrium payoff in the continuation game remains unchanged. In a stationary equilibrium this implies that these offers always yield a payoff of zero. This is inconsistent with these being equilibrium offers. If S_M gets only one offer, we use the standard argument in bargaining that it is better for the proposer to make an offer that gives the receiver a payoff equal to the discounted expected continuation payoff than waiting for the next period and discounting his own payoff. The formal proof¹⁴ requires some explicit calculations to verify these intuitions.

The third step considers the revision of beliefs about the informed seller. In any equilibrium, which involves the informed seller rejecting offers, she must also accept equilibrium offers with some positive probability to generate these revisions of beliefs. We show that for any $\delta \in (0, 1)$ and for any $\pi \in (0, 1)$, conditional on getting offers, the informed seller by rejecting equilibrium offers can generate a path of beliefs such that she is offered H in finite time. Hence, the second and the third step together imply that in any equilibrium, the game ends in finite time.

Next, we show that if players are patient enough, then in equilibrium both buyers cannot offer to both sellers with positive probability. If this were to happen then we can show that one of the buyers will deviate to make an unacceptable offer to the informed seller in order to increase his probability of being in a two-player game with S_M next period. Also, we show that every equilibrium has the feature that both buyers make offers to S_M with positive probability. This follows from the fact that if only one buyer makes offers to S_M , then from the third step we can conclude that the buyer offering to S_I can profitably deviate.

These conclusions allow us to infer that in any stationary equilibrium in the set $\Psi(\Pi^C)$, one buyer makes offers only to S_M and the other buyer randomises between making offers to S_I and S_M .

In the last part of the argument, we can show that as $\delta \rightarrow 1$, all price offers converge to H . This happens because in equilibrium, S_M always accepts an offer immediately, the buyer offering

¹⁴This is done in detail in appendix

to S_I is essentially facing a two-player game with the informed seller, in which we know the offers converge to H as $\delta \rightarrow 1$. In order to prevent a deviation by this buyer, price offers in equilibrium to S_M should also converge to H .

These results are summarised in the following proposition

Proposition 2 *Consider the set of stationary equilibria of the four-player game such that any equilibrium belonging to this set has the property that both buyers do not make offers only to the informed seller (S_I) on the equilibrium path. As the discount factor $\delta \rightarrow 1$, all price offers in any equilibrium belonging to this set converge to H*

The detailed proof of this proposition is formally outlined in Appendix D .

We will now argue that there does not exist any belief such that a stationary equilibrium is possible, where on the equilibrium path, both buyers offer only to the informed seller. The following lemma shows this

Lemma 1 *Let Π be the set of beliefs such that for $\pi \in \Pi$, it is possible to have a stationary equilibrium where both buyers offer only to S_I . The set Π is empty*

Proof. Suppose Π is non-empty. Consider any $\pi \in \Pi$. At this belief, it is possible to have a stationary equilibrium such that both buyers on the equilibrium path offer only to S_I . First, we will argue that the highest price offer in such a case is always strictly less than H for all $\delta \in (0, 1)$. This is because, in the continuation game S_M either faces a four-player game or a two player game. In the former case, either the equilibrium of the continuation game is in $\Phi(\Pi)$ or in $\Psi(\Pi^C)$. Hence, the maximum price offer she can get is bounded above by H . In the latter case the maximum price offer she can get is M . Since an informed seller always accepts an offer in equilibrium with positive probability, the latter case can occur with positive probability. This implies that if \mathcal{E}_π is the expected continuation payoff to S_M , then $\mathcal{E}_\pi < H - M$. Let \bar{p}_π be such that $\bar{p}_\pi - M = \delta \mathcal{E}_\pi < \delta(H - M)$. Hence, $\bar{p}_\pi < H$ for all $\delta \in (0, 1)$. If the maximum price offered to S_I exceeds \bar{p}_π , then the buyer making the highest offer to S_I can profitably deviate by making an offer to S_M ¹⁵

Let \bar{p} be the largest price offer, for any $\pi \in \Pi$, in any such equilibrium. (It is without loss of generality to assume that the maximum exists. This is because a supremum always exists and we can consider prices in the ϵ -neighborhood of this supremum). Clearly, as argued above, $\bar{p} < H$. As explained earlier, no equilibrium can involve offers that are rejected by both L and H types of S_I . Therefore, the L type must accept an offer with positive probability. This implies (by Bayes' Theorem and $\delta < 1$) that the sequence of prices must be increasing. Consider the offer of \bar{p} . There

¹⁵One can easily check that it is not possible in equilibrium to have that as $\delta \rightarrow 1$, the price offered to S_I goes to H . This is because as the price offered to S_I goes to H , the probability of acceptance goes to 1 . Hence, $\mathcal{E}_\pi < H - M$ and it is bounded away from $H - M$ as $\delta \rightarrow 1$. Thus, one of the buyers can profitably deviate

are two possibilities. Either the equilibrium of the continuation game (given the updated belief conditional on \bar{p} being rejected) is in $\Phi(\Pi)$ or it is in $\Psi(\Pi^C)$. In the former case, S_I should accept the offer with probability 1 and the updated belief is $\pi = 0$, where the equilibrium price offer must be $H > \bar{p}$. This means there exists a profitable deviation for the L -type S_I , for δ sufficiently high. For the latter case, if δ is high then from proposition (2) we know that for any stationary equilibrium all offers converge to H as $\delta \rightarrow 1$. Once again, this implies the existence of a profitable deviation for the L -type S_I . Hence, we cannot have Π non-empty. This concludes the proof. ■

It is to be noted that since Π is empty, there are no equilibria with both buyers offering only to S_I . Thus $\Psi(\Pi^C)$ is the set of all stationary equilibria of the four-player game with incomplete information. The main result of this subsection is summarised in the theorem below.

Theorem 1 *In any arbitrary stationary equilibrium of the four-player game, as the discount factor goes to 1, price offers in all transactions converge to H for all values of the prior $\pi \in [0, 1)$.*

Proof. The proof of the theorem follows directly from proposition (2) and lemma (1). ■

In the next subsection, we discuss some extensions by considering non-stationary equilibria and private offers.

4 Extensions

In this section we consider some possible extensions by considering a non-stationary equilibrium. We then discuss private offers.

4.1 A non-stationary equilibrium

We show that with public offers we can have a non-stationary equilibrium, so that the equilibrium constructed in the previous sections is not unique. This is based on using the stationary equilibrium as a punishment (the essence is similar to the *pooling equilibrium with positive profits* in [33]). The strategies sustaining this are described below. The strategies will constitute an equilibrium for sufficiently high δ , as is also the case for the stationary equilibrium.

Suppose for a given π , both the buyers offer M to S_M . S_M accepts this offer by selecting each seller with probability $\frac{1}{2}$. If any buyer deviates, for example by offering to S_I or making a higher offer to M , then all players revert to the stationary equilibrium strategies described above. If S_M gets the equilibrium offer of M from the buyers and rejects both of them then the buyers make the same offers in the next period and the seller S_M makes the same responses as in the current period.

Given the buyers adhere to their equilibrium strategies, the continuation payoff to S_M from rejecting all offers she gets is zero. So she has no incentive to deviate. Next, if one of the buyers offers slightly higher than M to S_M then it is optimal for her to reject both the offers. This is because on rejection next period players will revert to the stationary equilibrium play described above. Hence her continuation payoff is $\delta(E_\pi(p) - M)$, which is higher than the payoff from accepting.

Finally each buyer obtains an equilibrium payoff of $\frac{1}{2}(v - M) + \frac{1}{2}\delta v_B(\pi)$. If a buyer deviates then, according to the strategies specified, S_M should reject the higher offer if the payoff from accepting it is strictly less than the continuation payoff from rejecting (which is the one period discounted value of the payoff from stationary equilibrium). Hence if a buyer wants S_M to accept an offer higher than M then his offer p' should satisfy,

$$p' = \delta E_\pi(p) + (1 - \delta)M$$

The payoff of the deviating buyer will then be $\delta(v - E_\pi(p)) + (1 - \delta)(v - M)$. As $\delta \rightarrow 1$, $\delta(v - E_\pi(p)) + (1 - \delta)(v - M) \approx \delta(v - \bar{p}(\pi) + (1 - \delta)(v - M)) = \delta v_B(\pi) + (1 - \delta)(v - M)$.

For $\delta = 1$ this expression is strictly less than $\frac{1}{2}(v - M) + \frac{1}{2}\delta v_B(\pi)$, as $(v - M) > \delta v_B(\pi)$. Hence for sufficiently high values of δ this will also be true. Also if a buyer deviates and makes an offer in the range (M, p') then it will be rejected by S_M . The continuation payoff of the buyer will then be $\delta v_B(\pi) < \frac{1}{2}(v - M) + \frac{1}{2}\delta v_B(\pi)$. Hence we show that neither buyer has any incentive to deviate.

4.2 Private offers

In this subsection, we consider a variant of the extensive form of the four-player incomplete information game in which offers are private to the recipient and the proposer. This means in each period a seller observes only the offer(s) she gets and a buyer does not know what offers are made by the other buyer or received by the sellers.

The equilibrium notion here is that of a *public perfect Bayesian equilibrium (PPBE)*. That is, in equilibrium, strategies can condition only on the public history, (which is the set of players remaining in a particular period) and the public belief. If we suitably define the play off the equilibrium path, then we can sustain the equilibrium characterised for the four-player incomplete information game with public offers as a PPBE of the game with private offers. The following proposition describes this.

