Total positivity of sums, Hadamard products and Hadamard powers: Results and counterexamples

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Abstract

We show that, for Hankel matrices, total nonnegativity (resp. total positivity) of order r is preserved by sum, Hadamard product, and Hadamard power with real exponent $t \geq r-2$. We give examples to show that our results are sharp relative to matrix size and structure (general, symmetric or Hankel). Some of these examples also resolve the Hadamard critical-exponent problem for totally positive and totally nonnegative matrices.

Key Words and Phrases: Totally positive matrix, Totally nonnegative matrix, Hankel matrix, Stieltjes moment problem, Hadamard product, Hadamard power, Hadamard critical exponent.

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1 Introduction

A matrix M of real numbers is called *totally nonnegative* (TN) if every minor of M is nonnegative, and *totally positive* (TP) if every minor of M is positive. More generally, M is called *totally nonnegative of order* r (TN_r) if every minor of M of size $\le r$ is nonnegative, and *totally positive of order* r (TP_r) if every minor of M of size $\le r$ is positive.[1] Of course, for m-by-n matrices, $\mathrm{TN} = \mathrm{TN}_r$ and $\mathrm{TP} = \mathrm{TP}_r$ where $r = \min(m,n)$. Background information on totally nonnegative and totally positive matrices and their applications can be found in [2, 9, 14, 16, 24, 27].

It is an immediate consequence of the Cauchy–Binet formula that the product of two TN_r (resp. TP_r) matrices is TN_r (resp. TP_r). However, other natural matrix operations do not in general preserve total nonnegativity. For instance, it is well known (and easy to see by example) that the sum of two TP matrices need not even be TN_2. The situation is slightly (but not much) better when the matrices are symmetric. Likewise, it has been known for over 40 years that the Hadamard (entrywise) product of two TN (resp. TP) matrices is always TN_2 (resp. TP_2) but need not be TN_3 [26, p. 163]. Once again, the situation is slightly (but not much) better when the matrices are symmetric. In this paper we shall give counterexamples illustrating the various possibilities and showing the sharpness of each positive result.

The situation changes radically, however, for Hankel matrices, i.e. square matrices $A = (a_{ij})$ in which a_{ij} depends only on $i + j$. The Hankel matrices form an important subclass of symmetric matrices, and they arise in numerous applications [13, 18, 21, 29, 31, 32]. It is easy to see (Lemma 2.7 below) that a matrix is Hankel if and only if every contiguous submatrix is symmetric. Here we will exploit this fact to show that, for Hankel matrices, total nonnegativity — and more generally, total nonnegativity of order r — *is* preserved by sum and by Hadamard product. We will also show that total nonnegativity of order r is preserved under Hadamard powers with an arbitrary real exponent $t \ge r - 2$.

One important motivation for this investigation was the connection between the Stieltjes moment problem [1, 31] and the total positivity of Hankel matrices. It is well known that an *infinite* Hankel matrix $A = (a_{i+j})_{i,j=0}^{\infty}$ is totally nonnegative if and only if the underlying sequence $\boldsymbol{a} = (a_n)_{n=0}^{\infty}$ is a Stieltjes moment sequence (i.e. the moments of a positive measure on $[0, \infty)$): "only if" follows immediately from the standard positive-definiteness criterion for Stieltjes moment sequences [31, Theorem 1.3], while "if" follows by a simple Vandermonde-matrix argument [15, p. 460, Théorème 9] [27, Theorem 4.4]. This equivalence immediately implies that, for *infinite* Hankel matrices, total nonnegativity is preserved by sum and by Hadamard product. We therefore wondered whether the same result would hold when infinite Hankel matrices are replaced by finite ones, or when TN is replaced by TN_r. It is satisfying to know that the answer to both questions is yes.

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[1] **Warning:** Some authors (e.g. [2, 24, 27, 30, 32]) use the terms "totally positive" and "strictly totally positive" for what we have termed "totally nonnegative" and "totally positive", respectively. So it is very important, when seeing any claim about "totally positive" matrices, to ascertain which sense of "totally positive" is being used! (This is especially important because many theorems in this subject require the *strict* concept for their validity: see e.g. Section 2.1 below.)

Finally, some of our counterexamples also settle the Hadamard critical-exponent problem [23] for TN or TP matrices that are general, symmetric or Hankel.

2 Preliminaries

In this section we review some known results that will be used as tools in the remainder of the paper.

2.1 Inferring total positivity from a proper subset of minors

We write $[n] = \{1, \ldots, n\}$. A subset $I \subseteq [n]$ is called *contiguous* if it is an interval (i.e. $i, k \in I$ and $i < j < k$ imply $j \in I$). A subset $I \subseteq [n]$ is called *initial* if it is contiguous and contains 1.

If $A = (a_{ij})_{1 \le i \le m, 1 \le j \le n}$ is an m-by-n matrix and $I \subseteq [m]$, $J \subseteq [n]$, we denote by A_{IJ} the submatrix of A corresponding to the rows I and the columns J, all kept in their original order. The submatrix A_{IJ} (and the corresponding minor $\det A_{IJ}$) is called *contiguous* if $|I| = |J|$ and both I and J are contiguous; it is called *initial* if $|I| = |J|$ and both I and J are contiguous and at least one of them is initial. Note that each matrix entry is the lower-right corner of exactly one initial submatrix; so an m-by-n matrix has mn initial submatrices.

The following important result [17, Theorem 4.1] allows one to infer total positivity from a rather small subset of minors:

Theorem 2.1. [17] *Let A be an m-by-n matrix. If all the initial minors of A are positive, then A is TP.*

Proofs can be found in [9, Theorem 3.1.4] and [27, Theorem 2.3]. See also [12] for a combinatorial reinterpretation of Theorem 2.1 in the case $m = n$, as well as a generalization to some other sets of minors of the same cardinality $mn = n^2$.

