O-MINIMAL FLOWS ON ABELIAN VARIETIES.

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Abstract. Let A be an abelian variety over $\mathbb{C}$ of dimension n and $\pi\colon \mathbb{C}^n \longrightarrow A$ be the complex uniformisation. Let X be an unbounded subset of $\mathbb{C}^n$ definable in a suitable o-minimal structure. We give a description of the Zariski closure of $\pi(X)$.

1. Introduction.

Let A be a complex abelian variety of dimension n. Write $A = \mathbb{C}^n/\Lambda$ where $\Lambda \subset \mathbb{C}^n$ is a lattice and let $\pi\colon \mathbb{C}^n \longrightarrow A$ be the uniformisation map.

A subvariety V of A is called *weakly special* if $V = P + B$ where P is a point of A and B is an abelian subvariety. The abelian Ax-Lindemann-Weierstrass theorem is the following.

Theorem 1.1. *Let Y be a complex algebraic subset of $\mathbb{C}^n$. The components of the Zariski closure of $\pi(Y)$ are weakly special subvarieties.*

This theorem is due to Ax (see [1] and [2]) and plays an important role in the new proof by Pila and Zannier of the Manin-Mumford conjecture [7]. Note that the paper [7] provides a different proof of the abelian Ax-Lindemann-Weierstrass theorem. For a proof close in spirit to the contents of this paper, see Section 9 of [5]. In reality, in this statement, Y can be taken to be only *semialgebraic* ($\mathbb{C}^n$ being identified with $\mathbb{R}^{2n}$).

The aim of this paper is to investigate the Zariski closure of the sets $\pi(X)$ where X is definable in an o-minimal structure which is a much wider class of objects. We refer to the book [12] for the notion of a set definable in an o-minimal structure, in particular the structures $\mathbb{R}_{an}$ and $\mathbb{R}_{an,exp}$ (this last structure is actually defined and studied in [4]). Just recall that $\mathbb{R}_{an}$ is the o-minimal structure generated by the restricted analytic functions and $\mathbb{R}_{an.exp}$ is additionally generated by the graph of the real exponential. For a subset Σ of A, we denote by $Zar(\Sigma)$ its Zariski closure.

To be able to prove anything, we will need to make certain additional assumptions. Firstly, the set X will be assumed to be *unbounded.*

The necessity of this condition can be demonstrated by the following example. Let $\mathcal{F}$ be a connected bounded fundamental domain for the action of Λ on $\mathbb{C}^n$. The restriction of π to $\mathcal{F}$ is definable in $\mathbb{R}_{an}$. Let V be any algebraic subvariety of A and let $\widetilde{V} = \pi^{-1}(V) \cap \mathcal{F}$. Then $\widetilde{V}$ is definable in $\mathbb{R}_{an}$ and $Zar(\pi(\widetilde{V})) = V$.

However, when X is an unbounded real analytic manifold, we prove the following.

Theorem 1.2. *Let X be an unbounded real analytic manifold of $\mathbb{C}^n = \mathbb{R}^{2n}$ definable in an o-minimal structure which is an extension of $\mathbb{R}_{an}$.*

Let $V = Zar(\pi(X))$. For any point P of $\pi(X)$ there is a positive dimensional abelian subvariety B_P of A such that $P + B_P$ is contained in V.

In particular, V contains a Zariski dense set of positive dimensional weakly special subvarieties.

To investigate general definable sets X, we will also impose some mild restrictions on the o-minimal structure. Let $\mathcal{S}$ be an o-minimal structure over $\mathbb{R}$, containing $\mathbb{R}_{an}$ and whose definable sets admit an analytic stratification (as defined in [12], Chapter 3). This condition holds for most 'usual' o-minimal structures, for example $\mathbb{R}_{an}$ and $\mathbb{R}_{an,exp}$ (see [4]). We fix such a structure $\mathcal{S}$ and in what follows and by definable, we will mean 'definable in $\mathcal{S}$'.

