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Algebraic flows on Shimura varieties

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Abstract. In this paper we prove two results on algebraic flows on Shimura varieties. One is the so-called 'logarithmic' Ax-Lindemann theorem. The other concerns the closure of the image of a totally geodesic sub-variety of a symmetric space by the uniformisation map.

Contents

1. Introduction

In the paper [11] we formulated some conjectures about algebraic flows on abelian varieties and proved certain cases of these conjectures. The purpose of this paper is twofold. We first prove the 'inverse Ax-Lindemann theorem' (see details below). We then prove a result analogous to one of the main results of [11] in the hyperbolic (Shimura) case about the topological closure of the images of totally geodesic subvarieties of the symmetric spaces uniformising Shimura varieties.

Let (G, X) be a Shimura datum and X^+ be a connected component of X. This X^+ is a Hermitian symmetric domain. Recall from [12], Sect. 2.1 that a realisation $\mathcal{X}$ of X^+ is a complex quasi-projective variety $\mathcal{X}$ with a transitive holomorphic action

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of $G(\mathbb{R})^+$ such that for any $x_0 \in \mathcal{X}$, the orbit map $\psi_{x_0} : G(\mathbb{R})^+ \longrightarrow \mathcal{X}$ mapping g to gx_0 is semi-algebraic. There is a natural notion of a morphism of realisations. By [12], lemma 2.1, any realisation of X^+ has a canonical semi-algebraic structure and any morphism of realisations is semi-algebraic.

In what follows we fix a realisation $\mathcal{X}$ of X^+ and by a slight abuse of language still call this realisation X^+. It is an immediate consequence of Lemma 2.1 of [12] that all the conjectures and statements that follow are independent of the chosen realisation.

In view of the lemma B1 of [4], we define an algebraic subset Y of X^+ to be a closed analytic, semi-algebraic subset of X^+. Given an irreducible analytic subset $\Theta \subset X^+$, we define the Zariski closure of Θ to be the analytic component containing Θ of the smallest algebraic subset of X^+ containing Θ. We denote this closure by $Zar(\Theta)$. Furthermore, for a subset Θ of a complex algebraic variety, we denote by $Zar(\Theta)$ an irreducible component of the Zariski closure of Θ. For a subset Θ of $\mathbb{C}^n$, we denote by $\overline{\Theta}$ the closure of Θ for the Archimedean topology.

We can now state some results and conjectures.

The classical formulation of the hyperbolic Ax-Lindemann theorem is as follows:

Theorem 1.1. (Hyperbolic Ax-Lindemann theorem) *Let S be a Shimura variety and $\pi : X^+ \longrightarrow S$ be the uniformisation map. Let Z be an algebraic subvariety of S and Y a maximal algebraic subvariety of $\pi^{-1}(Z)$. Then $\pi(Y)$ is a weakly special subvariety of S.*

For compact Shimura varieties this theorem was proved by the present authors (see [10]). Pila and Tsimerman (see [6]) proved this theorem for $\mathcal{A}_g$. It was finally proved in full generality by Klingler and the present authors (see [14]). We will see (see Proposition 5.1) that Theorem 1.1 is equivalent to:

Theorem 1.2. (Hyperbolic Ax-Lindemann theorem, version 2.) *Let Z be any irreducible algebraic subvariety of X^+ then the Zariski closure of $\pi(Z)$ is weakly special.*

In the second section we define a notion of a weakly special subvariety of X^+. This is a complex analytic subset Θ of X^+ such that there exists a semi-simple algebraic subgroup F of $G(\mathbb{R})^+$ and a point $x \in X^+$ satisfying certain conditions such that $\Theta = F \cdot x$.

In Sect. 3 of this paper we prove an 'inverse' Ax-Lindemann theorem (a question asked by D. Bertrand).

Theorem 1.3. (The inverse Ax-Lindemann) *Let $\pi : X^+ \longrightarrow S$ be the uniformisation map. Let Y be an algebraic subvariety of S and let Y' be an analytic component of $\pi^{-1}(Y)$. The Zariski closure $Zar(Y')$ of Y' is a weakly special subvariety.*

The idea of the proof is as follows. We first reduce ourselves to the case where Y' is not contained in a proper weakly special subvariety of X^+. We write $S = \Gamma\backslash X^+$ where Γ is an arithmetic subgroup of $G(\mathbb{Q})$ acting 'algebraically' on X^+. Using a classical theorem about monodromy, we show that $Zar(Y')$ is stable by a Zariski-dense subgroup of G which implies that $Zar(Y') = X^+$ and thus concludes the proof.

In [11], we formulated two conjectures on algebraic flows on abelian varieties and proved partial results towards these conjectures. An attempt to formulate conjectures of this type in the context of Shimura varieties displays new phenomena that we intend to investigate in the future. We however prove a result which may be seen as a generalisation in the context of Shimura varieties of one of the main results of [11]. To state our result we need to introduce a few notations.