Proposition 3 *There exists a $\delta' < 1$ such that for all $\delta > \delta'$, there exists a PPBE of the four-player incomplete information game with private offers. The equilibrium has the following features:*

- (i) *On the equilibrium path, the play is identical to the equilibrium described in proposition 1.*

(ii) Off the equilibrium path, the play is defined in appendix (G)

Proof. We refer to the proof of proposition (1) in order to ensure that on the equilibrium path, no player has any incentive to deviate. Appendix(G) shows that behaviour described for off the the equilibrium path constitute mutual optimal play by the players. ■

The main difference between private and public offers lies in the nature of the deviations that are observable. In particular, a buyer cannot observe a deviation in offers made by the other buyer. In this model, we specifically need to take into account three kinds of deviations by the players and the associated observables. While this is outlined formally in the appendix, here we provide an informal discussion.

All these three deviations are from the buyer (B_2) who is supposed to make offers only to S_M . First, we consider the situation when B_2 makes an unacceptable offer to the seller S_M . The second case is when B_2 makes an offer to the informed seller but it is less than the equilibrium offer. Finally, we need to consider the situation when B_2 deviates and makes an offer to the informed seller but the offer is higher than the equilibrium offer. Any of these deviations matters only when following it, all four players are present when the equilibrium play would have resulted in S_M , at least, leaving with a buyer. This means that the buyer who is making offers to both sellers with positive probability (B_1) has made an equilibrium offer to the informed seller that has been rejected. In that case, if all players are present, B_1 can detect that some deviation has taken place. However, he does not know the exact kind of deviation. In such a situation, suppose B_1 regards such deviations as mistakes but believes different mistakes take place with different orders of probability. We suppose that the first case (B_2 makes an unacceptable offer to S_M) occurs with the highest order of probability. In the first case, as well as the second, the beliefs of both buyers are identical, though revised, after the deviation. In the third case, (B_2 makes a higher-than-equilibrium offer to S_I), the deviating buyer becomes more pessimistic, compared to B_1 , about the valuation of the informed seller (he now believes it to be H since an offer higher than the one that makes the L type indifferent has been rejected). This third type of deviation is accorded the lowest order of probability. Therefore, as the probability of any mistake goes to zero, a deviation is almost surely believed to be of the first kind. The deviating buyer B_2 , who offers only to S_M on the equilibrium path, does not do worse by playing an equilibrium strategy with actions corresponding to the belief of B_1 .

In case of public targeted offers, while proving that the stationary equilibrium outcome is unique, we did not use the fact that each seller while responding observes the other seller's offer. This implies that any stationary equilibrium of the public targeted offers game is a particular public perfect Bayesian equilibrium of the game with private targeted offers

We now argue that no non-stationary equilibrium with public offers can be established as an equilibrium with private offers. To begin with, it is to be noted that it is not possible to establish

the equilibrium described in the previous subsection in a model with private offers. This is because the equilibrium depends crucially on “overbidding” by one of the buyers to S_M being detected by the other buyer and seller, who then condition their future play on this deviation. But with private offers, this deviation is not detectable, so the switch to the “punishment phase” is not possible.

To be more general, any non-stationary equilibrium depends on the expected future changes due to publicly observed deviations. With private offers, the only deviation that is observed results in either some subset of players leaving or all players remaining. In the first case, the continuation game will be a 2 player game. These continuation games have unique equilibria. This implies that in any non-stationary equilibrium, there has to be a possible off-path play where all four players remain. However, in that case, the player who deviated needs to be detected for proper specification of continuation play after a deviation. This is not possible with private offers.

5 Conclusion

To summarise, we have first construct a stationary PBE, thus demonstrating existence. For this equilibrium, as $\delta \rightarrow 1$, the price in all transactions converge to H and the game ends “almost immediately”. We then show that any stationary equilibrium must have qualitative characteristics similar to the equilibrium we initially describe, so that the unique stationary PBE outcome has the property that , as $\delta \rightarrow 1$, the prices in both transactions go to the highest seller value H , with the buyers making offers. This is reminiscent of the Coase Conjecture, though the setting here is that of a stylised small market, rather than one of bilateral bargaining or such bargaining with an exogenously fixed outside option. With *private offers*, the uniqueness result is for public perfect Bayes’ equilibrium, since stationarity does not have any bite in this context.

Unlike the bilateral bargaining case, non-stationary equilibria exist in this model with very different characteristics and we show one such.

In our future research we intend to address the issue of having two privately informed sellers and to extend this model to more agents on both sides of the market.

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Appendix

A Equilibrium of the four-player game with incomplete information

A.1 Proof of Proposition 1

Proof. We prove this proposition in steps. First, we derive the equilibrium for a given value of π by assuming that there exists a threshold δ^* , such that if δ exceeds this threshold then for each value of π , a stationary equilibrium

as described exists. Later on we will prove this existence result.

To formally construct the equilibrium for different values of π , we need the following lemma which we label as the *Competition Lemma*, following the terminology of [10], though they proved it for a different model.

Consider the following sequences for $t \geq 1$:

$$\bar{p}_t = v - [(v - \delta^t H)\alpha + (1 - \alpha)\delta(v - \bar{p}_{t-1})] \quad (2)$$

$$p'_t = M + \delta(1 - \alpha)(\bar{p}_{t-1} - M) \quad (3)$$

where $\alpha \in (0, 1)$ and $\bar{p}_0 = H$.

Lemma 2 *There exists a $\delta' \in (0, 1)$, such that for $\delta > \delta'$ and for all $t \in \{1, \dots, N(\delta)\}$, we have,*

$$\bar{p}_t > p'_t$$

Proof.

$$\begin{aligned} \bar{p}_t - p'_t &= v - [(v - \delta^t H)\alpha + (1 - \alpha)\delta(v - \bar{p}_{t-1})] - M \\ &\quad - \delta(1 - \alpha)(\bar{p}_{t-1} - M) \\ &= (1 - \delta)(v - M) + \alpha(\delta^t H - \delta M - (1 - \delta)v) \end{aligned}$$

We need to show that the second term is always positive. Note that the coefficient of α is increasing in δ and is positive at $\delta = 1$. Take $t = N^*$, where N^* is the upper bound on the maximum equilibrium duration of the two player game with one sided asymmetric information. For $t = N^*$, $\exists \delta' < 1$ such that the term is positive whenever $\delta > \delta'$. Since this is true for $t = N^*$, it will be true for all lower values of t .

For any $\delta < 1$, $N(\delta) \leq N^*$. Hence, for all $t \in \{1, \dots, N(\delta)\}$,

$$\bar{p}_t > p'_t$$

whenever $\delta > \delta'$.

This concludes the proof of the lemma.

■

Fix a $\delta > \delta^*$. Suppose the current prior is $\pi \in (0, 1)$.¹⁶ Consider the sequence $d_\tau(\delta) = \{0, d_1, d_2, \dots, d_t, \dots\}$, which is identical with the similar sequence in the two-player game with asymmetric information. This implies that there exists a $t \geq 0$ ($t \leq N^*$) such that $\pi \in [d_t, d_{t+1})$. Next, we evaluate $v_B(\pi)$ (from the two player game). Define $\bar{p}(\pi)$ as,

$$\bar{p}(\pi) = v - v_B(\pi)$$

Define $p'_l(\pi)$ as,

$$p'_l(\pi) = M + \delta(1 - a(\pi))[E_{d_{t-1}}(p) - M] \quad (4)$$

where $E_{d_{t-1}}(p)$ represents the expected price offer to S_M in equilibrium when the probability that S_I is of the low type is d_{t-1} . $a(\pi)$ is the probability with which the informed seller accepts an equilibrium offer. From (4), we can posit that in equilibrium, $p'_l(\pi)$ is the minimum acceptable price for S_M , if she gets only one offer.

¹⁶ $\pi = 0$ is the complete information case with a H seller.

For $\pi = d_{t-1}$, the maximum price offer to S_M (according to the conjectured equilibrium) is $\bar{p}(d_{t-1})$. This implies that $E_{d_{t-1}}(p) \leq \bar{p}(d_{t-1})$.

Since $a(\pi) \in (0, 1)$, from Lemma (2) we can infer that $\bar{p}(\pi) > p'_l(\pi)$. Suppose there exists a $p_l(\pi) \in (p'_l(\pi), \bar{p}(\pi))$ such that,

$$p_l(\pi) = (1 - \delta)M + \delta E_\pi(p)$$

We can see that $p_l(\pi)$ represents the minimum acceptable price offer for S_M in the event she gets two offers. (Note that if S_M rejects both offers, the game goes to the next period with π remaining the same.)

From the conjectured equilibrium behaviour, we derive the following¹⁷ :

1. B_1 makes offers to S_I with probability $q(\pi)$, where

$$q(\pi) = \frac{v_B(\pi)(1 - \delta)}{(v - p'_l(\pi)) - \delta v_B(\pi)} \quad (5)$$

B_1 offers $\delta^t H$ to S_I . With probability $(1 - q(\pi))$, he makes offers to S_M . The conditional distribution of offers to S_M , given B_1 makes an offer to this seller is

$$F_1^\pi(s) = \frac{v_B(\pi)[1 - \delta(1 - q(\pi))] - q(\pi)(v - s)}{(1 - q(\pi))[v - s - \delta v_B(\pi)]} \quad (6)$$

We can check that $F_1^\pi(p_l(\pi)) > 0$ and $F_1^\pi(\bar{p}(\pi)) = 1$. This confirms that B_1 puts a mass point at $p_l(\pi)$.

