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# Magnus pairs in, and free conjugacy separability of, limit groups

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**Abstract** There are limit groups having non-conjugate elements whose images are conjugate in every free quotient. Towers over free groups are freely conjugacy separable.

**Keywords** Geometric group theory · Limit groups · Residual properties · Logic · Conjugacy

**Mathematics Subject Classification** 20F65 · 20E05 · 57M07

## 1 Introduction

This paper is concerned with the problem of finding free quotients of finitely generated groups in which non-conjugate elements have non-conjugate images. If a finitely generated group $G$ is not residually free, then there will be non-trivial elements that will always be sent to (conjugates of) the identity. If $G$ is residually free then it canonically embeds into a direct product of limit groups $P = L_1 \times \cdots \times L_n$ and every homomorphism to a free group factors through one of the projections $P \twoheadrightarrow L_i$.

It is therefore natural to restrict our attention to the class of limit groups. A group is *freely conjugacy separable* if for any pair $u, v \in G$ of non-conjugate elements there is a homomorphism $G \to \mathbb{F}$ to a free group $\mathbb{F}$ such that the images of $u$ and $v$ in $\mathbb{F}$ are non-conjugate.

Throughout this paper $\mathbb{F}$ will denote a non-abelian free group, $\mathbb{F}_n$ will denote a non-abelian free group of rank $n$, and $\mathbb{F}(X)$ will denote the free group on the basis $X$.

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We give two different types of examples of limit groups which are not freely conjugacy separable for different reasons. In Sect. 2 we produce a limit group $L$ with elements $u, v$ such that the cyclic groups $\langle u\rangle, \langle v\rangle$ are non-conjugate, but whose normal closures $\langle\langle u\rangle\rangle$ and $\langle\langle v\rangle\rangle$ coincide. We call such a pair of elements a *Magnus pair* (see Definition 2.1.) Such elements must have conjugate images in any free quotient by a theorem of Magnus [17]. See [3,4] for earlier generalizations to closed surface groups.

In Sect. 3 we construct a limit group which is a double of a free group over a cyclic group generated by a $C$-test word (see Definition 3.1). These limit groups, *$C$-doubles*, are low rank and we are able to construct their Makanin-Razborov diagrams and observe the failure of free conjugacy separability directly. These groups were also found by Heil [8], who published a preprint while this paper was in preparation.

**Definition 1.1** A sequence of homomorphisms $\{\phi_i : G \to H\}$ is *discriminating* if for every finite subset $P \subset G \setminus \{1\}$ there is some $N$ such that for all $j \geq N$, $1 \notin \phi_j(P)$.

**Definition 1.2** A finitely generated group $L$ is a *limit group* if there is a discriminating sequence of homomorphisms $\{\phi_i : L \to \mathbb{F}\}$, where $\mathbb{F}$ is a free group.

**Theorem A** *The class of limit groups is not freely conjugacy separable.*

This should be seen in contrast to the fact that limit groups are conjugacy separable [6]. Lioutikova [14], proved that iterated centralizer extensions (see Definition 4.3) of a free group $\mathbb{F}$ are freely conjugacy separable. It is a result of of Kharlampovich and Miasnikov [11] that all limit groups embed in to iterated centralizer extensions. Moreover by [7, Theorem 5.3], almost locally free groups [7, Definition 4.2] cannot have Magus pairs. This class includes the class of limit groups which are $\forall\exists$-equivalent to free groups. The class of iterated centralizer extensions and the class of limit groups $\forall\exists$-equivalent to free groups are contained in the class of towers, also known as NTQ groups. We generalize these results to the class of towers with the following strong free conjugacy separability result:

**Theorem B** *Let $\mathbb{F}$ be a non-abelian free group and let $G$ be a tower over $\mathbb{F}$ (see Definition 4.3). There is a discriminating sequence of retractions $\{\phi_i : G \twoheadrightarrow \mathbb{F}\}$, such that for any finite subset $S \subset G$ of pairwise non-conjugate elements, there is some $N$ such that for all $j \geq N$ the elements of $\phi_j(S)$ are pairwise non-conjugate in $\mathbb{F}$. Similarly for any indivisible $\gamma \in L$ with cyclic centralizer there is some $M$ such that for all $k \geq M$, $r_k(\gamma)$ is indivisible.*

Theorem B also settles [7, Question 7.1], which asks if arbitrarily large collections of pairwise nonconjugate elements can have pairwise nonconjugate images via a homomorphism to a free group. The proof of Theorem B is in Sect. 4 and follows from [12,19].

In Sect. 5, we analyze the failure in free conjugacy separability of our limit group with a Magnus pair and show that it is very different from the $C$-double constructed in Sect. 3. This motivates two natural questions about Magnus pairs in limit groups. Finally, we show that free conjugacy separability does not isolate towers within the class of limit groups.

## 2 A limit group with a Magnus pair

Consider the fundamental group of the graph of spaces $\mathbb{U}$ given in Fig. 1. We pick elements $u, v \in \pi_1(\mathbb{U})$ corresponding to the similarly labelled loops given in Fig. 1 and we also consider groups $\pi_1(\Sigma_u), \pi_1(\Sigma_v)$ to be embedded into $\pi_1(\mathbb{U})$.

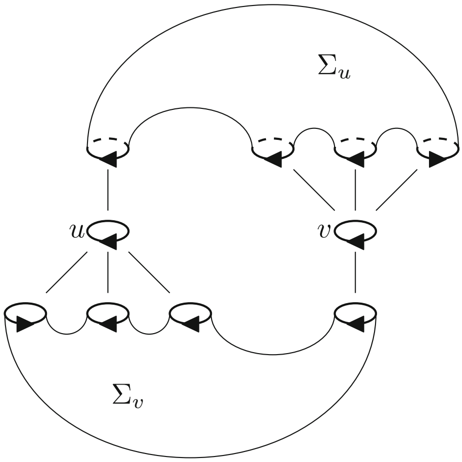



**Fig. 1** The graph of spaces $\mathbb{U}$. The attaching maps are of degree 1 and the black arrows show the orientations

**Definition 2.1** Let $G$ be a group, and let $\sim_\pm$ be the equivalence relation $g \sim_\pm h$ if and only if $g$ is conjugate to $h$ or $h^{-1}$, and denote by $[g]_\pm$ the $\sim_\pm$ equivalence class of $g$. A *Magnus pair* is a pair of $\sim_\pm$ classes $[g]_\pm \neq [h]_\pm$ such that $\langle\langle g \rangle\rangle = \langle\langle h \rangle\rangle$.

Note that if $h \sim_\pm g$ then $\langle\langle g \rangle\rangle = \langle\langle h \rangle\rangle$, but that the converse does not necessarily hold. The failure of the reverse implication is exactly witnessed by Magnus pairs. To save notation we will say that $g$ and $h$ are a Magnus pair if the classes $[g]_\pm$ and $[h]_\pm$ form a Magnus pair.

