Toward the Enumeration of Maximal Chains in the Tamari Lattices

by

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A Dissertation Presented in Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

Approved May 2016 by the
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August 2016

ABSTRACT

The Tamari lattices have been intensely studied since they first appeared in Dov Tamari's thesis around 1952. He defined the n-th Tamari lattice $\mathcal{T}_n$ on bracketings of a set of $n+1$ objects, with a cover relation based on the associativity rule in one direction. Despite their interesting aspects and the attention they have received, a formula for the number of maximal chains in the Tamari lattices is still unknown. The purpose of this thesis is to convey my results on progress toward the solution of this problem and to discuss future work.

A few years ago, Bergeron and Préville-Ratelle generalized the Tamari lattices to the m-Tamari lattices. The original Tamari lattices $\mathcal{T}_n$ are the case $m = 1$. I establish a bijection between maximum length chains in the m-Tamari lattices and standard *m-shifted* Young tableaux. Using Thrall's formula, I thus derive the formula for the number of maximum length chains in $\mathcal{T}_n$.

For each $i \geq -1$ and for all $n \geq 1$, I define $\mathcal{C}_i(n)$ to be the set of maximal chains of length $n+i$ in $\mathcal{T}_n$. I establish several properties of maximal chains (treated as tableaux) and identify a particularly special property: each maximal chain may or may not possess a *plus-full-set*. I show, surprisingly, that for all $n \geq 2i+4$, each member of $\mathcal{C}_i(n)$ contains a plus-full-set. Utilizing this fact and a collection of maps, I obtain a recursion for $\#\mathcal{C}_i(n)$ and an explicit formula based on predetermined initial values. The formula is a polynomial in n of degree $3i+3$. For example, the number of maximal chains of length n in $\mathcal{T}_n$ is $\#\mathcal{C}_0(n) = \binom{n}{3}$.

I discuss current work and future plans involving certain equivalence classes of maximal chains in the Tamari lattices. If a maximal chain may be obtained from another by swapping a pair of consecutive edges with another pair in the Hasse diagram, the two maximal chains are said to differ by a *square move*. Two maximal chains are said to be in the same equivalence class if one may be obtained from the other by making a set of square moves.

To Kendra

ACKNOWLEDGMENTS

I can't thank the members of my thesis committee enough for the education I have received while attending Arizona State University. Their commitment to teaching and knowledge of subject material has made my stay at ASU a great experience, and has provided me with the necessary tools to pursue a career in education and continued research.

The decision to pursue a PhD in mathematics was a bit of a surprise for my family. I thank them, especially my wife, for their continued support of which none of this would have been possible.

Thank you to ASU for all of the financial support I have received as a TA and RA, and for their generosity in research and travel grants.

I would like to extend many thanks to my colleagues Mahir Can and Kevin Treat for our numerous enlightening conversations. It has been a great pleasure working with both of them on my current research.

A special thanks is due to my advisor Dr. Susanna Fishel. I took an introductory combinatorics class with her when I first started at ASU. Subsequently, she introduced me to the Tamari lattices in my first summer of research. Her careful guidance led me to the topics in this thesis including current research, perhaps the most exciting thus far.

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INTRODUCTION

1.1 Posets

The following can be found in Stanley 2012, Chapter 3.

A *partially ordered set*, or *poset*, is a set P together with a binary relation $\leq$ such that for all $x, y, z \in P$,

1. (*reflexivity*) $x \leq x$,
2. (*antisymmetry*) $x \leq y$ and $y \leq x$ implies $x = y$,
3. (*transitivity*) $x \leq y$ and $y \leq z$ implies $x \leq z$.

If $x \leq y$ but $x \neq y$, we say that x is strictly less than y and denote $x < y$ or $y > x$. If $x < y$ and there does not exist z such that $x < z < y$, then we say that x is *covered by* y or y *covers* x and denote $x \lessdot y$ or $y \gtrdot x$. The *Hasse diagram* of a finite poset P is a graph of the vertices of P with edges the cover relations in P, such that if $x \lessdot y$ then x is drawn below y. Figure 1 is the poset of positive divisors of 120, where $x \leq y$ if and only if x is a divisor of y.

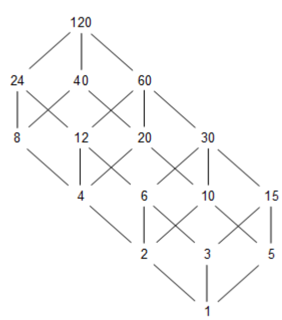

Figure 1: *The Hasse Diagram of the Poset of Positive Divisors of* 120

If Q is a subset of a poset P and there is a partial ordering on Q such that $x \leq y$ in Q implies that $x \leq y$ in P, then Q is called a *weak subposet* of P. If P also agrees with Q as sets, then P is called a *refinement* or *extension* of Q.

If Q is a subset of a poset P and there is a partial ordering on Q such that for $x, y \in Q$, $x \leq y$ in Q if and only if $x \leq y$ in P, we call Q an *induced subposet* of P. A *subposet* of P implicitly means an induced subposet. If $x \le y$ in P, the subposet $[x, y] = \{z \in P \mid x \le z \le y\}$ of P is called a (closed) *interval*. The number of intervals in a poset P is simply the number of pairs $x, y \in P$ satisfying $x \le y$.

If $x \le y$ or $y \le x$, we say that x and y are *comparable*; otherwise they are *incomparable*. A subset of a poset in which any two elements are comparable is called a *chain*. A *maximal chain* is a chain that cannot be extended to a larger chain by adding any other element of the poset. A chain C of a poset P is *saturated* if there does not exist $z \in P - C$ such that $x < z < y$ for some $x, y \in C$ and such that $C \cup \{z\}$ is a chain. The *length* of a finite chain C is $l(C) = \#C - 1$. In Figure 1, $2 < 12$ is a chain but is not saturated, $2 \lessdot 4 \lessdot 12$ is saturated but is not maximal, and $1 \lessdot 3 \lessdot 6 \lessdot 12 \lessdot 60 \lessdot 120$ is a maximal chain of length 5.

An element x in a poset P satisfying $x \le y$ for all $y \in P$ is called a $\hat{0}$. Similarly, an element x in a poset P satisfying $x \ge y$ for all $y \in P$ is called a $\hat{1}$. The poset in Figure 1 has a $\hat{0}$ and a $\hat{1}$, where $\hat{0} = 1$ and $\hat{1} = 120$.

In the event that every maximal chain of a poset P has the same length n, we say that the poset is *graded of rank* n. In this case, P is equipped with a unique *rank function* $\rho : P \to \{0, 1, \ldots, n\}$ such that $\rho(x) = 0$ for each minimal element $x \in P$ and $\rho(y) = \rho(x) + 1$ whenever $x \lessdot y$ in P. An element $x \in P$ such that $\rho(x) = i$ is said to have *rank* i. The poset in Figure 1 is graded of rank 5.

Two posets P and Q are *isomorphic* if there exists a bijection $\phi : P \to Q$ such that for all $x, y \in P$, $x \le y$ in P if and only if $\phi(x) \le \phi(y)$ in Q. The *dual* of a poset P is the poset P^* defined on the same set as P such that $x \le y$ in P if and only if $x \ge y$ in P^*. P is called *self-dual* if P is isomorphic to P^*.

Let P be a poset and $x, y \in P$. If $x, y \le z$ for some $z \in P$ then we call z an *upper bound* of x and y. A *least upper bound* of x and y is an upper bound $u \in P$ of x and y such that every upper bound $z \in P$ of x and y satisfies $u \le z$. If a least upper bound of x and y exists, it is denoted $x \vee y$, called the *join* of x and y. Similarly, a *lower bound* and the *greatest lower bound* of x and y are defined. If a greatest lower bound of x and y exists, it is denoted $x \wedge y$, called the *meet* of x and y. In the event that each pair of elements in P have a least upper bound and a greatest lower bound then P is called a *lattice*. The poset in Figure 1 is a lattice. The join and meet of any two elements is their least common multiple and greatest common divisor, respectively.

A lattice P is *distributive* if it satisfies the distributive laws: for all $x, y, z \in P$,

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z),$$
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

(Each of these laws implies the other.) Every finite distributive lattice is graded.

1.2 Young Diagrams and Tableaux

A *partition* of a positive integer n is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers summing to n. A *Young diagram* of *shape* λ is a left-justified collection of boxes having λ_j boxes in the j-th row. We adopt the *English notation* where rows are indexed downward. The *empty partition* $\lambda = (0)$ is associated with the *null diagram* having no boxes.

A *Dyck path* of *length* $2n$ is a path on the square grid of north and east steps from $(0,0)$ to (n,n) which never goes below the line $y = x$. Necessarily, every Dyck path begins with a north step and ends with an east step, and has an equal number of both types of steps. The number of Dyck paths of length $2n$ is the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

There is a natural bijective correspondence between Dyck paths of length $2n$ and Young diagrams contained in the Young diagram of staircase shape $(n-1, \ldots, 1)$. Roughly speaking, a Dyck path gives the silhouette of the Young diagram. The Dyck path that starts with n north steps followed by n east steps corresponds to the null diagram. The Dyck path that alternates between north and east steps corresponds to the Young diagram of staircase shape $(n-1, \ldots, 1)$. Figure 2 is the set of $C_4 = 14$ Dyck paths of length 8 and corresponding Young diagrams.

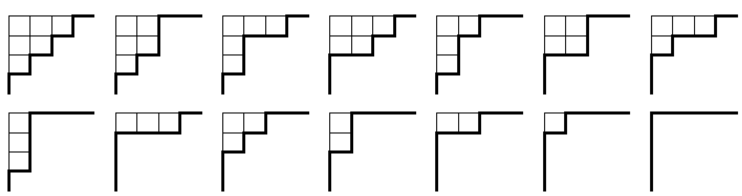

Figure 2: *Bijective Correspondence of Dyck Paths and Young Diagrams*

A filling of a Young diagram with letters from some alphabet is called a *Young tableau* or *tableau* for short. A *standard* Young tableau, abbreviated SYT, is a filling from a totally ordered set such that rows and columns are strictly increasing. A *semistandard* Young tableau, abbreviated SSYT, is a filling from a totally ordered set such that rows are weakly increasing and columns are strictly increasing. There are deep ties of SYT and SSYT to the study of the Schur functions, symmetric group, and representation theory. See Fulton and Harris 1991, Fulton 1997, Stanley 1999, Sagan 2001, Björner and Stanley 2010.

The 16 SYT of shape $(3, 2, 1)$ are shown in Figure 3. There is a simple method used to enumerate SYT of a given shape (Frame, Robinson, and Thrall 1954). The *hook length* of a box b in a Young diagram is defined to be

$$
\begin{aligned}
&= \text{number of boxes in the row to the right of } b \\
&\quad + \text{ number of boxes in the column below } b \\
&\quad + 1.
\end{aligned}
$$

The formula for the number of SYT of shape λ is $n!/H_\lambda$, where H_λ is the product of hook lengths over all boxes in the Young diagram of λ and n is its number of boxes. The *hook graph* $H[\lambda]$ is the filling of the Young diagram of shape λ of hook lengths. Figure 4 is $H[3,2,1]$, from which the number of standard fillings of shape $(3,2,1)$ is $6!/45 = 16$.

$$
\begin{array}{ccc}1&2&3\\4&5&\\6&&\end{array}\quad
\begin{array}{ccc}1&2&3\\4&6&\\5&&\end{array}\quad
\begin{array}{ccc}1&2&4\\3&5&\\6&&\end{array}\quad
\begin{array}{ccc}1&2&4\\3&6&\\5&&\end{array}\quad
\begin{array}{ccc}1&2&5\\3&4&\\6&&\end{array}\quad
\begin{array}{ccc}1&2&6\\3&4&\\5&&\end{array}\quad
\begin{array}{ccc}1&2&5\\3&6&\\4&&\end{array}\quad
\begin{array}{ccc}1&2&6\\3&5&\\4&&\end{array}
$$

$$
\begin{array}{ccc}1&3&4\\2&5&\\6&&\end{array}\quad
\begin{array}{ccc}1&3&4\\2&6&\\5&&\end{array}\quad
\begin{array}{ccc}1&3&5\\2&4&\\6&&\end{array}\quad
\begin{array}{ccc}1&3&6\\2&4&\\5&&\end{array}\quad
\begin{array}{ccc}1&3&5\\2&6&\\4&&\end{array}\quad
\begin{array}{ccc}1&3&6\\2&5&\\4&&\end{array}\quad
\begin{array}{ccc}1&4&5\\2&6&\\3&&\end{array}\quad
\begin{array}{ccc}1&4&6\\2&5&\\3&&\end{array}
$$

Figure 3: *SYT of Shape* $(3,2,1)$

$$
\begin{array}{ccc}5&3&1\\3&1&\\1&&\end{array}
$$

Figure 4: $H[3,2,1]$

$$
\begin{array}{ccc}1&2&3\\&4&5\\&&6\end{array}\quad
\begin{array}{ccc}1&2&4\\&3&5\\&&6\end{array}
$$

Figure 5: *Standard Shifted Tableaux of Shape* $(3,2,1)$

$$
\begin{array}{ccc}5&4&3\\&3&2\\&&1\end{array}
$$

Figure 6: $G[3,2,1]$

A *shifted Young diagram* of shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a diagram of distinct parts $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ such that each successive row begins one cell to the right of the row above. A *shifted tableau* is a filling of a shifted Young diagram. The 2 standard shifted tableau for $\lambda = (3,2,1)$ are shown in Figure 5. There is a similar method used to enumerate standard shifted tableaux (Knuth 1973; Thrall 1952). The *generalized hook length* of a box b in a shifted Young diagram is defined to be

$$
\begin{aligned}
&= \text{number of boxes in the row to the right of } b \\
&\quad + \text{ number of boxes in the column below } b \\
&\quad + \text{ number of boxes in the row just below the bottom box in the column of } b \\
&\quad + 1.
\end{aligned}
$$

The formula for the number of standard shifted tableaux of shape λ is $n!/G_\lambda$, where G_λ is the product of generalized hook lengths over all boxes in the Young diagram of λ and n is its number of boxes. The *generalized hook graph* $G[\lambda]$ is the filling of the shifted Young diagram of shape λ of generalized hook lengths. Figure 6 is $G[3,2,1]$, from which the number of shifted standard tableaux of shape $(3,2,1)$ is $6!/360 = 2$.

1.3 The Tamari Lattices

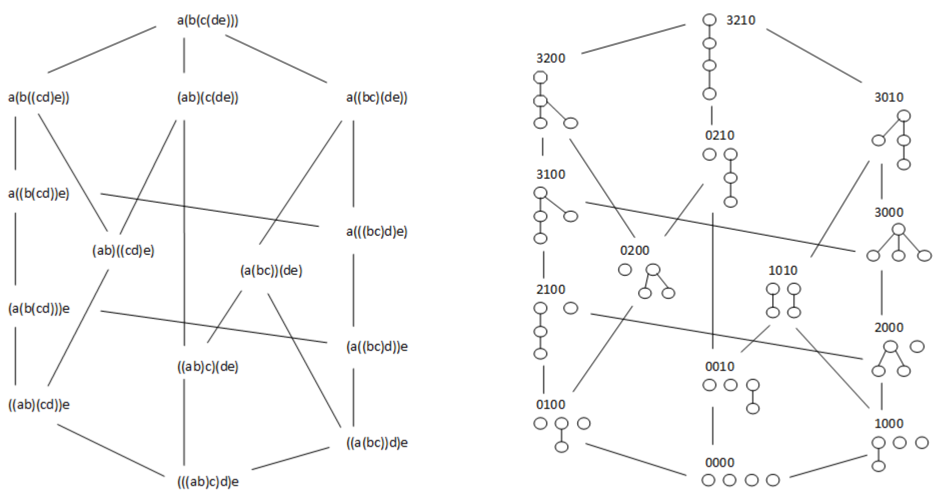

Figure 7: *Tamari's Original Representation of $\mathcal{T}_4$ with Cover Relation $(xy)z \lessdot x(yz)$ (to the left), and Knuth's Representation by Forests and Scope Sequences (to the right)*

Tamari 1962 introduced the poset $\mathcal{T}_n$ on bracketings of a set of $n+1$ objects, with a cover relation based on the associativity rule in one direction. Figure 7 (to the left) is the Hasse diagram of $\mathcal{T}_4$, based on his original interpretation. Friedman and Tamari 1967 proved the lattice property for this family of posets in 1967.

The number of vertices in $\mathcal{T}_n$ is the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Triangulations of a convex $(n+2)$-gon, noncrossing partitions of $[n]$ and Dyck paths of length $2n$ are examples of over 200 combinatorial structures counted by the Catalan sequence (Stanley 2008). Because of the many combinatorial structures that are counted by the Catalan sequence, the Tamari lattice is studied in various representations. Huang and Tamari 1972 describe $\mathcal{T}_n$ as the poset of of n-tuples $a_1, \ldots, a_n$ satisfying $i \le a_i \le n$ and $i \le j \le a_i$ implies $a_j \le a_i$, ordered coordinatewise. They gave a considerably simpler proof of the lattice property of $\mathcal{T}_n$ than in Friedman and Tamari 1967.

Knuth 2006 describes $\mathcal{T}_n$ as the set of forests on n nodes. As a forest is traversed from left to right as depicted in Figure 8 (to the left), the order in which nodes are encountered is called *preorder*, and the nodes are labeled as such. Each forest is assigned a *scope sequence* $s_1, \ldots, s_n$ such that node k in preorder has s_k descendants. If two forests $F, F' \in \mathcal{T}_n$ have scope sequences $s_1, \ldots, s_n$ and $s'_1, \ldots, s'_n$, respectively, then $F \leq F'$ if and only if $s_j \leq s'_j$ for all $1 \leq j \leq n$. See Figure 7 (to the right) for $\mathcal{T}_4$ by forests and their scope sequences.

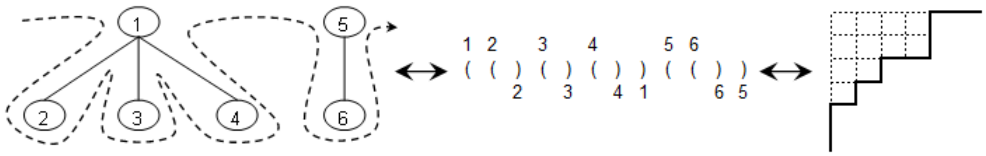

Figure 8: *A Forest on* 6 *Nodes with Scope Sequence* 300010 *Corresponds to Well Formed Parentheses and a Dyck Path*

Each n-node forest corresponds uniquely to a string of well formed parentheses and vice versa. This is a string of n pairs of left and right parentheses such that any left subset of the string never has more right parentheses than left parentheses. As the forest is traversed in Figure 8, write a left (respectively, right) parenthesis when a node is encountered on its left (respectively, right) side. In this manner, each node in the forest is matched to a pair of left and right parentheses. In turn, the correspondence between n pairs of well formed parentheses and Dyck paths of length $2n$ is bijective. As the parentheses string is read from left to right, take a north (respectively, east) step when encountering a left (respectively, right) parenthesis to obtain the Dyck path.

If P is a Dyck path and L is the line segment (of slope one) that joins the endpoints of P, then P is said to be *prime* (Bernardi and Bonichon 2009) if P intersects L only at the endpoints of P. Of the Dyck paths in Figure 2, only the last five in the second row are prime, corresponding to the Young diagrams contained in the Young diagram of shape $(2, 1)$. A Dyck path of length $2n$ has exactly n *prime Dyck subpaths* (Bernardi and Bonichon 2009), each uniquely determined by its beginning north step. In Figure 9, for the given Dyck path of length 8, its 4 prime Dyck subpaths are accentuated. The line segment joining the end points of each of the prime subpaths is shown.

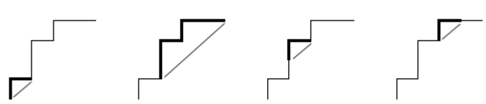

Figure 9: *Prime Dyck Subpaths*

Bernardi and Bonichon 2009 represent $\mathcal{T}_n$ as the set of Dyck paths of length $2n$, and express the covering relations in these terms. Let P and P' be two Dyck paths. Then P' covers P if and only if there exists an east step e in P, followed by a north step, such that P' is obtained from P by swapping e and the prime Dyck subpath following it. In the examples of Figure 10, the east step of P referenced is grayed and the prime Dyck subpath following it is accentuated.

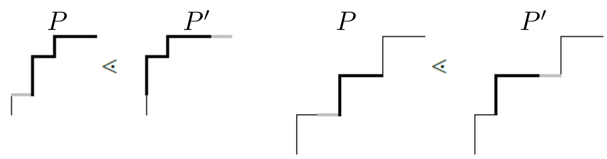

Figure 10: *The Covering Relation in the Tamari Lattices: Dyck Paths*

In a very slight twist from Bernardi and Bonichon 2009, I view the elements of $\mathcal{T}_n$ as Young diagrams contained in the Young diagram of staircase shape $(n-1,\ldots,1)$. In this way, maximal chains may be easily identified as Young tableaux, which are well suited for my purposes. The Hasse diagrams for $\mathcal{T}_3$ and $\mathcal{T}_4$ in terms of Young diagrams are shown in Figure 11.

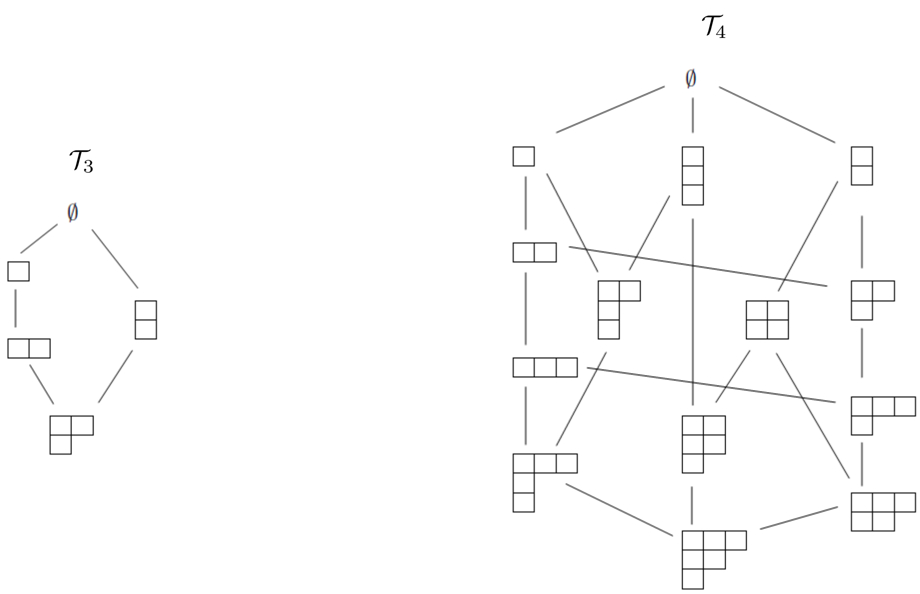

Figure 11: *Hasse Diagrams for $\mathcal{T}_3$ and $\mathcal{T}_4$ by Young Diagrams*

$\mathcal{T}_n$ is also represented as triangulations of an $(n+2)$-gon (Edelman and Reiner 1996), or binary trees on n nodes (Châtel and Pons 2015), and may be represented on virtually any Catalan set of objects.

Recently Bergeron and Préville-Ratelle 2012, while working on higher dimensional Catalan polynomials, generalized the Tamari posets to the m-Tamari posets $\mathcal{T}_n^{(m)}$ (the case $m=1$ is the original Tamari lattice $\mathcal{T}_n$). Shortly after, Bousquet-Mélou, Fusy, and Préville-Ratelle 2011 proved the m-Tamari posets are lattices.

They define vertices of $\mathcal{T}_n^{(m)}$ as paths on the square grid consisting of north and east steps, starting at $(0,0)$ and ending at (mn,n). In addition, each path lies strictly above (but may touch) the line $y = \frac{x}{m}$. For my purposes (through the natural correspondence), the vertices of $\mathcal{T}_n^{(m)}$ are the Young diagrams which fit inside the Young diagram of shape $(m(n-1), m(n-2), \ldots, m)$. Figure 12 is $\mathcal{T}_3^{(2)}$. In Chapters 2 and 3, I relate the covering relation for $\mathcal{T}_n^{(m)}$ and $\mathcal{T}_n$, respectively.