In fact a weaker result, due to Fekete [10] in 1912, would suffice for our applications:

Theorem 2.2. [10] *Let A be an m-by-n matrix. If all the contiguous minors of A are positive, then A is TP.*

Please note that Theorems 2.1 and 2.2 cannot be extended to TN matrices: for instance [6, p. 23], all the contiguous minors of the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ are nonnegative, and $\det A = 1$, but some of the 2-by-2 noncontiguous minors equal -1. (It follows that there is no way of perturbing A so that all the initial minors are positive.) Moreover, the lower-triangular matrix [5, p. 86] [6, p. 23] $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ and the symmetric matrix $C = \begin{bmatrix} \sqrt{2} & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & \sqrt{2} \end{bmatrix}$ have the same property. (Note that the lower-left 3-by-3 corner of B and C is A.) So it does not help here to assume that the matrix is triangular or symmetric. (But it *does* help to assume that the matrix is Hankel: see Theorem 3.2(a) below.)

In this paper we shall need an extension of Theorems 2.1 and 2.2 to total positivity of order r, which we state as follows:

Theorem 2.3. [9, Corollary 3.1.7] *Let A be an m-by-n matrix. Suppose that all the initial minors of A of size $\le r-1$ are positive, and that all the contiguous minors of A of size r are positive. Then A is TP_r.*

Since this result is stated in [9] without proof, it is perhaps useful to include a proof here. The first step is to establish the following weakened version of Theorem 2.3:

Lemma 2.4. *Let A be an m-by-n matrix. Suppose that all the contiguous minors of A of size $\le r$ are positive. Then A is TP_r.*

The statement above is essentially [9, Corollary 3.1.6], but for the reader's convenience we give the proof (which is slightly streamlined compared to the one given in [9]):

Proof of Lemma 2.4. Let $I \subseteq [m]$ be any contiguous set of rows of cardinality r, and let $J \subseteq [n]$ be any contiguous set of columns of cardinality r. Then Theorem 2.2 tells us that the submatrices $A_{I[n]}$ and $A_{[m]J}$ are TP. Now let $I' = \{i_1, \ldots, i_r\} \subseteq [m]$ be any collection of r rows, and let $B = A_{I'[n]}$ be the corresponding submatrix. Note that any r-by-r contiguous submatrix of B must lie in some collection of r consecutive columns, hence be a submatrix of some $A_{[m]J}$. So any such submatrix of B must be TP; and applying Theorem 2.2 again, we deduce that B is TP. But since any square submatrix of A of size $\le r$ is a submatrix of some such matrix B, we conclude that A is TP_r. $\square$

Proof of Theorem 2.3. By Theorem 2.1, the submatrix $A_{[r][n]}$ consisting of the initial r rows of A is TP. Now let B be the submatrix of A obtained by deleting the first row. Every initial minor of size $\le r-1$ of B is either an initial minor of A or a minor of $A_{[r][n]}$, hence positive; and by hypothesis the contiguous minors of size r of B are positive. So by Theorem 2.1 again, the submatrix $B_{[r][n]}$ consisting of the initial r rows of B is TP. Continuing in this manner, we can show that every submatrix of A consisting of r consecutive rows is TP. Lemma 2.4 then implies that A is TP_r. $\square$

Our applications will in fact require only Lemma 2.4, not the stronger Theorem 2.3.

2.2 Density of total positivity within total nonnegativity

Since many important properties of TP matrices (like those in the preceding subsection) do not extend to TN matrices, it is very useful to be able to approximate TN matrices by TP matrices, or more generally to approximate TN_r matrices by TP_r matrices. It is a well-known fact [27, Theorem 2.6] that the m-by-n TP matrices are dense in the m-by-n TN matrices. The same proof establishes, *mutatis mutandis*, the corresponding result for TP_r and TN_r:

Theorem 2.5. *The set of m-by-n TP_r matrices is dense in the set of m-by-n TN_r matrices.*

Since the proof of [27, Theorem 2.6] preserves symmetry, we can also assert:

Theorem 2.6. *The set of n-by-n symmetric TP_r matrices is dense in the set of n-by-n symmetric TN_r matrices.*

The corresponding result for Hankel matrices will be proven in Corollary 3.4 below.

2.3 Total positivity of Hankel matrices

An m-by-n matrix $A = (a_{ij})_{1 \le i \le m,\, 1 \le j \le n}$ is said to be a *Hankel matrix* if a_{ij} depends only on $i+j$, i.e. $a_{ij} = a_{i'j'}$ whenever $i+j = i'+j'$. Hankel matrices are characterized combinatorially by the following simple but important fact:

Lemma 2.7. *For an m-by-n matrix A, the following are equivalent:*

(a) A is Hankel.

(b) Every contiguous submatrix of A is Hankel.

(c) Every contiguous submatrix of A is symmetric.

(d) Every contiguous 2-by-2 submatrix of A is symmetric.

Proof. (a) $\Longrightarrow$ (b): Let A be a Hankel matrix, and consider a contiguous submatrix A_{IJ} where $I = \{i, i+1, \ldots, i+k\}$ and $J = \{j, j+1, \ldots, j+k\}$. Then the (s,t) entry of A_{IJ} is $a_{i+s-1, j+t-1}$; and this equals the (s',t') entry whenever $s+t = s'+t'$, by virtue of the Hankel property of A.

(b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) is obvious.

(d) $\Longrightarrow$ (a): If every contiguous 2-by-2 submatrix of A is symmetric, we have $a_{i+1,j} = a_{i,j+1}$ for all $i \in [m-1]$ and $j \in [n-1]$. By combining these facts for different i, j, it is easily seen that A is Hankel. $\square$

In the remainder of this paper, we shall consider only *square* Hankel matrices (i.e. $m = n$).

