On Choosability and Paintability of Graphs

by

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ABSTRACT

Let $G = (V, E)$ be a graph. A *list assignment* L for G is a function from V to subsets of the natural numbers. An *L-coloring* is a function f with domain V such that $f(v) \in L(v)$ for all vertices $v \in V$ and $f(x) \neq f(y)$ whenever $xy \in E$. If $|L(v)| = t$ for all $v \in V$ then L is a *t-list assignment*. The graph G is t-choosable if for every t-list assignment L there is an L-coloring. The least t such that G is t-choosable is called the list chromatic number of G, and is denoted by $\mathrm{ch}(G)$. The complete multipartite graph with k parts, each of size s is denoted by K_{s*k}. Erdős et al. suggested the problem of determining $\mathrm{ch}(K_{s*k})$, and showed that $\mathrm{ch}(K_{2*k}) = k$. Alon gave bounds of the form $\Theta(k \log s)$. Kierstead proved the exact bound $\mathrm{ch}(K_{3*k}) = \lceil \frac{4k-1}{3} \rceil$. Here it is proved that $\mathrm{ch}(K_{4*k}) = \lceil \frac{3k-1}{2} \rceil$.

An online version of the list coloring problem was introduced independently by Schauz and Zhu. It can be formulated as a game between two players, Alice and Bob. Alice designs lists of colors for all vertices, but does not tell Bob, and is allowed to change her mind about unrevealed colors as the game progresses. On her i-th turn Alice reveals all vertices with i in their list. On his i-th turn Bob decides, irrevocably, which (independent set) of these vertices to color with i. For a function l from V to the natural numbers, Bob wins the *l-game* if eventually he colors every vertex v before v has had $l(v)+1$ colors of its list revealed by Alice; otherwise Alice wins. The graph G is *l-online choosable* or *l-paintable* if Bob has a strategy to win the l-game. If $l(v) = t$ for all $v \in V$ and G is l-paintable, then G is t-paintable. The *online list chromatic number* of G is the least t such that G is t-paintable, and is denoted by $\mathrm{ch}^{\mathrm{OL}}(G)$. Evidently, $\mathrm{ch}^{\mathrm{OL}}(G) \geq \mathrm{ch}(G)$. Zhu conjectured that the gap $\mathrm{ch}^{\mathrm{OL}}(G) - \mathrm{ch}(G)$ can be arbitrarily large. However there are only a few known examples with this gap equal to one, and none with larger gap. This conjecture is explored in this thesis. One of the obstacles is that there are not many graphs whose exact list coloring number is

known. This is one of the motivations for establishing new cases of Erdős' problem. Here new examples of graphs with gap one are found, and related technical results are developed as tools for attacking Zhu's conjecture.

The square G^2 of a graph G is formed by adding edges between all vertices at distance 2. It was conjectured that every graph G satisfies $\chi(G^2) = \mathrm{ch}(G^2)$. This was recently disproved for specially constructed graphs. Here it is shown that a graph arising naturally in the theory of cellular networks is also a counterexample.

To my beloved family.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Chapter 1

INTRODUCTION

1.1 Preliminary Definitions and Notation

Let $G = (V, E)$ be a graph. A *vertex coloring* of the graph G is a map $f : V \to S$ such that $f(x) \neq f(y)$ whenever $xy \in E$. A *k-coloring* is a vertex coloring $f : V \to [k]$. The smallest integer k such that G has a k-coloring is called the *chromatic number* of G; it is denoted by $\chi(G)$. Similarly, we have definitions for *edge coloring, edge chromatic number* by replacing the words "graph G" with the words "the line graph of G" in the above definitions of *vertex coloring* and *chromatic number*, respectively.

A *list assignment* L for G is a function $L : V \to 2^{\mathbb{N}}$, where $\mathbb{N}$ is the set of natural numbers and $2^{\mathbb{N}}$ is the power set of $\mathbb{N}$. For a pair (G, L), *list size function* l is defined as a function $l : V \to \mathbb{N}$: $l(v) := |L(v)|$. If $l(v) = k$ for all vertices $v \in V$, then L is a *k-list assignment* for G. An *L-coloring* f from a list assignment L is a function $f : V \to \mathbb{N}$ such that $f(v) \in L(v)$ for all vertices $v \in V$ and $f(x) \neq f(y)$ whenever $xy \in E$. The graph G is *L-colorable* if there exists an L-coloring of G; it is *k-choosable* if it is L-choosable for all k-list assignments L. The *list chromatic number* or *choice number* of G, denoted $\mathrm{ch}(G)$, is the smallest integer k such that G is k-choosable. The general list coloring problem may consider list assignments with uneven list sizes, that is, for the pair (G, l) and $x, y \in V(G)$, it may be that $l(x) \neq l(y)$.

Introduced by Schauz (2009), *online choosability* or *paintability* is a coloring game played between two players Alice and Bob on a graph $G = (V, E)$ and a *list size function* $l : V \to \mathbb{N}$. Let V_i denote the vertex set at the start of round i; so $V_1 = V$. At round i, Alice selects a nonempty set of vertices $A_i \subseteq V_i$, and Bob selects an

	4	4	4	4		4	4		4	4
5	**4**	**4**	**4**	**4**		**4**	**8**	6	**4**	**4**
6	**8**	**8**	**8**	**8**		**8**	**4**	**5**	**8**	**8**

Figure 1.1: An example graph with uneven list sizes: left is a multipartite graph with 1 part of size 2 and 4 parts of size 3. For the part of size 2, the list sizes are 5 and 6. For each part of size 3, the list sizes are 4, 4, 8, respectively. Alice presents a set consisting of one vertex of part size 2 with list size 5, and two vertices from each part of size 3 with list sizes 4 and 8, respectively. The right picture is identical to the left up to isomorphism.

independent set $B_i \subseteq A_i$. Then B_i is deleted from the graph so that $V_{i+1} = V_i \setminus B_i$, and the rounds are continued until $V_n = \emptyset$. Alice's goal is to present some vertex v more than $l(v)$ times, while Bob's goal is to choose every vertex before it has been presented $l(v) + 1$ times. We say that G is *on-line l-choosable* if for any strategy of Alice, Bob has a strategy such that any vertex $v \in V$ is in at most $l(v)$ sets A_i. The graph G is *on-line k choosable* if G is on-line l-choosable, where $l(v) = k$ for all $v \in V$. The *on-line choice number*, denoted $\mathrm{ch}^{\mathrm{OL}}(G)$, is the least k such that G is on-line k-choosable. The general online list coloring problem may consider list size function with uneven list sizes.

Let L be a list assignment. Let $L\neg\alpha$ be the result of deleting color α from every list of L. For a set of vertices $S \subseteq V$ let $\mathcal{L}(S) = \{L(x) : x \in S\}$, $L(S) = \bigcap \mathcal{L}(S)$, $W(S) = \bigcup \mathcal{L}(S)$, and $l(S) = |L(S)|$. For a set S and element x we use the notation $S + x = S \cup \{x\}$ and $S - x = S \setminus \{x\}$.

In this thesis, lots of complete multipartite graphs with uneven list sizes will be considered. As a consequence, figures with natural numbers are drawn to represent these graph with uneven list size functions. Specifically, let $G = K_{1*k_1,2*k_2,\ldots}$ denote the complete multipartite graph with k_i parts of size i. Organize all the parts and let $V_i = \{v_{1,i}, \ldots, v_{t,i}\}$ be a part of size t. For a list size function l, denote the pair (G, l) by an $1 * k_1, 2 * k_2, \ldots$ array, where i, j-entry is $l(v_{i,j})$. For paintability, at round i, Alice presents the set A_i that is marked with boldface. See Figure 1.1 for an example.

1.2 Choosability

The original coloring problem can be viewed as a restriction of the exact same set of colors for each vertex in the list coloring problem. Thus $\text{ch}(G) \geq \chi(G)$ for any graph G. However, the chromatic number and the choice number for same graph can be very different. The first example known is as follows:

Example 1 (Erdős *et al.* (1980))**.** $K_{3,3}$ is 2-colorable but not 2-choosable: Since $K_{3,3}$ is a complete bipartite graph, we can color one part with a single color and the other part with a second color. Besides, we cannot use one color to color the whole graph since it is not independent set. Thus $\chi(K_{3,3}) = 2$.

Let L be the list assignment satisfying the following conditions. Assume 1,2,3 are all the possible colors that can be assigned to $v \in K_{3,3}$. For any v, $L(v) \in \{\{1,2\}, \{2,3\}, \{1,3\}\}$. Besides, for any two different vertices in each part, their lists are not equal.

Now for L, it can be observed that for each part, at least 2 colors need to be used, and the colors used for each part must be distinct. Thus there are at least 4 colors used throughout the graph. However, since there are only 3 colors total in the lists, this is impossible. Therefore, $\text{ch}(K_{3,3}) \geq 3$.

The study of list coloring was initiated by Vizing (1976) and by Erdős *et al.* (1980). It is a generalization of two well studied areas of combinatorics—graph coloring and transversal theory. Restricting the list assignment to a constant function, yields ordinary graph coloring; restricting the graph to a clique yields the problem of finding a system of distinct representatives (SDR) for the family of lists.

There are not many graphs whose exact choice number is known. However, there are some amazingly elegant results that add to the subject's charm. Here some other famous results are shown.

Theorem 2 (Thomassen (1994)). *Every planar graph is 5-choosable.*

Voigt (1993) proved that this is tight, that is, there is a planar graph with $\mathrm{ch}(G) > 4$.

Theorem 3 (Galvin (1995)). *For any bipartite graph,* $\mathrm{ch}(G) = \Delta(G)$*, where* Δ *is the maximum degree of the graph.*

Besides, choice number grows as average degree grows, which does not hold for chromatic number:

Theorem 4 (Alon (1993a)). *Let* G *be a graph with average degree at least* d*. If* s *is an integer and*

$$d > 4\binom{s^4}{s}\log(2\binom{s^4}{s})$$

then $\mathrm{ch}(G) > s$*.*

More results about list coloring for multipartite graph are stated in Section 1.4.

1.3 Background for Paintability

The game formulation in Section 1.1 hides the on-line nature of the problem. Another way of thinking about it is that Alice has secretly assigned lists of colors to all the vertices. At round i she reveals all vertices whose list contains color i, and

Bob colors an independent set of them with color i. In this formulation it is clear that $\text{ch}^{\text{OL}}(G) \geq \text{ch}(G)$ for all graphs G.

A "real-world" problem can describe the difference between choice number and online choice number. Suppose a company has lots of jobs in various cities. There are also some workers for these jobs, but not all workers are qualified for all jobs. A worker can be hired to do all the jobs in one city that he is qualified for, but cannot do jobs in two different cities. Then there are two questions:

(1) Can the workers be assigned to the jobs? This is a choice number problem for complete multipartite graphs.

(2) Now suppose the workers apply for jobs one at a time, and at the time they apply they must be immediately assigned to their jobs. This is the on-line choice number problem for complete multipartite graphs.

The next example shows that choice number and online choice number of a graph can differ. Let $\Theta_{r,s,t}$ be a graph that consists of two distinguished nodes a and b that are joined by three disjoint paths of length r, s, t.

Example 5 (Carraher *et al.* (2014))**.**

$$\text{ch}(\Theta_{2,2,4}) = 2 < 3 \leq \text{ch}^{\text{OL}}(\Theta_{2,2,4})$$

The above assertion will follow from Lemma 6 and 7.

Lemma 6 (Erdős *et al.* (1980))**.**

$$\text{ch}(\Theta_{2,2,2m}) = 2 \textit{ for all } m \in \mathbb{N}$$

Proof. Denote the vertices for the $a - b$ path of length $2m$ by $a = a_1, a_2, ..., a_{2m+1} = b$ and the remaining two vertices by c, d. See Figure 1.2. Assume L is an arbitrary 2-list assignment. Now we have two cases:

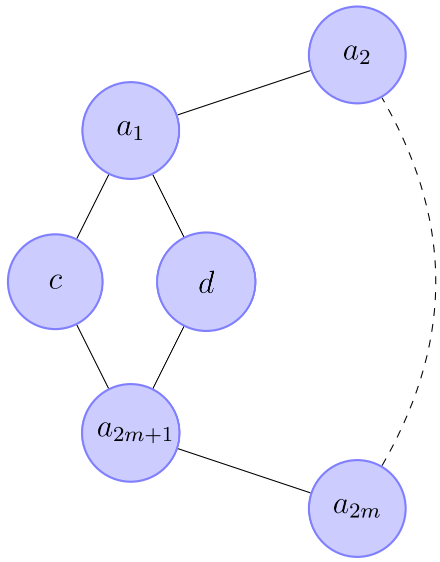


Figure 1.2: The Graph $\Theta_{2,2,2m}$

Case 1: $L(a_i) = \{\alpha, \beta\}$ for all $1 \le i \le 2m+1$. Then choose α for all a_i with i odd and choose β for all a_i with i even. Then we can complete the choice with a color in $L(c) - \alpha$, and a color in $L(d) - \alpha$.

Case 2: There exists i such that $L(a_i) \neq L(a_{i+1})$. Then tentatively we choose $\alpha_i \in L(a_i) \smallsetminus L(a_{i+1})$ for a_i, and then choose $\alpha_{i-1} \in L(a_{i-1}) - \alpha_i$ for a_{i-1}, ..., and finally $\alpha_1 \in L(a_1) - \alpha_2$ for a_1. Now we have four subcases. (1) $\alpha_1 \in L(a_{2m+1})$. Then we can color c with some color $\beta_1 \in L(c) - \alpha_1$ and d with some color $\beta_2 \in L(d) - \alpha_1$. Color a_{2m+1} with α_1. Then we can continue choosing $\alpha_{2m} \in L(a_{2m}) - \alpha_1$ for a_{2m}, ..., and finally $\alpha_{i+1} \in L(a_{i+1}) - \alpha_{i+2}$ for a_{i+1}. This completes our choice. (2) $\alpha_1 \notin L(a_{2m+1})$ and $\alpha_1 \notin L(c) \cap L(d)$. We can assume $\alpha_1 \notin L(c)$. Then we color d with $\beta_1 \in L(d) - \alpha_1$ and color a_{2m+1} with some color $\alpha_{2m+1} \in L(a_{2m+1}) - \beta_1$. Then we can color c with some color in $L(c) - \alpha_1 - \alpha_{2m+1}$, since $\alpha_1 \notin L(c)$. Then we can continue choosing $\alpha_{2m} \in L(a_{2m}) - \alpha_1$ for a_{2m}, ..., and finally $\alpha_{i+1} \in L(a_{i+1}) - \alpha_{i+2}$ for a_{i+1}. This completes our choice. (3) $(\alpha_1 \notin L(a_{2m+1})) \wedge (\alpha_1 \in L(c) \cap L(d)) \wedge (L(a_{2m+1}) \neq L(c) \cup L(d) - \alpha_1)$. Then we can color a_{2m+1} with $\alpha_{2m+1} \in L(a_{2m+1}) \smallsetminus (L(c) \cup L(d) - \alpha_1)$. Then we can color c

with some color in $L(c) - \alpha_{2m+1} - \alpha_1$ since $\alpha_{2m+1} \notin L(c)$. Similarly we can continue choosing $\alpha_{2m} \in L(a_{2m}) - \alpha_1$ for a_{2m}, ..., and finally $\alpha_{i+1} \in L(a_{i+1}) - \alpha_{i+2}$ for a_{i+1}. This completes our choice. (4) Otherwise, we have $L(c) = \{\alpha_1, \tau\}$, $L(d) = \{\alpha_1, \gamma\}$, $L(a_{2m+1}) = \{\tau, \gamma\}$. In this case, we color c, d with α_1. Then we can continue choosing $\beta_1 \in L(a_1) - \alpha_1$, $\beta_2 \in L(a_2) - \beta_1$, ..., and finally $\beta_{2m+1} \in L(a_{2m+1}) - \beta_{2m}$. As $\beta_{2m+1} \neq \alpha_1$ from our assumption. This completes our choice.

□

Lemma 7 (Carraher *et al.* (2014))**.**

$$\mathrm{ch}^{\mathrm{OL}}(\Theta_{2,2,4}) \neq 2$$

Proof. Figure 1.3 describes a strategy for Alice. Let $G = \Theta_{2,2,4}$ and $f(v) = 2$ for any $v \in V(G)$. The top left graph depicts the initial game position (G, f), and Alice's first move. The numbers inside the nodes represent the size of $f(v)$ for the corresponding vertex v. The nodes inside the box represent the vertices that Alice presents on here first move.

As play progresses Bob chooses certain vertices presented by Alice and passes over others. When a vertex is chosen its position is removed from the next graph (and the deleted vertex is marked as a small grey square in this example). When he passes over a vertex its list size is decreased by one. The arrows between the graphs point to the possible new game positions that arise from Bob's choice.

For example, after Bob's first move there are two possible graphs, provided Bob chooses a maximal independent set. It is shown in the second column of the first row and the first column of the third row, along with Alice's second move respectively. Now Bob has one possible response that are pointed to by the arrow for the positions of the first row.

Eventually, Alice forces one of two positions (G, f) such that G is not f-choosable, and Bob, being a gentleman, resigns. $\square$

1.4 Historical Results for Choosability of Multipartite Graph

In this section, I introduce some recent results on choice number for complete multipartite graph.

Erdős *et al.* (1980) suggested determining the choice number of uniform complete multipartite graphs. Recall that $K_{1*k_1,2*k_2\dots}$ denote the complete multipartite graph with k_i parts of size i. Since K_{1*k} is a clique and K_{s*1} is an independent set, these cases are trivial. Alon (1993b) proved the general bounds $c_1 k \log s \leq \mathrm{ch}(K_{s*k}) \leq c_2 k \log s$ for some constants $c_1, c_2 > 0$. This was tightened by Gazit and Krivelevich (2006).

Theorem 8 (Gazit and Krivelevich (2006))**.** $\mathrm{ch}(K_{s*k}) = (1 + o(1))\frac{\log s}{\log(1+1/k)}$.

The next well-known example provides the best lower bounds for small values of s by generalizing Example 1.

Example 9. $\mathrm{ch}(K_{s*k}) \geq \lceil \frac{2(s-1)k-s+2}{s} \rceil$: Let $G = K_{s*k}$ have parts $\{X_1, \dots, X_k\}$ with $X_i = \{v_{i,1}, \dots, v_{i,s}\}$. We will construct an $(l-1)$-list assignment L from which G cannot be colored. Equitably partition $C := [2k-1]$ into s parts $C_1, \dots, C_s$. Define a list assignment L for G by $L(v_{i,j}) = C \smallsetminus C_j$. Then each list has size at least

$$2k - 1 - \left\lceil \frac{2k-1}{s} \right\rceil = \left\lfloor \frac{2ks - s - 2k + 1}{s} \right\rfloor = \left\lceil \frac{2(s-1)k - 2s + 2}{s} \right\rceil = l - 1.$$

Consider any color $\alpha \in C$. Then $\alpha \in C_i$ for some $i \in [s]$. So $\alpha \notin L(x_{i,j})$ for every $j \in [k]$. Thus any L-coloring of G uses at least two colors for every part X_j. Since vertices in distinct parts are adjacent, they require distinct colors. As there are k parts this would require $2k > |C|$ colors, which is impossible.