Consider an algebraic subset Θ of X^+. In general, instead of (as in the hyperbolic Ax-Lindemann case) being interested in the Zariski closure of $\pi(\Theta)$, we look at the usual topological closure $\overline{\pi(\Theta)}$. We define a notion of *real weakly special* subvariety roughly as the image of $H(\mathbb{R}) \cdot x$ where H is a semisimple subgroup of G satisfying certain conditions and x is a point of X^+. Let K_x be the stabiliser of x in $G(\mathbb{R})^+$. In the case where $H(\mathbb{R})^+ \cap K_x$ is a maximal compact subgroup, a real weakly special subvariety of S is a *real* totally geodesic subvariety of S. Notice that in this case the homogeneous space $H(\mathbb{R})^+/H(\mathbb{R})^+ \cap K_x$ is a real symmetric space. In the case where x viewed as a morphism from $\mathbb{S}$ to $G_{\mathbb{R}}$ factors through $H_{\mathbb{R}}$, the corresponding real weakly special subvariety has Hermitian structure and in fact is a weakly special subvariety in the usual sense. We also note that given a real weakly special subvariety Z of S, there is a canonical probability measure μ_Z attached to Z which is the pushforward of the Haar measure on $H(\mathbb{R})^+$, suitably normalised to make it a probability measure.

In this paper we prove the following theorem.

Theorem 1.4. *Let Θ be a complex totally geodesic subvariety of X. Then the topological closure $\overline{\pi(\Theta)}$ is a real weakly special subvariety.*

Recall that a complex totally geodesic subvariety of X^+ is of the form $F \cdot x$ where F is a semisimple real Lie group subject to certain conditions and x is a point of X such that $F \cap K_x$ is a maximal compact subgroup of F.

The proof of Theorem 1.4 is a more-or-less direct consequence of Ratner's theorem adapted to Shimura varieties by Clozel and Ullmo. This theory is explained in detail in Sect. 4.

In certain cases, for example when the centraliser of F in $G(\mathbb{R})$ is trivial, we are able to show that $\overline{\pi(\Theta)}$ is actually a (complex) weakly special subvariety. This condition is satisfied in many cases. For example in the case of $\mathrm{SL}_2(\mathbb{R})$ diagonally embedded into a product of copies of $\mathrm{SL}_2(\mathbb{R})$. In particular this answers in the affirmative the question of Jonathan Pila which was the following. Consider the subset Z of $\mathbb{H} \times \mathbb{H}$ which is

$$Z = \{(\tau, g\tau) : \tau \in \mathbb{H}\}$$

where $g \in \mathrm{SL}_2(\mathbb{R}) \backslash \mathrm{SL}_2(\mathbb{Q})$. Is the image of Z dense in $\mathbb{C} \times \mathbb{C}$?

The proof of Theorem 1.4 relies on the results of Ratner (see [7]) on closure of unioptent one parameter subgroups in homogeneous spaces.

2. Weakly special subvarieties and monodromy

2.1. Monodromy

Let (G, X) be a Shimura datum. Recall that G is a reductive group over $\mathbb{Q}$ such that G^{ad} has no $\mathbb{Q}$-simple factor whose real points are compact and X is a $G(\mathbb{R})$-conjugacy class of a morphism $x : \mathbb{S} \longrightarrow G_{\mathbb{R}}$ where $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$. The morphism x is required to satisfy Deligne's conditions (see 2.1.1 of [3]) which imply that connected components of X are Hermitian symmetric domains. There is a natural notion of morphisms of Shimura data. We fix a connected component X^+ of X and we let $\Gamma = G(\mathbb{Q})_+ \cap K$ where $G(\mathbb{Q})_+$ is the stabiliser of X^+ in $G(\mathbb{Q})$. Let S be $\Gamma\backslash X^+$ and $\pi : X^+ \longrightarrow S$ be the natural morphism.

To (G, X), one associates the adjoint Shimura datum (G^{ad}, X^{ad}) with a natural morphism $(G, X) \longrightarrow (G^{ad}, X^{ad})$ induced by the quotient map $G \longrightarrow G^{ad}$. Notice that this map identifies X^+ with a connected component of X^{ad}. We have the following description of weakly special (or totally geodesic) subvarieties (see Moonen [5], Theorem 4.3):

Theorem 2.1. *A subvariety Z of S is totally geodesic if and only if there exists a sub-datum (M, X_M) of (G, X) and a product decomposition*

$$(M^{ad}, X_M^{ad}) = (M_1, X_1) \times (M_2, X_2)$$

and a point y_2 of X_2 such that $Z = \pi(X_1^+ \times y_2)$ for a component X_1^+ of X_1.

Note that $X_M^{ad,+} = X_1^+ \times X_2^+$ (with a suitable choice of connected components) is a subspace of X^+.