2. B_2 offers $p'_l(\pi)$ to S_M with probability $q'(\pi)$, where

$$q'(\pi) = \frac{v_B(\pi)(1 - \delta)}{(v - p_l(\pi)) - \delta v_B(\pi)} \quad (7)$$

With probability $(1 - q'(\pi))$, he makes offers to S_M by randomizing his offers in the support $[p_l(\pi), \bar{p}(\pi)]$. The conditional distribution of offers is given by

$$F_2^\pi(s) = \frac{v_B(\pi)[1 - \delta(1 - q'(\pi))] - q'(\pi)(v - s)}{(1 - q'(\pi))[v - s - \delta v_B(\pi)]} \quad (8)$$

This completes the derivation. Appendix(A.2) describes the off-path behaviour of players and shows that it sustains the equilibrium play.

Next, we show that there exists a δ^* such that $\delta' < \delta^* < 1$ and for all $\delta > \delta^*$, an equilibrium as described above exists for all values of $\pi \in [0, 1)$. This is shown by proving the following two lemmas.

Lemma 3 *If $\pi \in [0, d_1)$, then the equilibrium of the game is identical to that of the benchmark case.*

Proof. From the equilibrium of the two player game with one sided asymmetric information, we know that for $\pi \in [0, d_1)$, buyer always offers H to the seller and the seller accepts this with probability 1. Hence, this game is identical to the game between a buyer of valuation v and a seller of valuation H , with the buyer making the offers. Thus, in the four-player game, we will have an equilibrium which is identical to the one described in the benchmark case. We conclude the proof by assigning the following values:

$$p'_l(\pi) = M \text{ and } \bar{p}(\pi) = H \text{ for } \pi \in [0, d_1)$$

¹⁷We obtain these by using the indifference relations of the players when they are using randomized behavioural strategies.

■

Lemma 4 *If there exists a $\bar{\delta} \in (\delta', 1)$ such that for $\delta \geq \bar{\delta}$ and for all $t < T$ ($T < N^*$) an equilibrium as described in the current Proposition exists for $\pi \in [0, d_t(\delta))$, then there exists a $\delta_T^* \geq \bar{\delta}$ such that, for all $\delta \in (\delta_T^*, 1)$, such an equilibrium also exists for $\pi \in [d_T(\delta), d_{T+1}(\delta))$.*

Proof. We only need to show that there exists a $\delta_T^* \geq \bar{\delta}$ such that for all $\delta > \delta_T^*$ and for all $\pi \in [d_T(\delta), d_{T+1}(\delta))$, there exists a $p_l(\pi) \in (p_l'(\pi), \bar{p}(\pi))$ with

$$p_l(\pi) = (1 - \delta)M + \delta E_\pi(p)$$

From now on we will write d_T instead of $d_T(\delta)$. For each $\delta \in (\delta', 1)$, we can construct $d(\delta)$ and the equilibrium strategies as above for any $p_l = x \in (p_l'(\pi), \bar{p}(\pi))$. Function $G(x)$ is constructed as

$$G(x) = x - [\delta E_\pi^x(p) + (1 - \delta)M]$$

We can infer from ([7]) that the function $G(\cdot)$ is monotonically increasing in x . Since $E_\pi^x(p) < \bar{p}(\pi)$,

$$\lim_{x \rightarrow \bar{p}(\pi)} G(x) > 0$$

Next, we have

$$G(p_l'(\pi)) = p_l'(\pi) - [\delta E_\pi^{p_l'(\pi)}(p) + (1 - \delta)M]$$

By definition, $E_\pi^{p_l'(\pi)}(p) > p_l'(\pi)$. So for $\delta = 1$, $G(p_l'(\pi)) < 0$. Since $G(\cdot)$ is a continuous function, there exists a $\delta_T^* \geq \bar{\delta}$ such that for all $\delta > \delta_T^*$, $G(p_l'(\pi)) < 0$. By invoking the *Intermediate Value Theorem* we can say that for a given $\delta > \delta_T^*$, there exists a unique $x^*(\pi) \in (p_l'(\pi), \bar{p}(\pi))$ such that $G(x^*(\pi)) = 0$. This $x^*(\pi)$ is our required $p_l(\pi)$.

This concludes the proof. ■

From Lemma (3), we know that for any $\delta \in (0, 1)$, an equilibrium exists for $\pi \in [0, d_1]$.¹⁸ Using Lemma (4), we can obtain δ_t^* for all $t \in \{1, 2, \dots, N^*\}$. δ^* is defined as:

$$\delta^* = \max_{1, \dots, N^*} \delta_t^*$$

We can do this because N^* is finite. Lemma (3) and (4) now guarantee that whenever $\delta > \delta^*$, an equilibrium as described in the current proposition exists for all $\pi \in [0, 1)$.

This concludes the proof of the proposition. ■

A.2 Off-path behaviour of the 4 player game with incomplete information(public offers)

Suppose B_2 adheres to his equilibrium strategy. Then the off-path behaviour of B_1 and that of L -type S_I , while B_1 makes an offer greater than $\delta^t H$ to S_I , are the same as in the 2-player game with incomplete information. If B_1 's offer to S_I is less than $\delta^t H$ then the off-path behaviour of the L -type S_I is described in the following manner. If B_2 's offer to S_M is in the range $[p_l(\pi), \bar{p}(\pi)]$, then the L -type S_I behaves in the same way as in the 2-player game. Consider the case when B_2 offers $p_l'(\pi)$ to S_M . If the offer to S_I from B_1 is less than $[q(\pi)\delta^{t+1}H + (1 - q(\pi))\delta^{t+2}H]$, then both types reject the offer and S_M also rejects the offer. If the offer is greater than $[q(\pi)\delta^{t+1}H + (1 - q(\pi))\delta^{t+2}H]$

¹⁸Note that d_1 is independent of δ

but less than $[q(d_t)\delta^t H + (1 - q(d_t))\delta^{t+1}H]$ then the L -type S_I rejects it with a probability such that the updated belief is d_t . S_M also rejects the offer. In the following period(s), the offer to S_I is either $\delta^{t-1}H$ or a randomised offer between $\delta^t H$ or $\delta^{t-1}H$. Randomisations are done in a manner such that S_I is indifferent between accepting or rejecting the offer. If the offer is greater than $[q(d_t)\delta^t H + (1 - q(d_t))\delta^{t+1}H]$, the offer is accepted by L -type S_I with a probability such that the updated belief is d_{t-1} . S_M randomises between accepting and rejecting the offer of $p'_l(\pi)$. In the following period(s), offer to S_I is either $\delta^{t-1}H$ or a randomised offer between $\delta^{t-1}H$ and $\delta^{t-2}H$. These randomisations are such that S_I is indifferent between accepting and rejecting the offer. For high values of δ , B_1 does not have any incentive to deviate.

Next, suppose B_2 makes an unacceptable offer to S_M , (which is observable to S_I) and B_1 makes an equilibrium offer to S_I . The L -type S_I rejects this offer with a probability that takes the updated belief to d_{t-1} . If S_I rejects this equilibrium offer and next period both the buyers make offers to S_M , then two periods from now, the remaining buyer offers $\delta^{t-2}H$ (the buyer is indifferent between offering $\delta^{t-1}H$ and $\delta^{t-2}H$ at $\pi = d_{t-1}$) to S_I . Thus the expected continuation payoff to S_I from rejection is $\delta(q(d_{t-1})\delta^{t-1}H + \delta(1 - q(d_{t-1}))\delta^{t-2}H) = \delta^t H$. This implies that the L -type S_I is indifferent between accepting and rejecting an offer of $\delta^t H$ if he observes S_M to get an unacceptable offer.

Now consider the case when B_2 deviates and makes an offer to S_I . It is assumed that if S_I gets two offers then she disregards the lower offer.

Suppose B_1 makes an equilibrium offer to S_I and B_2 deviates and offers something less than $\delta^t H$ to S_I . S_I 's probability of accepting the equilibrium offer (which is the higher offer in this case) remains the same. If S_I rejects the higher offer (which in this case is the offer of $\delta^t H$ from B_1) and next period both the buyers make offers to S_M , then two periods from now, the remaining buyer offers $\delta^{t-2}H$ to S_I .

If B_2 deviates and offers $p^o \in (\delta^t H, \delta^{t-1}H]$ to S_I , then S_I rejects this with a probability that takes the updated belief to d_{t-2} . If S_I rejects this offer then next period if B_1 offers to S_I , he offers $\delta^{t-2}H$. If both B_1 and B_2 make offers to S_M then two periods from now the remaining buyer randomises between offering $\delta^{t-2}H$ and $\delta^{t-3}H$ to S_I (conditional on S_I being present). Randomisations are done in a manner to ensure that the expected continuation payoff to S_I from rejection is p^o . It is easy to check that for high values of δ , this can always be done. Lastly, if B_2 deviates and offers to S_I and B_1 offers to S_M (according to his equilibrium strategy), then the off-path specifications are the same as in the 2-player game with incomplete information.