**Lemma 2.2** *The elements $u$ and $v$ in $\pi_1(\mathbb{U})$ are a Magnus pair.*

*Proof* The graph of spaces given in Fig. 1 gives rise to a cyclic graph of groups splitting $D$ of $\pi_1(\mathbb{U})$. The underlying graph $X$ has 4 vertices and 8 edges where the vertex groups are $\langle u \rangle$, $\langle v \rangle$, $\pi_1(\Sigma_u)$, and $\pi_1(\Sigma_v)$. Now note that $\pi_1(\Sigma_u)$ can be given the presentation

$$\pi_1(\Sigma_u) = \langle a, b, c, d \mid abcd = 1 \rangle = \langle a, b, c \rangle$$

and that the incident edge groups have images $\langle a \rangle$, $\langle b \rangle$, $\langle c \rangle$, $\langle abc \rangle = \langle d \rangle$. Without loss of generality $v^{\pm 1}$ is conjugate to $a$,$b$, and $c$ in $\pi_1(\mathbb{U})$ and $u^{\pm 1}$ is conjugate to $d = abc$ in $\pi_1(\mathbb{U})$ which means that $u \in \langle\langle v \rangle\rangle$ and, symmetrically considering $\Sigma_v$, $v \in \langle\langle u \rangle\rangle$.

On the other hand, the elements $\langle a \rangle$, $\langle b \rangle$, $\langle c \rangle$, $\langle abc \rangle$ are pairwise non-conjugate in $\langle a, b, c \rangle$. By inspecting the action on the Bass–Serre tree, $u$ and $v$ are clearly non-conjugate, and are therefre form a Magnus pair. □

## 2.1 Strict homomorphisms to limit groups

**Definition 2.3** Let $G$ be a finitely generated group and let $D$ be a 2-acylindrical cyclic splitting of $G$. We say that a vertex group $Q$ of $D$ is *quadratically hanging (QH)* if it satisfies the following:

- $Q = \pi_1(\Sigma)$ where $\Sigma$ is a compact surface such that $\chi(\Sigma) \leq -1$, with equality only if $\Sigma$ is orientable or $\partial(\Sigma) \neq \emptyset$.
- The images of the edge groups incident to $Q$ correspond to the $\pi_1$-images of $\partial(\Sigma)$ in $\pi_1(\Sigma)$.

**Definition 2.4** Let $G$ be torsion-free group. A homomorphism $\rho\colon G \to H$ is *strict* if there some 2-acylindrical abelian splitting $D$ of $G$ such that the following hold:

- $\rho$ is injective on the subgroup $A_D$ generated by the incident edge groups of each each abelian vertex group $A$ of $D$.
- $\rho$ is injective on each edge group of $D$.
- $\rho$ is injective on the "envelope" $\hat{R}$ of each non-QH, non-abelian vertex group $R$ of $D$, where $\hat{R}$ is constructed by first replacing each abelian vertex group $A$ of $D$ by $A_D$ and then taking $\hat{R}$ to be the subgroup generated by $R$ and the centralizers of the edge groups incident to $R$.
- The $\rho$-images of QH subgroups are non-abelian.

This next Proposition is a restatement of [5, Proposition 4.21] in our terminology. It is also given as Exercise 8 in [2,21].

**Proposition 2.5** *If $L$ is a limit group, and $G$ is a finitely generated group with a strict homomorphism $\rho : G \to L$, then $G$ is also limit group.*

### 2.2 $\pi_1(\mathbb{U})$ is a limit group but it is not freely conjugacy separable

Consider the sequence of continuous maps given in Fig. 2. The space on the top left obtained by taking three disjoint tori, identifying them along the longitudinal curves as shown, and then surgering on handles $H_1$, $H_2$ is homeomorphic to the space $\mathbb{U}$. A continuous map from $\mathbb{U}$ to the wedge of three circles is then constructed by filling in and collapsing the handles to arcs $h_1$, $h_2$, identifying the tori, and then mapping the resulting torus to a circle so that the image of the longitudinal curve $u$ (or $v$, as they are now freely homotopic inside a torus) maps with degree 1 onto a circle in the wedge of three circles.

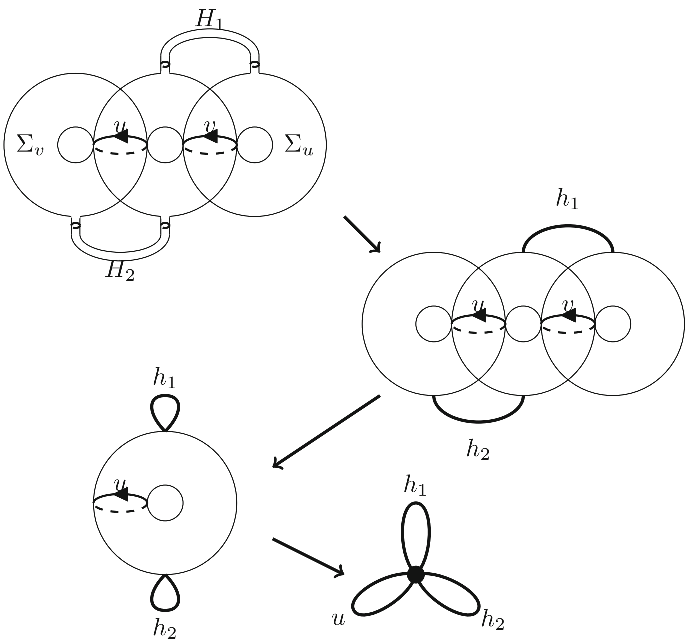



**Fig. 2** A continuous map from $\mathbb{U}$ to the wedge of three circles. The space on the top left is homeomorphic to $\mathbb{U}$. This can be seen by cutting along the curves labelled $u$, $v$

**Lemma 2.6** *The homomorphism $\pi_1(\mathbb{U}) \to \mathbb{F}_3$ given by the continuous map in Fig. 2 is onto, the vertex groups $\pi_1(\Sigma_v)$, $\pi_1(\Sigma_u)$ have non-abelian image and the edge groups $\langle u\rangle$, $\langle v\rangle$ are mapped injectively.*

*Proof* The surjectivity of the map $\pi_1(\mathbb{U}) \to \mathbb{F}_3$ as well as the injectivity of the restrictions to $\langle u\rangle$, $\langle v\rangle$ are obvious. Note moreover that the image of $\pi_1(\Sigma_u)$ contains (some conjugate of) $\langle u, h_1 u h_1^{-1}\rangle$ and is therefore non-abelian, the same is obviously true for the image of $\pi_1(\Sigma_v)$. □

The final ingredient is a classical result of Magnus.

