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Zariski closures and subgroup separability

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Abstract The main result of this article is a refinement of the well-known subgroup separability results of Hall and Scott for free and surface groups. We show that for any finitely generated subgroup, there is a finite dimensional representation of the free or surface group that separates the subgroup in the induced Zariski topology. As a corollary, we establish a polynomial upper bound on the size of the quotients used to separate a finitely generated subgroup in a free or surface group.

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1 Introduction

Given an algebraically closed field Ω, a finite dimensional Ω-vector space V, a finitely generated group Γ, and a homomorphism $\rho\colon \Gamma \to \mathrm{GL}(V)$, we have the subspace topology on $\rho(\Gamma)$ coming from the Zariski topology on $\mathrm{GL}(V) < \mathrm{End}(V)$. The pullback of this topology to Γ under ρ is called the Zariski topology associated to ρ. The primary goal of this article is to establish separability properties for Γ by using Zariski topologies associated to finite dimensional representations. The foundational

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result was established by Mal'cev [17] who proved that if $\rho\colon \Gamma \to \mathrm{GL}(V)$ is injective (e.g. Γ is linear), then Γ is residually finite.

We say Γ is **subgroup separable** (also called **LERF**) if every finitely generated subgroup is closed in the profinite topology. Our main result shows that finitely generated subgroups of free and surface groups can be separated in the Zariski topology associated to a representation that depends on the subgroup.

Theorem 1.1 *Let Γ be a free group with rank $r > 1$ or the fundamental group of a closed surface Σ_g with genus $g > 1$. If Δ_0 is a finitely generated subgroup of Γ, then there exists a faithful representation $\rho_{\Delta_0}\colon \Gamma \to \mathrm{GL}(V)$ such that $\overline{\rho_{\Delta_0}(\Delta_0)} \cap \rho_{\Delta_0}(\Gamma) = \rho_{\Delta_0}(\Delta_0)$, where $\overline{\rho_{\Delta_0}(\Delta_0)}$ is the Zariski closure of $\rho_{\Delta_0}(\Delta_0)$. That is, Δ_0 is closed in the Zariski topology associated to ρ_{Δ_0}.*

If a finitely generated subgroup Δ_0 is Zariski closed in the sense above and $\gamma \notin \Delta_0$, then there is a homomorphism $\varphi\colon \Gamma \to Q$ such that $|Q| < \infty$ and $\varphi(\gamma) \notin \varphi(\Delta_0)$. Letting $\Lambda = \Delta_0 \cdot \ker(\varphi) < \Gamma$, we see that Λ is a finite index subgroup of Γ of index at most $|Q|$ with $\Delta_0 \leq \Lambda$ and $\gamma \notin \Lambda$.

Corollary 1.2 *Let Γ be a free group with rank $r > 1$ or the fundamental group of a closed surface Σ_g with genus $g > 1$, and let $\mathfrak{X}$ be a finite generating set for Γ with $||\cdot||_{\mathfrak{X}}$ the associated norm. If $\Delta_0 < \Gamma$ is a finitely generated subgroup, then there exists a constant $D > 0$ such that for each $\gamma \in \Gamma - \Delta_0$, there exists a homomorphism $\varphi\colon \Gamma \to Q$ with $\varphi(\gamma) \notin \varphi(\Delta_0)$ and $|Q| \leq ||\gamma||_{\mathfrak{X}}^D$. Letting $\Lambda = \Delta_0 \cdot \ker(\varphi)$, Λ is a finite index subgroup of Γ, of index at most $|Q| \leq ||\gamma||_{\mathfrak{X}}^D$, such that $\Delta_0 \leq \Lambda$ and $\gamma \notin \Lambda$. Moreover, the index of the normal core of the subgroup Λ is bounded above by $|Q|$.*

Deducing Corollary 1.2 from Theorem 1.1 is straightforward and uses methods from [6]. The constant D explicitly depends on the subgroup Δ_0 and the dimension of V in Theorem 1.1. For a general finite index subgroup, the crude upper bound for the index of the normal core is factorial in the index of the subgroup. It is for this reason that we include the statement regarding the normal core of Λ at the end of Corollary 1.2.

Recently, several effective separability results have been established; see [2–6,8,9, 12–15,20–23,26]. Most relevant here are the papers [9,20] where bounds on the index of the separating subgroups for free and surface groups given. We compare the bounds of Corollary 1.2 to the results in [9,20] in Sect. 6.