We can without loss of any generality assume the group Γ to be neat, i.e. the stabiliser of each point of X^+ in Γ to be trivial (replacing Γ by a subgroup of finite index changes nothing to the property of a subvariety to be weakly special). We also assume the group K to be a product of compact open subgroups K_p of $G(\mathbb{Q}_p)$. This also causes no loss of generality. Fix a point x of the smooth locus Z^{sm} and $\widetilde{x} \in \pi^{-1}(x) \cap Z^{sm}$. We let M be the Mumford–Tate group of $\widetilde{x}$ and call it the generic Mumford–Tate group on Z. This gives rise to the monodromy representation

$$\rho^m : \pi_1(Z^{sm}, x) \longrightarrow \Gamma$$

whose image we denote by Γ^m. As a consequence of Theorem 1.4 of [5], we have an inclusion $\Gamma^m \subset M^{der}(\mathbb{Q}) \cap \Gamma$. Furthermore the Zariski closure of Γ^m is a normal subgroup of M^{der}. We call Γ^m the monodromy group attached to Z.

We summarise the situation in the following theorem.

Theorem 2.2. *Let (G, X) be Shimura datum, K a compact open subgroup of $G(\mathbb{A}_f)$ which is a product of compact open subgroups K_p of $G(\mathbb{Q}_p)$. Let $\Gamma := G(\mathbb{Q})_+ \cap K$ (assumed neat).*

Let $S = \Gamma\backslash X^+$ and Z an irreducible subvariety of S. Let M be the generic Mumford–Tate group on Z and X_M the $M(\mathbb{R})$-conjugacy class of x.

Let $\Gamma^m \subset M^{der}(\mathbb{Q}) \cap \Gamma$ be the monodromy group attached to Z as described above.

Let $(M^{ad}, X_M^{ad}) = (M_1, X_1) \times (M_2, X_2)$ *as in Theorem 4.3 of* [5]. *In particular* M_1 *is the image of the neutral component of the Zariski closure of* Γ^m *in* M^{ad}.

Let $K_M^{ad} = K_1 \times K_2$ *be a compact open subgroup containing the image of* $K_M = M(\mathbb{A}_f) \cap K$ *in* $M^{ad}(\mathbb{A}_f)$ *(here* K_i*s are compact open subgroups of* $M_i(\mathbb{A}_f)$*). We let* X_i *be the* $M_i(\mathbb{R})$*-conjugacy classes of* x.

Let $S_M \subset S$ *be a connected component of the image of* $Sh_{M(\mathbb{A}_f)\cap K}(M, X_M)$ *in* S *containing* Z.

Let S_i *(*$i = 1, 2$*) be appropriate components of* $Sh_{K_i}(M_i, X_i)$ *and* $S_M \longrightarrow S_1 \times S_2$ *be the natural map.*

The image of Z *in* $S_1 \times S_2$ *is of the form* $Z_1 \times \{z\}$ *(see Theorem 4.3 of* [5]*) where* Z_1 *is a subvariety of* S_1 *whose monodromy is Zariski dense in* M_1*and* z *is a point of* S_2.

2.2. *Weakly special subvarieties of* X^+

In this section we give a precise description of totally geodesic (weakly special) subvarieties of X^+.

Let (G, X) be a Shimura datum and X^+ a connected component of X. For the purposes of this section, we can without loss of generality assume that G is a semi-simple group of adjoint type. This is because there is a natural identification between connected components of X^+ and a connected component of X^{ad}.

The group G has no $\mathbb{Q}$-simple factors whose real points are compact and there is a morphism $x_0 : \mathbb{S} \longrightarrow G_{\mathbb{R}}$ such that $X^+ = G(\mathbb{R})^+ \cdot x_0$. Furthermore x_0 satisfies the following conditions such that

1. The adjoint representation Lie$(G_{\mathbb{R}})$ is of type $\{(-1, 1), (0, 0), (1, -1\}$. In particular $x_0(\mathbb{G}_{m,\mathbb{R}})$ is trivial.
2. The involution $x_0(\sqrt{-1})$ of $G_{\mathbb{R}}$ is a Cartan involution.

We will now describe totally geodesic subvarieties of X^+ (that we will naturally call weakly special).

Proposition 2.3. *Let* Z *be a totally geodesic complex subvariety of* X^+. *There exists a semi-simple real algebraic subgroup* F *of* $G_{\mathbb{R}}$ *without compact factors and some* $x \in X$ *such that* x *factors through* $FZ_G(F)^0$ *such that* $Z = F(\mathbb{R})^+ \cdot x$. *Conversely, let* F *be a semi-simple real algebraic subgroup of* $G_{\mathbb{R}}$ *without compact factors and let* $x \in X$ *such that* x *factors through* $FZ_G(F)^0$. *Then* $F(\mathbb{R})^+ \cdot x$ *is a totally geodesic subvariety of* X^+.