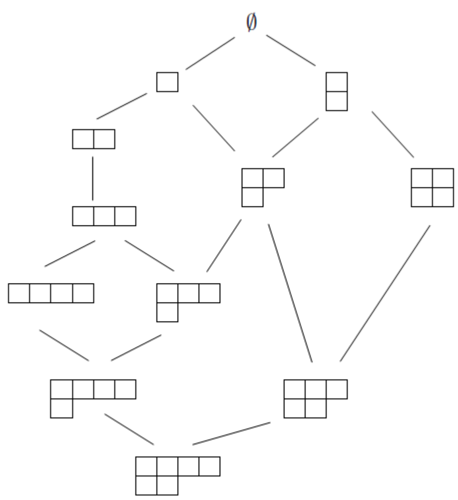

Figure 12: *The* 2*-Tamari Lattice* $\mathcal{T}_3^{(2)}$

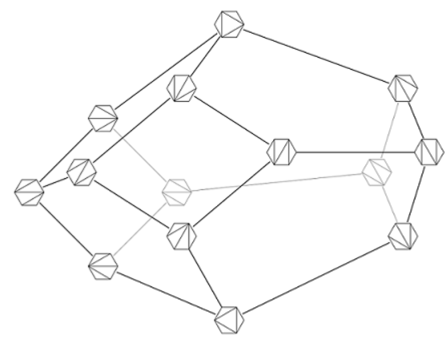

Figure 13: *The* 3*-dimensional Associahedron* K_5

The Hasse diagram of $\mathcal{T}_n$ is the 1-skeleton of the $(n-1)$-dimensional associahedron (or Stasheff polytope) K_{n+1} (Devadoss and Read 2001; Björner and Wachs 1997; Reading 2006). See Figure 13 for K_5. A k-*dissection*, $0 \leq k \leq n-1$, of an $(n+2)$-gon is a partition of the $(n+2)$-gon into $k+1$ polygons by k non-crossing diagonals. An $(n-1)$-dissection of an $(n+2)$-gon is a *triangulation*. The vertices of K_{n+1} (and often in the literature $\mathcal{T}_n$, see Reading 2012) correspond to the triangulations of an $(n+2)$-gon. Edges in K_{n+1} are determined by the ability to obtain one triangulation from another by a single *diagonal flip*. In general, the k-dissection of an $(n+2)$-gon corresponds to an $(n-1-k)$-dimensional face of K_{n+1}.

I close this section by recording some properties satisfied by the family of Tamari lattices. For many other interesting properties, see Geyer 1994, Knuth 2006, Markowsky 1992.

- $\mathcal{T}_n$ is not graded, so not a distributive lattice ($n \geq 3$) (Markowsky 1992; Knuth 2006).
- $\mathcal{T}_n$ is semidistributive (Knuth 2006; Urquhart 1978). A lattice P is called *join semidistributive* if for all $x, y, z \in P$,

$$x \vee y = x \vee z \text{ implies } x \vee y = x \vee (y \wedge z),$$

and is called *meet semidistributive* if for all $x, y, z \in P$,

$$x \wedge y = x \wedge z \text{ implies } x \wedge y = x \wedge (y \vee z).$$

When both of these properties are satisfied, the lattice is called *semidistributive*.

- $\mathcal{T}_n$ is self-dual (Knuth 2006).
- $\mathcal{T}_n$ is extremal in the sense of Markowsky 1992 (a lattice is *extremal* if the number of join-irreducibles, number of meet-irreducibles and length of the longest maximal chain are equal). A *join-irreducible* element $x \neq \hat{0}$ in a lattice is an element that cannot be written $x = y \vee z$ for $y, z < x$. A *meet-irreducible* element $x \neq \hat{1}$ in a lattice is an element that cannot be written $x = y \wedge z$ for $y, z > x$. In a finite lattice, join-irreducible (respectively, meet-irreducible) elements are the ones covering (respectively, covered by) exactly one element (Stanley 2012, Chapter 3).
- $\mathcal{T}_n$ is complemented (Grätzer 2003). A lattice P having a $\hat{0}$ and a $\hat{1}$ is *complemented* if for each $x \in P$, there exists $y \in P$ (y may depend on x) such that $x \vee y = \hat{1}$ and $x \wedge y = \hat{0}$ (Stanley 2012, Chapter 3).
- The Hasse diagram of $\mathcal{T}_n$ is an $(n-1)$-regular graph. (When $\mathcal{T}_n$ is viewed on triangulations, there are $n-1$ diagonal flips available at each vertex.)

1.4 Related Lattices to the Tamari Lattices

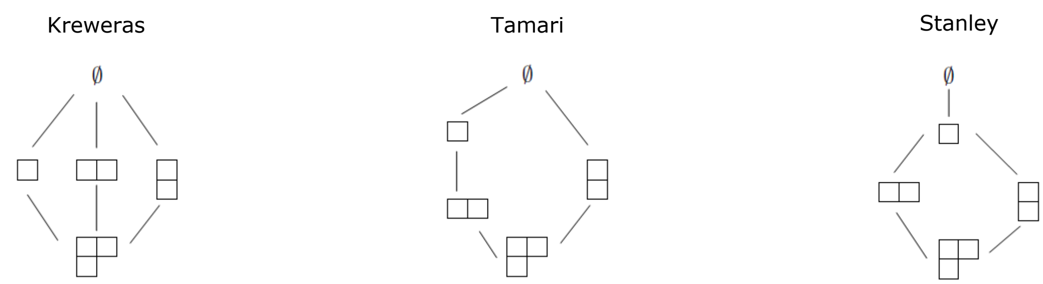

Figure 14: *Hasse Diagrams for Kreweras, Tamari and Stanley Lattices of Order* 3

The Kreweras and Stanley lattices are two other noted families of lattices defined on Catalan sets of objects. A set partition of $[n]$ is *noncrossing* if for all $i < j < k < l \in [n]$, if i, k are in a block together and j, l are in a block together then i, j, k, l are all in the same block. For example, in the set partitions of $[4]$, $13 \mid 24$ is the only one that is not noncrossing. Traditionally, the Kreweras lattice of order n is defined on the noncrossing set partitions of $[n]$ (of which there are C_n) ordered by *refinement* (Kreweras 1972; Knuth 2006; Fomin and

Reading 2007). A partition π is a refinement of a partition π' if each block in π is a subset of some block in π'. Of the set partitions of $[n]$, the minimal and maximal elements in the Kreweras lattice are $1 \mid 2 \mid \cdots \mid n$ and $12 \cdots n$, respectively. The Stanley lattice of order n is the set of Dyck paths of length $2n$ where $P \leq P'$ if and only if P' lies above P (Bernardi and Bonichon 2009). When considered on the same Catalan set of objects as in Figure 14, the Stanley lattice is a refinement of the Tamari lattice, which is a refinement of the Kreweras lattice (Knuth 2006; Bernardi and Bonichon 2009).

The *simple transpositions* S in the symmetric group $\mathcal{S}_n$ are the adjacent transpositions $s_i = (i\ i+1)$, $i \in [n-1]$. Each permutation in $\mathcal{S}_n$ can be written as a word in S. I adopt the convention that multiplication happens from right to left. For example, the permutation 321, in one-line notation, can be written in various ways as $321 = s_1 s_2 s_1 = s_1 s_1 s_1 s_2 s_1 = s_2 s_1 s_2$. A word which is minimum in length over all words for $w \in \mathcal{S}_n$ is called a *reduced word* (or *reduced decomposition* or *reduced expression*) for w. The *length* of $w \in \mathcal{S}_n$, denoted $l(w)$ is the length of a reduced word for w. The *(right) weak (Bruhat) order* on $\mathcal{S}_n$ is the transitive closure of the covering relation: $w \lessdot u$ if and only if $u = ws$ for some $s \in S$ and $l(u) = l(w) + 1$. The minimum element $\hat{0}$ is the identity permutation $e = 12 \cdots n$ and the maximum element $\hat{1}$ is $n\, n-1 \cdots 1$. The permutations in $\mathcal{S}_n$ together with the weak order form a lattice which is also denoted $\mathcal{S}_n$.

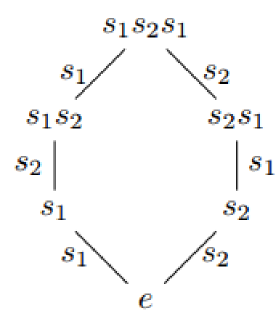

Figure 15: *The Hasse Diagram of* $\mathcal{S}_3$

Figure 15 is the Hasse diagram for $\mathcal{S}_3$, where the edge from w to ws is labeled with s. Björner and Wachs 1997 showed that $\mathcal{T}_n$ is a sublattice of $\mathcal{S}_n$ of 312-avoiding permutations and that there is an order preserving projection from $\mathcal{S}_n$ to $\mathcal{T}_n$. Reading 2006 generalized this projection by defining the Cambrian lattices as quotients of the weak order. The sudden outcropping of Catalan related lattices which stem from the single pre-Cambrian example of the Tamari lattice is the analogy he uses in naming the Cambrian lattices, identifying with the sudden increase in the fossil record of the Cambrian layer of rocks.

1.5 Related Enumeration Results

The search for enumeration formulas, whether for maximal chains or intervals, *etc.*, are traditional problems for any family of posets. The pursuit of solutions often leads to relationships with other combinatorial structures and a better understanding of the poset at hand.

1.5.1 Chain and Antichain Enumeration

Theorem 1.1 (Dilworth 1950) Define a subset S of a poset P as independent if every two distinct elements of S are non-comparable, and define S to be dependent if it contains two distinct elements which are comparable. Let every set of $k+1$ elements of a partially ordered set P be dependent while at least one set of k elements is independent. Then P is a set sum of k disjoint chains.

Now suppose P is a finite poset. Let A be an antichain and C be a partition of P into chains. Notice that any antichain may have at most one vertex from each chain in C, thus $\#A \leq \#C$. Thus for a finite poset P, an equivalent way of stating Theorem 1.1 is that the size of the largest antichain in P is the minimum number of chains that cover P.

Mirsky derived the dual form of Theorem 1.1 (allowing for the empty set to be both a chain and an antichain).

Theorem 1.2 (Mirsky 1971) Let P be a partially ordered set, and m a natural number. If P possesses no chain of cardinal $m+1$, then it can be expressed as the union of m antichains.

Similarly, for a finite poset P, an equivalent way of stating Theorem 1.2 is that the size of the largest chain in P is the minimum number of antichains that cover P.

Greene and Kleitman 1976 showed that for any finite poset P, there exists a partition $\lambda(P) = (\lambda_1(P), \lambda_2(P), ...)$ such that the sum of the first k parts of $\lambda(P)$ equals the maximum size of a union of k chains of P. In fact, remarkably, the sum of the first k parts of the conjugate of $\lambda(P)$ equals the maximum size of a union of k antichains of P (Greene 1976). Note $\lambda_1(P)$ equals the length of the longest maximal chain plus one in P.

Early, a student of Stanley, deduced formulas for $\lambda_2(\mathcal{T}_n)$, $\lambda_3(\mathcal{T}_n)$. He studied vertices of $\mathcal{T}_n$ as in Huang and Tamari 1972.

Theorem 1.3 (Early 2004) For $n > 5$, $\lambda_2(\mathcal{T}_n) = \lambda_1(\mathcal{T}_n) - 4$. For $n > 6$, $\lambda_3(\mathcal{T}_n) = \lambda_2(\mathcal{T}_n) - 2$.

1.5.2 Number of Intervals

Intervals in the Kreweras lattice are in bijection to ternary trees (Kreweras 1972; Edelman 1982), while those in the Stanley lattice are pairs of noncrossing Dyck paths (Sainte-Catherine and Viennot 1986).

Not too long ago, Chapoton enumerated the number of intervals in $\mathcal{T}_n$, finding this to be the number of planar triangulations (*i.e.*, maximal planar graphs).

Theorem 1.4 (Chapoton 2005/07) The number of intervals in the Tamari lattice $\mathcal{T}_n$ is

$$\frac{2(4n+1)!}{(n+1)!(3n+2)!}.$$

Subsequently, Chapoton's result was generalized to the m-Tamari lattices.

Theorem 1.5 (Bousquet-Mélou, Fusy, and Préville-Ratelle 2011) The number of intervals in $\mathcal{T}_n^{(m)}$ is

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n+m}{n-1}.$$

Very recently, Châtel and Pons 2015 introduced new combinatorial objects, the interval posets, and Chapoton, Chatel, and Pons n.d. described new bijections of intervals in the Tamari lattice.

1.5.3 Maximal Chain Enumeration

Maximal chains in $\mathcal{S}_n$ are in bijective correspondence with the reduced words of the permutation $\hat{1} = n\, n-1 \cdots 1$. Stanley 1984 showed that the number of these is the number of SYT of staircase shape $(n-1, \ldots, 1)$, which is

$$\frac{\binom{n}{2}!}{1^{n-1}3^{n-2}\cdots(2n-5)^2(2n-3)^1} \tag{1.1}$$

by applying the hook length formula.

When the Stanley lattice of order n is considered on Young diagrams contained in the Young diagram of staircase shape $(n-1, \ldots, 1)$, the order is by containment as seen in Figure 14. This gives a simple bijection between its maximal chains and SYT of staircase shape $(n-1, \ldots, 1)$, the number of which is again equation (1.1) (Knuth 2006).

Maximal chains in the Kreweras lattice on noncrossing partitions of $[n]$, are in bijection with factorizations of an n-cycle as the product of $n-1$ transpositions (Knuth 2006; Kreweras 1972), the number of which is n^{n-2} (Dénes 1959). This is also the number of parking functions of length $n-1$ (Foata and Riordan 1974; Stanley 1997), and the number of trees on n labeled vertices (Cayley 1889).

1.6 Maximal Chain Enumeration in the Tamari Lattices

A natural question arises: What is the number of maximal chains in the Tamari lattices? Although in related lattices the number of maximal chains is known, quoting Knuth, "The enumeration of such paths in Tamari lattices remains mysterious." Despite their interesting aspects and the attention they have received, a formula for the number of maximal chains in the Tamari lattices is still unknown. The purpose of this thesis is to convey my results on progress toward the solution of this problem and to discuss future work. The complexity of this problem is largely due to the fact that the Tamari lattices are not graded, *i.e.*, they have maximal chains of varying lengths. The unique shortest maximal chain in $\mathcal{T}_n$ has length $n-1$, while the longest ones have length $\binom{n}{2}$ (Markowsky 1992; Knuth 2006). Besides what is written in this dissertation, I am unaware of any results pertaining to the enumeration of maximal chains in the Tamari lattices.

Keller 2011 introduced *green mutations* and maximal sequences of such mutations, called *maximal green sequences*. In certain cases, maximal green sequences are in bijection with maximal chains in the Tamari lattice or the Cambrian lattice (a generalization of the Tamari lattice; see Reading 2006) (Keller n.d.; Garver and Musiker n.d.). Garver and Musiker n.d. list several applications of maximal green sequences to representation theory and physics, and the problems of enumeration and classification of such sequences are noted interests.

Chapter 2 is my paper "*Chains of maximum length in the Tamari lattice*", coauthored with Susanna Fishel (Fishel and Nelson 2014). Here I present what I envision as first steps towards the enumeration of maximal chains. Our work suggests not only that the original problem is not intractable, but that its solution may have an interpretation in terms of representation theory. This chapter is an offshoot of a long-standing project of Fishel, along with Grojnowski, on the combinatorics of higher dimensional Catalan polynomials. As mentioned, Bergeron and Préville-Ratelle 2012, also while working on higher dimensional Catalan polynomials, generalized the Tamari lattices to the m-Tamari lattices $\mathcal{T}_n^{(m)}$. The original Tamari lattices are the case $m = 1$. Some of the results in the first part of this chapter concern the generalized lattice, whereas the results in the second half are for the $m = 1$ case. Part of our strategy for finding the number of maximal chains is to focus on each length separately. Here, we determine the number of chains in the Tamari lattice $\mathcal{T}_n$ of maximum length. We do this using a simple bijection to shifted tableaux, then use Thrall's formula (Knuth 1973) to count those.

Chapter 3 is my paper "*A recursion on maximal chains in the Tamari lattices*" (Nelson n.d.). The subject here pertains to the following definition. For each $i \geq -1$ and for all $n \geq 1$, $\mathcal{C}_i(n)$ is the set of maximal chains of length $n+i$ in $\mathcal{T}_n$. I establish several properties of maximal chains (treated as tableaux) and identify a particularly special property: each maximal chain may or may not possess a *plus-full-set*. I show, surprisingly, that for all $n \geq 2i+4$, each member of $\mathcal{C}_i(n)$ contains a plus-full-set. Utilizing this fact and a collection of maps, which take maximal chains in $\mathcal{C}_i(n)$ to $\mathcal{C}_i(n+1)$, I obtain a recursion for $\#\mathcal{C}_i(n)$ and an explicit formula based on predetermined initial values. The formula is a polynomial in n of degree $3i+3$. For example, the numbers of maximal chains of lengths $n-1$, n and $n+1$ in $\mathcal{T}_n$ are $\#\mathcal{C}_{-1}(n) = 1$, $\#\mathcal{C}_0(n) = \binom{n}{3}$ and $\#\mathcal{C}_1(n) = 2\binom{n+1}{5} + 10\binom{n+1}{6}$, respectively. This result is a considerable generalization of results in Knuth 2006 and Markowsky 1992, where it is shown that there is one maximal chain of shortest length $n-1$.

In Chapter 4, I discuss current work and future plans in collaboration with Susanna Fishel, Kevin Treat and Mahir Can. We are investigating equivalence classes of maximal chains in the Tamari lattices and exciting new posets which arise. Preliminary findings suggest that these posets have many wonderful properties. This approach provides for alternative tools in which to study the Tamari lattices and corresponding associahedra. Our hope is that studying these new posets will lend insight to the enumeration of maximal chains in $\mathcal{T}_n$.

Chapter 2

CHAINS OF MAXIMUM LENGTH IN THE TAMARI LATTICE

In this chapter, we focus on the chains with maximum length in the Tamari lattices. We establish a bijection between the maximum length chains in the Tamari lattice and the set of standard shifted tableaux of staircase shape. We thus derive an explicit formula for the number of maximum length chains, using the Thrall formula for the number of shifted tableaux. We describe the relationship between chains of maximum length in the Tamari lattice and certain maximal chains in weak Bruhat order on the symmetric group, using standard Young tableaux. Additionally, recently Bergeron and Préville-Ratelle introduced a generalized Tamari lattice. Some of the results mentioned above carry over to their generalized Tamari lattice.

2.1 Background

The Tamari lattice is both a quotient and a sublattice of the weak order on the symmetric group (Björner and Wachs 1997; Reading 2006). The study of maximal chains in the weak order on $\mathcal{S}_n$ has proven to be extremely fruitful. The Stanley symmetric functions, balanced labelings, and dual equivalence arose from work on the weak order on $\mathcal{S}_n$ (Stanley 1984; Edelman and Greene 1987; Haiman 1992). It was with this rich history in mind that we began our investigation of the maximal chains in the Tamari lattices. The work of Stanley and Edelman-Greene shows that the chains in weak order have a meaning in representation theory: they index a basis for an irreducible representation of the symmetric group. We hope for a similar interpretation here, and the appearance of shifted tableaux supports this. To this end, we begin the study of the relationship between the maximal chains in the weak order and the maximal chains in the Tamari lattice. We seek to understand the interplay between the modified Robinson-Schensted algorithm of Edelman and Greene 1987 and the Tamari order, via c-sorting.

The outline of this chapter follows. In Section 2.2 we review the definition of the Tamari lattice, using the m-Tamari generalization in Bergeron and Préville-Ratelle 2012. We do not discuss the relationship of this definition to the original definition in terms of bracketings (Tamari 1962). We describe how to assign a tableau to each maximal chain in the lattice in Section 2.3. It is this correspondence which allows us to enumerate the chains of maximum length. In Section 2.4 we relate the Edelman-Greene bijection between maximal chains in weak order on the symmetric group and standard Young tableaux of staircase shape (Edelman and Greene

1987) to our work. We characterize the maximal chains in $\mathcal{S}_n$ which are maximum length chains in $\mathcal{T}_n$ when we view $\mathcal{T}_n$ as an induced subposet of $\mathcal{S}_n$, in order to show that the Edelman-Greene bijection encodes the Tamari order in this case.

2.2 The Tamari Lattice and its Covering Relation

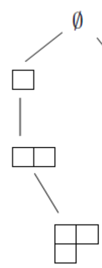

Figure 16: *The Hasse Diagram of the Poset* $\mathcal{T}_3$

A partially ordered set, or poset, is a lattice if every pair of elements has a least upper bound, the join, and a greatest lower bound, the meet. We consider the Tamari lattice here; the Tamari lattice of order n is denoted $\mathcal{T}_n$. See Figure 16 for $\mathcal{T}_3$. $\mathcal{T}_n$ has several equivalent definitions. In Tamari 1962, the vertices are proper bracketings of $n+1$ symbols and the cover relation is given by the associative rule. Knuth 1973 describes $\mathcal{T}_n$ as a poset of forests on n nodes, Björner and Wachs 1997 give the definition in terms of the scope sequences of these forests, and Bernardi and Bonichon 2009 phrases the definition in terms of Dyck paths. It is this last form of the definition that Bergeron and Préville-Ratelle 2012 generalizes. We make a very slight change and give their generalized definition in terms of certain partitions, which are better for our purposes.

Definition 2.1 An (m, n)-Dyck partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is an integer sequence such that

1. $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n = 0$ and
2. for each i, we have $\lambda_i \leq m(n-i)$.

It is well-known (Dvoretzky and Motzkin 1947) that there are the m-Catalan number $\frac{1}{nm+1}\binom{(m+1)n}{n}$ of these partitions. The (m, n)-Dyck partitions are the vertices for the m-Tamari lattice; the case $m = 1$ is the original Tamari lattice.

Definition 2.2 (Bergeron and Préville-Ratelle 2012). Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be an (m, n)-Dyck partition. For each i between 1 and n, there is a unique $k = k(\lambda, i) \leq i$ such that

1. $\lambda_j - \lambda_i < m(i - j)$ for $j = k, \ldots, i - 1$ and
2. either $k = 1$ or $\lambda_{k-1} - \lambda_i \geq m(i - k + 1)$.

Suppose $\lambda_i > \lambda_{i+1}$ and $k = k(\lambda, i)$. Set $\lambda \lessdot \mu$, where $\mu = (\lambda_1, \ldots, \lambda_{k-1}, \lambda_k - 1, \ldots, \lambda_i - 1, \lambda_{i+1}, \ldots, \lambda_n)$. The m-Tamari lattice $\mathcal{T}_n^{(m)}$ is the set of (m, n)-Dyck partitions, together with the transitive closure of this covering relation $\lambda \lessdot \mu$.

Friedman and Tamari 1967 and also Huang and Tamari 1972 showed that $\mathcal{T}_n = \mathcal{T}_n^{(1)}$ is a lattice. Bousquet-Mélou, Fusy, and Préville-Ratelle 2011 showed that $\mathcal{T}_n^{(m)}$ is a lattice and is in fact isomorphic to a sublattice of $\mathcal{T}_{nm}$. At the top of the lattice is the empty partition and at the bottom is the partition $((n-1)m, \ldots, m)$.

If a poset has an element x with the property that $y \leq x$ for all y in the poset, we denote that element by $\hat{1}$. Similarly, $\hat{0}$ is the element below all others if such an element exists. In a poset with $\hat{0}$ and $\hat{1}$, a *maximal chain* is a sequence of elements $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{N-1} \lessdot x_N = \hat{1}$.

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be an integer partition. Its *Young diagram* is an array of boxes, where there are λ_i boxes in row i. The (n, m)-Dyck partitions are the partitions whose Young diagram fits inside the diagram of $(m(n-1), m(n-2), \ldots, m)$. A *tableau of shape* λ is a filling of the Young diagram of λ by positive integers, where the entries strictly increase along rows and weakly along columns. It is a *standard Young tableau* if there are no repeated entries.

2.3 Maximum Length Chains in $\mathcal{T}_n^{(m)}$ and Tableaux

To each maximal chain C in $\mathcal{T}_n^{(m)}$, we associate a tableau $\Psi(C)$ of shape $(m(n-1), \ldots, m)$.

Definition 2.3 Let $C = \{\hat{1} = x_r \gtrdot \ldots \gtrdot x_1 \gtrdot x_0 = \hat{0}\}$ be a maximal chain. As the chain is traversed from top to bottom, boxes are added. Fill the boxes added in moving from x_j to x_{j-1} with $r - j + 1$. The resulting tableau is $\Psi(C)$.

Every maximal chain is assigned a tableau and those of maximum length a standard Young tableaux. See Figure 17. Ψ is injective. For the rest of this chapter, we focus on chains of maximum length.

Figure 17: *The tableau on the left corresponds to the chain on the left in Figure 16, and the tableau on the right to the chain on the right. The tableau on the left can be shifted by one, the one on the right cannot.*

Definition 2.4 Let m be a positive integer and $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition such that $\lambda_i - \lambda_{i+1} \geq m$, for i from 1 to $k-1$. Then the m-shifted diagram of λ is obtained from the usual diagram of λ by moving the i-th row $m(i-1)$ boxes to the right, for $i > 1$. An m-shifted tableau of shape λ is a filling of the m-shifted diagram of λ by positive integers such that the entries strictly increase along rows and weakly along columns. It is standard if no entry is repeated and the entries are from $\{1, 2, \ldots, \mid \lambda \mid\}$.