2 Preliminaries

Complex algebraic groups. Given a complex algebraic group $\mathbf{G} < \mathrm{GL}(n, \mathbf{C})$, there exist polynomials $Q_1, \dots, Q_r \in \mathbf{C}[X_{i,j}]$ such that

$$\mathbf{G} = \mathbf{G}(\mathbf{C}) = V(Q_1, \dots, Q_r) = \left\{ X \in \mathbf{C}^{n^2} \ : \ Q_k(X) = 0,\ k = 1, \dots, r \right\}.$$

We refer to the polynomials $Q_1, \dots, Q_r$ as defining polynomials for $\mathbf{G}$. We will say that $\mathbf{G}$ is K-defined for a subfield $K \subset \mathbf{C}$ if there exists defining polynomials

$Q_1, \dots, Q_r \in K[X_{i,j}]$ for $\mathbf{G}$. For a complex affine algebraic subgroup $\mathbf{H} < \mathbf{G} < \mathrm{GL}(n, \mathbf{C})$, we will pick the defining polynomials for $\mathbf{H}$ to contain a defining set for $\mathbf{G}$ as a subset. Specifically, we have polynomials $Q_1, \dots, Q_{r_\mathbf{G}}, Q_{r_\mathbf{G}+1}, \dots, Q_{r_\mathbf{H}} \in \mathbf{C}[X_{i,j}]$ such that

$$\mathbf{H} = V(Q_1, \dots, Q_{r_\mathbf{H}}), \quad \mathbf{G} = V(Q_1, \dots, Q_{r_\mathbf{G}}). \tag{1}$$

If $\mathbf{G}$ is defined over a number field K with associated ring of integers $\mathscr{O}_K$, we can find polynomials $Q_1, \dots, Q_r \in \mathscr{O}_K[X_{i,j}]$ as a defining set by clearing denominators. In the case when $K = \mathbf{Q}$ and $\mathscr{O}_K = \mathbf{Z}$, these are multivariable integer polynomials.

Spaces of representations. For a fixed finite set $\mathfrak{X} = \{x_j\}_{j=1}^t$ with associated free group $F(\mathfrak{X})$ and any group G, the set of homomorphisms from $F(\mathfrak{X})$ to G, denoted by $\mathrm{Hom}(F(\mathfrak{X}), G)$, can be identified with G^t. For any point $(g_1, \dots, g_t) \in G^t$, we have an associated homomorphism $\varphi_{(g_1,\dots,g_t)} \colon F(\mathfrak{X}) \to G$ given by $\varphi_{(g_1,\dots,g_t)}(x_i) = g_i$. For any word $w \in F(\mathfrak{X})$, we have a function $\mathrm{Eval}_w \colon \mathrm{Hom}(F(\mathfrak{X}), G) \to G$ defined by $\mathrm{Eval}_w(\varphi_{(g_1,\dots,g_t)}) = \varphi_{(g_1,\dots,g_t)}(w) = w(g_1, \dots, g_t)$. For a finitely presented group Γ, we fix a finite presentation $\langle \mathfrak{X}; \mathfrak{R} \rangle$ where $\mathfrak{X} = \{\gamma_1, \dots, \gamma_t\}$ is a generating set (as a monoid) and $\mathfrak{R} = \{r_1, \dots, r_{t'}\}$ is a finite set of relations. If $\mathbf{G}$ is a complex affine algebraic subgroup of $\mathrm{GL}(n, \mathbf{C})$, the set $\mathrm{Hom}(\Gamma, \mathbf{G})$ of homomorphisms $\rho \colon \Gamma \to \mathbf{G}$ can be identified with an affine algebraic subvarieity of $\mathbf{G}^t$. Specifically

$$\mathrm{Hom}(\Gamma, \mathbf{G}) = \left\{(g_1, \dots, g_t) \in \mathbf{G}^t \ : \ r_j(g_1, \dots, g_t) = I_n \text{ for all } j\right\}. \tag{2}$$

If Γ is finitely generated, $\mathrm{Hom}(\Gamma, \mathbf{G})$ is an affine algebraic variety by the Hilbert Basis Theorem.