We will now show that B_2 has no incentive to deviate. Suppose he makes an unacceptable offer to S_M . His expected discounted payoff from deviation is given by,

$$\mathcal{D} = q(\pi)[\delta\{a(\pi)(v - M) + (1 - a(\pi))v_B(d_{t-1})\}] + (1 - q(\pi))\delta v_B(\pi) \quad (9)$$

From (4) we know that,

$$p'_l(\pi) < M + \delta(1 - a(\pi))[\bar{p}(d_{t-1}) - M]$$

as $E_{d_{t-1}} < \bar{p}(d_{t-1})$. Hence we have,

$$p'_l(\pi) < M + \delta(1 - a(\pi))[(v - M) - (v - \bar{p}(d_{t-1}))]$$

Rearranging the above terms we get,

$$(v - p'_l(\pi)) > \delta\{a(\pi)(v - M) + (1 - a(\pi))v_B(d_{t-1})\} + (1 - \delta)(v - M) \quad (10)$$

By comparing (9) and (10) we have,

$$q(\pi)(v - p'_l(\pi)) + (1 - q(\pi))\delta v_B(\pi) > \mathcal{D}$$

The L.H.S of the above relation is B_2 's equilibrium payoff, as he puts a mass point at $p'_l(\pi)$. Hence, he has no incentive to make an unacceptable offer to S_M .

Next, suppose B_2 deviates and makes an offer of p^o to S_I such that $p^o \in (\delta^t H, \delta^{t-1} H]$. B_2 's payoff from deviation is:

$$\Gamma_H = q(\pi)[(v - p^o)a'(\pi) + (1 - a'(\pi))\delta v_B(d_{t-2})] + (1 - q(\pi))[(v - p^o)a(\pi) + (1 - a(\pi))\delta v_B(d_{t-1})]$$

where $a'(\pi)$ is the probability with which B_2 's offer is accepted by S_I in the event when both B_1 and B_2 make offers to S_I and B_2 's offer is in the range $(\delta^t H, \delta^{t-1} H]$. From our above specification it is clear that $a'(\pi) > a(\pi)$, where $a(\pi)$ is the acceptance probability of an equilibrium offer to S_I . This is also very intuitive. In the contingency when B_1 makes an equilibrium offer to S_M and B_2 's out of the equilibrium offer to S_I is in the range $(\delta^t H, \delta^{t-1} H]$, the acceptance probability is equal to $a(\pi)$. This is the equilibrium acceptance probability. In this case if the L -type S_I rejects an offer, then next period he will get an offer with probability 1. However if both B_1 and B_2 make offers to S_I and B_2 's offer is in the range $(\delta^t H, \delta^{t-1} H]$ then the L -type S_I accepts this offer with a higher probability. This is because, on rejection, there is a positive probability that S_I might not get an offer in the next period. This explains why $a'(\pi) > a(\pi)$.

Since $p^o > p'_l(\pi)$ ¹⁹ and $\bar{p}(d_{t-2}) > p'_l(\pi)$ ²⁰, we have

$$v - p'_l(\pi) > (v - p^o)a'(\pi) + (1 - a'(\pi))\delta v_B(d_{t-2}) \quad (11)$$

Also, since $p^o > \delta^t H$, we have

$$(v - p^o)a(\pi) + (1 - a(\pi))\delta v_B(d_{t-1}) < v_B(\pi)$$

The expression $[(v - p^o)a(\pi) + (1 - a(\pi))\delta v_B(d_{t-1}) - \delta v_B(\pi)]$ is strictly negative for $\delta = 1$. From continuity, we can say that for sufficiently high values of δ , $(v - p^o)a(\pi) + (1 - a(\pi))\delta v_B(d_{t-1}) < \delta v_B(\pi)$. This implies that

$$(v - p'_l(\pi))q(\pi) + (1 - q(\pi))\delta v_B(\pi) > \Gamma_H$$

The L.H.S of the above inequality is the equilibrium payoff of B_2 . Similarly if B_2 deviates and makes an offer to S_I such that his offer p^0 is in the range $[\delta^{t+1} H, \delta^t H)$, the payoff from deviation is

$$\begin{aligned} \Gamma_L &= q(\pi)[\delta\{a(\pi)(v - M) + (1 - a(\pi))v_B(d_{t-1})\}] \\ &+ (1 - q(\pi))[(v - p^0)a''(\pi) + (1 - a''(\pi))\delta v_B(d_t)] \end{aligned}$$

From the 2-player game we know that $[(v - p^0)a''(\pi) + (1 - a''(\pi))\delta v_B(d_t)] < v_B(\pi)$. Also, from the previous analysis we can posit that $(v - p'_l(\pi)) > \delta\{a(\pi)(v - M) + (1 - a(\pi))v_B(d_{t-1})\}$. Thus, for sufficiently high values of δ , $(v - p'_l(\pi))q(\pi) + (1 - q(\pi))\delta v_B(\pi) > \Gamma_L$.

Hence, B_2 has no incentive to deviate and make an offer to S_I .

¹⁹For sufficiently high values of δ this will always be the case.

²⁰Since $\bar{p}(d_{t-2}) > \bar{p}(\pi) > p'_l(\pi)$.

B Asymptotic result

We will show that in the equilibrium characterised in Proposition 1, as $\delta \rightarrow 1$, price offers in all transactions converge to H .

Appendix (C) shows that as $\delta \rightarrow 1$, $p'_l(\pi)$ reaches a limit which is strictly less than H . Since $v_B(\pi) \rightarrow (v - H)$ as $\delta \rightarrow 1$, from (5) we have

$$q(\pi) \rightarrow 0 \text{ as } \delta \rightarrow 1$$

From (6) we have,

$$1 - F_1^\pi(s) = \frac{\bar{p}(\pi) - s}{(1 - q(\pi))[v - s - \delta v_B(\pi)]}$$

We have shown that $q(\pi) \rightarrow 0$ as $\delta \rightarrow 1$. Hence as $\delta \rightarrow 1$, for s arbitrarily close to $\bar{p}(\pi)$, we have

$$1 - F_1^\pi(s) \approx \frac{\bar{p}(\pi) - s}{\bar{p}(\pi) - s} = 1$$

Hence, the distribution collapses and $p_l(\pi) \rightarrow \bar{p}(\pi)$. From the expression of $p_l(\pi)$, we know that $p_l(\pi) \rightarrow E_\pi(p)$ as δ goes to 1. Thus, we can conclude that as δ goes to 1, $E_\pi(p) \rightarrow H$ for all values of π .

Since $G(\bar{p}(\pi)) > 0$, there will be a threshold of δ such that for all δ higher than that threshold, we have $G(\delta\bar{p}(\pi)) > 0$. Thus $p_l(\pi)$ is bounded above by $\delta\bar{p}(\pi)$ for high values of δ . From (7), we can then infer that

$$q'(\pi) = \frac{1}{\frac{v}{v_B(\pi)} + \frac{\delta\bar{p}(\pi) - p_l(\pi)}{(1-\delta)v_B(\pi)}}$$

Since $p_l(\pi)$ is bounded above by $\delta\bar{p}(\pi)$ for high values of δ , $q'(\pi) \rightarrow 0$ as δ goes to 1.

Thus, we conclude that as δ goes to 1, prices in all transactions in the equilibrium characterised go to H .

C Limit of $p'_l(\pi)$

We first prove the following lemma

Lemma 5 *For a given $\pi > d_1$, the acceptance probability $a(\pi, \delta)$ of an equilibrium offer is increasing in δ and has a limit $\bar{a}(\pi)$ which is less than 1.*

Proof. The acceptance probability $a(\pi, \delta)$ of an equilibrium offer is equal to $\pi\beta(\pi, \delta)$, where $\beta(\pi, \delta)$ is the probability with which the L -type S_I accepts an equilibrium offer. From the updating rule we know that $\beta(\pi, \delta)$ is such that the following relation is satisfied:

$$\frac{\pi(1 - \beta(\pi, \delta))}{\pi(1 - \beta(\pi, \delta)) + (1 - \pi)} = d_{t-1}(\delta)$$

From the above expression, we get

$$\beta(\pi, \delta) = \frac{\pi - d_{t-1}(\delta)}{\pi(1 - d_{t-1}(\delta))}$$

Since d_t^s have a limit as δ goes to 1, so does $\beta(\pi, \delta)$. Therefore, $a(\pi, \delta)$ also has a limit $\bar{a}(\pi)$ which is less than 1 for $\pi \in (0, 1)$. ■

The above lemma implies that as $\delta \rightarrow 1$, $p'_l(\pi)$ reaches a limit which is strictly less than H and greater than M .

D Proof of proposition 2

Proof. We prove this proposition in steps, through a series of lemmas. First, we show that for any equilibrium belonging to the set of equilibria considered, the following lemma holds.

Lemma 6 *For any $\pi \in (0, 1)$, it is never possible to have a stationary equilibrium in the set of equilibria considered such that both buyers offer only to S_M on the equilibrium path.*

Proof. Suppose it is the case that there exists a stationary equilibrium in the game with four players such that both buyers offer only to S_M . Both buyers should have a distribution of offers to S_M with a common support²¹ $[\underline{s}(\pi), \bar{s}(\pi)]$. The payoff to each buyer should then be $(v - \bar{s}(\pi)) = v_4(\pi)$ (say). Let $v_B(\pi)$ be the payoff obtained by a buyer when his offer to S_M gets rejected. This is the payoff a buyer obtains by making offers to the informed seller in a two player game.