Next we introduce the notion of *essential Zariski closure*. Let X be an unbounded definable set as before. For $R > 0$, let $B(0, R)$ be the open unit ball of centre 0 and radius R. The variation of the sets $\pi(X \cap B(0, R))$ when R varies is what we call an *o-minimal flow*. We show that for R large enough, the Zariski closure of the set $\pi(X \backslash (X \cap B(0, R)))$ is constant. We call this the *essential* Zariski closure of $\pi(X)$ and denote it by $Zaress(\pi(X))$.

For an abelian subvariety B of A, write $V_B \subset \mathbb{C}^n$ for the tangent space to B at the origin and p_B for the projection $\mathbb{C}^n \longrightarrow V_B$.

We prove the following:

Theorem 1.3. *Let X be an unbounded definable subset of $\mathbb{C}^n$. Let V be $Zaress(\pi(X))$.*

For each point P, in $\pi(X)$, there exists a positive dimensional abelian subvariety B_P of A such that $P + B_P$ is contained in V.

In particular, V contains a Zariski dense set of positive dimensional weakly special subvarieties.

We prove a characterisation of subvarieties of an abelian variety containing a Zariski dense set of weakly special subvarieties (see proposition 4.1). Let V be such a subvariety. Our proposition 4.1 shows that there exist abelian subvarieties B and B' of A such that $A = B + B'$ and $B \cap B'$ is finite, $V = B + V'$ where V' is a subvariety of B'.

We deduce the following.

Theorem 1.4. *Assume that X is a definable subset of $\mathbb{C}^n$ such that for all abelian subvarieties B of A, $p_B(X)$ is unbounded. Then components of $Zaress(\pi(X))$ are weakly special.*

The strategy of the proof of the theorem 1.2 relies on the theory of o-minimality and Pila-Wilkie counting theorem. Let X be as in the statement and V be the Zariski closure of $\pi(X)$. Using a suitable definable set and applying Pila-Wilkie theorem, we show that there exists a positive dimensional semi-algebraic set $W \subset \mathbb{C}^n = \mathbb{R}^{2n}$ such that $X + W$ is contained in $\pi^{-1}(V)$. Applying the Ax-Lindemann-Weierstrass theorem, we then show that for any P of $\pi(X)$, there exists a weakly special subvariety $P + B_P \subset V$.

Finally, we would like to point out one possible application of our theorem.

Recall the following theorem of Bloch-Ochiai (see Chapter 9 of [3]) which is proved using Nevanlinna theory.

Theorem 1.5. *Let A be an abelian variety and $f\colon \mathbb{C} \longrightarrow A$ be a non-constant holomorphic map. Then the Zariski closure of $f(\mathbb{C})$ is a translate of an abelian subvariety.*

Theorem 1.4 imples some cases of theorem 1.5.

Consider for example $A = \mathbb{C}^n/\Lambda$ (where Λ is a lattice such that A is a simple abelian variety) and $f\colon \mathbb{C} \longrightarrow A$ given by $f(z) = (z, \dots, z, e^z, \dots, e^z) \mathrm{mod} \Lambda$ with s factors of z and r times of e^z with $r + s = n$. Then consider the set $X \subset \mathbb{C}^n$ given by

$$X = \{(x + iy, \dots, x + iy, e^x e^{iy}, \dots, e^x e^{iy}) : x \in \mathbb{R}, y \in [0, 2\pi]\}.$$

Clearly X is unbounded and definable in $\mathbb{R}_{an,exp}$ and its image in A is contained in $f(\mathbb{C})$. By theorem 1.4, the Zariski closure of $f(\mathbb{C})$ is A (since A is simple).

It is not however always possible to "extract" such a definable unbounded set X from $f(\mathbb{C})$ as the example of $(e^z, e^{iz}) \subset \mathbb{C}^2$ shows. Indeed, in this example, for any subset $Y \subset \mathbb{C}$ such that $f(Y)$ is definable, both the real and imaginary parts of $z \in Y$ must be bounded.