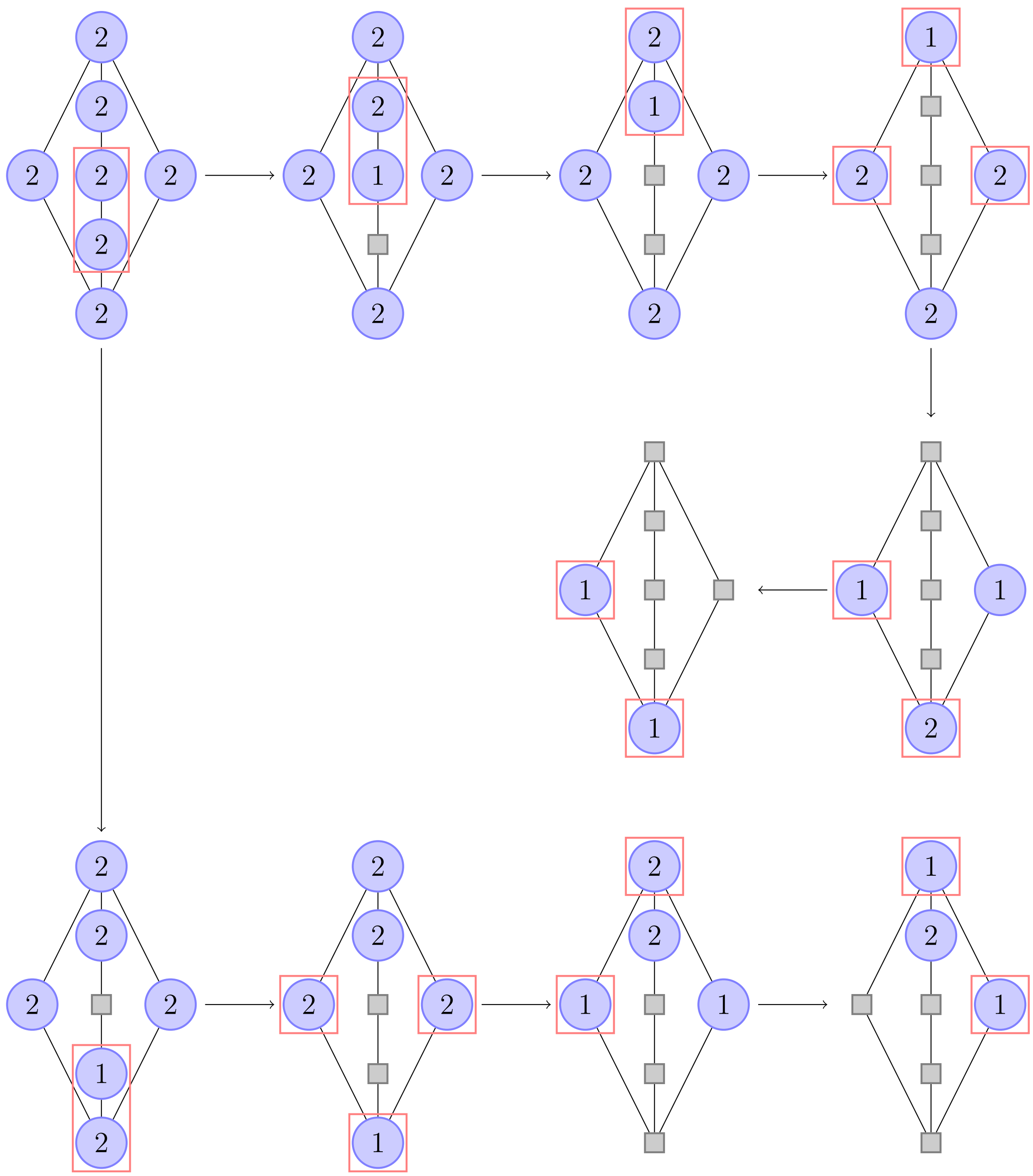


Figure 1.3: The Strategy for Alice Demonstrating $\mathrm{ch}^{\mathrm{OL}}(\Theta_{2,2,4}) \geq 3$

Restricting the question of Erdős *et al.* (1980), we ask for those integers s such that:

$$(\forall k \in \mathbb{Z}^+) \left[\mathrm{ch}(K_{s*k}) = l(s,k) := \left\lceil \frac{2(s-1)k - s + 2}{s} \right\rceil \right]. \quad (1.1)$$

The cases $s = 2$, $s = 3$ and $s = 4$ have been solved:

Theorem 10 (Erdős *et al.* (1980))**.** *All positive integers k satisfy* $\mathrm{ch}(K_{2*k}) = k$.

Proof. First notice that $\mathrm{ch}(K_{2*k}) \geq \chi(K_{2*k}) \geq \chi(K_k) = k$. Thus it suffices to show that $\mathrm{ch}(K_{2*k}) \leq k$.

We prove by contradiction. Let k be the least integer such that $\mathrm{ch}^{\mathrm{OL}}(K_{2*k}) > k$. Let L be a list assignment. Now we claim:

$$(*)\ L(a) \bigcap L(b) = \emptyset \text{ for any part } X = \{a, b\} \subseteq K_{2*k}.$$

If (*) is not true, then there exists some part $X = \{a, b\}$ such that we can choose one color $\alpha \in L(a) \cap L(b)$. Color a, b with α. Let $L' = L \neg \alpha$. Then we have $|L'(v)| \geq k - 1$ for any vertex $v \in K_{2*k} \smallsetminus X$. As $K_{2*k} \smallsetminus X$ has $k - 1$ parts and is not L-colorable, $K_{2*k} \smallsetminus X$ is not L'-colorable. Thus $K_{2*k} \smallsetminus X$ is not $k - 1$ choosable. This contradicts the minimality of k. So (*) holds.

Now we want to prove that $|S| \leq |L(S)|$ for any $S \subseteq V(G)$. Then by Hall's theorem, there is a L-coloring for G. If $1 \leq |S| \leq k$, we can find a vertex $v \in S$. $|W(S)| \geq |L(v)| = k \geq |S|$. If $|S| > k$, then there exists a part $X = \{a, b\} \subseteq S$. By (*), $|W(S)| \geq |W(X)| = |L(a)| + |L(b)| - |L(a) \bigcap L(b)| = 2k - 0 = 2k \geq |G| \geq |S|$. By using Hall's Theorem, we can find a L-coloring for graph G. Since L is an arbitrary k-list assignment, G is k-choosable. □

Theorem 11 (Kierstead (2000))**.** *All positive integers k satisfy* $\mathrm{ch}(K_{3*k}) = \lceil \frac{4k-1}{3} \rceil$.

Recently, Kozik *et al.* (2014) gave a very different proof of Theorem 11. The following more general result appears in Ohba (2004).

Theorem 12 (Ohba (2004))**.** $\mathrm{ch}(K_{1*k_1,3*k_3}) = \max\{k, \lceil\frac{n+k-1}{3}\rceil\}$, *where* $k = k_1 + k_3$ *and* $n = k_1 + 3k_3$.

The next example shows that the largest s satisfying (1.1) is at most 14.

Example 13. $\mathrm{ch}(K_{15*k}) \geq l := 2k$: Let $G = K_{s*k}$ have parts $\{X_1, \ldots, X_k\}$ with $X_i = \{v_{i,1}, \ldots, v_{i,s}\}$. We will construct an $(l-1)$-list assignment L from which G cannot be colored. Equitably partition $C := [3k-1]$ into 6 parts $C_1, \ldots, C_6$, and fix a bijection $f : [15] \to \binom{[6]}{2}$. Define a list assignment L for G by

$$L(v_{i,j}) = C \smallsetminus \bigcup\{C_h : h \in f(i)\}.$$

Then each list has size at least

$$3k - 1 - 2\left\lceil\frac{3k-1}{6}\right\rceil = 2k - 1 = l - 1.$$

Consider any two colors $\alpha, \beta \in C$. Then $\alpha, \beta \in \bigcup\{C_h : h \in f(i)\}$ for some $i \in [15]$. So $\alpha, \beta \notin L(x_{i,j})$ for every $j \in [k]$. Thus any L-coloring of G uses at least the colors for every part X_j. Since $3k > |C|$, this is impossible.

Yang (2003) proved $\lceil\frac{3k}{2}\rceil \leq \mathrm{ch}(K_{4*k}) \leq \lceil\frac{7k}{4}\rceil$, and Noel *et al.* (2015) improved the upper bound to $\lceil\frac{5k-1}{3}\rceil$. A result we proved is that (1.1) holds for $s = 4$. To prove this theorem We first extract a simple proof of Theorem 11 from Noel *et al.* (2014), and then elaborate on it.

Theorem 14 (Kierstead *et al.* (2014))**.** $\mathrm{ch}(K_{4*k}) = l(4, k) := \lceil\frac{3k-1}{2}\rceil$.

Another result shows that when the number of vertices is near chromatic number, the graph G is *chromatic-choosable*, which is defined as $\chi(G) = \mathrm{ch}(G)$. The conjecture was first raised by Ohba (2004) and proved by Noel *et al.* (2015).

Theorem 15 (Noel *et al.* (2015))**.** *If* $|V(G)| \leq 2\chi(G) + 1$, *then* $\mathrm{ch}(G) = \chi(G)$.

Actually, the same paper proved a more generalized result:

Theorem 16 (Noel *et al.* (2015)). *For any graph G with $|V(G)| = n$ and $\chi(G) = k$, $\mathrm{ch}(G) \geq \max\{k, \lceil \frac{n+k-1}{3} \rceil\}$*

1.5 Historical Results for Paintability of Multipartite Graph

Since $\mathrm{ch}^{\mathrm{OL}} \geq \mathrm{ch}$, choosability can provide a natural lower bound for paintability. It turns out that many results of paintability are same from that of choosability. Several results are listed as follows.

Theorem 17 (Schauz (2009)). *Every planar graph is 5-paintable.*

Theorem 18 (Schauz (2009)). *For any bipartite graph, $\mathrm{ch}^{\mathrm{OL}}(G) = \Delta(G)$, where Δ is the maximum degree.*

However, there is still a difference between choosability and paintability for some graphs. One of our goals is to understand such graphs. One example is the complete multipartite graph: not all complete multipartite graphs have $\mathrm{ch}^{\mathrm{OL}} > \mathrm{ch}$, but there are some with this property. Our results through this line are shown in Chapter 3. Here we list some historical results.

Example 19 (Zhu (2009)).

$$\mathrm{ch}(K_{2*(k-1),3}) = k < k + 1 = \mathrm{ch}^{\mathrm{OL}}(K_{2*(k-1),3}).$$

for any $k \geq 3$.

From Thereom 15, we have $\mathrm{ch}(K_{2*(k-1),3}) = k$, since $\chi(K_{2*(k-1),3}) = k$ and $|V(G)| \leq 2k + 1$. The latter part $\mathrm{ch}^{\mathrm{OL}}(K_{2*(k-1),3}) = k + 1$ follows from the next three lemmas.

Lemma 20 (Zhu (2009))**.** *Let G be a complete multipartite graph with each part of size at most 2. Let $\mathcal{A}$ be the set of the parts of size 1, $\mathcal{B}$ be the set of the parts of size 2 such that $V(G) = \bigcup \mathcal{A} \cup \bigcup \mathcal{B}$. Let k_1, k_2 be the cardinalities of $\mathcal{A}, \mathcal{B}$, respectively. Suppose $\mathcal{A}$ is ordered as $\mathcal{A} = (A_1, A_2, ..., A_{k_1})$. Then G is online L-choosable for any list assignment L satisfying the following conditions:*

$$|L(v)| \geq k_2 + i \qquad \text{for all } 1 \leq i \leq k_1 \text{ and } v \in \mathcal{A},$$

$$|L(v)| \geq k_2 \qquad \text{for all } v \in B \in \mathcal{B},$$

$$\sum_{v \in B} |L(v)| \geq 2k_2 + k_1 \qquad \text{for all } B \in \mathcal{B}.$$

Proof. The proof goes by induction on $|V(G)|$. If $|V(G)| = 1$, then $k_1 = 1, k_2 = 0$. The only vertex v has $|L(v)| \geq 1$. Thus the graph is online L-choosable.

Suppose the conclusion holds when $|V(G)| < n$. Now $|V(G)| = n$. Alice chooses a set $S \subseteq V(G)$, Bob will choose an independent set $T \subseteq S$ based on several different cases. After Bob deletes the set, we call the new graph G', the new list assignment L', the new set of parts as $\mathcal{A}', \mathcal{B}'$ and their new cardinalities k_1', k_2'.

In the following cases, each case assumes the previous cases fail.

Case 1: S contains some part $B \in \mathcal{B}$. Then Bob chooses the part B. Now $k_1' = k_1, k_2' = k_2 - 1$. We have $|L'(v)| \geq k_2 - 1 + i = k_2' + i$ for all $v \in A_i \in \mathcal{A}$. Similarly for $v \in B \in \mathcal{B}$, $|L'(v)| \geq k_2 - 1 = k_2'$. For any $B \in \mathcal{B}$, $\sum_{v \in B} |L'(v)| \geq 2k_2 + k_1 - 2 = 2k_2' + k_1'$.

Case 2: S contains some vertex $v \in \bigcup \mathcal{B}$ with $|L(v)| = k_2$. In this case Bob chooses the vertex v. Now we have $\mathcal{A}' = \mathcal{A} + (B - v)$ with $B - v$ placed last in $\mathcal{A}'$, $\mathcal{B}' = \mathcal{B} - B$, $k_1' = k_1 + 1$ and $k_2' = k_2' - 1$. For any part $A = \{w\} \neq (B - v) \in \mathcal{A}$, $|L(v)| \geq k_2 + i - 1 = k_2' + i$, For $w \in B - v$, we have $|L(w)| \geq 2k_2 + k_1 - k_2 = k_2 + k_1 = k_2' + k_1'$. For any $w \in B \in \mathcal{B}'$, we have $|L(w)| \geq k_2 - 1 = k_2'$. Also $\sum_{w \in B} |L(w)| \geq 2k_2 + k_1 - 1 = 2k_2' + k_1'$.

Case 3: S contains some vertex $v \in \bigcup \mathcal{A}$. Then Bob chooses such a part A where t is minimum. Now $\mathcal{A}' = \mathcal{A} - A$ with order unchanged, $\mathcal{B}' = \mathcal{B}$, $k_1' = k_1 - 1, k_2' = k_2$. For any $1 \le i < t$ and $w \in A_i'$, we have $|L'(w)| = |L(w)|$. For $t \le i \le k_1'$, $|L'(w)| \ge k_2 + i + 1 - 1 = k_2' + i, w \in A_i'$. Now for $\mathcal{B}$, since Case 2 fails, any $w \in B \in \mathcal{B}$ satisfies $|L(w)| \ge k_2 + 1$. Hence $|L'(w)| \ge k_2 = k_2'$. $\sum_{w \in B} |L'(w)| \ge 2k_2 + k_1 - 1 = 2k_2' + k_1'$ for all $w \in B \subseteq \mathcal{B}$.

Case 4: S contains some vertex $v \in \bigcup \mathcal{B}$. Then Bob chooses such a vertex $v \in \bigcup \mathcal{B}$. Now we have $\mathcal{A}' = \mathcal{A} + (B - v)$ with $B - v$ placed at the beginning of $\mathcal{A}'$, $\mathcal{B}' = \mathcal{B} - B$, $k_1 = k_1' - 1$ and $k_2 = k_2' + 1$. Thus for any part $A = \{w\} \neq (B - v) \in \mathcal{A}$, $|L'(w)| \ge k_2 + i = k_2' + (i+1)$. For $w \in B - v$, we have $|L'(w)| \ge k_2 = k_2' + 1$. For any other $w \in B' \in \mathcal{B}'$, we have $|L(w)| \ge k_2 - 1 = k_2'$. Also $\sum_{w \in B'} |L(w)| \ge 2k_2 + k_1 - 1 = 2k_2' + k_1'$.

Since all the above cases satisfy $|V(G')| < n$ and all inequalities in the statement of the Lemma. Then by induction hypothesis, we are done. □

Lemma 21.

$$\mathrm{ch}^{\mathrm{OL}}(K_{2*(k-1),3}) \le k + 1$$

Proof. From Lemma 20, we have $\mathrm{ch}^{\mathrm{OL}}(K_{2*k}) = k$. To show $\mathrm{ch}^{\mathrm{OL}}(K_{2*(k-1),3}) \le k + 1$, we observe that there is one more vertex in $K_{2*(k-1),3}$ than in K_{k*2}. We call this vertex v. Then we can use the same winning strategy for Bob, except Bob deletes the extra vertex v immediately when Alice chooses it. Since the list size has been increased by 1, Bob can afford this round and still win. Hence $\mathrm{ch}^{\mathrm{OL}}(K_{2*(k-1),3}) \le k + 1$. □

Lemma 22.

$$\mathrm{ch}^{\mathrm{OL}}(K_{2*(k-1),3}) \ge k + 1$$

Proof. We prove the lemma by showing $\mathrm{ch}^{\mathrm{OL}}(K_{2*(k-1),3})$ is not k-online choosable.

Base step: $k = 3$. Figure 1.4 describes the strategy for Alice. Let $G = K_{2*(k-1),3}$ and $f(v) = k$ for any $v \in V(G)$. The top left matrix depicts the initial game position (G, f), and Alice's first move. The positions in the matrix correspond to the vertices of $K_{2,2,3}$ arranged so that vertices in the same part correspond to positions in the same column. The order of vertices within a column is irrelevant, as is the order of the columns. The numbers represent the size of $f(v)$ for the corresponding vertex v. The sequence of numbers represents a function f. The bold positions represent the vertices that Alice presents on first move.

As play progresses Bob chooses certain vertices presented by Alice and passes over others. When a vertex is chosen its position is removed from the next matrix (and the positions in its column of the remaining vertices and the order of the columns may be rearranged). When he passes over a vertex its list size is decreased by one (and its position in its column and the order of the columns may change). The arrows between the matrices point to the possible new game positions that arise from Bob's choice, not counting equivalent positions and assuming that Bob always chooses a maximal independent set.

For example, after Bob's first move there are two possible game positions, provided Bob chooses a maximal independent set. It is shown in the second column of the first row and the second column of the third row, along with Alice's second move respectively. Now Bob has three possible responses that are pointed to by two arrows for the positions of the first row.

Eventually, Alice forces one of four positions (G, f) such that G is not f-choosable, and Bob, being a gentleman, resigns.

Induction step: $k = 3$. Figure 1.5 describes the strategy for Alice. The description is similar to that of base step. The only difference is the position at the third

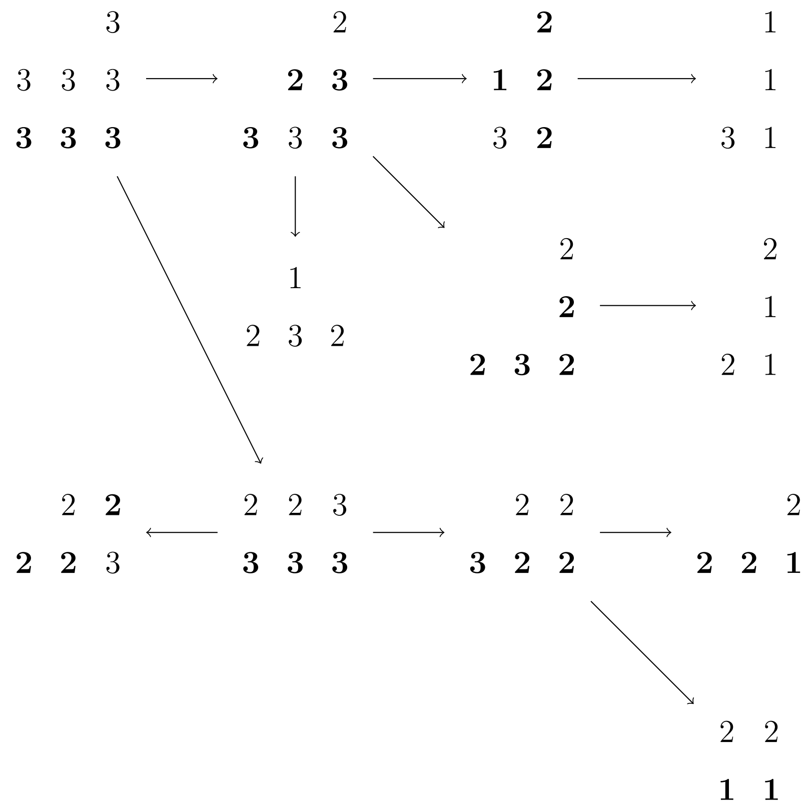


Figure 1.4: Strategy for Alice Demonstrating $\mathrm{ch}^{\mathrm{OL}}(K_{2,2,3}) \geq 4$

column of the first row is done by induction hypothesis as the graph is $G' = K_{2*(k-2),3}$ and $|L(v)| = k-1$ for each $v \in G$. We leave out some choices for Bob in some graphs, since these choices will lead to a graph with k-clique with list size function values in this clique at most $k-1$. □

Interestingly, it seems that in the above graphs, the choice numbers are much harder to find than the online choice numbers.

An analogue of the choosability version of Theorem 16 was raised naturally by Zhu (2009). The conjecture, however, has not been proved or disproved:

Conjecture 23 (Zhu (2009))**.** If $|V(G)| \leq 2\chi(G)$, then $\mathrm{ch}^{\mathrm{OL}}(G) = \chi(G)$.