**Theorem 2.7** [17] *The free group $\mathbb{F}$ has no Magnus pairs.*

**Proposition 2.8** *$\pi_1(\mathbb{U})$ is a limit group. For every homomorphism $\rho\colon \pi_1(\mathbb{U}) \to \mathbb{F}$ the images $\rho(u)$, $\rho(v)$ of the elements $u$, $v$ given in Lemma 2.2 are conjugate in $\mathbb{F}$ even though the pair $u, v$ are not conjugate in $\pi_1(\mathbb{U})$.*

*Proof* Lemma 2.6 and Proposition 2.5 imply that $\pi_1(\mathbb{U})$ is a Limit group. Lemma 2.2 and Theorem 2.7 imply that, for every homomorphism $\pi_1(\mathbb{U}) \to \mathbb{F}$ to a free group $\mathbb{F}$, the image of $u$ must be conjugate to the image of $v^{\pm 1}$ even though $u \not\sim_{\pm} v$. □

## 3 A different failure of free conjugacy separability

We now construct another limit group $L$ that is not freely conjugacy separable, but for a completely different reason.

**Definition 3.1** (*$C$-test words* [9]) A non-trivial word $w(x_1, \ldots, x_n)$ is a *$C$-test word* in $n$ letters for $\mathbb{F}_m$ if for any two $n$-tuples $(A_1, \ldots, A_n)$, $(B_1, \ldots, B_n)$ of elements of $\mathbb{F}_m$ the equality $w(A_1, \ldots, A_n) = w(B_1, \ldots, B_n) \neq 1$ implies the existence of an element $S \in \mathbb{F}_m$ such that $B_i = S A_i S^{-1}$ for all $i = 1, 2, \ldots, n$.

**Theorem 3.2** [9, Main Theorem] *For arbitrary $n \geq 2$ there exists a non-trivial indivisible word $w_n(x_1, \ldots, x_n)$ which is a $C$-test word in $n$ letters for any free group $\mathbb{F}_m$ of rank $m \geq 2$.*

**Definition 3.3** (*Doubles and retractions*) Let $\mathbb{F}(x, y)$ denote the free group on two generators, let $w = w(x, y)$ denote some word in $\{x, y\}^{\pm 1}$. The amalgamated free product

$$D(x, y; w) = \langle \mathbb{F}(x, y), \mathbb{F}(r, s) \mid w(x, y) = w(r, s)\rangle$$

is the *double of* $\mathbb{F}(x, y)$ *along* $w$. The homomorphism $\rho\colon D(x, y; w) \twoheadrightarrow \mathbb{F}(x, y)$ given by $r \mapsto x, s \mapsto y$ is the *standard retraction.*

**Definition 3.4** Let $u \in \mathbb{F}(x, y) \leq D(x, y; w)$, but with $u \not\sim_{\pm} w^n$ for any $n$, be given by a specific word $u(x, y)$. Its *mirror image* is the distinct element $u(r, s) \in \mathbb{F}(r, s) \leq D(x, y; w)$. $u(x, y)$ and $u(r, s)$ form a *mirror pair.*

It is obvious that mirror pairs are not $\sim_{\pm}$-equivalent. Let $w$ be a $C$-test word and let $L = D(x, y; w)$. It is well known that any such double is a limit group. We will call $L$ a *$C$-double*.

**Lemma 3.5** *The $C$-double $L$ cannot map onto a free group of rank more than 2.*

*Proof* $w$ is not primitive in $\mathbb{F}(x, y)$ therefore by [20] $L = D(x, y; w)$ is not free. Theorem 3.2 specifically states that $w$ is not a proper power. It now follows from [15, Theorem 1.5] that $D(w)$ cannot map onto $\mathbb{F}_3$. □

The proof of the next theorem amounts to analyzing a Makanin-Razborov diagram. We refer the reader to [8] for an explicit description of this diagram.

**Theorem 3.6** *For any map $\phi\colon L \to \mathbb{F}$ from a C-double to some free group, if $u(x, y) \in \mathbb{F}(x, y)$ lies in the commutator subgroup $[\mathbb{F}(x, y), \mathbb{F}(x, y)]$, but is not conjugate to $w^n$ for any n, then the images $\phi\,(u(x, y))$ and $\phi\,(u(r, s))$ of mirror pairs are conjugate. In particular the limit group L is not freely conjugacy separable. Furthermore mirror pairs $u(x, y), u(r, s)$ do not form Magnus pairs.*

*Proof* To answer this question we must analyze all maps for $L$ to a free group. By Lemma 3.5, any such map factors through a surjection onto $\mathbb{F}_2$, or factors through $\mathbb{Z}$.

**Case 1**: $\phi(w) = 1$. In this case the factor $\mathbb{F}(x, y)$ does not map injectively, it follows that its image is abelian. It follows that $\phi$ factors through the free product

$$\pi_{ab}\colon D(x, y; w) \to \mathbb{F}(x, y)^{\mathrm{ab}} * \mathbb{F}(r, s)^{\mathrm{ab}}.$$

In this case all elements of the commutator subgroups of $\mathbb{F}(x, y)$ and $\mathbb{F}(r, s)$ are mapped to the identity and therefore have conjugate images.

**Case 2**: $\phi(w) \neq 1$. In this case the factors $\mathbb{F}(x, y), \mathbb{F}(r, s) \leq D(x, y; w)$ map injectively. Indeed, since their image is nonabelian, their image is onto a non-abelian free group generated by two elements, therefore a free group of rank two; thus the restriction of the map is injective by the Hopf property. By Theorem 3.2, since $w$ is a C-test word and $\phi(w(x, y)) = \phi(w(r, s))$, there is some $S \in \mathbb{F}_2$ such that $S\phi(x)S^{-1} = \phi(r)$ and $S\phi(y)S^{-1} = \phi(s)$. Suppose now that $w(x, y)$ mapped to a proper power, then by [1, Main Theorem] $w(x, y) \in \mathbb{F}(x, y)$ is part of a basis, which is impossible. It follows that the centralizer of $\phi\,(w)$ is $\langle\phi(w)\rangle$ so that $S = \phi(w)^n$. Therefore $\phi(r) = w^n\phi(x)w^{-n}$ and $\phi(s) = w^n\phi(y)w^{-n}$; so mirror pairs are mapped to conjugates and, in particular, mirror pairs in the commutator subgroup of $\mathbb{F}(x, y)$ and $\mathbb{F}(r, s)$ are mapped to conjugates of the same elements.

We now show that a mirror pair $u(x, y)$ and $u(r, s)$ is not a Magnus pair. Consider the quotient $D(x, y; w)/\langle\langle u(x, y)\rangle\rangle$. By using a presentation with generators and relations, the group canonically splits as the amalgamated free product

$$(\mathbb{F}(x, y)/\langle\langle u(x, y)\rangle\rangle) *_{\langle\overline{w}\rangle} \left(\mathbb{F}(r, s)/\langle\langle w^n\rangle\rangle\right)$$

where $\langle w^n\rangle = \langle w\rangle \cap \langle\langle u\rangle\rangle$ and $\overline{w}$ is the image of $w$ in $\langle w\rangle/\langle w^n\rangle$. Now if $\langle\langle u(x, y)\rangle\rangle = \langle\langle u(r, s)\rangle\rangle$ then we must have $D(x, y; w)/\langle\langle u(r, s)\rangle\rangle = D(x, y; w)/\langle\langle u(x, y)\rangle\rangle$. This implies $\mathbb{F}(r, s)/\langle\langle(u(r, s))\rangle\rangle = \mathbb{F}(r, s)/\langle\langle w^n\rangle\rangle$, which implies by Theorem 2.7 that $u(r, s) \sim_{\pm} w^n$, which is a contradiction. □

It seems likely that failure of free conjugacy separability should typically follow from C-test word like behaviour, rather than from existence of Magnus pairs.