Another (counter)-example is the following. Define the iterated exponential function $exp_n(x)$ by $exp_1 = exp$ and $exp_n = exp \circ exp_{n-1}$. By Proposition 9.10 of [4], a definable function in $\mathbb{R}_{an,exp}$ is bounded by $exp_n(x^m)$ for some n, m. Therefore a graph of a function which 'grows faster' than any exp_n will not satisfy the assumptions of our theorems.

Note that it is a long-standing open problem whether there exists an o-minimal structure containing a "super-exponential" function.

We conclude this introduction with an open question in the spirit of [10]. It concerns the topological closure of $\pi(X)$ rather than Zariski closure. Recall from [10] that a *real* weakly special subvariety is defined to be a translate of a real subtorus of A (hence not necessarily algebraic).

Conjecture 1.6. *Let X be, as before, an unbounded definable real analytic manifold. We denote by $\overline{\pi(X)}$ the topological closure of $\pi(X)$.*

There exists a real analytic submanifold V of A containing a dense subset of real weakly special subvarieties such that

$$\overline{\pi(X)} = \pi(X) \cup V.$$

In section 4, we prove a characterisation of subvarieties of abelian varieties containing a Zariski dense subset of weakly special subvarieties, namely that such a subvariety is a union of weakly special ones. We believe this result and our argument to be of independent interest.

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2. Proof of theorem 1.2.

In this section we assume that X is an unbounded real analytic submanifold of $\mathbb{C}^n = \mathbb{R}^{2n}$ definable in some o-minimal structure which contains $\mathbb{R}_{an}$. Let V be the Zariski closure of $\pi(X)$ in A.

2.1. **A definable set and point counting.**

The contents of this section are essentially a reproduction of the arguments of Orr from Section 9 of [5] with slight adjustments.

In this section we define a certain definable set associated with X and, using Pila-Wilkie theorem, show that this set contains a positive dimensional semi-algebraic subset.

Choose a fundamental set $\mathcal{F}$ for the action of Λ on $\mathbb{C}^n$ such that $X \cap \mathcal{F}$ is non-empty. We choose $\mathcal{F}$ to be an open connected subset of $\mathbb{C}^n$ such that $\overline{\mathcal{F}}$ is compact and Λ-translates of $\overline{\mathcal{F}}$ cover $\mathbb{C}^n$. The set $\mathcal{F}$

is an 'open parallelepided'. Since $\mathcal{F}$ is an open subset of $\mathbb{C}^n$, we have that $\dim(X \cap \mathcal{F}) = \dim(X)$. Let $\widetilde{V}$ be $\mathcal{F} \cap \pi^{-1}V$. This is a definable set since the o-minimal structure contains $\mathbb{R}_{an}$ and π restricted to $\mathcal{F}$ is definable in $\mathbb{R}_{an}$.

Consider the definable set

$$\Sigma = \{x \in \mathbb{C}^n : \dim(X + x) \cap \widetilde{V} = \dim(X)\}.$$

We have the following lemma:

Lemma 2.1. *If $\lambda \in \Lambda$ and $X \cap (\mathcal{F} - \lambda) \neq \emptyset$, then $\lambda \in \Sigma$.*

Proof. From Λ-invariance of $\pi^{-1}V + \lambda = \pi^{-1}V$, we see that for λ as in the statement (in particular for $\lambda \in \Lambda$), $X + \lambda \subset \pi^{-1}V$.

It follows that

$$(X + \lambda) \cap \widetilde{V} = (X + \lambda) \cap \mathcal{F}.$$

As $\mathcal{F} - \lambda$ is an open subset of $\mathbb{C}^n$, we see that

$$\dim(X \cap (\mathcal{F} - \lambda)) = \dim(X) = \dim((X + \lambda) \cap \mathcal{F})$$

The conclusion follows. □

Fix a basis $\lambda_1, \ldots, \lambda_{2n}$ of Λ. Then $\Lambda \otimes \mathbb{Q}$ is identified with $\mathbb{Q}^{2n}$. We define the height of an element $\lambda = \sum a_i \lambda_i \in \Lambda$ ($a_i \in \mathbb{Z}$) as

$$H(\lambda) = \max(|a_1|, \ldots, |a_{2n}|).$$

This height thus coincides with the usual height on $\mathbb{Q}^n$.