This also means that we cannot find any counterexample so that $\mathrm{ch}^{\mathrm{OL}}(G) >$

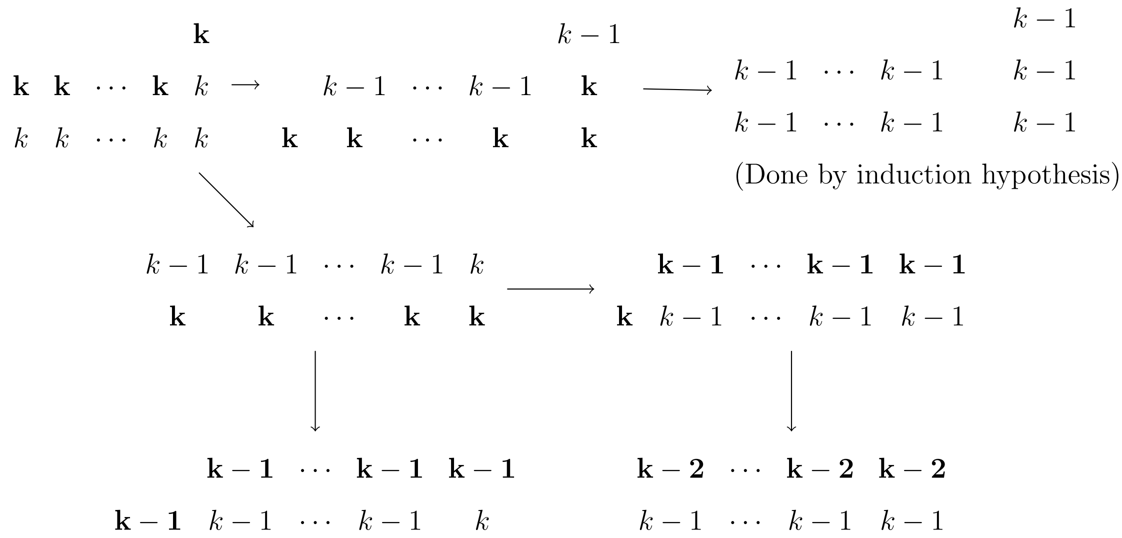


Figure 1.5: Strategy for Alice Demonstrating $\mathrm{ch}^{\mathrm{OL}}(K_{2*(k-1),3}) \geq k+1$

$\mathrm{ch}(G)+1$ for all graphs G with $|V(G)| \leq 2\chi(G)$ if Conjecture 23 is true.

It seems to be interesting to determine how large the difference can be between choosability and paintability. The conjecture is as follows:

Conjecture 24. There is a graph G with $\mathrm{ch}^{\mathrm{OL}}(G) - \mathrm{ch}(G) > 1$,

Much work is done towards solving Conjecture 24 in Chapter 3.

There are efforts made by other people. The following theorem shows a upper bound for online choosability of K_{3*k}.

Theorem 25 (Kozik *et al.* (2014))**.**

$$\mathrm{ch}^{\mathrm{OL}}(K_{3*k}) \leq \lfloor 3k/2 \rfloor$$

However, it is not known if the bound is tight or not.

Chapter 2

CHOICE NUMBER OF MULTIPARTITE GRAPH OF PART SIZE 4

2.1 Set-up

In this chapter, we prove Theorem 14. To start a proof, we first extract the technique used by Noel *et al.* (2015). From the technique, we deduce a simple proof of Theorem 11 in Section 2.2. After generalizing the technique, we finally prove Theorem 14 in Section 2.3.

Fix $s, k \in \mathbb{Z}^+$. Let $G = (V, E) = K_{s*k}$, and let $\mathcal{P}$ be the partition of V into k independent s-sets. Let $l = l(k, s) = \lceil \frac{(s-1)2k-s+2}{s} \rceil$, and consider any l-list assignment L for G. Put $C^* = \bigcup_{x \in V} L(x)$. Let $L\neg\alpha$ be the result of deleting α from every list of L.

We may write $x_1 \dots x_t$ for the subpart $S = \{x_1, \dots, x_t\} \subseteq X \in \mathcal{P}$; when we use this notation we implicitly assume the x_i are distinct. Also set $\overline{S} = X \smallsetminus S$. For a set of vertices $S \subseteq V$ let $\mathcal{L}(S) = \{L(x) : x \in S\}$, $L(S) = \bigcap \mathcal{L}(S)$, $W(S) = \bigcup \mathcal{L}(S)$, and $l(S) = |L(S)|$. The operation of replacing the vertices in S by a new vertex v_S with the same neighborhood as S is called *merging.* The new vertex v_S is said to be *merged*; vertices that are not merged are called *original.* When merging a set S we also create a list $L(v_S) = L(S)$.

For a color $\alpha \in C^*$, let $|X, \alpha| = |\{x \in X : \alpha \in L(x)\}|$ be the number of times α appears in the lists of vertices of X, $N_i(X) = \{\alpha \in C^* : |X, \alpha| = i\}$ be the set of colors that appear exactly i times in the lists of vertices in X, $n_i(X) = |N_i(X)|$, $N(X) = N_2(X) \cup N_3(X)$, and $n(X) = |N(X)|$. Let $\sigma_i(X) = \sum\{l(I) : I \subseteq X \wedge |I| = i\}$ and $\mu_i(X) = \max\{l(I) : I \subseteq X \wedge |I| = i\}$.

For a set S and element x we use the notation $S+x = S\cup\{x\}$ and $S-x = S\smallsetminus\{x\}$. The range of a function f is denoted by $\mathrm{ran}(f)$.

The following lemma was proved independently by Kierstead (2000), and by Reed and Sudakov (2005), Reed and Sudakov (2002), and named by Rabern.

Lemma 26 (Small Pot Lemma)**.** *If $\mathrm{ch}(G) > r$ then there exists a list assignment L such that G has no L-coloring, all lists have size r, and their union has size less than $|V(G)|$.*

If s does not satisfy (1.1) then there is a minimum counterexample k with $\mathrm{ch}(K_{s,k}) > l(s,k)$. By the Small Pot Lemma, this is witnessed by a list assignment L with $|\bigcup\{L(x) : x \in V(G)\}| < |V|$. We always assume L has this property.

Lemma 27. *If G is a minimum counterexample then every part X of G satisfies $L(X) = \emptyset$.*

Proof. Otherwise there exists a list assignment L, a color α, and a part X such that $\alpha \in L(X)$. Color each vertex in X with α, set $G' = G - X$, and put $L' = L\neg\alpha$. Then L' witnesses that $k-1$ is a smaller counterexample, a contradiction. □

By Lemma 27, $n_s(X) = 0$ for each part $X \in \mathcal{P}$. So by the Small Pot Lemma, $|W(X)| = \sum_{i=1}^{s-1} n_i(X) < sk$. Also $\sum_{i=1}^{s-1} i\, n_i(X) = sl$ is the number of occurrences of colors in the lists of vertices of X. Thus

$$\sum_{i=1}^{s-1}(i-1)n_i(X) \geq sl - |W(X)| \geq s(l-k)+1. \tag{2.1}$$

2.2 A Short Proof of Theorem 11

Now we warm-up by giving a short proof extracted from Noel *et al.* (2015) of Theorem 11.

Proof of Theorem 11. Let $s = 3$, $l = l(3, k) = \mathrm{ch}(K_{3*k}) = \lceil \frac{4k-1}{3} \rceil$, and assume G is a counterexample with k minimal. Then $k > 1$. By Lemma 27, $n_3(X) = 0$ for all $X \in \mathcal{P}$. We obtain a contradiction by L-coloring G. First we use the following steps to partition V into sets of vertices that will receive the same color. Then we *merge* each set I into a single vertex v_I, and assign v_I the set of colors in $L(I)$. Finally we apply Hall's Theorem to choose a system of distinct representatives (SDR) for these new lists; this induces an L-coloring of G.

Step 1. Partition $\mathcal{P}$ into a set $\mathcal{R}$ of $l - k$ *reserved* parts and a set $\mathcal{U} = \mathcal{P} \smallsetminus \mathcal{R}$ of $2k - l$ *unreserved* parts.

Step 2. Choose $\mathcal{U}_1 \subseteq \mathcal{U}$ to maximize $\nu = \sum_{X \in \mathcal{U}_1} \mu_2(X)$ subject to the constraint $|\mathcal{U}_1| \leq \mu_2(X)$ for all $X \in \mathcal{U}_1$. Set $u_1 = |\mathcal{U}_1|$. For each $X \in \mathcal{U}_1$ choose a pair $I_X \subseteq X$ with $l(I_X) \geq u_1$ maximum. Put $\mathcal{U}_2 = \mathcal{U} \smallsetminus \mathcal{U}_1$ and $u_2 = |\mathcal{U}_2|$.

The maximality of ν implies

$$\mu_2(X) \leq u_1 \text{ for all } X \in \mathcal{U}_2, \tag{2.2}$$

as otherwise adding X to $\mathcal{U}_1$, and deleting one part $Y \in \mathcal{U}_1$ with $\mu_2(Y) = u_1$, if such a part Y exists, would increase ν.

Step 3. Using (2.1), each part $X \in \mathcal{P}$ satisfies

$$n_2(X) \geq 3(l-k) + 1 = 3 \left\lceil \frac{k-1}{3} \right\rceil + 1 \geq k - 1 + 1 = k.$$

Form an SDR f for $\{L(v_{I_X}) : X \in \mathcal{U}_1\} \cup \{N(X) : X \in \mathcal{R}\}$ by greedily choosing representatives for the first family and then for the second family. For each $X \in \mathcal{R}$ choose a pair $I_X \subseteq X$ so that $f(x) \in L(I_X)$.

Step 4. For each $X \in \mathcal{U}_1 \cup \mathcal{R}$, merge I_X to a new vertex v_{I_X}, let $z_X \in X \smallsetminus I_X$, and set $X' = \{v_{I_X}, z_X\}$. If $X \in \mathcal{U}_2$, set $X' = X$. This yields a graph G' with parts $\mathcal{P}' = \{X' : X \in \mathcal{P}\}$, and list assignment L.

Next we use Hall's Theorem to prove that $\{L(x) : x \in V(G')\}$ has an SDR. For this it suffices to prove:

$$|S| \leq \left|\bigcup\{L(x) : x \in S\}\right| \text{ for every } S \subseteq V(G'). \tag{2.3}$$

To prove (2.3), let $S \subseteq V(G')$ be arbitrary, and set $W = W(S) := \bigcup\{L(x) : x \in S\}$. We consider several cases in order, always assuming all previous cases fail.

Case 1: There exists $X \in \mathcal{P}$ with $|S \cap X'| = 3$. Then $X' = X \in \mathcal{U}_2$, $u_2 \geq 1$, and $|S| \leq |G'| \leq 2k + u_2$. By (2.2), $u_1 \geq \mu_2(X) \geq \sigma_2(X)/3$. Using inclusion-exclusion, and Lemma 27,

$$\begin{aligned} |W| &\geq |W(X)| \geq \sigma_1(X) - \sigma_2(X) + \sigma_3(X) \geq 3l - 3u_1 = 3l - 3(2k - l - u_2) \\ &\geq 6(l-k) + 3u_2 \geq (2k-2) + (2 + u_2) \geq 2k + u_2 \geq |S|. \end{aligned}$$

Case 2: There is $X \in \mathcal{U}_2$ with $|S \cap X'| = 2$. Since $u_1 = 2k - l - u_2 < 2k - l$, By (2.2),

$$|W| \geq |W(S \cap X)| \geq 2l - l(S \cap X) \geq 2l - u_1 \geq 2l - (2k - l - u_2) \geq 3l + 1 - 2k \geq 2k \geq |S|.$$

Case 3: There is $X \in \mathcal{U}_1$ with $|S \cap X'| = 2$. As $|S| \leq 2k - u_2 = l + u_1$ and $L(v_{I_X} z_X) = L(X) = \emptyset$,

$$|W| \geq |W(S \cap X')| \geq l(v_{I_X}) + l(z_X) - l(v_{I_X} z_X) \geq u_1 + l \geq |S|.$$

Case 4: S has an original vertex. Then $|S| \leq 2k - u_1 - u_2 = l \leq |W|$.

Case 5: All vertices of S have been merged. Then $|S| \leq |f(S)| \leq |W|$.

□

2.3 Proof of Theorem 14

In this section we prove our main result, Theorem 14. The case when k is odd is considerably more technical. Casual or first time readers may wish to avoid these additional details; the proof is organized so that this is possible. In particular, in the even case Step 7(b), Step 11, Lemma 28(b), and Lemma 32 are not involved. Furthermore, only the first conclusion of Lemma 13 that k is odd (in the bad case covered by its hypothesis) is used. Let $b \in \{0,1\}$ with $b \equiv k \bmod 2$ and $l = l(4,k) = \lceil \frac{3k-1}{2} \rceil$. We often use the partition $k = (2k-l) + (l-k)$ of the integer k, and note that $2k - l = l - k + b$.

Proof of Theorem 14. Our set-up is the same as in the proof of Theorem 11. Let $s = 4$, $l = l(4,k)$, and $G = K_{4*k}$. The theorem is trivial if $k = 1$. Let $k > 1$ be a minimal counterexample, and let L be an l-list assignment for G with $|W(V)| \leq 4k-1$ and $L(X) = \emptyset$ for all parts $X \in \mathcal{P}$. Again we partition V into sets of vertices that will receive the same color, and then find an SDR for the induced list assignment that in turn induces an L-coloring of G. See Figure 2.1.

Step 1. Reserve notation for a partition $\mathcal{P} = \mathcal{R} \cup \mathcal{U}$ of V with $|\mathcal{R}| = l-k$, $|\mathcal{U}| = 2k-l$, $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$, and $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3 \cup \mathcal{U}_4$, where $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4$ are to be defined in the following steps.

Step 2. Choose $\mathcal{U}_1 \subseteq \mathcal{P}$ so that $|\mathcal{U}_1| \leq 2k-l$, for every $X \in \mathcal{U}_1$ there is a pair $I_X \subseteq X$ with (*) $l(I_X), l(\overline{I}_X) \geq k$, and subject to these constraints $|\mathcal{U}_1|$ is maximum. For each $X \in \mathcal{U}_1$ fix I_X witnessing (*). Let $u_1 := |\mathcal{U}_1|$. Then:

$$\text{If } u_1 < 2k-l \text{ then } (\forall X \in \mathcal{P} \smallsetminus \mathcal{U}_1)(\forall I \subseteq X)[|I| = 2 \rightarrow \min\{l(I), l(\overline{I})\} \leq k-1]. \quad (2.4)$$

Step 3. Choose $\mathcal{U}_2 \subseteq \mathcal{P} \smallsetminus \mathcal{U}_1$ so that $|\mathcal{U}_2| \leq 2k-l-u_1$ and $|\mathcal{U}_2| \leq \mu_3(X)$ for all $X \in \mathcal{U}_2$; subject to this let $\nu_3 = \sum_{X \in \mathcal{U}_2} \mu_3(X)$ be maximum. Let $u_2 = |\mathcal{U}_2|$. If $\mathcal{U}_2 \neq \emptyset$

we select a part $\dot{Z} \in \mathcal{U}_2$; else $\dot{Z} = \emptyset$. For each $X \in \mathcal{U}_2$ choose a triple $I_X \subseteq X$ with $l(I_X) \geq u_2$ maximum. Since ν_3 cannot be increased:

$$\text{If } u_1 + u_2 < 2k - l \text{ then } (\forall X \in \mathcal{U}_3 \cup \mathcal{U}_4 \cup \mathcal{R})[\mu_3(X) \leq u_2]. \tag{2.5}$$

Step 4. Choose $\mathcal{R}_1 \subseteq \mathcal{P} \smallsetminus (\mathcal{U}_1 \cup \mathcal{U}_2)$ so that $|\mathcal{R}_1| \leq l - k$ and there is a family of sets $\{I_X : X \in \mathcal{R}_1\}$ such that (*) $I_X \subseteq X$, $|I_X| = 3$, and there is an SDR f_1 of $\mathcal{L}(M_1)$, where $M_1 := \{v_{I_X} : X \in \mathcal{U}_2 \cup \mathcal{R}_1\}$; subject to this constraint, choose $\mathcal{R}_1$ with $|\mathcal{R}_1|$ maximum. Fix $\{I_X : X \in \mathcal{R}_1\}$, f_1 and M_1 satisfying (*). Let $C_1 = \operatorname{ran}(f_1)$ and $r_1 := |\mathcal{R}_1|$. Then:

$$\text{If } r_1 < l - k \text{ then } (\forall X \in \mathcal{U}_3 \cup \mathcal{U}_4 \cup \mathcal{R}_2 \cup \mathcal{R}_3)[N_3(X) \subseteq C_1]. \tag{2.6}$$

Moreover, by Lemma (27), $L(T) \cap L(T') = \emptyset$ for any two triples $T, T' \subseteq X$, and so

$$\text{If } r_1 < l - k \text{ then } (\forall X \in \mathcal{U}_3 \cup \mathcal{U}_4 \cup \mathcal{R}_2 \cup \mathcal{R}_3)[\sigma_3(X) \leq u_2 + r_1]. \tag{2.7}$$

Step 5. Choose $\mathcal{U}_3 \subseteq \mathcal{P} \smallsetminus (\mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{R}_1)$ so that $|\mathcal{U}_3| \leq 2k - l - u_1 - u_2$ and $l - k + u_2 + |\mathcal{U}_3| \leq \mu_2(X)$ for all $X \in \mathcal{U}_3$; subject to this constraint let $\nu_5 = \sum_{X \in \mathcal{U}_3} \mu_2(X)$ be maximum. Let $u_3 = |\mathcal{U}_3|$. Since ν_5 cannot be increased:

$$\text{If } u_1 + u_2 + u_3 < 2k - l \text{ then } (\forall X \in \mathcal{U}_4 \cup \mathcal{R}_2 \cup \mathcal{R}_3)[\mu_2(X) \leq l - k + u_2 + u_3]. \tag{2.8}$$

For all $X \in \mathcal{U}_3$ choose a pair $I_X = xy \subseteq X$ with $l(I_X) \geq l - k + u_2 + u_3$ maximum; subject to this choose I_X so that $\Delta_1(I_X) := l(I_X) - l(\overline{I}_X)$ is maximum. Set $\Delta_2(I_X) := 2u_2 - l(xyz) - l(xyw)$, where $zw = \overline{I}_X$.

Step 6. Choose $\mathcal{R}_2 \subseteq \mathcal{P} \smallsetminus (\mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3 \cup \mathcal{R}_1)$ so that $|\mathcal{R}_2| \leq l - k - r_1$ and $\sigma_2(X) - \sigma_3(X) \geq 5(l - k) + 2u_1 + 2u_2 + u_3 + r_1 + |\mathcal{R}_2|$ for all $X \in \mathcal{R}_2$; subject to this constraint let $\nu_6 = \sum_{X \in \mathcal{R}_2} (\sigma_2(X) - \sigma_3(X))$ be maximum. Set $r_2 = |\mathcal{R}_2|$. Then:

$$\begin{aligned} &\text{If } r_1 + r_2 < l - k \text{ then } (\forall X \in \mathcal{U}_4 \cup \mathcal{R}_3) \\ &\qquad [\sigma_2(X) - \sigma_3(X) \leq 5(l - k) + 2u_1 + 2u_2 + u_3 + r_1 + r_2]. \end{aligned} \tag{2.9}$$

Step 7. Choose $\mathcal{R}_3 \subseteq \mathcal{P} \smallsetminus (\mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3 \cup \mathcal{R}_1 \cup \mathcal{R}_2)$ with $|\mathcal{R}_3| = l - k - r_1 - r_2$. Let $r_3 = |\mathcal{R}_3|$. If $r_3 = 0$ then go to Step 8. Otherwise, for $I \subseteq X \in \mathcal{R}_3$, put $L^1(I) = L(I) \smallsetminus C_1$, $l^1(I) = |L^1(I)|$, and if $|I| = 2$, let $\Delta_1^1(I) = l^1(I) - l^1(\overline{I})$. By Lemma 28, all 2-sets $I \subset X$ satisfy

$$W(\overline{I} + v_I) \geq 3k - b + \Delta_1^1(I_X) - u_2 - r_1. \tag{2.10}$$

Set $\dot{r} = 0$ and do one of (a) or (b) below.

(a) If $b = 0$ or there are $X \in \mathcal{R}_3$ and a 2-set $I \subset X$ with $\Delta_1^1(I) \geq 1$ then choose $\dot{Y}$ and a 2-set $I_{\dot{Y}} \subset \dot{Y}$, so that $\Delta_1^1(I)$ is maximum. By Lemma 28(a) there is a family $\mathcal{J} =\{I_X : X \in \mathcal{R}_3 - \dot{Y}\}$ such that $I_X \subseteq X$, $|I_X| = 2$, $\Delta_1^1(I_X) \geq 0$, and for all $X \in \mathcal{R}_3$; and $\mathcal{L}(M_2 \cup \{v_{I_X} : X \in \mathcal{R}_3 \cup \mathcal{U}_3\})$ has an SDR f_2 extending f_1. Fix such $\mathcal{I} = \mathcal{J} + I_{\dot{Y}} = \{I_X : X \in \mathcal{R}_3\}$ and f_2, and set $M_2 = M_1 \cup \{v_{I_X} : X \in \mathcal{R}_3 \cup \mathcal{U}_3\}$ and $C_2 = \mathrm{ran}(f_2)$.

