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PEBBLING IN SPLIT GRAPHS*

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Abstract. Graph pebbling is a network optimization model for transporting discrete resources that are consumed in transit: the movement of 2 pebbles across an edge consumes one of the pebbles. The pebbling number of a graph is the fewest number of pebbles t so that, from any initial configuration of t pebbles on its vertices, one can place a pebble on any given target vertex via such pebbling steps. It is known that deciding whether a given configuration on a particular graph can reach a specified target is NP-complete, even for diameter 2 graphs, and that deciding whether the pebbling number has a prescribed upper bound is Π_2^{P}-complete. On the other hand, for many families of graphs there are formulas or polynomial algorithms for computing pebbling numbers; for example, complete graphs, products of paths (including cubes), trees, cycles, diameter 2 graphs, and more. Moreover, graphs having minimum pebbling number are called Class 0, and many authors have studied which graphs are Class 0 and what graph properties guarantee it, with no characterization in sight. In this paper we investigate an important family of diameter 3 chordal graphs called split graphs; graphs whose vertex set can be partitioned into a clique and an independent set. We provide a formula for the pebbling number of a split graph, along with an algorithm for calculating it that runs in $O(n^\beta)$ time, where $\beta = 2\omega/(\omega+1) \cong 1.41$ and $\omega \cong 2.376$ is the exponent of matrix multiplication. Furthermore we determine that all split graphs with minimum degree at least 3 are Class 0.

Key words. pebbling number, split graphs, class 0, graph algorithms, complexity

AMS subject classifications. 05C85, 68Q17, 90C35

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1. Introduction.

Graph pebbling is a network optimization model for transporting discrete resources that are consumed in transit: while 2 pebbles cross an edge of a graph, only one arrives at the other end as the other is consumed (or spent on toll, one can imagine). This operation is called a *pebbling step*. The basic questions in the subject revolve around deciding whether a particular configuration of pebbles on the vertices of a graph can *reach* a given *root* vertex via pebbling steps (for this reason, graph pebbling is carried out on connected graphs only). If a configuration can reach r, it is called r-*solvable*, and r-*unsolvable* otherwise.

Various rules for pebbling steps have been studied for years and have found applications in a wide array of areas. One version, dubbed *black and white* pebbling, was applied to computational complexity theory in studying time-space tradeoffs (see [15, 28]), as well as to optimal register allocation for compilers (see [30]). Connections have also been made to pursuit and evasion games and graph searching (see [21, 27]). Another version (*black* pebbling) is used to reorder large sparse matrices to minimize in-core storage during an out-of-core Cholesky factorization scheme (see [12, 22, 24]). A third version yields results in computational geometry in the rigidity of graphs,

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matroids, and other structures (see [13, 31]). The rule we study here originally produced results in combinatorial number theory and combinatorial group theory (the existence of zero sum subsequences—see [4, 11]) and have recently been applied to finding solutions in p-adic diophantine equations (see [23]). Most of these rules give rise to computationally difficult problems, which we discuss for our case below.

We follow fairly standard graph terminology (e.g., [32]), with a graph $G = (V, E)$ having $n = n(G)$ vertices $V = V(G)$ and having edges $E = E(G)$. The *eccentricity* $\mathsf{ecc}(G, r)$ for a vertex $r \in V$ equals $\max_{v\in V} \mathsf{dist}(v, r)$, where $\mathsf{dist}(x, y)$ denotes the length (number of edges) of the shortest path from x to y; the *diameter* $\mathsf{diam}(G) = \max_{r\in V} \mathsf{ecc}(G, r)$. When G is understood we will shorten our notation to $\mathsf{ecc}(r)$.

The most studied graph pebbling parameter, the one investigated here, is the *pebbling number* $\pi(G) = \max_{r\in V} \pi(G, r)$, where $\pi(G, r)$ is defined to be the minimum number t so that every configuration of size at least t is r-solvable. The *size* $|C|$ of a configuration $C : V \to \mathbb{N} = \{0, 1, \ldots\}$ is its total number of pebbles $\sum_{v\in V} C(v)$. Simple lower bounds like $\pi(G) \geq n$ (sharp for complete graphs, cubes, and, probabilisticaly, almost all graphs) and $\pi(G) \geq 2^{\mathsf{diam}(G)}$ (sharp for paths and cubes, among others) are easily derived. Graphs satisfying $\pi(G) = n$ are called *Class* 0 and are a topic of much interest (e.g., [2, 3, 5, 6, 9, 10]). Surveys on the topic can be found in [16, 17, 19], and include variations on the theme such as k-pebbling, fractional pebbling, optimal pebbling, cover pebbling, and pebbling thresholds, as well as applications to combinatorial number theory and combinatorial group theory.

Computing graph pebbling numbers is difficult in general. The problem of deciding whether a given configuration of pebbles on a graph can reach a particular vertex was shown in [14, 20] to be NP-complete (via reduction from the problem of finding a perfect matching in a 4-uniform hypergraph). The problem of deciding whether a graph G has a pebbling number of at most k was shown in [14] to be Π_2^{P}-complete.[1]

On the other hand, pebbling numbers of many graphs are known: for example, cliques, trees, cycles, cubes, diameter 2 graphs, graphs of connectivity exponential in its diameter, and others. In particular, in [26] the pebbling number of a diameter 2 graph G was determined to be n or $n + 1$. Moreover, [5] characterized those graphs having $\pi(G) = n + 1$ (a slight error in the characterization was corrected by [3]). All such connectivity 1 graphs have $\pi(G) = n+1$. The smallest such 2-connected graph is the *near-Pyramid* on six vertices, which is the 6-cycle (r, a, p, c, q, b) with an extra two or three of the edges of the triangle (a, b, c) (the *Pyramid* has all three). All diameter 2 graphs with pebbling number $n+1$ can be described by adding simple structures to the near-Pyramid. It was shown in [14] that one can recognize such graphs in quartic time.

Here we begin to study for which graphs their pebbling numbers can be calculated in polynomial time. Aiming for tree-like structures (as was considered in [6]), one might consider chordal graphs of various sorts. Moving away from diameter 2, one might consider diameter 3 graphs; recently in [29], the tight upper bound of $\lfloor 3n/2\rfloor+2$ has been shown for this class. Combining these two thoughts, we study split graphs in this paper, and find that their pebbling numbers can be calculated quickly, in fact, in $O(n^{1.41})$ time.[2]

Split graphs can be described by adding simplicial vertices (*cones*) to a fixed clique. In other words, a graph is a *split* graph if its vertices can be partitioned into

[1] That is, complete for the class of problems computable in polynomial time by a $\mathsf{co\text{-}NP}$ machine equipped with an oracle for an NP-complete language.

[2] Here $\beta \cong 1.41$ satisfies $\beta = 2\omega/(\omega+1)$, where $\omega \cong 2.376$ is the exponent of matrix multiplication.

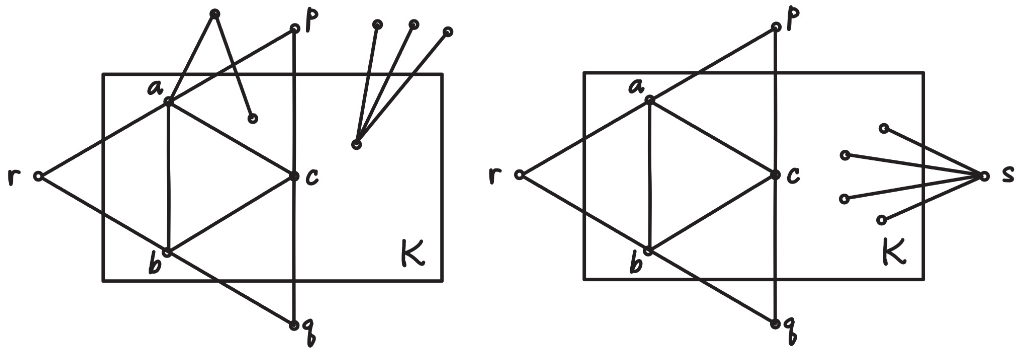

FIG. 1. *Examples of Pereyra (left) and Phoenix (right) graphs.*

an independent set S and a clique K. Notice that the Pyramid is a split graph with clique $\{a, b, c\}$ and cone vertices r, p, and q. The Pyramid plays a key role in the theory of split graphs. However, the Pyramid has diameter 2, and we are interested in diameter 3 split graphs.

It turns out that Pereyra and Phoenix graphs (which we define below and which necessarily contain the Pyramid) are important for our work (see Figure 1). We say that G has a Pyramid if there exist three cone vertices with degree 2 whose neighborhoods do not have the Helly property (that is, their neighborhoods form a triangle). We say that the subgraph induced by the closed neighborhoods of the three cone vertices is a Pyramid of G. If r is one of the three cone vertices, we say it is an r-Pyramid. A graph G is called r-*Pereyra* if it has an r-Pyramid, none of whose vertices is a cut vertex of G. Denote by $\delta^*(G, r)$ the minimum degree among all vertices at maximum distance from r. A graph G is r-*Phoenix* if it is r-Pereyra, $\mathsf{ecc}(r) = 3$, and $\delta^*(G, r) \geq 4$. A *Pereyra* (resp., *Phoenix*) graph is r-Pereyra (resp., r-Phoenix) for some r.