Proposition 2.2. *There exists $T_0 \geq 0$ such that for all $T \geq T_0$,*

$$|\{x \in \Sigma \cap \Lambda : H(x) \leq T\}| \geq T/2.$$

Proof. This is essentially Lemma 9.1 of [5].

The first observation is that if x_1 and x_2 are two points of Λ such that $X \cap (\mathcal{F} - x_1)$ and $X \cap (\mathcal{F} - x_2)$ are both non-empty, then $\Sigma \cap \Lambda$ contains at least one point of height h for every h between $H(x_1)$ and $H(x_2)$.

Note that X is path-wise connected in the Euclidean topology. Let C be a path from a point in $X \cap (\mathcal{F} - x_2)$ to a point in $X \cap (\mathcal{F} - x_2)$.

When C crosses over from $\mathcal{F} - u_1$ to to an adjacent domain $\mathcal{F} - u_2$, the heights of u_1 and u_2 change by at most one.

It follows that for any h between $H(x_1)$ and $H(x_2)$, there is a $u \in \Lambda$ of height $\leq h$ such that $X \cap (\mathcal{F} - u)$ is not empty. This u belongs to $\Sigma \cap X$.

By assumption X is unbounded. Thus as x varies in Λ such that $X \cap \mathcal{F} - x$ is non-empty, $h(x)$ goes to infinity.

It follows that there is an h_0 such that for any $h > h_0$, $\Sigma \cap \Lambda$ contains at least one point of height h.

Take $T_0 = 2h_0$. □

Remark 2.3. *The referee has pointed out to us that Tsimerman, in* [11]*, has made a similar observation. Namely, that in a similar setting an unbounded analytic set should intersect 'a lot of fundamental domains'.*

We now use the following theorem of Pila and Wilkie ([6], Theorem 1.8).

For a definable subset $\Theta \subset \mathbb{R}^n$, we define Θ^{alg} to be the union of all positive dimensional semi-algebraic subsets contained in Θ. We define Θ^{tr} to be $\Theta \backslash \Theta^{alg}$.

Theorem 2.4 (Pila-Wilkie)**.** *Let Θ be a subset of $\mathbb{R}^n$ definable in an o-minimal structure. Let $\epsilon > 0$. There exists a constant $c = c(\Theta, \epsilon)$ such that for any $T \geq 0$,*

$$|\{x \in \Theta^{tr} \cap \mathbb{Q}^n : H(x) \leq T\}| \geq cT^{\epsilon}.$$

From Proposition 2.2 it now follows that $\Sigma^{alg} \cap \Lambda$ is not empty.

Let W be a connected positive dimensional semi-algebraic subset contained in Σ. For each w in W, $\dim((X+w) \cap \widetilde{V}) = \dim(X)$ and hence an analytic component of $(X+w) \cap \mathcal{F}$ is contained in $\pi^{-1}V$. By analytic continuation, we see that $X + w \subset \pi^{-1}V$. We have proved:

Proposition 2.5. *With the notations and assumptions of this section, there exists a positive dimensional semialgebraic subset W such that*

$$X + W \subset \pi^{-1}V.$$

2.2. **Final argument.** We use the following lemma whose proof can for example be found in [5], Lemma 8.1.

Lemma 2.6. *Let $\mathcal{Z}$ be a connected complex analytic subset of $\mathbb{C}^g$. Let $\mathcal{X}$ be a connected irreducible semialgebraic set contained in $\mathcal{Z}$. Then there is a complex algebraic variety $\mathcal{Y}$ such that $\mathcal{X} \subset \mathcal{Y} \subset \mathcal{Z}$.*

By proposition 2.5 and the above lemma, we see that for any $x \in X$, there exists a positive dimensional complex algebraic subset Y_x containing X and contained in $\pi^{-1}(V)$. By the abelian Ax-Lindemann-Weierstrass theorem 1.1, the Zariski closure of $\pi(Y_x)$ is a union of weakly special subvarieties of V. Therefore, V contains a subvariety of the form $P + B_P$ where $P = \pi(x)$ and B_P is a positive dimensional abelian subvariety of A. This finishes the proof of theorem 1.2.