(b) Otherwise $b = 1$ and every $X \in \mathcal{R}_3$ and every 2-set $I \subset X$ satisfies $\Delta_1^1(I) = 0$. Let $\dot{Y}$ be any class in $\mathcal{R}_3$. By Lemma 28(b), there is a 2-set $I_{\dot{Y}} \subset \dot{Y}$ such that $\mathcal{I}$ can be chosen so that $\mathcal{L}(M_2 \cup \{v_{I_X} : X \in \mathcal{R}_3 \cup \mathcal{U}_3\} + v_{\overline{I}_{\dot{Y}}})$ has an SDR h extending f_1. Let $M_2 = M_1 \cup \{v_{I_X} : X \in \mathcal{R}_3 \cup \mathcal{U}_3\}$, f_2 equal h restricted to M_2, and $C_2 = \mathrm{ran}(f_2)$. (b*) If $u_1 = 0 = r_2$ then Step 9 is degenerate; set $f_3 = h$, $M_3 = M_2 + v_{\overline{I}_{\dot{y}}}$, $C_3 = \mathrm{ran}(f_3)$, and $\dot{r} = 1$.

Step 8. Put $\mathcal{U}_4 := \mathcal{U} \smallsetminus (\mathcal{R} \cup \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3)$, and $u_4 := |\mathcal{U}_4|$.

Step 9. Using Lemma 30, choose a family $\mathcal{I} = \{I_X : X \in \mathcal{R}_2\}$ such that $I_X \subseteq X$ and $|I_X| = 2$ for all $X \in \mathcal{R}_2$, and $\mathcal{L}(M_2 \cup \{v_{I_X} : X \in \mathcal{U}_1 \cup \mathcal{R}_2\} \cup \{v_{\overline{I}_X} : X \in \mathcal{U}_1 \cup \mathcal{R}_2\})$ has an SDR f_3 that extends f_2. Set $M_3 = M_2 \cup \{v_{I_X}, v_{\overline{I}_X} : X \in \mathcal{U}_1 \cup \mathcal{R}_2\}$ and $C_3 = \mathrm{ran}(f_3)$.

Step 10. Let $G' := (V', E')$ be the graph obtained from G by merging each I_X with $X \in \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3 \cup \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$ and each $\overline{I}_X$ with $X \in \mathcal{U}_1 \cup \mathcal{R}_2$. Note that this does not include $\overline{I}_{\dot{Y}}$ even if Step 7(b*) is executed. For a part X, let X' be the corresponding part in G', and set $\mathcal{P}' = \{X' : X \in \mathcal{P}\}$.

Step 11. Set $\dot{u} = \ddot{u} = 0$. If k is odd ($b = 1$) then we merge one more pair of vertices under the following special circumstances:

(a) there exists $X \in \mathcal{U}_4$ with $|W(X)| < |G'|$. Fix such an $X = \dot{X}$. By Lemma 31, $u_1 = 0 = r_3$ and there is a pair $I_{\dot{X}} \subseteq \dot{X}$ such that (i) f_3 can be extended to an SDR f of $\mathcal{L}(M)$, where $M := M_3 + v_{I_{\dot{X}}}$; (ii) $|W(\{v_{I_{\dot{X}}}, v\})| \geq 2k - 1$, and if equality holds then $|W(\{v_{I_{\dot{X}}}, v\} \cup \dot{Z}') \cup C_4| \geq 2k$ for both $v \in \overline{I}_{\dot{X}}$; and (iii) $W(\overline{I}_{\dot{X}} + v_{I_{\dot{X}}}) \geq |G'| - 1$. Merge $I_{\dot{X}}$ and set $\dot{u} = 1$.

(b) condition (a) fails and there exist $X \in \mathcal{U}_4$ and $xyz \subseteq X$ with

$$|W(xyz \cup \dot{Z}')| \leq 2k + u_4 - 1 < |W(X)|.$$

Fix such an $X = xyzw = \ddot{X}$. By Lemma 32 there is a pair $I_{\ddot{X}} \subseteq xyz$ such that (i) f_3 can be extended to an SDR f of $\mathcal{L}(M)$, where $M := M_3 + v_{I_{\ddot{X}}}$; (ii) $|W(\{v_{I_{\ddot{X}}}, v\})| \geq 2k$ for $v \in xyz \smallsetminus I_{\ddot{X}}$ and $|W(\{v_{I_{\ddot{X}}}, w\})| \geq 2k - 1 + u_3$; and (iii) $|W(\overline{I}_{\ddot{X}} + v_{I_{\ddot{X}}})| \geq 2k + u_4$. Merge $I_{\ddot{X}}$ and set $\ddot{u} = 1$.

Step 12. Recall that G' is the graph obtained after the first ten steps. Let H be the final graph obtained by this merging procedure, including $\overline{I}_{\dot{Y}}$ if Step 7(b*) is executed. (If $b = 0$, and possibly otherwise, $H = G'$). Also let M be the final set of merged vertices, f be the final SDR of $\mathcal{L}(V(H))$, and $C = \mathrm{ran}(f)$.

Recall that f_i is an SDR for $\mathcal{L}(M_i)$ with $\mathrm{ran}(f_i) = C_i$, M is the final set of merged vertices, f is the final SDR of $\mathcal{L}(M)$, and $C = \mathrm{ran}(f)$. Also, depending on whether

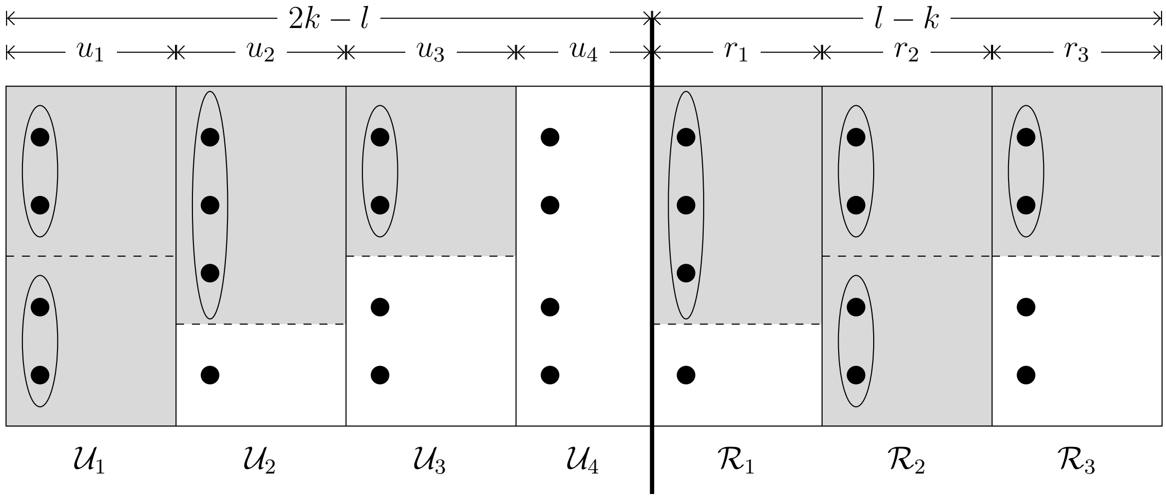


Figure 2.1: The ovals indicate sets of vertices that have been merged before Step 11 to form G'.

Step 7(b*) is executed, $M_3 = M_2 + v_{I_{\dot{Y}}}$ or $M_3 = M_2 \cup \{v_{I_X}, v_{\overline{I}_X} : X \in \mathcal{U}_1 \cup \mathcal{R}_2\}$. The following table summarizes some of the notation from the algorithm before Step 11.

i	$X \in \mathcal{U}_i$	$X \in \mathcal{U}_i$	$X \in \mathcal{R}_i$	M_i
1	$\lvert I_X\rvert = 2$	$l(I_X), l(\overline{I}_X) \geq k$	$\lvert I_X\rvert = 3$	$\{v_{I_X} : X \in \mathcal{U}_2 \cup \mathcal{R}_1\}$
2	$\lvert I_X\rvert = 3$	$l(I_X) \geq u_2$	$\lvert I_X\rvert = 2$	$M_1 \cup \{v_{I_X} : X \in \mathcal{U}_3 \cup \mathcal{R}_3\}$
3	$\lvert I_X\rvert = 2$	$l(I_X) \geq l - k + u_2 + u_3$	$\lvert I_X\rvert = 2$	$M_2 \cup \{v_{I_X}, v_{\overline{I}_X} : X \in \mathcal{U}_1 \cup \mathcal{R}_2\} + v_{I_{\dot{Y}}}$

Table 2.1: Notation and Facts from the Algorithm

Our next task is to state and prove the four lemmas on which the algorithm is based. The first lemma is used for Step 7. The statement of the lemma uses the notation from that step.

Lemma 28. *Every 2-set $I \subset X \in \mathcal{R}_3$ satisfies*

$$W(\overline{I}_X + v_{I_X}) \geq 3k - b + \Delta_1^1(I_X) - u_2 - r_1. \tag{2.11}$$

Furthermore:

(a) For every $Y \in \mathcal{R}_3$ and 2-set $I_Y \subset Y$ with $L^1(I_Y) \neq \emptyset$ and $\Delta^1_1(I_Y) \geq 0$ there is a family $\mathcal{I} = \{I_X : X \in \mathcal{R}_3\}$ such that $I_X \subseteq X$, $|I_X| = 2$, and $\Delta^1_1(I_X) \geq 0$ for all $X \in \mathcal{R}_3$; and $\mathcal{L}(M_1 \cup \{v_{I_X} : X \in \mathcal{R}_3 \cup \mathcal{U}_3\})$ has an SDR f_2 extending f_1.

(b) Furthermore, if (H) $W(\overline{I} \cup L(I)) \leq 3k - 1 - u_2 - r_1$ for all $X \in \mathcal{R}_3$ and all 2-sets $I \subset X$, then for some 2-set $I_Y \subset Y$, the family $\mathcal{I}$ and SDR f_2 can be chosen so that there is an SDR f_3 of $\mathcal{L}(M_1 \cup \{v_{I_X} : X \in \mathcal{R}_3 \cup \mathcal{U}_3\} + v_{\overline{I}_X})$ extending f_2.

Proof. Consider any $X = xyzw \in \mathcal{R}_3$. For $I \subseteq X$, let $l^2(I) = |L(I) \cap C_1|$. Then $l(I) = l^1(I) + l^2(I)$. First we prove (2.11) with $I = xy$. By (2.6), $l^1(wxy) = 0 = l^1(zxy)$, and by Lemma 27, $l(X) = 0$, and so $l^2(xy) - l^2(wxy) - l^2(zxy) \geq 0$. Thus

$$\begin{aligned} |W(wz + v_{xy})| &= l^1(w) + l^1(z) + l^1(xy) - l^1(wz) - l^1(wxy) - l^1(zxy) + \\ &\quad l^2(w) + l^2(z) + l^2(xy) - l^2(wz) - l^2(wxy) - l^2(zxy) \\ &\geq 2l + \Delta^1_1(xy) - l^2(wz) \geq 3k - b + \Delta^1_1(xy) - u_2 - r_1. \end{aligned}$$

Let $A(X) = N_1(X) \smallsetminus C_1$. By (2.6), $N_3(X) \subseteq C_1$. So $A(X)$ is the set of colors available for coloring a merged pair of vertices from X, and $L^1(I) = L(I) \smallsetminus C_1 = L(I) \cap A(X)$ for all pairs $I \subseteq X$. Moreover $\{L^1(I) : I \subseteq X \wedge |I| = 2\}$ is a partition of $A(X)$. For each color $\alpha \in A(x)$, set $I(\alpha) = \{x \in X : \alpha \in L(x)\}$. As $A(X) \subseteq N_2(X)$, $|I(\alpha)| = 2$. Let $B(X) = \{\alpha \in A(X) : \Delta^1_1(I(\alpha)) \geq 0)\}$. Colors $\beta \in B(X)$, do not conflict with f_1 and can be representatives for $L(I(\beta))$, while $\Delta^1_1(I(\beta)) \geq 0$.

Setting $X = wx_1x_2x_3$, and using $N_3(X) \subseteq C_1$,

$$|A(X)| = \sum_{i=1}^{3}(l^1(wx_i) + l^1(\overline{wx}_i)) \leq 2\sum_{i=1}^{3}\max(l^1(wx_i), l^1(\overline{wx}_i)) = 2|B(X)|. \quad (2.12)$$

By (2.1)

$$n_2(X) + 2n_3(X) \geq 4l - |W(X)| \geq 4(l - k) + 1 \geq 2k - 1. \quad (2.13)$$

As $N_3(X) \subseteq C_1$, $n_3(X) \leq |C_1| = u_2 + r_1$, and

$$\begin{aligned} |A(X)| &\geq n_2(X) + n_3(X) - |C_1| \geq n_2(X) + 2n_3(X) - n_3(X) - |C_1| \quad (2.14) \\ &\geq 2k - 1 - (2u_2 + 2r_1) \geq 2r_3 + 2u_3 - 1. \end{aligned}$$

By (2.12), $|B(X)| \geq \lceil |A(X)|/2 \rceil \geq r_3 + u_3$.

For (a), we construct the family $\mathcal{I}$ and SDR f_2 by a greedy algorithm. Start with the special class $Y \in \mathcal{R}_3$, and its preassigned subset I_Y. As $L^1(I_Y) \neq \emptyset$, and $\Delta_1^1(I_Y) \geq 0$ there is a color $\alpha \in L^1(I_Y) \cap B(Y)$. Let $f_2(L(v_{I_Y})) = \alpha$. Next process the $X \in \mathcal{R}_3 - Y$ one at a time. When X is considered, at most $r_3 - 1$ of the $r_3 + u_3$ colors of $B(X)$ have been used. Let β be an unused color, set $I_X = I(\beta)$, and put $f_2(L(v_{I_X})) = \beta$. Finally consider the $Z \in \mathcal{U}_3$. Recall that I_Z has been assigned in Step 3 so that $l(I_Z) \geq l - k + u_2 + u_3 \geq u_2 + r_1 + u_3 + r_3$. So there is a color $\gamma \in L(I_Z) \smallsetminus C_1$ that has not been used for any previous choices. Set $f_2(L(v_{I_Z})) = \gamma$.

For (b), suppose (H) holds. Then $\Delta_1^1(I) = 0$ for all $X \in \mathcal{R}_3$ and all 2-sets $I \subset X$. Thus $|A(X)|$ is even and $A(X) = B(X)$ for all $X \in \mathcal{R}_3$. Again we use a greedy procedure. First choose representatives for each $L(v_{I_Z})$ with $Z \in \mathcal{U}_3$. Also, for each representative α of $L(v_{I_Z}), Z \in \mathcal{U}_3$, remove α from all lists $L^1(I), I \subset X \in \mathcal{R}_3$; and for bookkeeping also remove some additional colors so that for each set $X = xyzw \in \mathcal{R}_3$, the new lists $L^-(I)$ satisfy

$$\begin{aligned} |L^-(xy)| &= |L^-(wz)|,\ |L^-(xz)| = |L^-(wy)|,\ |L^-(xw)| = |L^-(yz)|,\ \text{and} \\ r_3 &= |L^-(wx)| + |L^-(wy)| + |L^-(wz)|. \end{aligned}$$

Finish the construction by first choosing a 2-set $I_Y \subset Y$ with $L^-(I_Y) \neq \emptyset$, and setting $f_2(L(v_{I_Y})) = \alpha \in L^-(I_Y)$ and $f_3(L(v_{\overline{I}_Y})) = \beta \in L^-(\overline{I}_Y)$. Then for each $X \in \mathcal{R}_3$, greedily choose a 2-set $I_X \subset X$ so that $L^-(I_X)$ has an unused color γ and set $f_2(L(v_{I_X})) = \gamma$. This is possible since $\sum_{I \subset X, |I|=2} |L^-(I)| = 2r_3$. $\square$

The next lemma is used in Step 9. The statement of the lemma uses the notation from that step. We will need the following easy claim.

Claim 29. Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ be the three partitions of a 4-set X into pairs. For all $I_1 \in \mathcal{P}_1, I_2 \in \mathcal{P}_2, I_3 \in \mathcal{P}_3$ there exists $v \in X$ such that either (i) $v \in I_1 \cap I_2 \cap I_3$ or (ii) $v \notin I_1 \cup I_2 \cup I_3$.

Lemma 30. *There is a family $\mathcal{I} = \{I_X : X \in \mathcal{R}_2\}$ such that $I_X \subseteq X$ and $|I_X| = 2$ for all $X \in \mathcal{R}_2$, and $\mathcal{L}(M_3 \cup \{v_{I_X} : X \in \mathcal{U}_1 \cup \mathcal{R}_2\} \cup \{v_{\overline{I}_X} : X \in \mathcal{U}_1 \cup \mathcal{R}_2\})$ has an SDR f_4 that extends f_3.*

Proof. Each $X \in \mathcal{U}_1$ satisfies $L(I_X), L(\overline{I}_X) \geq k$ by Step 2 and $L(I_X) \cap L(I_{\overline{X}}) = \emptyset$ by Lemma 27. Thus $|L(I_X) \smallsetminus C_3|, |L(\overline{I}_X) \smallsetminus C_3| \geq k - u_2 - u_3 - r_1 - r_3 \geq u_1$. By Theorem 10, $\{L(I_X) \smallsetminus C_3, L(\overline{I}_X) \smallsetminus C_3 : X \in \mathcal{U}_1\}$ has an SDR, and so f_3 can be extended to an SDR g for $\mathcal{L}(M_3')$, where $M_3' := M_3 \cup \{I_X, \overline{I}_X : X \in \mathcal{U}_1\}$. Let

$$C^g = \operatorname{ran}(g). \tag{2.15}$$

Then $|C^g| = 2u_1 + u_2 + u_3 + r_1 + r_3$.

Next consider any $X \in \mathcal{R}_2$. Let $A(X) = N_2(X) \smallsetminus C^g$. Again by Theorem 10 it suffices to show:

$$(\exists I_X \subseteq X)[|I_X| = 2 \wedge |L(I_X) \cap A(X)| \geq r_2 \wedge |L(\overline{I}_X) \cap A(X)| \geq r_2]. \tag{2.16}$$

Observe $\sigma_2(X) = n_2(X) + 3n_3(X)$ and $\sigma_3(X) = n_3(X)$. So $n(X) = n_2(X) + n_3(X) = \sigma_2(X) - 2\sigma_3(X)$. By (2.6), $N_3(X) \subseteq C^g$ and $\sigma_3(X) \leq u_2 + r_1$. So by (2.9),

$$\begin{aligned} n(X) &= \sigma_2(X) - 2\sigma_3(X) \geq 5(l-k) + 2u_1 + 2u_2 + u_3 + r_1 + r_2 - (u_2 + r_1) \\ &\geq 5(l-k) + 2u_1 + u_2 + u_3 + r_2 \end{aligned} \tag{2.17}$$

and

$$\begin{aligned}|A(X)| &= |N_2(X) \setminus C^g| = |N_2(X) \cup N_3(X) \setminus C^g| \geq n(X) - |C^g| \\ &\geq 5(l-k) + 2u_1 + u_2 + u_3 + r_2 - (2u_1 + u_2 + u_3 + r_1 + r_3) \\ &\geq 5(l-k) - r_1 + r_2 - r_3 \geq 4(l-k) + 2r_2. \end{aligned} \tag{2.18}$$

Suppose (2.16) fails. Then for each of the three partitions of X into pairs, there is a pair uv with $|L(uv) \cap A(X)| \leq r_2 - 1$. Using Claim 29, there exists $w \in X$ such that either (i) $|L(vw) \cap A(X)| \leq r_2 - 1$ for all $v \in X - w$ or (ii) $|L(uv) \cap A(X)| \leq r_2 - 1$ for all $u, v \in X - w$.

If (i) holds then

$$|L(w) \cap N(X)| \leq |C^g| + \sum_{v \in X - w} |L(vw) \cap A(X)| \leq |C^g| + 3r_2 - 3. \tag{2.19}$$

For all $v \in X - w$,

$$\text{(a) } |L(v) \cap N(X)| \leq l \text{ and (b) } |L(v) \cap N(X)| \leq \sum_{u \in X - v} l(uv) \leq 3(l-k). \tag{2.20}$$

Using (2.19) and (2.20a),

$$2n(X) \leq |L(w) \cap N(X)| + \sum_{v \in X - w} |L(v) \cap N(X)| \leq (|C^g| + 3r_2 - 3) + 3l. \tag{2.21}$$

By (2.17), (2.15) and (2.21) we have

$$\begin{aligned} 10(l-k) + 4u_1 + 2u_2 + 2u_3 + 2r_2 &\leq 3l + 2u_1 + u_2 + u_3 + r_1 + r_3 + 3r_2 - 3 \\ (6l - 9k + 3) + 2u_1 + u_2 + u_3 &\leq k - l + r_1 + r_2 + r_3 = 0. \end{aligned} \tag{2.22}$$

Since $6l - 9k = -3b$, both $b = 1$ and $0 = u_1 = u_2 = u_3$. By (2.8),

$$\mu_2(X) \leq l - ku_2 + u_3 = l - k.$$

Using this with (2.20b) in (2.21) to strengthens the estimate in (2.22) yields the contradiction:

$$\begin{aligned} 10(l-k)+2r_2 &\leq 9(l-k)+(|C^g|+3r_2-3) \\ l-k &\leq r_1+r_2+r_3-3 < l-k. \end{aligned}$$

Thus (ii) holds. So

$$|A(X)| \leq l(w) + \sum_{uv \subseteq X-w} |L(uv) \cap A(X)| \leq l + 3(r_2 - 1). \tag{2.23}$$

Using (2.18) and (2.23),

$$\begin{aligned} 4(l-k)+2r_2 &\leq |A(X)| \leq l+3(r_2-1) \\ 3l-4k+3 &\leq r_2. \end{aligned}$$

As $2l-3k=-b$, this yields the contradiction $l-k+2 \leq r_2 \leq l-k$. □

The next Lemma is used in Step 11(a). Recall that $\dot{Z}$ is defined in Step 3, and Δ_1 and Δ_2 are defined in Step 5.