Proof. Let F be a semi-simple real algebraic subgroup of $G_{\mathbb{R}}$ without compact factors and let $x \in X$ such that x factors through $H := FZ_G(F)^0$. Then

$$Z_G(H(\mathbb{R})) \subset Z_G(x(\sqrt{-1})).$$

As $Z_G(x(\sqrt{-1}))$ is a compact subgroup of $G(\mathbb{R})$ so is $Z_G(H(\mathbb{R}))$. By using [13] lemma 3.13 we see that H is reductive.

Then the proof of [13] lemma 3.3 shows that $X_H := H(\mathbb{R})^+ \cdot x$ is an Hermitian symmetric subspace of X^+. We give the arguments to be as self contained as possible.

As Lie($H_{\mathbb{R}}$) is a sub vector space of Lie($G_{\mathbb{R}}$) the Hodge weights of Lie($H_{\mathbb{R}}$) are $\{(-1, 1), (0, 0), (1, -1)\}$. Then using Deligne [3] 1.1.17 we just need to prove that $x(\sqrt{-1})$ induces a Cartan involution of H^{ad}. As the square of $x(\sqrt{-1})$ is in the centre of $H(\mathbb{R})$, by Deligne [3] 1.1.15, it is enough to check that $H_{\mathbb{R}}$ admits a faithful real $x(\sqrt{-1})$-polarizable representation (V, ρ). We may take $V = \mathrm{LieG}_{\mathbb{R}}$ for the adjoint representation and the $x(\sqrt{-1})$-polarization induced from the Killing form $B(X, Y)$.

Then $H_{\mathbb{R}}$ is the almost direct product $H_{\mathbb{R}} \simeq F F_1^{nc} F_1^c$ where F_1 is either trivial or semi-simple without compact factors and F_1^c is reductive with $F_1^c(\mathbb{R})$ compact. If F_1^{nc} is trivial $X_F^+ = X_H^+$ is Hermitian symmetric. If F_1^{nc} is not trivial, we have a decomposition $X_H^+ = X_F^+ \times X_{F_1^{nc}}^+$ is a product of Hermitian subspaces and we have the natural identification of X_F^+ with $X_F^+ \times \{x_1\}$ where x_1 is the projection of x on $X_{F_1^{nc}}^+$. In any case X_F^+ is Hermitian symmetric and totally geodesic in X^+.

Conversely a totally geodesic subvariety of X^+ is of the form $X_F^+ = F(\mathbb{R})^+ \cdot x$ for a semi-simple subgroup $F_{\mathbb{R}}$ of $G_{\mathbb{R}}$ without compact factors. Let $T_x(X_F^+) \subset T_x(X^+)$ be the tangent space of X_F^+ at x. Let $U^1 \subset \mathbb{S}$ be the unit circle. The complex structure on $T_x(X^+)$ is given by the adjoint action of $x(U^1)$. If X_F is a complex subvariety, then $T_x(X_F^+)$ is stable by $x(U^1)$. Using Cartan decomposition we see that $x(U^1) = x(\mathbb{S})$ normalizes F. Let $F_1 = x(\mathbb{S})F$, then F_1 is reductive and is contained in $F Z_G(F)^0$. It follows that x factors through $F Z_G(F)^0$. □

Definition 2.4. An algebraic group H over $\mathbb{Q}$ is said to be of type $\mathcal{H}$ if its radical is unipotent and if $H/R_u(H)$ is an almost direct product of $\mathbb{Q}$ simple factors H_i with $H_i(\mathbb{R})$ non-compact. Furthermore we assume that at least one of these factors not to be trivial.

Let $H \subset G$ be a subgroup of type $\mathcal{H}$ and let us assume that G is of adjoint type. We will now explain how to attach a Hermitian symmetric space X_H to a group of type $\mathcal{H}$ and explain that X_H is independent of the choice of a Levi subgroup in H.

The domain X^+ is the set of maximal compact subgroups of $G(\mathbb{R})^+$. Let $x \in X^+$, we denote by K_x the associated maximal compact subgroup of $G(\mathbb{R})^+$. Let H be a subgroup of type $\mathcal{H}$ and let L be a Levi subgroup of H. We have a Levi decomposition $H = R_u(H) \cdot L$. Assume that $K_x \cap L(\mathbb{R})^+$ is a maximal compact subgroup of $L(\mathbb{R})^+$. Then $X_L^+ = L(\mathbb{R})^+ \cdot x \subset X^+$ is the symmetric space associated to L and is independent of the choice of $x \in X^+$ such that $K_x \cap L(\mathbb{R})^+$ is a maximal compact subgroup of $L(\mathbb{R})^+$. Let $X_H^+ := R_u(H) X_L(\mathbb{R})^+$, then X_H^+ is independent of the chosen Levi decomposition of H. This can be seen as follows. The Levi subgroups of H are conjugate by an element of $R_u(H)$. Let L' be a Levi of H and $w \in R_u(H)$ such that $L' = wLw^{-1}$. Let $x' = w \cdot x$. Then $K_{x'}$ is a maximal compact subgroup of $G(\mathbb{R})^+$ such that $K_{x'} \cap L'(\mathbb{R})^+$ is a maximal compact subgroup of $L'(\mathbb{R})^+$ and

$$R_u(H) \cdot X_{L'}^+ = R_u(H) \cdot L'(\mathbb{R})^+ \cdot x' = R_u(H) \cdot wL(\mathbb{R})^+ \cdot x = R_u(H) \cdot X_L^+.$$