Consider any $s \in [\underline{s}(\pi), \bar{s}(\pi)]$ and one of the buyers (say B_1). If the distributions of the offers are given by F_i for buyer i , then we have

$$(v - s)F_2(s) + (1 - F_2(s))\delta v_B(\pi) = v - \bar{s}(\pi)$$

This follows from the buyer B_1 's indifference condition.²²

Since in equilibrium, the above needs to be true for any $s \in [\underline{s}(\pi), \bar{s}(\pi)]$, we must have $v - \bar{s}(\pi) > \delta v_B(\pi)$. The above equality then gives us

$$F_2(s) = \frac{(v - \bar{s}(\pi)) - \delta v_B(\pi)}{(v - s) - \delta v_B(\pi)}$$

Since $v - \bar{s}(\pi) > \delta v_B(\pi)$, for $s \in [\underline{s}(\pi), \bar{s}(\pi))$, we have $v - s > v - \bar{s}(\pi) > \delta v_B(\pi)$. This implies

$$F_2(\underline{s}(\pi)) > 0$$

Similarly, we can show that

$$F_1(\underline{s}(\pi)) > 0$$

In equilibrium, it is not possible for both the buyers to put mass points at the lower bound of the support. Hence, S_M cannot get two offers with probability 1. This concludes the proof of the lemma. ■

For any equilibrium belonging to the set of equilibria we are considering, we know that S_M must get at least one offer with positive probability. The above lemma implies that S_I also gets at least one offer with a positive probability. We will now argue that for any equilibrium in the set of equilibria considered, S_M always accepts an equilibrium offer immediately. This is irrespective of whether S_M gets one offer or two offers.

To show this formally, consider such an equilibrium. We first define the following. Given a π , let $p_i(\pi)$ be the minimum acceptable price to the seller S_M in the event she gets i ($i = 1, 2$) offer(s) in the considered equilibrium. We have

²¹If the upper bounds are not equal, then the buyer with the higher upper bound can profitably deviate. On the other hand, if the lower bounds are different, then the buyer with the smaller lower bound can profitably deviate.

²²If there exists a stationary equilibrium where both buyers offer to S_M only, then the lower bound of the common support of offers is not less than the minimum acceptable price to S_M in the candidate stationary equilibrium. To see this, let $p_2^2(\pi) = (1 - \delta)M + \delta E_p^2(\pi)$. Suppose the lower bound of the support is strictly less than $p_2^2(\pi)$. Let $z(\pi)$ be the probability with which each buyer's offer is strictly less than $p_2^2(\pi)$. If $v_4^2(\pi)$ is the payoff to the buyers in this candidate equilibrium, the expected payoff to the buyer from making an offer strictly less than $p_2^2(\pi)$ is $z(\pi)\delta v_4^2(\pi) + (1 - z(\pi))\delta v_B(\pi)$. In equilibrium, we must have $z(\pi)\delta v_4^2(\pi) + (1 - z(\pi))\delta v_B(\pi) = v_4^2(\pi)$. Either $v_B(\pi) > v_4^2(\pi)$ or $v_B(\pi) \leq v_4^2(\pi)$. In the former case the equality does not hold for values of δ close to 1 and in the later case the equality does not hold for any value of $\delta < 1$.

$$p_1(\pi) - M = (1 - (\alpha(\pi))\delta[E_p(\tilde{\pi}) - M])$$

$E_p(\tilde{\pi})$ is the price corresponding to the expected equilibrium payoff to the seller S_M in the event she rejects the offer and the informed seller does not accept the offer. It is evident that when the seller S_M is getting one offer, the informed seller is also getting an offer. Here $\alpha(\pi)$ is the probability with which the informed seller accepts the offer and $\tilde{\pi}$ is the updated belief.

Similarly, we have

$$p_2(\pi) - M = \delta[E_p(\pi) - M]$$

where $E_p(\pi)$ is the price corresponding to the expected equilibrium payoff to S_M in the event she rejects both offers. In appendix E we argue that $E_p(\pi) > M$. The following lemma has the consequence that S_M always accepts an equilibrium offer (or highest of the equilibrium offers) immediately.

Lemma 7 *For any $\pi < 1$, if we restrict ourselves to the set of equilibria considered, then in any arbitrary equilibrium, it is never possible for a buyer to make an offer to S_M , which is strictly less than $\min\{p_1(\pi), p_2(\pi)\}$.*

Proof. Suppose the conclusion of the lemma does not hold, so there is such an equilibrium. Let the payoff to the buyers from this candidate equilibrium of the four-player game be $v_4(\pi)$. In appendix F we argue that $v_4(\pi) < v - p_2(\pi)$. Let $v_B(\pi)$ be the payoff the buyer gets by making offers to S_I in a two-player game.

Consider the buyer who makes the lowest offer to S_M . We label this buyer as B_1 and the lowest offer as $\underline{p}(\pi)$, where $\underline{p}(\pi) < \min\{p_1(\pi), p_2(\pi)\}$. Let $q(\pi)$ be the probability with which the other buyer makes an offer to the seller S_I . Let $\gamma(\pi)$ be the probability with which the other buyer, conditional on making offers to the seller S_M , makes an offer which is less than $p_2(\pi)$. Finally, $\alpha(\pi)$ is the probability with which the informed seller accepts an offer if the other buyer makes an offer to her. Since B_1 's offer of $\underline{p}(\pi)$ to S_M is always rejected, the payoff to B_1 from making such an offer is

$$\{q(\pi)\delta\{\alpha(\pi)(v - M) + (1 - \alpha(\pi))(v - E_p^b(\tilde{\pi}))\} + (1 - q(\pi))\delta\{\gamma(\pi)v_4(\pi) + (1 - \gamma(\pi))v_B(\pi)\}$$

where $E_p^b(\tilde{\pi})$ is such that $(v - E_p^b(\tilde{\pi}))$ is the expected equilibrium payoff to the buyer if the updated belief is $\tilde{\pi}$. We first argue that $(v - E_p^b(\tilde{\pi}))$ is less than or equal to $(v - E_p(\tilde{\pi}))$. This is because since $(E_p(\tilde{\pi}) - M)$ is the expected equilibrium payoff to the seller S_M when the belief is $\tilde{\pi}$, there is at least one price offer by the buyer, which is greater than or equal to $E_p(\tilde{\pi})$. Hence, we have

$$\begin{aligned} \delta\{\alpha(\pi)(v - M) + (1 - \alpha(\pi))(v - E_p^b(\tilde{\pi}))\} &\leq \delta\{\alpha(\pi)(v - M) + (1 - \alpha(\pi))(v - E_p(\tilde{\pi}))\} \\ &\Rightarrow (v - p_1(\pi)) - \delta\{\alpha(\pi)(v - M) + (1 - \alpha(\pi))(v - E_p^b(\tilde{\pi}))\} \\ &\geq (v - p_1(\pi)) - \delta\{\alpha(\pi)(v - M) + (1 - \alpha(\pi))(v - E_p(\tilde{\pi}))\} \end{aligned}$$

Since, $(v - p_1(\pi)) - \delta\{\alpha(\pi)(v - M) + (1 - \alpha(\pi))(v - E_p(\tilde{\pi}))\} = (1 - \delta)(v - M) > 0$, we have

$$(v - p_1(\pi)) - \delta\{\alpha(\pi)(v - M) + (1 - \alpha(\pi))(v - E_p^b(\tilde{\pi}))\} > 0$$

There are two possibilities. Either $p_1(\pi) < p_2(\pi)$ or $p_2(\pi) < p_1(\pi)$. If $p_2(\pi) > p_1(\pi)$, then the buyer can profitably deviate by making an offer of $p_1(\pi)$. The payoff from making such an offer is

$$q(\pi)(v - p_1(\pi)) + (1 - q(\pi))\{\gamma(\pi)\delta v_4(\pi) + (1 - \gamma(\pi))\delta v_B(\pi)\}$$

Since $(v - p_1(\pi)) - \delta\{\alpha(\pi)(v - M) + (1 - \alpha)(v - E_p^b(\tilde{\pi}))\} > 0$, we can infer that this constitutes a profitable deviation by the buyer.

Next, consider the case when $p_2(\pi) < p_1(\pi)$. In this situation, the buyer can profitably deviate by making an offer of $p_2(\pi)$. The payoff from making such an offer is

$$\{q(\pi)\delta\{\alpha(\pi)(v - M) + (1 - \alpha)(v - E_p^b(\tilde{\pi}))\} + (1 - q(\pi))\{\gamma(\pi)(v - p_2(\pi)) + (1 - \gamma(\pi))\delta v_B(\pi)\}$$

Since $v_4(\pi) < (v - p_2(\pi))$, this constitutes a profitable deviation by the buyer.

This concludes the proof of the lemma. ■

There are two immediate conclusions from the above lemma. First, if $p_2(\pi) < p_1(\pi)$, then it can be shown that if δ is high enough, then in equilibrium, no buyer should offer anything less than $p_1(\pi)$. To show this, suppose at least one of the buyers makes an offer which is less than $p_1(\pi)$ and consider the buyer who makes the lowest offer to S_M . Let $\gamma_1(\pi)$ be the probability with which the other buyer, conditional on making offers to S_M , makes an offer which is less than $p_1(\pi)$. The payoff to the buyer by making the lowest offer to S_M is

$$\{q(\pi)\delta\{\alpha(\pi)(v - M) + (1 - \alpha)(v - E_p^b(\tilde{\pi}))\} + (1 - q(\pi))\delta\{v_B(\pi)\}$$

However, if he makes an offer of $p_1(\pi)$ then the payoff is

$$\{q(\pi)(v - p_1(\pi)) + (1 - q(\pi))\{\gamma_1(\pi)(v - p_1(\pi)) + (1 - \gamma_1(\pi))\delta v_B(\pi)\}$$

We know that as $\delta \rightarrow 1$, $v_B(\pi) \rightarrow v - H$. Since $p_1(\pi) < H$, this implies that for high δ , $\gamma_1(\pi)(v - p_1(\pi)) + (1 - \gamma_1(\pi))\delta v_B(\pi) > \delta v_B(\pi)$. Hence, for high δ , this constitutes a profitable deviation by the buyer.