Lemma 31. *Suppose $X = xyzw \in \mathcal{U}_4$ and $|W(X)| < |G'|$. Then $b=1$, $u_1 = 0 = r_3$, $u_2+u_3 \geq 1$, and there exists a pair $J \subseteq X$ such that:*

1. *$L(J) \nsubseteq C_3$;*
2. *for both $v \in \overline{J}$, both $|W(\{v_J, v\})| \geq 2k-1$, and if $|W(\{v_J, v\})| = 2k-1$* then *$|W(\{v_J, v\} \cup \dot{Z}') \cup C_3| \geq 2k$;*
3. *$|W(\overline{J}+v_J)| \geq |G'|-1$; in particular $|W(X)| \geq |G'|-1$.*

Proof. Now $|G'| = 3k - u_1 - u_2 + u_4 - r_1 - r_2$. Observe that

$$\sigma_2(X) - \sigma_3(X) \geq 5(l-k) + 2u_1 + 2u_2 + u_3 + r_1 + r_2 + 1, \tag{2.24}$$

since otherwise inclusion-exclusion yields the contradiction:

$$\begin{aligned} |W(X)| &= \sigma_1(X) - \sigma_2(X) + \sigma_3(X) \\ &\geq 4l - 5(l-k) - 2u_1 - 2u_2 - u_3 - r_1 - r_2 \\ &\geq 3k + (2k - l - u_1 - u_2 - u_3) - u_1 - u_2 - r_1 - r_2 \\ &\geq 3k + u_4 - u_1 - u_2 - r_1 - r_2 = |G'| > |W(X)|. \end{aligned}$$

By (2.24) and (2.9), $r_1 + r_2 = l - k$ and $r_3 = 0$. Consider any pair $I = xy \subseteq X$. Then

$$\begin{aligned} |W(\overline{I} + v_I)| &\geq l(xy) + l(z) + l(w) - l(xyz) - l(xyw) - l(zw) \\ &= 2l - 2u_2 + \Delta_1(I) + \Delta_2(I) \\ |G'| - |W(\overline{I} + v_I)| &\leq b - 2u_1 + (u_1 + u_2 + u_4 - l + k) - \Delta_1(I) - \Delta_2(I) \\ 1 &\leq 2b - 2u_1 - u_3 - \Delta_1(I) - \Delta_2(I). \end{aligned} \tag{2.25}$$

By (2.25), $\Delta_1(I) + \Delta_2(I) \leq 1$. As $\Delta_1(I) = -\Delta_1(\overline{I})$, and $\Delta_2(I), \Delta_2(\overline{I}) \geq 0$ by (2.5), we could choose I with $\Delta_1(I) + \Delta_2(I) \geq 0$. So $b = 1$, $u_1 = 0$, $u_3 \leq 1$, and

$$1 \leq |G| - |W(\overline{I} + v_I)| \leq 2 - u_3 - \Delta_1(I) - \Delta_2(I) \leq 2. \tag{2.26}$$

Furthermore, using $\Delta_1(I) = -\Delta_1(\overline{I})$ and (2.5) again,

$$0 \leq 4u_2 - \sigma_3(X) = \Delta_2(I) + \Delta_2(\overline{I}) = \Delta_1(I) + \Delta_2(I) + \Delta_1(\overline{I}) + \Delta_2(\overline{I}) \leq 2. \tag{2.27}$$

By (2.24), $r_1 + r_2 = l - k$, $\sigma_2(X) \leq 6\mu_2(X)$, (2.8), and $\sigma_3 = 4u_2 - \Delta_2(I) - \Delta_2(\overline{I})$,

$$1 + 6(l-k) + 2u_2 + u_3 + \sigma_3(X) \leq \sigma_2(X) \leq 6(l - k + u_2 + u_3) \tag{2.28}$$

$$1 + u_3 + 6(l - k + u_2) - \Delta_2(I) - \Delta_2(\overline{I}) \leq \sigma_2(X) \leq 6(l - k + u_2 + u_3). \tag{2.29}$$

By (2.28) $u_2 + u_3 \geq 1$. So the first three assertions of the lemma have been proved. It remains to find a pair $J \subseteq X$ satisfying (1–3).

First suppose $u_3 = 1$. By (2.26), $\Delta_1(I) + \Delta_2(I) = 0$ for all pairs $I \subseteq X$. So $\Delta_1(I) \leq 0$ and $\Delta_1(\overline{I}) \leq 0$. As $\Delta_1(I) = -\Delta_1(\overline{I})$, this implies $\Delta_1(I) = 0 = \Delta_1(\overline{I})$. So $\Delta_2(I) = 0 = \Delta_2(\overline{I})$. By (2.29) and (2.8), there exists a pair $I \subseteq X$ with $l(I) \geq l - k + u_2 + u_3$. As $\Delta_1(I) = 0$, $l(\overline{I}) \geq l - k + u_2 + u_3$. Thus, using Lemma 27 and (2.5),

$$|W(\{v_I, v_{\overline{I}}\})| = l(I) + l(\overline{I}) \geq 2(l - k + u_2 + u_3) > 2(l - k) + u_2 + u_3 \geq |C_3|.$$

Pick $J \in \{I, \overline{I}\}$ such that $L(J) \nsubseteq C_3$. Then (1) holds. For (2), let $v' \in \overline{J}$. Using (2.5),

$$|W(\{v_J, v'\})| = l(J) + l(v') - l(J + v') \geq 2l - k + u_2 + u_3 - u_2 \geq 2k.$$

Thus (2) holds. As $u_3 = 1$, (2.26) implies (3).

Otherwise $u_3 = 0$. Then $u_2 \geq 1$, and so $\dot{Z} \neq \emptyset$. Put $C_0 := C_3 \cup W(\dot{Z}')$. By Step 3 and Lemma 27, $|C_0| \geq |W(\dot{Z}')| \geq l + u_2$. Call a vertex $x \in X$ *bad* if $|L(x) \cup C_0| \leq 2k - 1$; otherwise x is *good*. If x is bad then $|C_0 \smallsetminus L(x)| \leq 2k - 1 - l = l - k$. If another vertex y is also bad, then using (2.8) and (2.26),

$$\begin{aligned} l - k + u_2 \geq l(xy) &\geq |L(xy) \cap C_0| \geq |C_0| - |C_0 \smallsetminus L(x)| - |C_0 \smallsetminus L(y)| \\ &\geq l + u_2 - 2(l - k) \geq l - k + u_2 + 1, \end{aligned}$$

a contradiction. So at most one vertex of X is bad.

Call a pair $I \subseteq X$ *bad* if $L(I) \subseteq C_3$; otherwise I is *good*. Note that a good pair satisfies (1). Using (2.27), (2.29) and $u_3 = 0$ yields

$$6(l - k + u_2) - 1 \leq \sigma_2 \leq 6(l - k + u_2) \tag{2.30}$$

so by (2.8), every pair $I \subseteq X$ satisfies

$$l - k + u_2 - 1 \leq l(I) \leq l - k + u_2.$$

If the upper bound is sharp then call I *normal*; otherwise call I *abnormal*. By (2.30) there is at most one abnormal pair. If I is normal then $l(\overline{I}) \leq l(I)$; so $\Delta_1(I) \geq 0$.

By (2.5), every triple $T \subseteq X$ satisfies $l(T) \leq u_2$. If equality holds then call T *normal*; otherwise call T *abnormal*; if $l(T) \leq u_2 - 2$ then call T *very abnormal*. Suppose two pairs $I, J \subseteq T$ are both bad. At least one, say I, is normal. As $u_1 = 0 = u_3$,

$$\begin{aligned} 2(l-k)+u_2 &\geq |C_3| \geq |L(I) \cup L(J)| \geq l-k+u_2+l(J)-l(I\cup J) \qquad (2.31) \\ l(T) &= l(I \cup J) \geq l(J)-l+k = \begin{cases} u_2 & \text{if } J \text{ is normal} \\ u_2-1 & \text{if } J \text{ is abnormal} \end{cases}. \end{aligned}$$

So an abnormal triple contains at most one bad, normal pair, and a very abnormal triple contains at most one bad pair.

A pair I contained in an abnormal triple satisfies $\Delta_2(I) \geq 1$. Suppose J is a good, normal pair contained in an abnormal triple T with $w \in X \smallsetminus T$. Then $\Delta_1(J) + \Delta_2(J) \geq 1$. So J satisfies (3) by (2.26). Also, for both $v \in X \smallsetminus J$,

$$|W(v_J, v)| = l(J) + l(v) - l(J+v) \geq \begin{cases} 2l-k+u_2-(u_2-1) = 2k & \text{if } v \in T \smallsetminus J \\ 2l-k+u_2-u_2 = 2k-1 & \text{if } v = w \end{cases}.$$

We have proved the following observation.

Observation 1. *If $J \subseteq T \subseteq X$ and $w \in X \smallsetminus T$, where J is a good, normal pair and T is an abnormal triple, then J satisfies (1–3) provided w is good or $C_0 \nsubseteq W(v_J, w)$.*

By (2.29) and (2.27), $1 \leq \Delta_2(I) + \Delta_2(\overline{I}) \leq 2$. As $\sigma_3 = 4u_2 - \Delta_2(I) - \Delta_2(\overline{I})$, we have $\sigma_3(X) = 4u_2 - 2$ or $\sigma_3(X) = 4u_2 - 1$. In the first case either there are two abnormal triples or there is a very abnormal triple. In the second case, there is one abnormal triple, and there are no abnormal pairs since equality holds in (2.28).

First suppose there are two abnormal triples. Choose an abnormal triple T so that if there is a bad vertex then it is in T. As T contains three pairs, of which at most one is abnormal, and at most one is bad and normal, T contains a good, normal pair J. Say $J = yz$, $T = xyz$, and $w \in X \setminus T$. Then w is good, and thus J satisfies (1–3) by Observation 1.

Otherwise, let $T = xyz$ be the only abnormal triple and $w \in X \setminus T$. There is at most one abnormal pair, and only if T is very abnormal. So T contains at most one bad pair.

Now suppose T has two good, normal pairs xy and yz. By Observation 1, some $J \in P := \{xy, yz\}$ satisfies (1–3), unless $C_0 \subseteq L(J) \cup L(w)$ for both $J \in P$.

$$C_0 \subseteq (L(xy) \cup L(w)) \cap (L(yz) \cup L(w)) = L(T) \cup L(w).$$

As T is abnormal, and using $u_1 = u_3 = r_3 = 0$, this yields the contradiction

$$l + u_2 \leq |C_0| \leq |L(T) \cup L(w)| = u_2 - 1 + l.$$

Otherwise, T does not contain two good, normal pairs. So T is very abnormal. As T has at least two normal pairs and at most one bad pair, it has a bad normal pair (say) xz and a good normal pair (say) $J = yz$ is normal. Since xz is bad, $L(xz) \subseteq C_0$. Now

$$\begin{aligned} |C_0 \setminus W(\{v_J, w\})| &\geq |L(xz) \setminus (L(w) \cup L(J))| \geq l(xz) - l(xzw) - l(xzy) \\ &\geq l - k + u_2 - u_2 - (u_2 - 2) \geq 2, \end{aligned}$$

and so $C_0 \not\subseteq W(v_J, w)$. So J satisfies (1–3) by Observation 1. □

The next Lemma is needed for Step 11(c).

Lemma 32. *Suppose $b = 1$ and $X = xyzw \in \mathcal{U}_4$. If*

$$|W(xyz)| \leq 2k + u_4 - 1 < |W(X)| \tag{2.32}$$

then $u_1 = 0$ *and there exists a pair* $J \subseteq X$ *such that:*

1. $L(J) \nsubseteq C_3$;

2. $|W(\{v_J, v\})| \geq 2k$ *for* $v \in xyz \smallsetminus J$ *and* $|W(\{v_J, w\})| \geq 2k - 1 + u_3$*; and*

3. $|W(\overline{J} + v_J)| \geq 2k + u_4$.

Proof. Consider a pair $vv' \subseteq xyz$. By (2.8) and (2.32),

$$\begin{aligned} 2k + u_4 - 1 &\geq |W(xyz)| \geq |W(vv')| \geq l(v) + l(v') - l(vv') \\ &\geq 2l - (l - k + u_2 + u_3) \geq 3k - 1 - k + u_1 + u_4 \\ &\geq 2k + u_1 + u_4 - 1. \end{aligned}$$

So $u_1 = 0$, $l(vv') = l - k + u_2 + u_3$, and $W(xyz) = W(vv')$. Since vv' is arbitrary, every color in $W(xyz)$ appears in at least two of the lists $L(x)$, $L(y)$, $L(z)$. So $W(\{v_J, v\}) = W(xyz)$ and $|W(\{v_J, v\})| \geq 2k$ for every pair $J \subseteq xyz$ and vertex $v \in xyz \smallsetminus J$. As $|C_3| < 2k \leq |W(xyz)|$, there is a pair $J \subseteq xyz$ with $L(J) \nsubseteq C_4$. Furthermore, by (2.5),

$$|W(\{v_J, w\})| \geq l(J) + l(w) - l(J + w) \geq l - k + u_2 + u_3 + l - u_2 = 2k - 1 + u_3.$$

Finally, as $W(\{v_J, v\}) = W(xyz)$ for $v \in xyz \smallsetminus J$, and using (2.32),

$$|W(\overline{J} + v_J)| = |W(\{v_J, v\}) \cup W(w)| = |W(xyzw)| \geq 2k + u_4.$$

□

The next lemma completes the proof of our main theorem. The reader should keep Figure 2.1 and Table 2.1 in mind.

Lemma 33. *G' is L-choosable.*

Proof. First observe that if k is even then $b = \dot{u} = \ddot{u} = \dot{r} = 0$ and $H = G'$. In this case the following argument is much simpler.

We will prove $\mathcal{L}(V(H))$ has an SDR. Using Hall's Theorem it suffices to show $|S| \leq |W| := \left|\bigcup_{x \in S} L(x)\right|$ for every $S \subseteq V(H)$. Suppose for a contradiction that $|S| > |W|$ for some $S \subseteq V(H)$. We consider several cases. Each case assumes the previous cases fail.

Case 1: There is $X \in \mathcal{U}_4$ with $|S \cap X| = 4$. Since $|W| < |S| \leq |H| \leq |G'|$, Lemma 31 yields $b = 1$ and $|G'| - 1 = |W(X)| < |S| \leq |H|$. Furthermore, Step 11(a) is executed, and so $|H| \leq |G'| - 1$, a contradiction.

Case 2: There exists $X = xyzw \in \mathcal{U}_3$ with $|S \cap X'| = 3$. Since Case 1 fails,

$$|S| \leq |H| \leq 3k - u_1 - u_2 - r_1 - r_2 - \dot{r}. \tag{2.33}$$

Say $I_X = xy$. By Step 5, $\Delta_1(xy) \geq 0$ and $l(xyz) + l(xyw) = 2u_2 - \Delta_2(xy)$. By (2.7), $l(xyz) + l(xyw) \leq u_2 + r_1$ if $r_1 < l - k$; and it also holds if $r_1 = l - k$ since $u_2 \leq 2k - l - 1 \leq l - k$ and $l(xyz) + l(xyw) \leq 2u_2$. Thus

$$\begin{aligned} |W| &\geq |W(X')| \geq l(xy) + l(z) + l(w) - l(xyz) - l(xyw) - l(zw) \\ &= 2l + \Delta_1(xy) - 2u_2 + \Delta_2(xy) = 3k - b + \Delta_1(xy) - 2u_2 + \Delta_2(xy) \\ &\geq 3k - b + \Delta_1(xy) - u_2 - r_1 \geq |S| - b \geq |W|. \end{aligned} \tag{2.34}$$

So equality holds throughout. Thus (2.33) is sharp, $b = 1$, $u_1 = r_2 = \dot{r} = \Delta_1(xy) = 0$,

$$|W| = 3k - 1 - u_2 - r_1,$$

and $r_1 \leq u_2$ since $\Delta_2 \geq 0$. So Step 7(a) is not executed. Since $\dot{r} = 0$, Step 7(b*) is not executed. As $u_2 = 0 = r_1$, $r_3 = 0$ and Step 7 is degenerate. As $X \in \mathcal{U}_3$,

$$l - k = r_1 \leq u_2 \leq 2k - l - u_3 - u_4 \leq l - k - u_4.$$

So $u_4 = 0$, and by Step 5,

$$k = l - k + u_2 + u_3 \leq l(xy) = l(\overline{xy}) + \Delta_1(xy) = l(\overline{xy}).$$

By (2.4) this contradicts $u_1 = 0$.

Case 3: There exists $X = wxyz \in \mathcal{R}_3$ with $|S \cap X'| = 3$. Say $I_X = xy$. As the previous cases fail, $|S| \leq 3k - u_1 - u_2 - u_3 - r_1 - r_2 - \dot{r}$. By Step 7, $\Delta_1^1(xy) \geq 0$, and by Lemma 28,

$$|W| \geq |W(X')| \geq 3k - b + \Delta_1^1(xy) - u_2 - r_1 \geq |S| - b \geq |W|.$$

Thus $b = 1$, $0 = r_2 = u_1 = \dot{r}$, $|S| = 3k - 1 - u_2 - r_1$, and $\dot{Y}' \subseteq S$. As $\dot{r} = 0$, Step 7(b*) is not executed. Thus, as $0 = r_2 = u_1$ and $r_3 \neq 0$, Step 7(a) is executed. So $\Delta_1^1(I_{\dot{Y}}) \geq 1$. This yields the contradiction

$$|W| \geq |W(\dot{Y}')| \geq 3k - 1 + \Delta_1^1(I_{\dot{Y}}) - u_2 - r_1 \geq 3k - u_2 - r_1.$$

Case 4: There exists $X \in \mathcal{U}_4$ with $|S \cap X'| = 3$. As the previous cases fail,

$$|S| \leq 2k + u_4 - \dot{u} - \ddot{u}$$

Let $xy \subseteq (S \cap X') \smallsetminus M$. By (2.8),

$$\begin{aligned} |W| &\geq l(x) + l(y) - l(xy) \geq 3k - b - (l - k + u_2 + u_3) \\ &= 2k + (2k - l) - (u_2 + u_3) - b \geq 2k + u_1 + u_4 - b \geq |S| - b \geq |W|. \end{aligned}$$

So $b = 1$, $0 = u_1 = \dot{u} = \ddot{u}$, $|W| = 2k + u_4 - 1$, and $|S| = 2k + u_4$. Thus S has exactly two vertices in every class of $\mathcal{P}' \smallsetminus \mathcal{U}_4'$ and exactly three vertices in every class of $\mathcal{U}_4'$. In particular, $\dot{Z}' \subseteq S$. Since $\dot{u} = \ddot{u} = 0$, we have $X = X'$. As Step 11(a) is not executed, $|W(X)| \geq |G'| \geq |S| = 2k + u_4$. Thus, as Step 11(b) is not executed, we have the contradiction

$$|W| \geq |W((S \cap X) \cup \dot{Z}')| \geq 2k + u_4 = |S|.$$

Case 5: There exists $X \in \mathcal{U}_1$ with $|S \cap X'| = 2$. Then $v_{I_X}, v_{\overline{I}_X} \in S$. As the previous cases fail, $|S| \leq 2k$. Now

$$|W| \geq L(v_{I_X}) + L(v_{\overline{I}_X}) \geq 2k \geq |S|.$$

Case 6: There exists $X \in \mathcal{U}_3$ with $|S \cap X'| = 2$. Say $S \cap X' = vv'$. As the previous cases fail, $|S| \leq 2k - u_1$. If (say) $v, v' \notin M$ then $\overline{I}_X = vv'$. By (2.4), $l(\overline{I}_X) \leq k - 1$. So

$$|W(vv')| \geq l(v) + l(v') - l(vv') \geq 2l - (k-1) \geq 2k \geq |S|.$$

Otherwise (say) $v = v_{xy}$; so $I_X = xy$ and $v' = z \notin M$. By Step 5, $l(v_{xy}) \geq l - k + u_2 + u_3$, and $l(xyz) \leq u_2$ by (2.5). So

$$\begin{aligned} |W(vv')| &\geq l(v_{xy}) + l(z) - l(xyz) \\ &\geq l - k + u_2 + u_3 + l - u_2 \geq 2k - b + u_3 \geq 2k \geq |S|. \end{aligned}$$

Case 7: There exists $X \in \mathcal{U}_4$ with $|S \cap X'| = 2$. Say $S \cap X' = vv'$. If possible, choose X so that $S \cap X' \cap M = \emptyset$. As the previous cases fail, $|S| \leq 2k - u_1 - u_3$. If $v, v' \notin M$ then, using (2.5),

$$\begin{aligned} |W(vv')| &= l(v) + l(v') - l(vv') \geq 2l - (l - k + u_2 + u_3) \\ &= 2k - b + u_1 + u_4 \geq 2k \geq |S|. \end{aligned} \tag{2.35}$$

Otherwise $b = 1$, and (say) $v \in M$. By Step 11, either $v = v_{I_{\dot{X}}}$ or $v = v_{I_{\ddot{X}}}$. By Lemmas 31 and 32, $u_1 = 0$.