Like the Pyramid, an r-Pereyra graph having $\mathsf{ecc}(r) = 2$ has pebbling number one more than "normal"; that is, it is an exception to how most of the graphs in its class behave. On such G, the configuration that places 3 pebbles on p and q, 0 pebbles on r, a, b, and c, and 1 everywhere else is r-unsolvable, showing that $\pi(G, r) \geq n + \mathrm{x} + 1$, where x is the number of cut vertices of G.[3] (In the course of proving Theorem 5, one finds that this configuration is the unique r-unsolvable configuration of size $n + \mathrm{x}$ on G.) We will find analogous behavior for r-Phoenix graphs as well.

We note that the pebbling number of a split graph can always be realized at a cone root.

PROPOSITION 1. *If G is a split graph, then there is some cone vertex r such that $\pi(G) = \pi(G, r)$.*

Proof. We choose S to be maximal and let $r' \in K$. Because S is maximal we know that r' has a cone neighbor, say r. We show that $\pi(G, r) \geq \pi(G, r')$ by proving that if C' is an r'-unsolvable configuration, then there is an r-unsolvable configuration C with $|C| = |C'|$.

Suppose that C' is r'-unsolvable and define C by $C(r) = C'(r') = 0$, $C(r') = C'(r) \leq 1$, and $C(x) = C'(x)$ for all other x. For the sake of contradiction suppose that σ is a minimal sequence of pebbling moves from C that places a pebble on r. If

[3]Note that we use the roman font x as a number, while the math font x will denote a vertex.

ever a step of σ moves a pebble to r', then a subsequence of σ places a pebble on r' from C' restricted to $G - r$, a contradiction. Hence, no pebbles ever reach r' in σ. Moreover, no pebbles ever leave r' in σ since that would require a pebble to reach r' first. Thus σ moves a pebble to r from C restricted to $G - r'$, and the final step of σ moves a pebble from some $r'' \neq r'$ to r. Let σ' be the same as σ except with the step from r'' to r replaced by the step from r'' to r'. Then σ' places a pebble on r' from C' (because no pebble on r is used) restricted to $G - r$, a contradiction. ∎

Notationally, we abbreviate $\mathsf{deg}(x)$ by d_x. We also abbreviate $N(x)$ by N_x (so that $d_x = |N_x|$), with $[N_x]$ denoting $N_x \cup \{x\}$. If $v \in S$, we define $K_v = K - N_v$. We denote the set of cut vertices of G by X, with $\mathrm{x} = |X|$. For a set U of vertices, we write $C(U) = \sum_{x \in U} C(x)$, and define $U^i = \{u \in U \mid C(u) = i\}$. For a list of vertices, we denote $C(x_1, \ldots, x_k) = (C(x_1), \ldots, C(x_k))$. We say that a graph is r-*(semi)greedy* if every configuration of size at least $\pi(G, r)$ has a (semi)greedy r-solution; that is, every pebbling step in the solution decreases (does not increase) the distance of the moved pebble to r. Note that any step from a cone vertex to one of its neighbors is semigreedy.

Proposition 2. *Every split graph is semigreedy.*

Proof. We show that any sequence of pebbling steps that places a pebble on r can be converted to one which is semigreedy. Let σ be such a sequence—remove unnecessary steps so that it is minimal— and suppose that it is not semigreedy. Then there are vertices u and v with $\mathsf{dist}(r, u) < \mathsf{dist}(r, v)$ for which σ moves a pebble from u to v. Because $\mathsf{dist}(r, u) > 1$ and $\mathsf{dist}(r, v) \leq 3$, we have $\mathsf{dist}(r, u) = 2$ and $\mathsf{dist}(r, v) = 3$, so $v \in S$. By minimality σ must move a pebble from v to some neighbor $w \neq u$. However, the replacement of the two steps u to v and v to w by the single step u to w creates an r-solvable sequence with fewer nonsemigreedy steps. Repeating this process produces a semigreedy r-solution. ∎

We begin by outlining in section 2 a rather new technique for finding upper bounds on π using weight functions. From there we prove pebbling number results in the case that $\mathsf{ecc}(r) = 2$. We prepare in section 4 preliminary lemmas that will be used in section 5 to prove pebbling results for the $\mathsf{ecc}(r) = 3$ case. In section 6 we prove Theorem 18, which gives an exact formula for the pebbling number of split graphs. Using this we prove our main result, Corollary 19, which shows that pebbling numbers for split graphs can be calculated in polynomial time. From this analysis we learn that all split graphs with minimum degree at least 3 are Class 0. We end with some comments and conjectures in section 7.

2. The weight function lemma.

In this section we describe a tool developed in [18] for calculating upper bounds for pebbling numbers of graphs that will be useful in delivering a quick proof of Theorem 4.

Let G be a graph and let T be a subtree of G, with at least two vertices, rooted at vertex r. For a vertex $v \in V(T)$ let v^+ denote the *parent* of v; i.e., the T-neighbor of v that is one edge closer to r (we also say that v is a *child* of v^+). We call T a *strategy* when we associate with it a nonnegative, nonzero *weight function*[4] $\mathsf{w} : V(T) \to \mathbb{Q}$ with the property that $\mathsf{w}(r) = 0$ and $\mathsf{w}(v^+) \geq 2\mathsf{w}(v)$ for every other vertex that is not a neighbor of r (and $\mathsf{w}(v) = 0$ for vertices not in T). We extend w to a function on configurations by defining $\mathsf{w}(C) = \sum_{v \in V} \mathsf{w}(v)C(v)$. Now denote by $\mathbf{T}$ the configuration with $\mathbf{T}(r) = 0$, $\mathbf{T}(v) = 1$ for all $v \in V(T)$, and $\mathbf{T}(v) = 0$

[4]The definition in [18] uses $\mathbb{N}$, although Lemma 3 clearly holds with rationals $\mathbb{Q}$ as well.

everywhere else. The following was proven in [18].

Lemma 3 (weight function lemma). *Let T be a strategy of G rooted at r, with associated weight function* w. *Suppose that C is an r-unsolvable configuration of pebbles on $V(G)$. Then* $\mathsf{w}(C) \leq \mathsf{w}(\mathbf{T})$.

The manner in which one uses this lemma to obtain a pebbling number upper bound is as follows. If we have several strategies $T_1, \ldots, T_m$ of G, each rooted at r, with associated weight functions $\mathsf{w}_1, \ldots, \mathsf{w}_m$ and configurations $\mathbf{T}_1, \ldots, \mathbf{T}_m$, then we can define the accumulated weight function $\mathsf{w} = \sum_{i=1}^{m} \mathsf{w}_i$ and the accumulated configuration $\mathbf{T} = \sum_{i=1}^{m} \mathbf{T}_i$, and have that $\mathsf{w}(C) \leq \mathsf{w}(\mathbf{T})$ for every r-unsolvable configuration C. Moreover, if it so happens that $\mathsf{w}(v) \geq 1$ for all $v \in V - \{r\}$, then we also have $|C| \leq \mathsf{w}(C)$, from which follows $\pi(G, r) \leq \lfloor \mathsf{w}(\mathbf{T}) + 1 \rfloor$.

The use of the weight function lemma is a little bit like the pigeonhole principle. If a configuration has more pebbles than the upper bound provided by a set of strategies on a graph G with root r, then the inequality for some strategy fails. Thus the configuration can be solved on the tree corresponding to that strategy. One use of this is the following.

Suppose that the set of strategies is given by trees that faithfully represent the distances from r in G; for example, breadth-first search trees. Further assume that the upper bound generated by the strategies equals $\pi(G, r)$. Then because every minimal solution on a tree is greedy on that tree, it is also greedy in G. This would prove that G is r-greedy. We use this argument in Theorem 4 below.

3. Eccentricity two.

For a split graph G define $\mathrm{x}^r = |X - \{r\}|$.

Theorem 4. *If $r \in K$, then G is r-greedy and $\pi(G, r) = n + \mathrm{x}^r$.*

Proof. The lower bound is given by the configuration having 0 on r and every cut vertex, 3 on one leaf per vertex in $X - \{r\}$, and 1 everywhere else. The upper bound can be proved by using the weight function lemma as follows.

For every neighbor r' of r we define a strategy $T_{r'}$. If $r' \in X$, then give it weight 2. Include all of its neighbors outside of K, giving them weight 1 each. If $r' \notin X$, then give it weight 1. For every vertex s not yet in some strategy (necessarily not in K; also $d_s \geq 2$), choose neighbors s' and s'' and include s in both strategies $T_{s'}$ and $T_{s''}$ with weight 1/2 each. The resulting sum of strategies has weight 2 on every vertex in $X - \{r\}$, and weight 1 everywhere else. Hence $\pi(G, r) \leq n + \mathrm{x}^r$.