Secondly, if $p_1(\pi) < p_2(\pi)$, then only one buyer can make an offer with positive probability that is less than $p_2(\pi)$. This is because, any buyer who makes an offer to S_M in the range $(p_1(\pi), p_2(\pi))$ can get the offer accepted when the seller S_M gets only one offer. In that case the offer can still get accepted if it is lowered and that will not alter the outcomes following the rejection of the offer. Hence, the buyer can profitably deviate by making a lower offer. Thus, in equilibrium if a buyer has to offer anything less than $p_2(\pi)$ to the seller S_M , then it has to be equal to $p_1(\pi)$. However, in equilibrium both buyers cannot put mass points at $p_1(\pi)$. This shows that only one buyer can make an offer to S_M which is strictly less than $p_2(\pi)$.

Hence, we have argued that all offers to S_M are always greater than or equal to $p_1(\pi)$ and in the event S_M gets two offers, both offers are never below $p_2(\pi)$. This shows that S_M always accepts an equilibrium offer immediately.

We will now show that for any equilibrium in the set of equilibria considered, the informed seller by rejecting equilibrium offers for a finite number of periods can take the posterior to 0. This is shown in the following lemma.

Lemma 8 *Suppose we restrict ourselves to the set of equilibria considered. Given a π and δ , there exists a $T_\pi(\delta) > 0$ such that conditional on getting offers, the informed seller can get an offer of H in $T_\pi(\delta)$ periods from now by rejecting all offers she gets in between. $T_\pi(\delta)$ depends on the sequence of equilibrium offers and corresponding strategies of the responders in the candidate equilibrium. $T_\pi(\delta)$ is uniformly bounded above as $\delta \rightarrow 1$.*

Proof. To prove the first part of the lemma, we show that in the candidate equilibrium, rejection of offers by the informed seller can never lead to an upward revision of the belief.²³ If it does, then it implies that the offer is

²³We consider updating in equilibrium. Since this is about a candidate equilibrium, out of equilibrium events could only arise from non-equilibrium offers made by the buyers. However, if we were to follow the definition of the PBE, then no player's action should be treated as containing information about things which that player does not know (*no-signalling-what-you-don't-know*). Hence, these out of equilibrium events cannot lead to change in beliefs.

such that the H -type S_I accepts the offer with a positive probability and the L -type S_I rejects it with a positive probability. Since the H -type accepts the offer with a positive probability, this means that the offer must be greater than or equal to H (let this offer be equal to $p_h \geq H$) and we have

$$p_h - H \geq \delta(E' - H)$$

where E' is the price corresponding to the expected equilibrium payoff to the H -type S_I next period. Then,

$$\begin{aligned} p_h &\geq \delta(E') + (1 - \delta)H \Rightarrow p_h - L \geq \delta(E') + (1 - \delta)H - L \\ &\Rightarrow p_h - L \geq \delta(E' - L) + (1 - \delta)(H - L) > \delta(E' - L) \end{aligned}$$

This shows that the L -type S_I should accept p_h with probability²⁴ 1. This is a contradiction to our supposition that the L -type S_I rejects with some positive probability. Thus, the belief revision following a rejection must be in the downward direction. It cannot be zero since in that case it implies that both types reject with probability 1. This is not possible in equilibrium.

Thus, in equilibrium, S_I (L -type) should always accept an offer with a positive probability. This proves the first part of the lemma.

To show that the number of rejections required to get an offer of H is uniformly bounded above as $\delta \rightarrow 1$, we need to show that it cannot happen that the acceptance probabilities of any sequence of equilibrium offers to S_I are not uniformly bounded below as $\delta \rightarrow 1$.

In the equilibrium considered, if only one buyer makes offers to S_I , then the claim of the lemma holds. This is because of the fact that S_M always accepts an equilibrium offer immediately and hence, S_I on rejecting an offer knows that the continuation game will be a two-player game with one-sided asymmetric information. Thus, by invoking the finiteness result of the two-player game with one-sided asymmetric information, we know that S_I can take the posterior to 0 by rejecting equilibrium offers for finite number of periods.

Consider equilibria where more than one buyer makes offers to S_I . Given the set of equilibria we have considered and the results already proved, we can posit that in such a case, either one of the buyers is making offers only to S_I and the other is randomising between making offers to S_I and S_M , or both buyers are offering to both sellers with positive probabilities.

Let p_l be the minimum offer which gets accepted by S_I with positive probability in an equilibrium where two buyers offer to S_I with positive probability. We will now argue that there exists a possible outcome such that S_I gets only one offer and the offer is equal to p_l . When one of the buyers is making offers to S_I only, then p_l must be the lower bound of the support of his offers. In the second case, when both buyers with positive probability make offers to S_I and S_M , with positive probability S_I gets only one offer. Thus, there exists an instance that S_I gets the offer of p_l only.

When S_I gets the offer of p_l only, then she knows that by rejecting that she gets back a two-player game, which has the finiteness property. Thus there exists a $\tilde{T}(\delta) > 0$ such that S_I is indifferent between getting p_l now and H in $\tilde{T}(\delta)$ periods from now. This implies

$$p_l - L = \delta^{\tilde{T}(\delta)}(H - L)$$

From the finiteness property of the two player game with one sided asymmetric information, we know that $\tilde{T}(\delta)$ is

²⁴This follows from the fact that from next period onwards, L -type S_I can always adopt the optimal strategy of the H -type S_I . Hence, following a rejection of the offer p_h , the expected equilibrium payoff to the L -type S_I is $\leq E' - L$

uniformly bounded above as $\delta \rightarrow 1$.

Suppose there is a sequence of equilibrium offers such that the acceptance probabilities of the offers are not bounded below as $\delta \rightarrow 1$. Let p be the initial offer of that particular sequence. $p \geq p_l$. For a given δ , let $T(\delta) > 0$ be such that, given the acceptance probabilities of the sequence of offers, by rejecting p and subsequent equilibrium offers, S_I can get H in $T(\delta)$ periods from now. Hence, the L -type S_I should be indifferent between getting p now and H in $T(\delta)$ time periods from now. As per our supposition, $T(\delta)$ is not uniformly bounded above as $\delta \rightarrow 1$.

Then, we can find a $\delta^h < 1$ such that for all $\delta \in (\delta^h, 1)$, we have $T(\delta) > \tilde{T}(\delta)$. This gives us

$$\delta^{T(\delta)}(H - L) < \delta^{\tilde{T}(\delta)}(H - L) = p_l - L \leq p - L$$

Hence, the L -type S_I is not indifferent between getting p now and H in $T(\delta)$ time periods from now, contrary to our assumption.

Hence, as $\delta \rightarrow 1$, probabilities of acceptance of any sequence of equilibrium offers are bounded below. This concludes the proof of the lemma. ■

The above lemma shows that any stationary equilibrium in the set of equilibria considered possess the finiteness property. We will now show that we cannot have both buyers offering to both sellers with positive probability. This is argued in the following lemma.

Lemma 9 *In any equilibrium belonging to the set of equilibria considered, if players are patient enough then both buyers cannot make offers to both sellers with positive probability.*

Proof.

From the arguments of lemma (8), we know that in an arbitrary stationary equilibrium, any offer made to the informed seller should get accepted by the low type with a positive probability bounded away from 0. Suppose there exists a stationary equilibrium of the four-player game where both buyers offer to both sellers with a positive probability. Hence, in equilibrium, if the informed seller gets offer(s), then she either gets two offers or one offer. Since S_M always accepts an offer in equilibrium immediately, S_I knows that on rejecting an offer(s) she will get another offer in at most two periods from now. Hence, from lemma (8) we infer that if the informed seller gets one offer, then the L -type S_I can expect to get an offer of H in at most $T_1(\pi) > 0$ time periods from now, by rejecting all offers she gets in between. Similarly, if the informed seller gets two offers then the L -type S_I by rejecting both offers can expect to get an offer of H in at most $T_2(\pi) > 0$ time periods from now by rejecting all offers she gets in between. As we have argued in lemma (8), both $T_1(\pi)$ and $T_2(\pi)$ are bounded above as $\delta \rightarrow 1$. Thus, any offer s to the informed seller in equilibrium should satisfy

$$s \geq \delta^{T_1(\pi)}H + (1 - \delta^{T_1(\pi)})L \equiv s_1(\delta)$$

and

$$s \geq \delta^{T_2(\pi)}H + (1 - \delta^{T_2(\pi)})L \equiv s_2(\delta)$$

It is clear from the above that as $\delta \rightarrow 1$, both $s_1(\delta) \rightarrow H$ and $s_2(\delta) \rightarrow H$. Hence, if there is a support of offers to S_I in equilibrium, then the support should collapse as $\delta \rightarrow 1$.

We will now argue that for δ high enough but $\delta < 1$, the support in equilibrium cannot have two or more points.

Suppose it is possible that the support of offers to S_I has two or more points. This implies that the upper bound and the lower bound of the support are different from each other. Let $\underline{s}(\pi)$ and $\bar{s}(\pi)$ be the lower and upper bound of the support respectively.