The case $m = 1$ is the usual shifted tableau.

Theorem 2.5 The number of chains of maximum length in the m-Tamari lattice $\mathcal{T}_n^{(m)}$ is equal to the number of standard m-shifted tableaux of shape $(m(n-1), m(n-2), \ldots, m)$.

Proof. $\mathcal{T}_n$ has a chain of length $\binom{n}{2}$, and this is the maximum length (Markowsky 1992). In $\mathcal{T}_n^{(m)}$, consider the chain whose vertex x_i is the (m, n)-Dyck partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, where

$$\lambda_j = \begin{cases} m(n-j) & \text{if } m\binom{n}{2} - i \geq \sum_{h=1}^{j} m(n-h), \\ m\binom{n}{2} - i - \sum_{h=1}^{j-1} m(n-h) & \text{otherwise.} \end{cases}$$

This is a chain of length $m\binom{n}{2}$ and there can be no longer ones.

Let N denote $m\binom{n}{2}$. A maximum length chain $C = \{\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{N-1} \lessdot x_N = \hat{1}\}$ in $\mathcal{T}_n^{(m)}$ is one where the Young diagram for x_j has exactly one square more than the Young diagram for x_{j+1}, for j from 0 to $N-1$. This is only possible if $k(\lambda, i) = i$ in Definition 2.2 whenever $x_j = (\lambda_1, \ldots, \lambda_n)$ and $x_{j+1} = (\lambda_1, \ldots, \lambda_{k(\lambda,i)-1}, \lambda_{k(\lambda,i)} - 1, \ldots, \lambda_i - 1, \lambda_{i+1}, \ldots, \lambda_n)$. Thus for each cover in the maximum length chain, when a box is removed from row i of x_j, we have $i = 1$ or $\lambda_{i-1} - \lambda_i \geq m$. To express this in terms of the entries of the tableau $\Psi(C)$, let b_{gh} denote the entry in row g and column h. Then the condition $i = 1$ or $\lambda_{i-1} - \lambda_i \geq m$ becomes $i = 1$ or $b_{i,h} > b_{i-1,h+m}$.

This is exactly the property that the entries in a tableau must have if we are to be able to shift that tableau by m.

Conversely, given such a tableau, the conditions on the entries guarantee that it represents a maximum length chain in $\mathcal{T}_n^{(m)}$. $\square$

Corollary 2.6 The number of chains of length $\binom{n}{2}$ in $\mathcal{T}_n$ is

$$\binom{n}{2}!\frac{(n-2)!(n-3)!\cdots 1!}{(2n-3)!(2n-5)!\cdots 1!}.$$

Proof. Use the Thrall formula (Thrall 1952) for the number of shifted tableaux of shape $(n-1, n-2, \ldots, 1)$.

□

There are 2^{n-1} partitions with distinct parts whose Young diagram is contained within the Young diagram for $(n-1, \ldots, 1)$. These partitions label the vertices which appear in a chain of maximum length.

2.4 Maximal Chains in Weak Bruhat Order

Let $\mathcal{S}_n$ be the symmetric group, with simple transpositions S. Each w in $\mathcal{S}_n$ can be written as a word in S, and any word for w which is minimal in length among words for w is called a *reduced expression* for w. The *length* of w is the length of any reduced expression for w and is denoted $\ell(w)$. To define the *support* of w, written Sup(w), let $s_{i_1} \cdots s_{i_{l(w)}}$ be any reduced word for w. Then Sup$(w) \subseteq S$ is the set $\{s_{i_1}, s_{i_2}, \ldots, s_{i_{\ell(w)}}\}$. Any reduced expression for w can be transformed into any other by a sequence of braid relation transformations, which means that Sup(w) is independent of the reduced expression for w. The *(right) weak (Bruhat) order* on $\mathcal{S}_n$ is the transitive closure of the cover relation $w \lessdot ws$ whenever $s \in S$ and $\ell(w) < \ell(ws)$. See Figure 18. The identity permutation is the minimum element $\hat{0}$, and the maximum element $\hat{1}$ is w_0, where $w_0(i) = n - i + 1$. The symmetric group, together with the weak order, form a lattice, which we also denote by $\mathcal{S}_n$.

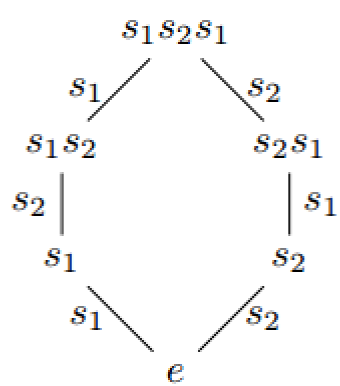

Figure 18: *The Hasse Diagram of the Poset $\mathcal{S}_3$, with Edge from w to ws Labeled by s.*

All maximal chains in $\mathcal{S}_n$ have the same length, $N = \binom{n}{2}$. Stanley 1984 conjectured and proved using symmetric functions that the number of maximal chains in $\mathcal{S}_n$ is the number of standard Young tableaux of shape $(n-1, n-2, \ldots, 1)$. Edelman and Greene 1987 re-proved this result using a bijection, described here in Section 2.4.1. On the other hand, Björner and Wachs 1997 and later Reading 2006 showed that $\mathcal{T}_n$ can be considered as a sublattice of $\mathcal{S}_n$ and that there is an order preserving projection from $\mathcal{S}_n$ to $\mathcal{T}_n$. We describe Reading's approach in Section 2.4.2. In Section 2.4.3, Theorem 2.12, we describe how the Edelman-Greene bijection is related to the function Ψ from Section 2.3 by the inclusion of $\mathcal{T}_n$ in $\mathcal{S}_n$. Note that the Stanley lattice refines $\mathcal{T}_n$ (Bernardi and Bonichon 2009) and has the same number of maximal chains as $\mathcal{S}_n$. However, we are interested in comparing $\mathcal{T}_n$ to $\mathcal{S}_n$, instead of to the Stanley lattice, because of the algebraic structure inherent to $\mathcal{S}_n$.

2.4.1 Maximal Chains in $\mathcal{S}_n$ and Standard Young Tableaux

A maximal chain in $\mathcal{S}_n$ is of the form $C = \{\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{N-1} \lessdot x_N = \hat{1}\}$, where $x_{k-1}s_{i_k} = x_k$ and $\ell(x_{k-1}) + 1 = \ell(x_k)$, for $1 \leq k \leq N$. Each maximal chain can be seen as a reduced word $s_{i_1} \cdots s_{i_N}$ for w_0, and we will use both ways of indicating a chain.

Edelman and Greene 1987 defines a bijection between maximal chains in $\mathcal{S}_n$ and standard Young tableaux of shape $(n-1, \ldots, 1)$. The authors define an analog of the Robinson-Schensted-Knuth correspondence for reduced expressions. Here we assume the reader is familiar with the usual Robinson-Schensted-Knuth insertion and review the Coxeter-Knuth insertion from Edelman and Greene 1987.

Definition 2.7 (Edelman and Greene 1987, Definition 6.20, Coxeter-Knuth insertion). Suppose that T is a tableau with rows $T_1, T_2, \ldots, T_j$ and x_0 is to be inserted into T_1. For each $i \geq 0$ add x_i to row T_{i+1}, bumping (perhaps) x_{i+1} to the next row, using the usual Robinson-Schensted-Knuth insertion, except in the following special case. If x_i bumps x_{i+1} from row T_{i+1}, $x_{i+1} = x_i + 1$, and x_i is already present in T_{i+1}, the value of x_i in T_{i+1} is changed from x_i to x_{i+1}.

Also in their words, if x is inserted into a row containing $x\ x+1$, a copy of $x+1$ is bumped to the next row, but the original $x\ x+1$ remains unchanged.

For the bijection: begin with a maximal chain in $\mathcal{S}_n$, written as a reduced expression $s_{i_1} \cdots s_{i_N}$ for w_0. Coxeter-Knuth insert its indices in reverse into the empty tableau to obtain a pair (P, Q) of tableaux of shape $(n-1, n-2, \ldots, 1)$. The Edelman-Greene bijection matches the chain with the standard Young tableau Q. See Figure 19.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad\qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$

Figure 19: *The tableau on the left corresponds to the chain $s_1s_2s_1$ on the left in Figure 18, and the tableau on the right to the chain $s_2s_1s_2$ on the right. The tableau on the left can be shifted by one, the one on the right cannot.*

2.4.2 c-Sorting

Björner and Wachs 1997 show that $\mathcal{T}_n$ is induced by weak order on a certain set of permutations, 312-avoiding permutations, and is also a quotient of weak order. For the latter, they give a projection from $\mathcal{S}_n$ to $\mathcal{T}_n$. Reading 2006 introduced c-sorting words, which generalize the 312-avoiding permutations. He used them to define Cambrian lattices and thereby generalized the Tamari lattice to any Weyl group W and Coxeter word $w \in C$. He defined a projection which generalizes the Björner-Wachs projection. The Tamari lattice is the case $c = s_1s_2\cdots s_{n-1}$ Cambrian lattice. We use the Reading description of $\mathcal{T}_n$ as a sublattice of $\mathcal{S}_n$, although in some sense it is more general than we need, because it explicitly describes the Tamari lattice in terms of elements of S, the simple transpositions.

Throughout this chapter, c will be the Coxeter word $s_1s_2\ldots s_{n-1}$. For a set $K = \{a_1 < a_2 < \ldots < a_r\} \subset [n]$, let c_K denote $s_{a_1}s_{a_2}\ldots s_{a_r}$. There may be many different ways to write $w \in \mathcal{S}_n$ as a reduced subword of $c^\infty = ccccc\ldots$. The *c-sorting word* of $w \in \mathcal{S}_n$ is the reduced subword of c^∞ for w which is lexicographically first, as a sequence of positions in c^∞. The c-sorting word for w can be written as $c_{K_1}c_{K_2}\cdots c_{K_p}$, where p is minimal for the property

$$w = c_{K_1}c_{K_2}\cdots c_{K_p} \quad \text{and} \quad \ell(w) = \sum_{i=1}^{p} |K_i|.$$

An element $w \in \mathcal{S}_n$ with c-factorization $c_{K_1}c_{K_2}\cdots c_{K_p}$ is called *c-sortable* if $K_p \subset K_{p-1} \subset \ldots \subset K_1$.

Example 2.8 In $\mathcal{S}_4$, let $w = 3241$ and $v = 4132$. w has the three reduced expressions: $s_1s_2s_3s_1$, $s_1s_2s_1s_3$, and $s_2s_1s_2s_3$. Its c-sorting word is $s_1s_2s_3s_1 = c_{\{1,2,3\}}c_{\{1\}}$ and it is c-sortable. v also has three reduced expressions: $s_3s_2s_3s_1$, $s_2s_3s_2s_1$, and $s_3s_2s_1s_3$. Its c-sorting word is $s_2s_3s_2s_1 = c_{\{2,3\}}c_{\{2\}}c_{\{1\}}$ and it is not c-sortable.

Reading 2006 defines the map $\pi_\downarrow^c$, which takes a element w of $\mathcal{S}_n$ to the maximal c-sortable word below w. See Figure 20. In the case $c = s_1s_2\ldots s_{n-1}$, this map is used to show that the Tamari lattice $\mathcal{T}_n$ is a lattice quotient of $\mathcal{S}_n$. Reading considers $\mathcal{T}_n$ as an induced sublattice of $\mathcal{S}_n$ and labels the elements of $\mathcal{T}_n$ by c-sortable words. Maximum length chains in $\mathcal{T}_n$ can be identified with certain maximal chains in $\mathcal{S}_n$.

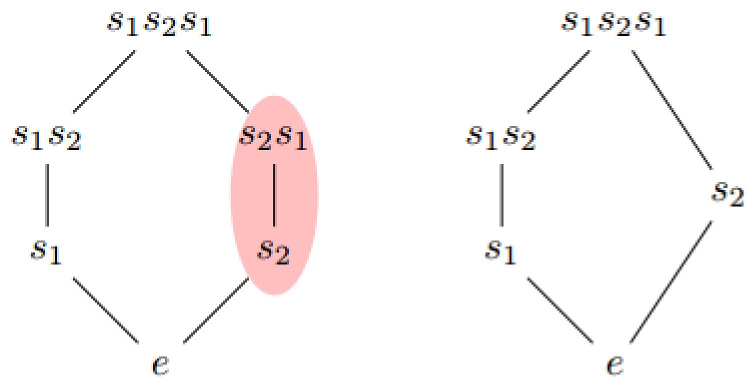

Figure 20: *The Hasse Diagrams of the Posets $\mathcal{S}_3$ and $\mathcal{T}_3$, with Vertices Now Labeled by c-sorting Words. Under $\pi_\downarrow^c$, the vertices s_2 and s_2s_1 are both mapped to the same vertex in $\mathcal{T}_3$, the vertex labeled with s_2.*

We will need the following lemma to characterize reduced expressions for w_0 in $\mathcal{S}_n$ coming from maximal length chains in $\mathcal{T}_n$. It is a very slightly modified version of a lemma from Hohlweg, Lange, and Thomas 2011.

Lemma 2.9 (Hohlweg, Lange, and Thomas 2011, Lemma 2.6) Suppose w and ws_k are both c-sortable and $\ell(ws_k) = \ell(w) + 1$. Suppose w's c-factorization is $c_{K_1} \cdots c_{K_p}$. Then the c-factorization of ws_k is either $c_{K_1} \cdots c_{K_p} c_{\{k\}}$ or $c_{K_1} \cdots c_{K_i \cup \{k\}} \cdots c_{K_p}$.

If $ws_k = c_{K_1} \cdots c_{K_i \cup \{k\}} \cdots c_{K_p}$, then i is uniquely determined and s_k commutes with every s_h for $h \in K_{i+1} \cup M$, where $M = \{m \in K_i | m > k\}$.

Proof. We include, almost verbatim, the proof from Hohlweg, Lange, and Thomas 2011 for completeness.

If $k \in K_p$, the c-factorization of ws_k is simply $c_{K_1} \cdots c_{K_p} c_{\{k\}}$, so assume $k \notin K_p$. Let $c_{L_1} \cdots c_{L_q}$ be the c-factorization of ws_k. Since $\ell((ws_k)s_k) < \ell(ws_k)$, by the exchange property (see Björner and Brenti 2005) there is a unique i, $1 \leq i \leq q$ and $r \in L_i$, such that $w = c_{L_1} \cdots c_{L_i \setminus \{r\}} \cdots c_{L_q}$.

Case 1. Suppose $i = 1$; that is, 1 is the unique index such that

$$w = c_{L_1 \setminus \{r\}} c_{L_2} \cdots c_{L_q}. \tag{2.1}$$

First we show that r is not a member of K_1. Suppose, for a contradiction, that $r \in K_1 = \mathrm{Sup}(w)$. Since $c_{L_1 \setminus \{r\}} c_{L_2} \cdots c_{L_q}$ is reduced and $L_2 \supseteq \cdots \supseteq L_q$ is nested, we have $r \in L_2$. Hence

$$K_1 = \mathrm{Sup}(w) = (L_1 \setminus \{r\}) \cup L_2 = L_1 \cup L_2 = L_1.$$

Thus $c_{L_2} \cdots c_{L_q}$ and $c_{K_2} \cdots c_{K_q} s_k$ are reduced expressions for some $\hat{w} \in W$ and $\ell(\hat{w} s_k) < \ell(\hat{w})$. The exchange condition implies the existence of a unique index j, $2 < j < q$ and $t \in L_j$ such that

$$\hat{w} = c_{L_2} \cdots c_{L_j \setminus \{t\}} \cdots c_{L_q}.$$

In other words,

$$w = c_{L_1}\hat{w} = c_{L_1}c_{L_2}\cdots c_{L_j\setminus\{t\}}\cdots c_{L_q}$$

is reduced. But this contradicts the uniqueness of $i = 1$ in equation (2.1). So $r \notin K_1$. Since $r \notin K_1$, we know $r \notin$ Sup(w) and $r \in$ Sup(ws_k), which shows that $k = r$. Because $s_k \notin$ Sup(w), in order to rewrite sw_k in its c-sorting form, we must have commuted s_k from the right to the left; i.e. s_k commutes with s_h for all $h \in K_2 \cup M$.

Case 2. Suppose $i > 1$. Then $K_1 =$ Sup$(w) = L_1$. Set $\nu := \min(p, i-1)$ and iterate the argument for ${c_{L_1}}^{-1}, {c_{L_2}}^{-1}{c_{L_1}}^{-1}w, \ldots$ to conclude $L_j = K_j$ for $1 \le j \le \nu$. If $\nu = p$ then $i = q = p+1$ and $L_i \setminus \{r\} = \emptyset$. So $L_i = \{j\} \subseteq L_{i-1} = K_p$, which contradicts $k \notin K_p$. Thus $\nu = i-1$ for some $i \le p$ and $L_j = K_j$ for $1 \le j \le i-1$. Thus, we may assume $i = 1$, and we are done by Case 1. $\square$

In a chain of maximum length in $\mathcal{T}_n$, the factorizations change by exactly one transposition as we move up the chain. Thus Lemma 2.9, combined with the fact that s_j and s_{j-1} do not commute, has the following corollary.

Corollary 2.10 Let v be a vertex in a chain of maximum length in $\mathcal{T}_n$ and $c = s_1s_2\ldots s_{n-1}$. Suppose that v's c-factorization is $c_{K_1}\cdots c_{K_p}$. Then if j is a member of K_i, either $j-1 \in K_i$ or $j = 1$. Additionally, if $j \le n-i$ and $j \notin K_i$, then $j-1 \notin K_{i+1}$.

Let λ be a partition. A *lattice permutation of shape* λ is a sequence $a_1a_2\ldots a_N$ in which i occurs λ_i times and such that in any left factor $a_1\ldots a_j$, the number of i's is at least as great as the number of $i+1$'s. See Stanley 1999. We combine Lemma 2.9 with Corollary 2.10 to obtain Corollary 2.11.

Corollary 2.11 Let $s_{i_1}\ldots s_{i_N}$ be a reduced word for w_0 coming from a maximum length chain in $\mathcal{T}_n$. Let $C = i_1i_2\ldots i_N$ and $C^R = i_N\ldots i_1$. Then both C and C^R are lattice permutations for the partition $\lambda = (n-1, n-2, \ldots, 1)$.

2.4.3 Maximal Length Chains in Tamari Order and Weak Bruhat Order

Theorem 2.12 and the results from Section 2.4 explain that the modified Robinson-Schensted algorithm encodes the Tamari order when used on certain chains in the weak Bruhat order on $\mathcal{S}_n$.

Theorem 2.12 Let $C = \{\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{N-1} \lessdot x_N = \hat{1}\}$ be a maximum length chain in $\mathcal{T}_n$. Each x_i is c-sortable, so it may also be considered as a chain in $\mathcal{S}_n$. As a chain in $\mathcal{T}_n$, C maps to a standard Young tableau T as in Section 2.3. As a chain in $\mathcal{S}_n$, it maps to a standard Young tableau T' as in Section 2.4. Then we have $T = T'$.

Proof. First we will define a bijection p between c-sortable words and (m,n)-Dyck partitions for $m = 1$ and show that it respects the covering relation in $\mathcal{T}_n$. We then show that if $w \lessdot ws_{i_k}$, then $p(ws_{i_k})$ is the partition $p(w)$ with column i_k shortened by 1. Lastly we show that if we insert i_k into the tableau produced by $s_{i_N} \ldots s_{i_{k+1}}$, we lengthen column i_k by 1. See Figure 21.

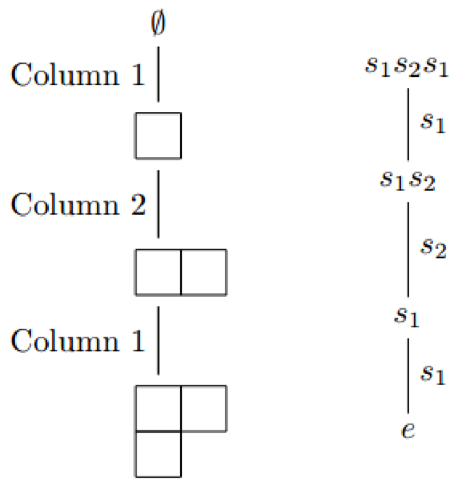

Figure 21: *On the left is the single maximum length chain in $\mathcal{T}_3$, and on the right is the maximum length chain in $\mathcal{S}_3$ that it corresponds to under p. If 121 is inserted into $\emptyset$ using Coxeter-Knuth insertion, the sequence of shapes we obtain is the same as those in the chain on the left.*

Given a c-sortable word w, with c-factorization $c_{K_1} \ldots c_{K_p}$, let $p(w)$ be the partition where column i has length $n - i - \#$ of occurrences of i in w's factorization. See also Bandlow and Killpatrick 2001, Krattenthaler 2001, Fulmek 2003. The partition $p(w)$ can also been described as follows. Let X be the partition $(n-1, n-2, \ldots, 1)$, the minimal element of $\mathcal{T}_n$. The k-th diagonal of X is the set of boxes (i,j) such that $i + j - 1 = k$. Label the boxes in column i by i. Then $p(w)$ is the partition X with the box labeled i in diagonal k removed if and only if $i \in K_k$. Then since $K_p \subset K_{p-1} \subset \ldots \subset K_1$, $p(w)$ is a partition.

If $w \lessdot ws_j$, then by Lemma 2.9 and the definition of p, we have that $p(ws_j)$ is the partition $p(w)$ with column j shortened by 1. Write $p(w) = (a_1, \ldots, a_n)$ and $p(ws_j) = (a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_n)$, where $a_i = j$. The box removed from the diagram of $p(w)$ was on diagonal $i+j-1$. Since this box was present in $p(w)$, we know that $j \notin c_{K_{i+j-1}}$ in the c-factorization of w. Thus by Corollary 2.10, $j+1 \notin c_{K_{i+j-1}}$, so that $a_{i-1} > a_i$ and $p(w) \lessdot p(ws_j)$ according to Definition 2.2.

Let $C = s_{i_1} \ldots s_{i_N}$ be a maximal chain in $\mathcal{S}_n$ which is also a maximal length chain in $\mathcal{T}_n$. Suppose

$s_{i_N}, \ldots, s_{i_{k+1}}$ have been inserted as in Section 2.4 to form a tableau P of shape λ. For ease of notation, write j for s_j. Let x, y, and z be the number of occurrences of $i_k - 1$, i_k, and $i_k + 1$,respectively, in $s_{i_N} \ldots s_{i_{k-1}}$. Since both C and C^R are lattice words and j appears a total of $N - j$ times in C, we have that $x = y + 1$ and $y = z$. Thus, by induction, column $i_k - 1$ has length x and both column i_k and $i_k + 1$ have length $x - 1$. In row h of P, we have $i_k + h - 2$, $i_k + h - 1$, and $i_k + h$ in columns $i_k - 1$, i_k, and $i_k + 1$, respectively. Again, by induction, when $i_k + h - 1$ is inserted into row h, for $h < x$, $i_k + h$ will be bumped into row $h + 1$. Finally, at row x, $i_k + x - 1$ will settle in column i_k, finishing the proof. $\square$

Chapter 3

A RECURSION ON MAXIMAL CHAINS IN THE TAMARI LATTICES

The Tamari lattices have been intensely studied since their introduction by Dov Tamari around 1960. However oddly enough, a formula for the number of maximal chains is still unknown. This is due largely to the fact that maximal chains in the n-th Tamari lattice $\mathcal{T}_n$ range in length from $n-1$ to $\binom{n}{2}$. We treat vertices in the lattice as Young diagrams and identify maximal chains as certain Young tableau. For each $i \geq -1$, we define $\mathcal{C}_i(n)$ as the set of maximal chains in $\mathcal{T}_n$ of length $n+i$. We give a recursion for $\#\mathcal{C}_i(n)$ and an explicit formula based on predetermined initial values. The formula is a polynomial in n of degree $3i+3$. For example, the number of maximal chains of length n in $\mathcal{T}_n$ is $\#\mathcal{C}_0(n) = \binom{n}{3}$. The formula has a combinatorial interpretation in terms of a special property of maximal chains.

3.1 Background

In this chapter, our focus pertains to the following definition.

Definition 3.1 For each $i \geq -1$ and for all $n \geq 1$, $\mathcal{C}_i(n)$ is the set of maximal chains of length $n+i$ in $\mathcal{T}_n$.