In studying Hankel matrices it is often convenient to number the rows and columns from 0 to $n-1$ rather than from 1 to n, as this facilitates the connection with the Stieltjes moment problem. Thus, an n-by-n Hankel matrix is of the form $A = (a_{i+j})_{0 \le i,j \le n-1}$ for some sequence of numbers $a_0, a_1, \ldots, a_{2n-2}$. Assuming that $n \ge 2$, let us also define $A' = (a_{i+j+1})_{0 \le i,j \le n-2}$, i.e. A' is the $(n-1)$-by-$(n-1)$ submatrix of A in its upper right (or lower left) corner.

The conditions for a finite Hankel matrix to be TN or TP are slightly delicate, because they involve the theory of the truncated Stieltjes moment problem [7]. But the conditions for an infinite Hankel matrix to be TN or TP are quite simple, and this is all we shall need here; indeed, we shall need only the TP case. Given an infinite sequence $\boldsymbol{a} = (a_k)_{k=0}^{\infty}$ and an integer $m \ge 0$, let us define the m-shifted n-by-n Hankel matrix $H_n^{(m)}(\boldsymbol{a}) = (a_{i+j+m})_{0 \le i,j \le n-1}$ and the m-shifted infinite Hankel matrix $H_\infty^{(m)}(\boldsymbol{a}) = (a_{i+j+m})_{i,j \ge 0}$. We then have the following well-known result:

Theorem 2.8. *For a sequence $\boldsymbol{a} = (a_k)_{k=0}^{\infty}$ of real numbers, the following are equivalent:*

(a) $H_\infty^{(0)}(\boldsymbol{a})$ *is totally positive. [Equivalently,* $H_n^{(0)}(\boldsymbol{a})$ *is totally positive for all n.]*

(b) Both $H_\infty^{(0)}(\boldsymbol{a})$ *and* $H_\infty^{(1)}(\boldsymbol{a})$ *are positive-definite. [Equivalently,* $H_n^{(0)}(\boldsymbol{a})$ *and* $H_n^{(1)}(\boldsymbol{a})$ *are positive-definite for all n.]*

(c) The leading principal minors $\Delta_n = \det H_n^{(0)}(\boldsymbol{a})$ *and* $\Delta'_n = \det H_n^{(1)}(\boldsymbol{a})$ *are positive for all n.*

(d) There exists a positive measure μ on $[0, \infty)$*, whose support is an infinite set, such that* $a_k = \int x^k \, d\mu(x)$ *for all* $k \ge 0$.

Indeed, (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) is trivial, and (b) $\Longleftrightarrow$ (c) is Sylvester's criterion for positive-definiteness; (c) $\Longleftrightarrow$ (d) is the standard criterion for the Stieltjes moment problem [31, Theorem 1.3]; and (d) $\Longrightarrow$ (a) follows by a simple Vandermonde-matrix argument [15, p. 460, Théorème 9] [27, Theorem 4.4].

3 Sums

The sum of two TN_1 matrices is trivially TN_1, and the sum of a TN_1 matrix and a TP_1 matrix is trivially TP_1. But simple examples show that the sum of two TP matrices need not be TN_2, even if *one* of the two matrices is symmetric.

The sum of *two* symmetric 2-by-2 TN matrices is TN: the 1-by-1 minors are covered by the trivial argument, and the 2-by-2 determinant is nonnegative because the sum of two positive-semidefinite matrices is positive-semidefinite. (Since a 2-by-2 symmetric matrix is automatically Hankel, this result is a special case of Corollary 3.3 below.) But the corresponding assertion fails already for 3-by-3 symmetric matrices, even when one of the two matrices is Hankel:

Example 3.1. [3, p. 21] The symmetric matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and the Hankel matrix $J = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ are TN, but $I + J = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ is not even TN_2.

Moreover, as the reader can easily verify, these two input matrices can be perturbed slightly to make them TP while preserving the symmetry and the Hankel property. ■

But if *both* input matrices are Hankel, we have a positive result:

Theorem 3.2.

(a) Let A be a Hankel matrix, all of whose contiguous minors of size $\le r$ are nonnegative. Then A is TN_r.

(b) Let A and B be Hankel matrices, all of whose contiguous minors of size $\le r$ are nonnegative. Then $A+B$ is TN$_r$.

(c) Let A (resp. B) be a Hankel matrix, all of whose contiguous minors of size $\le r$ are nonnegative (resp. positive). Then $A+B$ is TP$_r$.

Proof. Let A be a Hankel matrix, all of whose contiguous minors of size $\le r$ are nonnegative, and let B be a Hankel matrix, all of whose contiguous minors of size $\le r$ are positive. Then [20, Theorem 7.2.5] every contiguous submatrix of size $\le r$ of A (resp. B) is positive-semidefinite (resp. positive-definite). It follows that every contiguous submatrix of size $\le r$ of $A+B$ is positive-definite, and hence in particular has a positive determinant. By Lemma 2.4, $A+B$ is TP$_r$. This proves (c).

But now, given A, we can take B to be any TP Hankel matrix: that is, by Theorem 2.8 we can take B to be the Hankel matrix associated to any Stieltjes moment sequence of infinite support (for instance, $a_k = k!$ or $a_k = \lambda^{k^2}$ with $\lambda > 1$). Then $A + \epsilon B$ is TP$_r$ for all $\epsilon > 0$, hence A is TN$_r$. This proves (a).

Finally, let A and B be Hankel matrices, all of whose contiguous minors of size $\le r$ are nonnegative. Then every contiguous submatrix of size $\le r$ of A or B is positive-semidefinite, so the same holds for $A+B$. Applying (a) to $A+B$, we obtain (b). $\square$

Corollary 3.3.

(a) The sum of two TN$_r$ Hankel matrices is TN$_r$.

(b) The sum of a TN$_r$ Hankel matrix and a TP$_r$ Hankel matrix is TP$_r$.