If $v = v_{I_{\dot{X}}}$ then Step 11(a) was executed. So (i) $u_1 = 0 = r_3$, (ii) $|W(vv')| \geq 2k-1$, and (iii) if $|W(vv')| = 2k - 1$ then $|W(vv' \cup \dot{Z}') \cup C_3| \geq 2k$. Since

$$2k \geq |S| > |W| \geq |W(vv')| \geq 2k - 1,$$

we have $|S| = 2k$ and $u_3 = 0$. Thus S contains exactly two vertices of each part $Y' \in \mathcal{P}'$. As at most one class of $\mathcal{U}_4$ has merged vertices, the choice of X implies $u_4 = 1$; thus $u_2 = l - k$. Also, $\dot{Z}' \subseteq S$. Since $u_3 = 0 = r_3$, $M_3 \subseteq S$. So $|W| \geq |W(vv' \cup \dot{Z}') \cup C_4| \geq 2k$, a contradiction.

Otherwise $v = v_{I_{\ddot{X}}}$. Then Step 11(c) was executed. As only one part in $\mathcal{U}_4$ can have contracted vertices, $X = \ddot{X} = xyzw \in \mathcal{U}_4$ with (say) $I_{\ddot{X}} = xy$ and $v' = v_x y$. Then

$$|W(xyz \cup \dot{Z}')| \leq 2k + u_4 - 1 < |W(\ddot{X})|,$$

$|W(\{v_{xy}, w\})| \geq 2k - 1 + u_3$, and $|W(\{v_{xy}, z\})| \geq 2k$. So we are done, unless $v' = w$ and

$$2k \geq |S| > |W(\{v_{xy}, w\})| \geq 2k - 1 + u_3.$$

Thus $u_1 = 0 = u_3$ and $|S| = 2k$. Again S contains exactly two vertices of each class $Y' \in \mathcal{P}'$, and the choice of X implies $u_4 = 1$. So $u_2 = l - k$. Also, $\dot{Z}' \subseteq S$. As $|W(\ddot{X})| > |W(xyz \cup \dot{Z}')|$, we have $|L(w) \smallsetminus W(xyz \cup \dot{Z}')| \geq 1$. So we have the contradiction

$$|W| \geq |W(\{v_{xy}, w\} \cup \dot{Z}')| \geq W(\dot{Z}') + 1 = l + u_2 + 1 = 2l - k + 1 = 2k.$$

Case 8: There exists $X = xyzw \in \mathcal{U}_2$ with $|S \cap X'| = 2$. Say $S \cap X' = \{v_I, w\}$. As the previous cases fail, $|S| \leq 2k - u_1 - u_3 - u_4 = l + u_2$. Since $L(xyz) \cap L(w) = \emptyset$, we have

$$|W| \geq |W(X')| \geq l(xyz) + l(w) \geq u_2 + l \geq |S|.$$

Case 9: Otherwise. As the previous cases fail,

$$|S| \leq u_1 + u_2 + u_3 + u_4 + 2|\mathcal{R}| = l.$$

As $\mathcal{L}(M)$ has an SDR, there is a vertex $x \in S \smallsetminus M$. Thus $|W| \geq l(x) = l \geq |S|$.

□

This completes the proof of Theorem 14. □

Chapter 3

ONLINE CHOICE NUMBER

3.1 Introduction

From Theorem 8, $\mathrm{ch}(K_{s*k}) = (1+o(1))\frac{\log s}{\log(1+1/k)}$. Thus for a complete multipartite graph K_{s*k} with large s, we have $\mathrm{ch}(K_{s*k}) \gg \chi(K_{s*k})$. Then it is natural to ask if there exists a graph G with $\mathrm{ch}^{\mathrm{OL}}(G) \gg \mathrm{ch}(G)$. However, this problem seems hard to solve. Now it is even unknown if there exists a graph G with $\mathrm{ch}^{\mathrm{OL}}(G) - \mathrm{ch}(G) > 1$. Therefore, We mainly explore the following open problems in this chapter:

Problem 34. (1) Is there a graph G, a vertex $v \in V(G)$ and the list size functions l, l' where $l'(v) = l(v) + 1$ and $l'(u) = l(u)$ for any $u \in V(G) - v$, such that G is l-choosable, but G is not l'-paintable?

(2) Is there a graph that is k-choosable but not $k+1$-paintable?

(3) Is there a graph $G = (V, E)$ and the list size functions l, l' where $l'(v) = l(v)+1$ for any $v \in V(G)$, such that G is l-choosable, but G is not l'-paintable?

The chapter is organized as follows. We show that $c_1 k \log s < \mathrm{ch}^{\mathrm{OL}}(G) < c_2 k \log s$ for some constant c_1, c_2 for any complete multipartite graph $G = K_{s*k}$ in Section 3.2. Then we prove that solving Problem 34(2) is equivalent to solving Problem 34(3) in Section 3.3. Then we show that for $G = K_{4*3}$, $\mathrm{ch}^{\mathrm{OL}}(G) > \mathrm{ch}(G) > \chi(G)$ in Section 3.4. At last, we show a class of examples for solving Problem 34(1) in Section 3.5.

A computer program was written to calculate the paintability of different complete multipartite graphs. Through the program, we can specify a complete multipartite graph G and its list size function l. It works by exhaustively presenting and choosing sets to check if G is l-paintable or not. The program itself can only verify a graph with 15 nodes and list sizes 7 in reasonable time. However, it still provides some useful information.

3.2 Probabilistic Method

Theorem 35. *Let $s, k > 0$. There exists $c_1, c_2 > 0$ such that for any complete multipartite graph K_{s*k}, $c_1 k \log s \le \mathrm{ch}^{\mathrm{OL}}(K_{s*k}) \le c_2 k \log s$.*

Proof. The lower bound follows from the fact that $\mathrm{ch}^{\mathrm{OL}}(K_{s*k}) \ge \mathrm{ch}(K_{s*k}) \ge c_1 k \log s$, where the first part is trivial and the second part follows from Alon (1993b). Here we prove $\mathrm{ch}^{\mathrm{OL}}(K_{s*k}) \le c_2 k \log s$ for some c_2.

Let c_2 be some constant to be decided later. Set a list function $f : V \to \mathbb{N}$ to be $f(v) = r = \lceil c_2 k \log s \rceil$. Let $V_1, V_2, \dots, V_k$ be the parts of $G = K_{s*k}$. For each $1 \le i \le k$, $n_i = |V_i|$ and $V_i = \{v_{1i}, v_{2i}, \dots, v_{si}\}$.

The idea is to prove that for each strategy Alice presents, Bob has a strategy to win. As Bob can choose at least one vertex for each round, there are at most $V(G) = sk$ rounds. Thus probability argument works as we have finite number of rounds. Therefore we are done if we can design a strategy for Bob such that with a positive probability, all the vertices in K_{3*k} are chosen by him before it has been presented $r + 1$ times.

We design a strategy for Bob: whatever Alice presents a vertex set $S \subseteq V(G)$, Bob chooses an maximum independent set $S_i = V_i \bigcap S$ where i is chosen with probability $1/k$ independently. Now we define an event A_{ij} for each $1 \le i \le s, 1 \le j \le k$:

$$A_{ij} : v_{ij} \text{ is presented } r + 1 \text{ times before it is chosen by Bob.}$$

A_{ij} happens when Bob does not choose the V_i for each time $v_{ij} \in S$. However, since the probability of choosing V_i is fixed as p_i, we have

$$P(A_{ij}) \leq (1 - \frac{1}{k})^{c_2 k \log s} \leq \frac{1}{s^{c_2}}$$

Therefore

$$P(\bigcup A_{ij}) \leq \sum_{i,j} P(A_{ij}) \leq \frac{sk}{s^{c_2}}$$

Case 1: if $s \geq k$, then we can choose $c_2 > 2$, then $P(\bigcup A_{ij}) \leq s^{2-c_2} < 1$. Thus G is r-online choosable.

Case 2: if $k > s$, then we divide the set of all parts into 2 classes $\{\mathcal{C}_1, \mathcal{C}_2\}$ so that each class includes $k/2$ parts. Now for each round Alice presents a set, Bob chooses a number $j \in \{1, 2\}$ from a uniform distribution with probability 1/2. Then Bob will only choose a maximum independent set from $\mathcal{C}_j$. Now we prove at last, any vertex $v \in V(G)$ receives roughly one half of original chance that can be chosen by Bob.

Let the original list size function be f^0 where $f^0(v) = r_0 := f(v)$ for any $v \in V$ and the new function be f^1 where $f_(v)$ represents the possible number of rounds Bob can choose i whenever $v \in S_i \subset V(G)$. Then since Bob chooses for each round randomly and independently, thus $f^1(v)$ is a Binomial variable. By Chernoff bound,$P(\sum_{i=1}^{n} X_i < pn - pn^{2/3}) \leq e^{-n^{1/3}/2}$ for n Binomial variable X_i where $P(X_i = 1) = p(1 \leq i \leq n)$ we have

$$P(f^1(v) < \frac{r_0}{2} - \frac{r_0^{2/3}}{2}) \leq e^{-\frac{1}{2} r_0^{1/3}}$$

Therefore, the probability that there exists $v \in V$, such that $f^1(v) \leq \frac{r_0}{2} - \frac{r_0^{2/3}}{2}$ is smaller than $kse^{-\frac{1}{2} r_0^{1/3}} \leq k^2 e^{-\frac{1}{2} c_2^{\frac{1}{3}} k^{\frac{1}{3}} (\log s)^{\frac{1}{3}}} \ll 1$ (The last inequality follows form the fact that with any constant c, $e^n \gg n^c$ as $n \to \infty$). Thus let $r_1 = \min f^1(v)$, then we can ensure with almost probability 1, we have

$$r_1 \geq \frac{1}{2} r_0 - \frac{1}{2} r_0^{2/3}.$$

We then can repeatly use the technique for j iterations until

$$\frac{r_0}{2^j} \geq k.$$

Now we prove after j iterations, we don't lose too many choices for any vertex.

Since for any i, we have

$$r_{i+1} \geq \frac{r_i}{2} - \frac{r_i^{\frac{2}{3}}}{2}$$

Let $z_i = r_i^{\frac{1}{3}}$, then

$$z_{i+1}^3 \geq \frac{1}{2}(z_i^3 - z_i^2) \geq \frac{1}{2}(z_i^2(z_i - 1)) \geq \frac{(z_i - 1)^3}{2}$$

since $z_i \geq 1$. This implies

$$z_{i+1} \geq \frac{z_i - 1}{2^{\frac{1}{3}}}$$

Let $t_i = z_i + x$ where $x = \frac{1+x}{2^{\frac{1}{3}}}$. By solving the euqation we can get $x = \frac{1}{2^{\frac{1}{3}}-1} < 4$. As

$$t_{i+1} - x \geq \frac{t_i - x - 1}{2^{\frac{1}{3}}},$$

it follows that

$$t_{i+1} \geq \frac{t_i}{2^{\frac{1}{3}}}$$

which implies

$$t_j \geq \frac{t_0}{2^{\frac{j}{3}}} = \frac{z_0 + x}{2^{\frac{j}{3}}} \geq \frac{z_0}{2^{\frac{j}{3}}} = (\frac{r_0}{2^j})^{1/3}$$

Thus

$$s_j = (t_j - x)^3 \geq (t_j - 4)^3 = ((\frac{r_0}{2^j})^{\frac{1}{3}} - 4)^3 = \frac{r_0}{2^j} - O((\frac{r_0}{2^j})^{\frac{2}{3}})$$

Thus

$$r_j \geq \frac{r_0}{2^{j+1}} = \frac{c_2}{2}\frac{k}{2^j}\log s$$

If we choose c_2 big enough, then we can apply Case 1 to online color each of the smaller subgraphs $K_{s*(\frac{k}{2^j})}$. And since we predetermine the subgraph we want to choose from for each round, we have a proper online coloring of the whole graph. We are done. □

3.3 An Idea of Solving Problem 34(2)

In this section we prove that solving Problem 34(2) is equivalent to solving Problem 34(3). Since (2) $\to$ (3) is trivial, we only prove (3) $\to$ (2) here.

Lemma 36. *Let $G = (V, E)$ be a graph and $f, g : V \to \mathbb{N}$. Suppose G is f-choosable but not g-paintable such that $g(v) - f(v) = c$ for any $v \in V(G)$ where c is a constant greater than 0. Let $m = \max_{v \in V(G)} f(v)$ and $m' = m + c$. Then we can find a G' that is m-choosable but not m'-paintable.*

Proof. First we introduce notation as follows. Let $n = |G|$. Let $l = \min_{v \in V(G)} f(v)$, $l' = \min_{v \in V(G)} g(v)$, and $S = \{v \in V(G) : f(v) = l\}$. Denote $H(v, G)$ as a graph whose vertex set is $V(G) + v$ and whose edge set is every possible edge vw where $w \in V(G)$. Now we prove the conclusion by induction on $t = m - l$.

Base step: If $t = 0$, then $f(v) = m$ for any $v \in V(G)$. Therefore G is already a m-choosable but not m'-paintable graph.

Induction step: $t > 0$. In this case, we want to construct a new graph G^* from G such that G^* is f^*-choosable but not g^*-paintable with $t^* = m^* - l^* < t$. The construction is as follows: for G, we define $f^+ : V \to \mathbb{N}$ as

$$f^+(v) = \begin{cases} f(v) + 1 & \text{if } v \in S \\ f(v) & \text{otherwise} \end{cases}$$

We define $g^+ : V \to \mathbb{N}$ as $g^+(v) = f^+(v) + c$ for any $v \in V$,

Now let $G^* = (m(c+1))G + K_m + E$ where E is an edge set defined as follows: for each copy G_i of G, there is one and only one vertex $v \in V(K_m)$ so that $E \cap E(H(v, G_i)) \neq \emptyset$ and $E \cap E(H(v, G_i)) = \{e = vw \in E(H(v, G_i)) : w \in S\}$. For each $v \in V(K_m)$, it connects to $c + 1$ copies of G with edges in E. Now we define $f^*, g^* : V \to \mathbb{N}$ as:

$$f^*(v) = \begin{cases} f^+(v) & \text{if } v \in (m(c+1))G \\ m & \text{if } v \in V(K_m) \end{cases}$$

and $g^*(v) = f^*(v) + c$ for any $v \in G$.

Now we show that G^* is f^*-choosable, but not g^*-choosable.

First we prove G^* is f^*-choosable. Since $f^*(v) = m$ for any $v \in V(K_m)$, we can obtain a proper coloring for K_m. Now from our construction, any copy of G satisfies the property that for any $w \in V(G)$, $|E(w, K_m)| = 1$ when $w \in S$ and $|E(w, K_m)| = 0$ otherwise. Therefore, after deleting possible color c that is used in K_m, we still have enough choices in $L(w)$, i.e. $|L(w)\backslash c| \geq f(w)$. Thus the copy G is f^*-choosable by assuming K_m is already colored. Thus is true for each copy of G. Hence G^* is f^*-choosable.

To prove G^* is not g^*-paintable, Alice presents the following strategy. She picks a copy of G. Then she presents $S \subset G$ and the vertex $v \in K_m$ so that $E(S, v) \neq \emptyset$. Now Bob is forced to choose a subset of the set S, since otherwise, Alice can present her winning strategy for the copy since $g^*(v) = g(v)$ for any $v \in G\backslash S$ and $g^*(v) - 1 = g(v)$ for any $v \in S$. Now for each round, Alice repeats by picking one copy and the set S until all copies of G have been picked once. Since each time Bob is forced to choose S, there are only $g^*(v) - (c+1) = m-1$ choices remaining in any $v \in K_m$. However K_m is $m-1$ colorable and thus not paintable. Then Alice finds a winning strategy: G^* is not g^*-paintable.

Now from our construction, we have $\min_{v \in G^*} f^*(v) = l + 1$. Therefore $t^* = m^* - l^* = m - (l-1) = t - 1 < t$. We are done by induction hypothesis. □

3.4 Another Graph where Paintability Differs from Its Choosability

By Theorem 14, $\text{ch}(K_{4*3}) = 4$. Using a computer we have checked that $\text{ch}^{\text{OL}}(K_{4*3}) = 5$, but do not have a readable argument to verify the upper bound. Here we prove the lower bound. Notice that this is a graph with $\text{ch}^{\text{OL}} > \text{ch} > \chi$.

Theorem 37. $\text{ch}^{\text{OL}}(K_{4*3}) \geq 5$.

Proof. Figure 3.1 describes a strategy for Alice. The top left matrix depicts the initial game position, and Alice's first move. The positions in the matrix correspond to the vertices of K_{4*3} arranged so that vertices in the same part correspond to positions in the same column. The order of vertices within a column is irrelevant, as is the order of the columns. The numbers represent the size of the list of each corresponding vertex. The sequence of numbers represents a function f. The shaded positions represent the vertices that Alice presents on here first move.

As play progresses Bob chooses certain vertices presented by Alice and passes over others. When a vertex is chosen its position is removed from the next matrix (and the positions in its column of the remaining vertices and the order of the columns may be rearranged). When he passes over a vertex its list size is decreased by one (and its position in its column and the order of the columns may change). The arrows between the matrices point to the possible new game positions that arise from Bob's choice, not counting equivalent positions and omitting clearly inferior positions for Bob. In particular we assume Bob always chooses a maximal independent set.

For example, after Bob's first move there is only one possible game position, provided Bob chooses a maximal independent set. It is shown in the second column of the first row, along with Alice's second move. Now Bob has two possible responses that are pointed to by two arrows. Also consider the matrix in the third row and third column. There are three nonequivalent responses for Bob, but choosing the

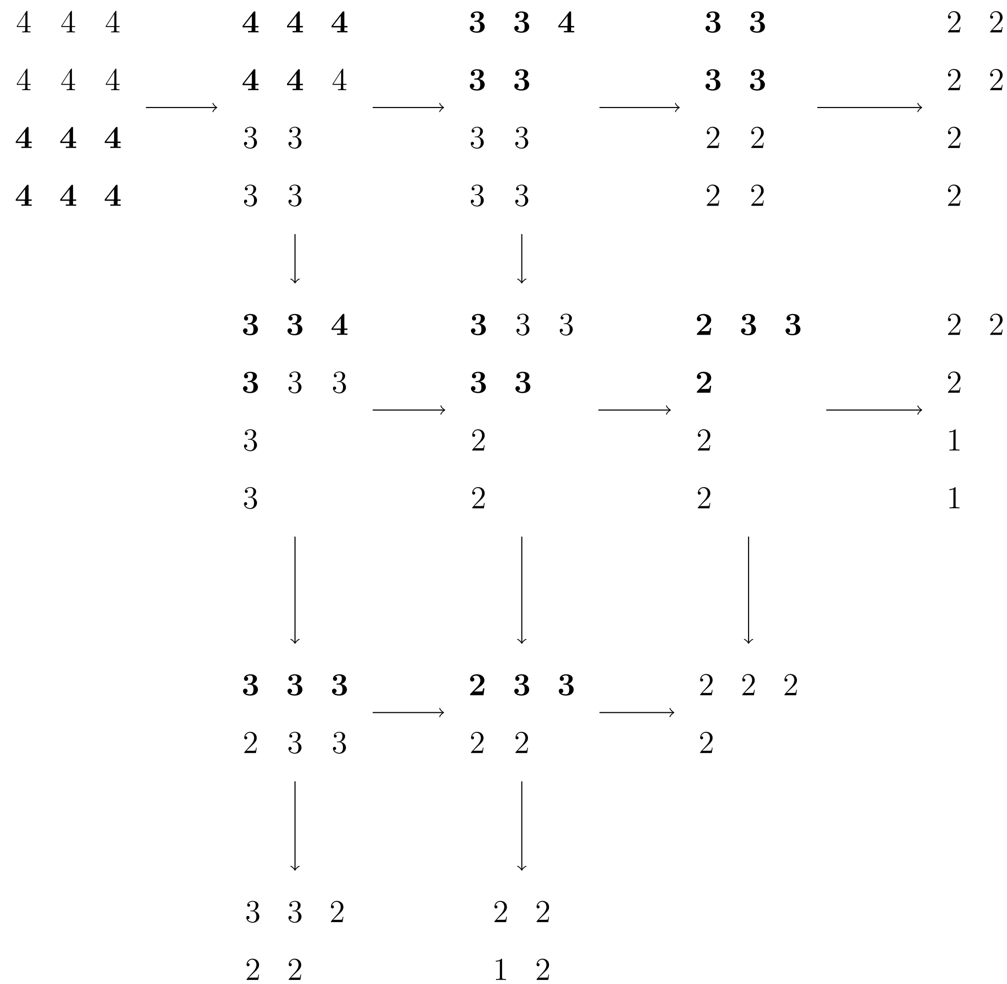


Figure 3.1: Strategy for Alice Demonstrating $\mathrm{ch}^{\mathrm{OL}}(K_{4*3}) \geq 5$.

offered vertex in the second column of the matrix results in a position that is inferior to choosing the offered vertex in the first column. So this option is not shown.