Some values of $\#\mathcal{C}_i(n)$ are given in Table 1. The main result of this chapter is Theorem 3.32: we give a recursion for $\#\mathcal{C}_i(n)$ and an explicit formula based on predetermined initial values. The formula is a polynomial in n of degree $3i+3$. For example, the number of maximal chains of length $n-1$ in $\mathcal{T}_n$ is $\#\mathcal{C}_{-1}(n) = 1$, while the number of length n is $\#\mathcal{C}_0(n) = \binom{n}{3}$. For a given n, a column of the table lists the numbers of maximal chains in $\mathcal{T}_n$ by length. In the column indexed by $n = 4$, the numbers of maximal chains of lengths $3, 4, 5$ and 6 in $\mathcal{T}_4$ are $\#\mathcal{C}_{-1}(4) = 1$, $\#\mathcal{C}_0(4) = 4$, $\#\mathcal{C}_1(4) = 2$ and $\#\mathcal{C}_2(4) = 2$, respectively.

Bernardi and Bonichon 2009 rewrote the covering relation in $\mathcal{T}_n$ in terms of Dyck paths. We find it useful to work mainly from the perspective of Young diagrams, but rely on properties of both sets. We present basic terminology and the covering relation in Section 3.2.

We rely on two maps: ψ and $\phi^r_{i,n}$. We use ψ to identify maximal chains in $\mathcal{T}_n$ with certain Young tableaux. We obtain an expression for the number of maximal chains using $\phi^r_{i,n}$, which takes a maximal chain in $\mathcal{C}_i(n)$ to one in $\mathcal{C}_i(n+1)$. Because of ψ, we may express $\phi^r_{i,n}$ as a map on tableaux. Sections 3.3 and 3.4 are devoted to ψ and $\phi^r_{i,n}$, respectively. In Section 3.3, after defining ψ and establishing basic properties, we enter into more technical material, which plays a role in verifying properties of $\phi^r_{i,n}$. A maximal chain in the image of ψ may or may not possess a "plus-full-set"; see Definition 3.16. In Section 3.4, we define $\phi^r_{i,n}$ where r determines the domain and codomain in terms of plus-full-sets. The focus of this section and a key ingredient leading up to our main objective is the fact that $\phi^r_{i,n}$ is bijective (Theorem 3.22).

Table 1: $\#\mathcal{C}_i(n)$: *Number of Maximal Chains in* $\mathcal{T}_n$ *of Length* $n+i$

length \ n	1	2	3	4	5	6	7	8	9
n - 1	1	1	1	1	1	1	1	1	1
n			1	4	10	20	35	56	84
n + 1				2	22	112	392	1,092	2,604
n + 2				2	22	232	1,744	9,220	37,444
n + 3					18	382	4,474	40,414	280,214
n + 4					13	348	8,435	123,704	1,321,879
n + 5					12	456	12,732	276,324	4,578,596
n + 6						390	17,337	550,932	12,512,827
n + 7						420	21,158	917,884	29,499,764
n + 8						334	27,853	1,510,834	62,132,126
n + 9						286	33,940	2,166,460	120,837,274
n + 10							41,230	3,370,312	221,484,557
n + 11							45,048	4,810,150	393,364,848
n + 12							50,752	7,264,302	666,955,139
n + 13							41,826	10,435,954	1,134,705,692
n + 14							33,592	15,227,802	1,933,708,535
n + 15								20,089,002	3,316,121,272
n + 16								27,502,220	5,604,687,775
n + 17								32,145,952	9,577,349,974
n + 18								36,474,460	15,969,449,634
n + 19								30,474,332	26,387,217,370
n + 20								23,178,480	41,902,119,016
n + 21									65,076,754,954
n + 22									93,803,013,648
n + 23									131,664,410,706
n + 24									158,363,393,996
n + 25									179,041,479,392
n + 26									150,158,648,356
n + 27									108,995,910,720
Totals	1	1	2	9	98	2,981	340,549	216,569,887	994,441,978,397

In Section 3.5, we gather more on properties and consequences of $\phi^r_{i,n}$ and tie our results together to write a recursive formula for $\#\mathcal{C}_i(n)$. $\phi^r_{i,n}$ takes a maximal chain to one with one more plus-full-set (Proposition 3.24). This enables us to write every maximal chain, which has a plus-full-set, uniquely in terms of one with one less plus-full-set. This may be extended to a unique representation in terms of a maximal chain with no plus-full-sets (Corollary 3.26). By relating this unique representation for a maximal chain to specific plus-full-sets that it contains (Proposition 3.28), we obtain an expression for $\#\mathcal{C}_i(n)$; see equation (3.5). For each $i \geq -1$, there exists a maximal chain in $\mathcal{C}_i(2i+3)$ containing no plus-full-sets (Lemma 3.30), but surprisingly, for all $n \geq 2i+4$, each maximal chain in $\mathcal{C}_i(n)$ has a plus-full-set (Theorem 3.31). We utilize these latter two facts to refine our expression for $\#\mathcal{C}_i(n)$ and achieve our main objective in Theorem 3.32.

In this section, we present basic terminology of Young diagrams and Dyck paths and the covering relation in the Tamari lattice.

A *partition* of a positive integer l is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ of positive integers summing to l. A *Young diagram* of *shape* λ is a left-justified collection of boxes having λ_j boxes in the j-th row. Rows and columns of the diagram begin with an index of one. We adopt the *English notation*, in which rows are indexed downward. The *empty partition* $\lambda = (0)$ is associated with the *null diagram*, having no boxes. λ is of *staircase shape* if each successive part in the partition is one less than the previous, ending with the last part equal to one. A *Young tableau* or *tableau* for short is obtained by filling a Young diagram, typically with a set (or multiset) of positive integers. In a Young diagram or tableau, we denote the box in the x-th row and y-th column by (x, y). In a tableau T we denote its label by $T(x, y)$.

A *Dyck path* of *length* $2n$ is a path on the square grid of north and east steps from $(0, 0)$ to (n, n) which never goes below the line $y = x$. Necessarily, every Dyck path begins with a north step and ends with an east step, and has an equal number of both types of steps. The *height* of a Dyck path is its number of north steps. In Bernardi and Bonichon 2009, vertices in $\mathcal{T}_n$ are interpreted as the set of Dyck paths of length $2n$, the number of which is the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

There is a natural bijective correspondence between the set of Dyck paths of length $2n$ and a set of Young diagrams, to which we identify the set of vertices in $\mathcal{T}_n$: roughly speaking, a Dyck path gives the silhouette of the Young diagram. For $n \geq 2$, this is the set of Young diagrams contained in the Young diagram of staircase shape $(n-1, \dots, 1)$. $\mathcal{T}_1$ is comprised of a single vertex, the null diagram. Figure 22 is the set of $C_4 = 14$ Dyck paths of length 8 and corresponding Young diagrams.

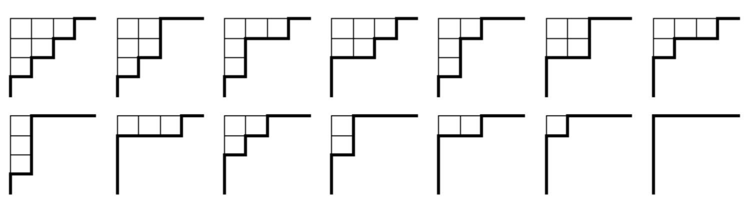

Figure 22: *The Vertices of $\mathcal{T}_4$ in Terms of Dyck Paths and Young Diagrams*

Remark 3.2 Let $Y \in \mathcal{T}_n$ be a Young diagram. For each $m \geq n$, $Y \in \mathcal{T}_m$ and Y corresponds to exactly one

Dyck path of length $2m$. In the other direction, any Dyck path (regardless of length) corresponds to exactly one Young diagram.

Definition 3.3 If P is a Dyck path and L is the line segment (of slope one) that joins the endpoints of P, then P is said to be *prime* if P intersects L only at the endpoints of P. (We word the definition of prime Dyck paths differently than in Bernardi and Bonichon 2009, but our definition has the same meaning.)

Of the Dyck paths in Figure 22, only the last five in the second row are prime, corresponding to the Young diagrams contained in the Young diagram of shape $(2, 1)$. A Dyck path of length $2n$ has exactly n *prime Dyck subpaths*, each uniquely determined by its beginning north step. In Figure 23, for the given Dyck path of length 8, its 4 prime Dyck subpaths are accentuated. The line segment joining the end points of each of the prime subpaths is shown. As required in Proposition 3.12, we characterize pairs of prime Dyck subpaths as follows.

Proposition 3.4 If Q and R are two distinct prime Dyck subpaths of a Dyck path, then exactly one of the following characterizes Q and R:

1. $Q \cap R = \emptyset$, i.e., Q and R have no common points.
2. Q and R intersect in a single point.
3. $Q \subsetneq R$, i.e., Q is a proper subpath of R.
4. $R \subsetneq Q$.

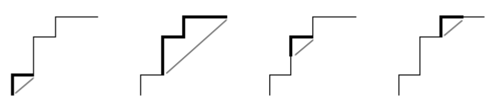

Figure 23: *Prime Dyck Subpath Examples*

For an example of (1), take Q and R to be the prime Dyck subpaths in the first and third examples of Figure 23, respectively. For (2), let Q and R be the subpaths in the first and second examples, respectively. For (3), let Q and R be the subpaths in the third and second examples, respectively.

The notion of prime is intimately tied to the covering relation in the Tamari lattices. We need to extend this notion.

Definition 3.5 Let Y be a Young diagram and P a corresponding Dyck path. If B is a box of Y, whose right vertical edge e lies on P, the *prime path of* B is the prime Dyck subpath of P beginning with e. We call a box of Y, whose bottom horizontal edge and right vertical edge lie on P, a *corner box*.

Figure 24 shows two examples of Definition 3.5, each with the prime path of B drawn. In the second example, B is a corner box.

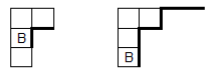

Figure 24: *Prime Path of a Box B Examples*

We give two versions of the covering relation. The second, in terms of Young diagrams, follows from the correspondence of Dyck paths. If x and y are two elements of a poset, we denote x is covered by y (or y covers x) as $x \lessdot y$.

Proposition 3.6 (Bernardi and Bonichon 2009, Proposition 2.1) Covering relation in the Tamari lattices, Dyck paths. Let P and P' be Dyck paths. Then P' covers P if and only if there exists an east step e in P, followed by a north step, such that P' is obtained from P by swapping e and the prime Dyck subpath following it.

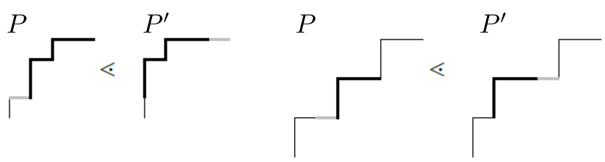

Figure 25: *The Covering Relation in the Tamari Lattices: Dyck Paths*

Proposition 3.7 Covering relation in the Tamari lattices, Young diagrams. Let Y and Y' be Young diagrams. Then Y' covers Y if and only if there exists a corner box B in Y, such that Y' is obtained from Y by removing each box whose right vertical edge lies on the prime path of B.

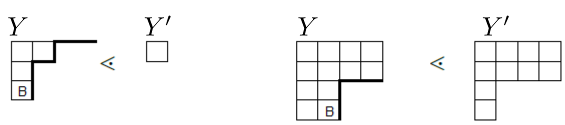

Figure 26: *The Covering Relation in the Tamari Lattices: Young Diagrams*

Examples of the covering relation for Dyck paths in Figure 25 correspond to the examples for Young diagrams in Figure 26. For the Dyck paths, the east step of P referenced in the proposition is grayed and the prime Dyck subpath following it is accentuated. For the Young diagrams, the prime path of the corner box B in Y is drawn.

The Hasse diagrams for $\mathcal{T}_3$ and $\mathcal{T}_4$ are shown in Figure 27. The maximal element $\hat{1}$ in $\mathcal{T}_n$ is the null diagram. For $n \geq 2$, the minimal element $\hat{0}$ has staircase shape $(n-1, \ldots, 1)$.

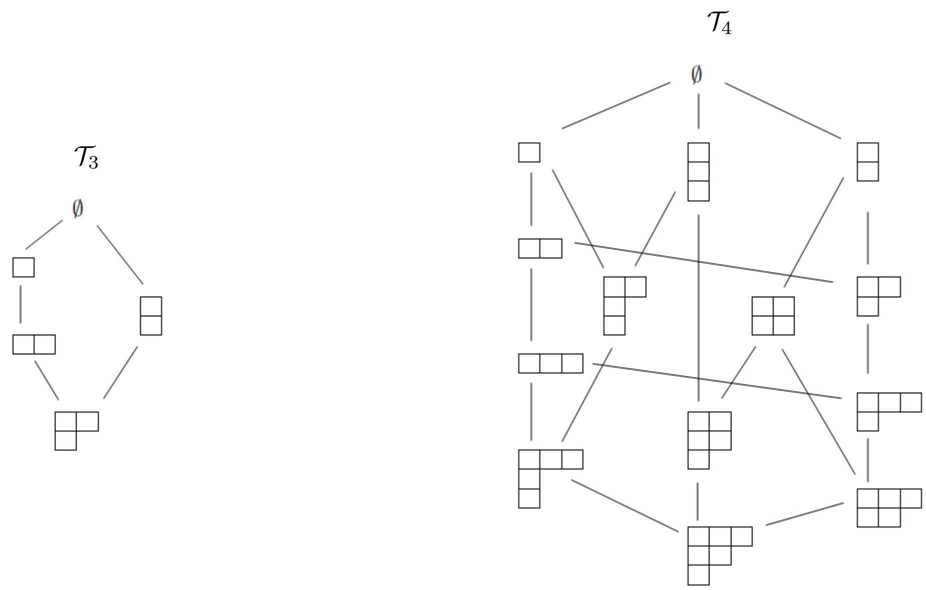

Figure 27: *Hasse Diagrams for $\mathcal{T}_3$ and $\mathcal{T}_4$ in Terms of Young Diagrams*

3.3 Representation of Maximal Chains

In this section, we relate an efficient approach to work with maximal chains (or saturated chains) in the Tamari lattices, through the map ψ. In particular, ψ assigns a maximal chain in $\mathcal{T}_n$ uniquely to a tableau. We establish basic properties, then enter into more technical material which plays an important role in verifying properties of the map $\phi_{i,n}^r$ defined in Section 3.4.

Definition 3.8 Let $C = (\hat{1} = Y_0 \gtrdot Y_1 \gtrdot \cdots \gtrdot Y_l)$ be a saturated chain under the Tamari order, whose maximal element is the null diagram. As C is traversed upwards in the Hasse diagram, boxes are removed in each transition. For each $r \in [l]$, label the boxes of Y_l corresponding to the boxes removed in the transition $Y_{r-1} \gtrdot Y_r$ with r. The resulting tableau, of the same shape as Y_l, is $\psi(C)$. A ψ*-tableau* is an element in the image of ψ. The *length* of a ψ-tableau is the number of its distinct entries.

Examples of ψ-tableaux are shown in Figure 28. The nine maximal chains of $\mathcal{T}_4$ are shown in Figure 29. We defined ψ on maximal chains in Definition 2.3.

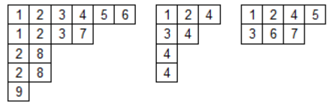

Figure 28: ψ-*tableaux*

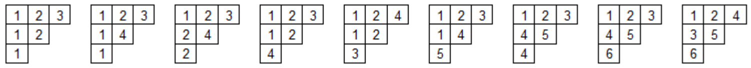

Figure 29: *The Nine Maximal Chains of* $\mathcal{T}_4$ *via* ψ

Remark 3.9 Let T be a ψ-tableau of length l. Then, $T = \psi(C)$ for some $C = (\hat{1} = Y_0 \gtrdot Y_1 \gtrdot \cdots \gtrdot Y_l)$. The length of T is the length of C. For each $r \in \{0, 1, \ldots, l\}$, Y_r is the Young diagram of boxes (x, y) of T, satisfying $T(x, y) \leq r$. Thus ψ is injective.

Moreover, for each $n \geq 1$ and for each Young diagram $Y \in \mathcal{T}_n$, ψ extends to a bijection of sets between saturated chains in $\mathcal{T}_n$ of length l whose minimal element is Y and maximal element is $\hat{1}$, and ψ-tableaux of length l and of the same shape as Y.

Since we identify vertices in $\mathcal{T}_n$, $n \geq 2$, as Young diagrams contained in the Young diagram of staircase shape $(n-1, \ldots, 1)$, ψ induces the following examples of bijective correspondences between:

- maximal chains in $\mathcal{T}_n$ of length $n+i$, *i.e.*, $\mathcal{C}_i(n)$, and ψ-tableaux of length $n+i$ and of staircase shape $(n-1, \ldots, 1)$,
- maximal chains in $\mathcal{T}_n$ and ψ-tableaux of staircase shape $(n-1, \ldots, 1)$,
- saturated chains in $\mathcal{T}_n$ ending with $\hat{1}$ and ψ-tableaux which fit in the Young diagram of staircase shape $(n-1, \ldots, 1)$.

Note, $\mathcal{T}_1$ has exactly one maximal chain, its length being zero. This is the unique maximal chain of $\mathcal{C}_{-1}(1)$, corresponding to the ψ-tableau of zero length, *i.e.*, the null diagram. We move forward now to relate basic properties of ψ-tableaux.

Definition 3.10 Let T be a ψ-tableau of length $l > 0$ and let $r \in [l]$. We call the boxes labeled with r an r-*set*. We say the r-set *begins* in its box of minimum row index and *ends* in its box of maximum row index. (This is well-defined since each row of labels of T is strictly increasing.) The *minor diagonal* of a Young diagram of staircase shape $(n-1, \ldots, 1)$, $n \geq 2$, is the set of boxes $(j, n-j)$, $j \in [n-1]$.

Proposition 3.11 Basic properties of ψ-tableaux. Let T be a ψ-tableau of length $l > 0$ and let $\psi^{-1}(T) = (\hat{1} = Y_0 \gtrdot Y_1 \gtrdot \cdots \gtrdot Y_l)$. Let $r \in [l]$ and suppose the r-set ends in (x, y) of T. Let k be the number of boxes in the r-set.

1. The r-set in T corresponds to the boxes removed from Y_r in the transition $Y_{r-1} \gtrdot Y_r$ (Definition 3.8), those boxes in Y_r whose right vertical edges lie on the prime path of the corner box (x, y) (Proposition 3.7), and consist of the last box in each j-th row where $x - k + 1 \le j \le x$.
2. The height of the prime path of the corner box (x, y) in Y_r equals k.
3. If (a, b) is a box of T labeled with r, then in Y_r, the prime path of (a, b) is a subpath of the prime path of (x, y) (Proposition 3.4).
4. $k > 1$ if and only if $r = T(x, y) = T(x - 1, y)$.
5. Each row of labels of T is strictly increasing when read left to right, and each column is weakly increasing when read top to bottom.
6. If T is of staircase shape, then there are no repeat labels in the minor diagonal of T.

We reference statements in Proposition 3.11 frequently in upcoming material. They follow immediately from basic definitions and the covering relation. The remaining material of this section is more technical and is necessary to verify properties of the map $\phi^r_{i,n}$ defined in the next section.

Proposition 3.12 Let $C = (\hat{1} = P_0 \gtrdot P_1 \gtrdot \cdots \gtrdot P_l)$ be a saturated chain in $\mathcal{T}_n$ whose maximal element is $\hat{1}$, where the P_j are Dyck paths of length $2n$. Fix an integer $k \in [n]$. Let h_j, $0 \le j \le l$, be the height of the prime Dyck subpath which begins with the k-th north step of P_j. Then, $h_0 \ge h_1 \ge \cdots \ge h_l$.

Proof. By induction on l, it suffices to show $h_{l-1} \ge h_l$. Let Q be the prime Dyck subpath of P_l, beginning with the k-th north step. Let R be the prime Dyck subpath of P_l that shifts to the left one unit in the transition $P_{l-1} \gtrdot P_l$ (see Proposition 3.6). We have a few cases to check as outlined in Proposition 3.4. If $Q = R$ or $Q \subsetneq R$ or $R \subsetneq Q$ or $Q \cap R = \emptyset$, then $h_{l-1} = h_l$. We're left with the case where $Q \cap R$ is a single point. If R ends at the same point where Q begins, then $h_{l-1} = h_l$. On the other hand, if Q ends at the same point where R begins, then the trailing east step of Q swaps with R in the transition $P_{l-1} \gtrdot P_l$. In this case, $h_{l-1} > h_l$. $\square$

Figure 30 is an example of a maximal chain in terms of Dyck paths P_j of length 8. As referenced in Proposition 3.12, the sequences $(h_0, h_1, h_2, h_3, h_4)$ for $k \in \{1, 2, 3, 4\}$ are $(4, 4, 2, 1, 1)$, $(3, 1, 1, 1, 1)$, $(2, 2, 2, 2, 1)$, $(1, 1, 1, 1, 1)$, respectively.

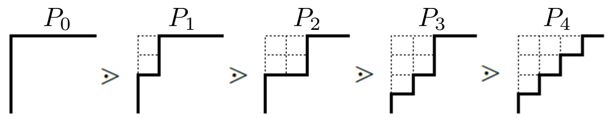

Figure 30: *Decreasing Heights Example*

Theorem 3.13 Let T be a ψ-tableau in which its d-th row has a nonzero total number of boxes b. Suppose that the height of the prime path of (d, b) is d. Obtain T' from T by shifting any rows of index greater than d down one row and repeating the d-th row in the $(d+1)$-th row. Then T' is a ψ-tableau.

Proof. Let $\psi^{-1}(T) = (\hat{1} = Y_0 \gtrdot Y_1 \gtrdot \cdots \gtrdot Y_l)$ and let $r = T(d, b)$.

Suppose there is an s-set in T such that $s > r$. By property (5) of Proposition 3.11, since $s > r = T(d, b)$, the s-set does not contain a box in the d-th row of T, and by property (1), there are two cases. If the s-set in T ends in a row of index less than d, it has the same positioning as the s-set in T'. Otherwise, if the s-set in T begins in a row of index greater than d, it has the same orientation relative to (d, b) as the s-set in T' relative to $(d+1, b)$. By nature of the covering relation, the collection of j-sets in T', $j > r$, has a valid labeling as inherited from T.

We now show that the r-set in T' has a valid labeling. Obtain U and U' from T and T', respectively, by removing any j-sets , $j > r$. Note U is a ψ-tableau. Let B and B' be the corner boxes in which the r-set ends in U and U', respectively. B' has one greater row index than B, which has that at least d. B' and B share the same column index. By assumption, the height of the prime path of (d, b) in Y_l is d, and since U has the same shape as Y_r, the height of the prime path of (d, b) in U is d (Proposition 3.12). The prime path of (d, b) is a subpath of the prime path of B in U (property (3) of Proposition 3.11), thus the r-set in U begins in its first row. By property (5) of Proposition 3.11 and our construction, the last box in each row of index less than or equal to that of B and B', in U and U', respectively, is labeled with r. The height of the prime path of (d, b) in U' is d, as U has this property. In U', the prime path of (d, b) is a subpath of the prime path of $(d+1, b)$ and the prime path of $(d+1, b)$ is a subpath of the prime path of B'. The latter follows because in U, the prime path of (d, b) is a subpath of the prime path of B. These observations imply that the r-set in U' is precisely those boxes whose right vertical edges lie on the prime path of B'. Therefore, the r-set in U' (and thus in T') has a valid labeling.

Further alter U and U' by removing the r-set from each. U is a ψ-tableau. If $b = 1$, $U = U'$ and thus T' is a ψ-tableau; otherwise, suppose $b > 1$. Again by Proposition 3.12, the height of the prime path of $(d, b-1)$ in U is d, and furthermore, U' is obtained from U as described in the theorem. By induction, U' is a ψ-tableau. Thus also T' is a ψ-tableau. $\square$

Figure 31 illustrates an example of Theorem 3.13 for $d = 5$ and $b = 2$. The prime path of (d, b) in T is drawn.

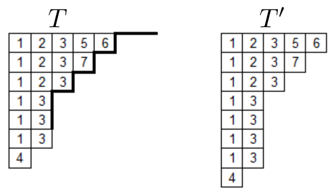

Figure 31: *Repeating a Row Example:* $d = 5$, $b = 2$

Lemma 3.14 Let T be a ψ-tableau in which its d-th and $(d+1)$-th rows have the same nonzero total number of boxes b. Then, for each y satisfying $1 \leq y \leq b$, $T(d, y) = T(d+1, y)$.

Proof. We show that $T(d, b) = T(d+1, b)$, then are done by induction. Let $\psi^{-1}(T) = (\hat{1} = Y_0 \gtrdot Y_1 \gtrdot \cdots \gtrdot Y_l)$ and let $r = T(d, b)$. For each Young diagram Y_j which contains $(d+1, b)$, Y_j must contain (d, b), and the prime path of (d, b) is a proper subpath of the prime path of $(d+1, b)$. This implies that $(d+1, b)$ and (d, b) are removed in the same transition, *i.e.*, $Y_{r-1} \gtrdot Y_r$. $\square$

Theorem 3.15 (Converse of Theorem 3.13) Let T be a ψ-tableau in which its d-th and $(d+1)$-th rows have the same nonzero total number of boxes b. Suppose that the height of the prime path of (d, b) is d. Obtain T' from T by deleting the $(d+1)$-th row and shifting any rows of index greater than $d+1$ up one row. Then T' is a ψ-tableau. Moreover, T' has the same length as T.