## 4 Towers are freely conjugacy separable

**Definition 4.1** Let $G$ be a group. A *regular quadratic extension* of $G$ is an extension $G \leq H$ such that

- $H$ splits as a fundamental group of a graph of groups with two vertex groups: $H_{v_1} = G$ and $H_{v_2} = \pi_1(\Sigma)$ where $H_{v_2}$ is a QH vertex group (See Definition 2.3.)
- There is a retraction $H \twoheadrightarrow G$ such that the image of $\pi_1(\Sigma)$ in $G$ is non abelian.

We say that $\Sigma$ is the *surface associated to the quadratic extension*. And note that if $\partial\Sigma = \emptyset$ then $H = G * \pi_1(\Sigma)$.

**Definition 4.2** Let $G$ be a group. An *abelian extension by the free abelian group* $A$ is an extension $G \le G *_{\langle u\rangle} (\langle u\rangle \oplus A) = H$ where $u \in G$ is such that either its centralizer $Z_G(u) = \langle u\rangle$, or $u = 1$. In the case where $u = 1$ the extension is $G \le G * A$ and it is called a *singular abelian extension*.

**Definition 4.3** Let $\mathbb{F}$ be a (possibly trivial) free group. A *tower of height n over* $\mathbb{F}$ is a group $G$ obtained from a sequence of extensions

$$\mathbb{F} = G_0 \le G_1 \le \cdots \le G_n = G$$

where $G_i \le G_{i+1}$ is either a regular quadratic extension or an abelian extension. The $G_i's$ are the *levels* of the tower $G$ and the sequence of levels is a *tower decomposition*. A tower consisting entirely of abelian extensions is an *iterated centralizer extension.*

**Definition 4.4** Let $\mathbb{F} = G_0 \le \cdots \le G_n = G$ be a tower decomposition of $G$. We call the graphs of groups decomposition of $G_i$ with one vertex group $G_{i-1}$ and the other vertex group a surface group or a free abelian group as given in Definitions 4.1 and 4.2 the $i$th *level decomposition.*

Towers appear as NTQ groups in the work of Kharlampovich and Miasnikov, and as $\omega$-residually free towers, as well as completions of strict resolutions in the the work of Sela. It is a well known fact that towers are limit groups [10]. This also follows easily from Proposition 2.5 and the definitions.

**Proposition 4.5** *Let $G$ be a tower of height $n$ over $\mathbb{F}$. Then $G$ is discriminated by retractions $G \to G_{n-1}$. $G$ is also discriminated by retractions onto $\mathbb{F}$.*

Following Definition 1.15 of [19] we have:

**Definition 4.6** Let $G$ be a tower. A *closure* of $G$ is another tower $G^\star$ with an embedding $\theta\colon G \hookrightarrow G^\star$ such that there is a commutative diagram

$$\begin{array}{ccccccccc}
G_0 & \le & G_1 & \le & \cdots & \le & G_n & = & G \\
\downarrow \| & & \downarrow & & & & \downarrow & & \\
G_0 & \le & G_1^\star & \le & \cdots & \le & G_n^\star & = & G^\star
\end{array}$$

where the injections $G_i \hookrightarrow G_i^\star$ are restrictions of $\theta$ and the horizontal lines are tower decompositions. Moreover the following must hold:

1. If $G_i \le G_{i+1}$ is a regular quadratic extension with associated surface $\Sigma$ such that $\partial\Sigma$ is "attached" to $\langle u_1\rangle, \ldots, \langle u_n\rangle \le G_i$ then $G_i^\star \le G_{i+1}^\star$ is a regular quadratic extension with associated surface $\Sigma$ such that $\partial\Sigma$ is "attached" to $\langle\theta(u_1)\rangle, \ldots, \langle\theta(u_n)\rangle \le G_i^\star$, in such a way that $\theta\colon G_i \hookrightarrow G_i^\star$ extends to a monomorphism $\theta\colon G_{i+1} \hookrightarrow G_{i+1}^\star$ which maps the vertex group $\pi_1(\Sigma)$ surjectively onto the vertex group $\pi_1(\Sigma) \le G_{i+1}^\star$.

2. If $G_i \leq G_{i+1}$ is an abelian extension then $G_i^\star \leq G_{i+1}^\star$ is also an abelian extension. Specifically (allowing $u_i = 1$) if $G_{i+1} = G_i *_{\langle u_i \rangle} (\langle u_i \rangle \oplus A_i)$, then $G_{i+1}^\star = G_i^\star *_{\langle \theta(u_i) \rangle} (\langle \theta(u_i) \rangle \oplus A_i')$. Moreover we require the embedding $\theta \colon G_{i+1} \to G_{i+1}^\star$ to map $\langle u_i \rangle \oplus A_i$ to a finite index subgroup of $\langle \theta(u_i) \rangle \oplus A_i'$.

We will now state one of the main results of [12,19] but first some explanations of terminology are in order. Towers are groups that arise as completed limit groups corresponding to a strict resolution and the definition of closure corresponds to the one given in [19]. We also note that our requirement on the Euler characteristic of the surface pieces given in Definitions 2.3 and 4.1 ensures that our towers are coordinate groups of *normalized* NTQ systems as described in the discussion preceding [12, Lemma 76] we also point out that a *correcting embedding* as described right before [12, Theorem 12] is in fact a closure in the terminology we are using.

We now give an obvious corollary (in fact a weakening) of [19, Theorem 1.22], or [12, Theorem 12]; they are the same result. Let $X, Y$ denote fixed tuples of variables.

**Lemma 4.7** *[∀∃-lifting Lemma] Let $\mathbb{F}$ be a fixed non-abelian free group and let*

$$G = \langle \mathbb{F}, X \mid R(\mathbb{F}, X) \rangle$$

*be a standard finite presentation of a tower over $\mathbb{F}$. Let $W_i(X, Y, \mathbb{F}) = 1$ and $V_i(X, Y, \mathbb{F}) \neq 1$ be (possibly empty) finite systems of equations and inequations (resp.) If the following holds:*

$$\mathbb{F} \models \forall X \exists Y \left( R(\mathbb{F}, X) = 1 \to \bigvee_{i=1}^{m} \left( W_i(X, Y, \mathbb{F}) = 1 \wedge V_i(X, Y, \mathbb{F}) \neq 1 \right) \right)$$

*then there is an embedding $\theta \colon G \hookrightarrow G^\star$ into some closure such that*

$$G^\star \models \exists Y \bigvee_{i=1}^{m} \left( W_i(\theta(X), Y, \mathbb{F}) = 1 \wedge V_i(\theta(X), Y, \mathbb{F}) \neq 1 \right)$$

*where $X$ and $\mathbb{F}$ are interpreted as the corresponding subsets of $G = \langle \mathbb{F}, X \mid R(\mathbb{F}, X) \rangle$*

In the terminology of [19] we have $G = \langle \mathbb{F}, X \rangle$ and $G^\star = \langle \mathbb{F}, X, Z \rangle$ for some collection of elements $Z$. Let $Y = (y_1, \ldots, y_k)$ be a tuple of elements in $G^\star$ that witness the existential sentence above. A collection of words $y_i(\mathbb{F}, X, Z) =_{G^*} y_i$ is called a set of *formal solution in $G^\star$*. According to [12, Definition 24] the tuple $Y \subset G^\star$ is an *R-lift*.