3. Cell decomposition and essential closure.

In this section we consider an unbounded definable set $X \subset \mathbb{C}^n$. We refer to section 8 of [4] for the definition of a real analytic cell. What is relevant to us is that a real analytic cell in $\mathbb{R}^n$ is a definable real analytic submanifold, definable-analytically isomorphic to $\mathbb{R}^m$ for some $m \leq n$. By Theorem 8.9 of [4], there is a finite number of analytic cells $X_1, \dots, X_k$ such that X is a disjoint union of the X_k.

Proposition 3.1. *The essential closure $Zaress(\pi(X))$ is the union of $Zar(\pi(X_i))$ where X_is are the unbounded cells.*

Proof. We start with a lemma.

Lemma 3.2. *Let Z be a real analytic manifold in $\mathbb{C}^n$ and $U \subset Z$ an open subset.*

Then

$$Zar(\pi(U)) = Zar(\pi(Z))$$

In particular, if Z is an analytic unbounded submanifold of $\mathbb{C}^n$, then

$$Zaress(\pi(Z)) = Zar(\pi(Z))$$

Proof. One inclusion is obvious.

Write $Zar(\pi(U)) \subset \mathbb{P}^m$ for some m and let $s \in H^0(\mathbb{P}^m, \mathcal{O}(l))$ for $l \geq 1$ such that s is zero on $\pi(U)$. Then $s \circ \pi$ is zero on U and by analytic continuation $s \circ \pi$ is zero on Z. It follows that s is zero on $\pi(Z)$, hence $Zar(\pi(Z)) \subset Zar(\pi(U))$. □

Let $X = X_1 \coprod \dots \coprod X_k$ be a cell decomposition of X. For R large enough, $X \cap B(0, R)$ contains the union of all the bounded cells in the above decomposition.

We have

$$Zaress(\pi(X)) = \bigcup_{\{i : X_i \text{unbounded}\}} Zaress(\pi(X_i)).$$

By Lemma 3.2, for an unbounded cell X_i,

$$Zaress(\pi(X_i)) = Zar(\pi(X_i)).$$

The result follows.

□

4. Characterisation of subvarieties containing a dense set of weakly special subvarieties.

In this section we prove a proposition which we believe to be of independent interest.

Let A be an abelian variety and V a subvariety of A. Define the stabiliser of V as

$$\mathrm{Stab}(V) = \{P \in A : P + V = V\}.$$

Recall that for an abelian subvariety B of A, there exists an abelian subvariety B' such that $A = B + B'$ and $B \cap B'$ is finite. We always refer to B and B' as above.

Proposition 4.1. *Let V be an irreducible subvariety of A.*

(1) *Assume* $\dim \mathrm{Stab}(V) > 0$.
Then there exists abelian subvarieties B and B' of A such that $A = B + B'$ and $V = B + V'$ where V' is a subvariety of B'.

(2) *Assume that* $\mathrm{Stab}(V)$ *is finite. Then the set of positive dimensional weakly special subvarieties contained in V is not Zariski dense.*

(3) *Assume again that* $\mathrm{Stab}(V)$ *is finite. Let Σ be the set of all positive dimensional weakly special subvarieties contained in V.*
For an abelian subvariety $B \subset A$, denote by B' an abelian subvariety such that $A = B + B'$.
There exists a finite set $B_1, \dots, B_r$ of abelian subvarieties of A and $W_1, \dots, W_r$ of subvarieties of B_i' such that

$$Zar(\Sigma) = \bigcup_{i=1}^{r} B_i + W_i.$$

Proof. Assume $\dim \mathrm{Stab}(V) > 0$ and let B be the neutral component of $\mathrm{Stab}(V)$.