Proof. Let $\psi^{-1}(T) = (\hat{1} = Y_0 \gtrdot Y_1 \gtrdot \cdots \gtrdot Y_l)$ and let $r = T(d, b)$. By Lemma 3.14, the d-th and $(d+1)$-th rows of T have identical labels. Thus, T' has the same length as T.

Suppose there is an s-set in T such that $s > r$. If the s-set in T ends in a row of index less than d, it has the same positioning as the s-set in T'. Otherwise, if the s-set in T begins in a row of index greater than $d+1$, it has the same orientation relative to $(d+1, b)$ as the s-set in T' relative to (d, b). The collection of j-sets in T', $j > r$, has a valid labeling as inherited from T.

We now show that the r-set in T' has a valid labeling. Obtain U and U' from T and T', respectively, by removing any j-sets, $j > r$. Note U is a ψ-tableau. Let B and B' be the corner boxes in which the r-sets end in U and U', respectively. B' has one less row index than B, which has that at least $d+1$. B' and B share the same column index. As in the proof of Theorem 3.13, U has the same shape as Y_r, the height of the prime path of (d, b) in U is d, and the prime path of (d, b) is a subpath of the prime path of B in U. Thus the r-set in U begins in the first row. The last box in each row of index less than or equal to that of B and B', in U and U', respectively, is labeled with r. The height of the prime path of (d, b) in U' is d, as U has this property. In U', the prime path of (d, b) is a subpath of the prime path of B'. This follows because in U, the prime path of $(d + 1, b)$ is a subpath of the prime path of B. These observations imply that the r-set in U' is precisely those boxes whose right vertical edges lie on the prime path of B'. Therefore, the r-set in U' (and thus in T') has a valid labeling.

The rest of the proof follows exactly the last paragraph in the proof of Theorem 3.13. $\square$

Going forward, we identify chains by corresponding ψ-tableaux.

3.4 Plus-full-sets and the Map $\phi^r_{i,n}$

This section is devoted to $\phi^r_{i,n}$. A maximal chain in the image of ψ may or may not possess a "plus-full-set". $\phi^r_{i,n}$ takes a maximal chain in $\mathcal{C}_i(n)$ to one in $\mathcal{C}_i(n+1)$, where r determines the domain and codomain in terms of plus-full-sets. Because of ψ, we may express $\phi^r_{i,n}$ as a map on tableaux. The main focus of this section and a key ingredient leading up to Theorem 3.32 is the fact that $\phi^r_{i,n}$ is bijective (Theorem 3.22).

Definition 3.16 Let C be a maximal chain in $\mathcal{T}_n$, $n \geq 2$. Let l be its length and let $r \in [l]$. If the r-set begins in the first row and ends in the minor diagonal of C, then we call the r-set an r-*full-set*, or more generally a *full-set*. In this case, there is a unique box in the minor diagonal of C labeled with r (property (6) of Proposition 3.11), *i.e.*, there is a unique $k \in [n-1]$ satisfying $C(k, n-k) = r$. We call the r-set an r^+-*full-set*, or more generally a *plus-full-set*, if each of the following is met:

1. The r-set in C is a full-set.
2. The box $(k, n-k)$ in which the r-set ends, is such that $k = n - 1$, or the southwest neighbor of $(k, n-k)$ has a label less than r, *i.e.*, $k \in [n-2]$ and $C(k+1, n-k-1) < C(k, n-k) = r$.

The number of full-sets in a maximal chain C of length $l \geq 0$ is $\#\{j \in [l] \mid \text{the } j\text{-set in } C \text{ is a full-set}\}$ (the null diagram has zero full-sets).

Definition 3.17 For each $r \in [n+i]$, $\mathcal{S}^r_i(n)$ is the set of $C \in \mathcal{C}_i(n)$ satisfying:

1. the r-set in C is a plus-full-set, and
2. for each $j \in [r-1]$, the j-set in C is not a plus-full-set.

Remark 3.18 Condition (2) ensures that the $\mathcal{S}_i^r(n)$, $r \in [n+i]$, are disjoint subsets of $\mathcal{C}_i(n)$. We denote a disjoint union of sets with $\biguplus$.

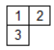

Figure 32: $\mathcal{C}_0(3)$

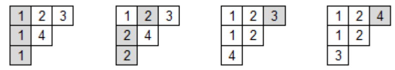

Figure 33: $\mathcal{C}_0(4) = \biguplus_{r \in [4]} \mathcal{S}_0^r(4)$

$\mathcal{C}_0(3)$ consists of a single maximal chain, call it C, shown in Figure 32. Each of the 1-set and 3-set in C is not a full-set, so is not a plus-full-set. The 2-set is a full-set, but is not a plus-full-set ($3 = C(2,1) > C(1,2) = 2$). It follows that each of the subsets $\mathcal{S}_0^1(3)$, $\mathcal{S}_0^2(3)$ and $\mathcal{S}_0^3(3)$ of $\mathcal{C}_0(3)$ is the empty set.

Each of the subsets $\mathcal{S}_0^1(4)$, $\mathcal{S}_0^2(4)$, $\mathcal{S}_0^3(4)$, $\mathcal{S}_0^4(4)$ of $\mathcal{C}_0(4)$, consists of exactly one maximal chain, listed respectively in Figure 33. The plus-full-set that qualifies each maximal chain is grayed. $\mathcal{C}_0(4)$ is the disjoint union of its subsets $\mathcal{S}_0^r(4)$, $r \in [4]$. In Theorem 3.31, we show that for all $n \geq 2i+4$, $\mathcal{C}_i(n)$ is the disjoint union of nonempty subsets $\mathcal{S}_i^r(n)$, $r \in [3i+4]$.

Definition 3.19 For each triple i, n and r, satisfying $i \geq -1$, $n \geq 1$ and $0 \leq r \leq n+i$, we define the map

$$\phi_{i,n}^r : \{C \in \mathcal{C}_i(n) \mid \forall j \in [r],\, C \notin \mathcal{S}_i^j(n)\} \to \mathcal{S}_i^{r+1}(n+1)$$

as follows. Suppose C is an element of the domain. If there exists $k \in [n-1]$ such that $C(k, n-k) \leq r$, then let d be minimal for k; otherwise, set $d = n$. Perform the following iterative steps on C to obtain $\phi_{i,n}^r(C)$:

1. In each row of index less than d, shift boxes with labels greater than r to the right by one box (leaving one unlabeled box in that row to the left of shifted boxes).
2. If $d \in [n-1]$, shift any rows of index greater than d down one row and repeat the d-th row in the $(d+1)$-th row.
3. Add the unlabeled box $(d, n-d+1)$ to the diagram.
4. Increment any labels greater than r by one.
5. Label the unlabeled boxes with $r+1$.

It's not clear that $\phi_{i,n}^r$ maps into the range as defined, but the main focus of this section is to show that $\phi_{i,n}^r$ is actually bijective. For a particular maximal chain $C \in \mathcal{C}_1(5)$, examples of $\phi_{1,5}^r(C)$ are shown in Figure 34. C has no plus-full-sets, so for each r satisfying $0 \leq r \leq 6$, C is an element of the domain of $\phi_{1,5}^r$. So we compute $\phi^r_{1,5}(C)$ for each r in the noted range. The columns show the resulting diagram after each step, the last being $\phi^r_{1,5}(C)$. Boxes shifted in step (1) are grayed under Step 1 in the examples. Boxes shifted in step (2) are grayed and the borders of repeated rows are accentuated under Step 2. The border of the unlabeled box added in step (3) is accentuated under Step 3. Boxes whose labels are incremented in step (4) are grayed under Step 4. Boxes labeled with $r+1$ in step (5) are grayed under Step 5.

Remark 3.20 The condition on the domain of $\phi^r_{i,n}$, $\forall j \in [r]$, $C \notin \mathcal{S}^j_i(n)$, is equivalent to the condition, $\forall j \in [r]$, the j-set in C is not a plus-full-set.

Theorem 3.21 If C is in the domain of $\phi^r_{i,n}$, then $\phi^r_{i,n}(C) \in \mathcal{S}^{r+1}_i(n+1)$.

Proof. Let ϕ denote $\phi^r_{i,n}$, and suppose C is in the domain of ϕ. We first show $\phi(C) \in \mathcal{C}_i(n+1)$. By our choice of d, each box of row index less than d in the minor diagonal of C has a label greater than r. Thus, steps (1)-(3) ensure that $\phi(C)$ has the staircase shape $(n, \ldots, 1)$. Additionally, steps (3)-(5) result in $\phi(C)$ having length $n+1+i$, thus we only need to show that $\phi(C)$ is a ψ-tableau.

Suppose there is an s-set in C such that $s > r$. If $d = n$, the s-set ends in a row of index less than d. If $d \in [n-1]$, then since $C(d, n-d) \leq r < s$, the s-set does not contain a box in the d-th row of C. Thus we obtain the following two cases. If the s-set in C ends in a row of index less than d, it has the same orientation relative to $(1, n-1)$ as the $(s+1)$-set in $\phi(C)$ relative to $(1, n)$, due to steps (1) and (4) (see the grayed boxes under Step 1 in the examples). Otherwise, if the s-set in C begins in a row of index greater than d, it has the same orientation relative to $(d, n-d)$ as the $(s+1)$-set in $\phi(C)$ relative to $(d+1, n-d)$, due to steps (2) and (4) (for $r = 3$ see the 4-set under C and the 5-set under Step 4, also for $r = 5$ see the 6-set under C and the 7-set under Step 4 in the examples). The collection of j-sets in $\phi(C)$, $j > r+1$, has a valid labeling as inherited from C.

Next, we look at the $(r+1)$-set in $\phi(C)$. Obtain C' from the transformed diagram after step (4) is completed, by removing any j-sets, $j > r+1$. No boxes in C' are labeled with $r+1$. By our choice of d, for each $j \in [d-1]$, $C(j, n-j) > r$. Thus, due to steps (1) and (4), for each $j \in [d-1]$, C' does not have the box $(j, n-j+1)$ and the last box in the j-th row of C' is unlabeled. $(d, n-d+1)$ is the only other unlabeled box in C'. These observations, by Definition 3.3, imply that the height of the prime path of $(d, n-d+1)$ in C' is d and the unlabeled boxes in C' are precisely those whose right vertical edges lie on the prime path of $(d, n-d+1)$. Thus, the boxes labeled with $r+1$ in step (5) have valid labels. Furthermore, the $(r+1)$-set in $\phi(C)$ is a plus-full-set as follows. It begins in the first row of $\phi(C)$. If $d = n$, it ends in

$(n,1)$. If $d \in [n-1]$, it ends in $(d, n-d+1)$ and by way of steps (2)-(5),

$$\phi(C)(d+1, n-d) = \phi(C)(d, n-d) = C(d, n-d) < r+1 = \phi(C)(d, n-d+1).$$

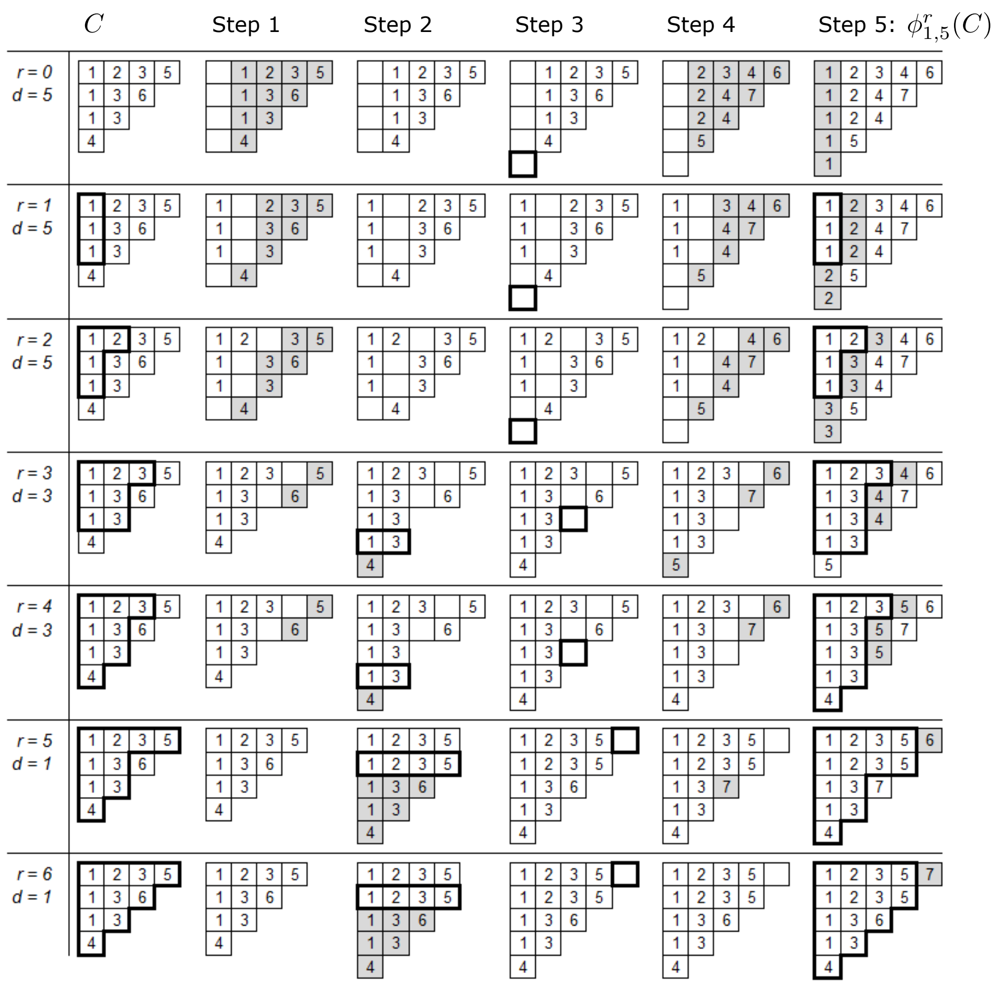

Figure 34: *Examples of* $\phi^r_{1,5}$

Obtain T from C and T' from $\phi(C)$ by removing any j-sets, $j > r$. (In the last six examples, T and T' are the accentuated bordered areas under C and $\phi(C)$, respectively.) T is a ψ-tableau, and if we show that T' is a ψ-tableau, then we have proved that $\phi(C)$ is a ψ-tableau. If $r = 0$, then each of T and T' is the null diagram (as in the first example). More generally, if $d = n$ (as in the first three examples), then $T = T'$, so assume $d \in [n-1]$. By our choice of d, T has the box $(d, n-d)$, but for each $j \in [d-1]$, does not have the box $(j, n-j)$. By Definition 3.3, the height of the prime path of $(d, n-d)$ in T is d. T satisfies the conditions of Theorem 3.13, and T' is obtained here as it is obtained in the theorem. Thus T' is a ψ-tableau.

Furthermore, for $d \in [n-1]$, the d-th row of C agrees with the $(d+1)$-th row of $\phi(C)$, and any entry in such rows is an entry in the first row of both C and $\phi(C)$.

To prove $\phi(C) \in \mathcal{S}_i^{r+1}(n+1)$, it remains to show that for each $j \in [r]$, $\phi(C)$ has no j^+-full-set. So assume $r \geq 1$ and let $j \in [r]$. We claim that C has a j-full-set if and only if $\phi(C)$ has a j-full-set as follows. Each box in the minor diagonal of C and $\phi(C)$ of row index less than d and $d+1$, respectively, has a label greater than r, which restricts where a j-full-set may end. For $d \leq a \leq n-1$, step (2) implies that $C(a, n-a) = j$ if and only if $\phi(C)(a+1, n-a) = j$. In that case, if C has a j-full-set or if $\phi(C)$ has a j-full-set, then j must be an entry in the d-th row of C or the $(d+1)$-th row of $\phi(C)$, respectively, and by the last sentence of the previous paragraph, j is an entry in the first row of both C and $\phi(C)$. Thus our claim is settled. So suppose $\phi(C)$ has a j-full-set such that $\phi(C)(a+1, n-a) = j$ and $d \leq a \leq n-1$. Then C has a j-full-set such that $C(a, n-a) = j$ which, by Remark 3.20, is not a plus-full-set. Therefore, $a \in [n-2]$ and $C(a+1, n-a-1) > C(a, n-a) = j$. If $C(a+1, n-a-1) > r$, then steps (2) and (4) imply that $\phi(C)(a+2, n-a-1) = C(a+1, n-a-1)+1$; otherwise, step (2) implies that $\phi(C)(a+2, n-a-1) = C(a+1, n-a-1)$. In either case

$$\phi(C)(a+2, n-a-1) \geq C(a+1, n-a-1) > C(a, n-a) = \phi(C)(a+1, n-a) = j,$$

thus the j-full-set in $\phi(C)$ is not a plus-full-set. □

Theorem 3.22 For each triple i, n and r, satisfying $i \geq -1$, $n \geq 1$ and $0 \leq r \leq n+i$, $\phi_{i,n}^r$ is bijective.

Proof. Let ϕ denote $\phi_{i,n}^r$. Suppose $\phi(C_1) = \phi(C_2) = C \in \mathcal{S}_i^{r+1}(n+1)$. Let the d-values in the definition of ϕ for C_1 and C_2 be d_1 and d_2, respectively. Then $C(d_1, n-d_1+1) = r+1 = C(d_2, n-d_2+1)$, as seen in steps (3) and (5) of Definition 3.19. However, as noted in Definition 3.16, there is exactly one box in the minor diagonal of C labeled with $r+1$, thus $d_1 = d_2$. Furthermore, with knowledge of d satisfying $r+1 = C(d, n-d+1)$, the steps of ϕ are invertible. Therefore, $C_1 = C_2$ and ϕ is injective.

To prove ϕ is surjective, suppose $C \in \mathcal{S}_i^{r+1}(n+1)$. Ascertain the unique $d \in [n]$ satisfying $C(d, n-d+1) = r+1$ and perform the following iterative steps on C to obtain $\phi^{-1}(C)$:

1. Erase labels of $r+1$ (leaving unlabeled boxes).
2. Decrement any labels greater than $r+1$ by one.
3. Remove the box $(d, n-d+1)$ from the diagram.
4. If $d \in [n-1]$, delete the $(d+1)$-th row and shift any rows of index greater than $d+1$ up one row.
5. In each row of index less than d, shift boxes with labels greater than r to the left by one box (overlaying the unlabeled box in that row).

We first show $\phi^{-1}(C) \in \mathcal{C}_i(n)$. The $(r+1)$-full-set in C is a set of d boxes, one in each j-th row, $j \in [d]$, ending in $(d, n-d+1)$. Any boxes in the same row as one in the $(r+1)$-set, positioned to the left have labels r or less, while those positioned to the right have labels greater than $r+1$. Thus, each box of row index less than d, in the minor diagonal of C, has a label greater than $r+1$. The steps involved in ϕ^{-1} then ensure that $\phi^{-1}(C)$ has one less box in each row than C. Moreover, there are no unlabeled boxes in $\phi^{-1}(C)$ due to step (1) because of steps (3) and (5). It remains to show that $\phi^{-1}(C)$ is a ψ-tableau of length $n+i$.

Suppose there is an s-set in C such that $s > r+1$. If $d = n$, then $C(n,1) = r+1$ and the s-set ends in a row of index less than d. If $d \in [n-1]$, by condition (2) of Definition 3.16, $C(d+1, n-d) < C(d, n-d+1) = r+1$, thus neither the d-th nor $(d+1)$-th rows of C contains a box in the s-set. We obtain the following two cases. If the s-set in C ends in a row of index less than d, it has the same orientation relative to $(1,n)$ as the $(s-1)$-set in $\phi^{-1}(C)$ relative to $(1, n-1)$, due to steps (2) and (5). Otherwise, if the s-set in C begins in a row of index greater than $d+1$, it has the same orientation relative to $(d+1, n-d)$ as the $(s-1)$-set in $\phi^{-1}(C)$ relative to $(d, n-d)$, due to steps (2) and (4). The collection of j-sets in $\phi^{-1}(C)$, $j > r$, has a valid labeling as inherited from C.

Obtain T from C and T' from $\phi^{-1}(C)$ by removing any j-sets, $j > r$. T is a ψ-tableau, and if we show that T' is a ψ-tableau of the same length as T, then we have proved that $\phi^{-1}(C)$ is a ψ-tableau of length $n+i$. If $d = n$, then $T = T'$, so assume $d \in [n-1]$. As noted above, $C(d+1, n-d) < C(d, n-d+1) = r+1$, thus the last box in the d-th row of T is $(d, n-d)$ and the last box in the $(d+1)$-th row is $(d+1, n-d)$. The height of the prime path of $(d, n-d)$ in T is d, which follows by Proposition 3.12 and the fact that the $(r+1)$-full-set in C ends in $(d, n-d+1)$. T satisfies the conditions of Theorem 3.15, and T' is obtained here as it is obtained in the theorem. Thus, T' is a ψ-tableau having the same length as T. Furthermore, for $d \in [n-1]$, the d-th row of $\phi^{-1}(C)$ agrees with the $(d+1)$-th row of C, and any entry in such rows is an entry in the first row of both $\phi^{-1}(C)$ and C.

It remains to show, by Remark 3.20, that for each $j \in [r]$, $\phi^{-1}(C)$ has no j^+-full-set. So assume $r \geq 1$ and let $j \in [r]$. We claim that $\phi^{-1}(C)$ has a j-full-set if and only if C has a j-full-set as follows. Each box in the minor diagonal of $\phi^{-1}(C)$ and C of row index less than d and $d+1$, respectively, has a label greater than r, which restricts where a j-full-set may end. For $d \leq a \leq n-1$, the last sentence of the previous paragraph and step (4) imply that $\phi^{-1}(C)(a, n-a) = j$ if and only if $C(a+1, n-a) = j$. In that case, if $\phi^{-1}(C)$ has a j-full-set or if C has a j-full-set, then j must be an entry in the d-th row of $\phi^{-1}(C)$ or the $(d+1)$-th row of C, respectively, and by the last sentence of the previous paragraph, j is an entry in the first row of both $\phi^{-1}(C)$ and C. Thus our claim is settled. So suppose $\phi^{-1}(C)$ has a j-full-set such that $\phi^{-1}(C)(a, n-a) = j$ and

$d \le a \le n-1$. Then C has a j-full-set such that $C(a+1, n-a) = j$ which, by Definition 3.17, is not a plus-full-set. Therefore, $a+1 \in [n-1]$ and $C(a+2, n-a-1) > C(a+1, n-a) = j$. If $C(a+2, n-a-1) > r+1$, then steps (2) and (4) imply that $\phi^{-1}(C)(a+1, n-a-1) = C(a+2, n-a-1) - 1 > r$; otherwise, step (4) implies that $\phi^{-1}(C)(a+1, n-a-1) = C(a+2, n-a-1) > j$. In either case

$$\phi^{-1}(C)(a+1, n-a-1) > j = \phi^{-1}(C)(a, n-a),$$

thus the j-full-set in $\phi^{-1}(C)$ is not a plus-full-set. □

Corollary 3.23 For each triple i, n and r, satisfying $i \ge -1$, $n \ge 1$ and $0 \le r \le n+i$,

$$\#\mathcal{S}_i^{r+1}(n+1) = \#\mathcal{C}_i(n) - \sum_{j=1}^{r} \#\mathcal{S}_i^j(n).$$

Proof. This is a direct implication of Theorem 3.22, recalling Remark 3.18. □

3.5 A Formula for the Number of Maximal Chains of Length $n+i$ in $\mathcal{T}_n$

The technical work in verifying the bijectivity of $\phi_{i,n}^r$ is complete! In this section, we gather more on properties and consequences of this map and tie our results together to write a recursive formula for $\#\mathcal{C}_i(n)$. $\phi_{i,n}^r$ takes a maximal chain to one with one more plus-full-set (Proposition 3.24). This enables us to write every maximal chain which has a plus-full-set, uniquely in terms of one with no plus-full-sets (Corollary 3.26). By relating this unique representation for a maximal chain to specific plus-full-sets that it contains (Proposition 3.28), we obtain an expression for $\#\mathcal{C}_i(n)$; see equation (3.5). For each $i \ge -1$, there exists a maximal chain in $\mathcal{C}_i(2i+3)$ containing no plus-full-sets (Lemma 3.30), but surprisingly, for all $n \ge 2i+4$, each maximal chain in $\mathcal{C}_i(n)$ has a plus-full-set (Theorem 3.31). We utilize these latter two facts to refine our expression for $\#\mathcal{C}_i(n)$ and show that it is a polynomial of degree $3i+3$ in Theorem 3.32.

Proposition 3.24 Each C in the domain of $\phi_{i,n}^r$ has one less plus-full-set than its image has, and for $j > r$, C has a j^+-full-set if and only if $\phi_{i,n}^r(C)$ has a $(j+1)^+$-full-set.