This shows that the space X_H^+ is independent of the choice of the Levi.

Definition 2.5. A real weakly special subvariety of S is a real analytic subset of S of the form

$$Z = \Gamma \cap H(\mathbb{R})^+ \backslash H(\mathbb{R})^+ \cdot x$$

where H is an algebraic subgroup of G of type $\mathcal{H}$ and $x \in X^+$.

In the case where $K_x \cap L(\mathbb{R})^+$ is a maximal compact subgroup of $L(\mathbb{R})^+$ for some Levi subgroup of H, $H(\mathbb{R})^+/K_x \cap H(\mathbb{R})^+$ is a real symmetric space.

We have the following proposition.

Proposition 2.6. *Let Z be a real weakly special subvariety of S. Then the Zariski closure $Zar(Z)$ of Z is weakly special.*

Proof. By definition, Z is of the form $Z = H(\mathbb{R})^+ \cdot x$ where H is a group of type $\mathcal{H}$. Let S_M be as in Theorem 2.2 the smallest special subvariety containing $Zar(Z)$.

Let $S_1 \times S_2$ be the product of Shimura varieties as in Theorem 2.2 such that the image of $Zar(Z)$ in $S_1 \times S_2$ is of the form $Z_1 \times \{z\}$ where Z_1 is a subvariety of S_1 whose monodromy Γ_1^m is Zariski dense in M_1 and z is a Hodge generic point of S_2.

To prove that $Zar(Z)$ is weakly special, it is enough to show that $Z_1 = S_1$. In what follows, we replace S by S_1 and Z by Z_1. The monodromy of $Zar(Z)$ is hence now Zariski dense in G.

For any $q \in H(\mathbb{Q})^+$, we have that $Z \subset T_q Z$, therefore

$$Z \subset Zar(Z) \cap T_q(Zar(Z)).$$

Since $Zar(Z) \cap T_q(Zar(Z))$ is algebraic, we have

$$Zar(Z) \subset Zar(Z) \cap T_q(Zar(Z))$$

and therefore, for each $q \in H(\mathbb{Q})$ we have

$$Zar(Z) \subset T_q(Zar(Z)).$$

Let T be a non-trivial subtorus of H.

Let us recall the notion of the Nori constant $C(V)$ of a Hodge generic subvariety V of a Shimura variety S as in Sect. 2.1 such that the monodromy of V is Zariski dense in G. We refer to [15], Sect. 4 for details.

There exists an integer $C(V) > 0$ such that the following holds. Let $g \in G(\mathbb{Q})^+$ and $p > C(V)$. Assume that for all $l \neq p$, g_l is in K_l. Then $T_g(V)$ is irreducible.

We apply this for $V = Zar(Z)$. Let $p > C(Zar(Z))$ and $q \in T(\mathbb{Q})$ given by Lemma 6.1 of [15]. Then $T_q(Zar(Z))$ is irreducible and the orbits of $T_q + T_{q^{-1}}$ are dense in S. Therefore $Zar(Z) = T_q(Zar(Z))$ and $Zar(Z)$ contains a dense subset of S. This implies that $Zar(Z) = S$ as required. □

3. The inverse Ax-Lindemann

Let $S = \Gamma\backslash X^+$ as before and consider a realisation $X^+ \subset \mathbb{C}^n$ (in the sense of [13]). In particular X^+ is a semi-algebraic set and the action of $G(\mathbb{R})^+$ is semi-algebraic.

Let $\widetilde{Y}$ be a complex analytic subset of X^+. Then the Zariski closure $Zar(\widetilde{Y})$ of $\widetilde{Y}$ in $\mathbb{C}^n$ is an algebraic subset of $\mathbb{C}^n$ and $Zar(\widetilde{Y}) \cap X^+$ has finitely many analytic components. By slight abuse of notation, we refer to $Zar(\widetilde{Y}) \cap X^+$ as the *Zariski closure of* $\widetilde{Y}$ and still denote it by $Zar(\widetilde{Y})$.

The components of $Zar(\widetilde{Y})$ are algebraic in the sense that they are analytic and semi-algebraic subsets of X^+. We refer to the Appendix B of [4] for more on these notions.