Corollary 3.4. *The TP$_r$ Hankel matrices are dense in the TN$_r$ Hankel matrices.*

Corollary 3.5. *Let $A = (a_{i+j})_{0\le i,j\le n-1}$ be an n-by-n Hankel matrix, and define $A' = (a_{i+j+1})_{0\le i,j\le n-2}$. Then A is TN$_r$ (resp. TP$_r$) if and only if every principal minor of A and A' of size $\le r$ is nonnegative (resp. positive). In particular, A is TN (resp. TP) if and only if both A and A' are positive-semidefinite (resp. positive-definite).*

Proof of Corollary 3.5. This is an immediate consequence of Theorem 3.2(a) (resp. Lemma 2.4) together with the observation that every contiguous minor of A is a principal minor of either A or A'. $\square$

Remarks. 1. In the same way that Theorem 2.3 improves Lemma 2.4, one might hope to improve Theorem 3.2 by weakening the hypothesis on contiguous minors of size $\le r$ to "initial minors of size $\le r-1$ and contiguous minors of size r". But it turns out that this does *not* hold in general. Consider, for any $n \ge 3$, the n-by-n matrix A having 1 in the upper-left corner, -1 in the lower-right corner, and zeros elsewhere. Then A is Hankel; all its initial minors of size 1 are either 0 or 1, all its contiguous minors of size 2 are zero, and all its minors of size ≥ 3 are zero; but A is not even TN_1, much less TN_r for some or all $r \in [2, n]$.

2. In a general partially ordered commutative ring, the sum of two TN Hankel matrices can *fail* to be TN, even if one of the two matrices is a matrix of pure numbers: for instance, in the polynomial ring $\mathbb{R}[x]$ with the coefficientwise order, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & x \\ x & x^2 \end{bmatrix}$ are TN, but $\det(A + B) = 1 - 2x + x^2$ fails to be coefficientwise nonnegative. See [32] for further discussion. ■

Finally, let us return to general matrices, and pose the following question: Which m-by-n matrices A have the property that $A+B$ is TN whenever B is TN? The answer is as follows:

Theorem 3.6. *Let $A = (a_{ij})$ be an m-by-n matrix. Then the following are equivalent:*

(a) $A + B$ is TN whenever B is TN.

(b) $A + B$ is TP whenever B is TP.

(c_r) $A + B$ is TN_r whenever B is TN_r. [Here $r \geq 2$.]

(d_r) $A + B$ is TP_r whenever B is TP_r. [Here $r \geq 2$.]

(e) $A + B$ is TN_2 whenever B is TN.

(f) $A + B$ is TN_2 whenever B is TP.

(g) The upper-left and lower-right elements of A are nonnegative, and all other elements are zero.

Proof. Suppose first that all the elements of A are zero except possibly a_{11} and a_{mn}, and consider an r-by-r minor $\det{(A + B)_{IJ}}$ according to whether it contains the first row and column, the last row and column, both, or neither. If neither, then obviously $\det{(A + B)_{IJ}} = \det B_{IJ}$. If the first but not the last, then $\det{(A + B)_{IJ}} = \det B_{IJ} + a_{11} \det B_{I\setminus 1, J\setminus 1}$. If the last but not the first, then $\det{(A + B)_{IJ}} = \det B_{IJ} + a_{mn} \det B_{I\setminus m, J\setminus n}$. And if both, then then $\det{(A + B)_{IJ}} = \det B_{IJ} + a_{11} \det B_{I\setminus 1, J\setminus 1} + a_{mn} \det B_{I\setminus m, J\setminus n} + a_{11}a_{mn} \det B_{I\setminus\{1,m\}, J\setminus\{1,n\}}$. It follows that (g) implies all the other statements.

Conversely, suppose that $A + B$ is TN_2 whenever B is TN. Taking $B = 0$, we conclude that A must be TN_2 and in particular all its entries must be nonnegative. Furthermore, if (i, j) is any entry other than $(1, 1)$ or (m, n), then we can choose (i', j') to be either $(i - 1, j + 1)$ or $(i + 1, j - 1)$ and let B be the matrix with $b_{i'j'} = \lambda$ and all other entries zero. Then B is TN whenever $\lambda \geq 0$, and we have $\det(A + B)_{\{i,i'\},\{j,j'\}} = \det A_{\{i,i'\},\{j,j'\}} - \lambda a_{ij}$. Taking $\lambda \to +\infty$ we conclude that $a_{ij} \leq 0$. This proves (e) $\Longrightarrow$ (g). Finally, (f) $\Longrightarrow$ (e) is an immediate consequence of Theorem 2.5. □

The matrices A characterized in Theorem 3.6 could be termed the "additive core" of the TN matrices, by analogy with the "Hadamard core" studied in [4].

4 Hadamard product

If $A = (a_{ij})$ and $B = (b_{ij})$ are two matrices of the same size (say, m-by-n), their *Hadamard product* (or *entrywise product*) $A \circ B$ is the matrix with elements $(A \circ B)_{ij} = a_{ij} b_{ij}$. See [19, Chapter 5] for further information on the properties of the Hadamard product.

The Hadamard product of two TN_1 (resp. TP_1) matrices is trivially TN_1 (resp. TP_1). Moreover, the TN_2 and TP_2 cases are handled by the following easy positive result:

Lemma 4.1.

(a) The Hadamard product of two TN_2 matrices is TN_2.

(b) The Hadamard product of a TP_2 matrix with a $TP_1 \cap TN_2$ matrix is TP_2.

However, in general the Hadamard product of two TP matrices need not even be TN_3, as was observed in [26, p. 163]. The precise situation is as follows.