Proof. By definition, if C is in the domain of $\phi_{i,n}^r$, then for each $k \in [r]$, neither C nor $\phi_{i,n}^r(C)$ has a k^+-full-set. Of course, $\phi_{i,n}^r(C)$ has the $(r+1)^+$-full-set. So suppose there is $j > r$ such that C has a j^+-full-set. Then for a unique $b \in [n-1]$, $C(b, n-b) = j$. By Definition 3.16, either $b = n-1$, or $b \in [n-2]$ and $C(b+1, n-b-1) < C(b, n-b) = j$. Each box of row index less than b, in the minor diagonal of C, has a label greater than j. By our choice of d in Definition 3.19, always $d > b$.

If $b = n-1$ then $d = n$. Otherwise, suppose $b \in [n-2]$. If $C(b+1, n-b-1) \leq r$ then $d = b+1$, else if $C(b+1, n-b-1) > r$ then $d > b+1$. Regardless of whether $b = n-1$ or $b \in [n-2]$, steps (1) and (4) of Definition 3.19 imply that the j-full-set in C maps to a $(j+1)$-full-set in $\phi^r_{i,n}(C)$ ending in $(b, n-b+1)$. Furthermore, $\phi^r_{i,n}(C)(b+1, n-b) < \phi^r_{i,n}(C)(b, n-b+1) = j+1$ so that the $(j+1)$-full-set in $\phi^r_{i,n}(C)$ is a $(j+1)^+$-full-set. Since $\phi^r_{i,n}$ is bijective, this correspondence is bijective. $\square$

Definition 3.25 $\mathcal{N}_i(n)$ is the subset of maximal chains in $\mathcal{C}_i(n)$ having no plus-full-sets.

$\mathcal{C}_i(n)$ is a disjoint union of the $n+i+1$ subsets, $\mathcal{N}_i(n)$ and $\mathcal{S}^j_i(n)$, $j \in [n+i]$, *i.e.*,

$$\mathcal{C}_i(n) = \mathcal{N}_i(n) \biguplus \left(\biguplus_{j \in [n+i]} \mathcal{S}^j_i(n) \right). \tag{3.1}$$

Corollary 3.26 Suppose the number of plus-full-sets of some $C \in \mathcal{C}_i(n)$ is $t > 0$. Then, there exists a unique $\tilde{C}_1 \in \mathcal{C}_i(n-1)$ and a unique r_1, such that $C = \phi^{r_1}_{i,n-1}(\tilde{C}_1)$. This representation may be extended to obtain unique representations

$$\begin{aligned} C &= \phi^{r_1}_{i,n-1}(\tilde{C}_1) \\ &= \phi^{r_1}_{i,n-1}(\phi^{r_2}_{i,n-2}(\tilde{C}_2)) \\ &\vdots \\ &= (\phi^{r_1}_{i,n-1} \circ \phi^{r_2}_{i,n-2} \circ \cdots \circ \phi^{r_t}_{i,n-t})(\tilde{C}_t), \end{aligned}$$

until we arrive at $\tilde{C}_t \in \mathcal{N}_i(n-t)$.

Proof. The codomain of $\phi^r_{i,n-1}$ is $\mathcal{S}^{r+1}_i(n)$. As r varies, $0 \leq r \leq n-1+i$, the $\mathcal{S}^{r+1}_i(n)$ are disjoint subsets of $\mathcal{C}_i(n)$. C is an element of exactly one of the $\mathcal{S}^{r+1}_i(n)$, so there exists a unique r_1, such that $\phi^{r_1}_{i,n-1}$ has C in its codomain. Since $\phi^{r_1}_{i,n-1}$ is bijective, there exists a unique $\tilde{C}_1 \in \mathcal{C}_i(n-1)$, such that $C = \phi^{r_1}_{i,n-1}(\tilde{C}_1)$. By the proposition, the number of plus-full-sets in $\tilde{C}_1$ is $t-1$. If $t = 1$, then $\tilde{C}_1 \in \mathcal{N}_i(n-1)$; otherwise, $t-1 > 0$, and we may repeat until we arrive at $\tilde{C}_t \in \mathcal{N}_i(n-t) \subseteq \mathcal{C}_i(n-t)$. $\square$

Remark 3.27 A maximal chain in $\mathcal{T}_n$ has at most $n-1$ plus-full-sets, as bounded by the $n-1$ strictly increasing labels in the first row of its ψ-tableau.

By Corollary 3.26, the number of maximal chains in $\mathcal{C}_i(n)$ with exactly t plus-full-sets, $1 \leq t \leq n-1$, is the number of representations

$$(\phi^{r_1}_{i,n-1} \circ \phi^{r_2}_{i,n-2} \circ \cdots \circ \phi^{r_t}_{i,n-t})(\tilde{C}) \tag{3.2}$$

over t-tuples $(r_1, r_2, \ldots, r_t)$ and over $\tilde{C} \in \mathcal{N}_i(n-t)$. Each t-tuple $(r_1, r_2, \ldots, r_t)$ must satisfy restrictions imposed on the r_j in Definition 3.19:

Proposition 3.28 Suppose that $\tilde{C} \in \mathcal{N}_i(n-t)$ for some n and t, satisfying $1 \leq t \leq n-1$. A t-tuple $(r_1, r_2, \ldots, r_t)$ for the representation (3.2), must only satisfy $0 \leq r_1 \leq r_2 \leq \cdots \leq r_t \leq n-t+i$. The number of these, hence the number of representations (3.2), is $\binom{n+i}{t}$.

For a t-tuple $(r_1, r_2, \ldots, r_t)$ which satisfies the criteria, let $C = (\phi_{i,n-1}^{r_1} \circ \phi_{i,n-2}^{r_2} \circ \cdots \circ \phi_{i,n-t}^{r_t})(\tilde{C})$. The set of specific plus-full-sets in C is

$$\{j \mid C \text{ has a } j^+\text{-full-set}\} = \{r_1+1, r_2+2, \ldots, r_t+t\},$$

which is a t-element subset of $[n+i]$ unique to $(r_1, r_2, \ldots, r_t)$.

Proof. Consider a t-tuple $(r_1, r_2, \ldots, r_t)$ for the representation (3.2). By Definition 3.19, since $\tilde{C} \in \mathcal{N}_i(n-t)$, r_t for $\phi_{i,n-t}^{r_t}$ must only satisfy $0 \leq r_t \leq n-t+i$. We obtain $\phi_{i,n-t}^{r_t}(\tilde{C}) \in \mathcal{S}_i^{r_t+1}(n-t+1)$. The (r_t+1)-set in $\phi_{i,n-t}^{r_t}(\tilde{C})$ is its only plus-full-set. By definition, r_{t-1} for $\phi_{i,n-(t-1)}^{r_{t-1}}$ must only satisfy $0 \leq r_{t-1} \leq r_t$. Continuing in this manner, we find that $(r_1, r_2, \ldots, r_t)$ must only satisfy $0 \leq r_1 \leq r_2 \leq \cdots \leq r_t \leq n-t+i$. The standard trick is to make the substitution $r_k = u_k - k$, obtaining $0 < u_1 < u_2 < \cdots < u_t \leq n+i$. The t-tuples $(u_1, u_2, \ldots, u_t)$ which satisfy this are the t-element subsets of $[n+i]$. Moreover, by Proposition 3.24, $\{u_1, u_2, \ldots, u_t\} = \{r_1+1, r_2+2, \ldots, r_t+t\}$ is the set $\{j \mid C \text{ has a } j^+\text{-full-set}\}$ for $C = \left(\phi_{i,n-1}^{r_1} \circ \phi_{i,n-2}^{r_2} \circ \cdots \circ \phi_{i,n-t}^{r_t}\right)(\tilde{C})$. $\square$

Corollary 3.29 There is equal representation in $\mathcal{C}_i(n)$ over equal size subsets of $[n+i]$ in terms of specific plus-full-sets found in maximal chains: For each t-element subset $U \subseteq [n+i]$, such that $0 \leq t \leq n-1$,

$$\#\{C \in \mathcal{C}_i(n) \mid U = \{j \mid C \text{ has a } j^+\text{-full-set}\}\} = \#\mathcal{N}_i(n-t), \tag{3.3}$$

$$\#\{C \in \mathcal{C}_i(n) \mid U \subseteq \{j \mid C \text{ has a } j^+\text{-full-set}\}\} = \#\mathcal{C}_i(n-t). \tag{3.4}$$

Proof. The case $t = 0$ is trivial so assume $1 \leq t \leq n-1$.

For equation (3.3), let $S_1 = \{C \in \mathcal{C}_i(n) \mid U = \{j \mid C \text{ has a } j^+\text{-full-set}\}\}$. Suppose $U = \{u_1, u_2, \ldots, u_t\}$, where $0 < u_1 < u_2 < \cdots < u_t \leq n+i$. By Corollary 3.26 and Proposition 3.28, each $C \in S_1$ has the representation

$$\left(\phi_{i,n-1}^{u_1-1} \circ \phi_{i,n-2}^{u_2-2} \circ \cdots \circ \phi_{i,n-t}^{u_t-t}\right)(\tilde{C}),$$

for a unique $\tilde{C} \in \mathcal{N}_i(n-t)$, and this is an element of S_1 for every $\tilde{C} \in \mathcal{N}_i(n-t)$.

For equation (3.4), let $S_2 = \{C \in \mathcal{C}_i(n) \mid U \subseteq \{j \mid C$ has a j^+-full-set$\}\}$. Because of equation (3.3), $\#S_2$ depends only on $\#U = t$. It suffices to consider $U = [t]$. Suppose $C \in S_2$ has exactly $s \geq t$ plus-full-sets. Again by Corollary 3.26 and Proposition 3.28, there exists a unique $\tilde{C}_s \in \mathcal{N}_i(n-s)$ and a unique s-tuple $(r_1, r_2, \ldots, r_s)$, such that

$$C = \left(\phi_{i,n-1}^{r_1} \circ \phi_{i,n-2}^{r_2} \circ \cdots \circ \phi_{i,n-s}^{r_s}\right)(\tilde{C}_s),$$

where $0 \leq r_1 \leq r_2 \leq \cdots \leq r_s \leq n - s + i$ and $[t] \subseteq \{r_1 + 1, r_2 + 2, \ldots, r_s + s\}$. For each $k \in [t]$, we must have $r_k = 0$. Thus, each $C \in S_2$ has the representation

$$\left(\phi_{i,n-1}^{0} \circ \phi_{i,n-2}^{0} \circ \cdots \circ \phi_{i,n-t}^{0}\right)(\tilde{C}),$$

for a unique $\tilde{C} \in \mathcal{C}_i(n-t)$, and this is an element of S_2 for every $\tilde{C} \in \mathcal{C}_i(n-t)$. $\square$

An expression for $\#\mathcal{C}_i(n)$ is acquired from equation (3.3). (Note, the expression may be obtained directly from Proposition 3.28.)

$$\#\mathcal{C}_i(n) = \sum_{t=0}^{n-1} \binom{n+i}{t} \#\mathcal{N}_i(n-t) = \sum_{t=1}^{n} \binom{n+i}{t+i} \#\mathcal{N}_i(t). \tag{3.5}$$

The second expression follows from the first by reindexing and is refined in Theorem 3.32.

An expression for $\#\mathcal{N}_i(n)$ is obtained from equations (3.3), (3.4) and the principle of inclusion and exclusion.

$$\#\mathcal{N}_i(n) = \sum_{t=0}^{n-1} (-1)^t \binom{n+i}{t} \#\mathcal{C}_i(n-t) = \sum_{t=1}^{n} (-1)^{n-t} \binom{n+i}{t+i} \#\mathcal{C}_i(t). \tag{3.6}$$

The second expression follows from the first by reindexing.

In Theorem 3.31, the surprising fact is that for all $n \geq 2i + 4$, every maximal chain in $\mathcal{C}_i(n)$ has a plus-full-set. For this condition on $\mathcal{C}_i(n)$, we can do no better (the following lemma). These facts enable us to reach our main objective in Theorem 3.32.

Lemma 3.30 For each $i \geq -1$, $\#\mathcal{N}_i(2i+3) > 0$.

Proof. For $i = -1$, $\mathcal{C}_{-1}(1)$ consists of only the null diagram, so assume $i \geq 0$. A qualifying maximal chain in $\mathcal{T}_n$, $n = 2i + 3$, will have the staircase shape $(n-1, \ldots, 1) = (2i+2, \ldots, 1)$ and length $n + i = 3i + 3$. Let C be a Young diagram of the desired shape.

For each k, satisfying $0 \leq k \leq i$, let $C(n - (2k+1), 2k+1) = n + i - k$. Each r-set, where r is in the interval $[n, n+i]$, is labeled here and consists of a single box in the minor diagonal. Specifically, $C(n-1, 1) = n + i$ and $C(2, n-2) = n$, and every other box between $(n-1, 1)$ and $(2, n-2)$ gets a label.

For each k, satisfying $k \in [n-1]$, label the remaining unlabeled boxes in the k-th column with k. Each r-set, where $r \in [n-1]$, is labeled here and is a vertical column of boxes.

The resulting diagram is a ψ-tableau of the desired shape and length. Moreover, the full-sets of C end in the boxes $(n-(2k+2), 2k+2)$, $0 \le k \le i$, and for each k as specified, $C(n-(2k+1), 2k+1) > C(n-(2k+2), 2k+2)$, so that each full-set is not a plus-full-set. $\square$

Examples of maximal chains for Lemma 3.30, for $i \in \{0, 1, 2, 3\}$, are shown respectively in Figure 35.

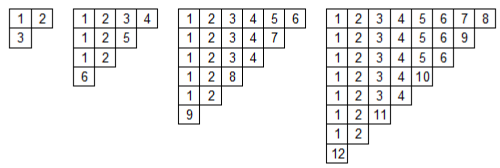

Figure 35: *Examples of Maximal Chains in* $\mathcal{N}_i(2i+3)$

Theorem 3.31 For each $i \ge -1$ and for all $n \ge 2i+4$,

1. $\#\mathcal{C}_i(n) > 0$,
2. every element of $\mathcal{C}_i(n)$ has a plus-full-set, i.e., $\#\mathcal{N}_i(n) = 0$,
3. $\mathcal{C}_i(n) = \biguplus_{j \in [3i+4]} \mathcal{S}_i^j(n)$, where each $\mathcal{S}_i^j(n)$ is nonempty.

Proof. (1) Since maximal chains in $\mathcal{T}_n$ range in length from $n-1$ to $\binom{n}{2}$, it suffices to show that $n-1 \le n+i \le \binom{n}{2}$. Since $i \ge -1$, $n-1 \le n+i$. Since $n \ge 2i+4 \ge 2$,

$$n + i \le \binom{n}{2} \Longleftrightarrow 0 \le n^2 - 3n - 2i,$$

and

$$n^2 - 3n - 2i \ge n^2 - 3n + 4 - n = (n-2)^2 \ge 0.$$

(2) Let $C \in \mathcal{C}_i(n)$. Since $n \ge 2$, C is not the null diagram. Suppose $n = 2i + 4 + l$ for some $l \ge 0$. The length of C, *i.e.*, the number of its distinct entries, is $n + i = 3i + 4 + l$. There are no repeat labels in the first row of $n - 1 = 2i + 3 + l$ boxes of C. Likewise, there are no repeat labels in the $n-1$ boxes in the minor diagonal. Let x be the number of full-sets in C and note that $(1, n-1)$ constitutes a full set. The combined number of distinct labels in the first row and minor diagonal is $x + 2(n-1-x) \le n+i$, thus

$$x \ge n - 2 - i = i + 2 + l.$$

Suppose two full-sets end in boxes in adjacent columns, say in $(k, n-k)$ and $(k+1, n-k-1)$. Then $C(k+1, n-k-1) < C(k, n-k)$, and thus C has an r^+-full-set for $r = C(k, n-k)$. On the other hand suppose no two full-sets end in adjacent columns. Then we require at least $(i+2+l)+(i+1+l) = 2i+3+2l$ boxes in the minor diagonal. Thus, $l = 0$, $n = 2i+4$, and of the $2i+3$ boxes in the minor diagonal, full-sets end in $i+2$ of them. But then one must end in $(n-1,1)$, resulting in an r^+-full-set for $r = C(n-1,1)$.

(3) Let $n = 2i+4+l$ for some $l \geq 0$. Let $C \in \mathcal{C}_i(n)$ and suppose its number of plus-full-sets is t. By Corollary 3.26, there exists a unique $\tilde{C} \in \mathcal{N}_i(n-t)$ and a unique t-tuple $(r_1, r_2, \ldots, r_t)$, such that $C = \left(\phi^{r_1}_{i,n-1} \circ \phi^{r_2}_{i,n-2} \circ \cdots \circ \phi^{r_t}_{i,n-t}\right)(\tilde{C})$. By (2), $n-t \leq 2i+3$, thus $t \geq l+1$. Since the length of C is $n+i = 3i+4+l$, there exists a j^+-full-set in C satisfying $j \leq 3i+4$. Thus, $C \in \mathcal{S}^r_i(n)$ for some $r \leq j \leq 3i+4$. We show by induction on n, that for each $j \in [3i+4]$, $\mathcal{S}^j_i(n)$ is nonempty. For the base case, $n = 2i+4$, let $C \in \mathcal{N}_i(2i+3)$. Then, for each r (as in Definition 3.19), satisfying $0 \leq r \leq 3i+3$, $\phi^r_{i,2i+3}(C) \in \mathcal{S}^{r+1}_i(2i+4)$. Now suppose the statement is true for n. By the inductive hypothesis, there exists $C \in \mathcal{S}^{3i+4}_i(n)$. We may choose any value of r (as in Definition 3.19), satisfying $0 \leq r \leq 3i+3$, to obtain $\phi^r_{i,n}(C) \in \mathcal{S}^{r+1}_i(n+1)$. □

Theorem 3.32 For each $i \geq -1$ and for all $n \geq 1$, the number of maximal chains in $\mathcal{T}_n$ of length $n+i$ is

$$\#\mathcal{C}_i(n) = \sum_{t=1}^{2i+3} \binom{n+i}{t+i} \#\mathcal{N}_i(t), \tag{3.7}$$

a polynomial in n of degree $3i+3$. The initial values of $\#\mathcal{N}_i(n)$, $n \in [2i+3]$, are

$$\#\mathcal{N}_i(n) = \sum_{t=1}^{n} (-1)^{n-t} \binom{n+i}{t+i} \#\mathcal{C}_i(t). \tag{3.8}$$

Proof. If $n \geq 2i+4$, then for all t satisfying $2i+4 \leq t \leq n$, $\#\mathcal{N}_i(t) = 0$, thus equation (3.5) reduces to equation (3.7). On the other hand, suppose $1 \leq n \leq 2i+3$. Then for all t satisfying $n < t \leq 2i+3$, $\binom{n+i}{t+i} = 0$, thus equation (3.7) reduces to equation (3.5). Equation (3.8) is equation (3.6).

The summand for $t = 2i+3$ in equation (3.7) is the one containing the term with the largest power of n, the term being

$$\frac{n^{3i+3}}{(3i+3)!} \#\mathcal{N}_i(2i+3).$$

By Lemma 3.30, this term is nonzero, thus $\#\mathcal{C}_i(n)$ is a polynomial of degree $3i+3$. □

The problem of enumerating $\mathcal{C}_i(n)$ is reduced to computing $\#\mathcal{N}_i(n)$, $n \in [2i+3]$. Values of $\#\mathcal{N}_i(n)$, for an index of i up to 5, are given in Table 2. For example, the number of maximal chains of length 14 in $\mathcal{T}_{11}$ is

$$\begin{aligned} \#\mathcal{C}_3(11) &= 18\binom{14}{8} + 220\binom{14}{9} + 1464\binom{14}{10} + 9240\binom{14}{11} + 15400\binom{14}{12}, \\ &= 18\binom{14}{6} + 220\binom{14}{5} + 1464\binom{14}{4} + 9240\binom{14}{3} + 15400\binom{14}{2}. \end{aligned} \tag{3.9}$$

According to Theorem 3.31, for all $n \geq 2i + 4 = 10$, each maximal chain in $\mathcal{C}_3(n)$ has a plus-full-set, so this follows for $n = 11$. The interpretation for equation (3.9) is that in $\mathcal{C}_3(11)$, the numbers of maximal chains having exactly 2, 3, 4, 5 and 6 plus-full-sets are $15400\binom{14}{2}$, $9240\binom{14}{3}$, $1464\binom{14}{4}$, $220\binom{14}{5}$ and $18\binom{14}{6}$, respectively. Moreover, in each subset of $\mathcal{C}_3(11)$ containing exactly j plus-full-sets, $2 \leq j \leq 6$, there is an equal number of maximal chains over all j-elements subsets of $[14]$ of particular sets of plus-full-sets.

Table 2: $\#\mathcal{N}_i(n)$: *Number of Maximal Chains in $\mathcal{T}_n$ of Length $n + i$ with No Plus-full-sets*

length \ n	1	2	3	4	5	6	7	8	9	10	11	12	13
n - 1	1												
n			1										
n + 1				2	10								
n + 2				2	8	112	280						
n + 3					18	220	1,464	9,240	15,400				
n + 4					13	218	5,322	42,592	281,424	1,121,120	1,401,400		
n + 5					12	324	8,052	142,944	1,714,700	12,180,168	65,985,920	190,590,400	190,590,400

Interpreting maximal chains in the Tamari lattice as ψ-tableaux has proven an efficient method of study. The pursuit of the formula for $\#\mathcal{C}_i(n)$ led to the plus-full-set property and some interesting combinatorics. Based on numerical evidence, we conclude this paper with a conjecture.

Conjecture 3.33 For all $i \geq -1$,

$$\#\mathcal{N}_i(2i+3) = \prod_{j=1}^{i+1} \binom{3j-1}{2},$$

and for all $i \geq 0$,

$$\#\mathcal{N}_i(2i+2) = \frac{i}{5} \prod_{j=1}^{i+1} \binom{3j-1}{2}.$$

Chapter 4

THE TAMARI BLOCK POSET

4.1 Background

In this chapter, I discuss current work and future plans in collaboration with Susanna Fishel, Kevin Treat and Mahir Can.

Recall, a *partition* of a positive integer n is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers summing to n. In certain situations, I relax the condition of positive integers to nonnegative integers. But it is understood that a partition having trailing zeros is the same as the one obtained by omitting those trailing zeros. A *Young diagram* of λ or of *shape* λ is a left-justified collection of boxes having λ_j boxes in the j-th row. The *empty partition* (0) is associated with the *null diagram* $\emptyset$ having no boxes. Often times which will be clear by the context, I abuse notation by identifying a partition λ with its associated Young diagram also denoted λ or vice versa. So for example, I often identify the shape of a maximal chain with a Young diagram.

Work involved in this chapter pertains to *pentagon* and *square intervals* in the Tamari lattice $\mathcal{T}_n$ (see Figure 36). As in the previous chapters, I treat vertices in $\mathcal{T}_n$ as the Young diagrams contained within the Young diagram of staircase shape $(n-1, \ldots, 1)$, and maximal chains are identified as Young tableaux.

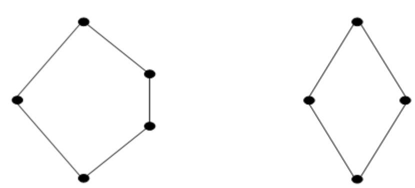

Figure 36: *Hasse Diagrams of Pentagon and Square Intervals in the Tamari Lattice*

Definition 4.1 A saturated chain M whose maximal element is the null diagram under the Tamari order is called a *top chain*. Again, I abuse notation by identifying M with its associated ψ-tableau also denoted M or vice versa (see Definition 3.8). The length and shape of M are denoted as $l(M)$ and $sh(M)$, respectively.

Definition 4.2 $\mathcal{M}_\lambda$ is the set of top chains of shape λ. $\mathcal{M}_n$ is the set of maximal chains in $\mathcal{T}_n$, so $\mathcal{M}_n = \mathcal{M}_{(n-1,\ldots,1)}$. Suppose M_1 and M_2 are top chains in $\mathcal{M}_\lambda$. If M_1 and M_2 agree except on a pentagon interval (respectively, square interval), where they differ, then M_1 has a *pentagon move* (respectively, *square move*) to M_2. If M_1 has a pentagon move to M_2 and $l(M_1) < l(M_2)$ (respectively, $l(M_1) > l(M_2)$), then M_1 has a *pentagon move up* (respectively, *pentagon move down*) to M_2.

A square move sends a top chain to one of equal length, whereas a pentagon move increases or decreases the length by one. This is easily seen by inspecting the sides that make up the square and pentagon intervals. Any two maximal chains in the Tamari lattice are connected by a sequence of maximal chains, where consecutive ones differ by a pentagon or square move (Barad 2008; Garver and McConville n.d.). This is similar to the weak order on $\mathcal{S}_n$ where maximal chains differ by hexagon or square moves, determined by braid and commuting relations (Björner et al. 1999; Williams n.d.), and intimately related to the subject of dual equivalence (Haiman 1992; Roberts 2014; Assaf 2015).