Consider a buyer who is making an offer to S_I . This buyer must be indifferent between making an offer of $\underline{s}(\pi)$ and $\bar{s}(\pi)$. Let $q(\pi)$ be the probability with which the other buyer makes an offer to S_I . Since in equilibrium S_M always accepts an offer immediately, the payoff from making an offer of $\underline{s}(\pi)$ to S_I is

$$\begin{aligned}\Pi_{\underline{s}(\pi)} &= (1 - q(\pi))[\alpha_{\underline{s}(\pi)}(v - \underline{s}(\pi)) + (1 - \alpha_{\underline{s}(\pi)})\delta v_B(\pi')] \\ &\quad + q(\pi)E_s\{\beta_\pi^s\delta(v - M) + (1 - \beta_\pi^s)\delta v_4(\pi_s'')\}\end{aligned}$$

$\alpha_{\underline{s}(\pi)}$ is the acceptance probability of $\underline{s}(\pi)$ when S_I gets the offer of $\underline{s}(\pi)$ only. β_π^s is the acceptance probability of the offer s to S_I when she gets two offers. $v_B(\cdot)$ and $v_4(\cdot)$ are the buyer's payoffs from the two-player incomplete information game and the four-player incomplete information game respectively. For the second term of the right-hand side, we have taken an expectation because when two offers are made, this buyer's offer of $\underline{s}(\pi)$ to S_I never gets accepted and the payoff then depends on the offer made by the other buyer. When S_I gets only one offer and rejects an offer of $\underline{s}(\pi)$, then the updated belief is π' ; π_s'' denotes the updated belief when S_I rejects an offer of $s \in (\underline{s}(\pi), \bar{s}(\pi)]$ and she gets two offers.

Similarly, the payoff from offering $\bar{s}(\pi)$ is

$$\Pi_{\bar{s}(\pi)} = (1 - q(\pi))[\alpha_{\bar{s}(\pi)}(v - \bar{s}(\pi)) + (1 - \alpha_{\bar{s}(\pi)})\delta v_B(\pi''')] + q(\pi)[\beta_{2\pi}(v - \bar{s}(\pi)) + (1 - \beta_{2\pi})\delta v_4(\pi^4)]$$

Here π''' is the updated belief when S_I gets one offer and rejects an offer of $\bar{s}(\pi)$. When S_I gets two offers and rejects an offer of $\bar{s}(\pi)$, the updated belief is denoted by π^4 . Note that if at all S_I accepts an offer, she always accepts the offer of $\bar{s}(\pi)$, if made. The quantity $\alpha_{\bar{s}(\pi)}$ is the probability with which the offer of $\bar{s}(\pi)$ is accepted by S_I when she gets one offer. When S_I gets two offers, then the offer of $\bar{s}(\pi)$ gets accepted with probability $\beta_{2\pi}$.

As argued above, $\bar{s}(\pi) \rightarrow H$ and $\underline{s}(\pi) \rightarrow H$ as $\delta \rightarrow 1$. This implies that $v_4(\pi) \rightarrow (v - H)$ as $\delta \rightarrow 1$. From the result of the two-player one-sided asymmetric information game, we know that $v_B(\pi) \rightarrow H$ as $\delta \rightarrow 1$. Since $v - M > v - H$, we have $\Pi_{\underline{s}(\pi)} > \Pi_{\bar{s}(\pi)}$ as $\delta \rightarrow 1$. From lemma (8) we can infer that both β_π^s and $\beta_{2\pi}$ are positive. Hence, there exists a threshold for δ such that if δ crosses that threshold, $\Pi_{\underline{s}(\pi)} > \Pi_{\bar{s}(\pi)}$. This is not possible in equilibrium. Thus, for high δ , the support of offers can have only one point. The same arguments hold for the other buyer as well. Hence, each buyer while offering to S_I has a one-point support. Next, we establish that both buyers should make the same offer. If they make different offers, then as explained before, for δ high enough the buyer making the higher offer can profitably deviate by making the lower offer. However, in equilibrium it is not possible to have both buyers making the same offer to S_I ²⁵

Hence, when agents are patient enough, in equilibrium both buyers cannot offer to both sellers with a positive probability. This concludes the proof of the lemma. ■

In the following lemma we show that in any stationary equilibrium of the four player game, as players get patient enough, S_M always gets offers from two buyers with a positive probability.

Lemma 10 *In any stationary equilibrium belonging to the set of equilibria considered, there exists a threshold of δ such that if δ exceeds that threshold, both buyers make offers to S_M with positive probability.*

Proof.

²⁵These arguments would also work even if the supports were not taken to be symmetric. In that case, let $\underline{s}(\pi)$ be the minimum of the lower bounds and $\bar{s}(\pi)$ be the maximum of the upper bounds. If these are associated with the same buyer, then same arguments hold. If not, then the buyer with the higher upper bound can profitably deviate by shifting its mass to $\underline{s}(\pi)$.

Suppose there exists a stationary equilibrium where S_M gets offers from only one buyer, say B_1 . There are two possibilities. Either S_I gets an offer from B_2 only or from both B_1 and B_2 with positive probability.

Consider the first possibility. From lemma (8), we know that in a stationary equilibrium, the informed seller should always accept an equilibrium offer with some positive probability. Thus, as $\delta \rightarrow 1$, the equilibrium payoff to B_2 approaches $v - H$ from above. On the other hand, B_1 knows that the informed seller is going to accept with some probability and hence under that situation if his offer to S_M gets rejected, he gets a two-player game with S_M . The payoff to B_1 from that two-player game is $(v - M)$. Thus, B_1 's continuation payoff from his offer getting rejected by S_M is strictly greater than $(v - H)$. This implies that the equilibrium payoff to B_1 should also be strictly greater than $(v - H)$. Hence, as δ exceeds a threshold, B_2 finds it profitable to deviate and make offers to S_M .

In the latter case, we know that B_1 offers to both S_I and S_M with positive probability and B_2 makes offers only to S_I . Therefore, using the result of lemma (8), if B_1 has to get an offer accepted by S_I , then for high values of δ that offer should be close to H and thus the payoff to B_1 from making offers to S_I should be close to $(v - H)$. However as argued, the payoff to B_1 from making offers to S_M is strictly greater than $(v - H)$. In equilibrium, the buyer has to be indifferent between making offers to S_I and S_M . Hence, it is not possible to have a stationary equilibrium where S_M gets offers from only one buyer. This concludes the proof. ■

From the characteristics of the restricted set of equilibria being considered, we know that S_M always gets an offer with a positive probability. The above lemma then allows us to infer that, in any stationary equilibrium of the four player game, both buyers should offer to S_M with positive probability. From our arguments and hypothesis, we know that both buyers cannot make offers to only one seller (S_I or S_M) and both buyers cannot randomise between making offers to both sellers. Hence, we can infer that one of the buyers has to make offers to S_M only and the other buyer should randomise between making offers to S_I and S_M .

The following lemma now shows that for any $\pi \in [0, 1)$, any equilibrium in this restricted set possesses the characteristic that the price offers to all sellers approach H as $\delta \rightarrow 1$.

Lemma 11 *For a given π , in any hypothesised equilibrium, price offers to all sellers go to H as $\delta \rightarrow 1$.*

Proof. Let $\bar{s}(\pi)$ be the upper bound of the support²⁶ of offers to S_M .

S_M always accepts an equilibrium offer immediately. Hence, if the L -type S_I rejects an equilibrium offer, she gets back a two-player game with one-sided asymmetric information. Thus, the buyer offering to S_I in a period must offer at least p^e such that

$$p^e - L = \delta(H - \epsilon - L) \Rightarrow p^e = (1 - \delta)L + \delta(H - \epsilon)$$

where $\epsilon > 0$ and $\epsilon \rightarrow 0$ as $\delta \rightarrow 1$.

Consider B_1 , who is randomising between making offers to S_I and S_M . When offering to S_I , B_1 must offer p^e and it must be the case that

$$(v - p^e)\alpha(\pi) + (1 - \alpha(\pi))\delta\{v - (H - \epsilon)\} = v - \bar{s}(\pi)$$

where $\alpha(\pi)$ is the probability with which the offer is accepted by the informed seller. This follows from the fact that B_1 must be indifferent between offering to S_I and S_M . The L.H.S of the above equality is the payoff to B_1 from offering to S_I and the R.H.S is the payoff to him from offering to S_M . Since in any hypothesized equilibrium, S_M

²⁶The upper bound of support of offers to S_M for both buyers should be the same. Else, the buyer with the higher upper bound can profitably deviate

always gets an offer in period 1 and S_M accepts an equilibrium offer immediately, S_I , by rejecting an equilibrium offer, always gets back a two-player game with one-sided asymmetric information. Hence, the payoff to the buyer from offering to S_I is the same as in the two-player game with one-sided asymmetric information. This implies that

$$(v - p^e)\alpha(\pi) + (1 - \alpha(\pi))\delta\{v - (H - \epsilon)\} = v_B(\pi)$$

Thus, we can conclude that $v_B(\pi) = v - \bar{s}(\pi)$.