Greediness follows because every strategy used is r-greedy. ∎

We recall from the theorem of [3, 5] that if G is a diameter 2 graph, then, if it has connectivity 1, we then have $\pi(G) = n + 1$, and if it is 2-connected, we then have $\pi(G) = n + 1$ if and only if G is a member of the following special class of graphs $\mathcal{F}$. First, $\mathcal{F}$ contains the Pyramid P, as well as $P - e$ for any edge e of the triangle (a, b, c). Notice that these graphs have the following *separation property*: $\{a, b\}$ separates r from c, $\{b, c\}$ separates q from a, and $\{a, c\}$ separates p from b. Next, $\mathcal{F}$ is closed by adding cones over pairs or triples from $\{a, b, c\}$. Finally, $\mathcal{F}$ is closed by adding edges between cone vertices, provided that we maintain the separation property. Thus, if G is a 2-connected split graph of diameter 2, then $G \in \mathcal{F}$ if and only if G is Pereyra. In particular, we obtain the diameter 2 case of Theorem 5 below.

For a cone vertex r, we have two cases since $\mathsf{ecc}(r) \in \{2, 3\}$. We first note that, in the case $\mathsf{ecc}(r) = 2$, every r-unsolvable configuration C has $C(v) \leq 3$ for all v. In particular, the solution moving pebbles directly to r from a vertex with $C(v) \geq 4$ is greedy. Recall that $\mathrm{x} = |X|$.

Theorem 5. *If r is a cone vertex with* $\mathsf{ecc}(r) = 2$, *then $\pi(G, r) = n + \mathrm{x} + \psi$, where $\psi = \psi(G, r)$ is 1 if G is r-Pereyra and 0 otherwise.*

Proof. The lower bound for non–r-Pereyra graphs is given by the following two unsolvable configurations having size $n+\mathrm{x}-1$. The first, when r is the only leaf, has 0 on r and its neighbor r', 3 on some $x \neq r'$, and 1 everywhere else. Otherwise, the second has 0 on r and every cut vertex, 3 on one leaf per vertex in $X - \{r\}$, and 1 everywhere else. For r-Pereyra graphs we place 0 on r, a, b, and c, 3 on p and q, and 1 everywhere else ($X = \emptyset$ because $\mathsf{ecc}(r) = 2$).

We first prove the upper bound directly for r-Pereyra graphs. If G is r-Pereyra, then $N_r = \{a, b\}$, and since $\mathsf{ecc}(r) = 2$, we have $\mathrm{x} = 0$ and $[N_x] \cap \{a, b\} \neq \emptyset$ for all x. If C is r-unsolvable of size $|C| = n+1$, then $C(r) = 0$ and some $C(x) \geq 2$ with, say, $a \sim x$. Thus $C(a) = 0$, and also $C(y) \leq 1$ for all $y \in N_a$. Now we have $n+1$ pebbles on $n-2$ vertices, which means there must be another vertex z, with $b \sim z \not\sim a$, having $C(z) \geq 2$, and so $C(b) = 0$. This puts the $n+1$ pebbles on just $n-3$ vertices, which can only happen if $C(r, a, b, x, z) = (0, 0, 0, 3, 3)$ and $C(y) = 1$ for all other y. But this allows us to solve r by moving a pebble from x to a, from z to a common neighbor of z and a and then to a, and finally from a to r. This contradiction means that every configuration of size $n+1$ is r-solvable.

Next, we prove the upper bound for non–r-Pereyra G. Let C be a configuration of size $n+\mathrm{x}$. We argue by induction on the number of cone vertices that C is r-solvable. The base case is any graph with at most two cones. Such a graph has diameter 2, for which we already noted in the paragraph following the proof of Theorem 4 that the result is true. Hence we may assume that there are at least three cones and, moreover, that $\mathsf{diam}(G) = 3$. This means that $d_r \geq 2$ because, otherwise, $\mathsf{ecc}(r) = 2$ would require that every vertex is adjacent to the neighbor of r. Moreover, $\mathsf{diam}(G) = 3$ implies that there are at least two cones different from r whose neighborhoods are disjoint.

If a cone vertex $v \neq r$ has the property that $G - v$ is r-Pereyra, then we say that v is *bad*; otherwise, it is a *good* cone vertex. Notice that a bad cone vertex is necessarily a leaf adjacent to a neighbor of r; in addition, it is the unique such leaf and $d_r = 2$.

Suppose, for the sake of contradiction, that C is not r-solvable. Let $v \neq r$ be any cone vertex, and define $G' = G - v$, with $C' = C$ on G' and $C'(v) = 0$. Also define $\mathrm{x}' = \mathrm{x}(G')$ and $\psi' = \psi(G')$. Because C' is r-unsolvable on G', we have $n - C(v) + \mathrm{x} = |C'| < \pi(G', r)$. By induction, $\pi(G', r) = (n-1) + \mathrm{x}' + \psi' \leq (n-1) + \mathrm{x}$ whether v is good or bad: if v is good, it holds because $\mathrm{x}' \leq \mathrm{x}$ and $\psi' = 0$, and if v is bad, it holds because $\mathrm{x}' = 0$, $\psi' = 1$, and $\mathrm{x} = 1$. Therefore, we may assume that $C(v) \geq 2$.

If $C(v) = 2$, then move a pebble from v to one of its neighbors to form C^*. Then C^* is a configuration on G' of size $n - 1 + \mathrm{x}$, which by induction is r-solvable. On the other hand, if $C(v) \geq 3$, then $C(v) = 3$. We can make the above argument for each cone vertex; thus we may assume that $C(v) = 3$ for every cone vertex different from r. Hence no neighbor of r is adjacent to more than one cone vertex, and every neighbor of r adjacent to some cone vertex must have no pebble. Furthermore, if some $x \in K$ has 2 pebbles, then we can move pebbles greedily from v to its common neighbor r' of r, from x to r', and then from r' to r. Hence we may assume that $C(x) \leq 1$ for all $x \in K$.

Recall that there are at least two cone vertices. If v is a cone vertex with neighbor v' having $C(v') \geq 1$, then move a pebble from another cone vertex u to its common neighbor u' of r. Then move a second pebble from v to v' to u' to r. Thus we must have $C(N_v) = 0$ for every cone vertex v.

We claim that the neighborhoods of cone vertices are pairwise disjoint. Indeed, suppose two cone vertices u and v have a common neighbor x. If there is a third cone vertex w (necessarily having 3 pebbles), then move one to its common neighbor w' of r. Then move pebbles from u and v to x, then from x to w' to r. Thus there are no other cone vertices. As mentioned above, if u and v are the only cone vertices, then N_u and N_v are disjoint. This proves the claim.

Now we may partition $G - r$ into closed neighborhoods of cone vertices and one extra part K' consisting of vertices of K adjacent to no cone. Notice that the above arguments show that $C([N_v]) = 3$ for every cone vertex v. Moreover, $3 = |[N_v]| + 1$ when $d_v = 1$ (i.e., $v' \in X$), and $3 \leq |[N_v]|$ otherwise. Also, $C(K') \leq |K'|$. Hence $|C| \leq n - 1 + \mathrm{x}$, a contradiction. ∎

We finish this section with a result that will be used to prove Theorem 13. Define $\pi_k(G, r)$ to be the minimum number of pebbles t so that from every configuration of size t one can move k pebbles to r (such a configuration is called *k-fold r-solvable*). For example, $\pi_1(G, r) = \pi(G, r)$.

Recall that $\mathrm{x}^r = |X - \{r\}|$.

THEOREM 6. *If $r \in K$ and $\delta = \delta^*(G, r)$, then*

$$\pi_2(G, r) = \begin{cases} n + \mathrm{x}^r + 4 & \textit{if} \quad \delta = 1, \\ n + 6 - \delta & \textit{if} \quad 1 < \delta < 4, \\ n + 2 & \textit{if} \quad \delta \geq 4. \end{cases}$$

Proof. Suppose $\delta = 1$. Choose s to be a vertex at distance 2 from r with $d_s = \delta$. The lower bound is given by the following configuration C of size $n + \mathrm{x}^r + 3$ that is not 2-fold r-solvable: we place 0 pebbles on r and each cut vertex, 7 on s, 3 on one leaf per vertex in $(X - \{r\}) - N_s$, and 1 everywhere else. Evidently, the only pebble that can reach r comes from four that are on s.

For the upper bound, we assume that C is a configuration of size $n + \mathrm{x}^r + 4$ that cannot place 2 pebbles on r. If we can place one pebble on r using at most 3 pebbling steps, then Theorem 4 says we can place another on r with the remaining $n + \mathrm{x}^r$ pebbles, so we suppose otherwise.