First, an easy positive result concerning 3-by-3 *symmetric* matrices:

Proposition 4.2. *The Hadamard product of two* 3*-by-*3 *TN (resp. TP) symmetric matrices is TN (resp. TP).*

Proof. The 1-by-1 minors are trivially nonnegative (resp. positive). The 2-by-2 minors are nonnegative (resp. positive) by Lemma 4.1. The 3-by-3 determinant is nonnegative (resp. positive) by the Schur product theorem [20, Theorem 7.5.3]. □

But the corresponding result fails for 3-by-3 nonsymmetric matrices, and for 4-by-4 symmetric matrices, as the following two examples demonstrate:

Example 4.3. [4] The 3-by-3 matrices $W = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $W^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ are TN, but $W \circ W^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is not (its determinant is -1). Moreover, by a suitable small perturbation we can take the two starting matrices to be TP.

It also does not help to assume that *one* of the two matrices is symmetric or even Hankel, when the other is nonsymmetric: for instance, if $A = \begin{bmatrix} a_0 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ and W is as above, then A is TP whenever $a_0 > 5$, but $A \circ W$ is TN only when $a_0 \geq 6$.

Moreover, by "exterior bordering" [9, Theorem 9.0.1] there exist *arbitrarily large* TP matrices A since that $A \circ A^{\mathrm{T}}$ is not even TN_3. ■

Example 4.4. The 4-by-4 symmetric matrices

$$A = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \tag{4.1}$$

are TN, but their Hadamard product

$$A \circ B = \begin{bmatrix} 4 & 2 & 1 & 0 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 0 & 1 & 2 & 4 \end{bmatrix} \tag{4.2}$$

has several negative 3-by-3 minors (for instance, the upper right 3-by-3 submatrix has determinant -6). Moreover, by a suitable small perturbation (using Theorem 2.6) we can take the two starting matrices to be TP.

It also does not help to assume that *one* of the two symmetric matrices is Hankel, when the other is not. For instance, the Hankel matrix

$$H = \begin{bmatrix} a_0 & a_1 & 2 & 1 \\ a_1 & 2 & 1 & 2 \\ 2 & 1 & 2 & a_1 \\ 1 & 2 & a_1 & a_0 \end{bmatrix} \tag{4.3}$$

is TP whenever $a_1 > 7$ and $a_0 > a_1^2 - 4a_1 + 5$. But

$$A \circ H = \begin{bmatrix} 2a_0 & 2a_1 & 2 & 1 \\ 2a_1 & 4 & 1 & 2 \\ 2 & 1 & 4 & 2a_1 \\ 1 & 2 & 2a_1 & 2a_0 \end{bmatrix} \tag{4.4}$$

fails to be TN_3 whenever $a_1 < 4 + \frac{3}{2}\sqrt{5} \approx 7.354102$; and similarly $B \circ H$ fails to be TN_3 whenever $a_1 < 4 + 2\sqrt{3} \approx 7.464102$. ■

But if *both* input matrices are Hankel, we again have a positive result:

Theorem 4.5.

(a) The Hadamard product of two TN_r Hankel matrices is TN_r.

(b) The Hadamard product of two TP_r Hankel matrices is TP_r.

PROOF. Suppose that A and B are TN_r (resp. TP_r) Hankel matrices. Then all their contiguous submatrices of size $\le r$ are positive-semidefinite (resp. positive-definite). Therefore, by the Schur product theorem [20, Theorem 7.5.3], all the contiguous submatrices of $A \circ B$ of size $\le r$ are positive-semidefinite (resp. positive-definite) and in particular have a nonnegative (resp. positive) determinant. By Theorem 3.2(a) (resp.

Lemma 2.4) it follows that $A \circ B$ is TN_r (resp. TP_r). [Alternatively, one could first prove (b) and then invoke Corollary 3.4 to deduce (a).] $\square$

Remarks. 1. In a general partially ordered commutative ring — for instance, in the polynomial ring $\mathbb{R}[x]$ with the coefficientwise order — the Hadamard product of two TN Hankel matrices can *fail* to be TN, even if one of the two matrices is a matrix of pure numbers. Furthermore, the Hadamard square of a TN Hankel matrix can fail to be TN. See [32] for details.

2. Many further results concerning total positivity and the Hadamard product can be found in [9, Chapter 8]. For instance, the Hadamard product of a TN matrix and a *tridiagonal* TN matrix is TN [9, Theorem 8.2.5]; and this result extends to TN_r, by the same proof. $\blacksquare$

5 Hadamard powers

If $A = (a_{ij})$ is a matrix and $t > 0$ is an integer, the *Hadamard power* (or *entrywise power*) $A^{\circ t}$ is defined to be the matrix with elements $(A^{\circ t})_{ij} = a_{ij}^t$. Moreover, if the matrix A has *nonnegative* real entries — as we shall assume henceforth — we can make this same definition for arbitrary *real* powers $t > 0$.

Note that each minor of $A^{\circ t}$ is an exponential polynomial $f(t) = \sum_{i=1}^{n} a_i \, e^{\lambda_i t}$, where we can assume that $a_1, \ldots, a_n$ are real and nonzero and $\lambda_1 < \ldots < \lambda_n$. Laguerre's rule of signs [25] [28, pp. 46–47, Problem V.77] [22] then states that the number of real zeros of f (counting multiplicity) is at most the number of sign changes in the sequence $a_1, \ldots, a_n$, and is also of the same parity.