**Proposition 4.8** *Let $G$ be a tower over a non abelian free group $\mathbb{F}$and let $S \subset G$ be a finite family of pairwise non-conjugate elements of $G$. There exists a discriminating family of retractions $\psi_i \colon G \twoheadrightarrow \mathbb{F}$ such that for each $\psi_i$ the elements of $\psi_i(S)$ are pairwise non-conjugate.*

*Proof* Suppose towards a contradiction that this was not the case. Then either there exists a finite subset $P \subset G \setminus \{1\}$ such that for every retraction $r \colon G \twoheadrightarrow \mathbb{F}$, $1 \in r(P)$ or the elements of $r(S)$ are not pairwise non-conjugate. If we write elements of $P$ and $S$ as fixed words $\{p_i(\mathbb{F}, X)\}$ and $\{s_j(\mathbb{F}, X)\}$ (resp.) then we can express this as a sentence. Indeed, consider first the formula:

$$\Phi_{P,S}(\mathbb{F}, X, t) = \left( \left[ \bigvee_{p_i \in P} p_i(\mathbb{F}, X) = 1 \right] \vee \left[ \bigvee_{(s_i, s_j) \in \Delta(S)} t^{-1} s_i(\mathbb{F}, X) t = s_j(\mathbb{F}, X) \right] \right)$$

where $\Delta(S) = \{(x, y) \in S \times S \mid x \neq y)\}$. In English this says that either some element of $P$ vanishes or two distinct elements of $S$ are conjugated by some element $t$. We therefore have:

$$\mathbb{F} \models \forall X \left[ (R(\mathbb{F}, X)) = 1) \rightarrow \exists t \Phi_{P,S}(\mathbb{F}, X, t) \right]. \tag{1}$$

It now follows by Lemma 4.7 that there is some closure $\theta\colon G \hookrightarrow G^\star$ such that

$$G^\star \models \exists t \Phi_{P,S}(\mathbb{F}, \theta(X), t).$$

Since $1 \notin P$ and $\theta$ is a monomorphisms none of the $p_i(\mathbb{F}, X)$ are trivial so

$$G^\star \models \exists t \left[ \bigvee_{(s_i, s_j) \in \Delta(S)} \left( t^{-1} s_i(\mathbb{F}, X) t = s_j(\mathbb{F}, X) \right) \right].$$

In particular there are elements $u, v \in G$ which are not conjugate in $G$ but are conjugate in $G^\star$. We will derive a contradiction by showing that this is impossible.

We proceed by induction on the height of the tower. If the tower has height 0 then $G = \mathbb{F}$ and the result obviously holds. Suppose now that the claim held for all towers of height $m \leq n$. Let $G$ have height $n$ and let $u, v$ be non-conjugate elements of $G$ let $G \leq G^\star$ be any closure and suppose that there is some $t \in G^\star \setminus G$ such that $tut^{-1} = v$.

Let $D$ be the $n$th level decomposition of $G^\star$ and let $T$ be the corresponding Bass–Serre tree. Let $T(G)$ be the minimal $G$-invariant subtree and let $D_G$ be the splitting induced by the action of $G$ on $T(G)$. By Definition 4.6 $D_G$ is exactly the $n$th level decomposition of $G$ and two edges of $T(G)$ are in the same $G$-orbit if and only if they are in the same $G^\star$-orbit. We now consider separate cases:

**Case 1:** *Without loss of generality $u$ is hyperbolic in the* $n$th *level decomposition of* $G$. If $v$ is elliptic in the $n$th level decomposition of $G$ then it is elliptic in the $n$th-level decomposition of $G^\star$ and therefore cannot be conjugate to $u$ which acts hyperbolically on $T$.

It follows that both $u, v$ must be hyperbolic elements with respect to the $n$th level decomposition of $G$. Let $l_u, l_v$ denote the axes of $u, v$ (resp.) in $T(G) \subset T$. Since $tut^{-1} = v$, we must have $t \cdot l_u = l_v$. Let $e$ be some edge in $l_u$ then by the previous paragraph $t \cdot e \subset l_v$ must be in the same $G$-orbit as $e$, which means that there is some $g \in G$ such that $gt \cdot e = e$, but again by Definition 4.6 the inclusion $G \leq G^\star$ induces a surjection of the edge groups of the $n$th level decomposition of $G$ to the edge groups of the $n$th level decomposition of $G^\star$, it follows that $gt \in G$ which implies that $t \in G$ contradicting the fact that $u, v$ were not conjugate in $G$.

**Case 2:** *The elements $u, v$ are elliptic in the* $n$th *level decomposition of* $G$. Suppose first that $u, v$ were conjugate into $G_{n-1}$, then the result follows from the fact that there is a retraction $G \twoheadrightarrow G_{n-1}$ and by the induction hypothesis. Similarly by examining the induced splitting of $G \leq G^\star$, we see that $u$ cannot be conjugate into $G_{n-1}$ and $v$ into the other vertex group of the $n$th-level decomposition. We finally distinguish two sub-cases.

**Case 2.1:** $G_{n-1} \leq G$ is an abelian extension by the free abelian group $A$ and $u, v$ are conjugate in $G$ into some free abelian group $\langle w \rangle \oplus A$. Any homomorphic image of $\langle w \rangle \oplus A$ in $\mathbb{F}$ must lie in a cyclic group, since $u \neq v$ in $G^\star$ and $G^\star$ is discriminated by retractions onto $\mathbb{F}$, there must be some retraction $r\colon G^\star \rightarrow \mathbb{F}$ such that $r(u) \neq r(v)$ which means that $u, v$ are sent to distinct powers of a generator of the cyclic subgroup $r(\langle w \rangle \oplus A)$. It follows that their images are not conjugate in $\mathbb{F}$ so $u, v$ cannot be conjugate in $G^\star$.