Definition 4.3 The *top chain graph* TG_λ is a simple graph whose vertices are elements of $\mathcal{M}_\lambda$. Two top chains are adjacent if and only if one may be obtained from the other by making a pentagon or square move. Edges are labeled with P or S accordingly.

Figure 37 is the top chain graph $TG_4 = TG_{(3,2,1)}$.

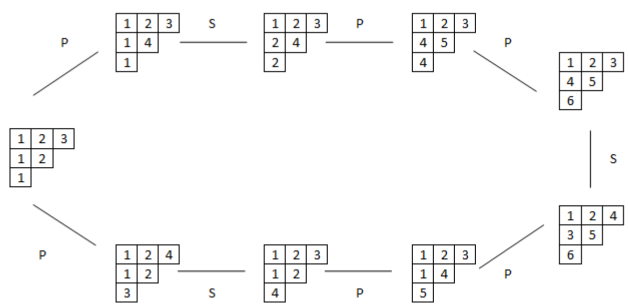

Figure 37: *The Top Chain Graph TG_4 of Maximal Chains in $\mathcal{T}_4$*

Definition 4.4 There is an equivalence relation on $\mathcal{M}_\lambda$ by $M_i \equiv M_j$ if and only if M_j can be obtained from M_i through a sequence of square moves. The equivalence classes are called *blocks*. The blocks of $\mathcal{M}_\lambda$ form a poset, called the *Tamari Block Poset* $\mathcal{TB}_\lambda$ ($\mathcal{TB}_n$ is the Tamari Block Poset for $\mathcal{M}_n$). A block is covered by another if and only if there is a pentagon move that sends a top chain in the former block to a longer one in the latter block. The length of a block B, denoted $l(B)$, is the length of a top chain contained in B.

Since square moves preserve length, every element in a given block B has the same length, thus $l(B)$ is well-defined. Since a pentagon move up increases the length by one, if B_1 and B_2 are two blocks such that B_1 is covered by B_2, *i.e.*, $B_1 \lessdot B_2$, then $l(B_2) = l(B_1) + 1$.

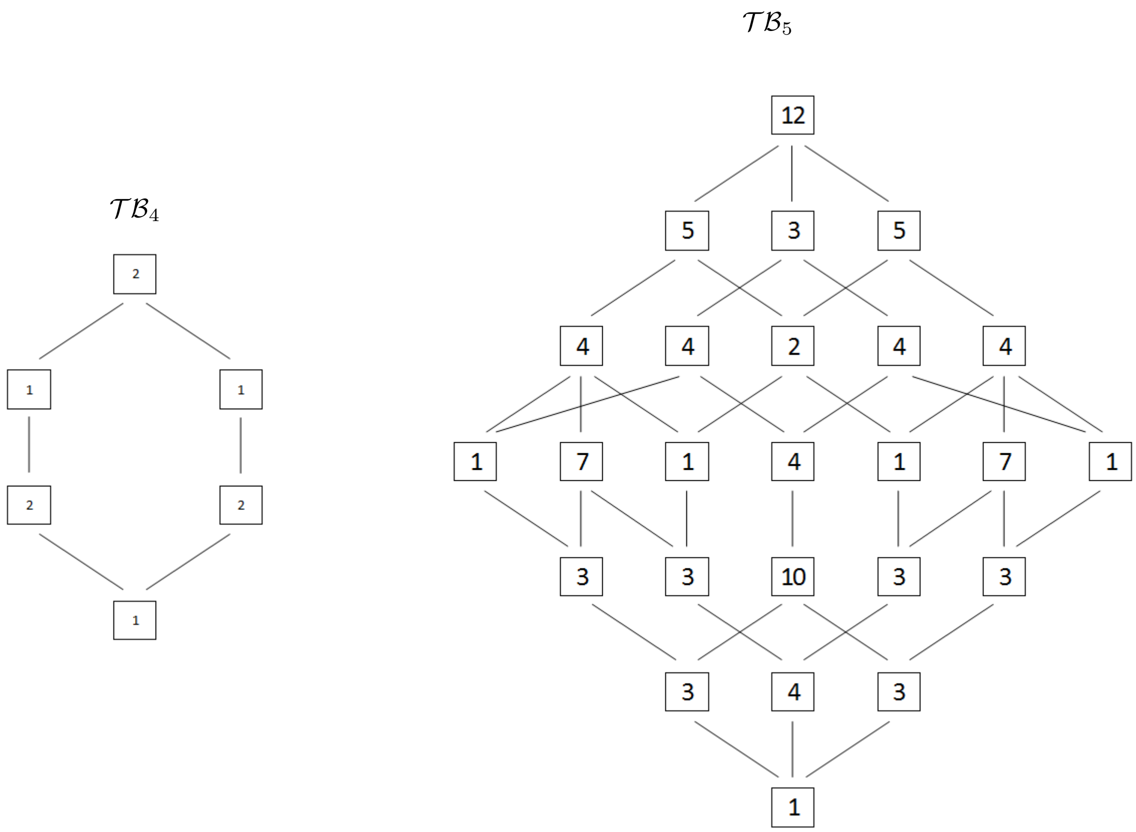

Figure 38: *Hasse Diagrams of the Tamari Block Posets Associated to $\mathcal{T}_4$ and $\mathcal{T}_5$*

Examples of Tamari Block Posets are shown in Figure 38, where sizes of the blocks are noted. There are 9 maximal chains in $\mathcal{T}_4$ (see Figure 37) which are partitioned into the 6 blocks of $\mathcal{TB}_4$. The unique minimum block of $\mathcal{TB}_4$ contains the unique shortest maximal chain in $\mathcal{T}_4$, while the unique maximum block contains the two longest maximal chains. There are 98 maximal chains in $\mathcal{T}_5$ which are partitioned into the 25 blocks of $\mathcal{TB}_5$. Similarly, there are unique minimum and maximum blocks of $\mathcal{TB}_5$, containing the unique shortest and longest maximal chains in $\mathcal{T}_5$, respectively.

After first analyzing the structure in several cases of $\mathcal{TB}_n$, we conjectured the following.

Conjecture 4.5 For each $n \geq 1$,

1. $\mathcal{TB}_n$ is graded of rank $\binom{n-1}{2}$ with the rank function $\rho(B) = l(B) - (n-1)$, and has elements $\hat{0}$ and $\hat{1}$.
2. $\mathcal{TB}_n$ is a lattice.
3. $\mathcal{TB}_n$ has vertical symmetry due to self-duality in the Tamari lattices. The "dual" of a block is a block.
4. $\#\mathcal{TB}_n$ is isomorphic to the higher Stasheff-Tamari posets on triangulations of the cyclic 3-polytope $C(n+2,3)$ (Edelman and Reiner 1996).

Conjecture 4.6 For each $i \geq 0$, the number of blocks in $\mathcal{TB}_n$ of rank i (consisting of maximal chains of length $n-1+i$) is a polynomial in n of degree i.

I address our work on Conjectures 4.5(1) and 4.5(2) in Sections 4.2 and 4.3, respectively. I discuss future plans in Section 4.4.

4.2 Conjecture 4.5(1)

In this section, I provide an overview of our proof of Conjecture 4.5(1).

Definition 4.7 Let $Y \neq \emptyset$ be a Young diagram. Let m be the maximum of $\{i+j \mid (i,j) \in Y\}$. We call the set of boxes $\{(i,j) \in Y \mid i+j = m\}$ the set of boxes in the *outer diagonal* of Y. (Note, each box in the outer diagonal is a corner box; see Definition 3.5.)

Recall that for a top chain M, the r-set of M is its set of boxes labeled with r (see Definition 3.10). The *length* of an r-set is its number of boxes.

Remark 4.8 There are no repeat labels in the boxes in the outer diagonal of a top chain. This is a property of boxes in the outer diagonal, but is not true for corner boxes in general. Furthermore, for each box in the outer diagonal of a top chain, there is an r-set which ends in that box.

Definition 4.9 Let M be a top chain. Record the entries in the boxes in the outer diagonal of M as read in the northeast direction, enclosed in parentheses. This is the first *word* of the *diagonal sentence* of M. Repeat this process with the rest of the diagonals working inward so that each diagonal represents a word, and so that each element of $[l(M)]$ appears only once in the diagonal sentence (the first time encountered).

For each shape λ, no two chains in $\mathcal{M}_\lambda$ have the same diagonal sentence. Diagonal sentence examples are shown in Figure 39.

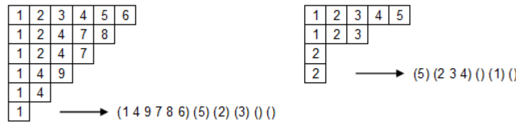

Figure 39: *Diagonal Sentence Examples*

Proposition 4.10 Let M be a top chain, say with $t > 0$ boxes in its outer diagonal. Then there is a top chain M' obtained from M through a sequence of square moves, such that the largest t entries in M are contained in the first word of the diagonal sentence of M'. Since the tableau obtained by removing the t r-sets which end in boxes in the outer diagonal of M' is a top chain, this process may be repeated.

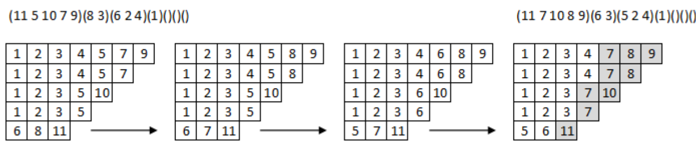

Figure 40: *Placing Largest Entries of a Top Chain in its Outer Diagonal through a Sequence of Square Moves*

The first top chain of Figure 40 has 5 boxes in its outer diagonal. The first word of its diagonal sentence $(11\,5\,10\,7\,9)$ is missing the entry 8. Through a series of square moves, the first word is transformed to $(11\,7\,10\,8\,9)$ in the final top chain.

Corollary 4.11 In $\mathcal{TB}_\lambda$, there is a unique block which contains the maximum length top chains and a unique block which contains the minimum length ones.

Proof. In minimum (respectively, maximum) length elements of $\mathcal{M}_\lambda$, each word of the diagonal sentence is strictly increasing (respectively, decreasing). This implies that over all minimum (respectively, maximum) length top chains we arrive at the same minimum (respectively, maximum) length chain by repeating the process described in the proposition. □

Top chains of minimum and maximum lexicographic diagonal sentence over all top chains in a block of $\mathcal{TB}_\lambda$ have special properties as follows.

Proposition 4.12 The top chains of minimum and maximum lexicographic diagonal sentence over all top chains in a block of $\mathcal{TB}_\lambda$ have a pentagon move down (if not a chain of minimum length) and a pentagon move up (if not a chain of maximum length), respectively.

Theorem 4.13 $\mathcal{TB}_\lambda$ is graded with rank function $\rho(B) = l(B) - m$, where m is the minimum length over top chains in $\mathcal{M}_\lambda$, and has elements $\hat{0}$ and $\hat{1}$.

Proof. Corollary 4.11 and Proposition 4.12 imply this. □

Conjecture 4.5(1) is a special case of Theorem 4.13. The block of $\mathcal{TB}_n$ containing the unique shortest maximal chain, of length $n-1$, has rank 0. The block containing the longest maximal chains, of length $\binom{n}{2}$, has rank $\binom{n}{2} - (n-1) = \binom{n-1}{2}$, which is the length of every maximal chain in $\mathcal{TB}_n$.

4.3 Conjecture 4.5(2)

In this section, I provide an overview of our proof of Conjecture 4.5(2), assuming Conjecture 4.23.

A milestone in our work, key to understanding properties of $\mathcal{TB}_\lambda$, was the characterization of blocks by a new statistic on $\mathcal{M}_\lambda$. This was accomplished by examining the particular makeup of maximal chains within blocks.

Definition 4.14 The *r-stat* of a top chain M, denoted $\pi(M)$, is an array such that the i-th row of $\pi(M)$ is the sequence of lengths of the r-sets of M that end in its i-th row. The shape of an r-stat π, denoted $sh(\pi)$, is the shape of a top chain whose r-stat is π (this is well-defined).

If two top chains have different shapes, then they have different r-stats. Thus $sh(\pi)$ is well-defined. In fact, given an r-stat π, it is a trivial task to construct its underlying Young diagram whose shape is $sh(\pi)$.

Figure 41 is an example of the construction of the r-stat $\pi(M)$ from a top chain M. For example, notice that the 2-set (of length 3) and the 7-set (of length 2) are the only r-sets in M which end in its 3-rd row. The 3-rd row of $\pi(M)$ consists of those lengths.

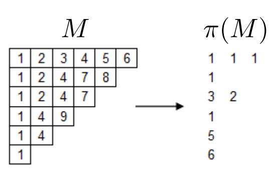

Figure 41: *Example of the r-stat*

Theorem 4.15 Two top chains in $\mathcal{M}_\lambda$ are in the same block if and only if they have the same r-stat.

Thus we obtain an encoding of $\mathcal{TB}_\lambda$ in terms of r-stats. Figure 42 is $\mathcal{TB}_5$.

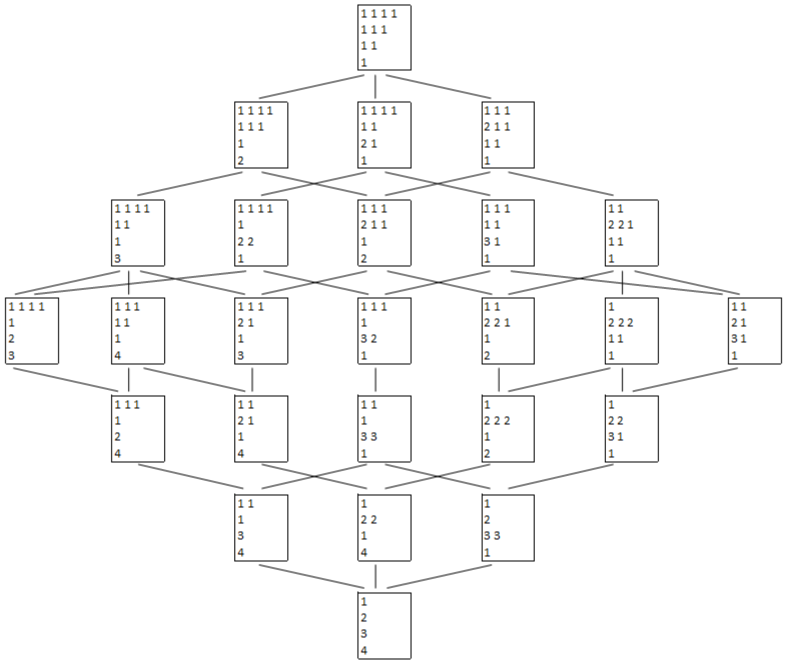

Figure 42: *Hasse Diagram of $\mathcal{TB}_5$ in Terms of r-stats*

In a top chain having its largest entries in its outer diagonal, the associated r-sets may be "peeled" to obtain a new top chain. Because of Proposition 4.10, there exists such a top chain in every block of $\mathcal{TB}_\lambda$ ($\lambda \neq \emptyset$). It follows that the "outer diagonal" entries of an r-stat may also be "peeled" to obtain a new r-stat. See Figure 43. This is formalized as follows.

Figure 43: *"Peeling" a Top Chain and an r-stat: the r-stats on the Right Correspond to the Top Chains on the Left.*

Definition 4.16 Let π be an r-stat of shape λ. Let $r_1 < r_2 < \cdots < r_n$ be the row indexes of the boxes in the outer diagonal of λ.

1. Define the *outer peel* of π as the n-tuple $p(\pi) = (p_1, \ldots, p_n)$ such that p_i is the last nonzero entry in the r_i-th row of π. (If $\lambda = \emptyset$, then $p(\pi) = ()$, *i.e.*, the 0-tuple.)
2. Define π_P as the r-stat obtained from π by removing said entries. (If $\lambda = \emptyset$, then $\pi_P = \pi$.)

3. Define the set $P_\lambda = \{p(\pi) \mid \pi$ is an r-stat of shape $\lambda\}$. (Note, $P_\lambda = \{p(\pi(M)) \mid M$ is a top chain of shape $\lambda\}$.)

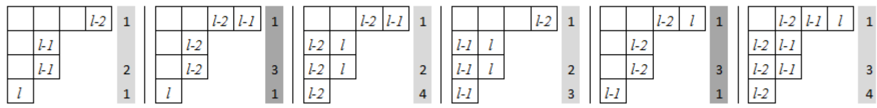

Figure 44: *Example of Outer Peels*

In Figure 44 there are 6 repeats of the Young diagram of shape $(4, 2, 2, 1)$, which has 3 boxes in its outer diagonal. If only top chains (of this shape) having their largest entries in their outer diagonals are considered, then there are $3! = 6$ possibilities in which to remove the 3 boxes. l denotes the length of the top chains, so l, $l-1$ and $l-2$ label the 6 possible orientations of the 3 r-sets which end in boxes in the outer diagonal. The lengths of those r-sets are listed to the right of each Young diagram. The second and fifth diagram result in the same set of lengths. The number of 3-tuples of the lengths of the r-sets that result is the 3-rd Catalan number $C_3 = 5$. By inspection,

$$P_{(4,2,2,1)} = \{(1, 2, 1), (1, 3, 1), (1, 2, 4), (1, 2, 3), (1, 3, 4)\}.$$

Proposition 4.17 Let $r_0 = 0$. Let $r_1 < r_2 < \cdots < r_n$ be the row indexes of the boxes in the outer diagonal of a Young diagram $\lambda \neq \emptyset$. Then, $\#P_\lambda$ is the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, and $(p_1, p_2, \ldots, p_n) \in P_\lambda$ if and only if for each $i \in [n]$,

1. $p_i \in \{r_i - r_j \mid 0 \leq j < i\}$, and
2. if $p_i = r_i - r_j$, then $p_k \leq r_k - r_j$ for $j < k < i$.

Proof. The proof is by induction on n. For the base case $n = 1$, conditions (1) and (2) provide for exactly one 1-tuple which is $(p_1) = (r_1)$, from which it's clear that $P_\lambda = \{(r_1)\}$ and $\#P_\lambda = 1 = C_1$.

For the inductive step suppose $n > 1$. Let $B_i = (r_i, c_i)$, $i \in n$, denote the boxes in the outer diagonal of λ. Let C be a top chain of shape λ. Let $\pi = \pi(C)$ and let $(p_1(\pi), \ldots, p_n(\pi)) = p(\pi)$. The r-set in C that ends in B_n must begin in one of the l-th rows where $l \in \{r_j + 1 \mid 0 \leq j < n\}$. Thus $r_n - p_n(\pi) + 1 \in \{r_j + 1 \mid 0 \leq j < n\}$ from which $p_n(\pi) \in \{r_n - r_j \mid 0 \leq j < n\}$. As the Hasse diagram of the Tamari lattice is traversed upwards on C, we consider the order in which the n boxes in its outer diagonal are removed.

Suppose $p_n(\pi) = r_n - r_0 = r_n$. This is the case if and only if B_n is removed only after removing every B_t where $t \in [n-1]$. Let Y be the Young diagram of the first r_{n-1} rows of λ. By the inductive hypothesis,

P_Y is the set of $(n-1)$-tuples as specified by conditions (1) and (2) for $i \in [n-1]$. Thus

$$\{(p_1, p_2, \ldots, p_n) \in P_\lambda \mid p_n = r_n\} = \{(p_1, p_2, \ldots, p_n) \mid (p_1, p_2, \ldots, p_{n-1}) \in P_Y, p_n = r_n\},$$

which agrees with the subset of n-tuples obtained from the proposition when $p_n = r_n$. Futhermore, by the inductive hypothesis, $\#\{(p_1, p_2, \ldots, p_n) \in P_\lambda \mid p_n = r_n\} = C_{n-1}$.

Suppose $p_n(\pi) = r_n - r_{n-1}$. This is the case if and only if B_n is removed prior to removing B_{n-1}. As above, let Y be the Young diagram of the first r_{n-1} rows of λ. By the inductive hypothesis, P_Y is the set of $(n-1)$-tuples as specified by conditions (1) and (2) for $i \in [n-1]$. Thus

$$\begin{aligned}&\{(p_1, p_2, \ldots, p_n) \in P_\lambda \mid p_n = r_n - r_{n-1}\}\\ &= \{(p_1, p_2, \ldots, p_n) \mid (p_1, p_2, \ldots, p_{n-1}) \in P_Y, p_n = r_n - r_{n-1}\},\end{aligned}$$

which agrees with the subset of n-tuples obtained from the proposition when $p_n = r_n - r_{n-1}$. Furthermore, $\#\{(p_1, p_2, \ldots, p_n) \in P_\lambda \mid p_n = r_n - r_{n-1}\} = C_{n-1}$.

Finally suppose $p_n(\pi) = r_n - r_s$ for some $s \in [n-2]$. This is the case if and only if B_n is removed prior to removing B_s and only after removing every B_t where $s < t < n$. Let Y_1 be the Young diagram of the first r_s rows of λ. Let Y_2 be the Young diagram of all rows of index l of λ such that $r_s < l \leq r_{n-1}$. By the inductive hypothesis, P_{Y_1} is the set of s-tuples as specified by conditions (1) and (2) for $i \in [s]$, and P_{Y_2} is the set of $(n-s-1)$-tuples $(p_{s+1}, p_{s+2}, \ldots, p_{n-1})$ such that for each $i \in [s+1, s+2, \ldots, n-1]$,

$p_i \in \{r_i - r_j \mid s \leq j < i\}$, and

if $p_i = r_i - r_j$, then $p_k \leq r_k - r_j$ for $j < k < i$.

Thus

$$\begin{aligned}&\{(p_1, p_2, \ldots, p_n) \in P_\lambda \mid p_n = r_n - r_s\}\\ &= \{(p_1, p_2, \ldots, p_n) \mid (p_1, p_2, \ldots, p_s) \in P_{Y_1}, (p_{s+1}, p_{s+2}, \ldots, p_{n-1}) \in P_{Y_2}, p_n = r_n - r_s\},\end{aligned}$$

which agrees with the subset of n-tuples obtained from the proposition when $p_n = r_n - r_s$. Furthermore, $\#\{(p_1, p_2, \ldots, p_n) \in P_\lambda \mid p_n = r_n - r_s\} = C_s C_{n-s-1}$.

Summing over all cases, by the well known recurrence for Catalan numbers,

$$\#P_\lambda = C_{n-1} + C_{n-1} + \sum_{i=1}^{n-2} C_i C_{n-i-1} = \sum_{i=0}^{n-1} C_i C_{n-i-1} = C_n.$$

$\square$

Corollary 4.18 Let $\lambda = (n, n-1, \ldots, 1)$, $n \geq 1$, be the staircase shape. Then, $(p_1, p_2, \ldots, p_n) \in P_\lambda$ if and only if for each $i \in [n]$,

1. $1 \leq p_i \leq i$, and
2. if $p_i = l$, then $p_{i-r} \leq l - r$ for $1 \leq r \leq l-1$.

Proof. The set of boxes in the outer diagonal of λ is $\{(i, n-i+1) \mid i \in [n]\}$. Substitute $r_t = t$ in Proposition 4.17 to obtain the equivalent statements. □

Proposition 4.17 provides for an inductive characterization of the r-stat. The set of n-tuples in Corollary 4.18 is a set of objects enumerated by the Catalan numbers (Stanley 1999, Exercise 6.19 (z)).

Figure 45 is meant to be a continuation of the example in Figure 44. There is a bijective correspondence between the 3-tuples of the lengths of the r-sets that arise and Young diagrams (shown below) that result from "peeling" the associated r-sets. This is Corollary 4.21.

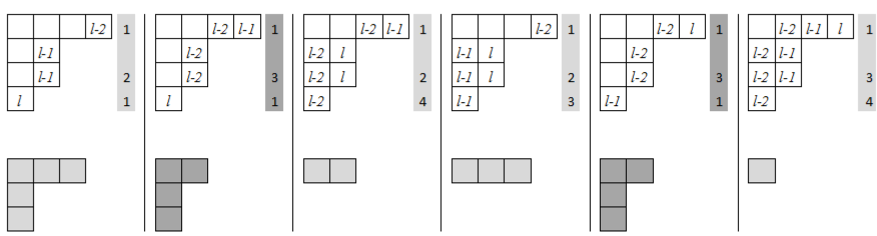

Figure 45: *Example of Outer Peels and Corresponding Shapes*

Definition 4.19 Let π be an r-stat of shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ and let $(\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_t) = sh(\pi_P)$.

1. Define the t-tuple $q(\pi) = (q_1, \ldots, q_t)$ such that $q_i = \lambda_i - \tilde{\lambda}_i$. (If $\lambda = \emptyset$, then $q(\pi) = ()$, *i.e.*, the 0-tuple.)
2. Define the set $Q_\lambda = \{q(\pi) \mid \pi$ is an r-stat of shape $\lambda\}$.