This means that $C(x) \leq 1$ for all $x \in K$, $C(x) \leq 3$ for all x, $C(N_x) = 0$ for all $x \in S^+ = S^2 \cup S^3$, and $N_x \cap N_y = \emptyset$ for all $x, y \in S^+$. Now every $x \in S^+$ satisfies $|[N_x]| + 1 \geq 3 \geq C(x) = C([N_x])$, with equality if and only if x is a leaf. Hence, with L denoting the set of leaves, $L^+ = L \cap S^+$ and $U = V - \cup_{x \in S^+}[N_x]$, we have

$$\begin{aligned} |C| &= \sum_{x \in L^+} C([N_x]) + \sum_{x \in S^+ - L^+} C([N_x]) + \sum_{x \in U} C(x) \\ &\leq \sum_{x \in L^+} (|[N_x]| + 1) + \sum_{x \in S^+ - L^+} |[N_x]| + (|U| - 1) \\ &\leq n + \mathrm{x}^r - 1, \end{aligned}$$

a contradiction.

Now suppose that $1 < \delta < 4$—notice that $\mathrm{x}^r = 0$ when $\delta > 1$. The lower bound comes from the configuration that places 7 on s, 0 on r and N_s, and 1 everywhere else, having size $n + 5 - \delta$. Once again, the only pebble that can reach r comes from four that are on s.

The very same upper bound argument above works here when $\delta = 2$, so we assume that $\delta = 3$, whereby C has size $n + 3$. Suppose C is not 2-fold r-solvable. Then by Theorem 4 we have $\pi_1(G, r) = n$; it must be that:

1. $C(r) = 0$;

2. $C(x) \le 1$ for every $x \in K$;
3. if $x \in S$ and $C(x) \ge 2$, then $C(N_x) = 0$;
4. (by induction) $C(x) \ge 2$ for every $x \in S - \{s\}$; and
5. if there exists a vertex $x \ne s$ at distance 2 from r with $d_x = \delta$, then $C(s) \ge 2$.

Now, if there exists $x \in S - \{s\}$, then by part 4 we have $C(x) \ge 2$, and by part 3 we have $C(N_x) = 0$. Let $h \in N_x$, $h \ne r$, and consider $G' = G - h$. Notice that $\delta^*(G', r) \ge \delta - 1 = 2$ so that, by induction, $\pi_2(G', r) = n - 1 + 6 - \delta^*(G', r) \le n + 3 = |C|$. Thus C is 2-fold r-solvable, a contradiction.

Otherwise, $S - \{s\} = \emptyset$, and we can assume $K = \{r\} \cup N_s$. It follows that $n = 5$, $|C| = 8$, and C is 2-fold r-solvable, a contraction.

Finally, suppose that $\delta \ge 4$. In this case the lower bound comes from the configuration with 3 on s, 0 on r, and 1 everywhere else, having size $n + 1$. Here, the only pebble that can reach r comes from two on s.

For the upper bound, let C be a configuration of size $n + 2$ that is not 2-fold r-solvable. Since, by Theorem 4, we have $\pi(G, r) = n$, it must be that $C(r) = 0$, and $C(x) \le 1$ for every $x \in K$. We will use induction on $|K|$, with the base cases being all split graphs G and roots r such that $\delta^*(G, r) < 4$, for which we have just shown that the result is true. Note that as $|K|$ decreases, eventually $\delta^* < 4$.

Let $x \in K - \{r\}$. Because $\delta \ge 4$, $G - x$ is connected and has no cut vertices except possibly r. Denote $\delta' = \delta^*(G - x, r)$. Notice that $\delta' \ge \delta - 1$, and so, by the inductive hypothesis,

$$\pi_2(G - x, r) = \begin{cases} n - 1 + 6 - 3 = n + 2 & \text{when } \delta' = 3, \\ n - 1 + 2 = n + 1 & \text{when } \delta' \ge 4. \end{cases}$$

This implies that if $C(x) = 0$, then C is 2-fold r-solvable, a contradiction.

Therefore, $C(x) = 1$ for every $x \in K - \{r\}$, thus $C(S) = n + 2 - |K_r| = |S| + 3$. This means that in S there is a vertex with at least 4 pebbles or there are two vertices with at least 2 pebbles each. In both cases we can place 2 pebbles on r, a contradiction which completes the proof. ▯

4. Eccentricity three. In the case that $\mathsf{ecc}(r) = 3$, define $D_3(r)$ to be the set of vertices at distance 3 from r, with $\delta = \delta^*(G, r)$, and let $s \in D_3(r)$ be chosen to have $d_s = \delta$. Denote by S the set of cone vertices of G, with $S_v = S - \{v\}$ and $S_{rs} = S - \{r, s\}$. Also, let $K_v = K - N_v$ and $K_{rs} = K_r - N_s$, and define $X_{rs} = X \cap K_{rs}$, with $\mathrm{x}_{rs} = |X_{rs}|$. Now let X_0 be the set of cut vertices of N_r adjacent to some cone vertex in S_r, with $\mathrm{x}_0 = |X_0|$. Note that $\mathrm{x}_{rs} > 0$ implies $d_s = 1$.

Define the following four functions:

$$\begin{aligned} t_{rs}(G, r) &= n + \mathrm{x}_{rs} + 6 - d_r - d_s, \\ t_r(G, r) &= n + \mathrm{x}_{rs} + 2 - d_r, \\ t_s(G, r) &= n + \mathrm{x}_{rs} + \mathrm{x}_0 + 2 - d_s, \\ t_0(G, r) &= n + \mathrm{x}_{rs} + \mathrm{x}_0, \end{aligned}$$

and let $t(G, r) = \max\{t_\alpha(G, r) \mid \alpha \in \{rs, r, s, 0\}\}$. Notice that t is well defined: the selection of vertex s does not change the value of t. Furthermore, the choice of S in the split representation of G does not influence t either. Also, if G is r-Phoenix, then $d_r = 2$, $\mathrm{x}_0 = \mathrm{x}_{rs} = 0$, and $d_s \ge 4$, which yields $t(G, r) = n$ in this instance.

Next, define the following four configurations C_α of sizes $|C_\alpha| = t_\alpha(G, r) - 1$.

C_{rs}: 0 on r, N_r, N_s, X_{rs}; 7 on s, 3 on one leaf per cut vertex in X_{rs}; and 1 everywhere else.

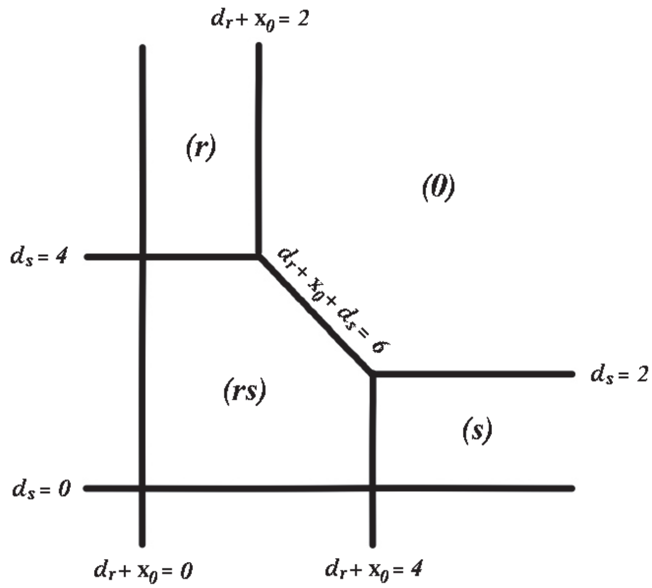

FIG. 2. *Graph of cases in Lemma* 8.

C_r: 0 on r, N_r, and X_{rs}; 3 on s and on one leaf per cut vertex in X_{rs}; and 1 everywhere else.

C_s: 0 on r, N_s, X_{rs}, and X_0; 3 on s and on one leaf per cut vertex in $X_{rs} \cup X_0$; and 1 everywhere else.

C_0: 0 on r, X_{rs}, and X_0; 3 on one leaf per cut vertex in $X_{rs} \cup X_0$; and 1 everywhere else.

Also, in the case that G is r-Phoenix, define the configuration C_P by placing 0 on $\{r, a, b, c\}$, 3 on p and q, and 1 everywhere else. Notice that C_P witnesses that $\pi(G) \geq n + 1$ for every r-Phoenix graph G.