**Case 2.2:** $G_{n-1} \leq G$ is a quadratic extension and $u$ and $v$ are conjugate in $G$ into the vertex group $\pi_1(\Sigma)$. Arguing as in Case 1 we find that if there is some $t \in G^\star$ such that $tut^{-1} = v$

then there is some $g \in G$ such that $gt$ fixes a vertex of $T(G) \subset T$ whose stabilizer is conjugate to $\pi_1(\Sigma)$. Again by the surjectively criterion in item 1. of Definition 4.6, $gt \in G$ contradicting the fact that $u, v$ were not conjugate in $G$. All the possibilities have been exhausted so the result follows. □

*Proof of Theorem B* Let $S_1 \subset S_2 \subset S_3 \subset \cdots$ be an exhaustion of representatives of distinct conjugacy classes of $G$ by finite sets. For each $S_j$ let $\{\psi_i^j\}$ be the discriminating sequence given by Proposition 4.8. We take $\{\phi_i\}$ to be the diagonal sequence $\{\psi_i^i\}$. This sequence is necessarily discriminating and the result follows. □

It is worthwhile to point out that *test sequences* given in the proof of [19, Theorem 1.18], or the *generic sequence* given in [12, Definition 44], because of their properties, must satisfy the conclusions of Theorem B. As an immediate consequence of the Sela's completion construction ([19, Definition 1.12]) or canonical embeddings into NTQ groups ([13, Sect. 7]) Theorem B implies the following:

**Corollary 4.9** *Let $L$ be a limit group and suppose that for some finite set $S \subset L$ there is a homomorphism $f : L \to \mathbb{F}$ such that:*

- *The elements of $f(S)$ are pairwise non-conjugate.*
- *There is a factorization*

$$f = f_m \circ f_{m-1} \circ \cdots \circ f_1$$

*such that each $f_i$ is a* strict *homomorphisms between limit groups (see Definition 2.4).*

*Then there is a discriminating sequence $\psi_i : L \to \mathbb{F}$ such that for all $i$ the elements $\psi_i(S)$ are pairwise non-conjugate.*

## 5 Refinements

### 5.1 $\pi_1(\mathbb{U})$ is almost freely conjugacy separable

The limit group $L$ constructed in Sect. 3 had an abundance of pairs of nonconjugate elements whose images had to have conjugate images in every free quotient. The situation is completely different for our Magnus pair group.

**Proposition 5.1** *$\langle u\rangle, \langle v\rangle \leq \pi_1(\mathbb{U})$ are the only maximal cyclic subgroups of $\pi_1(\mathbb{U})$ whose conjugacy classes cannot be separated via a homomorphism to a free group $\pi_1(\mathbb{U}) \to \mathbb{F}$.*

*Proof* We begin by embedding $\pi_1(\mathbb{U})$ into a hyperbolic tower. Let $\rho : \pi_1(\mathbb{U}) \twoheadrightarrow \mathbb{F}_3$ be the strict homomorphism given in Fig. 2. Consider the group

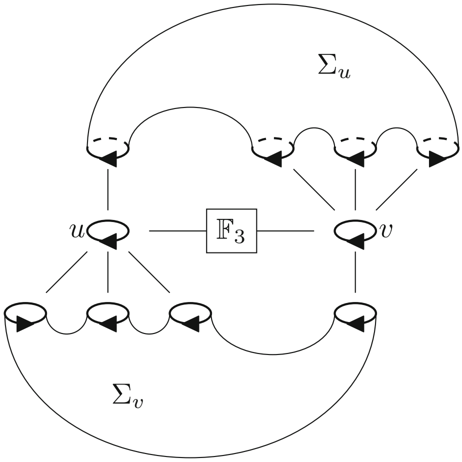
$$T = \langle \pi_1(\mathbb{U}), \mathbb{F}_3, s \mid u = \rho(u), svs^{-1} = \rho(v)\rangle.$$

This presentation naturally gives a splitting $D$ of $T$ given in Fig. 3. We have a retraction $\rho* : T \twoheadrightarrow \mathbb{F}_3$ given by

$$\rho* : \begin{cases} g \mapsto \rho(g); & g \in \pi_1(\mathbb{U}) \\ f \mapsto f; & f \in \mathbb{F}_3 \\ s \mapsto 1 \end{cases}$$

It therefore follows that $T$ is a hyperbolic tower over $\mathbb{F}_3$.

**Fig. 3** The splitting $D$ of $T$




Claim: *if* $\alpha, \beta \in \pi_1(\mathbb{U}) \leq T$ *are non-conjugate in* $\pi_1(\mathbb{U})$ *and* $\alpha, \beta$ *are not both conjugate to* $\langle u \rangle$ *or* $\langle v \rangle$ *in* $\pi_1(\mathbb{U})$ *then they are not conjugate in* $T$. If both $\alpha$ and $\beta$ are elliptic, then this follows easily from the fact that the vertex groups are malnormal in $T$. Also $\alpha$ cannot be elliptic while $\beta$ is hyperbolic. Suppose now that $\alpha, \beta$ are hyperbolic. Let $T$ be the Bass–Serre tree corresponding to $D$ and let $T' = T(\pi_1(\mathbb{U}))$ be the minimal $\pi_1(\mathbb{U})$ invariant subtree. Suppose that there is some $s \in T$ such that $s\alpha s^{-1} = \beta$, then as in the proof of Proposition 4.8 and Proposition we find that for some $g \in \pi_1(\mathbb{U})$ either $gs$ permutes two edges in $T'$ that are in distinct $\pi_1(\mathbb{U})$-orbits or it fixes some edge in $T'$. The former case is impossible and it is easy to see that the latter case implies that $gs \in \pi_1(\mathbb{U})$. Therefore we have a contradiction to the assumption that $\alpha, \beta$ are not conjugate in $\pi_1(\mathbb{U})$. The claim is now proved.

It therefore follows that if $\alpha, \beta \in \pi_1(\mathbb{U}) \leq T$ are as above, then by Theorem B there exists some retraction $r : T \twoheadrightarrow \mathbb{F}_3$ such that $r(\alpha), r(\beta)$ are non-conjugate. □

This construction gives an alternative proof to the fact that $\pi_1(\mathbb{U})$ is a limit group. The group $T$ constructed is a triangular quasiquadratic group and the retraction $\rho^*$ makes it non-degenerate, and therefore an NTQ group. $T$ and therefore $\pi_1(\mathbb{U}) \leq T$ are therefore limit groups by [10].

### 5.2 $C$-doubles do not contain Magnus pairs

Theorem B enables us to examine a $C$-double $L$ more closely.

**Proposition 5.2** *The $C$-double $L$ constructed in Sect. 3 does not contain a Magnus pair.*

*Proof* We need to show that if two elements $u, v$ of $L$ have the same normal closure in $L$ then they must be conjugate. Suppose that $u, v$ are both elliptic with respect to the splitting (as a double) of $L$ but not conjugate. By Theorem 3.2 if they are conjugate to a mirror pair $(u^g, v^h)$ for some $g, h \in L$ then they do not form a Magnus pair, i.e. they have separate normal closures. Otherwise there are homomorphisms $L \to \mathbb{F}$ in which $u, v$ have non-conjugate images, therefore by Theorem 2.7 the normal closures of their images are distinct; so $\langle\langle u \rangle\rangle \neq \langle\langle v \rangle\rangle$ as well.