$\mathrm{Hom}(\Gamma, \mathbf{G})$ also has a topology induced by the analytic topology on $\mathbf{G}^t$. There is a Zariski open subset of $\mathrm{Hom}(\Gamma, \mathbf{G})$ that is smooth in the this topology called the smooth locus, and the functions $\mathrm{Eval}_\gamma \colon \mathrm{Hom}(\Gamma, \mathbf{G}) \to \mathbf{G}$ are analytic on the smooth locus. For any subset $S \subset \Gamma$ and representation $\rho \in \mathrm{Hom}(\Gamma, \mathbf{G})$, $\overline{\rho(S)}$ will denote the Zariski closure of $\rho(S)$ in $\mathbf{G}$.

Effective separability functions. For a finitely generated group Γ with a fixed finite generating set $\mathfrak{X}$, we denote the associated norm by $||\cdot||_\mathfrak{X}$. Given a subgroup $\Delta_0 < \Gamma$ and $\gamma \in \Gamma - \Delta_0$, we define

$$\mathrm{D}_\Gamma(\Delta_0, \gamma) = \min\{[\Gamma : \Lambda] \ : \ \Delta_0 < \Lambda, \ \gamma \notin \Lambda\}.$$

When Δ_0 is separable in Γ, $\mathrm{D}(\Delta_0, \gamma) < \infty$ for all $\gamma \in \Gamma - \Delta_0$. The maximal value of $\mathrm{D}(\Delta_0, \gamma)$ ranging over all $\gamma \in \Gamma - \Delta_0$ with $||\gamma||_\mathfrak{X} \leq m$ will be denoted by $\mathbf{Sep}_\Gamma(\Delta_0, m)$. Note that in [9], $\mathrm{D}_\Gamma(\Delta_0, \gamma)$ and $\mathbf{Sep}_\Gamma(\Delta_0, m)$ are denoted by $D_\Gamma^{\Omega_{\Delta_0}}(\gamma)$ and $\mathbf{Sep}_{\Gamma,\mathfrak{X}}(\Delta_0, m)$, respectively.

Recall that for a pair of functions $f_1, f_2 \colon \mathbf{N} \to \mathbf{N}$, we say $f_1 \preceq f_2$ if there exists a constant $C > 0$ such that $f_1(m) \leq C f_2(Cm)$ for all m. When $f_1 \preceq f_2$ and $f_2 \preceq f_1$, we write $f_1 \approx f_2$. The function $\mathbf{Sep}_\Gamma(\Delta_0, m)$ above depends on the choice of the generating set $\mathfrak{X}$. However, it is straightforward to see verify that $\mathbf{Sep}_{\Gamma,\mathfrak{X}}(\Delta_0, m) \approx \mathbf{Sep}_{\Gamma,\mathfrak{X}'}(\Delta_0, m)$ holds for any finite generating sets $\mathfrak{X}, \mathfrak{X}'$ of Γ. We will suppress any dependence of the function $\mathbf{Sep}_\Gamma(\Delta_0, m)$ on the generating set $\mathfrak{X}$.

3 Evaluation maps

Throughout this section, Γ will be a finitely generated group and Δ_0 a finitely generated subgroup of Γ. For a complex affine algebraic group $\mathbf{G}$ and any representation $\rho_0 \in \mathrm{Hom}(\Gamma, \mathbf{G})$, we have the closed affine subvariety

$$\mathscr{R}_{\rho_0,\Delta_0}(\Gamma, \mathbf{G}) = \{\rho \in \mathrm{Hom}(\Gamma, \mathbf{G}) \ : \ \rho_0(\delta) = \rho(\delta) \text{ for all } \delta \in \Delta_0\}.$$

We say that ρ_0 **distinguishes** Δ_0 from γ if the restriction of Eval_γ to $\mathscr{R}_{\rho_0,\Delta_0}(\Gamma, \Delta)$ is non-constant, that is to say, there exists $\rho \in \mathrm{Hom}(\Gamma, G)$ such that $\rho|_{\Delta_0} = \rho_0$ and $\rho(\gamma) \neq \rho_0(\gamma)$. We say that ρ_0 **weakly distinguishes** Δ_0 in Γ, if ρ_0 distinguishes Δ_0 from γ for all $\gamma \in \Gamma - \Delta_0$. We say that ρ_0 **distinguishes** Δ_0 in Γ if for each finite set $S \subset \Gamma - \Delta_0$, there are $\rho, \rho' \in \mathscr{R}_{\rho_0,\Delta_0}(\Gamma, \mathbf{G})$ such that $\mathrm{Eval}_\gamma(\rho) \neq \mathrm{Eval}_\gamma(\rho')$ for all $\gamma \in S$. Finally, we say that ρ_0 **strongly distinguishes** Δ_0 in Γ if there are $\rho, \rho' \in \mathscr{R}_{\rho_0,\Delta_0}(\Gamma, \mathbf{G})$ such that $\rho(\gamma) \neq \rho'(\gamma)$ for all $\gamma \in \Gamma - \Delta_0$.