LEMMA 7. *Each C_α is r-unsolvable.*

Proof. For $\alpha \in \{s, 0\}$, C_α is r-unsolvable because the only pebbling moves available are from the cones with 3 pebbles to K, and after those no pebbling move is available. In C_r, the only move available is from s to some $v \in N_s$, and then from v along any path to some $u \in N_r$, at which point no more moves are available. In C_{rs}, the leaves with 3 pebbles can only move to their neighbors, at which point they stop. Then s can only move 3 to its neighbor, at which point it can travel along any path to some neighbor of r and stop there. Finally, as mentioned in the proof of Theorem 5, C_P is r-unsolvable on r-Pereyra graphs. ∎

LEMMA 8. *With the values of t_α defined above, we list when (if and only if) each is largest.*

(*rs*) $t_{rs} \geq t_\alpha$ *for all* $\alpha \in \{s, r, 0\}$ *when* $d_s \leq 4$, $d_r + \mathrm{x}_0 \leq 4$, *and* $d_r + d_s + \mathrm{x}_0 \leq 6$;

(*r*) $t_r \geq t_\alpha$ *for all* $\alpha \in \{rs, s, 0\}$ *when* $d_s \geq 4$ *and* $d_r + \mathrm{x}_0 \leq 2$;

(*s*) $t_s \geq t_\alpha$ *for all* $\alpha \in \{rs, r, 0\}$ *when* $d_r + \mathrm{x}_0 \geq 4$ *and* $d_s \leq 2$;

(0) $t_0 \geq t_\alpha$ *for all* $\alpha \in \{rs, s, r\}$ *when* $d_r + d_s + \mathrm{x}_0 \geq 6$, $d_r + \mathrm{x}_0 \geq 2$, *and* $d_s \geq 2$.

Proof. The proof is easy to check (see Figure 2). ∎

The next lemma shows how the function t changes when some vertex is removed. We say that a vertex v has a *false twin* if there exists v' nonadjacent to v such that

$N_v = N_{v'}$.

LEMMA 9. *Let $v \in S_{rs}$. Then:*

1. *If $d_v \geq 2$, then $t(G-v,r) = t(G,r) - 1$.*
2. *If $d_v = 1$ and v has at least one false twin different from r, then $t(G-v,r) = t(G,r) - 1$.*
3. *If $d_v = 1$ and r is the only false twin vertex of v, then $t(G-v,r) \leq t(G,r)-1$.*
4. *If $d_v = 1$, v has no false twins, and $N_v \subseteq X_{rs}$, then $t(G-v,r) = t(G,r) - 2$.*
5. *If $d_v = 1$, v has no false twins, and $N_v \subseteq X_0$, then $t(G-v,r) \leq t(G,r) - 1$.*

Proof. The proof follows from Lemma 8. ∎

COROLLARY 10. *If $v \in S_{rs}$, then $t(G-v,r) \leq t(G,r) - 1$.*

LEMMA 11. *If $d_r \geq 2$, $x \in N_r$, and $N_x \cap S = \{r\}$, then $t(G-x,r) \leq t(G,r)$.*

Proof. The proof follows from Lemma 8. ∎

LEMMA 12. *Let G be non–r-Phoenix, $\delta = \delta^*(G,r)$, and assume there exists $v \in S_r$ such that $G' = G - v$ is r-Phoenix. Then exactly one of the following statements is true.*

1. *v is the only vertex of G with degree 1, and $N_v \subseteq N_r$. In this case, $d_r(G) = d_r(G') = 2$, $\delta \geq 4$, $\mathrm{x}_{rs} = 0$, and $\mathrm{x}_0 = 1$; thus $t(G,r) = n+1$.*
2. *$\delta \leq 3$ and v is the only vertex of $D_3(r)$ with $d_v = \delta$. In this case,*

$$t(G,r) = \begin{cases} n+3 & \textit{if } \delta = 1, \\ n+4-\delta & \textit{if } 2 \leq \delta \leq 3. \end{cases}$$

In both cases, if $w \neq r$ is a cone vertex of an r-Pyramid of G, then $G - w$ is not r-Phoenix and $t(G-w,r) = t(G,r) - 1$.

Proof. This follows from the definition of r-Phoenix and from Lemma 8. ∎

THEOREM 13. *If r is a cone vertex with $\mathsf{ecc}(r) = 3$, then $\pi(G,r) = t(G,r) + \phi(G,r)$, where $\phi(G,r) = 1$ if G is r-Phoenix and 0 otherwise.*

5. Proof of Theorem 13.

The lower bound is given by Lemma 7. The upper bound follows by induction on $n = |V(G)|$. The theorem is trivially true if $n = 4$. Suppose that G is a graph with at least 5 vertices, r a cone vertex with $\mathsf{ecc}(r) = 3$, and C a configuration on G of size (without loss of generality) exactly $t = t(G,r) + \phi(G,r)$. We assume, for the sake of contradiction, that C is not r-solvable; in particular, $C(r) = 0$. Among vertices in $D_3(r)$, let s be chosen to have the minimum degree $\delta = \delta^*(G,r)$ and, among such vertices, having the maximum number of pebbles.

5.1. *G* is *r*-Phoenix.

Since G is r-Phoenix, then $t(G,r) = n$ and so $|C| = n+1$. Let $p \in S$ be a cone vertex of an r-Pyramid such that $N_p = \{a,c\}$. It is clear that $C(p) \leq 3$. By Lemma 9(1), $t(G-p,r) = t(G,r)-1$. Thus, by the inductive hypothesis, we have $\pi(G-p,r) = t(G-p,r) + \phi(G-p,r) = t(G,r) - 1 + \phi(G-p,r) \leq t(G,r) = n$.

If $C(p) = 2$, then we can move a pebble from p to N_p, and if $C(p) = 1$, then we do nothing. In each case we have created a configuration C' on $G-p$ of size n, which implies that C', and hence C is r-solvable, a contradiction. So we may assume that $C(p) = 3$ and, by an analogous argument, that $C(q) = 3$, where q is a cone vertex of the r-Pyramid such that $N_q = \{b,c\}$. Moreover, we can assume that p and q are the only cone vertices with degree 2 whose neighborhoods are $\{a,c\}$ or $\{b,c\}$. It follows that the graph $G-p$ is not r-Phoenix, so $\phi(G-p,r) = 0$.

Then, as above, we obtain $\pi(G-p,r) = t(G,r) - 1 + \phi(G-p,r) = t(G,r) - 1 = n-1$. Moving a pebble from p to N_p, we obtain a configuration C' on $G-p$ of size $n+1-3+1 = n-1$, which implies that C', and hence C is r-solvable, a contradiction.

5.2. G is not r-Phoenix. Since G is not r-Phoenix, we have $|C| = t(G,r) + \phi(G,r) = t(G,r) + 0 = t(G,r)$. We break this into two cases, subsections 5.2.1 and 5.2.2.

5.2.1. $G - v$ is r-Phoenix for some $v \in S_r$. We consider the two different cases of Lemma 12.

1. The first case of Lemma 12 has $t(G,r) = n+1$ and v at distance 2 of r; thus $C(v) \le 3$.
   (a) If $C(v) \le 2$, we obtain a configuration C' of $G-v$ with at least $|C|-1 = t(G,r) - 1 = n$ pebbles. Since $G - v$ is r-Phoenix, $t(G-v,r) = n-1$, then, by the inductive hypothesis, $\pi(G-v,r) = t(G-v,r)+\phi(G-v,r) = n-1+1 = n$. This means that C', and so C, is r-solvable, a contradiction.
   (b) If $C(v) = 3$, let $w \ne r$ be a cone vertex of an r-Pyramid having distance 2 from v. It is clear that $C(w) \le 1$; thus we obtain a configuration $|C'|$ of $G-w$ with at least $|C|-1 = t(G,r)-1 = n$ pebbles. By the observation at the end of Lemma 12, $t(G-w,r) = n+1-1 = n$ and $G-w$ is not r-Phoenix; then, by the inductive hypothesis, $\pi(G-w,r) = t(G-w,r) + \phi(G-w,r) = n+0 = n$. This means that C', and thus C, is r-solvable, a contradiction.
2. The second case of Lemma 12 has two options for $t(G,r)$, depending on the value of δ.
   (a) If $d_v = \delta = 1$, then $|C| = t(G,r) = n+3$. We can assume that $C(v) \le 7$.
      i. If $C(v) \le 6$, then, since by the inductive hypothesis $\pi(G-v,r) = t(G-v,r) + \phi(G-v,r) = n-1+1 = n$, it is easy to see that C is r-solvable, a contradiction.
      ii. If $C(v) = 7$, then let $w \ne r$ be a cone vertex of an r-Pyramid. It is clear that $C(w) \le 1$; thus we have a configuration C' on $G-w$ of size at least $n+2$. By the observation at the end of Lemma 12, $t(G-w,r) = n+3-1 = n+2$. Also, $G-w$ is not r-Phoenix so, by the inductive hypothesis, $\pi(G-w,r) = t(G-w,r) + \phi(G-w,r) = n+2+0 = n+2$. This means that C', and thus C, is r-solvable, a contradiction.
   (b) If $2 \le d_v = \delta \le 3$, then $|C| = t(G,r) = n+4-d$. Let p be a cone vertex of an r-Pyramid such that $N_p = \{a,c\}$ with $a \in N_r$. Since $G-p$ is not r-Phoenix, by Lemma 9 and the inductive hypothesis, we can assume that $C(p) = 3$. Thus we find the configuration C', equal to C on $G - \{w,r\}$, having size $|C|-3 = n+4-d-3 = n+1-d \ge n-2$. By Theorem 4 we have $\pi(G-\{p,r\},a) = n-2$, and so C' is a-solvable, implying that C is r-solvable, a contradiction.