If A is TN_1 (resp. TP_1), then trivially so is $A^{\circ t}$ for all real $t > 0$. Moreover, the following result is almost trivial:

Proposition 5.1. *If A is TN_2 (resp. TP_2), then so is $A^{\circ t}$ for all real $t > 0$.*

Less trivially, the $\mathrm{TN}_3/\mathrm{TP}_3$ case is handled by the following result [23, Theorem 4.2] (see also [9, pp. 179–180]):

Theorem 5.2. *If A is TN_3 (resp. TP_3), then so is $A^{\circ t}$ for all real $t \geq 1$.*

Here is a slightly simplified proof:

PROOF. It obviously suffices to prove the result for 3-by-3 matrices; and it suffices to prove the TP_3 case, since the TN_3 case then follows by Theorem 2.5. By row and column rescalings it suffices to consider $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & b \\ 1 & c & d \end{bmatrix}$. Such a matrix is TP_2 if and only if $a > 1$, $b > a$, $c > a$ and $ad > bc$. Then $\det(A^{\circ t}) = (ad)^t - d^t - (bc)^t + b^t + c^t - a^t$, which equals $(\log a \log d - \log b \log c)t^2 + O(t^3)$ near $t = 0$ and hence has at least a double root there; moreover, $\det(A^{\circ t}) \to +\infty$ as $t \to +\infty$. By Laguerre's rule of signs, $f(t) = \det(A^{\circ t})$ has precisely three real roots (note that the unknown ordering of b and c plays no role here because their coefficients in f have the same sign; likewise for d

and bc). And regardless of whether the third root lies at $t < 0$, $t = 0$ or $t > 0$ (which depends on the sign of $\log a \log d - \log b \log c$), $f(1) > 0$ implies $f(t) > 0$ for all $t > 1$. $\square$

The following example shows that Theorem 5.2 does not extend to $0 < t < 1$, even if the matrix is assumed to be Hankel:

Example 5.3. The matrix $A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 5 & 9 & 17 \end{bmatrix}$ is the 3-by-3 Hankel matrix associated to the Stieltjes moment sequence $a_n = 1^n + 2^n$ $(n \geq 0)$: all the 1-by-1 and 2-by-2 minors are positive, and $\det A = 0$. But

$$\det(A^{\circ t}) \;=\; 170^t - 162^t - 153^t + 2 \cdot 135^t - 125^t \;, \tag{5.1}$$

which is strictly negative for $0 < t < 1$.[2]

Moreover, by a small perturbation (using Corollary 3.4) we can make the matrix Hankel and TP and have $\det(A^{\circ t}) < 0$ for $\delta < t < 1 - \delta$, for arbitrarily small $\delta > 0$. $\blacksquare$

Let us now present some positive results for symmetric and Hankel matrices. Our key tool will be the following [11, Theorem 2.2]:

Theorem 5.4. [11] *Let $n \geq 2$, and let A be a symmetric positive-semidefinite (resp. positive-definite) n-by-n matrix with nonnegative real entries. Then, for all real $t \geq n-2$ ($t > 0$ if $n = 2$), the Hadamard power $A^{\circ t}$ is positive-semidefinite (resp. positive-definite).*

The non-strict ("positive-semidefinite") version of this result was proven in [11].[3] By a perturbation argument one can then deduce the strict version.[4]

Let us observe that the bound $t \geq n - 2$ in Theorem 5.4 cannot be improved, even if the matrix is Hankel and TP, as the following example shows:

Example 5.5. Let $n \geq 2$ and $u_1, \ldots, u_n \in \mathbb{R}$, and define for each $\epsilon > 0$ the n-by-n matrix $A_n(\epsilon) = (1 + \epsilon u_i u_j)_{i,j=1}^n$. This matrix is symmetric and positive-semidefinite; if the $u_1, \ldots, u_n$ are not all equal, it is of rank 2 (otherwise it is of rank 1); and if

[2] Proof: For $f(t) = \det(A^{\circ t})$, easy computations show that $t = 0$ is a double root and $t = 1$ is a simple root. Since the coefficient sequence has three sign changes, Laguerre's rule of signs implies that f has no other real zeros. Moreover, straightforward computations show that $f(t) < 0$ for t slightly less than 1 and for t slightly greater than 0; therefore $f(t) < 0$ for all $t \in (0, 1)$.

[3] **Warning:** In [11] the term "positive definite" is used for what is ordinarily called "positive-semidefinite".

[4] Given a positive-definite matrix A, choose $\epsilon > 0$ such that $A - \epsilon I$ is positive-semidefinite. Then $A^{\circ t} = (A - \epsilon I)^{\circ t} + D$ where $D = \operatorname{diag}\big(a_{ii}^t - (a_{ii} - \epsilon)^t\big)$ is positive-definite.

$0 \le u_1 < u_2 < \ldots < u_n$, then it is also TN $\cap$ TP$_2$. A straightforward computation (see Appendix A) shows that

$$\det A_n(\epsilon)^{\circ t} \;=\; \Biggl(\prod_{k=1}^{n-1} \frac{1}{k!}\Biggr)\Biggl(\prod_{1\le i<j\le n} (u_i-u_j)^2\Biggr)\Biggl(\prod_{k=0}^{n-2} (t-k)^{n-1-k}\Biggr)\epsilon^{n(n-1)/2} + O(\epsilon^{n(n-1)/2\,+1})\;. \tag{5.2}$$

Therefore, if $n \ge 3$ and the $u_1, \ldots, u_n$ are all distinct, then for any $t \in (n-3, n-2)$ we have $\det A_n(\epsilon)^{\circ t} < 0$ for all sufficiently small $\epsilon > 0$. More generally, if $t \in (m-3, m-2)$ for some integer $m \in [3, n]$, then $A_m(\epsilon)^{\circ t}$ — which is a leading principal submatrix of $A_n(\epsilon)^{\circ t}$ — has a negative determinant for small $\epsilon > 0$. So, for all noninteger $t \in (0, n-2)$, $A_n(\epsilon)^{\circ t}$ fails to be positive-semidefinite for small $\epsilon > 0$ (how small may depend on t).

In [11, p. 636] the authors chose $u_i = i$ and proved the failure of positive-semidefiniteness for noninteger $t \in (0, n-2)$ and small ϵ by a different method (computing the inner product $\mathbf{x}^{\mathrm{T}} A_n(\epsilon)^{\circ t}\,\mathbf{x}$ in power series in ϵ for a suitably chosen vector $\mathbf{x} \in \mathbb{R}^n$).