Lemma 3.1 *Let Γ be a finitely generated group, $\mathbf{G}$ is a complex algebraic group, and Δ_0 a finitely generated subgroup of Γ. If Δ_0 is strongly distinguished by a representation $\rho_0 \in \mathrm{Hom}(\Gamma, \mathbf{G})$, then there exists a representation $\Phi\colon \Gamma \to \mathbf{G} \times \mathbf{G}$ such that $\Phi(\Gamma) \cap \overline{\Phi(\Delta_0)} = \Phi(\Delta_0)$, where $\overline{\Phi(\Delta_0)}$ is the Zariski closure of $\Phi(\Delta_0)$ in $\mathbf{G} \times \mathbf{G}$.*

Proof By definition, there are representations $\rho, \rho' \in \mathscr{R}_{\rho_0,\Delta_0}(\Gamma, \mathbf{G})$ such that $\gamma(\rho) \neq \gamma(\rho')$ for all $\gamma \in \Gamma - \Delta_0$. Take $\Phi\colon \Gamma \to \mathbf{G} \times \mathbf{G}$ given by $\Phi = \rho \times \rho'$. By construction, $\Phi(\Delta_0) < \mathrm{Diag}(\mathbf{G})$ and $\Phi(\gamma) \notin \mathrm{Diag}(\mathbf{G})$ for all $\gamma \in \Gamma - \Delta_0$. In particular, $\overline{\Phi(\Delta_0)} < \mathrm{Diag}(\mathbf{G})$ since $\mathrm{Diag}(\mathbf{G})$ is Zariski closed. Hence, $\Phi(\Delta_0) = \overline{\Phi(\Delta_0)} \cap \Phi(\Gamma)$. □

Lemma 3.2 *Let Γ be a finitely generated group, $\mathbf{G}$ a complex algebraic group, and Δ_0 a finitely generated subgroup of Γ. If Δ_0 is distinguished by a representation $\rho_0 \in \mathrm{Hom}(\Gamma, \mathbf{G})$, then ρ_0 strongly distinguishes Δ_0.*

Proof We order $\Gamma - \Delta_0 = \{\gamma_1, \gamma_2, \ldots\}$ and for each $j \in \mathbf{N}$, define $S_j = \{\gamma_i\}_{i=1}^{j}$. As ρ_0 distinguishes Δ_0, for each $j \in \mathbf{N}$, there exists $\rho_j \in \mathrm{Hom}(\Gamma, \mathbf{G})$ such that $\rho_j(\delta) = \rho_0(\delta)$ for all $\delta \in \Delta_0$ and $\rho_j(\gamma_i) \neq \rho_0(\gamma_i)$ for all $1 \leq i \leq j$. Selecting a non-principal ultrafilter ω on $\mathbf{N}$, we have the associated ultraproduct representation $\rho_\omega\colon \Gamma \to \mathbf{G}$ (cf [10]). If $\gamma \in \Gamma - \Delta_0$, then $\rho_j(\gamma) \neq \rho_0(\gamma)$ for a cofinite set of $j \in \mathbf{N}$ and so $\rho_\omega(\gamma) \neq \rho_0(\gamma)$. Similar, if $\delta \in \Delta_0$, then $\rho_j(\delta) = \rho_0(\delta)$ for all $j \in \mathbf{N}$ and so $\rho_\omega(\delta) = \rho_0(\delta)$. In particular, ρ_0 strongly distinguishes Δ_0. □

Remark 3.3 Lemma 3.2 can also be proved using the Baire Category Theorem.