Suppose now that $u$ or $v$ is hyperbolic in $L$. Recall the generating set $x, y, r, s$ for $L$ given in Definition 3.3. Let $\mathbb{F} = \mathbb{F}(x, y)$ and consider the embedding into a centralizer extension, represented as an HNN extension

$$L \hookrightarrow \langle \mathbb{F}, t | t w(x, y) = w(x, y) t \rangle = \mathbb{F}*^t_{\langle w \rangle}$$
$$x \mapsto x, \quad y \mapsto y$$
$$r \mapsto t^{-1} x t, \quad s \mapsto t^{-1} y t$$

The stable letter $t$ makes mirror pairs conjugate in this bigger group. A hyperbolic element of $L$ can be written as a product of syllables

$$u = a_1(x, y) a_2(r, s) \ldots a_l(r, s)$$

with $a_1$ or $a_l$ possibly trivial. The image of $u$ in $\mathbb{F}*^t_{\langle w \rangle}$ is

$$u = a_1(x, y) \left(t^{-1} a_2(x, y) t\right) \ldots \left(t^{-1} a_l(x, y) t\right).$$

Consider the set of words of the form

$$w_1(x, y) \left(t^{-1} w_2(x, y) t\right) \ldots w_{N-1}(x, y) \left(t^{-1} w_N(x, y) t\right),$$

with $w_1$ or $w_N$ possibly trivial. This set is clearly closed under multiplication, inverses and passing to $\mathbb{F}^t_{\langle w \rangle}$-normal form. It follows that we can identify the image of $L$ with this set of words, which we call $t^{-1} * t$*-syllabic words*. Each factor $w_i(x, t)$ or $t^{-1} w_j(x, y) t$ is called a $t^{-1} * t$*-syllable*.

It is an easy consequence of Britton's Lemma that if $u$ is a hyperbolic, i.e. with cyclically reduced syllable length more than 1, $t^{-1} * t$-syllabic word and $g^{-1} u g$ is again $t^{-1} * t$-syllabic for some $g$ in $\mathbb{F}*^t_{\langle w \rangle}$ then $g$ must itself be $t^{-1} * t$-syllabic. Indeed this can be seen by cyclically permuting the $\mathbb{F}*^t_{\langle w \rangle}$-syllables of a cyclically reduced word $u$. We refer the reader to [16, Sect. IV.2] for further details about normal forms and conjugation in HNN extensions.

Suppose now that $u, v$ are non conjugate in $L$, but have the same normal closure in $L$. Since at least one of them is hyperbolic in $L$, it is clear from the embedding that its image must also be hyperbolic with respect to the HNN splitting $\mathbb{F}*^t_{\langle w \rangle}$. Now, since $\langle\langle u \rangle\rangle_L = \langle\langle v \rangle\rangle_L$, in the bigger group $\mathbb{F}*^t_{\langle w \rangle}$ we have:

$$\langle\langle u \rangle\rangle_{\mathbb{F}*^t_{\langle w \rangle}} = \langle\langle \langle\langle u \rangle\rangle_L \rangle\rangle_{\mathbb{F}*^t_{\langle w \rangle}} = \langle\langle \langle\langle v \rangle\rangle_L \rangle\rangle_{\mathbb{F}*^t_{\langle w \rangle}} = \langle\langle v \rangle\rangle_{\mathbb{F}*^t_{\langle w \rangle}}$$

By Theorem B or [14] centralizer extensions are freely conjugacy separable, therefore they cannot contain Magnus pairs. It follows that $u, v$ must be conjugate in the bigger $\mathbb{F}*^t_{\langle w \rangle}$. Let $g^{-1} u g =_{\mathbb{F}*^t_{\langle w \rangle}} v$. Now both $u$ and $v$ must be hyperbolic so it follows that $g$ must also be a $t^{-1} * t$-syllabic word; thus $g$ is in the image of $L$ of $\mathbb{F}*^t_{\langle w \rangle}$. Furthermore since the map $L \hookrightarrow \mathbb{F}*^t_{\langle w \rangle}$ is an embedding

$$g^{-1} u g =_{\mathbb{F}*^t_{\langle w \rangle}} v \Rightarrow g^{-1} u g =_L v,$$

contradicting the fact that $u, v$ are non conjugate in $L$. □

Perhaps the methods of the previous proof can be used and extended to address the following questions.

**Question 1** *Does a limit group contain only finitely many Magnus pairs up to automorphism?*

In fact, an even simpler question: "Does a limit group only contain finitely many Magnus pairs?" is open. In particular the Magnus pair constructed in $\pi_1(\mathbb{U})$ (see Sect. 2), viewed as an unordered pair (recall Definition 2.1), is Aut $(\pi_1(\mathbb{U}))$-invariant.

**Question 2** *Do Magnus pairs in limit groups arise from embedded groups of the form* $\pi_1(\mathbb{U})$*, given in Sect. 2? More precisely, does every Magnus pair in a limit group always arise from an embedding of the fundamental group of a graph of orientable, genus-zero surfaces, amalgamated along their boundaries?*

### 5.3 A non-tower limit group that is freely conjugacy separable

In this section we construct a limit group that is freely conjugacy separable but which does not admit a tower structure. Let $H \leq [\mathbb{F}, \mathbb{F}]$ be some f.g. malnormal subgroup of $\mathbb{F}$, e.g. $H = \langle aba^{-1}b^{-1}, b^{-2}a^{-1}b^2a \rangle \leq \mathbb{F}(a, b)$. And pick $h \in H \setminus [H, H]$ such that $H$ is *rigid* relative to $h$, i.e. $H$ has no non-trivial cyclic or free splittings relative to $\langle h \rangle$. Because $h \in [\mathbb{F}, \mathbb{F}]$ there is a quadratic extension

$$\mathbb{F} < \mathbb{F} *_{\langle h \rangle} \pi_1(\Sigma)$$

where $\Sigma$ has one boundary component and has genus $g = \text{genus}(h)$, in particular there is a retraction onto $\mathbb{F}$. Consider now the subgroup $L = H *_{\langle h \rangle} \pi_1(\Sigma)$.

**Proposition 5.3** *L as above is freely conjugacy separable.*

*Proof* Because $H \leq \mathbb{F}$ was chosen to be malnormal, an easy Bass–Serre theory argument (e.g. apply [16, Theorem IV.2.8]) tells us that $\alpha, \beta \in L$ are conjugate if and only if they are conjugate in $\mathbb{F} *_{\langle h \rangle} \pi_1(\Sigma)$. On the other hand by Theorem B, $\mathbb{F} *_{\langle h \rangle} \pi_1(\Sigma)$, and hence $L$, are freely conjugacy separable. □

**Definition 5.4** A splitting $\mathbb{X}$ is *elliptic* in a splitting $\mathbb{Y}$ if every edge group in $\mathbb{X}$ is conjugate into a vertex group of $\mathbb{Y}$. Otherwise we say $\mathbb{X}$ is hyperbolic in $\mathbb{Y}$.