We will now show that the upper bound of the support of offers to S_M is strictly greater than p^e . We have

$$\begin{aligned} (v - p^e) - \delta\{v - (H - \epsilon)\} &= v(1 - \delta) + \delta(H - \epsilon) - \delta(H - \epsilon) - (1 - \delta)L \\ &= (1 - \delta)(v - L) > 0 \end{aligned}$$

for $\delta < 1$. This implies that

$$v - p^e > \delta\{v - (H - \epsilon)\}$$

Since $(v - \bar{s}(\pi))$ is a convex combination of $v - p^e$ and $\delta\{v - (H - \epsilon)\}$, we have

$$v - p^e > v - \bar{s}(\pi) \Rightarrow \bar{s}(\pi) > p^e$$

Next, we will argue that as $\delta \rightarrow 1$, the support of offers to S_M from any buyer is bounded below by p^e . Consider a buyer who makes an offer of p^e to S_M in equilibrium. Then, if q^e is the probability with which this offer gets accepted, we have

$$(v - p^e)q^e + (1 - q^e)\delta v_B(\pi) = v - \bar{s}(\pi)$$

This follows since S_M always accepts an offer in equilibrium immediately, this buyer's offer to S_M gets rejected only when the other buyer also makes an offer to S_M .

This gives us,

$$\begin{aligned} q^e &= \frac{(v - \bar{s}(\pi)) - \delta v_B(\pi)}{(v - p^e) - \delta v_B(\pi)} = \frac{(1 - \delta)(v - \bar{s}(\pi))}{(v - p^e) - \delta(v - \bar{s}(\pi))} \\ &\Rightarrow q^e = \frac{1}{\frac{v}{v - \bar{s}(\pi)} + \frac{\delta \bar{s}(\pi) - p^e}{(1 - \delta)(v - \bar{s}(\pi))}} \end{aligned}$$

and

$$q^e \rightarrow 0 \text{ as } \delta \rightarrow 1$$

This shows that in equilibrium, as $\delta \rightarrow 1$, any offer to S_M that is less than or equal to p^e always gets rejected. Since we have argued earlier that in equilibrium, no buyer should make an offer to S_M that she always rejects, we can infer that the support of offers to S_M from any buyer is bounded below by p^e as δ approaches 1. Hence, in any arbitrary stationary equilibrium of this kind, the price offers to all sellers are bounded below by p^e as δ approaches 1. However, as $\delta \rightarrow 1$, $p^e \rightarrow H$. Hence, as $\delta \rightarrow 1$, the support of offers to S_M from any buyer collapses and hence price offers to all sellers converge to H . ■

Thus, we have shown that for any stationary equilibrium in the set of equilibria considered, one of the buyers randomises between making offers to S_M and S_I and the other buyer makes offers to S_M only. Further, as $\delta \rightarrow 1$, price offers in all transactions in these stationary equilibria go to H . This concludes the proof of the proposition. ■

E $E_p(\pi) > M$

Suppose not. This means that $E_p(\pi) = M$. The implication of this is that the seller S_M in equilibrium only gets offer(s) equal to M . Thus, in that case S_M can get offers from one buyer only and the offer is always equal to M . S_M always accepts this offer immediately as by rejecting she cannot get anything more. Hence, the equilibrium payoff of this buyer is $v - M$. The other buyer is making offers to S_I only. Since, S_M immediately accepts an offer, the payoff to her is never greater than $v_B(\pi)$ where $v_B(\pi)$ is the payoff to a buyer in the two-player game with one sided asymmetric information. As $\delta \rightarrow 1$, $v_B(\pi) \rightarrow H$. Hence, this buyer can profitably deviate by offering to S_M . Thus, we must have $E_p(\pi) > M$.

F $(v - p_2(\pi)) > v_4(\pi)$

Since $E_p(\pi) - M$ is the expected payoff to the seller S_M in equilibrium, there is at least one price offer by the buyer which is greater than or equal to $E_p(\pi)$. Hence, we must have $v_4(\pi) \leq (v - E_p(\pi))$. This gives us

$$(v - p_2(\pi)) = \delta(v - E_p(\pi)) + (1 - \delta)(v - M) > (v - E_p(\pi)) \geq v_4(\pi)$$

This concludes the proof.

G Off-path behaviour with private offers

The off-path behaviour of the players in the case of private offers, to sustain the equilibrium described in Proposition 3 are described as follows.

Specifically we need to describe the behaviour of the players in the following three contingencies.

- (i) B_2 makes an unacceptable offer to S_M .
- (ii) B_2 makes an offer of p^o to S_I such that $p^o < \delta^t H$.
- (iii) B_2 makes an offer of p^o to S_I such that $p^o > \delta^t H$.

We denote the above three contingencies by E_1 , E_2 and E_3 respectively. We now construct a particular belief system that sustains the equilibrium described in the text.

Suppose B_1 attaches probabilities λ, λ^2 and λ^3 ($0 < \lambda < 1$) to E_1 , E_2 and E_3 respectively. Thus, he thinks that B_2 is going to stick to his equilibrium behaviour with probability $[1 - (\lambda + \lambda^2 + \lambda^3)]$. On the other hand, in the event when S_M does not get any offer, she attaches probabilities $\frac{\lambda^2}{\lambda^2 + \lambda^3}$ and $\frac{\lambda^3}{\lambda^2 + \lambda^3}$ to E_2 and E_3 respectively.

If E_1 or E_2 occurs and B_1 makes an equilibrium offer to S_I , then S_I 's probability of accepting the equilibrium offer remains the same and two periods from now (conditional on the fact that the game continues until then), if B_2 is the remaining buyer he offers $\delta^{t-2}H$ to S_I . If E_3 occurs and all players are observed to be present, then next period B_2 's offers to S_M has the same support and follows the same distribution as in the case with belief equal to d_{t-1} . In any off-path contingency, if B_1 is the last buyer remaining (two periods from now) then he offers $\delta^{t-2}H$ to S_I .

The L -type S_I accepts an offer higher than $\delta^t H$ with probability 1 if she gets two offers. If she gets only one offer then the probability of her acceptance of out-of-equilibrium offers is the same as in the two-player game with incomplete information.

We will now argue that the off-path behaviour constitutes a sequentially optimal response by the players to the limiting beliefs as $\lambda \rightarrow 0$.

Suppose B_1 makes an equilibrium offer to S_I and it gets rejected. Although offers are private, each player can observe the number of players remaining. Thus, next period, if B_1 finds that all four players are present he infers that this is due to an out-of-equilibrium play by B_2 . Using Bayes' rule he attaches the following probabilities to E_1 , E_2 and E_3 respectively.

$$\begin{aligned} \frac{1}{1 + \lambda + \lambda^2} & \text{ to } E_1 \\ \frac{\lambda}{1 + \lambda + \lambda^2} & \text{ to } E_2 \\ \frac{\lambda^2}{1 + \lambda + \lambda^2} & \text{ to } E_3 \end{aligned}$$

As $\lambda \rightarrow 0$, the probability attached to E_1 goes to 1. Thus B_1 believes that his equilibrium offer of $\delta^t H$ to S_I was rejected and the updated belief is d_{t-1} . Also, when S_M does not get an offer, then next period if all four players are present then S_M 's belief is also d_{t-1} . In the case of E_1 or E_2 the beliefs of B_1 and B_2 coincide. However, in the case of E_3 they differ. Suppose E_3 occurs and B_1 's equilibrium offer to S_I gets rejected. Then next period all four players will be present and given L -type S_I 's behaviour, the belief of B_2 will be $\pi = 0$ and that of B_1 will be $\pi = d_{t-1}$. In that contingency it is an optimal response of B_2 to offer to S_M as in the case with belief equal to d_{t-1} . This is because he knows that B_1 is playing his equilibrium strategy with the belief d_{t-1} .

Next we will argue that the L -type S_I finds it optimal to accept an offer higher than $\delta^t H$ with probability 1, if she gets two offers. This is because in the event when she gets two offers she knows that rejection will lead the buyer B_1 to play according to the belief d_{t-1} and, two periods from now, the remaining buyer will offer $\delta^{t-2} H$ to S_I . Thus her continuation payoff from rejection is

$$\delta\{\delta^{t-1} H q(d_{t-1}) + \delta(1 - q(d_{t-1}))\delta^{t-2} H\} = \delta\{\delta^{t-1} H\} = \delta^t H$$

Hence she finds it optimal to accept an offer higher than $\delta^t H$ with probability 1.

We need to check that B_2 has no incentive to deviate and make an offer of p^o to S_I such that $p^o > \delta^t H$.

Suppose B_2 deviates and makes an offer of p^o to S_I such that $p^o > \delta^t H$. With probability $q(\pi)$, S_I will get two offers and B_2 's will be accepted with probability π . With probability $(1 - q(\pi))$, S_I will get only one offer. B_2 then gets a payoff of

$$(v - p^o)q(\pi)\pi + (1 - q(\pi))[(v - p^o)a(\pi) + (1 - a(\pi))\delta v_B(d_{t-1})]$$

For high values of δ we have $(v - p^o)a(\pi) + (1 - a(\pi))\delta v_B(d_{t-1}) < \delta v_B(\pi)$. Also for high values of δ , $p^o > p'_l(\pi)$. Thus²⁷,

$$\begin{aligned} v_B(\pi) &= (v - p'_l(\pi))q(\pi) + (1 - q(\pi))\delta v_B(\pi) \\ &> (v - p^o)q(\pi)\pi + (1 - q(\pi))[(v - p^o)a(\pi) + (1 - a(\pi))\delta v_B(d_{t-1})] \end{aligned}$$

Hence B_2 has no incentive to deviate and make an offer of p^o to S_I .

Lastly, to show that B_2 has no incentive to deviate and make an unacceptable offer to S_M or offer p^0 to S_I such that $p^0 < \delta^t H$ we refer to the analysis described in appendix A.2.

²⁷This is because B_2 puts a mass point at $p'_l(\pi)$