5.2.2. For every $x \in S_r$, $G - x$ is not r-Phoenix. Recall that $s \in D_3(r)$ has the maximum number of pebbles among those vertices of $D_3(r)$ having $d_s = \delta$.

1. Some $v \in S_{rs}$ has $C(v) \le 2$. We obtain a configuration C' of $G-v$ with at least $|C|-1 = t(G,r)-1$ pebbles. By Corollary 10, $t(G-v,r) \le t(G,r)-1$, so by the inductive hypothesis $\pi(G-v,r) = t(G-v,r) + \phi(G-v,r) \le t(G,r)-1+0 = t(G,r)-1$. This means that C', and hence C, is solvable, a contradiction.
2. Some $v \in S_{rs}$ has $C(v) \ge 4$ and every other $u \in S_{rs}$ has $C(u) \ge 3$. Notice that we can assume that $v \in D_3(r)$, that $C(x) \le 3$ for every $x \in S_{rs} - \{v\}$, and that $C(y) = 0$ and $N_y \cap S = \{r\}$ for all $y \in N_r$ (in particular, $\mathrm{x}_0 = 0$).

Let $r' \in N_r$ and assume that $d_r = 1$. By Theorem 6 we have

$$\pi_2(G-r,r') = \begin{cases} n-1+\mathrm{x}_{rs}+1+4 = n+\mathrm{x}_{rs}+4 & \text{if} \quad \delta = 1, \\ n-1+6-\delta = n+5-\delta & \text{if} \quad 1<\delta<4, \\ n-1+2 = n+1 & \text{if} \quad \delta \geq 4. \end{cases}$$

By Lemma 8 (since $\mathrm{x}_{rs} = 0$ when $\delta > 1$) we also have

$$t(G,r) = \begin{cases} n+\mathrm{x}_{rs}-1-1+6 = n+\mathrm{x}_{rs}+4 & \text{if} \quad \delta = 1, \\ n-1-\delta+6 = n+5-\delta & \text{if} \quad 1<\delta<4, \\ n-1+2 = n+1 & \text{if} \quad \delta \geq 4. \end{cases}$$

Thus C can place 2 pebbles on r', then 1 on r, a contradiction. It follows that we can assume that $d_r \geq 2$, so that the graph $G - r'$ is connected.
The configuration C', the restriction of C to $G - r'$, has size $|C'| = |C| = t(G,r)$. By Lemma 11, $t(G-r',r) \leq t(G,r)$. Since $G - r'$ is not r-Phoenix, we know from the inductive hypothesis that $\pi(G-r',r) = t(G-r',r) + \phi(G-r',r) = t(G-r',r) \leq t(G,r)$. This means that C', and therefore C, is solvable, a contradiction.

3. $S_{rs} = \emptyset$ or every $v \in S_{rs}$ has $C(v) = 3$.
   (a) $\mathrm{x}_0 \geq 1$: Let w be a leaf adjacent to $r' \in N_r$. By Theorem 4, $\pi(G-r-w,r') = n-2+\mathrm{x}_{rs}+\gamma$, where $\gamma = 1$ when $d_s = 1$ and $\gamma = 0$ otherwise. We move a pebble from w to r' and consider the configuration C', the restriction of C to $G-r-w$, of size $t(G,r)-3$. Notice that when $d_s = 1$ we have $t(G,r)-3 \geq t_s(G,r)-3 = n+\mathrm{x}_{rs}+\mathrm{x}_0-ds+2-3 = \pi(G-r-w,r')-\gamma+\mathrm{x}_0-d_s+1 \geq \pi(G-r-w,r')$, and that when $d_s > 1$ we have $t(G,r)-3 \geq t_0(G,r)-3 = n+\mathrm{x}_{rs}+\mathrm{x}_0-3 = \pi(G-r-w,r')-\gamma+\mathrm{x}_0-1 \geq \pi(G-r-w,r')$. Thus, in both cases, it is possible to move another pebble to r', a contradiction.
   (b) $\mathrm{x}_0 = 0$ and $\mathrm{x}_{rs} \geq 1$: Notice that in this case $C(s) \geq 3$. Let w be a leaf adjacent to $w' \in K_{rs}$.
      i. If w has no false twins, by the inductive hypothesis and Lemma 9(4), $\pi(G-w,r) = t(G-w,r) = t(G,r)-2$. We move a pebble from w to w' and consider the configuration C', the restriction of C to $G-w$ (except with $C'(w') = C(w')+1$), having size $t(G,r)-3+1 = t(G,r)-2 = \pi(G-w,r)$. This makes C', and hence C, r-solvable, a contradiction.
      ii. If w has a false twin, then we can assume that s has no false twins and $C(s) = 3$. Thus w can be chosen as s and the proof follows as above.
   (c) $\mathrm{x}_0 = 0$ and $\mathrm{x}_{rs} = 0$: Recall from Lemma 8 that in this case we have

$$t(G,r) = \begin{cases} n-d_r-d_s+6 & \text{if} \quad d_r \leq 4, \quad d_s \leq 4, \quad d_r+d_s \leq 6; & (rs) \\ n-d_r+2 & \text{if} \quad d_r \leq 2, \quad d_s \geq 4; & (r) \\ n-d_s+2 & \text{if} \quad d_r \geq 4, \quad d_s \leq 2; & (s) \\ n & \text{if} \quad d_r \geq 2, \quad d_s \geq 2, \quad d_r+d_s \geq 6. & (0) \end{cases}$$

   Furthermore, when $d_r = 1$ we have from Theorem 6 that $|C| = t(G,r) \geq \pi_2(G-r,r')$, where r' is the neighbor of r. Thus we can place 2 pebbles on r' and hence solve r, a contradiction. So we will assume hereafter that $d_r \geq 2$.

i. $C(N_r) > 0$: Then there exists $r' \in N_r$ with $C(r') = 1$. By Theorem 4, $\pi(G-r, r') = n - 1 + \gamma$, where $\gamma = 1$ when $d_s = 1$ and $\gamma = 0$ otherwise. We consider the configuration C', the restriction of C to $G - r$ (except with $C'(r') = 0$), having size $t(G, r) - 1$, which is at least $\pi(G - r, r')$ when $d_s > 1$. When $d_s = 1$ we see that $t(G, r) - 1 \geq t_s(G, r) - 1 = n - 1 + 2 - 1 = \pi(G - r, r')$. In either case, C' is r'-solvable, a contradiction.

ii. $C(N_r) = 0$: Define the sets

$$\begin{aligned} A_{rs} &= \{x \in S_{rs} \mid N_x \cap N_r \neq \emptyset, N_x \cap N_s \neq \emptyset\}, \\ A_r &= \{x \in S_{rs} \mid N_x \cap N_r \neq \emptyset, N_x \cap N_s = \emptyset\}, \\ A_s &= \{x \in S_{rs} \mid N_x \cap N_r = \emptyset, N_x \cap N_s \neq \emptyset\}, \text{ and} \\ A_0 &= \{x \in S_{rs} \mid N_x \cap N_r = \emptyset, N_x \cap N_s = \emptyset\}. \end{aligned}$$

Of course, $K^i = \emptyset$ for $i \geq 4$. Notice that, whenever $C(s) \geq 4$, $A_r \neq \emptyset$, $A_{rs} \neq \emptyset$, $K^1 \cap N_r \neq \emptyset$, or some pair of vertices $x, y \in S_{rs}$ satisfies $N_x \cap N_y \neq \emptyset$, we can assume that both $K^i = \emptyset$ for $i \geq 2$ and that either $A_0 = \emptyset$ or the sets $[N_x]$ for $x \in A_0$ are pairwise disjoint. We will analyze the possible intersections between the neighborhoods of the cone vertices to compare the number of vertices and the size of the configuration. We consider different cases depending on the number of pebbles in s. Let $K' = K - N(S)$.

A. $6 \leq C(s) \leq 7$: In this case $A_r = A_{rs} = A_s = \emptyset$. Thus $n = 1 + d_r + 1 + d_s + \sum_{x \in A_0} |[N_x]| + |K'| \geq 1 + d_r + 1 + d_s + 3|A_0| + |K^1|$. We also have $C(K) = |K^1|$, and so $|C| = 3|A_0| + C(s) + |K^1|$. Then $|C| = t(G, r) \geq n - d_r - d_s + 6 \geq 1 + d_r + 1 + d_s + 3|A_0| + |K^1| - d_r - d_s + 6 = |C| - C(s) + 8$. Thus $C(s) \geq 8$, a contradiction.