Theorem 3.1. (The inverse Ax-Lindemann) *Let $\pi : X^+ \longrightarrow S$ be the uniformisation map. Let Y be an algebraic subvariety of S and let Y' be an analytic component of $\pi^{-1}(Y)$. The Zariski closure of Y' is a weakly special subvariety.*

Proof. Let $\widetilde{Y}$ be an analytic component of Y'. As in the previous section we can replace S by S_1 and Y by Y_1 given by the Proposition 2.2. In doing this we attach the monodromy to a point $y \in Y^{sm}$ and $\widetilde{y} \in Y'$. Let Γ_Y be the monodromy group attached to Y. Notice that Γ_Y is the stabiliser of Y' in Γ. Then, with our assumptions, Γ_Y is Zariski dense in G.

Let $\alpha \in \Gamma_Y$. We have

$$\alpha Y' = Y'.$$

Therefore,

$$Zar(\alpha Y') = Zar(Y').$$

We also have

$$\alpha Zar(Y') \supset \alpha Y'.$$

As $\alpha Zar(Y')$ is algebraic, we have

$$\alpha Zar(Y') \supset Zar(\alpha Y').$$

The same argument with α^{-1} instead of α shows that the reverse inclusion holds and therefore

$$Zar(\alpha Y') = \alpha Zar(Y') = Zar(Y').$$

It follows that $Zar(Y')$ is stabilised by Γ_Y.

Consider the stabiliser G_Y of $Zar(Y')$ in $G(\mathbb{R})$. Since $Zar(Y')$ is semi-algebraic and the action of $G(\mathbb{R})^+$ on X^+ is semi-algebraic, G_Y is semi-algebraic. Furthermore, G_Y is analytically closed and hence is a real algebraic group. Since G_Y contains Γ_Y which is Zariski dense in $G_{\mathbb{R}}$, we see that $G_Y = G(\mathbb{R})^+$. It follows that $Zar(Y') = X^+$ as required. □

4. Facts from ergodic theory: Ratner's theory

In this section we recall some results from ergodic theory of homogeneous varieties to be used in the next section. This is known as Ratner's theory. The orginal paper by Ratner is [9]. Ratner's theory has been first applied to Shimura varieties by Clozel and Ullmo, see [2]. The contents of this section are mainly based on Section 3 of [2,12].

Let G be a semi-simple algebraic group over $\mathbb{Q}$. We can assume that G has no $\mathbb{Q}$-simple simple factors that are anisotropic over $\mathbb{R}$. This condition is satisfied by all groups defining Shimura data.

Let Γ be an arithmetic lattice in $G(\mathbb{R})^+$ and let $\Omega = \Gamma\backslash G(\mathbb{R})^+$.

We have already defined a subgroup $H \subset G$ of type $\mathcal{H}$, we now define a group of type $\mathcal{K}$.

Definition 4.1. Let $F \subset G(\mathbb{R})$ be a closed connected Lie subgroup. We say that F is of type $\mathcal{K}$ if

1. $F \cap \Gamma$ is a lattice in F. In particular $F \cap \Gamma\backslash F$ is closed in $\Gamma\backslash G(\mathbb{R})^+$. We denote by μ_F the F-invariant normalised measure on $\Gamma\backslash G(\mathbb{R})^+$.
2. The subgroup $L(F)$ generated by one-parameter unipotent subgroups of F acts ergodically on $F \cap \Gamma\backslash F$ with respect to μ_F.

For the purposes of this section, we in addition assume F to be semisimple.

The relation between types $\mathcal{K}$ and $\mathcal{H}$ is as follows (see [1], lemme 3.1 and 3.2):

Lemma 4.2. 1. *If H is of type $\mathcal{H}$, then $H(\mathbb{R})^+$ is of type $\mathcal{K}$.*
2. *It F is a closed Lie subgroup of $G(\mathbb{R})^+$ of type $\mathcal{K}$, then there exists a $\mathbb{Q}$ subgroup $F_{\mathbb{Q}}$ of G of type $\mathcal{H}$ such that $F = F(\mathbb{R})^+$.*

For a subset E of $G(\mathbb{R})$, we define the Mumford–Tate group $MT(E)$ of E as the smallest $\mathbb{Q}$-subgroup of G whose $\mathbb{R}$-points contain E. If F is a Lie subgroup of $G(\mathbb{R})^+$ of type $\mathcal{K}$, then by (2) of the above lemma, $MT(F) = F_{\mathbb{Q}}$ and it is of type $\mathcal{H}$.

We will make use of the following lemma, which is Lemma 2.4 of [12].