Proposition 4.20 Let $r_0 = 0$. Let $r_1 < r_2 < \cdots < r_n$ be the row indexes of the boxes in the outer diagonal of a Young diagram $\lambda \neq \emptyset$ of shape $(\lambda_1, \lambda_2, \ldots, \lambda_t)$. The map

$$\begin{aligned} f \quad : \quad & P_\lambda \to Q_\lambda \\ & (p_1, p_2, \ldots, p_n) \mapsto (q_1, q_2, \ldots, q_t) \end{aligned}$$

defined by

$$q_k = \#\{j \in [n] \mid r_j - p_j + 1 \leq k \leq r_j\} \tag{4.1}$$

is bijective, thus $\#Q_\lambda = C_n$. Furthermore, $(q_1, q_2, \ldots, q_t) \in Q_\lambda$ if and only if each of the following is met:

1. $q_{r_n} = 1$.
2. for each $i \in [n-1]$, $1 \leq q_{r_i} \leq q_{r_{i+1}} + 1$.
3. for each $i \in [n]$, $q_k = q_{r_i}$ for $r_{i-1} < k < r_i$.
4. $q_k = 0$ for $r_n < k \leq t$.

Proof. We first show that f is bijective. Let $B_i = (r_i, c_i)$, $i \in n$, denote the boxes in the outer diagonal of λ. Let C be a top chain of shape λ such that as the Hasse diagram of the Tamari lattice is traversed upwards on C, the n boxes on its outer diagonal are the first n removed. Let $\pi = \pi(C)$, $(p_1(\pi), \ldots, p_n(\pi)) = p(\pi)$, and $(q_1(\pi), \ldots, q_n(\pi)) = q(\pi)$. Let C_P be the top chain obtained by removing every r-set from C which ends in a box in its outer diagonal. Then $sh(C_P) = sh(\pi_P)$. Let $(\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_t) = sh(\pi_P)$. The number of boxes removed from each k-th row of C to obtain C_P is the number of r-sets ending in boxes in the outer diagonal of C which have a box in the k-th row, *i.e.*,

$$q_k(\pi) = \lambda_k - \tilde{\lambda}_k = \#\{j \in [n] \mid r_j - p_j(\pi) + 1 \leq k \leq r_j\}. \tag{4.2}$$

Thus, f is surjective. In particular, for each $i \in [n]$,

$$\begin{aligned} q_{r_i}(\pi) &= \#\{j \in [n] \mid r_j - p_j(\pi) + 1 \leq r_i \leq r_j\} \\ &= \#\{j \in [i, n] \mid r_j - p_j(\pi) + 1 \leq r_i\}. \end{aligned} \tag{4.3}$$

To show that f is injective, suppose C_1 and C_2 are top chains of shape λ with the additional criteria described above. Let $\pi_1 = \pi(C_1)$ and $\pi_2 = \pi(C_2)$. Let $(p_1, p_2, \ldots, p_n) = p(\pi_1)$ and $(t_1, t_2, \ldots, t_n) = p(\pi_2)$ and suppose $p(\pi_1) \neq p(\pi_2)$. Then $p_s \neq t_s$ for some $s \in [n]$. Choose s such that $p_j = t_j$ for all $j > s$, and without loss of generality, suppose $p_s > t_s$. The r-set in C_2 that ends in B_s begins in an $(r_l + 1)$-th row where $0 < l < s$, so $r_s - t_s + 1 = r_l + 1$. By condition (2) of Proposition 4.17,

$$t_s = r_s - r_l \Rightarrow t_k \leq r_k - r_l \text{ for } l < k < s \Rightarrow r_k - t_k + 1 > r_l \text{ for } l < k < s.$$

We obtain by equation (4.3)

$$\begin{aligned} q_{r_l}(\pi_2) &= \#\{j \in [l, n] \mid r_j - t_j + 1 \leq r_l\} \\ &= 1 + \#\{j \in [l+1, s] \mid r_j - t_j + 1 \leq r_l\} + \#\{j \in [s+1, n] \mid r_j - t_j + 1 \leq r_l\} \\ &= 1 + \#\{j \in [s+1, n] \mid r_j - t_j + 1 \leq r_l\} \end{aligned}$$

and

$$\begin{aligned} q_{r_l}(\pi_1) &= \#\{j \in [l, n] \mid r_j - p_j + 1 \le r_l\} \\ &= 1 + \#\{j \in [l+1, s] \mid r_j - p_j + 1 \le r_l\} + \#\{j \in [s+1, n] \mid r_j - p_j + 1 \le r_l\}. \end{aligned}$$

Since $p_j = t_j$ for all $j > s$,

$$\#\{j \in [s+1, n] \mid r_j - t_j + 1 \le r_l\} = \#\{j \in [s+1, n] \mid r_j - p_j + 1 \le r_l\},$$

thus

$$q_{r_l}(\pi_1) - q_{r_l}(\pi_2) = \#\{j \in [l+1, s] \mid r_j - p_j + 1 \le r_l\}.$$

The r-set in C_1 that ends in B_s begins in an (r_m+1)-th row where $0 \le m < l$. Thus $r_s - p_s + 1 = r_m + 1 < r_l + 1$, *i.e.*, $r_s - p_s + 1 \le r_l$, and therefore $q_{r_l}(\pi_1) - q_{r_l}(\pi_2) > 0$, whereby f is injective. This completes the proof that f is bijective.

Now suppose $(q_1(\pi), q_2(\pi), \ldots, q_t(\pi)) \in Q_\lambda$ for some r-stat π of shape λ. Let $(p_1(\pi), \ldots, p_n(\pi)) = p(\pi)$. We show that conditions (1) thru (4) are satisfied utilizing equations (4.2) and (4.3). For condition (1), $q_{r_n}(\pi) = \#\{n\} = 1$. For condition (2), suppose $i \in [n-1]$. We obtain

$$\begin{aligned} q_{r_i}(\pi) &= \#\{j \in [i+1, n] \mid r_j - p_j(\pi) + 1 \le r_i\} + 1, \\ q_{r_{i+1}}(\pi) &= \#\{j \in [i+1, n] \mid r_j - p_j(\pi) + 1 \le r_{i+1}\}. \end{aligned}$$

Thus $1 \le q_{r_i}(\pi)$, and since

$$\{j \in [i+1, n] \mid r_j - p_j(\pi) + 1 \le r_i\} \subseteq \{j \in [i+1, n] \mid r_j - p_j(\pi) + 1 \le r_{i+1}\},$$

it follows that $q_{r_i}(\pi) \le q_{r_{i+1}}(\pi) + 1$. For condition (3), suppose $i \in [n]$ and $r_{i-1} < k < r_i$. Then

$$q_{r_i}(\pi) = \#\{j \in [i, n] \mid r_j - p_j(\pi) + 1 \le r_i\},$$

$$\begin{aligned} q_k(\pi) &= \#\{j \in [n] \mid r_j - p_j(\pi) + 1 \le k \le r_j\} \\ &= \#\{j \in [i, n] \mid r_j - p_j(\pi) + 1 \le k\}. \end{aligned}$$

Since $k < r_i$, $q_k(\pi) \le q_{r_i}(\pi)$. Now suppose $r_j - p_j(\pi) + 1 \le r_i$. Then, since $r_j - p_j(\pi) + 1 \in \{r_l + 1 \mid 0 \le l < j\}$, $r_j - p_j(\pi) + 1 \le r_{i-1} + 1$. Thus

$$\{j \in [i, n] \mid r_j - p_j(\pi) + 1 \le r_i\} \subseteq \{j \in [i, n] \mid r_j - p_j(\pi) + 1 \le r_{i-1} + 1\},$$

and since $k \geq r_{i-1} + 1$,

$$\{j \in [i, n] \mid r_j - p_j(\pi) + 1 \leq r_{i-1} + 1\} \subseteq \{j \in [i, n] \mid r_j - p_j(\pi) + 1 \leq k\},$$

whereby $q_{r_i}(\pi) \leq q_k(\pi)$. Condition (4) is clear.

We've shown that Q_λ is a subset of the set of t-tuples defined by conditions (1) thru (4). The number of such t-tuples is the number of n-tuples $(q_{r_1}, q_{r_2}, \ldots, q_{r_n})$ which satisfy conditions (1) and (2). For each $(q_{r_1}, q_{r_2}, \ldots, q_{r_n})$, obtain the n-tuple $(a_1, a_2, \ldots, a_n)$ by letting $a_j = q_{r_{n-j+1}} - 1$, $j \in [n]$. The set of n-tuples obtained in this manner is the set of $(a_1, a_2, \ldots, a_n)$ that satisfy

$a_1 = 0$, and

$0 \leq a_{j+1} \leq a_j + 1,$

which is a set of objects enumerated by the Catalan numbers (Stanley 1999, Exercise 6.19 (u)). This completes our proof since also $\#Q_\lambda = C_n$. □

Corollary 4.21 Suppose $\pi, \tilde{\pi} \in \mathcal{TB}_\lambda$. Then $p(\pi) = p(\tilde{\pi})$ if and only if $sh(\pi_P) = sh(\tilde{\pi}_P)$.

Proof. This follows immediately since the map defined in the proposition is bijective. □

Corollary 4.22 Let $\lambda = (n, n-1, \ldots, 1)$, $n \geq 1$, be the staircase shape. Then $\{sh(\pi_P) \mid \pi$ is an r-stat of shape $\lambda\}$ is the set of Young diagrams which fit in the Young diagram of staircase shape $(n-1, \ldots, 1)$.

Proof. This also follows immediately since $\#\{sh(\pi_P) \mid \pi$ is an r-stat of shape $\lambda\} = \#Q_\lambda = C_n$ and each element of this set is a Young diagram which fits in the Young diagram of staircase shape $(n-1, \ldots, 1)$. (The number of the latter is also C_n.) □

We may form a partition of $\mathcal{TB}_\lambda$ by associating to each r-stat $\pi \in \mathcal{TB}_\lambda$ its outer peel $p(\pi)$. By way of Corollary 4.21, we obtain the same partition by associating to each r-stat π the Young diagram of shape $sh(\pi_P)$. Figure 46 is the partition of $\mathcal{TB}_5$ in terms of Young diagrams. Notice by Corollary 4.22 that we obtain every Young diagram which fits in the staircase shape $(3, 2, 1)$, which gives a partition of the r-stats into $C_4 = 14$ equivalence classes.

Our method of proof of the lattice property of $\mathcal{TB}_\lambda$ depends on the validity of the following conjecture.

Conjecture 4.23 Each equivalence class of the r-stats in $\mathcal{TB}_\lambda$ forms an interval in $\mathcal{TB}_\lambda$. The induced poset on the equivalence classes of r-stats in $\mathcal{TB}_n$ is isomorphic to the Tamari lattice $\mathcal{T}_{n-1}$.

Remark 4.24 Actually, the induced poset of the equivalence classes of r-stats in $\mathcal{TB}_5$ is isomorphic to the dual of $\mathcal{T}_4$, but recall $\mathcal{T}_n$ is self-dual.

I outline a proof of the lattice property of $\mathcal{TB}_n$ assuming this conjecture as follows. Because $\mathcal{TB}_n$ is finite and has a unique minimum element, to prove the lattice property, it suffices to show that any two elements have a join. (If any two elements have a join in a finite poset with unique minimum element, the meet of say x and y is the join of the set of all lowerbounds to both x and y, which is nonempty and finite.)

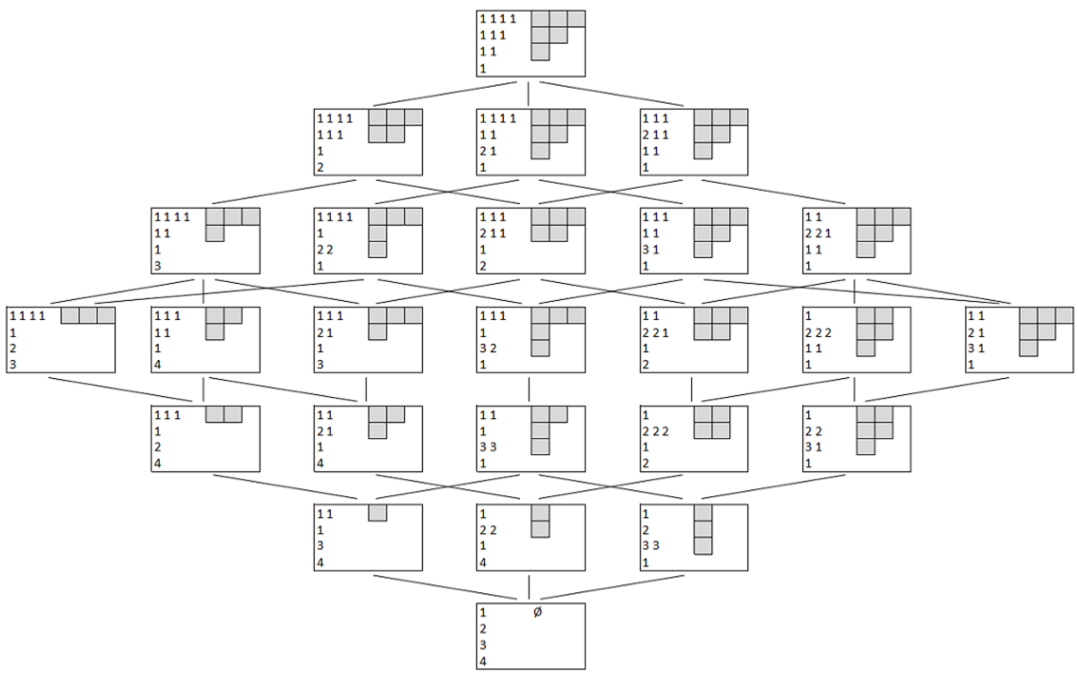

Figure 46: *Hasse Diagram of $\mathcal{TB}_5$ in Terms of r-stats and Associated Young Diagrams*

Suppose first that two r-stats x and y are in the same equivalence class. By an inductive argument, the join of x and y exists in the interval of that class. There is a technical proof to show that the join within the interval is the join in the whole poset.

Now suppose that two r-stats x and y in $\mathcal{TB}_n$ are in differing equivalence classes; see Figure 47. Since the induced poset of equivalence classes of r-stats is isomorphic to the dual of $\mathcal{T}_{n-1}$, we know the "equivalence class" join exists. Thus, if x and y have a join, it must be in the equivalence class which is the meet of their equivalence classes in $\mathcal{T}_{n-1}$, call this meet class Z. In the example, the meet of $x = (2, 1, 1)$ and $y = (1, 1)$ in $\mathcal{T}_4$ is $Z = (3, 2, 1)$. Consider the shortest paths from x and y up to their respective vertices in Z. The two end points in Z are well defined: there is a lemma that says that any pair of shortest paths from a vertex up to another equivalence class must end at the same vertex. In the example, there are two shortest paths from y to Z, but by the lemma they end at the same point. Again by induction, the join of x and y is the join of those two end points in the interval of Z. (Employ the proof which shows that the join within the interval is the join in the whole poset.)

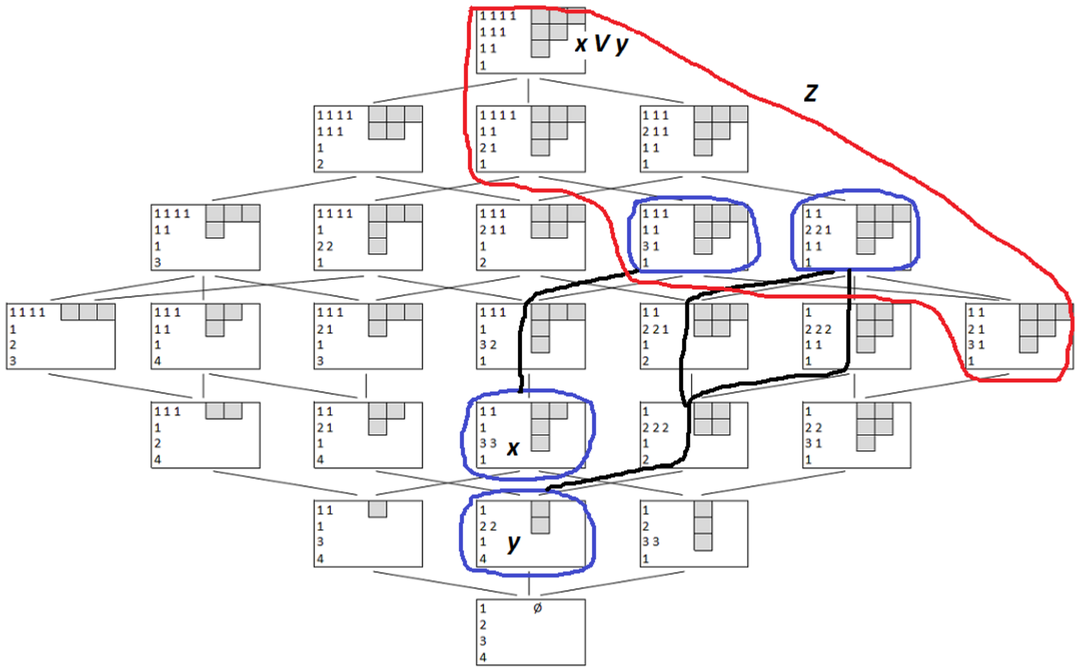

Figure 47: *Lattice Property Example in* $\mathcal{TB}_5$

4.4 Future Plans

It appears that $\mathcal{TB}_n$ is isomorphic to the higher Stasheff-Tamari posets on triangulations of the cyclic 3-polytope $C(n+2,3)$ (Edelman and Reiner 1996), which are already known to be graded lattices; see Conjecture 4.5(4). Establishing a connection would provide for a more geometric view of $\mathcal{TB}_n$. The number of triangulations of $C(n+2,3)$ is listed as an open problem in Rambau and Reiner 2012.

It is well known that the Tamari lattices are self-dual (Knuth 2006; Markowsky 1992). Each vertex in the Tamari lattice is associated to a dual vertex and this property extends to maximal chains. The notion of duality appears to extend further to certain statistics of maximal chains. For example, in Conjecture 4.5(3), the “dual” of a block is the set of maximal chains dual to the ones it contains, which we conjecture is the entire makeup of another block (the dual may be itself). This implies that the notion of dual extends to the r-stat (at least for the staircase shape). It appears that the dual vertex in the examples of Figure 38 is the one obtained by reflecting about the vertical line drawn down the middle of the Hasse diagram. I noticed similar patterns when working with the *plus-full-set* statistic in Chapter 3. The triangulations in Reading 2012, Figure 1, as a medium for studying the Tamari lattice, best portray the self-dual property. The dual of a triangulation in this particular style is obtained simply by rotating it 180 degrees on the vertical axis. I plan to study the dual map in terms of statistics it preserves, as I believe that it will play a role in enumeration.

Conjecture 4.6 bears a mysterious resemblance to my main result in Chapter 3: for each $i \geq 0$, the number of maximal chains of length $n-1+i$ in $\mathcal{T}_n$ is a polynomial in n of degree $3i$. I plan to address this by utilizing methods in that chapter and expect more to follow as this connection unfolds. My hope is that knowledge of the r-stat will lead to an explicit formula for $\#\mathcal{TB}_n$ (through a recursion, generating function, etc.) and lend insight to the enumeration of maximal chains in $\mathcal{T}_n$. As a further note, for each $i \geq 0$, the number of blocks in $\mathcal{TB}_n$ of rank $\binom{n-1}{2} - i$ (consisting of maximal chains of length $\binom{n}{2} - i$), also appears to be a polynomial of degree i.

The 2-dimensional faces of the associahedron are 4-gons and 5-gons, which are the square and pentagon intervals, respectively, in the Tamari lattices. (The Hasse diagram of $\mathcal{T}_n$ is the 1-skeleton of the associahedron.) There is an explicit formula for the number of them (Devadoss and Read 2001). Aspects of our work should translate to properties of the associahedron.

I plan to construct an associated block poset for the weak order on $\mathcal{S}_n$ and relate it to $\mathcal{TB}_n$. Since $\mathcal{T}_n$ is both a quotient and a sublattice of the weak order on $\mathcal{S}_n$ (where maximal chains in the weak order differ by squares or hexagons), I believe this could lead to some interesting results.

I plan to extend this work to the m-Tamari (see Bergeron and Préville-Ratelle 2012) and Cambrian lattices, and generalized permutohedra (see Postnikov 2009). I feel that there is a deeper meaning behind Conjecture 4.23 that may be abstracted to one or more of these other lattices.

Chapter 5

CONCLUSION

The Tamari lattices have been intensely studied since they first appeared in Dov Tamari's thesis around 1952. He defined the n-th Tamari lattice $\mathcal{T}_n$ on bracketings of a set of $n+1$ objects, with a cover relation based on the associativity rule in one direction. The number of vertices in $\mathcal{T}_n$ is the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. $\mathcal{T}_n$ is both a quotient and a sublattice of the weak order on the symmetric group $\mathcal{S}_n$, and its Hasse diagram is the 1-skeleton of the associahedron (or Stasheff polytope). Despite their interesting aspects and the attention they have received, a formula for the number of maximal chains in the Tamari lattices is still unknown. It is mainly this problem which I addressed in this dissertation. I reviewed terminology and discussed history and my motivation in Chapter 1. I conveyed my results on progress toward the solution of this problem in Chapters 2 and 3. I discussed current work and future plans in Chapter 4.

Because of the many combinatorial structures counted by the Catalan sequence, the Tamari lattices are studied in numerous equivalent representations. Throughout this dissertation, I interpret the set of vertices in $\mathcal{T}_n$ as the set of Young diagrams contained in the one of staircase shape $(n-1,\ldots,1)$. This approach provides for a method in which maximal chains are identified as Young tableaux, where I am better equipped to obtain results on enumeration.

I counted maximum length chains in Chapter 2, which is my paper "*Chains of maximum length in the Tamari lattice*", coauthored with Susanna Fishel (Fishel and Nelson 2014). I received permission to include this paper as indicated in the Appendix. I established a bijection between maximum length chains in $\mathcal{T}_n^{(m)}$ and standard m-*shifted* Young tableaux of shape $(m(n-1), m(n-2), \ldots, m)$. Using Thrall's formula, I thus derived the formula for the number of maximum length chains in $\mathcal{T}_n$ as

$$\binom{n}{2}!\,\frac{(n-2)!(n-3)!\cdots(2)!(1)!}{(2n-3)!(2n-5)!\cdots(3)!(1)!}.$$

In addition, I characterized the maximal chains in $\mathcal{S}_n$, which are maximum length chains in $\mathcal{T}_n$, when $\mathcal{T}_n$ is viewed as an induced subposet of $\mathcal{S}_n$.

I obtained recursive formulas for the number of maximal chains by length in Chapter 3, which is my paper "*A recursion on maximal chains in the Tamari lattices*" (Nelson n.d.). For each $i \geq -1$ and for all $n \geq 1$, I defined $\mathcal{C}_i(n)$ as the set of maximal chains of length $n+i$ in $\mathcal{T}_n$. I established properties of maximal chains (treated as tableaux) and identified a particularly special property: each maximal chain may or may not possess a *plus-full-set*. I showed that for all $n \geq 2i+4$, each member of $\mathcal{C}_i(n)$ contains a plus-full-set.

Utilizing this fact and a collection of maps which take maximal chains in $\mathcal{C}_i(n)$ to $\mathcal{C}_i(n+1)$, I obtained a recursion for $\#\mathcal{C}_i(n)$ and an explicit formula based on predetermined initial values. For each $i \geq -1$ and for all $n \geq 1$, the number of maximal chains in $\mathcal{T}_n$ of length $n+i$ is

$$\#\mathcal{C}_i(n) = \sum_{t=1}^{2i+3} \binom{n+i}{t+i} \#\mathcal{N}_i(t),$$

a polynomial in n of degree $3i+3$, where $\mathcal{N}_i(t)$ is the subset of $\mathcal{C}_i(t)$ having no plus-full-sets. This result is a considerable generalization of results of Knuth and Markowsky, where they show that there is one maximal chain of shortest length $n-1$.

Finally in Chapter 4, in collaboration with Susanna Fishel, Kevin Treat and Mahir Can, I discussed current work and future plans involving certain equivalence classes of maximal chains in the Tamari lattices and new posets which arise. I related many wonderful properties of these posets. My hope is that studying these new posets will lend insight to the enumeration of maximal chains in $\mathcal{T}_n$.

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APPENDIX A

ACKNOWLEDGMENT OF CO-AUTHORSHIP ON INCLUDED WORK

In mathematics authors are listed alphabetically. Chapter 2 of this dissertation is Fishel and Nelson 2014, published in 2014 in *Proceedings of the American Mathematical Society*. This paper was co-authored by the author of this dissertation and Susanna Fishel. Fishel has given her approval that this work be included in this dissertation.