B. $4 \leq C(s) \leq 5$: In this case $A_r = A_{rs} = \emptyset$. Moreover, $K^1 \subseteq N_s$, $|A_s| + |K^1| \leq 1$, and $N_x \cap N_y = \emptyset$ for all $\{x, y\} \subseteq S_{rs}$ $(x \neq y)$. This means that $|C| = 3|A_0| + 3|A_s| + |K^1| + C(s)$ and $n \geq 1 + d_r + 1 + d_s + \sum_{x \in A_0} |[N_x]| + |A_s| \geq d_r + d_s + 3|A_0| + 2 + |A_s|$. Together these imply that $|C| = t(G, r) \geq n - d_r - d_s + 6 \geq 3|A_0| + 8 + |A_s| = |C| - 2|A_s| - |K^1| - C(s) + 8$, and hence $C(s) \geq 8 - 2|A_s| - |K^1| \geq 6$, a contradiction.

C. $2 \leq C(s) \leq 3$:

I. If $A_r \neq \emptyset$, then $A_{rs} = A_s = \emptyset$, $K^i = \emptyset$ for $i \geq 2$, and $K^1 \subseteq N_{A_r} - N_r - N_s$.

$\star$ If $|A_r| \leq 2$, then $n \geq 1 + d_r + 1 + d_s + \sum_{x \in A_0} |[N_x]| + |A_r| + |K^1| \geq d_r + d_s + 2 + 3|A_0| + |A_r| + |K^1|$. Also $3|A_0| + 3|A_r| + C(s) + |K^1| = |C| \geq n - d_r - d_s + 6 \geq 8 + 3|A_0| + |A_r| + |K^1|$, which implies the contradiction that $C(s) \geq 8 - 2|A_r| \geq 4$.

$\star\star$ If $|A_r| \geq 3$, then $K^1 = \emptyset$ and $n \geq 1 + 1 + d_s + \sum_{x \in A_0 \cup A_r} |[N_x]| \geq d_s + 2 + 3|A_0| + 3|A_r|$. Thus $3|A_0| + 3|A_r| + C(s) = |C| \geq n - d_s + 2 \geq 4 + 3|A_0| + 3|A_r|$, which implies the contradiction that $C(s) \geq 4$.

II. If $A_r = \emptyset$ and $A_{rs} \neq \emptyset$, then A_{rs} contains exactly one vertex w and $K^1 \subseteq N_w$. In this case we see that the sets $[N_r]$, $[N_x]$

(for all $x \in A_0$), K^1, and $[N_s]$ are pairwise disjoint. Thus $|C| \leq 3 + |K^1| + 3|A_0| + C(s)$ and $|C| = t(G,r) \geq n - d_r - d_s + 6 \geq 1 + d_r + 1 + |K^1| + 3|A_0| + 1 + d_s - d_r - d_s + 6$, which implies $C(s) \geq 6$, a contradiction.

III. If $A_r = A_{rs} = \emptyset$, let $r' \in N_r$ and consider $G' = G - ([N_r] - r')$. Notice that if $\delta = 1$, then, by Theorem 6,

$$\pi_2(G', r') = \begin{cases} n - d_r + \mathrm{x}_{rs} + 1 + 4 & \text{if} \quad \delta = 1, \\ n - d_r + 6 - \delta & \text{if} \quad 1 < \delta < 4, \\ n - d_r + 2 & \text{if} \quad \delta \geq 4. \end{cases}$$

Since $C(N_r) = 0$, the restriction of C to G' has size

$$t(G,r) = \begin{cases} n + \mathrm{x}_{rs} + 5 - d_r & \text{if} \quad \delta = 1, d_r \leq 4, \\ n + \mathrm{x}_{rs} + 1 & \text{if} \quad \delta = 1, d_r \geq 4, \\ n + 4 - d_r & \text{if} \quad \delta = 2, d_r \leq 4, \\ n & \text{if} \quad \delta = 2, d_r \geq 4, \\ n + 3 - d_r & \text{if} \quad \delta = 3, d_r \leq 3, \\ n & \text{if} \quad \delta = 3, d_r \geq 3, \\ n + 1 & \text{if} \quad \delta \geq 4, d_r = 1, \\ n & \text{if} \quad \delta \geq 4, d_r \geq 2. \end{cases}$$

Thus C is 2-fold r'-solvable, hence r-solvable, a contradiction.

D. $C(s) \leq 1$: In this case, we have a configuration C' (the restriction of C to $G - s$) of size at least $|C| - 1 = t(G,r) - 1$ on the graph $G - s$. We will show that $\pi(G - s, r) \leq t(G,r) - 1$, implying that C', and hence C is r-solvable, a contradiction.

I. If r has eccentricity 2 in $G - s$ and $G - s$ is not Pereyra, then $\pi(G - s, r) = n - 1$. On the other hand $t(G,r) \geq n$.

II. If r has eccentricity 2 in $G - s$ and $G - s$ is r-Pereyra, then $\pi(G - s, r) = n - 1 + 1 = n$ and $d_r = 2$. Furthermore, $d_s \leq 3$ because G is not r-Phoenix. Hence $t(G,r) = n - 2 - d_s + 6 \geq n + 1$.

III. If r has eccentricity 3 in $G - s$, then, by the inductive hypothesis, $\pi(G - s, r) = t(G - s, r)$, since we know that $G - s$ is not r-Phoenix. Let $\delta' = \delta^*(G - s, r)$ and notice that, since any cone vertex of S_{rs} has 3 pebbles and s has just 1 pebble, then $d_s < \delta'$. We have from Lemma 8 that

$$t(G - s, r) = \begin{cases} n - d_r - \delta' + 5 & \text{if} \quad d_r \leq 4,\ \delta' \leq 4, & (rs)' \\ & \quad\quad \text{and } d_r + \delta' \leq 6; & \\ n - d_r + 1 & \text{if} \quad d_r \leq 2,\ \delta' \geq 4; & (r)' \\ n - \delta' + 1 & \text{if} \quad d_r \geq 4,\ \delta' \leq 2; & (s)' \\ n - 1 & \text{if} \quad d_r \geq 2,\ \delta' \geq 2, & (0)' \\ & \quad\quad \text{and } d_r + \delta' \geq 6. & \end{cases}$$

Observe that the only possible change of cases from G to $G - s$ is from (rs) to $(r)'$ or $(0)'$, or from (s) to $(0)'$. It is easy to see that in all cases $t(G - s, r) \leq t(G,r) - 1$.

This completes the proof. ∎

For $n = 2m$ (+1 if n is odd), define the *sun* S_n to be the split graph with $|K| = m$ and m leaves matched with the vertices of K (and an extra leaf joined to K

if necessary). According to Theorem 13 we have $\pi(S_n) = n + (m-2) + (6-1-1) = \lfloor 3n/2 \rfloor + 2$, showing that the pebbling bound for diameter 3 graphs given in [29] is tight.

6. Algorithms. We begin with a key construction for finding a Pyramid in a split graph G. Suppose that r is a cone vertex of G with $d_r = 2$. Then let X be the set of cut vertices of G, let W be the set of degree 2 vertices of G whose neighbors are in $G - X$, and define the graph $H = H(G)$ to have vertices $\cup_{v \in W} N_v$ and edges $\{N_v\}_{v \in W}$.

THEOREM 14. *Given a split graph G and root r, deciding whether G is r-Pereyra is a linear-time problem.*

Proof. Of course, G being r-Pereyra requires $d_r = 2$. The graph $H = H(G)$ takes linear time to construct. Then G is r-Pereyra if and only if H has a triangle including the edge N_r, which can be checked in linear time. ∎

COROLLARY 15. *If G is a split graph with root r, then $\pi(G, r)$ can be calculated in linear time.*

Proof. The set of cut vertices X of G is the neighborhood of the degree one cone vertices, and so can be calculated in linear time at the start. For $r \in K$, Theorem 4 determines $\pi(G, r)$ immediately. For a cone vertex r, we calculate its eccentricity in linear time via breadth-first search. If its eccentricity is 2, then Theorem 5 determines $\pi(G, r)$ in linear time from recognizing whether it is r-Pereyra or not. Otherwise, we have $\mathsf{ecc}(r) = 3$. In the breadth-first search we also learned of all cone vertices s at distance 3 from r. As we encounter each such s we keep track of the one having least degree. At the end we calculate $t(G, r)$ immediately from Lemma 8 and find $\pi(G, r)$ via Theorem 13. ∎

Finding a triangle in a graph is a well-known problem in combinatorial optimization. The best known algorithm is found in [1], below. Let $\omega \cong 2.376$ be the exponent of matrix multiplication, and define $\beta = 2\omega/(\omega + 1) \cong 1.41$.