Corollary 3.4 *Let Γ be a finitely generated group, $\mathbf{G}$ is a complex algebraic group, and Δ_0 a finitely generated subgroup of Γ. If Δ_0 is distinguished by a representation $\rho_0 \in \mathrm{Hom}(\Gamma, \mathbf{G})$, then there exists a representation $\Phi\colon \Gamma \to \mathbf{G} \times \mathbf{G}$ such that $\Phi(\Gamma) \cap \overline{\Phi(\Delta_0)} = \Phi(\Delta_0)$, where $\overline{\Phi(\Delta_0)}$ is the Zariski closure of $\Phi(\Delta_0)$ in $\mathbf{G} \times \mathbf{G}$.*

Proof Since Δ_0 is distinguished by ρ_0, it follows from Lemma 3.2 that Δ_0 is strongly distinguished by ρ_0. Hence, by Lemma 3.1, we obatin the desired representation $\Phi\colon \Gamma \to \mathbf{G} \times \mathbf{G}$. □

3.1 Twisting by automorphisms

Given an automorphism $\psi_0 \in \mathrm{Aut}(\Gamma)$, we define

$$\mathrm{Aut}_{\psi_0,\Delta_0}(\Gamma) = \left\{\psi \in \mathrm{Aut}(\Gamma) \ : \ \psi_{|\Delta_0} = (\psi_0)_{|\Delta_0}\right\}.$$

For each $\gamma \in \Gamma$, we have the function $\mathrm{Eval}_{\mathrm{Aut},\gamma} \colon \mathrm{Aut}(\Gamma) \to \Gamma$ defined by $\mathrm{Eval}_{\mathrm{Aut},\gamma}(\psi) = \psi(\gamma)$. We say that Δ_0 is **weakly ψ_0-distinguished in Γ** if $\mathrm{Eval}_{\mathrm{Aut},\gamma}$ is non-constant on $\mathrm{Aut}_{\psi_0,\Delta_0}(\Gamma)$ for all $\gamma \in \Gamma - \Delta_0$. We say that Δ_0 is **ψ_0-distinguished** if for any finite set S of $\Gamma - \Delta_0$, there are automorphisms $\psi_S, \psi'_S \in \mathrm{Aut}_{\psi_0,\Delta_0}(\Gamma)$ such that $\mathrm{Eval}_{\mathrm{Aut},\gamma}(\psi_S) \neq \mathrm{Eval}_{\mathrm{Aut},\gamma}(\psi'_S)$, i.e. $\psi_S(\gamma) \neq \psi'_S(\gamma)$, for all $\gamma \in S$. Finally, we say Δ_0 is **strongly ψ_0-distinguished** if there exist $\psi, \psi' \in \mathrm{Aut}_{\psi_0,\Delta_0}(\Gamma)$ such that $\psi(\gamma) \neq \psi'(\gamma)$ for all $\gamma \in \Gamma - \Delta_0$.

Lemma 3.5 *If Γ is a finitely generated group and Δ_0 is (weakly, strongly) ψ_0-distinguished in Γ, then for any complex algebraic group $\mathbf{G}$ and any injective representation $\rho \in \mathscr{R}(\Gamma, \mathbf{G})$, Δ_0 is (weakly, strongly) distinguished by $\rho \circ \psi_0$ in Γ.*

Proof For any $\psi, \psi' \in \mathrm{Aut}_{\psi_0,\Delta_0}(\Gamma)$ and $\rho \in \mathrm{Hom}(\Gamma, \mathbf{G})$, we have $(\rho \circ \psi)_{|\Delta_0} = (\rho \circ \psi')_{|\Delta_0}$. In particular, for each $\gamma \in \Gamma - \Delta_0$, there exists $\psi, \psi' \in \mathrm{Aut}_{\psi_0,\Delta_0}(\Gamma)$ such that $\mathrm{Eval}_{\mathrm{Aut},\gamma}(\psi) \neq \mathrm{Eval}_{\mathrm{Aut},\gamma}(\psi')$ since Δ_0 is weakly ψ_0-distinguished. As ρ is injective, $\rho(\psi(\gamma)) \neq \rho(\psi'(\gamma))$ and so $\mathrm{Eval}_\gamma(\rho \circ \psi) \neq \mathrm{Eval}_\gamma(\rho \circ \psi')$. By definition, $\rho \circ \psi, \rho \circ \psi' \in \mathscr{R}_{\rho\circ\psi_0,\Delta_0}(\Gamma, \mathbf{G})$ and so Δ_0 is weakly distinguished by $\rho \circ \psi_0$. The proof when Δ_0 is ψ_0-distinguished or strongly ψ_0-distinguished is identical. □

4 Proof of Theorem 1.1

Before proving Theorem 1.1, we require a pair of lemmas.