Let B' be an abelian subvariety such that $A = B + B'$ and let $\psi\colon A \longrightarrow A/B$ be the quotient. Let V' be $\psi|_{B'}^{-1}(\psi(V))$. Then

$$V = \{B + x : x \in V\} = \{B + x : x \in V'\} = B + V'.$$

This proves (1).

We will now prove (2). Assume that $\mathrm{Stab}(V)$ is finite. We start by reducing to the case where $\mathrm{Stab}(V) = \{0\}$. Let $A' = A/\mathrm{Stab}(V)$ and let $\phi\colon A \longrightarrow A'$ be the quotient map and let $V' = \phi(V)$. Note that $\phi^{-1}(V') = V + \mathrm{Stab}(V) = V$. We claim that $\mathrm{Stab}(V') = \{0\}$. Let $P \in \mathrm{Stab}(V')$ and $Q \in \phi^{-1}(P)$. We have

$$\phi(Q + V) = P + V' = V'$$

It follows that $Q+V \subset \phi^{-1}(V') = V$ and for dimension reasons $Q+V = V$. Hence $Q \in \mathrm{Stab}(V)$ and $P = \phi(Q) = 0$.

As the conclusion of (2) holds for V if and only if it holds for V', we may therefore assume that $\mathrm{Stab}(V) = \{0\}$.

For $m > 1$, consider the map

$$\phi_m \colon V^m \longrightarrow A^{m-1}$$

defined by

$$\phi_m(x_1, \ldots, x_m) = (x_1 - x_2, \ldots, x_m - x_{m-1}).$$

By [13], Lemma 3.1, there exists $m > 1$ such that the map ϕ_m is a generic embedding.

Let $P + B$ be a positive dimensional weakly special subvariety contained in V. Then $\phi_m((P + B)^m) = B^{m-1}$. The map ϕ_m is therefore not injective on $(P+B)^m$. Therefore V can not contain a Zariski dense set of positive dimensional subvarieties of the form $P + B$. This proves (2).

Let us now prove (3). Let Σ as in the statement, the set of all positive dimensional weakly special subvarieties contained in V and let W be a component of $Zar(\Sigma)$. Then W contains a Zariski dense set of weakly special subvarieties and by (2), $\mathrm{Stab}(W)$ is positive dimensional. It follows from (1) that $W = B + W'$ where B is an abelian subvariety of A and W' a subvariety of B'. Since $Zar(\Sigma)$ has finitely many components, the conclusion of (3) follows. □

Remark 4.2. *The geometric aspect of Lang's conjecture predicts that given a variety of general type V, the union of subvarieties, not of general type, is not Zariski dense. It is a known fact that a subvariety V of an abelian variety is of general type if and only if* $\mathrm{Stab}(V)$ *is finite. Therefore, our proposition 4.1 implies the geometric Lang's conjecture for subvarieties of abelian varieties.*

Remark 4.3. *This proposition is an abelian analogue of the result of the first author (see* [9]*) in the hyperbolic case which is proved by completely different methods.*

5. Proof theorems 1.3 and 1.4.

In this section we deduce theorems 1.3 and 1.4 from the preceding results.

Let A and X be as in the assumptions of Theorem 1.3. Let V be a component of the essential Zariski closure of $\pi(X)$.

In section 3 we have seen that $Zaress(\pi(X))$ is a finite union of Zariski closures of sets of the form $\pi(Y)$ where Y is an unbounded definable real analytic submanifold of $\mathbb{C}^n$. Therefore, the conclusion of theorem 1.3 follows from theorem 1.2.

Let now X be as in 1.4. By theorem 1.3, $V = Zaress(X)$ contains a Zariski dense set of positive dimensional weakly special subvarieties.

From proposition 4.1, we deduce that V is of the form $V = B + V'$ where B is a positive dimensional abelian subvariety of A and V' is a subvariety of B'. Reiterating the argument with B' and V', we conclude that components of V are weakly special.

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