ALGORITHM 16 (see [1, Theorem 3.5]). *Deciding whether a graph G with m edges contains a triangle, and finding one if it does, can be completed in $O(m^\beta)$ time.*

THEOREM 17. *Given a split graph G, there is an $O(n^{1.41})$ algorithm to decide whether G is Pereyra.*

Proof. We define $H = H(G)$ as above and see that G is Pereyra if and only if H has a triangle. Then Algorithm 16 decides this in $O(n^{1.41})$ time, since the number of edges of H is at most the number of vertices of G. ∎

THEOREM 18. *If G is a diameter 3 split graph, then $\pi(G)$ is given as follows.*

1. *If* $\mathrm{x} \geq 2$, *then*
$$\pi(G) = n + \mathrm{x} + 2.$$
2. *If* $\mathrm{x} = 1$, *then*
$$\pi(G) = \begin{cases} n + 5 - \delta^* & \text{if some leaf } r \text{ has } \mathsf{ecc}(r) = 3 \text{ and } \delta^* = \delta^*(G, r) \leq 4, \\ n + 1 & \text{otherwise.} \end{cases}$$
3. *If* $\mathrm{x} = 0$, *then*
$$\pi(G) = \begin{cases} n + 4 - \delta^* & \text{if there is a cone vertex } r \text{ with } d_r = 2, \mathsf{ecc}(r) = 3 \\ & \quad \text{and } \delta^* = \delta^*(G, r) \leq 3, \\ n + 1 & \text{if no such } r \text{ exists and } G \text{ is Pereyra,} \\ n & \text{otherwise.} \end{cases}$$

Proof. Recall from Proposition 1 that $\pi(G) = \pi(G, r)$ for some cone root r.

If $\mathrm{x} \geq 2$, then there exist leaves r and s at distance 3 from each other (in fact, if r is a leaf, then so is s). For every such r and s we have $t(G, r) = t_{rs}(G, r) = n + \mathrm{x}_{rs} + 6 - d_r - d_s$ from Lemma 8. Also, $\mathrm{x}_{rs} = \mathrm{x} - 2$ and $d_r = d_s = 1$, so that

$t(G, r) = n + \mathrm{x} + 2$ when r is a leaf. When $\mathsf{ecc}(r) = 3$ but r is not a leaf, we see that $t(G, r) \leq n + \mathrm{x} + 2$ (with equality if and only if $d_r = 2$, $d_s = 1$, and $x_0 = 0$). Finally, when $\mathsf{ecc}(r) = 2$ we have from Theorem 5 that $\pi(G, r) = n + \mathrm{x} + \psi < n + \mathrm{x} + 2$. Hence Theorem 13 implies $\pi(G) = n + \mathrm{x} + 2$.

If $\mathrm{x} = 1$, then G is not Phoenix. When $\mathsf{ecc}(r) = 2$, G is not Pereyra, and so Theorem 5 gives $\pi(G, r) = n + 1$. When $\mathsf{ecc}(r) = 3$, the cut vertex v is a neighbor of either r or s, and so $\mathrm{x}_{rs} = 0$. The function $t_r = n + 2 - d_r$ is maximized at $n + 1$ when r is a leaf, so $\pi(G) \geq n+1$. Obviously $t_0 \leq n+1$, and $t_s = n + \mathrm{x}_0 + 2 - d_s \leq n+1$, since $d_s = 1$ implies $\mathrm{x}_0 = 0$. The function t_{rs} is also maximized when r is a leaf. Indeed, if $v \notin N_r$ then s is a leaf. Then with $r' = s$ having corresponding $s' \in D_3(r')$ we have $t_{r's'} \geq t_{rs}$ because $d_{r'} = d_s = 1$ and $d_{s'} \leq d_r$. So we may assume that $v \in N_r$. If r is not a leaf, then let w be a leaf. But then with $r' = w$ having corresponding $s' \in D_3(r')$ we have $t_{r's'} > t_{rs}$ since $d_{r'} < d_r$ and $d_{s'} = d_s$. Thus we have $\pi(G) \geq n + 5 - d_s$ when r is a leaf and $s \in D_3(r)$ with $d_s = \delta^*$.

Finally, if $\mathrm{x} = 0$, we note from Lemma 8 and Theorem 13 that the only way to have $\pi(G, r) \geq n + 1$ when some cone vertex r has $\mathsf{ecc}(r) = 3$ is either via t_{rs} (with $d_r = 2$ and $d_s \leq 3$) or if G is r-Phoenix. When a cone vertex r has $\mathsf{ecc}(r) = 2$, then we have $\pi(G, r) = n + 1$ if G is r-Pereyra, by Theorem 5. Thus $\pi(G, r) = n$ in all other cases.

The above description can be reorganized as follows. Suppose that there is no cone vertex r with $d_r = 2$ and $s \in D_3(r)$ with $d_s = \delta^*(G, r) \leq 3$. If G is Pereyra, then it is r-Pereyra for some cone vertex r with $d_r = 2$. Now we know that either $\mathsf{ecc}(r) = 2$ or $\delta^*(G, r) \geq 4$, the latter case of which makes G r-Phoenix. In either case we get $\pi(G, r) = n + 1$. $\square$

It is apparent from Theorem 18 that for every $n \leq k \leq \lfloor 3n/2 \rfloor + 2$ there is an n-vertex split graph G with $\pi(G) = k$.

Corollary 19. *Calculating $\pi(G)$ when G is a split graph can be completed in $O(n^{1.41})$ time.*

Proof. Recall that we discover the value of x in linear time. So if $\mathrm{x} \geq 2$, then $\pi(G) = n + \mathrm{x} + 2$. When $\mathrm{x} = 1$ we let r be any leaf of G. Using breadth-first search from r we discover whether $D_3(r) \neq \emptyset$ and, if so, find $s \in D_3(r)$ with $d_s = \delta^*(G, r)$. Thus, in linear time we know $\pi(G)$.

Now, if $\mathrm{x} = 0$, we describe a linear algorithm either to find a cone vertex r with $d_r = 2$ and some $s \in D_3(r)$ having $d_s \leq 3$ or to conclude that none exist. For ease of notation, we write d_i for d_{v_i} and N_i for N_{v_i}.

In linear time we can reorder the vertices of G so that $d_i = 2$ for $1 \leq i \leq k$ and $d_i = 3$ for $k + 1 \leq i \leq l$. Initialize $\lambda(i)$ to be empty for every vertex v_i of G.

Then as i ranges from 1 to k we perform the following two steps. First, we add i to $\lambda(j)$ for each $v_j \in N_i$. Second, we check the size of $L_i = \cup_{v_j \in N_i} \lambda(j)$. If $|L_i| < i$, then we choose any $j \in \{1, \ldots, i\} - L_i$—for such a j we know that $N_i \cap N_j = \emptyset$—then set $r = v_j$ and $s = v_i$, and halt the algorithm. If $|L_i| \geq i$, then we continue to the next i.

If it is the case that r and s have not yet been found, we let i range from $k + 1$ to l, performing the following step each time. If $|L_i| < k$, then we choose any $j \in \{1, \ldots, k\} - L_i$—for such a j we know that $N_i \cap N_j = \emptyset$—then set $r = v_j$ and $s = v_i$, and halt the algorithm. If $|L_i| \geq k$, then we continue to the next i.

If we have not yet found r and s by now, they do not exist. This algorithm is linear because of the bounded degrees.

If r and s were found, then $\pi(G) = n + 6 - d_r - d_s$. If no such r and s exist, we use Theorem 17 to discover whether G is Pereyra, which takes $O(n^{1.41})$ time. If it is, then $\pi(G) = n + 1$, otherwise $\pi(G) = n$. ∎

7. Remarks. We begin by noting the following corollary to Theorem 18.

COROLLARY 20. *If G is a split graph with $\delta(G) \geq 3$, then G is Class* 0.

Proof. The first two instances of the x = 0 case of Theorem 18 require $\delta(G) = 2$. ∎

Note that this implies that every 3-connected split graph is Class 0. The analogous result with "split" replaced by "diameter 2" was proven in [5]. The full characterization of diameter 2, 2-connected, non-Class 0 graphs in [5] involves the appearance of a Pyramid, whereas for 2-connected, non-Class 0 split graphs, Pereyra and Phoenix graphs play a significant role.

With the similarities in structure and function mentioned above between Pyramid and Pereyra graphs, one wonders two things. First, in the diameter 2 case, it is possible to add edges between twin cone vertices (thus leaving the class of split graphs) without changing the pebbling number; is the same true for diameter 3? Second, what graph (or family of graphs) might appear in a formulation of the pebbling number of a 2-connected diameter 4 graph?

It is interesting that, while one can calculate the pebbling number of a diameter 2 graph in polynomial time, it was shown in [8] that it is NP-complete to decide whether a given configuration on a diameter 2 graph can solve a fixed root. (The same was proven more recently for planar graphs in [7]—the problem is polynomial for planar diameter 2 graphs.) In that context we offer the following.

PROBLEM 21. *Let C be a configuration on a split graph G with root r. Is it possible in polynomial time to determine if C is r-solvable?*

We also offer the following two conjectures.

CONJECTURE 22. *If G is chordal, then $\pi(G)$ can be calculated in polynomial time.*

CONJECTURE 23. *For fixed d, if $\mathsf{diam}(G) = d$, then $\pi(G)$ can be calculated in polynomial time.*

At the very least we believe that, for a chordal or fixed diameter graph G, it can be decided in polynomial time whether or not G is Class 0.

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