Eventually, Alice forces one of five positions (G, f) such that G is not f-choosable, and Bob, being a gentleman, resigns. $\square$

3.5 Towards Larger Gap between Paintability and Choosability

The following theorem comes from Kozik *et al.* (2014). Here we restate the theorem since it will lead to our new result.

Theorem 38 (Kozik *et al.* (2014))**.** *Let G be a complete multipartite graph with part size 1 and 3. Let $\mathcal{A}, \mathcal{S}, \mathcal{C}$ be a partition of the set of parts of G such that $\mathcal{A}$ and $\mathcal{S}$ contain only classes of size 1, $\mathcal{C}$ only contains classes of size 3. Let k_1, s, k_3 denote the cardinalities of classes $\mathcal{A}, \mathcal{S}, \mathcal{C}$, respectively, Suppose that $\mathcal{A}, \mathcal{S}$ are ordered, i.e. $\mathcal{A} = (A_1, \cdots, A_{k_1})$ and $\mathcal{S} = (S_1, \cdots, S_s)$. If $f : V(G) \to \mathbb{N}$ is a function for which the following conditions hold:*

$$
\begin{aligned}
f(v) &\geq k_3 + i && \text{for all } 1 \leq i \leq k_1 \text{ and } v \in A_i \\
f(v) &\geq 2k_3 + k_1 + i && \text{for all } 1 \leq i \leq s \text{ and } v \in S_i \\
f(v) &\geq k_3 && \text{for all } v \in C \in \mathcal{C} \\
f(u) + f(v) &\geq 2k_3 + k_1 && \text{for all } u, v \in C \in \mathcal{C} \\
\sum_{v \in C} &\geq 4k_3 + 2k_1 + s - 1 && \text{for all } C \in \mathcal{C}
\end{aligned}
$$

then G is f-choosable.

From the above theorem 38, we can derive Theorem 11 easily by setting list size function $f(v) = \lceil \frac{4k-1}{3} \rceil$ for any $v \in V(K_{3*k})$. Then every inequality is satisfied within Theorem 38. For the same reason, we can prove the following corollary.

Corollary 39. *Let $G = K_{3*k}$. Represent K_{3*k} with a $3 \times k$ array with (i, j)-entry $v_{i,j}$. If $f : V(G) \to \mathbb{N}$ is a list size function such that $f(v_{1,i}) = k, f(v_{2,i}) = k, f(v_{3,i}) = 2k - 1$ for any i, then G is f-choosable.*

In Example 19 we already stated that $\mathrm{ch}^{\mathrm{OL}}(K_{2*(k-1),3}) > k$ for any $k \geq 3$. Here we prove a simple corollary of the example:

Corollary 40. *Let $G = K_{3*k}$. If $k \geq 3$ and $f : V(G) \to \mathbb{N}$ is a list size function such that $f(v_{1,i}) = k, f(v_{2,i}) = k, f(v_{3,i}) = 2k - 1$, then G is not f-paintable.*

Proof. Alice first choose $A_1 = \{v_{3,1}, v_{3,2}, \ldots, v_{3,k}\}$. No matter what Bob chooses, we end up with isomorphic graphs with same list sizes. Without losing generality, we can assume $B_1 = \{v_{3,k}\}$. Then $G_1 = K_{2,3*(k-1)}$ with $f(v_{1,i}) = k, f(v_{2,i}) = k, f(v_{3,i}) = 2k-2$. Now continue the process, that is, $A_2 = \{v_{3,1}, \cdots, v_{3,k-1}\}, B_2 = \{v_{3,k-1}\}, G_2 = K_{2*2,3*(k-1)}$ with $f(v_{1,i}) = k, f(v_{2,i}) = k, f(v_{3,i}) = 2k-3$,..., $A_{k-1} = \{v_{3,1}, v_{3,2}\}$, $B_{k-1} = \{v_{3,2}\}$, $G_{k-1} = K_{2*(k-1),3}$ with $f(v) = k$ for any $v \in G_{k-1}$. From Lemma 22, G_{k-1} is not k-paintable. As Bob exhaustively chooses all independent sets for any A_i $(1 \leq i \leq k-1)$, G is not paintable. □

Now we state another improvement of Corollary 40. This improvement, along with Corollary 40, provides a graph G and two list size functions $f \leq g$, such that G is f-choosable but not g-paintable and there exists $v \in V(G)$ with $g(v) - f(v) > 1$. This solves Problem 34(1).

Theorem 41. *Let $G = K_{3*k}$. Represent K_{3*k} with a $3 \times k$ array with (i,j)-entry $v_{i,j}$. If $k \geq 4$ and $f : V(G) \to \mathbb{N}$ is a function that $f(v_{1,i}) = k, f(v_{2,i}) = k, f(v_{3,i}) = 2k$, then G in not f-paintable.*

Proof. Here we use a similar figure as in Theorem 37 proof to describe a winning strategy for Alice. The description then is the same. Here I omit the description and only show the figures.

Base step: $k = 4$. See Figure 3.2. There are graphs with some choices omitted for Bob, since these choices will lead to a clique such that the maximum list size function value is less than k,

Induction step: $k > 4$. See Figure 3.3. We do not show the choices of choosing an independent set for the parts of size 3 in the second graph of the first row, since it leads to a situation with k-clique and list size function value at most $k-1$ which is obviously not choosable. □

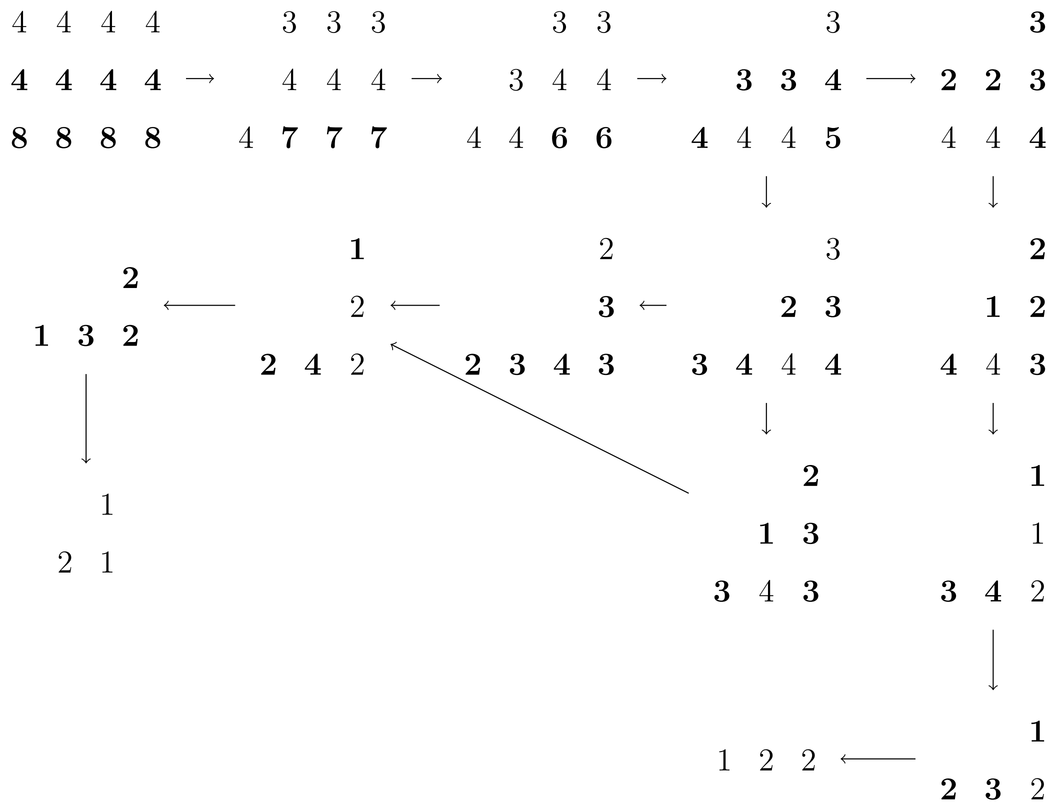


Figure 3.2: Strategy for Alice Demonstrating K_{3*4} is not f-paintable

We define critical paintable graph as follows:

Definition 42. *Let G be a graph and $g : V(G) \to \mathbb{N}$. G is g-critical paintable if and only if G is g-paintable and for any $v \in V(G)$, G is not g_v-paintable for any $v \in V(G)$ where g_v is defined as*

$$g_v(u) = \begin{cases} g(u) - 1 & \text{if } u = v \\ g(u) & \text{if } u \neq v \end{cases}$$

Proposition 43. *Let $G = K_{2*k}$. Let $f(v) = k$ for any $v \in V(G)$, then G is f-critical paintable.*

Proof. By definition, we just need to show that the leftmost graph in Figure 3.4 is not paintable. We prove it by induction. The base step $k = 1$ is trivial, since there

$$
\begin{array}{ccccc} k & k & k & \cdots & k \\ \mathbf{k} & \mathbf{k} & \mathbf{k} & \cdots & \mathbf{k} \\ \mathbf{2k} & \mathbf{2k} & \mathbf{2k} & \cdots & \mathbf{2k} \\ \multicolumn{5}{c}{(k \text{ parts})} \end{array}
\longrightarrow
\begin{array}{ccccc} & k-1 & k-1 & \cdots & k-1 \\ & \mathbf{k} & \mathbf{k} & \cdots & \mathbf{k} \\ \mathbf{k} & \mathbf{2k-1} & \mathbf{2k-1} & \cdots & \mathbf{2k-1} \\ \multicolumn{5}{c}{(k \text{ parts *})} \end{array}
$$

$$\downarrow$$

$$
\begin{array}{cccc} k-1 & k-1 & \cdots & k-1 \\ k-1 & k-1 & \cdots & k-1 \\ 2k-2 & 2k-2 & \cdots & 2k-2 \\ \multicolumn{4}{c}{(k-1 \text{ parts}} \\ \multicolumn{4}{c}{\text{Done by induction hypothesis)}} \end{array}
$$

Figure 3.3: Strategy for Alice Demonstrating K_{3*k} ($k \geq 4$) is not f-paintable.

$$
\begin{array}{ccccc} k-1 & k & k & \cdots & k \\ \mathbf{k} & \mathbf{k} & \mathbf{k} & \cdots & \mathbf{k} \\ \multicolumn{5}{c}{(k \text{ parts})} \end{array}
\longrightarrow
\begin{array}{ccccc} & k-1 & k-1 & \cdots & k-1 \\ k & \mathbf{k-1} & \mathbf{k} & \cdots & \mathbf{k} \\ \multicolumn{5}{c}{(k \text{ parts *})} \end{array}
$$

$$\downarrow$$

$$
\begin{array}{cccc} k-2 & k-1 & \cdots & k-1 \\ k-1 & k-1 & \cdots & k-1 \\ \multicolumn{4}{c}{(k-1 \text{ parts}} \\ \multicolumn{4}{c}{\text{Done by induction hypothesis)}} \end{array}
$$

Figure 3.4: Strategy for Alice Demonstrating K_{2*k} ($k \geq 2$) is g-critical paintable.

is one vertex with list size 0. The figure shows the induction step when $k > 1$. The description of the proof is as same as it is in the proof of Theorem 41. $\square$

Proposition 44. *Let $G = K_{2*k}$. Fix $u \in V(G)$. Let $f(v) = k$ for any $v \in V(G) - u$ and $f(u) = k - 1$. Then, when $k = 2$, G is not f-choosable; when $k > 2$, G is f-choosable.*

Proof. First suppose $k = 2$. Without losing generality, we can assume $V(G) = V_1 \cup V_2$ with $V_1 = \{v_{11}, v_{21}\}, V_2 = \{v_{21}, v_{22}\}$ where $v_{11} = u$. Now we assign a f-list assignment L as follows: $L(v_{11}) = \{1\}, L(v_{21}) = \{2, 3\}, L(v_{12}) = \{1, 2\}, L(v_{22}) = \{1, 3\}$. We prove G has no L-coloring. We try to obtain a coloring c. Since there is only one color in $L(v_{11})$, $c(v_{11}) = 1$. Since v_{12}, v_{22} are adjacent to v_{11}, $c(v_{12}) = 2, c(v_{22}) = 3$. But as $L(v_{21}) = \{2, 3\}$, we cannot finish the coloring.

Now suppose $k > 2$. Let the ith part of G be $V_i = \{v_{1i}, v_{2i}\}$. Again, without losing generality, we assume $v_{11} = u$. Suppose there is a list assignment L such that we cannot obtain a coloring from L. We claim that (*) $L(v_{11}) \cap L(v_{21}) = \emptyset$. If not, there exists $\alpha \in L(v_{11}) \cap L(v_{21})$. Then we color v_{11}, v_{21} with α and set $L'(v) = L(v) - \alpha$ for $v \in V(G) \backslash V_1$. Then as $\mathrm{ch}(K_{2*(k-1)})$ is $(k-1)$-choosable from Theorem 10, we obtain L-coloring. Contradiction. Thus $L(v_{11}) \cap L(v_{21}) = \emptyset$.

Also from Lemma 26, we have for any $V_i = \{v_{1i}, v_{2i}\}$, $|L(v_{1i}) \cup L(v_{2i})| \leq 2k - 1 < |V(G)|$. Then from inclusion-exclusion, $L(v_{1i}) \cap L(v_{2i}) \neq \emptyset$ for any $i \geq 2$. Assume $\alpha \in L(v_{12}) \cap L(v_{22})$. If $\alpha \in L(v_{21})$, from claim (*) we have $\alpha \notin L(v_{11})$. Then we can color v_{12}, v_{22} with α such that for any $v \in V(G) \backslash V_2$, $|L(v) - \alpha| \geq k - 1$. Again from Theorem 10, we obtain L-coloring. Thus $\alpha \in L(v_{11})$. In this case we still color v_{12}, v_{22} with α. Now by deleting possible α from $L(v)$ for any $v \in V(G)$, we obtain a new list assignment L' for $G' = G[V \backslash V_2]$ where $|L'(v_{11})| = k - 2, |L'(v_{21})| = k, |L'(v_{1i})| = |L'(v_{2i})| = k - 1$ for any $i \geq 2$.

Now we continue this process for $t \leq k-2$ steps by coloring the vertices in parts with a common color and deleting the color from the list of all the other vertices. At last, we end up with a pair (G^*, L^*) with $k-t$ parts. By reordering the parts, we can write $|L^*(v_{11})| = k-t-1, |L^*(v_{21})| = k, |L^*(v)| = k-t$ for any other $v \in V(G^*)$. Now there are two cases: (1) There exist $V_i (i \geq 2)$ such that $L(v_{1i}) \cap L(v_{2i}) \neq \emptyset$. Then $t = k-2$. Thus there are only two parts left in G^*. In this case, we color v_{11} by using $\beta \in L^*(v_{11})$, color v_{12} with the color in $\gamma \in L^*(v_{12}) - \beta$ and color v_{22} with the color in $\tau \in L^*(v_{22}) - \beta$. And then we can finish coloring v_{21} by using $\rho \in L^*(v_{21}) - \gamma - \tau$ as $k \geq 3$. (2) Otherwise we have for each part $V_i (i \geq 2)$, $L^*(v_{1i}) \cap L^*(v_{2i}) = \emptyset$. In this case we color v_{11} with some color $\beta \in L^*(v_{11})$ and delete β from other lists. Then we end up with a pair (G^{**}, L^{**}) such that $V(G^{**}) = V(G^*) - v_{11}, |L^{**}(v_{21})| = k, |L^{**}(V_{1i})| \geq k-t-1, |L^{**}(V_{2i})| \geq k-t$. Then G^{**} is L^{**}-colorable by Lemma 20. Thus G is L-colorable. □

The following theorem comes from Kozik *et al.* (2014).

Theorem 45 (Kozik *et al.* (2014))**.** *Let G be a complete multipartite graph with each part of size at most 3. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{S}$ be a partition of the set of parts of G such that $\mathcal{A}$ contains parts of size 1, $\mathcal{B}$ contains parts of size 2, $\mathcal{C}$ contains parts of size 3, $\mathcal{S}$ contains parts of size 1 or 2. Let k_1, k_2, k_3, s denote the cardinalities of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{S}$ respectively. Suppose $\mathcal{A}, \mathcal{S}$ are ordered i.e. $\mathcal{A} = (A_1, A_2, \ldots, A_{k_1})$ and $\mathcal{S} = (S_1, S_2, \ldots, S_s)$. For $1 \leq i \leq s$, let $v_s(i) = \sum_{1 \leq j < i} |S_i| + 1$. Suppose $f : V(G) \to \mathbb{N}$ is* *a function for which the following conditions hold:*

$$
\begin{aligned}
f(v) &\geq k_3 + k_2 + i && \text{for any } 1 \leq 1 \leq k_1 \text{ and } v \in A_i \\
f(v) &\geq 2k_3 + k_2 + k_1 + v_s(i) && \text{for any } 1 \leq i \leq s \text{ and } v \in S_i \\
f(v) &\geq k_3 + k_2 && \text{for any } v \in B \in \mathcal{B} \\
\sum_{v \in B} f(v) &\geq |V(G)| && \text{for any } V \in \mathcal{B} \\
f(v) &\geq k_3 + k_2 && \text{for any } v \in C \in \mathcal{C} \\
f(u) + f(v) &\geq |V(G)| - 1 && \text{for any } u, v \in C \in \mathcal{C} \text{ and } u \neq v \\
\sum_{v \in C} f(v) &\geq |V(G)| - 1 + k_3 + k_2 + k_1 && \text{for any } C \in \mathcal{C}
\end{aligned}
$$

Then G is f-choosable.

Combining with the above theorem we have:

Proposition 46. *Let $k \geq 3$ and $G = K_{1,2*(k-2),3}$. Let $X = \{v_1, v_2, v_3\}$ be the only part of size 3. Let $g : V(G) \to \mathbb{N}$ be as follows:*

$$
g(v) = \begin{cases} k & \text{if } v \neq v_1 \\ k-1 & \text{if } v = v_1 \end{cases}
$$

Then G is g-critical paintable.

Proof. Again we use a similar figure as in the proof of Theorem 37 to describe a winning strategy for Alice. The description then is mostly the same. Here I omit the description and add some more details.

From Theorem 45, we have G is g-paintable. Therefore, to prove that G is g-critical paintable, it suffices to show that all the cases in Figure 3.5 are not paintable:

$$\begin{array}{ccccc} & & & & k-1 \\ & k & \cdots & k & k \\ k-1 & k & \cdots & k & k \\ & \multicolumn{3}{c}{(k \text{ parts})} & \end{array} \qquad \begin{array}{ccccc} & & & & k-1 \\ & k-1 & \cdots & k & k \\ k & k & \cdots & k & k \\ & \multicolumn{3}{c}{(k \text{ parts})} & \end{array}$$

$$\begin{array}{ccccc} & & & & k-1 \\ & k & \cdots & k & k-1 \\ k & k & \cdots & k & k \\ & \multicolumn{3}{c}{(k \text{ parts})} & \end{array} \qquad \begin{array}{ccccc} & & & & k-2 \\ & k & \cdots & k & k \\ k & k & \cdots & k & k \\ & \multicolumn{3}{c}{(k \text{ parts})} & \end{array}$$

Figure 3.5: All the Cases Need to be Shown not Paintable for Proposition 46

Base step: $k = 3$. See Figure 3.6. In the figure, the four leftmost figures are all the cases we need to prove G is g-critical paintable for $k = 3$. To keep the simplicity of the figure, we leave out some of Bob's choices that leads to a graph containing a k-clique with maximum list size function value not exceeding $k-1$ in the clique, since K_k is not $k-1$ choosable. In the figure, one example is the graph in the first row and first column. If Bob chooses two boldfaced vertices from last part, the resulting graph has a 3-clique such that the maximum list size 2. In this case, we don't draw this graph.