**Theorem 5.5** [18, Theorem 7.1] *Let G be an f.p. group with a single end. There exists a reduced, unfolded $\mathbb{Z}$-splitting of G called a JSJ decomposition of G with the following properties:*

1. *Every canonical maximal QH (recall Definition 2.3) subgroup (CMQ) of G is conjugate to a vertex group in the JSJ decomposition. Every QH subgroup of G can be conjugated into one of the CMQ subgroups of G. Every non-CMQ vertex groups in the JSJ decomposition is elliptic in every $\mathbb{Z}$-splitting of G.*
2. *An elementary $\mathbb{Z}$-splitting $G = A *_C B$ or $G = A*_C$ which is hyperbolic in another elementary $\mathbb{Z}$-splitting is obtained from the JSJ decomposition of G by cutting a 2-orbifold corresponding to a CMQ subgroup of G along a weakly essential simple closed curve (s.c.c.).*
3. *Let $\Theta$ be an elementary $\mathbb{Z}$-splitting $G = A *_C B$ or $G = A*_C$ which is elliptic with respect to any other elementary $\mathbb{Z}$ splitting of G. There exists a G-equivariant simplicial map between a subdivision of $T_{\mathrm{JSJ}}$, the Bass–Serre tree corresponding to the JSJ decomposition, and $T_\Theta$, the Bass–Serre tree corresponding to $\Theta$.*
4. *Let $\Lambda$ be a general $\mathbb{Z}$-splitting of G. There exists a $\mathbb{Z}$-splitting $\Lambda_1$ obtained from the JSJ decomposition by splitting the CMQ subgroups along weakly essential s.c.c. on their corresponding 2-orbifolds, so that there exists a G-equivariant simplicial map between a subdivision of the Bass–Serre tree $T_{\Lambda_1}$ and $T_\Lambda$.*
5. *If* $\mathrm{JSJ}_1$ *is another JSJ decomposition of G, then there exists a G-equivariant simplicial map $h_1$ from a subdivision of $T_{\mathrm{JSJ}}$ to $T_{\mathrm{JSJ}_1}$, and a G-equivariant simplicial map $h_2$ from a subdivision of $T_{\mathrm{JSJ}_1}$ to $T_{\mathrm{JSJ}}$, so that $h_1 \circ h_2$ and $h_2 \circ h_1$ are G-homotopic to the corresponding identity maps.*

We note that item 5. of the above theorem describes the canonicity of a JSJ decomposition, and that requiring the existence of an equivariant simplicial map from a subdivision of a tree $S$ to a tree $T$ is the same as requiring an equivariant continuous map from $S$ to $T$ that sends vertices to vertices.

**Lemma 5.6** *The splitting $L = H *_{\langle h \rangle} \pi_1(\Sigma)$ is a cyclic JSJ splitting.*

*Proof* This is an elementary $\mathbb{Z}$ splitting of $L$, let's see how it can be obtained from the JSJ decomposition given in Theorem 5.5. Let $T_{\mathrm{JSJ}}$ denote the Bass–Serre tree of the JSJ decomposition and let $T$ denote the Bass–Serre tree of the splitting $L = H *_{\langle h \rangle} \pi_1(\Sigma)$. The factor $\pi_1(\Sigma)$ is a QH subgroup, so by item 1. of Theorem 5.5, the JSJ decomposition must contain a CMQ vertex group $\pi_1(\Sigma_M) \leq L$ where $\Sigma_M$ is some surface with boundary. By 4. of Theorem 5.5 $\pi_1(\Sigma) \leq L$ can be represented as a subsurface $\Sigma \subset \hat{\Sigma}_M$. Since $L$ is not a closed surface group the JSJ decomposition has at least 2 vertex groups.

By 4. of Theorem 5.5, we can cut the CMQ vertex group $\pi_1(\Sigma_M) \leq L$ along simple closed curves to get a new splitting with Bass–Serre tree $T_{\Lambda_1}$ such that $T_{\Lambda_1} \twoheadrightarrow T_{\mathrm{JSJ}}$ is obtained by perhaps collapsing edges dual to the simple closed curves and there is an $L$-equivariant continuous (but perhaps not simplicial) map $T_{\Lambda_1} \twoheadrightarrow T$. The subgroup $\pi_1(\Sigma)$ is a vertex group of $T_{\Lambda_1}$, and in particular the element $h \in \pi_1(\Sigma)$ acts elliptically on $T_{\Lambda_1}$. The subgroup $H$ must also act elliptically on $T_{\Lambda_1}$, for otherwise $H$ has a cyclic or free splitting relative to $h$, contradicting rigidity. Since the vertex groups of $T$ fix vertices of $T_{\Lambda_1}$, there is also a continuous map $T \twoheadrightarrow T_{\Lambda_1}$. It follows that $T_{\Lambda_1}$ has at most 2 (conjugacy classes of) maximal vertex groups so the map $T_{\Lambda_1} \twoheadrightarrow T_{\mathrm{JSJ}}$ is the identity (i.e. no CMQ subgroups cut along simple closed curves). It follows that we have $L$-equivariant maps $T \twoheadrightarrow T_{JSJ}$ and $T_{JSJ} \twoheadrightarrow T$. Therefore $L = H *_{\langle h \rangle} \pi_1(\Sigma)$ is a cyclic JSJ decomposition. □

**Proposition 5.7** *The limit group $L = H *_{\langle h \rangle} \pi_1(\Sigma)$ does not admit a tower structure.*

*Proof* Suppose towards a contradiction that $L$ was a tower, consider the last level:

$$L_{n-1} < L_n = L.$$

Since $L$ has no non-cyclic abelian subgroups $L_{n-1} < L$ must be a hyperbolic extension. This means that $L$ admits a cyclic splitting $\mathbb{D}$ with a vertex group $L_{n-1}$ and a QH vertex group $Q$. Since $L = H *_{\langle h \rangle} \pi_1(\Sigma)$ is a JSJ decomposition and $\pi_1(\Sigma)$ is a CMQ vertex group. By 1. and 4. of Theorem 5.5, the QH vertex group $Q$ must be represented as $\pi_1(\Sigma_1)$, where $\Sigma_1$ is a connected subsurface $\Sigma_1 \subset \Sigma$. It follows from 4. of Theorem 5.5 that the other vertex group must be $L_{n-1} = H *_{\langle h \rangle} \pi_1(\Sigma')$ where $\Sigma' = \Sigma \setminus \Sigma_1$.

Since $L_{n-1} < L$ is a quadratic extension there is a retraction $L \twoheadrightarrow L_{n-1}$. Note however that because $\Sigma'$ has at least two boundary components

$$H *_{\langle h \rangle} \pi_1(\Sigma') = H * \mathbb{F}_m$$

where $m = -\chi(\Sigma')$. Now since we have a retraction $L \twoheadrightarrow L_{n-1}$ there is are $x_i, y_i \in L_{n-1}$ such that

$$h = \prod_{i=1}^{g} [x_i, y_i]$$

But this would imply that $h \in [L_{n-1}, L_{n-1}]$ which is clearly seen to be false by abelianizing $H * \mathbb{F}_m$ and remembering that $h \notin [H, H]$. □

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