On the other hand, if we choose $u_i = 2^{i-1}$, then the matrix $A_n(\epsilon)$ is Hankel and TN $\cap$ TP$_2$. Moreover, by a small perturbation we can make the matrix Hankel and TP (by Corollary 3.4). Therefore, for each $n \ge 3$ and each noninteger $t \in (0, n-2)$, there exists an n-by-n TP Hankel matrix A such that one of the leading principal minors of $A^{\circ t}$ is negative, and in particular $A^{\circ t}$ fails to be positive-semidefinite. ■

Returning now to TN and TP for Hadamard powers, we have the following positive result for 4-by-4 *symmetric* matrices:

Proposition 5.6. *If A is a 4-by-4 TN (resp. TP) symmetric matrix, then so is $A^{\circ t}$ for all real $t \ge 2$.*

PROOF. The 1-by-1 minors are handled by the trivial argument; the 2-by-2 minors are handled by Proposition 5.1; the 3-by-3 minors are handled by Theorem 5.2; and the 4-by-4 determinant is handled by Theorem 5.4. □

Example 5.5 shows that Proposition 5.6 cannot be extended to any $t \in (0,1)\cup(1,2)$, even if the matrix is Hankel and TP. It follows, in the language of [23], that the Hadamard critical exponent for 4-by-4 *symmetric* TN or TP matrices is 2.

But for 4-by-4 nonsymmetric matrices, and for 5-by-5 symmetric matrices, even the Hadamard square ($t = 2$) does not in general preserve total nonnegativity, as we now proceed to show.

In [8, Example 1] an example was given of a 4-by-4 TP matrix whose Hadamard square has a negative determinant:

$$A \;=\; \begin{bmatrix} 1 & 11 & 22 & 20 \\ 6 & 67 & 139 & 140 \\ 126 & 182 & 395 & 445 \\ 12 & 138 & 309 & 376 \end{bmatrix}\;, \tag{5.3}$$

which has $\det(A\circ A) = -114904113$. Here is another example of the same phenomenon, in which moreover $\det(A^{\circ t}) < 0$ also for real $t > 1$:

Example 5.7. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+x & 1+2x & 1+3x \\ 1 & 1+2x & 1+4x & 1+6x \\ 1 & 1+3x & 1+8x & 1+14x \end{bmatrix}, \tag{5.4}$$

which fails to be symmetric only because $a_{34} \neq a_{43}$. All the 2-by-2 minors are of the form $ax+bx^2$ with $a > 0$ and $b \geq 0$; all the 3-by-3 minors are of the form cx^2 with $c \geq 0$; and $\det(A) = 0$. So A is coefficientwise TN in the polynomial ring $\mathbb{R}[x]$; in particular, it is TN for all $x \geq 0$. But $\det(A \circ A) = -16x^4 + 248x^5$, so $\det(A \circ A) < 0$ whenever $0 < x < 2/31$. Furthermore, for small x we have $\det(A^{\circ t}) = 2(t^3 - t^4)x^4 + O(x^5)$; so for every real $t > 1$ there exists $\delta_t > 0$ such that $\det(A^{\circ t}) < 0$ whenever $0 < x < \delta_t$. ■

And by perturbing a few coefficients, TN can be upgraded to TP:

Example 5.8. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+x & 1+2x & 1+3x \\ 1 & 1+2x & 1+(4+\epsilon)x & 1+(6+\frac{5}{2}\epsilon)x \\ 1 & 1+3x & 1+8x & 1+(14+\epsilon)x \end{bmatrix} \tag{5.5}$$

with $0 < \epsilon < 1$. All the 2-by-2 minors are again of the form $ax + bx^2$ with $a > 0$ and $b \geq 0$; but now all the 3-by-3 minors are of the form cx^2 with $c > 0$, and $\det A = \epsilon^2 x^3$. So A is TP for all $x > 0$ and $\epsilon \in (0, 1)$. And for small x,

$$\det(A^{\circ t}) = \epsilon^2 t^3 x^3 + \frac{1}{4}(8 - 70\epsilon - 59\epsilon^2 - 4\epsilon^3)(t^3 - t^4)x^4 + O(x^5). \tag{5.6}$$

Therefore, for every real $t > 1$ and small enough $\epsilon > 0$ (depending on t) there exists a nonempty interval of $x > 0$ such that $\det(A^{\circ t}) < 0$. ■

Examples 5.7 and 5.8 answer an open question from [23, p. 81], by showing that the Hadamard critical exponent for 4-by-4 TN or TP matrices is ∞. Moreover, by "exterior bordering" [9, Theorem 9.0.1] there exists, for any $n \geq 4$ and $t > 1$, a TP n-by-n matrix A such that $A^{\circ t}$ is not even TN_4. Therefore, for all $n \geq 4$ and $r \geq 4$, the Hadamard critical exponent for n-by-n TN_r or TP_r matrices is also ∞.

And here is an example that is almost Hankel:

Example 5.9. Consider the matrix

$$A = \begin{bmatrix} 1+3x & 1+6x & 1+14x & 1+36x \\ 1+6x & 1+14x & 1+36x & 1+98x \\ 1+14x & 1+36x & 1+98x & 1+276x \\ 1+36x & 1+98x & 1+284x & 1+842x \end{bmatrix}. \tag{5.7}$$

All the 2-by-2 minors are of the form $ax + bx^2$ with $a, b > 0$; all the 3-by-3 minors are of the form $cx^2 + dx^3$ with $c, d > 0$; and $\det(A) = 0$. So A is coefficientwise

TN in the polynomial ring $\mathbb{R}[x]$ and is TN for all $x \geq 0$. But for small x we have $\det(A^{\circ t}) = 28584(t^3 - t^4)x^4 + O(x^5)$; so for every real $t > 1$ there exists $\delta_t > 0$ such that $\det(A^{\circ t}) < 0$ whenever $0 < x < \delta_t$.