Induction step: $k > 3$. See Figure 3.7. Similarly, in this figure, the four leftmost figures are all the cases we need to prove G is g-critical paintable for $k > 3$. To keep the simplicity of the figure, we leave out some of Bob's choices that leads to a graph containing a k-clique with maximum list size function value not exceeding $k-1$ in the clique, since K_k is not $k-1$ choosable. In the figure, one example is the graph in the first row and first column. If Bob chooses two boldfaced vertices from last part, the resulting graph has a k-clique such that the maximum list size $k-1$. In this case, we don't draw this graph. All these graphs are marked with *. □

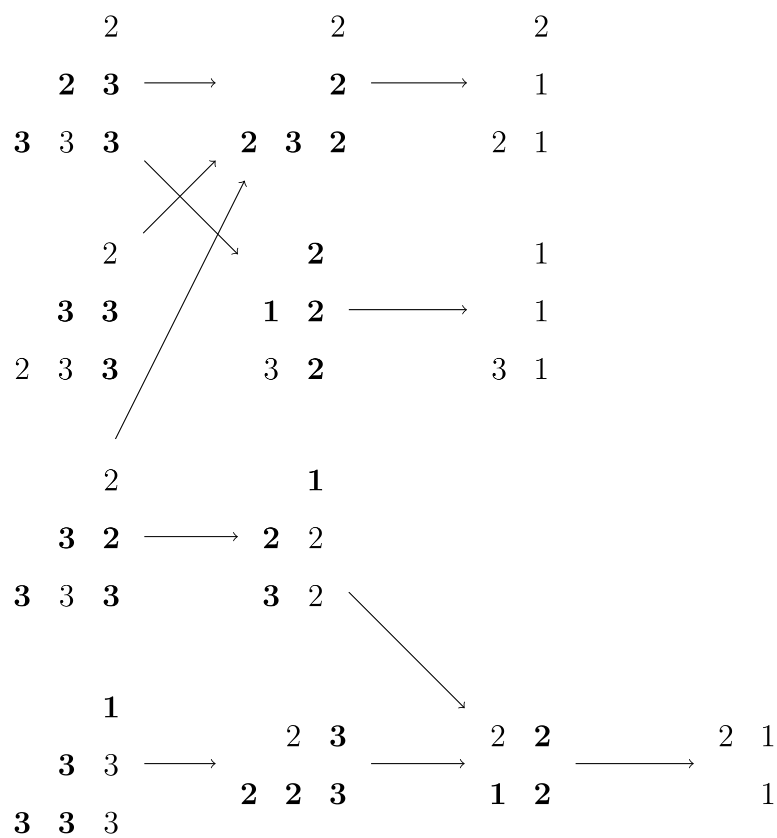


Figure 3.6: Base Step Strategy for Alice in Proposition 46

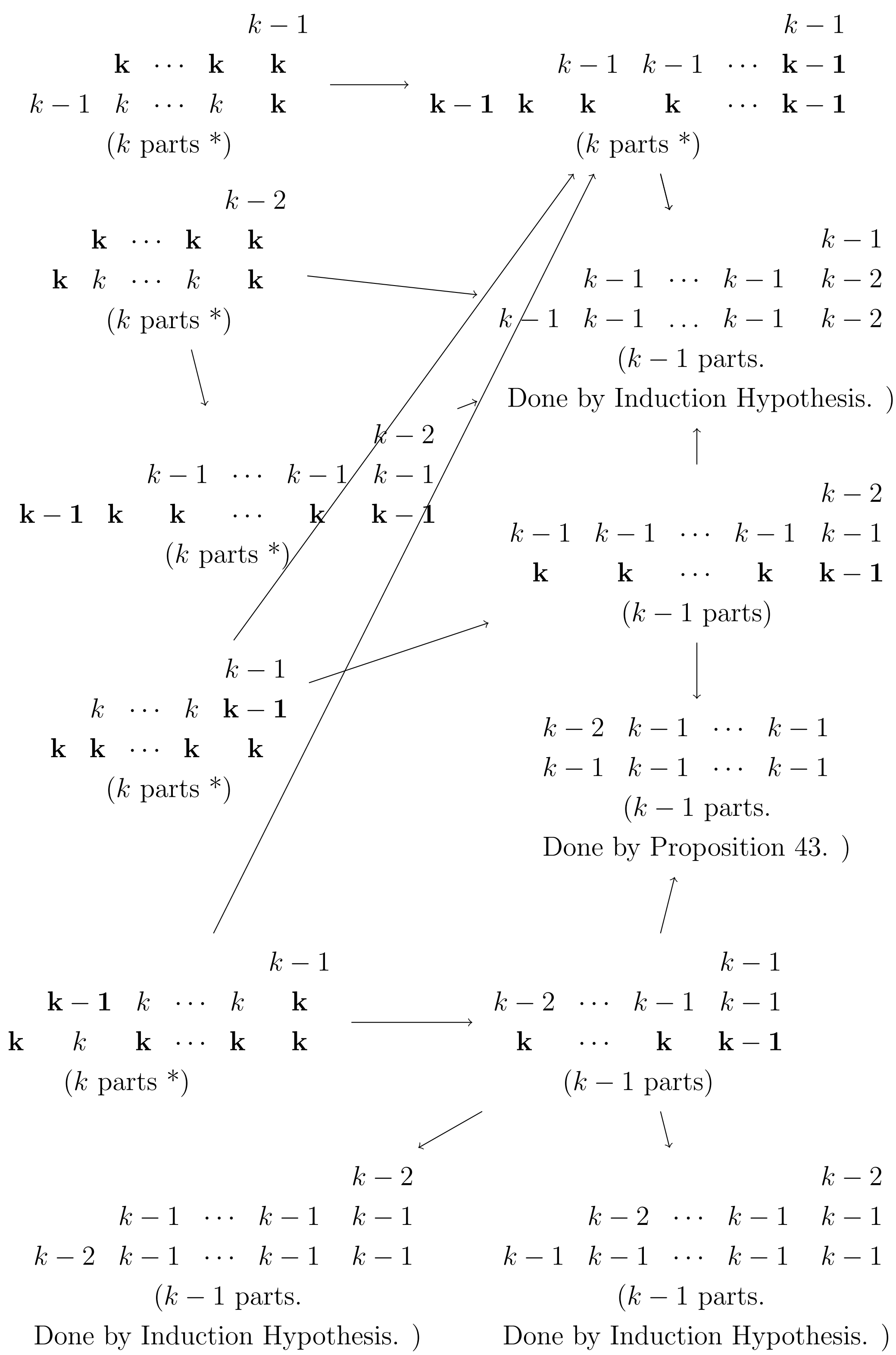


Figure 3.7: Induction Step Strategy for Alice in Proposition 46

Chapter 4

CHOICE NUMBER OF A SQUARE GRAPH

4.1 Notation and Background

Let $G = (V, E)$ be a graph. Let $d : V \times V \to G$ be defined so that for each pair $(u, v) \in V \times V$, $d(u, v)$ is the distance between vertex u and vertex v. Then the *distance k-graph* G^k is defined as $G^k = (V, E^k)$ with $E^k := \{(u, v)|(u \neq v \wedge d(u, v) \leq k\}$. Thus from the definition $G^1 = G$. G^2 is also called *square graph* of G. Let χ_k, ch_k be the chromatic number and choice number for distance-k graph, respectively. Kostochka and Woodall (2001) conjectured that distance 2-graph is chromatic choosable.

Conjecture 47 (Kostochka and Woodall (2001))**.** *For every graph G,* $\mathrm{ch}_2(G) = \chi_2(G)$.

Kim and Park (2015) disproved the conjecture.

Theorem 48 (Kim and Park (2015))**.** *For each prime $n \geq 3$, there exists a graph G such that* $\mathrm{ch}_2(G) - \chi(G) \geq n - 1$

The construction for the graph G is the above theorem involves Latin squares, which enables large independent sets. Kosar *et al.* (2014) generalizes the technique and prove the following theorem:

Theorem 49 (Kosar *et al.* (2014))**.** *There is a constant c such that for every $k \in \mathbb{N}$, there is an infinite family of graphs G such that* $\mathrm{ch}_k(G) \geq c\chi_k(G)\log(\chi_k(G))$.

Here we consider a graph G that arises in the study of cellular networks. We were asked by computer scientists to prove that $\mathrm{ch}_2(G) = \chi_2(G)$. To our surprise it turns out to be false. So it provides a more natural counterexample for the conjecture.

A *cellular graph* is a graph G such that each node of the graph represents a hexagonal cell, and two nodes have an edge between them if the corresponding cells have a boundary. See Figure 4.1 for visualization. A *straight line* is a subgraph of a cellular graph such that it consists of vertices and edges within a same line. A *triangular grid* is defined as a cellular network that is bounded by exactly three straight lines. In the next section, we prove that $\mathrm{ch}_2(G) \neq \chi_2(G)$.

Here is the background of cellular graph. In computer science, a 2-band buffering system is modeled as the interference does not extend beyond two cells away from the original cell. Then the channel assignment problem is described as follows. Each node has a fixed set of frequency channels where only a subset can be available at the given time for communication. Assume that only a subset of frequency channels are available for communication at each node. Now we try to determine the size of the smallest set of free channels in all the nodes such that each node can be assigned a channel.

The practical question above can be described as determining choices number of G^2. Thus, instead of the artificial graph found by Kim and Park (2015), we find another real-world example that contradicts Conjecture 47.

4.2 The Square Graph of Cellular Network is not List-chromatic Choosable

Let G^2 be the square graph of a cellular network. Then G^2 is 7-colorable by Sen *et al.* (1999). Here we prove:

Theorem 50 (Wang *et al.* (2015))**.** *Let G be a subgraph of a triangular grid. Then G^2 is not 7-choosable when G is sufficiently large.*

Proof. To prove the theorem, we need the following lemma first:

Lemma 51. *Any proper 7 coloring c of a sufficiently large G^2 satisfies the following*

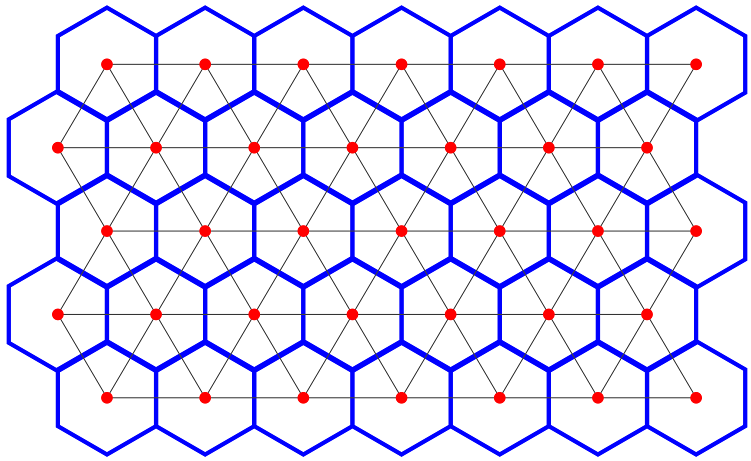


Figure 4.1: Blue hexagons are original cells which are represented by red nodes. Black segments are edges.

property: For any straight line $l \subset G$, if $v, w \in l$ with $c(v) = c(w)$ and v, w are not endpoints, then $d(v, w) \geq 7$.

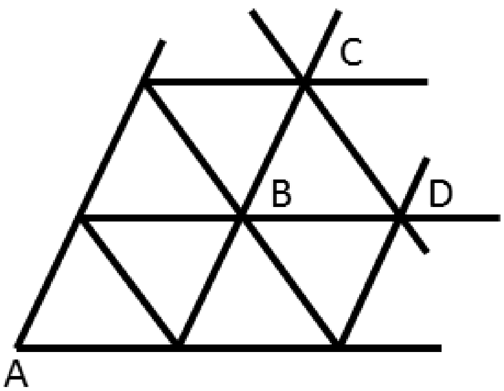
Proof. We first claim the following facts for c:

Claim 52. Either $c(A) = c(C)$ or $c(A) = c(D)$ where A, C, D are as in Figure 4.2a.

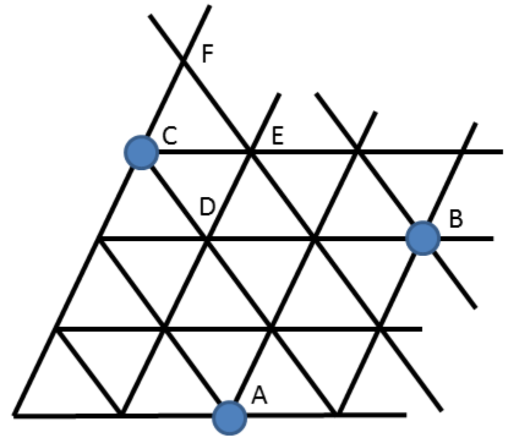
Denote the closed 1-neighborhood of B in G^2 as $N[B]$. Since $N[B]$ includes 7 vertices that forms a clique in G^2, all 7 colors must be used. Thus there exists some vertex $v \in N[B]$ such that $c(v) = c(A)$. However, for any $v \in N[B] \backslash \{C, D\}$, $vA \in E(G^2)$. Therefore the only possibilities are $c(A) = c(C)$ or $c(A) = c(D)$.

Claim 53. When A and B colored with the same color α, then C is also colored with α where A, B, C are as in Figure 4.2b.

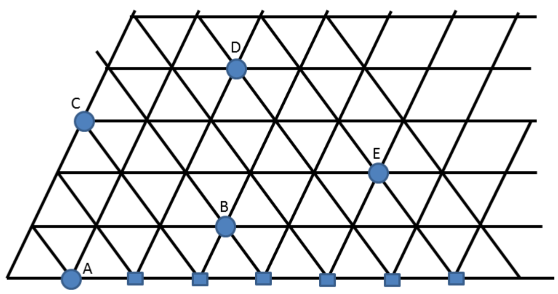
We apply Claim 52 to A, D and conclude that one of C and E receives the same color as A. Then we apply the same claim to B, E and conclude that one of C and



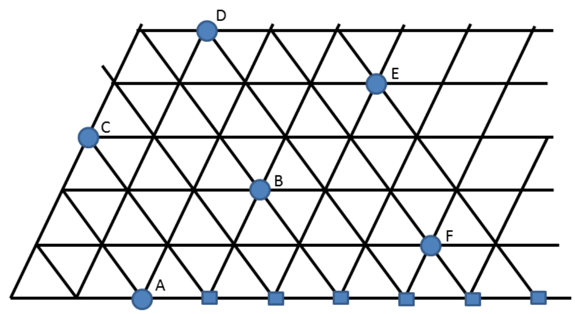
(a) Diagram for Proving Claim 52



(b) Diagram for Proving Claim 53



(c) Distribution of $c(A)$ in G^2



(d) Distribution of $c(A)$ in G^2

Figure 4.2: Diagram for Proving Claims in Theorem 50

F receives the same color as A. However, since $EF \in E(G^2)$, the only possibility is that C receives the same color as A.

Now we can prove the lemma. We want to see how $c(A)$ is distributed in G^2 under the two possibilities above. First, suppose the vertex at D's position in Figure 4.2a is colored with $c(A)$. Then we are in the case of Figure 4.2c. Once we color A, B with the same color, then C receives the same color as A and B by Claim 2. Then we can consecutively apply Claim 2 to color D, E. Now each square-marked node cannot be colored with $c(A)$ since it is either within distance 2 of B or E. Similarly, suppose the vertex at C's position in Figure 4.2a is colored with $c(A)$. We will have the similar case as Figure 4.2d. □

Notice from the lemma above, we can easily conclude that 7 different colors have to be used in any 7 consecutive nodes in a straight line in G^2.

Now we construct a such counterexample G^2 with 7-list-assignment L such that G^2 is not L-choosable. The graph G is a subgraph of a triangular grid that consists of 3 parts: a horizontal below T with length $7\binom{7}{4}^2+7$, the part G_a above T, and the part G_b below T. Now for each $v \in G_a$, we assign $L(v) = \{1, 2, 3, 4, 5, 6, 7\}$. For each $v \in G_b$, we assign $L(v) = \{8, 9, 10, 11, 12, 13, 14\}$. Now we choose a subset $M \subset T$ of size $\binom{7}{4}^2$ such that for any $a, b \in C$, we have $7 \mid d(a, b)$: assign lists to the vertices of M so that for every $S_1 \in \binom{[7]}{4}$ and $S_2 \in \binom{[7]}{3}$ there is $v \in M$ with $L(v) = S_1 \cup S_2$. Then for any $w \in M - v$, $L(v) \neq L(w)$. We can assign this since there are only $\binom{7}{4}\binom{7}{3} = \binom{7}{4}^2 = |M|$ possibilities.

Now we prove that there is no proper coloring from this list assignment. Suppose not. We now see what a proper coloring from L looks like. First we find a proper coloring from G_a, G_b, then from Lemma 51, we have a repeated permutation of 7 colors from $\{1, 2, 3, 4, 5, 6, 7\}$ on the bottom line of G_a. Similarly, we have another repeated permutation of 7 colors from $\{8, 9, 10, 11, 12, 13, 14\}$ on the top line of G_b. Consider the first $v \in M$. Say it has four neighbors on the bottom line of G_a have colors $\alpha, \beta, \gamma, \delta$. Put $S_1 = \{\alpha, \beta, \gamma, \delta\}$. Then every vertex in M is is adjacent to vertices with all colors in S_1. Similarly v has three neighbors in the top line of G_b. Say their neighbors have colors ρ, σ, τ. Set $S_2 = \{\rho, \sigma, \tau\}$. Then every vertex of M is adjacent to vertices with all colors in S_2. Also $S_1 \cap S_2 = \emptyset$. So $|S_1 \cup S_2| = 7$. Since there exists $w \in M$ with $L(w) = S_1 \cup S_2$, w cannot be colored. Thus G^2 cannot be colored.

□

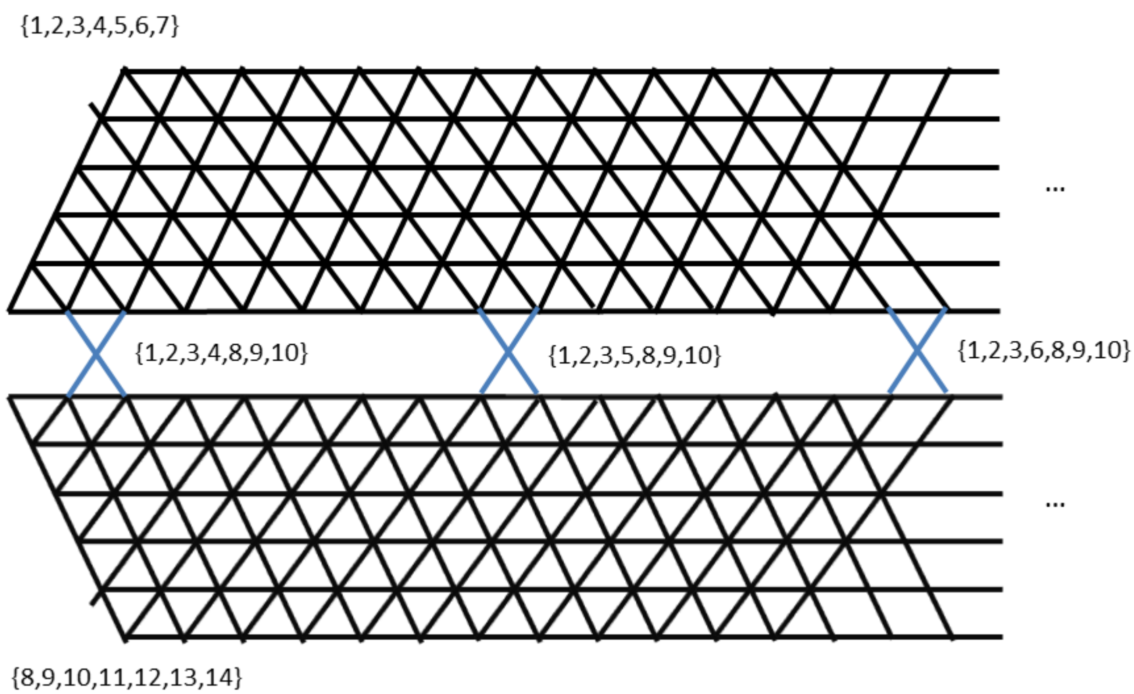


Figure 4.3: The Counterexample G^2 with the List Assignment L

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