Lemma 4.1 *If $\Lambda = \Delta_0 * \Delta$ with $\Delta_0 \neq \{1\}$, then there exists an automorphism of Λ whose set of fixed points is exactly Δ_0. In particular, Δ_0 is strongly ψ_0-distinguished, where ψ_0 is the identity automorphism.*

Proof We assume $\Lambda \neq \Delta_0$ as that case is trivial. Fix δ a nontrivial element in Δ_0. Define an automorphism $\psi \colon \Lambda \to \Lambda$ as being the identity on Δ_0 and $\psi(k) = \delta \cdot k \cdot \delta^{-1}$ for all $k \in \Delta$. Given $\gamma \in \Lambda - \Delta_0$, we have a reduced expression $\gamma = h_1 k_1 \cdots h_m k_m$, where $m \geq 1$, $h_i \in \Delta_0 - \{1\}$, and $k_i \in \Delta - \{1\}$, with the exception that h_1 or k_m could be trivial. Thus,

$$\begin{aligned}\psi(\gamma) = \psi(h_1 k_1 \cdots h_m k_m) &= (h_1\delta)k_1(\delta^{-1}h_2\delta)\cdots(\delta^{-1}h_m\delta)k_m(\delta^{-1}) \\ &= h'_1 k_1 h'_2 \cdots h'_m k_m \delta^{-1},\end{aligned}$$

where $h_i \neq h'_i$ for all $i = 1, \ldots, m$. Thus, $\psi(\gamma) \neq \gamma$ for all $\gamma \in \Lambda - \Delta_0$, so that the set of fixed points of ψ is exactly Δ_0. □

Lemma 4.2 *If Σ_g is a closed surface of genus $g > 1$ and Σ' is a compact, embedded, incompressible subsurface, then $\pi_1(\Sigma', p)$ is strongly ψ_0-distinguished in $\pi_1(\Sigma_g, p)$, where ψ_0 is the identity.*

Proof We assume $\pi_1(\Sigma_g) \neq \pi_1(\Sigma')$ as the alternative is trivial. We need $\psi \in \mathrm{Aut}_{\psi_0}(\pi_1(\Sigma_g), \pi_1(\Sigma'))$ with $\psi([\gamma]) \neq \psi_0([\gamma]) = [\gamma]$ for all $[\gamma] \in \pi_1(\Sigma_g) - \pi_1(\Sigma')$. Fixing $p \in \mathrm{Int}(\Sigma')$, for every $[\gamma]$ in $\pi_1(\Sigma_g, p) - \pi_1(\Sigma', p)$ and any loop c representing $[\gamma]$, we must have $c \cap \partial\Sigma' \neq \emptyset$. For each boundary component α_i, set $\tau_i : \Sigma_g \to \Sigma_g$ to be Dehn twist about α_i for $i = 1, \dots, b$, and note that τ_i induces an automorphism $\psi_i \in \mathrm{Aut}_{\psi_0}(\pi_1(\Sigma_g, p), \pi_1(\Sigma', p))$ defined by $\psi_i([\gamma]) = [\tau_i(\gamma)]$. Thus, for any $[\gamma] \in \pi_1(\Sigma_g) - \pi_1(\Sigma')$, $\psi_i([\gamma]) \neq [\gamma]$ for some i, and setting $\psi = \psi_b \circ \cdots \circ \psi_1$ completes the proof. ◻