Note how this example is constructed: we start from the 4-by-4 Hankel matrix $(1 + \alpha_{i+j}x)_{0 \leq i,j \leq 3}$ associated to the Stieltjes moment sequence $\alpha_n = 1^n + 2^n + 3^n$, which is coefficientwise TN in $\mathbb{R}[x]$ by a general result in [32]; we then modify this matrix by changing a_{32} and a_{33}.

By replacing 842 by $842 + \epsilon$ in the lower-right matrix entry, TN can be upgraded to TP analogously to Example 5.8. ■

We can now exhibit a 5-by-5 symmetric TP matrix whose Hadamard powers with $t > 1$ fail to be TN_4; indeed, we will choose this 5-by-5 symmetric matrix so that the 4-by-4 submatrix in its upper right corner is precisely the almost-Hankel matrix of Example 5.9:

Example 5.10. Consider the matrix

$$A \;=\; \begin{bmatrix} 1+2x & 1+3x & 1+6x & 1+14x & 1+36x \\ 1+3x & 1+6x & 1+14x & 1+36x & 1+98x \\ 1+6x & 1+14x & 1+36x & 1+98x & 1+276x \\ 1+14x & 1+36x & 1+98x & 1+284x & 1+842x \\ 1+36x & 1+98x & 1+276x & 1+842x & 1+2604x \end{bmatrix} . \tag{5.8}$$

All the 2-by-2 minors are of the form $ax + bx^2$ with $a, b > 0$; all the 3-by-3 minors are of the form $cx^2 + dx^3$ with $c, d > 0$; all the 4-by-4 minors are of the form $ex^3 + fx^4$ with $e, f \geq 0$; and $\det(A) = 0$. So A is coefficientwise TN in $\mathbb{R}[x]$ and is TN for all $x \geq 0$. But the 4-by-4 upper-right submatrix of $A^{\circ t}$ has a negative determinant in the circumstances discussed in Example 5.9. (Once again, TN can be upgraded to TP by a small perturbation.) ■

Example 5.10 shows that the Hadamard critical exponent for 5-by-5 symmetric TN or TP matrices is ∞; and by "exterior bordering" the same result holds for n-by-n symmetric TN_r or TP_r matrices for all $n \geq 5$ and $r \geq 4$.

Finally, for Hankel matrices we have the following positive result:

Theorem 5.11. *For every integer $r \geq 3$: If A is a TN_r (resp. TP_r) Hankel matrix, then so is $A^{\circ t}$ for all real $t \geq r - 2$.*

Proof. By Theorem 3.2(a) (resp. Lemma 2.4), it suffices to show that all the contiguous minors of $A^{\circ t}$ of size $\leq r$ are nonnegative (resp. positive). Since all the contiguous submatrices of A of size $\leq r$ are symmetric and positive-semidefinite (resp. positive-definite), this is an immediate consequence of Theorem 5.4. □

Example 5.5 shows that for every $n \geq r$ and every noninteger $t \in (0, r-2)$, there is an n-by-n TP Hankel matrix A such that one of the leading principal minors of $A^{\circ t}$ of size $\leq r$ is negative. So the bound $t \geq r - 2$ in Theorem 5.11 cannot be improved.

In other words, the Hadamard critical exponent for TN_r or TP_r Hankel matrices is exactly $r-2$.

Final remark. Most of the counterexamples in this paper were found by applying MATHEMATICA's function `FindInstance` to a suitably chosen Ansatz, sometimes followed by experimentation to find a simpler "nearby" example. Since in practice this works (with present-day hardware and software) only if the Ansatz has at most three or four parameters, considerable trial and error was sometimes needed to find a suitable Ansatz.

A Proof of equation (5.2)

Let $n \geq 2$ and $u_1, \ldots, u_n \in \mathbb{C}$, and set $M = \max\limits_{1 \leq i \leq n} |u_i|$. We consider the n-by-n matrix $A_n(\epsilon) = (1+\epsilon u_i u_j)_{i,j=1}^n$ for $\epsilon \in \mathbb{C}$. The binomial series for $(1+\epsilon u_i u_j)^t$ is convergent for $|\epsilon| < 1/M^2$ and yields

$$A_n(\epsilon)^{\circ t} \;=\; \sum_{k=0}^{\infty} \epsilon^k \, \frac{t^{\underline{k}}}{k!} \, \mathbf{u}^{[k]} \mathbf{u}^{[k]\mathrm{T}} \tag{A.1}$$

where $t^{\underline{k}} = t(t-1)\cdots(t-k+1)$ and $\mathbf{u}^{[k]} = (u_1^k, \ldots, u_n^k)$. We therefore have $A_n(\epsilon)^{\circ t} = VDV^{\mathrm{T}}$ where $V = (u_i^j)_{1 \leq i \leq n,\, j \geq 0}$ is the n-by-∞ Vandermonde matrix, and $D = \mathrm{diag}(\epsilon^j t^{\underline{j}}/j!)_{j \geq 0}$ is a diagonal matrix. The Cauchy–Binet formula then gives

$$\det A_n(\epsilon)^{\circ t} \;=\; \sum_J (\det V_{[n]J})^2 \prod_{j \in J} \frac{\epsilon^j t^{\underline{j}}}{j!} \;, \tag{A.2}$$

where the sum runs over n-element subsets $J \subseteq \mathbb{N}$. The smallest power of ϵ comes from $J = \{0, 1, \ldots, n-1\}$ and in this case $\det V_{[n]J}$ is the Vandermonde determinant $\prod\limits_{1 \leq i < j \leq n} (u_j - u_i)$. All other terms contribute higher powers of ϵ (and the coefficients are generalized Vandermonde determinants). This proves (5.2).

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