Lemma 4.3. *Let H be a $\mathbb{Q}$-algebraic subgroup of G with H^0 almost simple. Let L be an almost simple factor of $H^0_{\mathbb{R}}$. Then*

$$MT(L) = H^0$$

Let $\Omega = \Gamma\backslash G(\mathbb{R})^+$. Note that Ω carries a natural probability measure, the pushforward of the Haar measure on $G(\mathbb{R})^+$, normalised to be a probability measure (the volume of Ω is finite). For each F of type $\mathcal{K}$, there is a natural probability measure μ_F attached to F.

The following theorem is a consequence of results of Ratner.

Theorem 4.4. *Let $F = F(\mathbb{R})^+$ be a subgroup of $G(\mathbb{R})^+$ be a semi-simple group without compact factors.*

Let H be MT(F)*. The closure of $\Gamma\backslash\Gamma F$ in Ω is*

$$\Gamma\backslash\Gamma H(\mathbb{R})^+ = \Gamma \cap H(\mathbb{R})^+\backslash H(\mathbb{R})^+$$

Proof. By a result of Cartan ([8], Proposition 7.6) the group F is generated by its one-parameter unipotent subgroups.

A result of Ratner (see [7], Theorem 3), implies that the closure of $\Gamma\backslash\Gamma F$ in Ω is homogeneous i.e. there exists a Lie group H of type $\mathcal{K}$ such that

$$\overline{\Gamma\backslash\Gamma F} = \Gamma\backslash\Gamma H$$

By Lemme 2.1(c) of [1], there exists a $\mathbb{Q}$-algebraic subgroup $H_{\mathbb{Q}} \subset G$ such that

$$H(\mathbb{R})^+ = H$$

Since $F \subset H$, we have that $MT(F) \subset H$. On the other hand, by Lemme 2.2 of [1] (due to Shah), the radical of $MT(F)$ is unipotent which implies that $MT(F)$ is of type $\mathcal{H}$. It follows that $H_{\mathbb{Q}} = MT(F)$ which finishes the proof. □

5. Algebraic flows on Shimura varieties

5.1. Reformulation of the hyperbolic Ax-Lindemann theorem

Let (G, X) be a Shimura datum. Let K be a compact open subgroup of $G(\mathbb{A}_f)$, $\Gamma = G(\mathbb{Q})_+ \cap G(\mathbb{A}_f)$ and $S = \Gamma\backslash X^+$. Let $\pi : X^+ \to S$ be the uniformizing map. Without loss of any generality, in this section we assume that the group G is semisimple of adjoint type.

We first give a reformulation of the hyperbolic Ax-Lindemann conjecture in terms of algebraic flows.

Proposition 5.1. *The hyperbolic Ax-Lindemann conjecture is equivalent to the following statement. Let Z be any irreducible algebraic subvariety of X^+ then the Zariski closure of $\pi(Z)$ is weakly special.*

Proof. Let us assume that the hyperbolic Ax-Lindemann conjecture holds true. Let A be an irreducible algebraic subvariety of X^+ and V be the Zariski closure of $\pi(A)$. Let A' be a maximal irreducible algebraic subvariety of $\pi^{-1}(V)$ containing A. By the hyperbolic Ax-Lindemann conjecture $\pi(A')$ is a weakly special subvariety of V. As $A \subset \pi(A') \subset V$ and as $\pi(A')$ is irreducible algebraic we have $V = \pi(A')$. Therefore V is weakly special.

Let us assume that the statement of the proposition holds true. Let V be an irreducible algebraic subvariety of S. Let Y be a maximal irreducible algebraic subvariety of $\pi^{-1}(V)$. Then the Zariski closure V' of $\pi(Y)$ is weakly special. Moreover $V' \subset V$. Let W be an analytic component of $\pi^{-1}(V')$ containing Y. As V' is weakly special, W is irreducible algebraic. By maximality of Y we have $Y = W$. Therefore $\pi(Y) = V'$ is weakly special. □

5.2. An application of Ratner's theory

Let (G, X) be a Shimura datum and X^+ a connected component of X. We assume that G is semi-simple of adjoint type, which we do.

We now consider a totally geodesic (weakly special) subvariety Z of X^+. Recall that there exists a semi-simple subgroup $F(\mathbb{R})^+$ of G without almost simple compact factors and a point x such that x factors through $FZ_G(F)^0$.

Let α be the natural map $G(\mathbb{R})^+ \longrightarrow \Gamma\backslash G(\mathbb{R})^+$ and π_x be the map $\Gamma\backslash G(\mathbb{R})^+ \longrightarrow \Gamma\backslash X^+$ sending x to gx. Recall that $\pi\colon X^+ \longrightarrow \Gamma\backslash X^+$ is the uniformisation map. We have

$$\overline{\pi(Z)} = \overline{\pi_x \circ \alpha(F(\mathbb{R})^+)}$$

We let H be the Mumford–Tate group of $F(\mathbb{R})^+$. Recall that it is defined to be the smallest connected subgroup of G (hence defined over $\mathbb{Q}$) whose extension to $\mathbb{R}$ contains $F(\mathbb{R})^+$.