Proof of Theorem 1.1 Let Γ be either a free group of rank $r > 1$ or the fundamental group of a closed surface Σ_g of genus $g > 1$. Given a finitely generated subgroup Δ_0, if Γ is free, then by Hall [11], there exists a finite index subgroup $\Lambda < \Gamma$ with $\Lambda = \Delta_0 * \Delta$. If Γ is the fundamental group of a closed surface, then by Scott [25], there is a finite cover $P : \Sigma_{g_0} \to \Sigma_g$ such that $\Delta_0 < P_*(\pi_1(\Sigma_{g_0})) = \Lambda$. Moreover, there exists a embedded compact subsurface Σ_{Δ_0} of Σ_{g_0} with $\Delta_0 = \pi_1(\Sigma_{\Delta_0})$. In either case, we can apply Lemma 4.1 or Lemma 4.2 to see that Δ_0 is strongly ψ_0-distinguished in Λ. For any faithful representation $\rho_0 \in \mathrm{Hom}(\Lambda, \mathrm{GL}(2, \mathbf{C}))$, we see that ρ_0 strongly distinguishes Δ_0 by Lemma 3.5. By Corollary 3.4, we have $\Phi : \Lambda \to \mathrm{GL}(2, \mathbf{C}) \times \mathrm{GL}(2, \mathbf{C})$ such that $\Phi(\gamma) \in \mathrm{Diag}(\mathrm{GL}(2, \mathbf{C}))$ if and only if $\gamma \in \Delta_0$. Setting $d_{\Delta_0} = [\Gamma : \Lambda]$, we have the induced representation $\mathrm{Ind}_\Lambda^\Gamma(\Phi) : \Gamma \to \mathrm{GL}(2d_{\Delta_0}, \mathbf{C}) \times \mathrm{GL}(2d_{\Delta_0}, \mathbf{C})$. Taking $\rho = \mathrm{Ind}_\Lambda^\Gamma(\Phi)$, it follows by the construction of ρ and from the definition of induction that, $\rho(\gamma) \in \overline{\rho(\Delta_0)}$ if and only if $\gamma \in \Delta_0$. In particular, $\rho(\Delta_0) = \rho(\Gamma) \cap \overline{\rho(\Delta_0)}$ as needed, and ρ is faithful since ρ_0 is faithful. ◻

5 Proof of Corollary 1.2

The following basic result has been proven in [1,18,19].

Lemma 5.1 *Let $\mathbf{G} < \mathrm{GL}(n, \mathbf{C})$ be a $\overline{\mathbf{Q}}$-algebraic group, $\mathbf{H} < \mathbf{G}$ a $\overline{\mathbf{Q}}$-algebraic subgroup, $\Gamma < \mathbf{G}$ a finitely generated subgroup. If $\Delta_0 = \mathbf{H} \cap \Gamma$, then Δ_0 is closed in the profinite topology.*

We include a proof here as it is required in the proof of Corollary 1.2.

Proof Given $\gamma \in \Gamma - \Delta_0$, we require a homomorphism $\varphi : \Gamma \to Q$ with $|Q| < \infty$ and $\varphi(\gamma) \notin \varphi(\Delta_0)$. We first select polynomials $Q_1, \dots, Q_{r_\mathbf{G}}, \dots, Q_{r_\mathbf{H}} \in \mathbf{C}[X_{i,j}]$ satisfying (1). Since $\mathbf{G}, \mathbf{H}$ are $\overline{\mathbf{Q}}$-defined, we can select $Q_j \in \mathcal{O}_{K_0}[X_{i,j}]$ for some number field $K_0/\mathbf{Q}$. We fix a finite set $\{\gamma_1, \dots, \gamma_{r_\Gamma}\}$ that generates Γ as a monoid. In order to distinguish between the elements of Γ as an abstract group versus the explicit elements in $\mathbf{G}$, we set $\gamma = A_\gamma \in \mathbf{G}$ for each $\gamma \in \Gamma$. In particular, we have a representation $\rho_0 : \Gamma \longrightarrow \mathbf{G}$ given by $\rho_0(\gamma_t) = A_{\gamma_t}$. We set K_Γ to be the field generated over K_0 by the set of matrix entries $\{(A_{\gamma_t})_{i,j}\}_{t,i,j}$. It is straightforward to see that K_Γ is independent of the choice of the generating set for Γ. Since Γ is finitely generated, the field K_Γ has finite transcendence degree over $\mathbf{Q}$ and so K_Γ is isomorphic to a field of the form $K(T)$ where $K/\mathbf{Q}$ is a number field and $T = \{T_1, \ldots, T_d\}$ is a transcendental basis (see [24, Cor. 3.3.3]). For each A_{γ_t}, we have $(A_{\gamma_t})_{i,j} = F_{i,j,t}(T) \in K_\Gamma$. In particular, we can view the (i, j)-entry of the matrix A_{γ_t} as a rational function in d variables with coefficients in some number field K. Taking the ring R_Γ generated over $\mathcal{O}_{K_0}$ by the set $\left\{(A_{\gamma_t})_{i,j}\right\}_{t,i,j}$, R_Γ is obtained from $\mathcal{O}_K[T_1, \ldots, T_d]$ by inverting a finite number of integers and polynomials. Any ring homomorphism $R_\Gamma \to R$ induces a group homomorphism $\mathrm{GL}(n, R_\Gamma) \to \mathrm{GL}(n, R)$, and as $\Gamma < \mathrm{GL}(n, R_\Gamma)$, we obtain $\Gamma \to \mathrm{GL}(n, R)$. If $\gamma \in \Gamma - \Delta_0$, then there exists $r_{\mathbf{G}} < j_\gamma \leq r_{\mathbf{H}}$ such that $P_\gamma = Q_{j_\gamma}((A_\gamma)_{1,1}, \ldots, (A_\gamma)_{n,n}) \neq 0$. Using Lemma 2.1 in [6], we have a ring homomorphism $\psi_R \colon R_\Gamma \to R$ with $|R| < \infty$ such that $\psi_R(P_\gamma) \neq 0$. Setting $\rho_R \colon \mathrm{GL}(n, R_\Gamma) \to \mathrm{GL}(n, R)$, we assert that $\rho_R(\gamma) \notin \rho_R(\Delta_0)$. To see this, set $\overline{A}_\eta = \rho_R(\eta)$ for each $\eta \in \Gamma$, and note that $\psi_R(Q_j((A_\eta)_{1,1}, \ldots, (A_\eta)_{n,n})) = Q_j((\overline{A}_\eta)_{1,1}, \ldots, (\overline{A}_\eta)_{n,n})$. For each $\delta \in \Delta_0$, we know that $Q_{j_\gamma}((A_\delta)_{i,j}) = 0$ and so $Q_j((\overline{A}_\eta)_{1,1}, \ldots, (\overline{A}_\eta)_{n,n}) = 0$. However, by selection of ψ_R, we know that $\psi_R(P_\gamma) \neq 0$ and so $\rho_R(\gamma) \notin \rho_R(\Delta_0)$. □

5.1 Proof of Corollary 1.2

To prove Corollary 1.2, we combine Theorem 1.1 with Lemma 5.1. By Theorem 1.1, there exists a representation $\rho \colon \Gamma \to \mathrm{GL}(n, \mathbf{C})$ such that if $\mathbf{G} = \overline{\rho(\Gamma)}$ and $\mathbf{H} = \overline{\rho(\Delta_0)}$, then $\rho(\Delta_0) = \rho(\Gamma) \cap \mathbf{H}$. We can construct ρ_0 in the proof of Theorem 1.1 so that $\mathbf{G}$, $\mathbf{H}$ are both $\overline{\mathbf{Q}}$-defined. Consequently, we can use Lemma 5.1 to separate Δ_0 in Γ. In order to make Lemma 5.1 effective we need to bound the order of the ring R in terms of the word length of the element γ in the proof of Lemma 5.1. Lemma 2.1 from [6] bounds the size of R in terms of the coefficient size and degree of the polynomial P_γ. It follows from the discussion on pp. 412–413 of [6] that the coefficients and degree can be bounded in terms of the word length of γ, and the coefficients and degrees of the polynomials Q_j. As the functions Q_j are independent of the word γ, we see that there exists a constant D_0 such that $|R| \leq ||\gamma||^{D_0}$. By construction, the group Q needed in Corollary 1.2 for γ is a subgroup of $\mathrm{GL}(n, R)$ and so $|Q| \leq |R|^{n^2} \leq ||\gamma||^{D_0 n^2}$. Hence, we can take $D = D_0 n^2$.

6 Final remarks

The main contribution of Corollary 1.2 is that we establish polynomial bounds on the size of the normal core of the finite index subgroup Λ used in separating γ from Δ_0. The methods used in [9] give linear bounds in terms of the word length of γ on the index of the subgroup used in the separation but do not easily produce polynomial bounds for the normal core of that finite index subgroup. With care taken to make our argument optimal, we can obtain bounds on the index of the separating subgroup on the order of magnitude $C\,||\gamma||$ as well.

Finally, to what extent Theorem 1.1 can be generalized to other classes of groups is unclear. We make specific use of our settings but believe that the broad framework we present should work for a larger class of groups. That prompts the following question:

Question 6.1 *Does Theorem 1.1 hold when* Γ *is the fundamental group of a closed hyperbolic 3-manifold and* Δ_0 *is a finitely generated, geometrically finite subgroup? Does Theorem 1.1 hold when* Γ *is a right-angled Artin group and* Δ_0 *is a quasi-convex subgroup?*

By [9], separability of these subgroups can be done with finite index subgroups of polynomial index in $||\gamma||$. That is a necessary for Theorem 1.1 to hold. In the above question, we optimistically believe that this condition is sufficient.

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