By [8], Prop 7.6, the group $F(\mathbb{R})^+$ is generated by its one-parameter unipotent subgroups.

By Theorem 4.4, we conclude the following:

Proposition 5.2. *The closure of $\alpha(F(\mathbb{R})^+)$ in $\Gamma\backslash G(\mathbb{R})^+$ is $\Gamma \cap H(\mathbb{R})^+\backslash H(\mathbb{R})^+$.*

5.3. The closure in S

From the fact that the map π_x is proper and Proposition 5.2, we immediately deduce the following

Theorem 5.3. *The closure of $\pi(Z)$ in S is V, the image of $H(\mathbb{R})^+ \cdot x$ i.e. it is a real weakly special subvariety.*

In this section we examine cases where we can actually derive a stronger conclusion, namely:

1. The variety V from Theorem 5.3 is locally symmetric and hence real totally geodesic.
2. It has a Hermitian structure i.e. is a weakly special subvariety.

Theorem 5.4. *Assume $Z_G(F)$ is compact. Then V is a locally symmetric variety.*

Proof. It is enough to show that $H(\mathbb{R})^+ \cap K_x$ is a maximal compact subgroup of $H(\mathbb{R})^+$.

Notice that since $Z_G(F)$ fixes x, we have

$$Z_G(F) \subset K_x.$$

We follow Section 3.2 of [12].

Since K_x is a maximal compact subgroup of $G(\mathbb{R})^+$ such that $F(\mathbb{R})^+ \cap K_x$ is a maximal compact subgroup of $F(\mathbb{R})^+$, we have two Cartan decompositions:

$$G(\mathbb{R})^+ = P_x K_x \text{ and } F = (P_x \cap F) \cdot (K_x \cap F)$$

for a suitable parabolic subgroup P_x of $G(\mathbb{R})^+$.

We now apply Proposition 3.10 of [12] in our situation. We have a connected semi-simple group H such that $F \subset H_{\mathbb{R}}$. According to Proposition 3.10 of [12], there exists a Cartan decomposition

$$H(\mathbb{R}) = (P_x \cap H(\mathbb{R})) \cdot (K_x \cap H(\mathbb{R})).$$

In particular $K_x \cap H(\mathbb{R})$ is a maximal compact subgroup of $H(\mathbb{R})^+$. This finishes the proof. □

Theorem 5.5. *Assume that $Z_G(F)$ is trivial. Then V is a weakly special subvariety.*

Proof. In this case, x factors through F and therefore through $H_{\mathbb{R}}$. Let X_H be the $H(\mathbb{R})$-orbit of x. By lemma 3.3 of [12], (H, X_H) is a Shimura subdatum of (G, X) and therefore V is a weakly special subvariety. □

Example 5.6. We give examples where $Z_G(F)$ is neither trivial nor compact, but the closure of $\pi(Z)$ is nevertheless Hermitian.

Let G be an almost simple group over $\mathbb{Q}$. A typical example is $G = Res_{K/\mathbb{Q}} SL_{2,K}$ where K is a totally real field of degree n. Let F be an $\mathbb{R}$-simple factor of $G_{\mathbb{R}}$. In the above case F could be for example $SL_2(\mathbb{R})$ embedded as $SL_2(\mathbb{R}) \times \{1\} \times \cdots \times \{1\}$. Then the centraliser of F is not compact. However, by Lemma 2.4 of [12], the Mumford–Tate group of F is G and for any point x of X^+, the image of $F \cdot x$ in S is G.

Example 5.7. (*Products of two modular curves*) Consider $G = \mathrm{SL}_2 \times \mathrm{SL}_2$, $X^+ = \mathbb{H} \times \mathbb{H}$ and

$$Z = \{(\tau, g\tau), \tau \in \mathbb{H}\}.$$

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ and $\pi : \mathbb{H} \times \mathbb{H} \longrightarrow \Gamma\backslash X^+$.

Then, if $g \in G(\mathbb{Q})$, then the closure of $\pi(Z)$ is a special subvariety. It is the modular curve $Y_0(n)$ for some n.

If $g \notin G(\mathbb{Q})$, then $\pi(Z)$ is dense in $\Gamma\backslash X^+$. In this case the group $F(\mathbb{R})^+$ is $(h, ghg^{-1}) \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$.

Example 5.8. (*Rank one groups*) Here is another quite general example where $Z_G(F)$ is trivial and hence the closure of the image of $F(\mathbb{R})^+ \cdot x$ is a weakly special subvariety.

Suppose that the groups G is $U(n, 1)$. In this case X^+ is an open ball in $\mathbb{C}^n$. The real rank of G is one. Let F be the subgroup $U(m, 1)$ of $U(n, 1)$ (with $m \leq n$). Then the centraliser $Z_G(F)$ is trivial. Indeed, as the split torus is already contained in F, the centraliser must be compact.

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